FREE VIBRATIONS OF A VIERENDEEL GIRDER

by

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LIST OF SYMBOLS

A. Symbols Used in Virtual Displacement Equations

E	Young's Modulus
F_i	Force acting at the bottom of joint i
g	Acceleration of gravity
H_i	Axial force in girder i
h_i	Length of girder i
h^i	Conversion factor from I to Ic
I_i	Moment of inertia of girder i
I_c	Conversion parameter for I and J
J_i	Moment of inertia of column i
L	Height of all columns
L^i	Conversion factor from J to I_c
M	Bending moment acting on the girder
Q_i	Mass concentrated at bottom joint i
p	Fundamental frequency
R	Statical reaction at a support
T_i	Axial force in column i
U_i	Static deflection at bottom joint i
V_i	Vertical shear in girder i
x	Variable distance in the horizontal direction
y	Variable distance in the vertical direction

Y_i Horizontal shear in column i

Z Ordinate of the shape function

B. Symbols Used in Moment Distribution Equations

a^2 $\frac{EIg}{w}$

b Conversion factor for length

c Conversion factor for weight

d Conversion factor for moment of inertia

h Conversion factor for $k_o\ l_o$

kl Frequency parameter

K Stiffness factor

r Reaction at the far end of a beam due to a unit pulsating moment at the near end.

r^1 Reaction at the near end of a beam due to a unit pulsating moment at the near end.

w Uniform weight

X_i True displacement of column i

α Slope at one end of a beam due to a unit pulsating moment at the same end

β Slope at one end of a beam due to unit pulsating moment at the opposite end of the beam

φ Tabulated parameter for the non-displaced end of a member

ψ Tabulated parameter for the displaced end of a member

δ Oscillating displacement of unity

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III. INTRODUCTION

Vibrations occur in many engineering structures and frequently can be a controlling factor in the final design for a given structure. The basis for the design procedure can be the fundamental frequency of vibration of the structure. The determination of this value will vary from problem to problem and can be a laborious task for certain structures if one considers all factors entering into the governing differential equation.

The purpose of this thesis is to determine and compare the fundamental frequency of free vibrations of a parallel chord Vierendeel girder based upon two methods. The first method consists of the application of equations derived from the principle of virtual displacements for the particular rigid frame. The second method is the application of a numerical method of convergence introduced in a paper by Raithel [1], which is an adaptation of the Cross method of statical moment distribution.

IV. THE REVIEW OF LITERATURE

A statical analysis of Vierendeel type trusses based upon the principle of virtual work is presented in a paper by Young[2]. For the symmetrical case, the author writes an equation for the horizontal displacement of the midpoint of a typical interior column with respect to the midpoint of the adjoining column in terms of shears and bending moments using virtual work. Assuming this relative displacement is zero and using the basic equation of equilibrium at different sections, he reduces the original equation to the form from which the horizontal component of axial stress for any chord member may be conveniently calculated. At this point it should be noted that the equilibrium virtual loading consists of unit loads applied in the opposite horizontal direction at the midpoints of each column.

For a parallel chord truss with a constant moment of inertia, the equation reduces to the form:

$$H_{bw} = H_{aw} + 6\frac{L}{D}\sum_{1}^{a} H_w - 6\frac{L}{D^2} M_b + 3\frac{L^2}{D^2} V_{ab} \qquad \ldots.(a)$$

where,

H_{bw} = the horizontal shear at the center of column b

H_{aw} = the horizontal shear at the center of column a

L = length of girders

D = depth of truss

H_w = the total traverse shear in a column

M_b = external moment about joint b

V_{ab} = vertical shear in panel ab

Thus, an equation for each column can be written and from the resulting simultaneous equations, one can determine the horizontal shear in each web member.

For the unsymmetrical case, Young writes an equation for the relative horizontal displacement between two adjoining columns at their joints, considering the lower portion of the truss only; and an equation for the same points considering the upper portion of the truss. The two equations are equated to each other and the resulting equation reduces to the form of equation (a) for a parallel chord Vierendeel Truss.

The importance of dynamic loads in structural design is presented as a general review in a paper by Wilbur and Hansen(3). The authors discuss the causes of dynamic loadings such as, earthquakes, wind, man-made forces, and shock and blast waves from atomic bombs. An important fact brought out by the authors is that the designing engineer should be discreet in using empirical formulae pertaining to the dynamic analysis for a given structure. Thus, the use of static loads as equivalent dynamic loads for structures whose dynamic response may be considerably different from the static response can lead to a faulty design.

Many authors have assumed dynamic response of rigid frames is equivalent to the dynamic response of beams of varying cross sections. A single example of this is the tall frame treated as a cantilever beam. Glover(4) uses this technique in studying the earthquake response of a tall tower. Due to the translation of the support caused by the earthquake, he adds to the governing differential equation for a beam with fixed ends an additional term to account for a constant force applied to the support. A particular integral is added to the solution of the differential equation which represents the general shape of the static deflection of the beam under the constant loads produced by the acceleration. The governing differential equation is:

$$E\frac{\partial^2}{\partial x^2}\left(I\frac{\partial^2 y}{\partial x^2}\right) + K\frac{\partial^2}{\partial x^2}\left(I\frac{\partial^3 y}{\partial x^2 \partial t}\right) + \frac{\mu}{g}\frac{\partial^2 y}{\partial t^2} = -\frac{\mu}{g}X \qquad \ldots(6)$$

where

K = viscosity constant

μ = weight per unit of length of beam

X = acceleration due to a quake

The left side of the equation represents the usual form of a beam in vibration which considers damping and is equated to the force acting at the support resulting from the translation of the beam. The solution of the differential equation will contain constants of integration which are determined from the boundary conditions of the beam, and an initial condition that the beam is undeflected at time equal to zero.

When a structure oscillates in a state of free vibrations, the fact that no matter how irregular the disturbing forces may be, the structure seeks to adjust its deflected position to one of the mode shapes of free oscillation, is presented in a paper by Goldberg(5). By assuming an initial deflected position of a structure, one can find a new set of deflection values at each joint by using the slope-deflection formulae. Thus, the process reduces to a method of successive convergence from which a final assumption of deflections correspond reasonably well to the deflections found by the final analysis using the slope-deflection formulae. Finally, the energy method, which is based upon the assumption that when the kinetic energy is zero the potential energy is a maximum, and vice versa, and that the total energy remains constant, gives rise to the following equation for free vibrations:

$$p = \frac{1}{2\pi}\sqrt{\frac{g\sum Wy}{\sum Wy^2}} \qquad \ldots.(c)$$

where p is the fundamental frequency; g, the acceleration of gravity; W, the increments of weight; and y, the static deflections. It has been noted that this method is similar in form to the Stodola-Vianello method.

A method which utilizes converging approximations of the fundamental frequency of multi-story buildings has been published by Raithel[1] and an extended explanation and example problem pertaining to this method has been presented as a thesis by Swift[6]. The method is an adaptation of the Cross method of static moment distribution, and is discussed in detail and applied to an example problem in this thesis.

An unpublished method to determine the natural frequency of lateral vibrations for elastic beams has been suggested by Marcus* which utilizes the principle of virtual displacements. A thesis by Lee[7] has been written using this method for beams of constant and variable cross-sections, and the work includes several examples under various end conditions. A detailed explanation of this method will also be presented in this paper.

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* Dr. Henri Marcus, Consultant, Mechanics Division, Naval Research Laboratory, Washington, D. C.

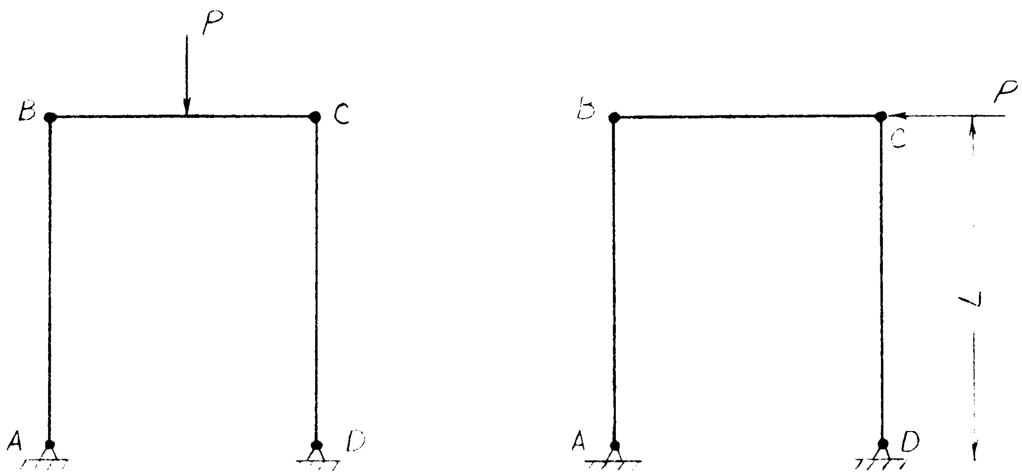
V. THE INVESTIGATION

A. The Principle of Virtual Displacements

1. Structural Stability of Rigid Frames

The statical analysis of rigid frames can be performed by the conventional methods of analysis, such as, moment distributions, slope-deflection equations, column analogy, or the classical methods based upon the principles of energy. For certain indeterminate problems, these methods may be laborious and thus, points of inflection are sometimes assumed at certain positions for a given structure. By assuming a sufficient number of points of inflection, the problem can be reduced to a form in which the equations of statics are sufficient to determine the shears and moments acting in the structure. Now, if one were to assume more points of inflection, or hinges, than unknowns for a given rigid frame, the stability of the structure would be questioned. That is, there would be extra equations to be satisfied for the problem.

