

THE INFLUENCE OF SECONDARY EFFECTS ON BEAMS ON
ELASTIC FOUNDATIONS AND VIBRATING BEAMS

by

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II. SYMBOLS

<u>Symbol</u>	<u>Definitions</u>	<u>Dimensions</u>
x, y, z	Rectangular coordinates	L
u, v, w	Components of displacements	L
b	Width of beam	L
h	Height of beam	L
q_1	Pressure on upper surface of beam	M/LT ²
q_2	Pressure on lower surface of beam	M/LT ²
$\sigma_x, \sigma_y, \sigma_z$	Normal components of stress	M/LT ²
$\tau_{xy}, \tau_{xz}, \tau_{yz}$	Shearing-stress components	M/LT ²
$\epsilon_x, \epsilon_y, \epsilon_z$	Unit elongations	
$\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$	Shearing-strain components	
k	Modulus of foundation	M/L ² T ²
M_x	Bending moment	ML ² /T ²
V_x	Total shear	ML/T ²
P_x	Total load	ML/T ²
α	Rotation of elements	
ν	Poisson's ratio	
G	Modulus of rigidity	M/LT ²
E	Modulus of elasticity	M/LT ²
ρ	Mass per unit volume	M/L ³

III. INTRODUCTION

The usual practice in many engineering problems is to neglect secondary effects. However, if secondary effects are to be neglected safely, a knowledge is required of their influence on the problem. In many instances, neglect of secondary effects introduces serious errors. The purpose of this thesis is to investigate the influence of secondary effects on beams on elastic foundations and vibrating beams. The secondary effects that are considered for beams resting on elastic foundations are shear and normal pressure; while the effects of shear, rotatory inertia, and lateral inertia are considered for vibrating beams.

The well-known classical theory^(1,2) for beams on elastic foundations is based on the assumption that the moment in the beam is proportional to the curvature of the neutral axis, and no effects of shear or normal pressure are included. Recently, a new theory was presented by Frederick⁽³⁾ which included the effects of shear and normal pressure. This theory is derived using a variational principle and the equations of linear elasticity. In the first part of this thesis the equations of Frederick are derived by considering only the equations of linear elasticity. The new theory is then applied to (a) a

semi-infinite beam with a moment on the end, and to (b) a finite beam with clamped ends deflected by equal loads on its ends. The deflection, moment, and shear for each beam is compared with the results obtained from the classical theory.

The classical results for a laterally vibrating beam with no secondary effects were obtained by Rayleigh⁽⁴⁾. An equation including shear and rotatory inertia was deduced by Timoshenko⁽⁵⁾. Other work has been done on this problem by Davies⁽¹²⁾, Jacobsen⁽¹³⁾, Goens⁽¹⁴⁾, and Kruszewski⁽⁷⁾, in which Timoshenko's equations were applied to particular problems. Traill-Nash and Collar⁽⁸⁾ derived the differential equation for a vibrating beam in a manner similar to Timoshenko, and applied it to beams with all the usual end conditions. One matter of unusual interest is that, at sufficiently high frequency, a complete new spectrum of natural frequencies appears when both shear and rotatory inertia effects are taken into account.

In the second part of this thesis an equation is derived for a freely vibrating beam which includes the effects of shear, rotatory inertia, and lateral inertia. This equation is deduced using the equations of motion in a manner similar to the procedure followed in the first part for a beam on an elastic foundation. A comparison is then made to show the influence of secondary

effects by considering a vertical wave in an infinite beam. The new spectrum of frequencies appears when shear and rotatory inertia or shear, rotatory inertia, and lateral inertia are included.

IV. REVIEW OF LITERATURE

The classical theory for beams resting on elastic foundations is given by Timoshenko⁽¹⁾ and Hetenyi⁽²⁾, but neither considers the influence of secondary effects. Frederick⁽³⁾ derives an equation from a variational principle and the equations of linear elasticity without the simplifying assumptions of zero shear deformation and zero normal pressure. The theory is then applied to an infinite beam resting on an elastic foundation with a single concentrated load, and the deflection and moment curves are compared with the classical theory. The theory indicates that the moment is always smaller than the one calculated by the classical theory, and the classical theory can be 100 per cent in error.

The classical theory for a vibrating beam was deduced by Rayleigh⁽⁵⁾; and although rotatory inertia was considered, the classical equation was principally studied. Timoshenko⁽⁶⁾ derives an equation for a laterally vibrating bar using the assumption that moment is proportional to the slope of the deflection curve, and then corrections are made to take into account the effects of shear and rotatory inertia. The correction for rotatory inertia and shear together is

approximately two per cent when the wave length is ten times longer than the depth of the beam.

Using the equation derived by Timoshenko, Davies⁽¹²⁾, Jacobsen⁽¹³⁾, Goens⁽¹⁴⁾, and Kruszewski⁽⁷⁾ apply it to particular problems. Davies applies it to the case of a loaded fixed-free bar, Jacobsen neglects rotatory inertia effects and studies vibration problems of buildings, and Goens investigates the vibration of a free-free bar. Kruszewski studies the simultaneous effects of rotatory inertia and shear on a cantilever beam and a free-free beam (antisymmetrical and symmetrical modes). Graphs of the ratio of natural frequency to natural frequency of beam excluding secondary effects versus coefficient of shear rigidity are given for different values of a coefficient of rotatory inertia.

Recently, Traill-Nash and Collar⁽⁸⁾ made a more complete study of the problem. The differential equation of the vibrating beam is derived in a method similar to Timoshenko's, and then the equation is put into a dimensionless form. Appropriate frequency equations are derived for all the usual end conditions; the modified equations are given when either or both the effects of shear and rotatory inertia are neglected. One matter of unusual interest is that, at sufficiently high frequencies, a complete new spectrum of natural frequencies appears when both shear and

rotatory inertia effects are taken into account. This appears to be the first time this spectrum has been noticed. Also, experimental results are compared with calculated natural frequencies for a free-free beam when the following effects are considered: (a) bending (B) only, (b) bending and shear (B + S), and (c) bending, shear, and rotatory inertia (B + S + RI). When the experimental and calculated frequency are compared, the mean error for (a) (B) is 21 per cent for fundamental and 112 per cent for first overtone, (b) (B + S) is 2 per cent for fundamental and 3 per cent for first overtone, and (c) (B + S + RI) is 1 per cent for fundamental and first overtone. It is concluded by Traill-Nash and Collar that shear effects should always be included in calculations of frequency of vibration.

V. INVESTIGATION

A. Influence of Secondary Effects on Beams on Elastic Foundations

1. Theory

Using the equations of linear elasticity, an approximate solution is derived for a rectangular beam resting on an elastic foundation. The displacements (u , v , w) and the stress (σ_x) are expanded by a Taylor's series in the z direction with second order terms neglected. Also, the usual Winkler-Zimmermann hypothesis for beams on elastic foundations is used. In order to change the equations of elasticity which are in terms of stresses and strains into equations containing only stress resultants, the equations of elasticity are integrated over the depth of the beam. Five equations are deduced which are solved simultaneously to give a differential equation for the deflection of the beam.

Consider a uniform beam shown in Figure 1(a) and an element of the beam, Figure 1(b). The positive direction for coordinates, normal pressure, shear, and moment are indicated by arrows.