However, for a given rigid frame with more hinges than required to reduce the problem from an indeterminate one to a determinate one, the direction of the loading will be an important factor regarding the stability of the structure. More specifically, consider the frame in Figure 1 (a). If one were to start at joint A and proceed to B, C, and D, calculating the shears in the frame at the joints using the equations of statics, one would find that the extra equation (for this particular case) $\sum M_D = 0$, would be automatically satisfied since the loading is



(a) (b)

Figure 1. Rigid Frames with Assumed Hinges

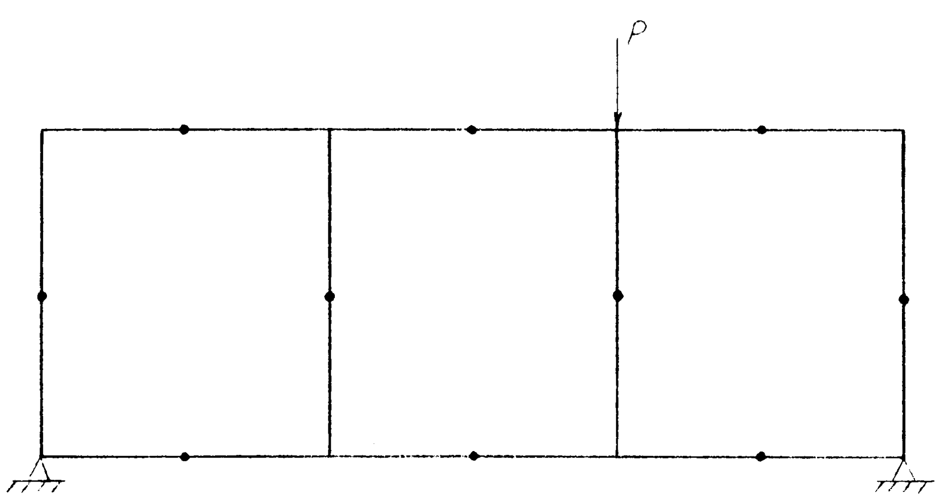


Figure 2. Structurally Stable Frame

vertical. However, for the frame in Figure 1 (b), this would not be the case since the applied load produces a moment equal to PL about the hinge at D, and therefore, this frame is unstable due to the horizontal loading.

For the rigid frame in Figure 2, where the points of inflection are located at the middle of each member, it can be shown that the structure is stable for all vertical loads, both symmetrically and unsymmetrically placed. Thus, if one were to proceed from joint to joint and to cut sections where desired to find the forces acting in the member, one would find that the shears, axial forces, and bending moments at the last joint would be in equilibrium, and thus all equations of equilibrium would be satisfied. In conclusion, for the unsymmetrical vertical loading of a frame similar to Figure 2, the structure is stable.

2. Basic Assumptions

A typical section of the parallel chord, Vierendeel girder to be analyzed is shown in Figure 3. The following assumptions are made to reduce the complexity of the problem:

a. Hinges are placed at the centers of all girders.

b. The vertical shears, V, in the upper and lower girders of a given panel are equal in value to each other.

c. The masses are concentrated at the lower chord joints.

d. When the truss deflects as a unit, the points of inflection in the columns are assumed to deflect only in a vertical direction.

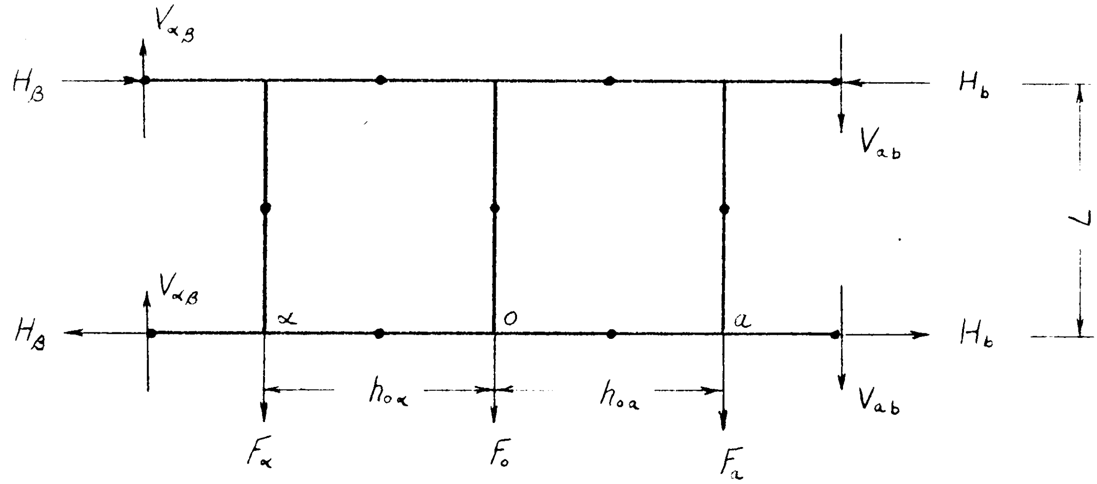


Figure 3. Typical Section of Girder

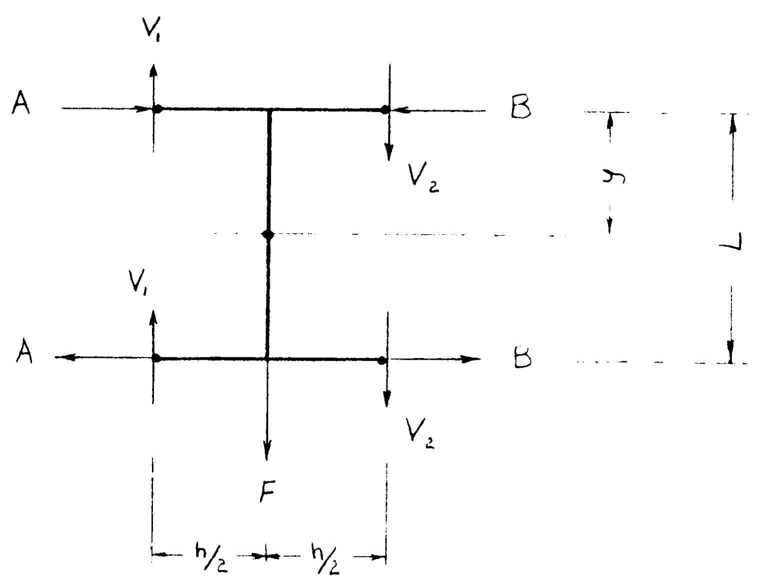


Figure 4. Free-body Diagram for Location of Column Hinge

If the distances between panel points are equal to h, a natural consequence of the first two assumptions will be a hinge at the middle of each column. This can be proved by considering a typical free-body diagram of an interior section shown in Figure 4. Thus, for the given section, the following equations of statics will locate the distance "Y" from the upper chord for the hinge in the column.

$$\Sigma V = 0 = 2V_1 - F - 2V_2$$

$$V_2 = V_1 - \frac{F}{2}$$

Sum the moments about the lower left hinge.

$$2\left(V_1 - \frac{F}{2}\right)h + F\frac{h}{2} + AL - BL = 0$$

$$\therefore A = B - 2V_1\frac{h}{L} + \frac{Fh}{2L}$$

Now take moments about the hinge in the column considering a section above the hinge.

$$\frac{h}{2}\left(V_1 - \frac{F}{2}\right) + V_1\frac{h}{2} - By + \left(B - 2V_1\frac{h}{L} + \frac{Fh}{2L}\right)y = 0$$

$$\frac{y}{L}\left(\frac{Fh}{2} - 2V_1h\right) = \frac{Fh}{4} - V_1h$$

$$\therefore y = \frac{\left(\frac{F}{4} - V_1\right)L}{\left(\frac{F}{2} - 2V_1\right)} = \frac{L}{2}$$

Since the distances between joints for a Vierendeel girder are usually equal, with the exception of the exterior panel, points of inflection at the middle of each column will be assumed for the general case.

It should be noted that all bending moments are assumed to act in a clockwise direction about the joint; all vertical shears in the girders are assumed to act clockwise about the joint; and all horizontal shears in the columns are assumed to act counterclockwise about the joint.

3. Derivation of Equations

a. Typical Interior Joint

Consider the typical interior panel in Figure 5 (a) showing the real forces acting upon the structure, and the equilibrium virtual loading shown in Figure 5 (b). Applying the principle of virtual displacements for this set of loading, and letting u denote the real vertical displacement at the lower joint, we have

$$\frac{u_\alpha}{h_{o\alpha}} - \frac{u_o}{h_{o\alpha}} - \frac{u_o}{h_{oa}} + \frac{u_a}{h_{oa}} = -\int_0^{L/2} \frac{Y_\alpha y^2 dy}{E J_\alpha L} + \int_0^{h_{o\alpha}} \frac{1}{E I_{o\alpha}} \left[\left(M_{o\alpha} - 2M_{o\alpha}\frac{x}{h_{o\alpha}}\right)\left(\frac{x}{h_{o\alpha}} - \frac{1}{2}\right)\right] dx$$

$$+ \int_0^{h_{oa}} \frac{1}{E I_{oa}} \left[\left(2M_{oa}\frac{x}{h_{oa}} - M_{oa}\right)\left(\frac{x}{h_{oa}} - \frac{1}{2}\right)\right] dx + \int_0^{L/2} \frac{Y_a y^2 dy}{E J_a L} \qquad \ldots (1)$$

Assuming a constant moment of inertia for each member, upon integration, equation (1) reduces to the following form:

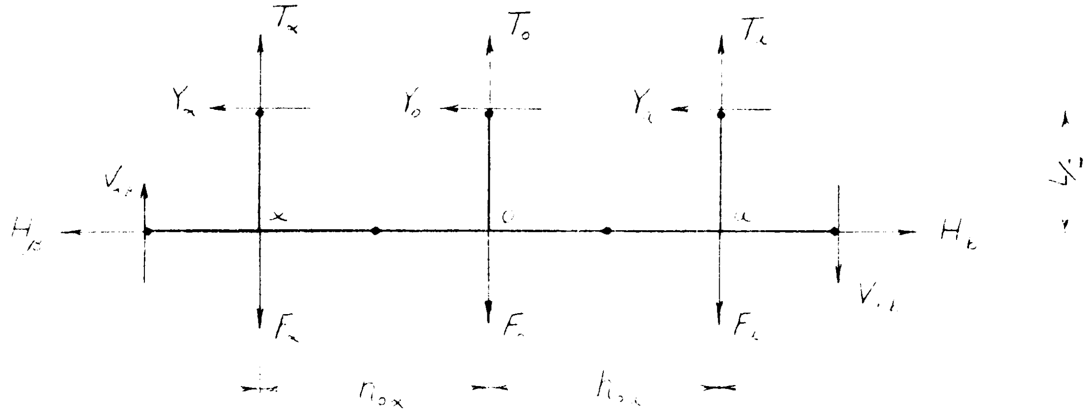
$$\frac{u_\alpha}{h_{o\alpha}} - \frac{u_o}{h_{o\alpha}} - \frac{u_o}{h_{oa}} + \frac{u_a}{h_{oa}} = \frac{Y_a L^2}{24 E J_a} + \frac{M_{oa} h_{oa}}{6 E I_{oa}} - \frac{Y_\alpha L^2}{24 E J_\alpha} - \frac{M_{o\alpha} h_{o\alpha}}{6 E I_{o\alpha}} \qquad \ldots (2)$$

Introduce the following relationships:

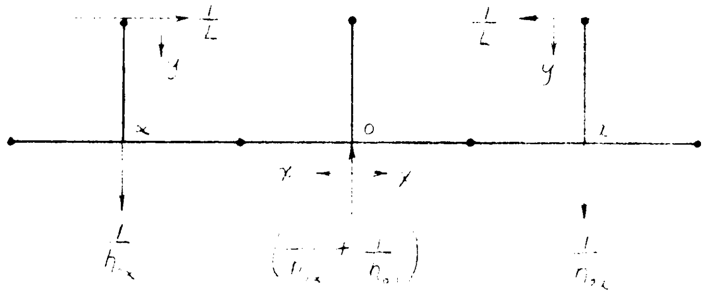
$$h' = \frac{I_c}{I} h \qquad\qquad L' = \frac{I_c}{J} L \qquad \ldots (a)$$

and let

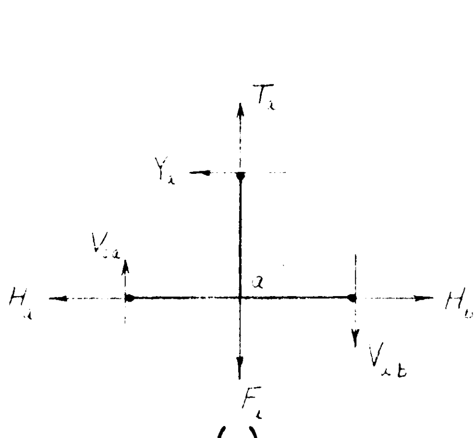
$$U_o = 6 E I_c \left(\frac{u_\alpha}{h_{o\alpha}} - \frac{u_o}{h_{o\alpha}} - \frac{u_o}{h_{oa}} + \frac{u_a}{h_{oa}}\right) \qquad \ldots (b)$$



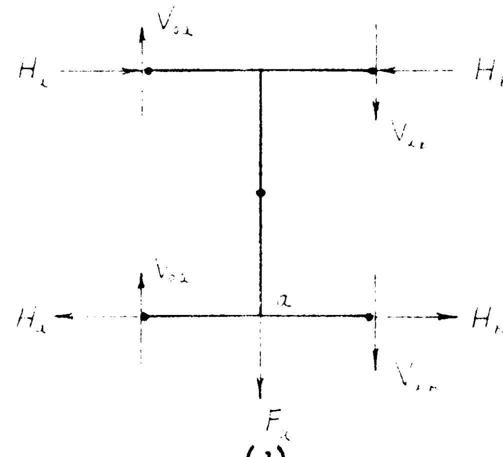
(a)



(b)



(c)



(d)

Figure 5. Forces on a Typical Interior Section of the Girder

Substitute the above relationships into equation (2) and the equation can now be written as:

$$U_o = \frac{Y_a L L_a'}{4} - \frac{Y_\alpha L L_\alpha'}{4} + M_{oa} h_{oa}' - M_{o\alpha} h_{o\alpha}' \qquad \ldots (3)$$

Equation (3) can be reduced in terms of vertical shears in the girders by considering the sections shown in Figures 5 (c) and (d).

Thus,

$$Y_a = V_{oa} \frac{h_{oa}}{L} + V_{ab} \frac{h_{ab}}{L} \qquad \ldots (c)$$

$$V_{ab} = V_{oa} - \frac{F_a}{2} \qquad \ldots (d)$$

Similar expressions can be written for the other joints. Since the hinges are at the center of each girder,

$$M_{oa} = V_{oa} \frac{h_{oa}}{L} \qquad \ldots (e)$$

Now substitute equations (c) and (e) for V and M into equation (3).

$$U_o = \left[V_{oa} h_{oa} + V_{ab} h_{ab} \right] \frac{L_a'}{4} - \left[V_{o\alpha} h_{o\alpha} + V_{\alpha\beta} h_{\alpha\beta} \right] \frac{L_\alpha'}{4} + \frac{V_{oa} h_{oa} h_{oa}'}{2} - \frac{V_{o\alpha} h_{o\alpha} h_{o\alpha}'}{2} \qquad \ldots (4)$$

Use relationships similar to equation (d) to reduce equation (4) in terms of the forces at the lower joint.

$$U_o = \left[V_{oa} h_{oa} + \left(V_{oa} - \frac{F_a}{2} \right) h_{ab} \right] \frac{L_a'}{4} - \left[\left(\frac{F_o}{2} + V_{oa} \right) h_{o\alpha} + \left(\frac{F_\alpha}{2} + \frac{F_o}{2} + V_{oa} \right) h_{\alpha\beta} \right] \frac{L_\alpha'}{4}$$

$$+ \frac{V_{oa} h_{oa} h_{oa}'}{2} - \left(V_{oa} + \frac{F_o}{2} \right) \frac{h_{o\alpha} h_{o\alpha}'}{2}$$

or

$$U_o = -\frac{F_a h_{ab} L_a'}{8} - \left(F_o h_{o\alpha} + F_\alpha h_{\alpha\beta} + F_o h_{\alpha\beta} \right) \frac{L_\alpha'}{8} - \frac{F_o h_{o\alpha} h_{o\alpha}'}{4}$$

$$+ V_{oa} \left(\frac{h_{oa} L_a'}{4} + \frac{h_{ab} L_a'}{4} - \frac{h_{o\alpha} L_\alpha'}{4} - \frac{h_{\alpha\beta} L_\alpha'}{4} + \frac{h_{oa} h_{oa}'}{2} \pm \frac{h_{o\alpha} h_{o\alpha}'}{2} \right)$$

$$\ldots (5)$$

For convenience, call

$$e_o = \frac{h_{oa}L'_a}{4} + \frac{h_{ab}L'_a}{4} - \frac{h_{o\alpha}L'_\alpha}{4} - \frac{h_{\alpha\beta}L'_\alpha}{4} + \frac{h_{oa}h'_{oa}}{2} - \frac{h_{o\alpha}h'_{o\alpha}}{2} \qquad \ldots\ldots (f)$$

Equation (5) may be rewritten as:

$$\frac{U_o}{e_o} = -\frac{F_a h_{ab} L'_a}{8e_o} - \left(F_o h_{o\alpha} + F_\alpha h_{\alpha\beta} + F_o h_{\alpha\beta}\right)\frac{L'_\alpha}{8e_o} - \frac{F_o h_{o\alpha} h'_{o\alpha}}{4e_o} + V_{oa} \qquad \ldots (5.1)$$

Another equation in terms of V_{oa} can be obtained in the following manner. Reduce Equation (4) in terms of $V_{o\alpha}$. Thus,

$$\frac{U_o}{e_o} = -\frac{F_\alpha h_{\alpha\beta} L'_\alpha}{8e_o} - \left(F_o h_{oa} + F_a h_{ab} + F_o h_{ab}\right)\frac{L'_a}{8e_o} - \frac{F_o h_{oa} h'_{oa}}{4e_o} + V_{o\alpha}$$

Now permute subscripts and the equation become:

$$\frac{U_a}{e_a} = -\frac{F_o h_{o\alpha} L'_o}{8e_a} - \left(F_a h_{ab} + F_b h_{bc} + F_a h_{bc}\right)\frac{L'_b}{8e_a} - \frac{F_a h_{ab} h'_{ab}}{4e_a} + V_{oa} \qquad \ldots (6)$$

Equate equations (5.1) and (6) based upon the relationship each equation gives for V_{oa}.

$$\frac{U_o}{e_o} - \frac{U_a}{e_a} = -\frac{F_a h_{ab} L'_a}{8e_o} - \left(F_o h_{o\alpha} + F_\alpha h_{\alpha\beta} + F_o h_{\alpha\beta}\right)\frac{L'_\alpha}{8e_o} - \frac{F_o h_{o\alpha} h'_{o\alpha}}{4e_o}$$
$$+ \frac{F_o h_{o\alpha} L'_o}{8e_a} + \left(F_a h_{ab} + F_b h_{bc} + F_a h_{bc}\right)\frac{L'_b}{8e_a} + \frac{F_a h_{ab} h'_{ab}}{4e_a} \qquad \ldots (7)$$

Up to this point only the static case for analysis has been considered. However, equation (7) consists of the desired terms for the problem of free vibrations. Thus, the displacement u may be represented as,

$$u = Z\left(A \sin pt + B \cos pt\right) \qquad \ldots (g)$$

where Z represents the ordinate of the shape function.

The forces F at each lower joint can also be reduced to a function of the frequency of vibration and time by letting the mass Q be concentrated at each lower joint for the given girder. Hence,

$$F = -Q\frac{\partial^2 u}{\partial t^2} = Qp^2 Z(A \sin pt + B \cos pt) \qquad \ldots (h)$$

Substitute equations (g) and (h) into equations (7); the following equation is obtained:

$$Z_\alpha\left[\frac{1}{e_o h_{o\alpha}} + \frac{Q_\alpha p^2 h_{\alpha\beta} L'_\alpha}{48 E I_c e_o}\right] + Z_o\left[-\frac{1}{e_o h_{o\alpha}} - \frac{1}{e_o h_{oa}} - \frac{1}{e_a h_{oa}} + \frac{Q_o p^2 L'_\alpha}{48 E I_c e_o}(h_{o\alpha} + h_{\alpha\beta})\right.$$

$$\left. + \frac{Q_o p^2 h_{o\alpha} h'_{o\alpha}}{24 E I_c e_o} - \frac{Q_o p^2 h_{o\alpha} L'_o}{48 E I_c e_a}\right] + Z_a\left[\frac{1}{e_o h_{oa}} + \frac{1}{e_a h_{oa}} + \frac{1}{e_a h_{ab}} - \frac{Q_a p^2 L'_b}{48 E I_c e_a}(h_{ab} + h_{bc})\right.$$

$$\left. - \frac{Q_a p^2 h_{ab} h'_{ab}}{24 E I_c e_a} + \frac{Q_a p^2 h_{ab} L'_a}{48 E I_c e_o}\right] + Z_b\left[-\frac{1}{e_a h_{ab}} - \frac{Q_b p^2 h_{bc} L'_b}{48 E I_c e_a}\right] = 0 \qquad \ldots (8)$$