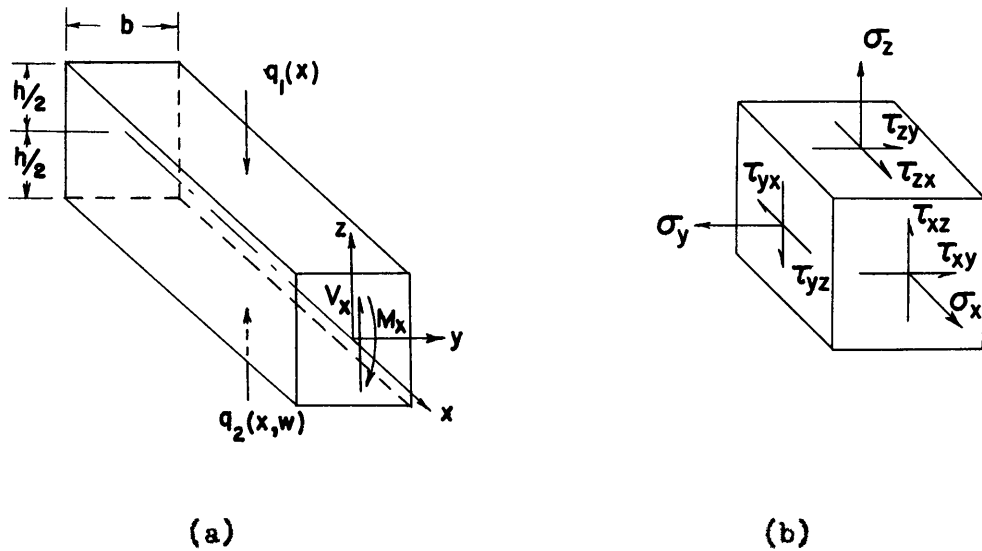


Figure 1
Notation and Coordinate System

The following assumptions were made

$$\sigma_x = \bar{\sigma}_x(x, y) + z \bar{\sigma}_{x1}(x, y) \quad (1a)$$

$$\tau_{xy} = 0 \quad (1b)$$

$$\sigma_y = 0 \quad (1c)$$

$$u = \bar{u}(x, y) + z \bar{u}_1(x, y) \quad (1d)$$

$$v = \bar{v}(x, y) + z \bar{v}_1(x, y) \quad (1e)$$

$$w = \bar{w}(x, y) + z \bar{w}_1(x, y) \quad (1f)$$

$$q_2 = -kw(x, -\frac{h}{2}) = -kw_B \quad (1g)$$

where (1a), (1d), (1e), and (1f) can be obtained by a Taylor's expansion of true function of x , y , and z in the z -direction

and neglecting terms of order two and higher in z . Assumption (1g) is the usual assumption that the reaction force of the foundation is proportional at every point to the deflection of the beam at that point.

The usual stress resultants

$$M_x = b \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x z \, dz \quad (2a)$$

$$V_x = b \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} \, dz \quad (2b)$$

$$P_x = b \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \, dz \quad (2c)$$

and the boundary conditions

$$\sigma_z \left(x, \frac{h}{2} \right) = -q_1(x) \quad (3a)$$

$$\sigma_z \left(x, -\frac{h}{2} \right) = -q_2(x,w) \quad (3b)$$

$$\tau_{xz} \left(x, \pm \frac{h}{2} \right) = 0 \quad (3c)$$

$$\tau_{yz} \left(x, \pm \frac{h}{2} \right) = 0 \quad (3d)$$

are now considered with the equilibrium equations from the theory of elasticity with no body forces.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \quad (4a)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0 \quad (4b)$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \quad (4c)$$

Multiplying (4a) by z , integrating with respect to z over the depth of the beam, and satisfying boundary condition (3c); it follows that

$$\frac{\partial M_x}{\partial x} - V_x = 0 \quad (5)$$

From (4b), (1b), (1c), and (3d) it follows that

$$\tau_{yz} = 0 \quad (\text{everywhere}) \quad (6)$$

Integrating (4c) over the depth of the beam and using boundary conditions (3a) and (3b), it follows that

$$\frac{\partial V_x}{\partial x} + (q_2 - q_1)b = 0 \quad (7)$$

Using assumption (1a), integrating over the depth of the beam, and assuming there are no loads on the ends of the beam, the following result is obtained

$$\bar{\sigma}_x = 0 \quad (8)$$

Multiplying assumption (1a) by z and integrating with respect to z over the depth of the beam, the following is obtained

$$\sigma_x = \frac{12 M_x z}{bh^3} \quad (9)$$

which is the usual equation used in elementary theory.

The remaining stresses τ_{xz} and σ_z are determined from (4a), (3d), (5), and (4c), (3a), (3b), (7), respectively; by integrating with respect to z from a point in the beam to an outer surface. These stresses are

$$\tau_{xz} = \frac{3V_x}{2bh} \left[1 - \left(\frac{z}{h/2} \right)^2 \right] \quad (10)$$

$$\sigma_z = - \left(\frac{q_1 + q_2}{2} \right) + (q_2 - q_1) \left[\frac{3z}{2h} - \frac{2z^3}{h^3} \right] \quad (11)$$

Integrating (1d) and (1c) over the depth of the beam it follows that (the displacement resultants are taken as zero)

$$\bar{u} = 0 \quad (12)$$

$$\bar{v} = 0 \quad (13)$$

If (1d) is multiplied by z and integrated with respect to z over the depth of the beam; and since \bar{u}_1 equals rotation of elements, α , the following is obtained

$$\alpha = \frac{12}{h^3} \int_{-\frac{h}{2}}^{\frac{h}{2}} u z dz \quad (14)$$

If the stress-strain relations (15) and (16)

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)] \quad (15)$$

$$\epsilon_z = \frac{\partial w}{\partial z} = \frac{1}{E} [\sigma_z - \nu(\sigma_x + \sigma_y)] \quad (16)$$

are used with (9) and (11), multiplied by z , and integrated with respect to z over the depth of the beam, it follows that

$$\frac{dQ}{dx} = \frac{M_x}{EI} - \frac{6V}{5Eh} (q_2 - q_1) \quad (17)$$

$$w_T - w_B = -\frac{h}{2E} (q_1 + q_2) \quad (18)$$

Using (1f), (1g), and (18); multiplying (16) by

$$\left[z - \frac{z^3}{3\left(\frac{h}{2}\right)^2} \right]; \text{ substituting (9) and (11) into (16); and}$$

integrating with respect to z over the depth of the beam; it follows that

$$w_B = \frac{1}{\left(1 + \frac{13}{35} \frac{hk}{E}\right)} \left[\bar{w} + \frac{9}{70} \frac{h}{E} q_1 - \frac{6}{5} \frac{V}{E} \frac{M_x}{bh} \right] \quad (19)$$

The quantity $\left[z - \frac{z^3}{3\left(\frac{h}{2}\right)^2} \right]$ is used so that the energy is kept a

minimum (see reference 3).