b. <u>Special End Condition</u>

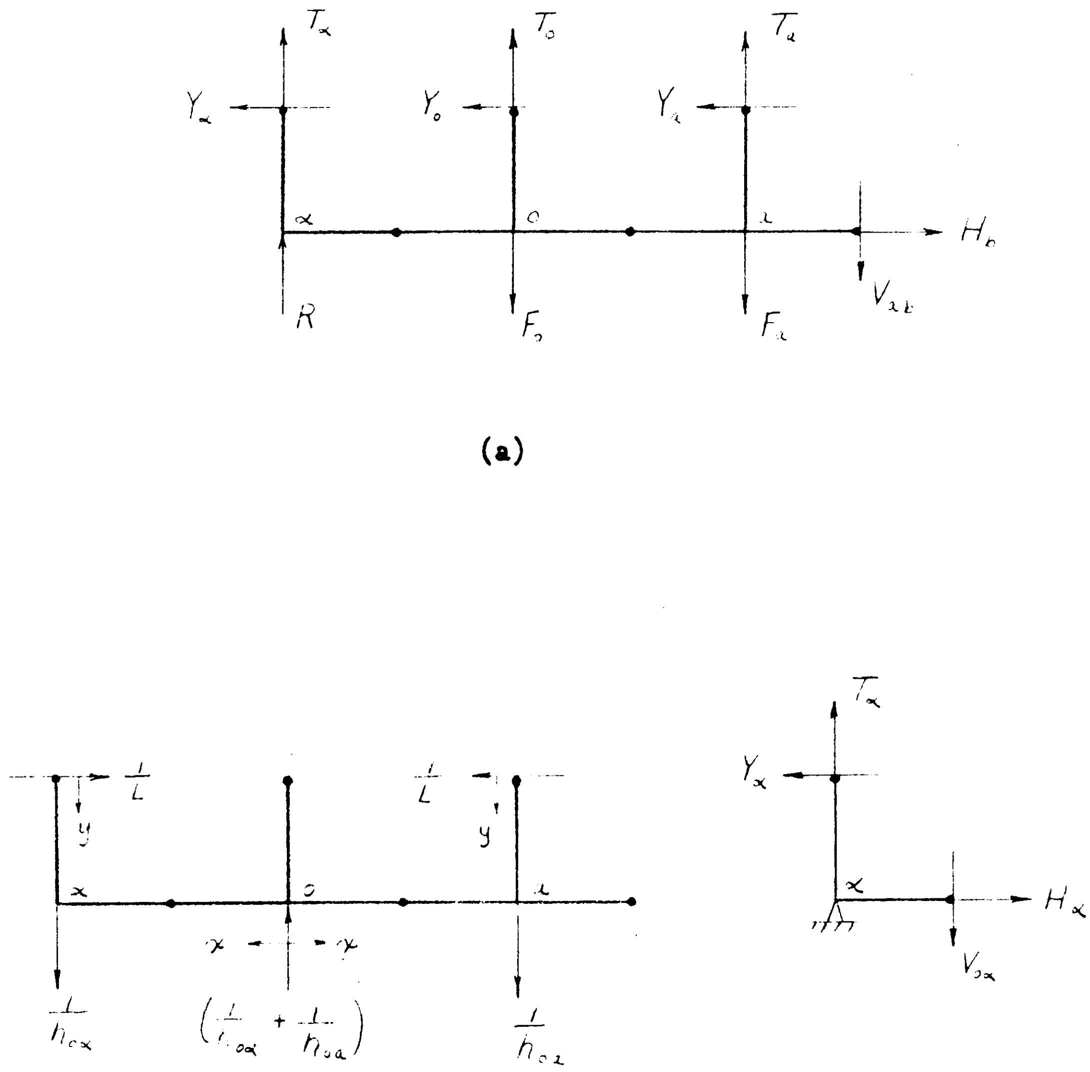
Special consideration must be given for the first joint from the support. Consider such a typical section shown in Figures 6 (a) and 6 (b). Apply the principle of virtual displacements for the equilibrium virtual loading in Figure 6 (b). Upon integration and substitution of equation (a), we have as before,

$$\frac{u_\alpha}{h_{o\alpha}} - \frac{u_o}{h_{o\alpha}} - \frac{u_o}{h_{oa}} + \frac{u_a}{h_{oa}} = \frac{Y_a L L'_a}{24 E I_c} - \frac{Y_\alpha L L'_\alpha}{24 E I_c} + \frac{M_{oa} h'_{oa}}{6 E I_c} - \frac{M_{o\alpha} h'_{o\alpha}}{6 E I_c} \qquad \ldots (9)$$

Since there is no vertical displacement at joint α, $u_\alpha = 0$.

Let

$$U_{o1} = 6 E I_c\left[\frac{u_o}{h_{o\alpha}} - \frac{u_o}{h_{oa}} + \frac{u_a}{h_{oa}}\right] \qquad \ldots (i)$$

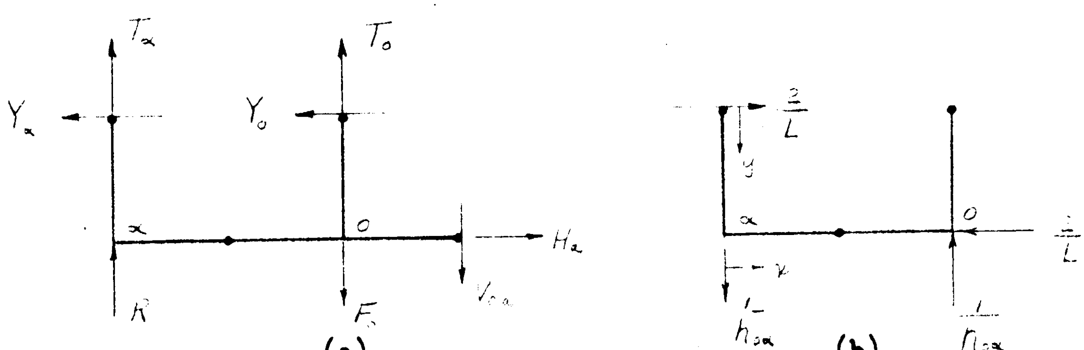


(a)

(b)

(c)

Figure 6. Forces on an End Section of the Girder



(a)

(b)

Figure 7. Section for Additional Equation on End Condition

Equation (9) is thus written:

$$U_{o1} = \frac{Y_a L L_a'}{4} - \frac{Y_\alpha L L_\alpha}{4} + M_{oa} h_{oa}' - M_{o\alpha} h_{o\alpha}' \qquad \dots (10)$$

From Figure 6 (c), by summing moments about the support, the following relationship holds:

$$Y_\alpha = V_{o\alpha} \frac{h_{o\alpha}}{L} \qquad \dots (j)$$

Now substitute equation (j) and expressions similar to equations (c), (d) and (e) into Equation (10).

$$U_{o1} = \left[\left(V_{o\alpha} - \frac{F_o}{2} \right) h_{oa} + \left(V_{o\alpha} - \frac{F_o}{2} - \frac{F_a}{2} \right) h_{ab} \right] \frac{L_a'}{4} - \frac{V_{o\alpha} h_{o\alpha} L_\alpha'}{4}$$
$$+ \left(V_{o\alpha} - \frac{F_o}{2} \right) \frac{h_{oa} h_{oa}'}{2} - \frac{V_{o\alpha} h_{o\alpha} h_{o\alpha}'}{2}$$

or

$$U_{o1} = \left[-F_o h_{oa} - F_o h_{ab} - F_a h_{ab} \right] \frac{L_a'}{8} - \frac{F_o h_{oa} h_{oa}'}{4}$$
$$+ V_{o\alpha} \left(\frac{h_{oa} L_a'}{4} + \frac{h_{ab} L_a'}{4} - \frac{h_{o\alpha} L_\alpha'}{4} + \frac{h_{oa} h_{oa}'}{2} - \frac{h_{o\alpha} h_{o\alpha}'}{2} \right) \qquad \dots (10.1)$$

Let

$$e_{o1} = \frac{h_{oa} L_a'}{4} + \frac{h_{ab} L_a'}{4} - \frac{h_{o\alpha} L_\alpha'}{4} + \frac{h_{oa} h_{oa}'}{2} - \frac{h_{o\alpha} h_{o\alpha}'}{2} \qquad \dots (k)$$

Thus,

$$\frac{U_{o1}}{e_{o1}} = -\left[F_o h_{oa} + F_o h_{ab} + F_a h_{ab} \right] \frac{L_a'}{8 e_{o1}} - \frac{F_a h_{oa} h_{oa}'}{4 e_{o1}} + V_{o\alpha} \qquad \dots (11)$$

To reduce equation (11) into the dynamic form, it will be desirable to eliminate the vertical girder shear, $V_{o\alpha}$. To do this, consider the real loading in Figure 7 (a) and a new equilibrium virtual loading in Figure 7 (b). Once again apply the principle of virtual displacements.

$$\frac{u_\alpha}{h_{o\alpha}} - \frac{u_o}{h_{o\alpha}} = \int_0^{h_{o\alpha}} \frac{1}{E I_{o\alpha}} \left[\left(2M_{o\alpha}\frac{x}{h_{o\alpha}} - M_{o\alpha}\right)\left(1 - \frac{x}{h_{o\alpha}}\right)\right] dx - \int_0^{L/2} \frac{2 Y_\alpha y^2 dy}{E J_\alpha L} \quad \ldots (12)$$

Since the support has no vertical movement $u_\alpha = 0$, it follows that upon integration, equation (12) reduces to the form:

$$\frac{u_o}{h_{o\alpha}} = \frac{Y_\alpha L^2}{12 E J_\alpha} + \frac{M_{o\alpha} h_{o\alpha}}{6 E I_{o\alpha}}$$

or

$$\frac{u_o}{h_{o\alpha}} = \frac{Y_\alpha L L'_\alpha}{12 E I_c} + \frac{M_{o\alpha} h'_{o\alpha}}{6 E I_c} \quad \ldots (13)$$

Reduce equation (13) to the equivalent vertical shear terms for Y and M. Hence,

$$12 E I_c \frac{u_o}{h_{o\alpha}} = V_{o\alpha} h_{o\alpha} L'_\alpha + V_{o\alpha} h_{o\alpha} h'_{o\alpha}$$

or

$$V_{o\alpha} = \frac{12 E I_c u_o}{h_{o\alpha}^2 (L'_\alpha + h'_{o\alpha})} \quad \ldots (14)$$

Substitute equation (14) into equation (11).