It can be seen that shearing-strains γ_{xy} and γ_{yz} are zero using (1b) and (6) with (20a) and (20c), while γ_{xz} is the usual equation (20b).

$$\gamma_{xy} = \frac{1}{G} \tau_{xy} = 0 \quad (20a)$$

$$\gamma_{xz} = \frac{1}{G} \tau_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \quad (20b)$$

$$\gamma_{yz} = \frac{1}{G} \tau_{yz} = 0 \quad (20c)$$

Multiplying (20b) by $\left[1 - \left(\frac{z}{h/2}\right)^2\right]$, (again to keep energy a minimum) and integrating over the depth of the beam it follows that

$$a + \frac{d\bar{w}}{dx} = \frac{6}{5Gbh} V_x \quad (21)$$

Using assumptions (1g) and solving (5), (7), (17), (19), and (21) for \bar{w} , M_x , and V_x , the following are obtained

$$\begin{aligned} \frac{d^4 \bar{w}}{dx^4} - \left[\frac{kh^2 d}{5EIC} \right] \frac{d^2 \bar{w}}{dx^2} + \left[\frac{kb}{EIC} \right] \bar{w} \\ = \left[\frac{bh^2(2+\nu)}{10EIC} \left(\frac{hk}{2E} + 1 \right) \right] \frac{d^2 q_1}{dx^2} - \left[\frac{b}{EIC} \left(\frac{hk}{2E} + 1 \right) \right] q_1 \end{aligned} \quad (22)$$

$$M_x = \frac{1}{k} D_1 \left(1 + \frac{kh}{2E} \right) q_1 + D_1 \bar{w} + D_2 \frac{d^2 \bar{w}}{dx^2} \quad (23)$$

$$V_x = \frac{1}{k} D_1 \left(1 + \frac{kh}{2E} \right) \frac{dq_1}{dx} + D_1 \frac{d\bar{w}}{dx} + D_2 \frac{d^3 \bar{w}}{dx^3} \quad (24)$$

$$\text{where } C = 1 + \frac{13 kh}{35E}$$

$$K = C + \frac{3\nu(2 + \nu) kh}{25E}$$

$$D_1 = \frac{bh^2k}{10K} (2 + \nu)$$

$$D_2 = -\frac{EIC}{K}$$

The complimentary solution to (22) is

$$\begin{aligned} \bar{w}_{\text{comp}} = & C_1 e^{\bar{a}x} \cos \bar{b}x + C_2 e^{\bar{a}x} \sin \bar{b}x + C_3 e^{-\bar{a}x} \cos \bar{b}x \\ & + C_4 e^{-\bar{a}x} \sin \bar{b}x \end{aligned} \quad (25)$$

where the following notation is used

$$\cos \omega = \frac{m^2}{2}$$

$$\bar{a} = \frac{1}{L} \cos \frac{\omega}{2}$$

$$\bar{b} = \frac{1}{L} \sin \frac{\omega}{2}$$

$$L^4 = \frac{EIC}{kb}$$

$$m^2 = \frac{L^2 kh^2 b}{5EIC}$$

The general solution to (22) can be found by finding a particular solution for the loading function q_1 , and adding the complimentary solution (25).

2. Semi-infinite Beam Problem

One way to investigate the influence of secondary effects on a beam resting on an elastic foundation is to solve particular problems and make a comparison with the classical theory. Therefore, two problems are studied; the first is a semi-infinite beam with a moment on the end (Figure 2).

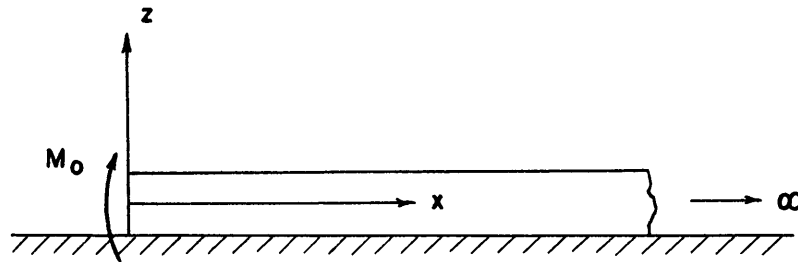


Figure 2

Semi-infinite Beam Problem

The boundary conditions are the following

$$\text{for } x = 0 \quad M_x = -M_0 \quad (26a)$$

$$V_x = 0 \quad (26b)$$

$$\text{for } x \rightarrow \infty \quad M_x = 0 \quad (26c)$$

$$V_x = 0 \quad (26d)$$

$$\bar{w} = 0 \quad (26e)$$

The solution to (22) is composed of the complimentary function and particular integral, but in this case the particular integral is zero. Therefore, equation (25) is the solution to this problem, where the constants remain to be determined. By boundary condition (26e), equation (25) reduces to

$$\bar{w} = C_3 e^{-\bar{a}x} \cos \bar{b}x + C_4 e^{-\bar{a}x} \sin \bar{b}x \quad (27)$$

Using (23) and (24) with boundary conditions (26a) and (26b), the constants C_3 and C_4 are determined. For $\nu = 1/4$ and $x = 0$, Equation (27) is plotted (Figure 3) using the dimensionless parameters $\frac{kh}{E}$ and $\frac{\bar{w} E b h}{M_0}$. The results obtained using the classical theory are also plotted in Figure 3.

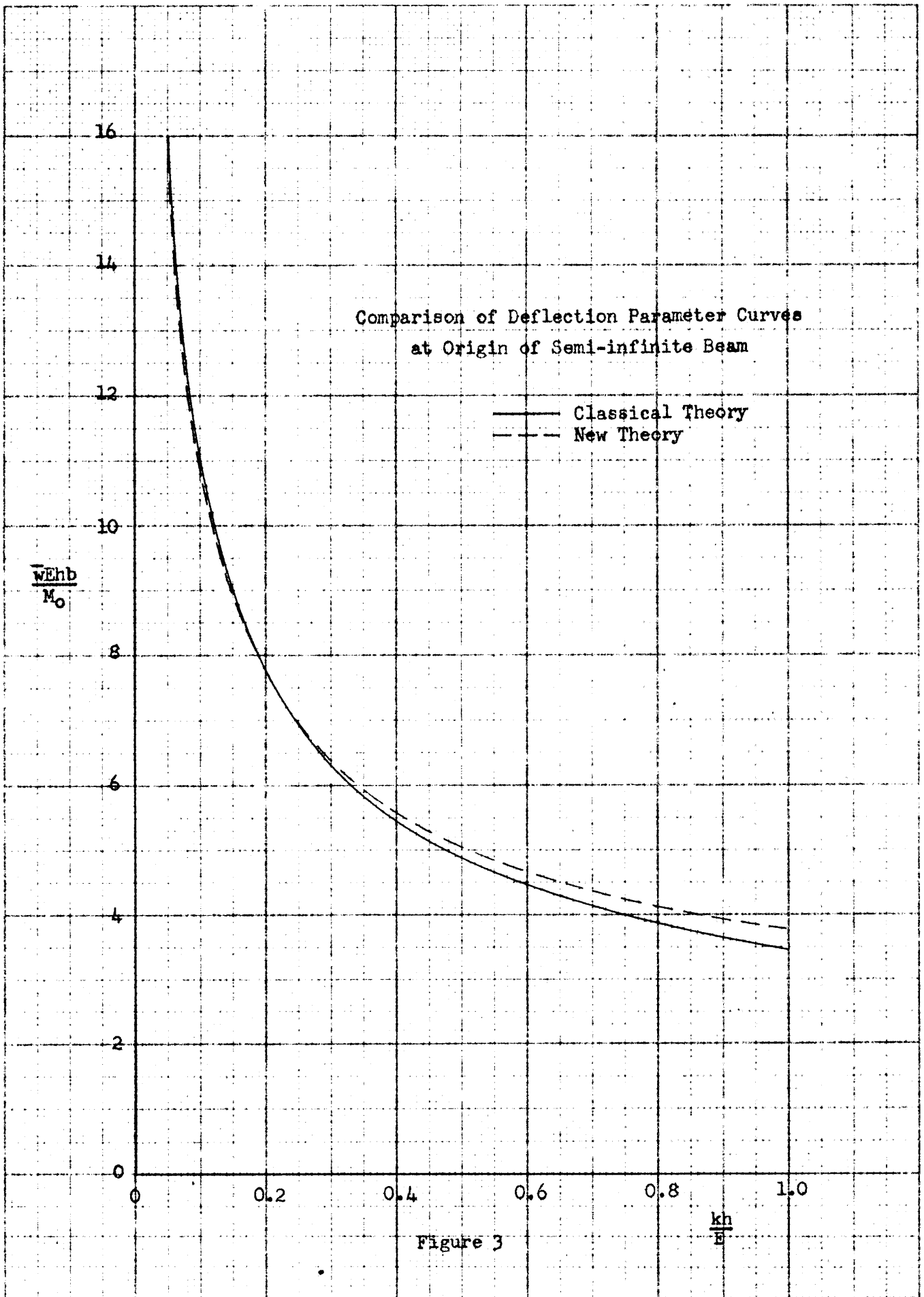


Figure 3

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With $\frac{kh}{E} = 1$, the deflection parameter, $\frac{\bar{w} Ehb}{M_0}$, along the beam in terms of $\frac{x}{h}$ can be determined from (27) and is plotted in Figure 4. Equation (23) is used to determine the moment parameter, M_x/M_0 along the beam (Figure 5); and the shear parameter, $\frac{V_x h}{M_0}$, is calculated from (24) (Figure 6). In Figures 4 to 6, the classical results are also given to show the influence of secondary effects.

Comparison of Deflection Parameter Curves Along
Semi-Infinite Beam Problem

— Classical Theory
- - - New Theory

$$\frac{kh}{2} = 1.0$$

$$\nu = \frac{1}{4}$$

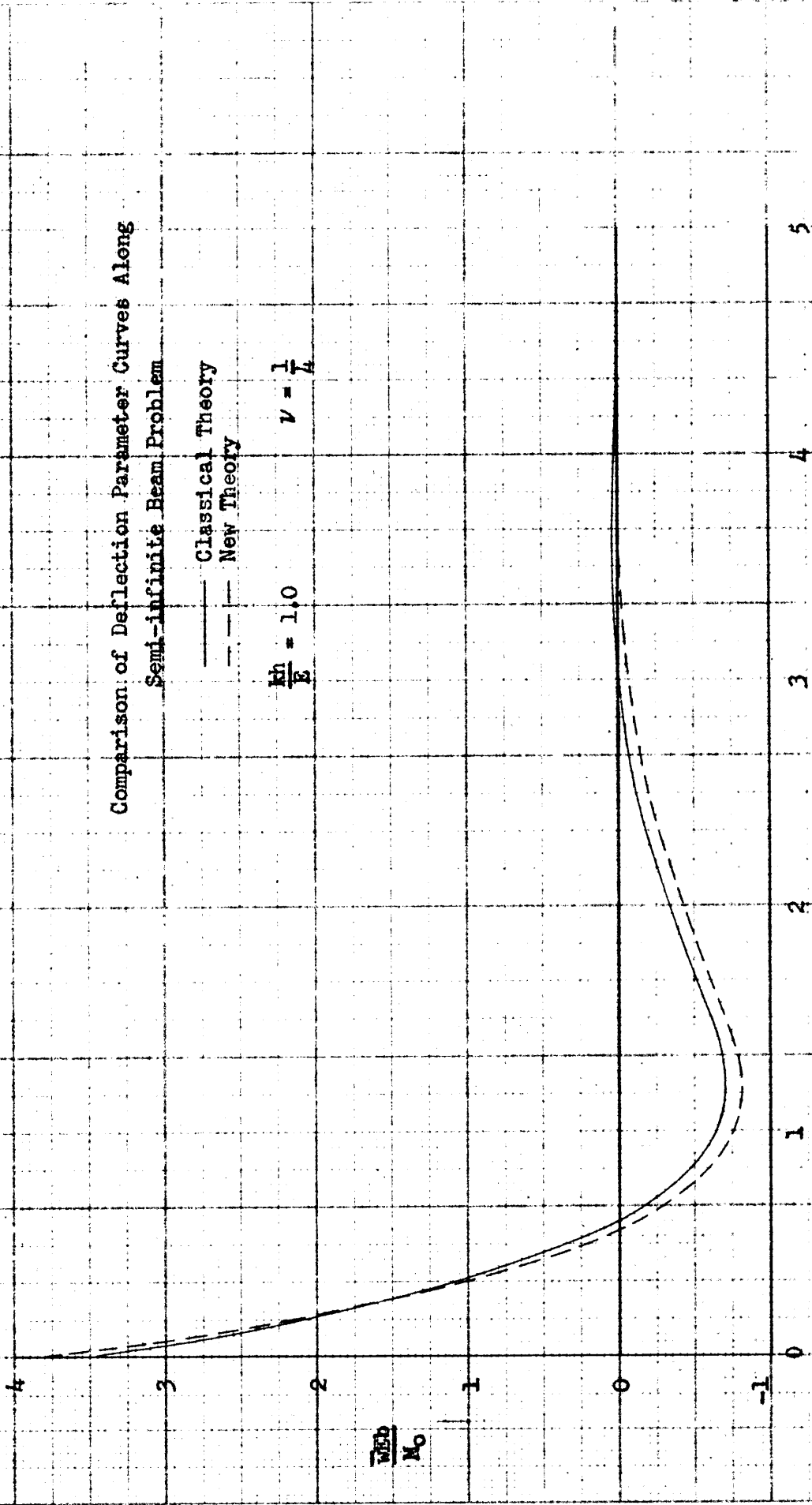


Figure 4

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Comparison of Moment Parameter Curves Along
Semi-Infinite Beam Problem