$$\frac{U_{o1}}{e_{o1}} = \frac{12 E I_c u_o}{h_{o\alpha}^2 (L'_\alpha + h'_{o\alpha})} - \left[F_o h_{oa} + F_o h_{ab} + F_a h_{ab}\right] \frac{L'_a}{8 e_{o1}} - \frac{F_a h_{oa} h'_{oa}}{4 e_{o1}} \quad \ldots (15)$$

By substituting equations (g) and (h) into equation (15), the desired dynamic form of the equation is obtained.

$$\frac{1}{e_{01}}\left[-\frac{Z_0}{h_{0\alpha}}-\frac{Z_0}{h_{0a}}+\frac{Z_a}{h_{0a}}\right]-\frac{2Z_0}{h_{0\alpha}^2(L'_\alpha+h'_{0\alpha})}=\frac{p^2}{6EI_c}\left\{\left[-Q_0Z_0h_{0a}-Q_0Z_0h_{ab}\right.\right.$$

$$\left.\left.-Q_aZ_ah_{ab}\right]\frac{L'_a}{8e_{01}}-\frac{Q_0Z_0h_{0a}h'_{0a}}{4e_{01}}\right\}$$

or

$$Z_0\left[-\frac{1}{e_{01}h_{0\alpha}}-\frac{1}{e_{01}h_{0a}}-\frac{2}{h_{0\alpha}^2(L'_\alpha+h'_{0\alpha})}+\frac{Q_0p^2L'_a}{48EI_ce_{01}}(h_{0a}+h_{ab})+\frac{Q_0p^2h_{0a}h'_{0a}}{24EI_ce_{01}}\right]$$

$$+Z_a\left[\frac{1}{e_{01}h_{0a}}+\frac{Q_ap^2h_{ab}L'_a}{48EI_ce_{01}}\right]=0 \qquad \ldots.(16)$$

Thus, for a parallel chord Vierendeel girder of n interior joints, one can now write n equations based upon the derived expressions of equations (8) and (16). There will be however (n + 1) unknowns since the frequency term, p^2, will be in each equation and since none of the Z values is zero, the determinant of the n equations is equated to zero.

It should be mentioned here that this method is somewhat similar to the Raleigh method for beams in that this latter method equates the maximum potential energy to the maximum kinetic energy as determined from an assumed deflection curve. If the correct deflection curve is assumed, the exact answer for the fundamental frequency is obtained. Likewise, in this method, by transposing the static equations to the dynamic equations in order to find the fundamental frequency, it is assumed that an anology exists between the static deflections and dynamic mode shapes.

As previously mentioned, for a girder of n interior joints, there will be n equations. However, the number of equations may be reduced by one-half for a symmetrical bridge, which is frequently the case, since $Z_1 = Z_{n-1}$, $Z_2 = Z_{n-2}$, etc. The resulting roots for p^2 found by solving the determinant will give the first, third, fifth, etc. mode frequencies. However, should $Z_1 = -Z_{n-1}$, $Z_2 = -Z_{n-2}$, etc. be assumed, the resulting roots for p^2 will be the even mode frequencies.

4. Special Consideration for the Equation of an Interior Joint

Figure 8 shows the number of panels for the Virendeel girder to be analyzed in this paper. It has been found from experience, that for this particular problem which has an odd number of interior panel points, a special equation for joint four should be derived. The typical sections and virtual loading as shown in Figure (5) will be the same for this joint, and the following relationships between the subscripts will be

$$\alpha = 3 \qquad o = 4 \qquad a = 5$$

Equations (5.1) can be written for the particular subscripts.

$$\frac{U_4}{e_4} = -\frac{F_5 h_{56} L_5'}{8e_4} - \left(F_4 h_{34} + F_3 h_{23} + F_4 h_{23}\right)\frac{L_3'}{8e_4} - \frac{F_4 h_{34} h_{34}'}{4e_4} + V_{45} \quad \ldots\ldots (L)$$

From symmetry,

$$V_{45} = -V_{56}$$

$$\therefore V_{45} = \frac{F_5}{2} + V_{56} = \frac{F_5}{4}$$

Substituting this expression for V_{45} into equation (L), we have,

$$U_4 = -\frac{F_5 h_{56} L_5'}{8} - \left(F_4 h_{34} + F_3 h_{23} + F_4 h_{23}\right)\frac{L_3'}{8} - \frac{F_4 h_{34} h_{34}'}{4} + \frac{F_5 e_4}{4}$$

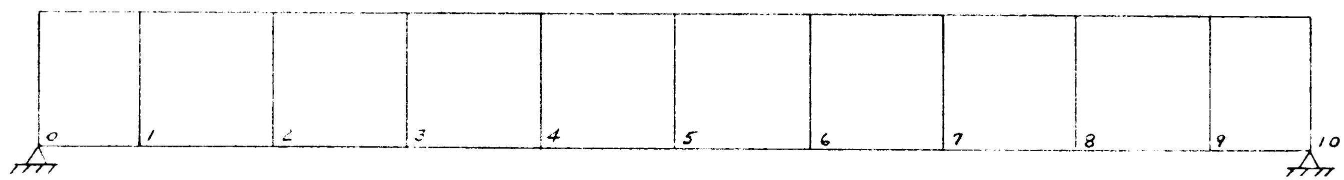


Figure 8. Side View of Analyzed Frame

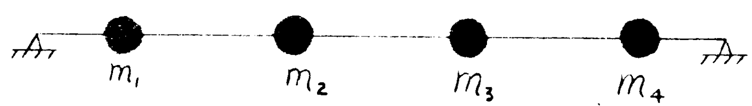


Figure 9. Weightless Beam with Concentrated Masses

This expression reduces to the following dynamic form:

$$Z_3\left[\frac{1}{h_{34}}+\frac{Q_3 p^2 h_{23} L_3'}{48 E I_c}\right]+Z_4\left[-\frac{1}{h_{34}}-\frac{1}{h_{45}}+\frac{Q_4 p^2 L_3'}{48 E I_c}(h_{23}+h_{34})+\frac{Q_4 p^2 h_{34} h_{34}'}{24 E I_c}\right]$$

$$+Z_5\left[\frac{1}{h_{45}}+\frac{Q_5 p^2 h_{56} L_5'}{48 E I_c}-\frac{Q_5 p^2 e_4}{24 E I_c}\right]=0$$

The reason for deriving this special equation lies in the fact that e_5 equals zero for the symmetrical girder that was analyzed. As a result, only the Z_4 and Z_5 terms remained in the equation for joint four, which would not give a solution to the determinant for the fundamental frequency.

5. Equations for the Special Case

For the special case where the moment of inertia of all columns, J, is equal; the moment of inertia of all girders, I, is equal; and the distance between all joints, h, is equal, the general equations can be reduced in size considerably. Since the value of all e's will be zero for the general expression of an interior joint as represented by equation (8), multiply all terms by e_o and then let $e_o = 0$. Note that nothing has been said about e_a. The resulting equation will be:

$$Z_\alpha\left[\frac{1}{h}+\frac{Q_\alpha p^2 h L'}{48 E I_c}\right]+Z_o\left[-\frac{2}{h}+\frac{Q_o p^2 h L'}{24 E I_c}+\frac{Q_o p^2 h h'}{24 E I_c}\right]+Z_a\left[\frac{1}{h}+\frac{Q_a p^2 h L'}{48 E I_c}\right]=0 \qquad \ldots\ldots (m)$$

Equation (m) could have been derived directly from equation (5) since V_{oa} in this latter equation would drop out due to the bracketed term being zero in value.

For the special end condition, multiply each term of equation (16) by e_{o1} where,

$$e_{o1} = \frac{hL'}{4}$$

Hence, equation (16) reduces to:

$$Z_o\left[-\frac{2}{h} - \frac{L'}{2h(L'+h')} + \frac{Q_o p^2 L'h}{24EI_c} + \frac{Q_o p^2 hh'}{24EI_c}\right] + Z_a\left[\frac{1}{h} + \frac{Q_a p^2 hL'}{48EI_c}\right] = 0 \qquad \ldots (n)$$

6. Additional Approximation of the Equations

In the example problem for which the derived equations were applied, the first four frequency modes were obtained. It was decided to neglect the shear term V_{oa} in equation (5) for the general interior point, and to neglect the shearing term $V_{o\alpha}$ in equation (10.1). Then, use the resulting equations for the problem and compare the final answers of the fundamental frequencies.

Thus, equation (5) is now written:

$$U_o = -\frac{F_a h_{ab} L'_a}{8} - \left(F_o h_{o\alpha} + F_\alpha h_{\alpha\beta} + F_o h_{\alpha\beta}\right)\frac{L'_\alpha}{8} - \frac{F_o h_{o\alpha} h'_{o\alpha}}{4} \qquad \ldots (o)$$

Transpose into the dynamic case and equation (o) reduces to the form:

$$Z_\alpha\left[\frac{1}{h_{o\alpha}} + \frac{Q_\alpha p^2 h_{\alpha\beta} L'_\alpha}{48EI_c}\right] + Z_o\left[-\frac{1}{h_{o\alpha}} - \frac{1}{h_{oa}} + \frac{Q_o p^2 L'_\alpha}{48EI_c}(h_{o\alpha}+h_{\alpha\beta}) + \frac{Q_o p^2 h_{o\alpha} h'_{o\alpha}}{24EI_c}\right]$$
$$+ Z_a\left[\frac{1}{h_{oa}} + \frac{Q_a p^2 h_{ab} L'_a}{48EI_c}\right] = 0 \qquad \ldots (17)$$

For the end condition, neglect the shear term $V_{o\alpha}$ in equation (10.1). Thus,

$$U_{o1} = \left[-F_o h_{oa} - F_o h_{ab} - F_a h_{ab}\right]\frac{L'_a}{8} - \frac{F_o h_{oa} h'_{oa}}{4} \qquad \ldots (p)$$

In the dynamic form, equation (p) becomes:

$$Z_o\left[-\frac{1}{h_{o\alpha}} - \frac{1}{h_{oa}} + \frac{Q_o p^2 L'_a}{48EI_c}(h_{oa}+h_{ab}) + \frac{Q_o p^2 h_{oa} h'_{oa}}{24EI_c}\right] +$$
$$+ Z_a\left[\frac{1}{h_{oa}} + \frac{Q_a p^2 h_{ab} L'_a}{48EI_c}\right] = 0 \qquad \ldots (18)$$

7. <u>Review of Procedure</u>

For convenience, the following steps list the procedure to be followed using the principle of virtual displacements.

a. Determine the concentrated masses at each lower joint.

b. Calculate h^1 for each girder and L^1 for each column.

c. Calculate e for each joint.

d. Apply the equations derived from the principle of virtual displacements.

e. From the equations obtained in step (d), set up the determinant and equate it to zero.

f. Solve the determinant for p^2.