— Classical Theory

- - - New Theory

$$\frac{kh}{E} = 1.0$$

$$\nu = \frac{1}{4}$$

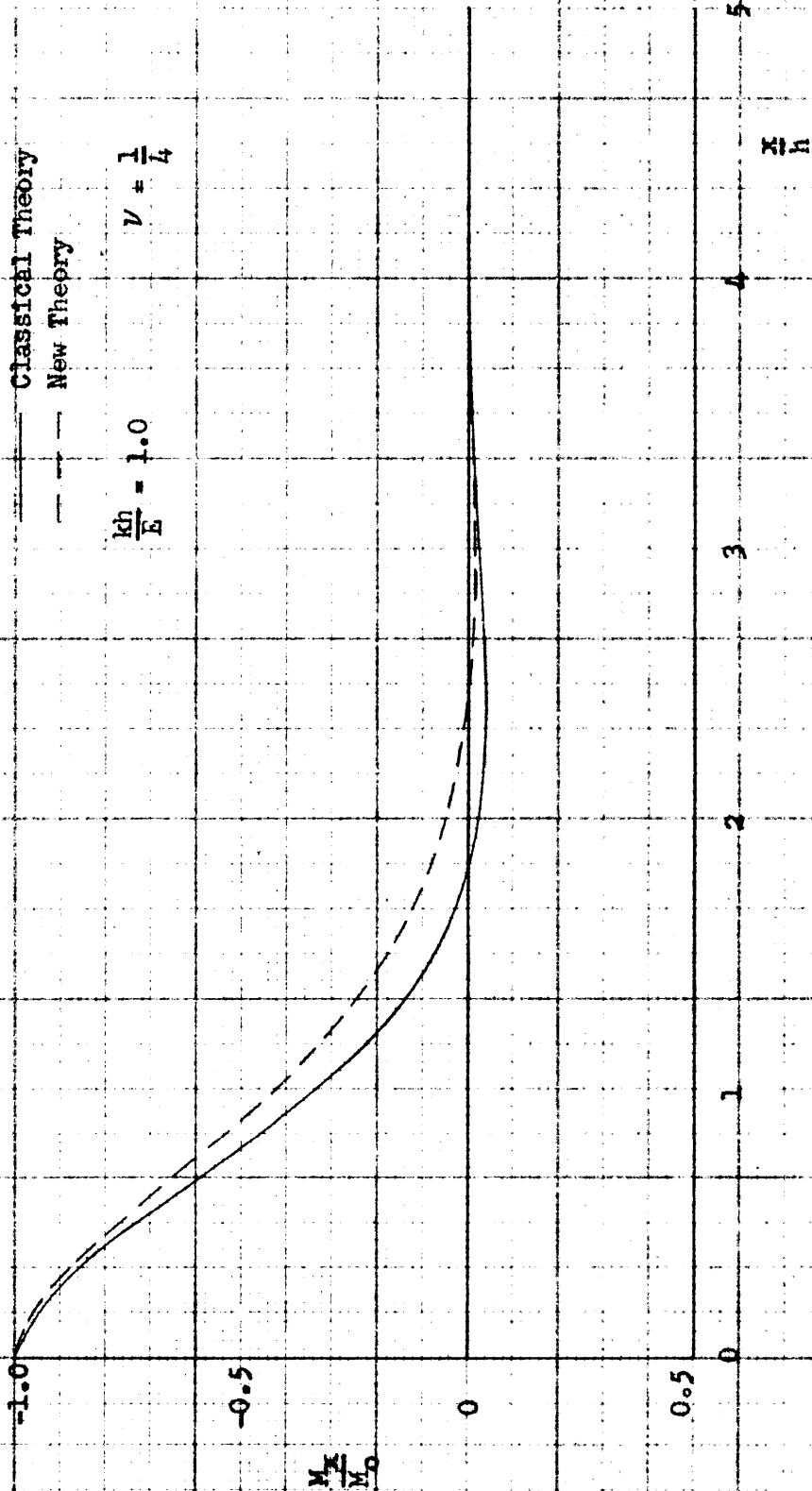


Figure 5

Comparison of Shear Parameter Curves Along
Semi-infinite Beam Problem

— Classical Theory
- - - New Theory

$$\frac{kh}{E} = 1.0$$

$$\nu = \frac{1}{4}$$

1.0

0.5

0

-0.5

$$\frac{V_x h}{N_0}$$

1

2

3

4

5

$$\frac{x}{h}$$

Figure 6

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20 X 20 REF. INCH

3. Finite Beam Problem

The second problem considered is a finite beam with clamped ends deflected by equal loads on its ends (Figure 7).

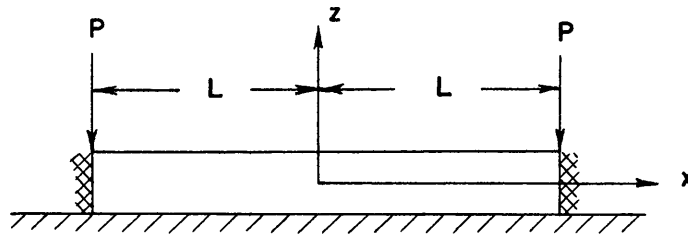


Figure 7

Finite Beam Problem

The boundary conditions are the following for $x = \pm L$

$$\bar{w} = \delta \tag{28a}$$

$$a = 0 \tag{28b}$$

In this case the particular integral is again zero; so equation (25) is the solution to this problem. Equation (25) can be written in terms of trigonometric and hyperbolic functions; and since the loading is symmetrical in this case, only the even terms will remain. Then, (25) can be written as

$$\bar{w} = A_1 \cos \bar{v}x \cosh \bar{a}x + A_4 \sin \bar{v}x \sinh \bar{a}x \quad (29)$$

Using the boundary conditions (28a) and (28b) with equations (21) and (24) the constants A_1 and A_2 are determined for

$$\frac{kh}{E} = 1.0 \quad (30a)$$

$$\nu = \frac{1}{4} \quad (30b)$$

$$\frac{L}{h} = 10 \quad (30c)$$

Then, (29) is put into dimensionless form having the parameters $\frac{x}{h}$ and $\frac{\bar{w}}{\delta}$ and is plotted in Figure 8. From (23) and (24) the moment parameter, $M_x/Ebh\delta$, and shear parameter, $V_x/Eb\delta$, are obtained and plotted in Figures 9 and 10, respectively. The results obtained using the classical theory are also shown in Figures 8 to 10.

The boundary conditions above are in terms of the deflection, δ , at the ends of the beam; but a relation can be determined between the load, P , and the deflection. The load is equal to the total shear, V_x , at the end of the beam; and using (24), a relation between the load and the deflection is easily obtained.

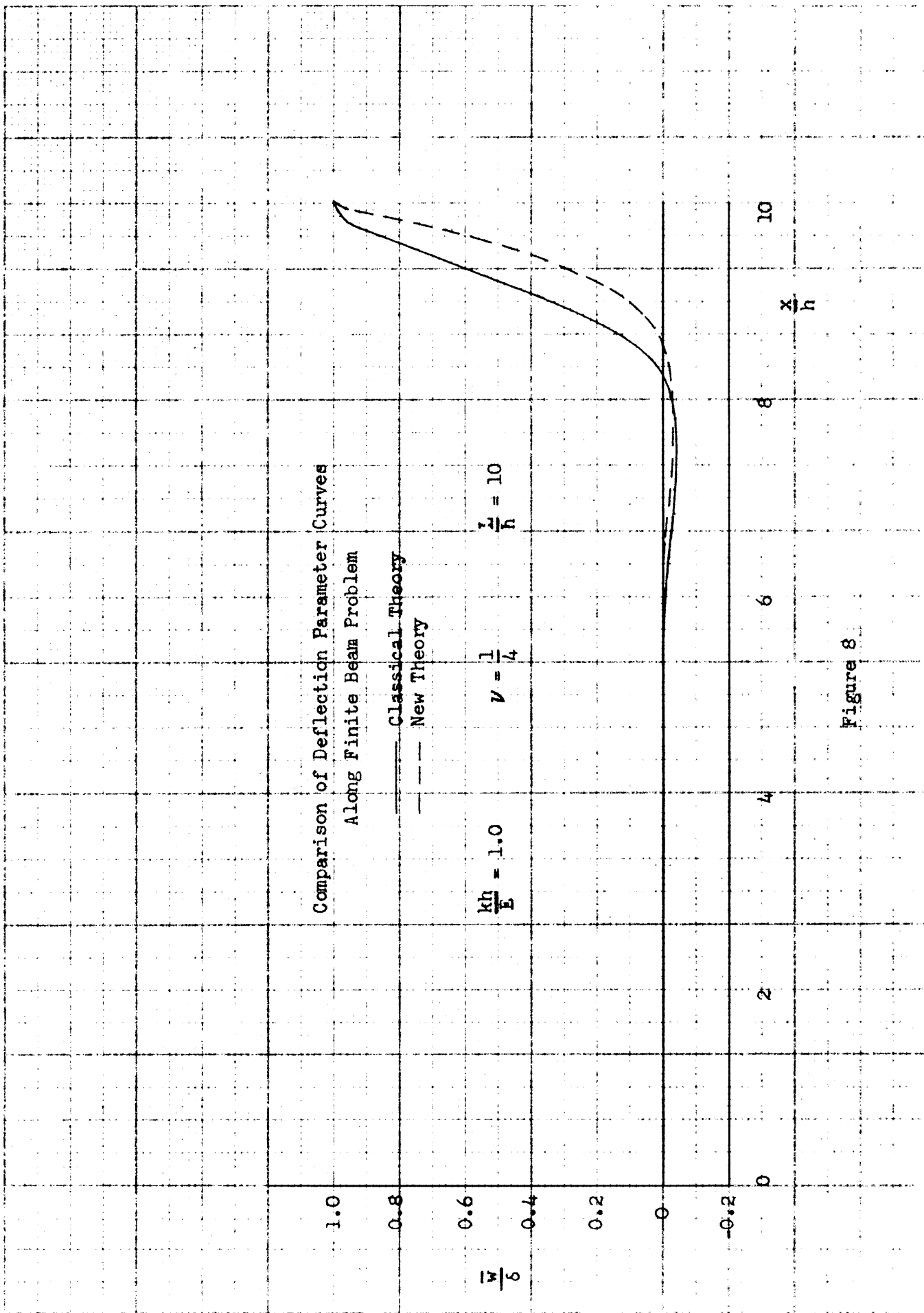


Figure 8

30 X 20 PER FOOT

Comparison of Moment Parameter Curves
Along Finite Beam Problem

— Classical Theory
- - - New Theory

$$\frac{kh}{E} = 1.0$$

$$v = \frac{1}{4}$$

$$\frac{l}{h} = 10$$

$$\frac{M_x}{Ebh^3}$$

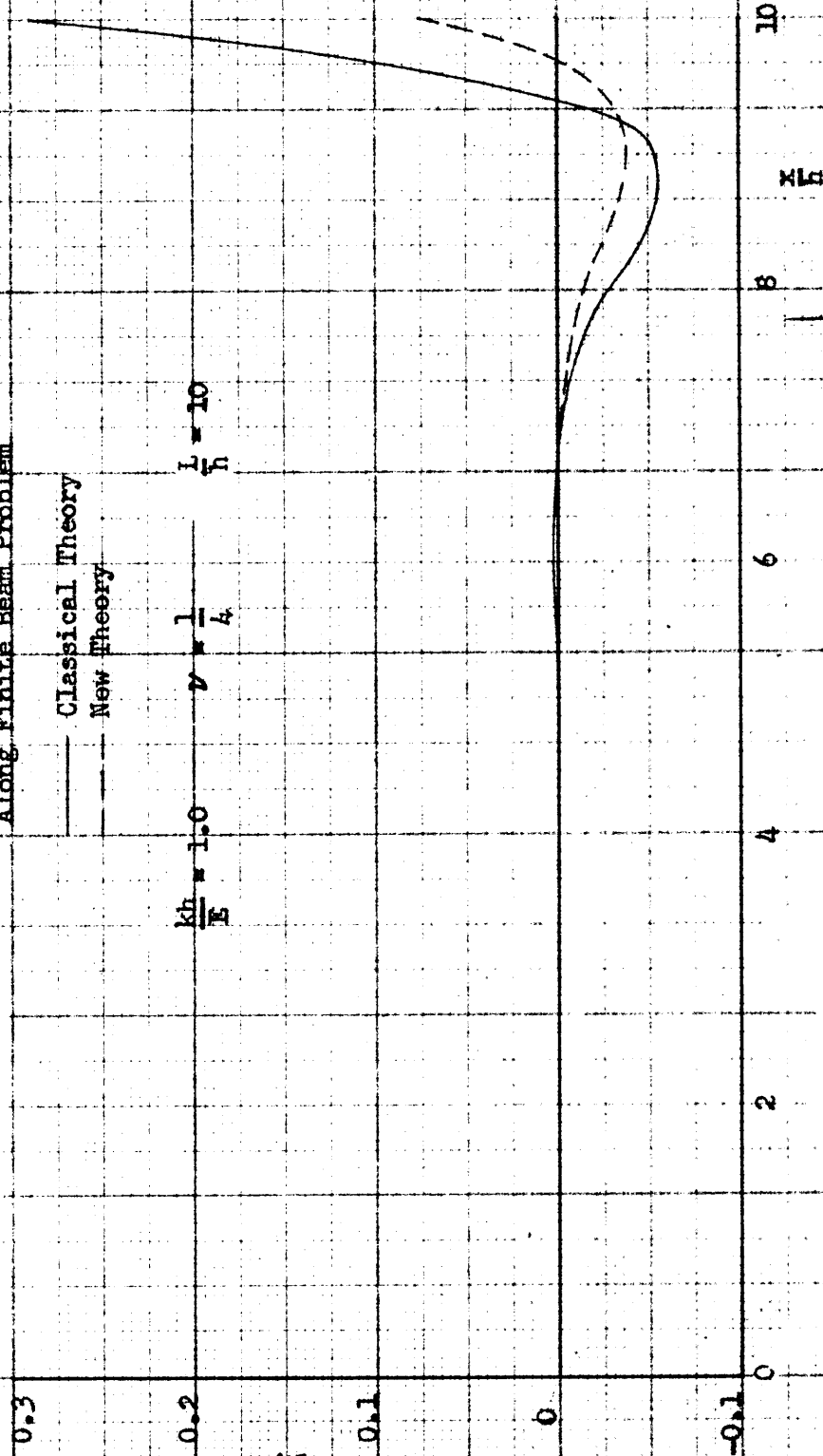


Figure 9

Comparison of Shear Parameter Curves
Along Finite Beam Problem

Classical Theory

--- New Theory

$\frac{kh}{E} = 1.0$

$\nu = \frac{1}{4}$

$\frac{b}{h} = 10$

$\frac{V_x}{Ed\delta}$

$\frac{x}{h}$

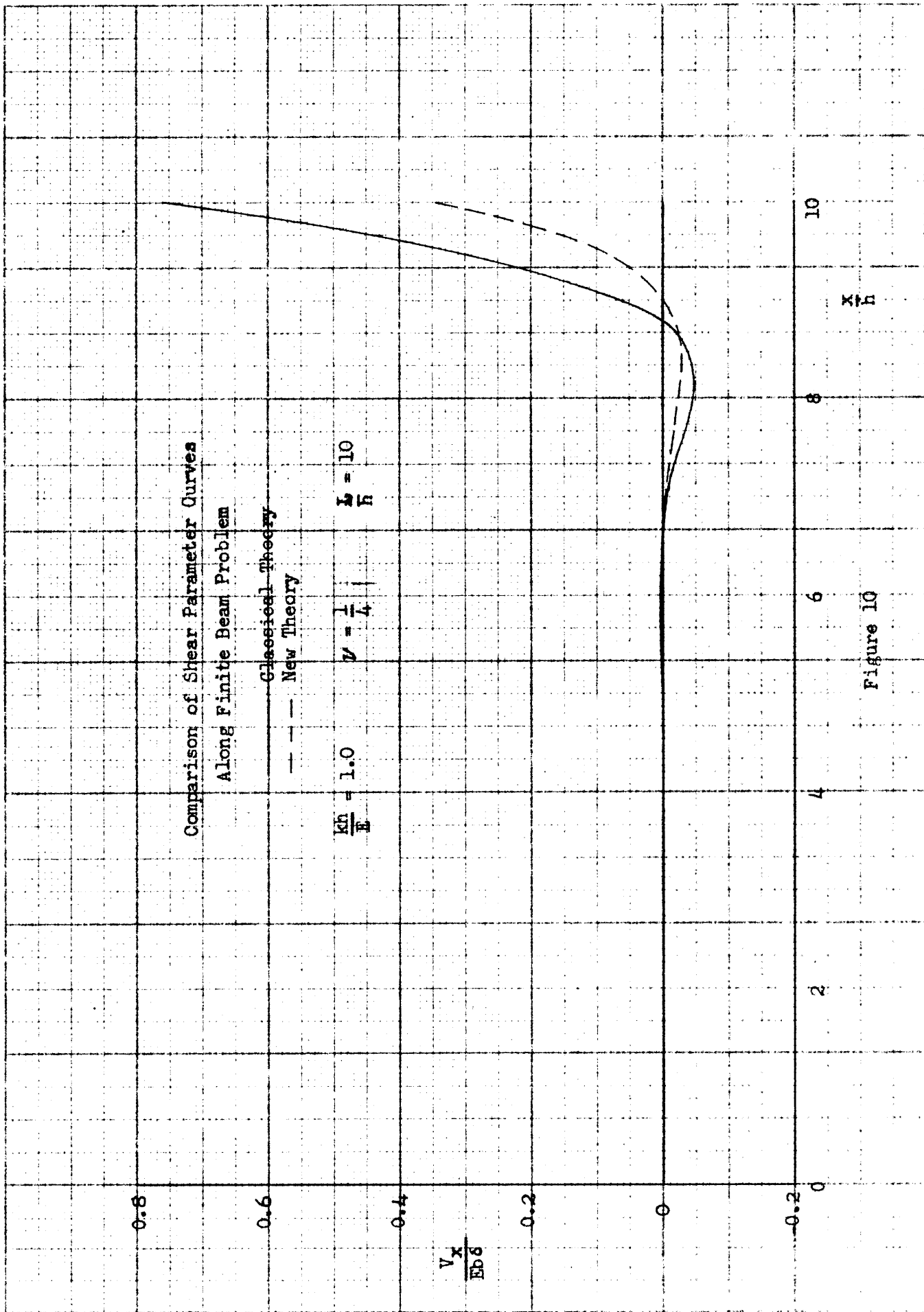


Figure 10

B. Influence of Secondary Effects on Vibrating Beams

1. Theory

Previously, the theory was developed for a static beam on an elastic foundation. Now, the theory will be developed for a freely-vibrating beam in a similar manner. As there is very little difference in the two derivations, only basic changes and final results will be given.

Since a freely-vibrating beam is being considered, the modulus of the foundation, k , is zero; and normal pressure, q_1 , is zero. All the other assumptions and boundary conditions remain the same. The equilibrium equations (4a) and (4c) are modified so that they become the equations of motion (31) and (32).

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = \rho \frac{\partial^2 u}{\partial t^2} \quad (31)$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = \rho \frac{\partial^2 w}{\partial t^2} \quad (32)$$

Following the same procedure as in the case of a beam on an elastic foundation, and taking into consideration the inertia terms in (31) and (32) the following are obtained

$$\frac{\partial M_x}{\partial x} = V_x + \rho \frac{h^3 b}{12} \frac{\partial^2 \alpha}{\partial t^2} \quad (33)$$

$$\frac{\partial V_x}{\partial x} = \rho b h \frac{\partial^2 \bar{w}}{\partial t^2} \quad (34)$$

$$\frac{\partial \alpha}{\partial x} = \frac{M_x}{EI} + \rho \frac{\nu}{5E} \frac{\partial^2 \bar{w}}{\partial t^2} + \rho \frac{\nu h^2}{10 E} \frac{\partial^3 \alpha}{\partial x \partial t^2} \quad (35)$$