B. Method of Dynamic Moment Distribution

1. Introduction

The basic principle that underlies Raithel's method is concerned with the resisting forces that are set up at each joint while a rigid frame is in vibration. As a preliminary procedure of explanation, consider the weightless beam with masses concentrated at various points, as shown in Figure 9 (a). If m_1 is displaced a unit amount in the vertical direction while all other masses are held in place, (this will be defined as an absolute displacement), and then released, a state of free vibration will exist in the system. The forces that act at m_1 are the inertia force of the mass itself and the shear, or resisting force, at this point, which is dependent upon the movement of all the masses and the absolute displacement of m_1.

If m_2 is absolutely displaced by a unit amount and released in free vibration, a new shear value will be set up at m_1 which will be a function of the movement of all the masses and the absolute displacement of m_2. Likewise, by performing the same operation for m_3 and m_4, additional resisting forces will be set up at m_1. Naturally, similar resisting forces will be set up at the other three masses.

For the actual case of the beam vibrating in its fundamental mode, by using the Principle of Superposition, an equation of forces at each mass can be written which equates the inertia force of the mass to the resisting shear due to each absolute displacement multiplied by the true displacement of each mass. Using the summation convention:

$$F_i = S_{ij} X_j \qquad i = 1, 2, 3, \ldots n$$
$$j = 1, 2, 3, \ldots n$$

where F_i represents the inertia force of each mass; S_{ij} is the shear at i due to a unit displacement of mass j; and X_j is the true displacement of mass j.

2. Application to a Rigid Frame

From the preceding discussion of the principle underlying Raithel's method, if the beam is replaced by a Vierendeel girder, the concentrated masses are replaced by columns, and an interesting problem ensues. It will be recognized that as each column is now absolutely displaced a unit amount and then released, dynamic fixed end moments and shears are set up at the respective girder ends, and in turn, are constantly distributing themselves throughout the vibrating frame. In turn, there are dynamic joint rotations due to this distribution of moments and shears. Also, since the shape of each member is changing with time, a non-uniform distribution of inertia force will exist for each member, which in turn will affect the end shears of each member. These special affects which arise for an oscillating rigid frame will be discussed in detail when the fixed end moment and shear expressions are derived. First, however, the dynamic carry-over factor and stiffness factor will be considered.

3. Solutions of the Governing Equation

For the dynamic case, the governing differential equation for lateral free vibrations of a beam of uniform weight w, is

$$EI \frac{\partial^4 y}{\partial x^4} - \frac{w}{g} \frac{\partial^2 y}{\partial t^2} = 0 \qquad \text{.... (a)}$$

Solve the differential equation by the method of variables separable, letting X be the shape function and T be the time function. The solution for X is

$$X = A \sin kx + B \cos kx + C \sinh kx + D \cosh kx \qquad \text{.... (b)}$$

where

$$a^2 = \frac{EIg}{w} \qquad k^4 = \frac{p^2}{a^2} \qquad \text{.... (c)}$$

The constants of integration will be determined for the given boundary conditions. Thus, for a simply supported beam with a pulsating moment(M sin pt) applied at the end of the beam as shown in Figure 10, theend conditions are:

$$X(0) = 0 \qquad X(L) = 0 \qquad X''(0) = 0 \qquad X''(L) = \frac{M}{EI}$$

Having solved for the constants of integration, the solution reduces to the form:

$$X = \frac{M}{2EIk^2}\left(\frac{\sin kx}{\sin kL} - \frac{\sinh kx}{\sinh kL}\right) \qquad \text{.... (19)}$$

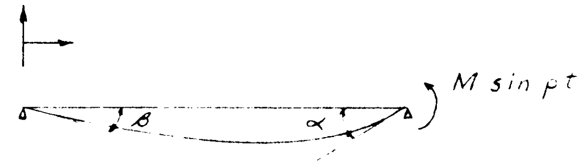


Figure 10. Derivation for α and β

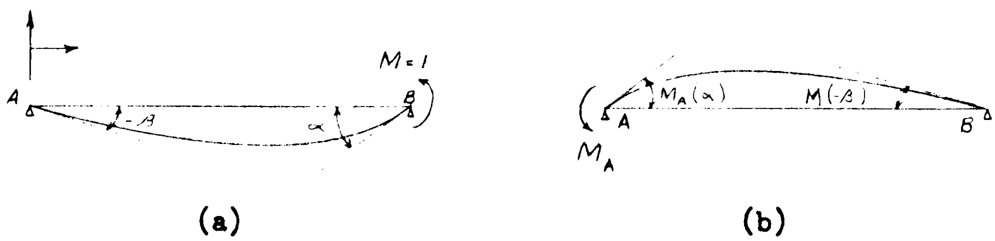


Figure 11. Derivation for Carry-Over Factor

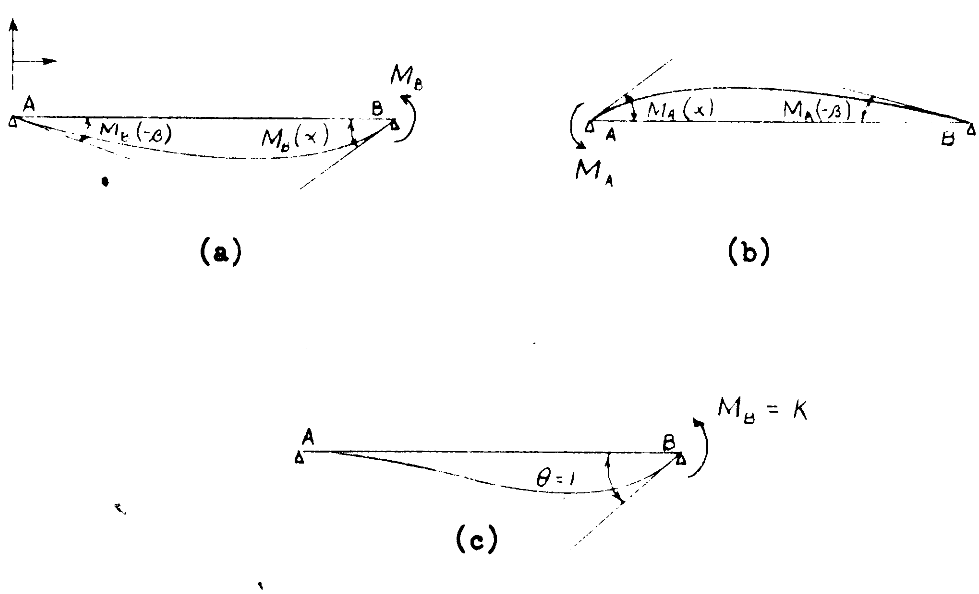


Figure 12. Derivation for K

The slopes at the end of the beam are designated α and β.
By taking the first derivative of equation (19), we obtain:

$$X'(0) = \beta = \frac{ML}{6EI} \psi \qquad \ldots (20)$$

where
$$\psi = \frac{3}{kL}\left(\frac{1}{\sin kL} - \frac{1}{\sinh kL}\right) \qquad \ldots (d)$$

$$X'(L) = \alpha = \frac{ML}{3EI} \phi \qquad \ldots (21)$$

where
$$\phi = \frac{3}{2kL}(\cot kL - \coth kL) \qquad \ldots (e)$$

The values of ψ and ϕ for various values of kl are tabulated in Table 1.

For future reference, let α be defined as the slope at one end of the beam due to a unit pulsating moment applied at that end, and β be defined as the slope at one end of the beam due to a unit pulsating moment at the opposite end.

4. Carry-Over and Stiffness Factor

For the dynamic case, the carry-over factor for a beam, whose ends are designated as A and B respectively, is defined as the ratio of the moment at A to the moment at B due to a unit pulsating moment at B, while end A is held fixed. For the beam shown in Figure 11 (a), it is observed that for a unit pulsating moment applied at B, a slope of ($-\beta$) is developed at A. To nullify this slope in order to satisfy the definition of dynamic carry-over, a moment M_A must be applied at end A as shown in Figure 11 (b). Now sum the two slopes at end A.

$$M_A(\alpha) - 1(\beta) = 0$$

$$\therefore M_A = \frac{\beta}{\alpha} \qquad \ldots (22)$$

Thus, the carry-over factor due to a unit pulsating moment at one end is defined by equation (22).

The dynamic stiffness for the same beam is defined as the pulsating moment required at end B to turn end B through a unit angle, keeping end A fixed. Referring to Figures 12 (a), (b) and (c), in order to satisfy the definition for stiffness, the moment M_A must be applied at end A to keep it as a fixed end after the pulsating moment M_B has been applied at end B. The resulting slope at end B can therefore be written as:

$$\theta = 1 = M_B(\alpha) - M_A(\beta)$$

From the definition of carry-over factor:

$$M_A = M_B \frac{\beta}{\alpha}$$

$$\therefore 1 = M_B\left(\alpha - \frac{\beta^2}{\alpha}\right) = M_B\left(\frac{\alpha^2 - \beta^2}{\alpha}\right)$$

$$M_B = \text{Stiffness Factor } K = \frac{\alpha}{\alpha^2 - \beta^2} \qquad \ldots\ldots (23)$$

5. <u>Fixed End Moments and Shears</u>

The fixed end moments and shears are found from equation (19). However, before proceeding to derive these expressions, due consideration must be given to the end shears which are affected by the unequal distribution of inertia force from the girder as the frame vibrates. Consider Figure 13 (a) which shows the forces acting on the girder and joints as one end is fixed and the other end displaced an amount δ .