$$\alpha = \frac{6}{5 G h b} V_x - \frac{\partial \bar{w}}{\partial x} \quad (36)$$

Solving (33), (34), (35), and (36) for \bar{w} the following is obtained

$$\begin{aligned} \frac{\partial^4 \bar{w}}{\partial x^4} + \rho \frac{b h}{EI} \frac{\partial^2 \bar{w}}{\partial t^2} + \frac{6 \rho^2}{5 E G} \frac{\partial^4 \bar{w}}{\partial t^4} + \left[-\frac{\rho}{E} + \frac{\rho \nu}{5 E} - \frac{6 \rho}{5 G} \right] \frac{\partial^4 \bar{w}}{\partial x^2 \partial t^2} \\ + \frac{3 \rho^2 \nu h^2}{25 E G} \frac{\partial^6 \bar{w}}{\partial x^2 \partial t^4} - \rho \frac{\nu h^2}{10 E} \frac{\partial^6 \bar{w}}{\partial x^4 \partial t^2} = 0 \end{aligned} \quad (37)$$

If $\nu \sigma_z$ is considered zero in equation (15), then the two sixth derivative terms and $\frac{\rho \nu}{5 E}$ term drop out of (37) which is the equation deduced by Timoshenko. The classical equation consists of the first two terms of (37) and these terms are due to bending. The term due to vertical deflection from shear is

$$-\frac{6\rho}{5G} \frac{\partial^4 \bar{w}}{\partial x^2 \partial t^2} \quad (38)$$

and the terms due to rotatory inertia are

$$\frac{6\rho^2}{5EG} \frac{\partial^4 \bar{w}}{\partial t^4} \quad (\text{from shearing}) \quad (39a)$$

$$-\frac{\rho}{E} \frac{\partial^4 \bar{w}}{\partial x^2 \partial t^2} \quad (\text{from bending}) \quad (39b)$$

The three other terms previously mentioned are effects of lateral inertia. The point that should be noticed here is that (39a) comes from the right side of (31). This clearly shows that (39a) is an effect of rotatory inertia due to shear rather than an effect of vertical shearing deflection which is not explained clearly by Timoshenko.

2. Comparison with Different Secondary Effects Included

In order to obtain a general comparison of the influence of secondary effects on a vibrating beam, a vertical wave (40) is considered in an infinite beam.

$$\bar{w} = A \sin \frac{2\pi}{\lambda} (x - ct) \quad (40)$$

Also, c_0 is defined as

$$c_0 = \sqrt{\frac{E}{\rho}} \quad (41)$$

which is the velocity of a longitudinal wave in a beam.

If bending terms alone are considered in (37), and equations (40) and (41) are substituted in (37); then it follows that for $\nu = \frac{1}{4}$,

$$\frac{h}{\lambda} = .5513 \frac{c}{c_0} \quad (42)$$

When the various secondary effects are included in (37), $\frac{h}{\lambda}$ can be written as some function of $\frac{c}{c_0}$. This is done for (a) bending and rotatory inertia (B + RI); (b) bending and shear (B + S); (c) bending, shear, and rotatory inertia (B + S + RI); and (d) bending, shear, rotatory inertia, and lateral inertia

(B + S + RI + LI). The results are plotted in Figure (11). The exact solution (plane strain) is also shown in Figure (11). This result was obtained by Lamb⁽¹⁰⁾ and used by Mindlin⁽⁹⁾.

It should be noticed that the second spectrum appears when either (B + S + RI) or (B + S + RI + LI) is considered. These two modes take place with different angles of rotation.

PROPAGATION OF WAVES IN INFINITE BEAM WITH VARIOUS SECONDARY EFFECTS INCLUDED

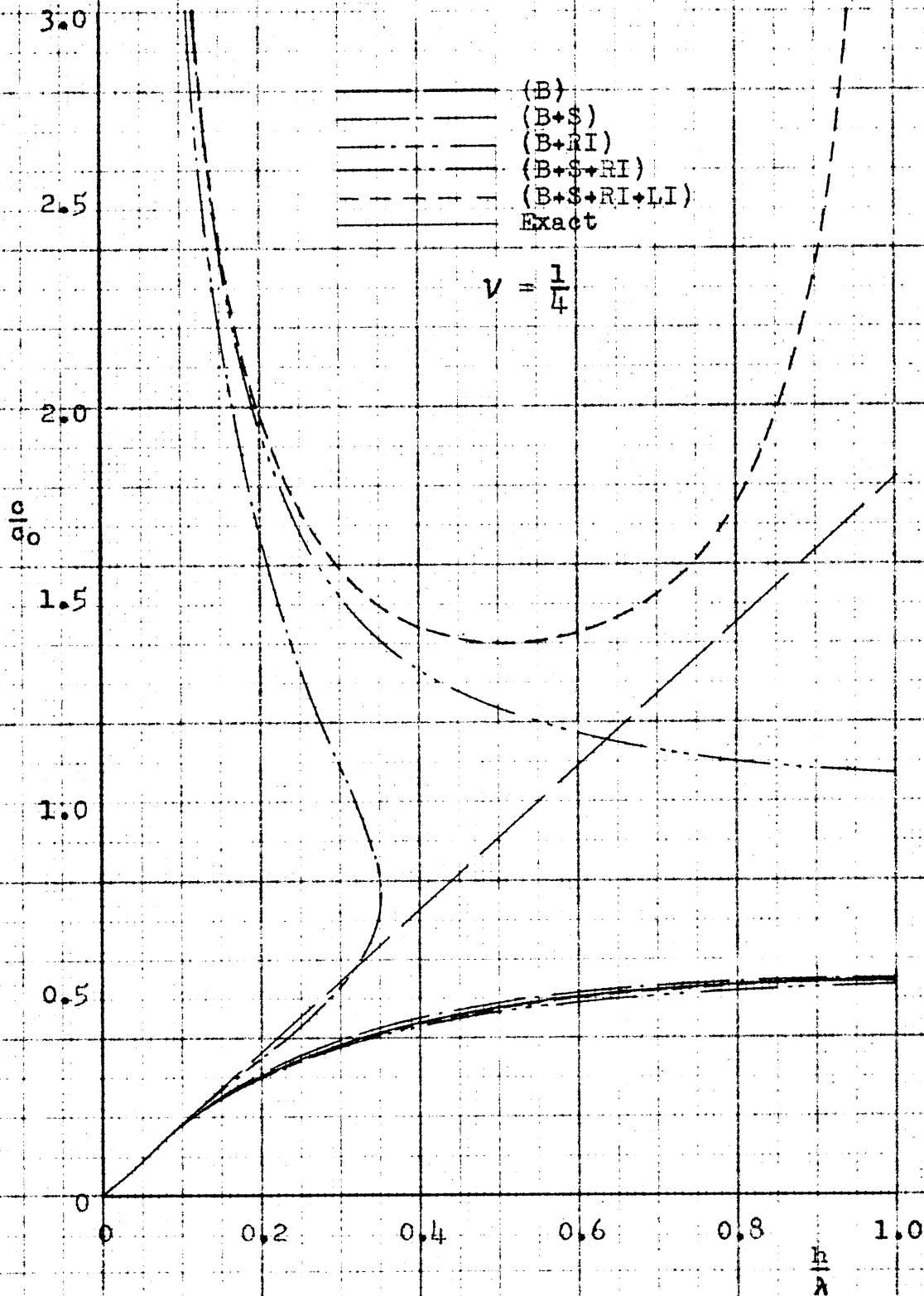


FIGURE 11

VI. DISCUSSION OF RESULTS AND CONCLUSIONS

The purpose of this thesis has been to investigate the influence of secondary effects for beams resting on elastic foundations and vibrating beams. Two problems have been presented for beams on elastic foundations and a general comparison has been made for an infinite vibrating beam.

For the semi-infinite beam on an elastic foundation, the greatest effects of shear and normal pressure occur when $\frac{kh}{E}$ is large (Figure 3). The maximum deflection parameter (Figure 4) is larger by the new theory, and the maximum moment parameters (Figure 5) are the same by the two theories due to boundary conditions. The maximum shear parameter (Figure 6) by the new theory is smaller than that by the classical theory.

For the finite beam problem, the maximum deflection parameter (Figure 8) occurs at the ends of the beam and is the same by both theories due to boundary conditions. The maximum moment (Figure 9) and shear (Figure 10) parameters by the classical theory are larger than those calculated by the new theory.

It is concluded that shear and normal pressure should be included when the height of the beam or the modulus of the foundation is large. For the two problems presented here and

for the problem by Frederick⁽³⁾, it is seen that the maximum deflection parameter is larger and the maximum moment and shear parameters are smaller than those given by the classical theory (boundary conditions may cause certain parameters to be equal). It should also be noticed that the classical theory gives a "safe" answer for stress in these examples.

The influence of secondary effects on a vibrating beam (Figure 11) are now considered. For the lower spectrum the results are nearly the same as the exact theory when the following secondary effects are included: (a) shear, (b) shear and rotatory inertia, and (c) shear, rotatory inertia, and lateral inertia. For other cases, the results agree with the exact theory for small values of $\frac{h}{\lambda}$ only. When $\frac{h}{\lambda}$ is large for a vibrating beam problem, effects of shear should be included. The other secondary effects would make the problem more complex, yet add little to improve the results. The lateral inertia terms might become important for a vibrating beam with large normal pressure, such as a beam on a foundation.

The upper spectrum appears only for (B + S + RI) and (B + S + RI + LI). For the lower spectrum, the beam vibrates mostly in shear for large values of $\frac{h}{\lambda}$ while for small values of $\frac{h}{\lambda}$ the beam is most in bending. One physical explanation

of the upper spectrum⁽⁸⁾ is that there is a resonant interaction between the rotatory inertia forces (resulting from bending motion) and shear forces. The investigation of the second spectrum is beyond the scope of this thesis and very little work has been done on it. Perhaps a further study of this problem will be made by someone.

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