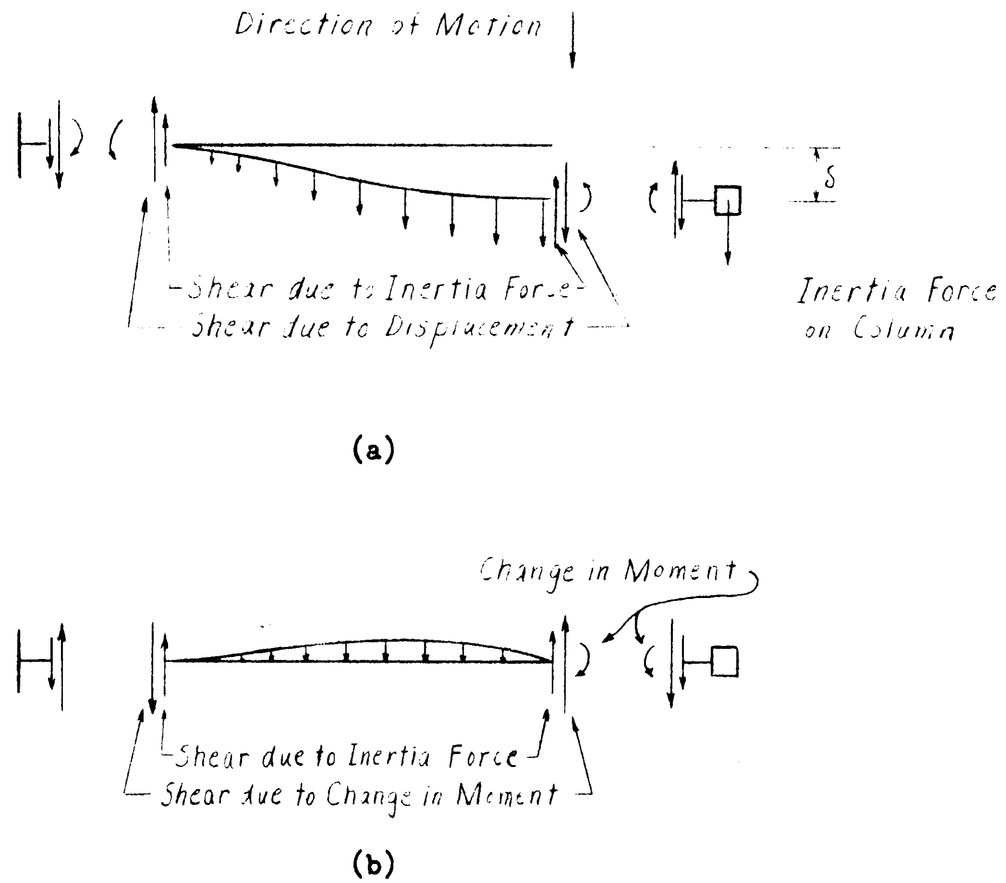


Figure 13. Effect of Change in Moment

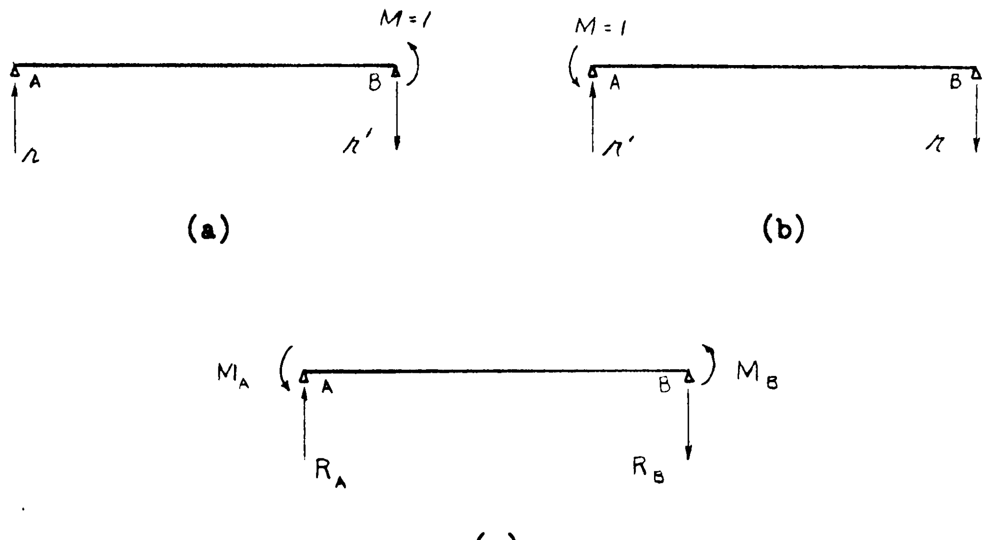


Figure 14. Derivation for r and r[1]

In the actual case, since the joints are allowed to rotate, the fixed end moments in the girder are distributed throughout the frame as the column translates. Thus, a change in the end moment of the girder will produce an unequal distribution of the inertia forces which depend upon the mass of the column. Figure 13 (b) shows the change in moment acting at B after joint B rotates, and also shows the resulting shears.

For the Vierendeel girder, both ends of a girder can have a change in moment which will affect the distribution of the inertia forces. Figures 14 (a) and (b) show simply supported beams acted upon by unit moments at any time. Let r^1 be the reaction at the end where the moment is applied, and r be the reaction at the opposite support. Figure 14 (c) shows the applied change in moments at supports A and B as M_A and M_B respectively. Thus, the reaction at A and B are:

$$R_A = M_A(r') + M_B(r) \qquad \text{..... (f)}$$

$$R_B = M_A(r) + M_B(r') \qquad \text{.... (g)}$$

The third derivative of equation (19) is:

$$EIX''' = -\frac{Mk}{2}\left(\frac{\cos kx}{\sin kL} + \frac{\cosh kx}{\sinh kL}\right) \qquad \text{.... (24)}$$

The values of the reactions r and r^1 in Figure 14 (a) will therefore be:

$$EIX'''(0) = r = -M\psi_5 \qquad \text{.... (25)}$$

$$EIX'''(L) = r' = -M\phi_5 \qquad \text{..... (26)}$$

where

$$\psi_5 = \frac{k}{2}\left(\frac{1}{\sin kL} + \frac{1}{\sinh kL}\right)$$

$$\varphi_5 = \frac{k}{2}\left(\cot kL + \coth kL\right)$$

The values of ψ_5 and φ_5 for various values of kl are tabulated in Table 2.

The values for the fixed end moments and shears will now be computed for a fixed - fixed beam due to a translation of one end by an amount δ . The boundary conditions are:

$$X(0) = 0 \qquad X(L) = \delta \qquad X'(0) = 0 \qquad X'(L) = 0$$

Upon evaluation of the constants of integration in equation (c), the solution for the shape function is:

$$X = \frac{\delta}{2(1-\cosh kL\cos kL)}\Big[(\sin kL + \sinh kL)(\sin kx - \sinh kx) + (\cos kL - \cosh kL)(\cos kx - \cosh kx)\Big] \quad \ldots(h)$$

The fixed end moments and shears are determined by taking the second and third derivatives of equation (h) and multiplying by EI in each case. By letting δ = 1, the resulting equations are:

$$_F M(0) = \frac{6EI}{L^2}\psi_2 \qquad \text{where} \quad \psi_2 = \frac{(kL)^2(\cos kL - \cosh kL)}{6(1-\cos kL\cosh kL)} \quad \ldots(27)$$

$$_F M(L) = \frac{6EI}{L^2}\varphi_2 \qquad \text{where} \quad \varphi_2 = \frac{(kL)^2(\sin kL \sinh kL)}{12(1-\cos kL\cosh kL)} \quad \ldots(28)$$

$$_F S(0) = \frac{12EI}{L^3}\psi_1 \qquad \text{where} \quad \psi_1 = \frac{(kL)^3(\sin kL + \sinh kL)}{12(1-\cos kL\cosh kL)} \quad \ldots(29)$$

$$_F S(L) = \frac{12EI}{L^3}\varphi_1 \qquad \text{where} \quad \varphi_1 = \frac{(kL)^3(\sin kL\cosh kL + \sinh kL\cos kL)}{12(1-\cos kL\cosh kL)} \quad \ldots(30)$$

Values for φ_1, ψ_1, φ_2 and ψ_2 are tabulated in Table 3 for various values of kl.

6. Conversion Factors for Physical Properties

For a rigid frame which is composed of members with various sizes and weights, it is convenient to select the moment of inertia, length and weight of a member which is most predominant throughout the structure. These values, however, need not correspond to the same member. Designate the above selected values as I_o, L_o and w_o. The corresponding values of the remaining members will be some percentage of the reference values. Thus, by letting the subscript i refer to any other member, the following relationships are established:

$$\frac{L_i}{L_o} = b_i \qquad \frac{w_i}{w_o} = c_i \qquad \frac{I_i}{I_o} = d_i \qquad \ldots (31)$$

From equations (b),

$$k^4 = \frac{p^2}{a^2} = \frac{p^2 w}{EIg}$$

Therefore, a $k_i\, L_i$ term with reference to $k_o\, L_o$ will be:

$$k_i L_i = b_i L_o \sqrt[4]{\frac{c_i w_o p^2}{E d_i I_o g}} = b_i \sqrt[4]{\frac{c_i}{d_i}}\; k_o L_o \qquad \ldots (32)$$

Let

$$\frac{k_i L_i}{k_o L_o} = h_i = b_i \sqrt[4]{\frac{c_i}{d_i}} \qquad \ldots (33)$$

7. Review of Procedure

The procedure to be followed using Raithel's method is outlined as follows:

a. Assume a value of k_oL_o.

b. Calculate the carry-over and stiffness factors for the members with reference to the tabulated values of φ and ψ.

c. Calculate the fixed end moments and shears and also the r and r^1 values.

d. Determine the change in moment at each joint by moment distribution for each column that is translated an amount $\delta = 1$.

e. Determine the final shears at each joint.

f. Write an equation of forces at each joint equating the inertia force of the column to the corresponding resisting forces at the joint.

g. From the equations of step (f), a determinant of high order is obtained. Solve the determinant using the Crout method* by substituting into $(k_oL_o)^4$, the assumed value of k_oL_o in step (a).

h. Assume a new value of k_oL_o and repeat the above procedure.

i. Plot the value of the determinant as ordinate and $(k_oL_o)^4$ as the abscissa. Where the curve crosses the abscissa is the answer of k_oL_o for the problem.

j. Now solve for the fundamental frequency using the equation:

$$p^2 = \frac{(k_o L_o)^4 E I_o g}{L_o^4 \; w_o}$$

*P. D. Crout, A Short Method for Evaluating Determinants and Solving Systems of Linear Equations with Real or Complex Coefficients, American Institute of Electrical Engineers, published 1941.

Some of the preceding steps in the above outline should be explained in detail. However, it is advisable at this time to consider the following notation for fixed end shears and moments. Figure 15 shows an arbitrary girder translated. The letter L refers to the end of the girder being displaced while the end that remains fixed is designated as the O end. The following convention will be used; fM (L) or f S (L) will refer to the fixed end moment or shear at the end of the girder displaced, while fM (o) and fS (o) will refer to the end held fixed.

The final shear computation in step (e) is found at a joint by adding to the fixed end shear the product of the r and r^1 values and their corresponding change in moment, as expressed by equations (f) and (g). The direction of shears is best determined by drawing a free body diagram of the moments and resulting shears for a particular member. Generally speaking, the shear resulting from a change in moment is subtracted from the fixed end shear.

The inertia force on the column, mentioned in step (f), is equal to the product of its mass and acceleration as it translates. Thus, for any translating column i:

$$F_i = (Mp^2X)_i = \frac{b_i c_i \, w_o L_o}{g} \frac{(k_o L_o)^4 E I_o g}{L_o^4} \frac{X_i}{w_o} = (k_o L_o)^4 b_i c_i \frac{E I_o}{L_o^3} X_i \qquad \ldots (34)$$

These forces are equated to the resisting forces at the corresponding joint as discussed in the introduction of this Section B. Thus:

$$(k_o L_o)^4 b_i c_i \frac{E I_o}{L_o^3} X_i = S_{ij} X_j \qquad \ldots (35)$$

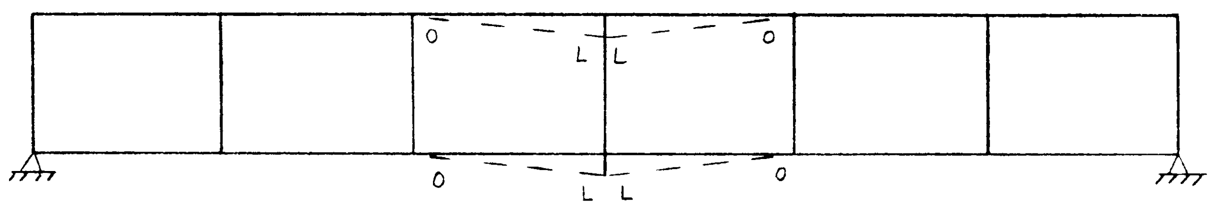


Figure 15. Convention for Fixed End Moments and Shears

Now consider in further detail steps (g), (h) and (i). Since Raithel's method is one of convergence, one can assume a value of k_oL_o and then solve the resulting determinant for the lowest root of $(k_oL_o)^4$; substitute this new value into the procedure and, in turn, find another term of k_oL_o. However, it has been found through experience that this convergence is rather slow and the solution for the smallest root of $(k_oL_o)^4$ is a rather tedious operation. Thus, according to step (g) it has been suggested that the resulting determinant be solved by the Crout method, which is a continuous operation on a computing machine, by substituting the assumed value of k_oL_o into $(k_oL_o)^4$. The graph that is plotted according to step (i) will give the value of k_oL_o which, if assumed and in turn substituted into the resulting determinant, will be the convergent value sought.

VI. APPLICATION TO A VIERENDEEL GIRDER

A. Method Based on the Principle of Virtual Displacements

The symmetrical Vierendeel girder for which the natural frequencies are calculated is shown in Figure 16. The analysis consists of two parts; one which is based on the derived equations considering the vertical shearing terms of the girders and a second which neglects the shearing terms.

It should be noted that only equations for joints one, two, three, four, and five were written due to the symmetry of the girder.

The first four natural frequencies were found in each case. Curves are shown in the appendix with the calculation data which represent the resulting terms of the determinant for various assumptions of p^2. Where the curve crosses the axis, the corresponding value of the abscissa is the value of p^2 which satisfies the determinant.

The following results were obtained for the frequencies in cycles per second.

	With V	Without V
p_1	10.3	9.3
p_2	27.6	24.0
p_3	34.4	33.9
p_4	52.6	48.8

B. Raithel's Method

Figure 17 shows the girder with the equivalent I, w, and L for each member. The moment of inertia, weight per unit length, and length of member A_1A_2 was chosen as the equivalent I_o, w_o and L_o.

The calculations were reduced considerably due to the symmetry of the girder. This was particularly noted for the moment distribution computations.

The values of k_oL_o which were assumed are: zero, 0.50, 0.75 and 0.90. The curve of the resulting determinant is shown in the computation data. The abscissa is crossed by the curve at 0.36. The resulting value of the first fundamental frequency is 13.8 cycles per second.

VII. DISCUSSION OF RESULTS

The results obtained for the Vierendeel girder based on the equations derived by the Principle of Virtual Displacements are listed once again.

	With V (cps)	Without V (cps)
p_1	10.3	9.3
p_2	27.6	24.0
p_3	34.4	33.9
p_4	52.6	48.8

For the first fundamental frequency, a difference in value of about 10% exists between that found with V and without V. For the higher mode frequencies, one would expect this difference to increase, but the results do not indicate such a trend. There is no apparent reason for this discrepency.

The value for the fundamental frequency calculated by Raithel's method was found to be 13.8 cycles per second. A difference of about 28% lies between this value and that found by the equations of virtual displacements considering the vertical shear V.

It should be noted that although Raithel's method is the more exact method of the two used in the preceding example, a considerable difference in time for calculating the fundamental frequency exists between the two methods. The time required to determine the first fundamental frequency based on the equations of virtual displacements for this particular problem required approximately eight hours. To find this same value using Raithel's method, required approximately 32 hours.

It has not been shown by Raithel's method how the higher mode frequencies can be found. However, such frequencies are a direct result from the equations based on virtual displacements.

As a suggestion for further work in deriving the equations by the Principle of Virtual Displacements, it might be advisable to write the equations based on the vertical displacements of the points of inflection of each column. The concentrated mass of each panel would be located at the center of the respective columns. This appears to be a better approximation than concentrating each mass at the lower joint.

VIII. ACKNOWLEDGMENTS

The author wishes to extend his sincere gratitude to Professor Grover L. Rogers who, as thesis advisor, gave generously his aid and advice to the author in order to overcome the obstacles and problems that were met while writing this thesis.

He also takes this opportunity to thank all of the professors at Virginia Polytechnic Institute who, through their class lectures, gave him the necessary background to write this thesis.

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XI. APPENDIX

A. Calculation of Fundamental Frequencies Using Principle of Virtual Displacements

1. Computation Data - With Shear Terms

Joint	Weight (lbs)	Mass $\left(\frac{lb-sec^2}{in}\right)$	J (in^4)	L^1 $(I_c \times 10^{-3})$	e (I_c)
0	—	—	2798	.0386	
1	6730	17.44	5454	.0198	.653
2	6083	15.76	3912	.0276	1.239
3	5237	13.57	2402	.0450	3.120
4	4427	11.47	1166	.0926	15.715
5	3770	9.77	290	.03724	0

Member	I (in^4)	h^1 $(I_c \times 10^{-3})$
01	2798	.0322
12	2798	.0343
23	2798	.0343
34	2798	.0343
45	2798	.0343

Joint 1

$$Z_1\left[-\frac{1}{e_1 h_{01}} - \frac{1}{e_1 h_{12}} - \frac{2}{h_{01}^2(L'_0+h'_{01})} + \frac{Q_1 p^2 L'_2}{48 E I_c e_1}(h_{12}+h_{23}) + \frac{Q_1 p^2 h_{12} h'_{12}}{24 E I_c e_1}\right]$$

$$+ Z_2\left[\frac{1}{e_1 h_{12}} + \frac{Q_2 p^2 h_{23} L'_2}{48 E I_c e_1}\right] = 0$$

$$Z_1\left[-2.3807 + \frac{14.39 p^2}{10^6}\right] + Z_2\left[1.0417 + \frac{2.90 p^2}{10^6}\right] = 0$$

Joint 2

$$Z_1\left[\frac{1}{e_2 h_{12}} + \frac{Q_1 p^2 h_{01} L_1'}{48 E I_c e_2}\right] + Z_2\left[-\frac{1}{e_2 h_{12}} - \frac{1}{e_2 h_{23}} - \frac{1}{e_3 h_{23}} + \frac{Q_2 p^2 L_1'}{48 E I_c e_2}(h_{01}+h_{12})\right.$$

$$\left. + \frac{Q_2 p^2 h_{12} h_{12}'}{24 E I_c e_2} - \frac{Q_2 p^2 h_{12} L_2'}{48 E I_c e_3}\right] + Z_3\left[\frac{1}{e_2 h_{23}} + \frac{1}{e_3 h_{23}} + \frac{1}{e_3 h_{34}} + \frac{Q_3 p^2 h_{34} L_3'}{48 E I_c e_2} - \frac{Q_3 p^2 L_4'}{48 E I_c e_3}(h_{34}+h_{45})\right.$$

$$\left. - \frac{Q_3 p^2 h_{34} h_{34}'}{24 E I_c e_3}\right] + Z_4\left[-\frac{1}{e_3 h_{34}} - \frac{Q_4 p^2 h_{45} L_4'}{48 E I_c e_3}\right] = 0$$

$$Z_1\left[1.0417 + \frac{2.16 p^2}{10^6}\right] + Z_2\left[-2.4970 + \frac{10.08 p^2}{10^6}\right] + Z_3\left[1.8690 - \frac{1.58 p^2}{10^6}\right] + Z_4\left[-.4137 - \frac{2.75 p^2}{10^6}\right] = 0$$

Joint 3

$$Z_2\left[\frac{1}{e_3 h_{23}} + \frac{Q_2 p^2 h_{12} L_2'}{48 E I_c e_3}\right] + Z_3\left[-\frac{1}{e_3 h_{23}} - \frac{1}{e_3 h_{34}} - \frac{1}{e_4 h_{34}} + \frac{Q_3 p^2 L_2'}{48 E I_c e_3}(h_{12}+h_{23})\right.$$

$$\left. + \frac{Q_3 p^2 h_{23} h_{23}'}{24 E I_c e_3} - \frac{Q_3 p^2 h_{23} L_3'}{48 E I_c e_4}\right] + Z_4\left[\frac{1}{e_3 h_{34}} + \frac{1}{e_4 h_{34}} + \frac{1}{e_4 h_{45}} + \frac{Q_4 p^2 h_{45} L_4'}{48 E I_c e_3} - \frac{Q_4 p^2 L_5'}{48 E I_c e_4}(h_{45}+h_{56})\right.$$

$$\left. - \frac{Q_4 p^2 h_{45} h_{45}'}{24 E I_c e_4}\right] + Z_5\left[-\frac{1}{e_4 h_{45}} - \frac{Q_5 p^2 h_{56} L_5'}{48 E I_c e_4}\right] = 0$$

$$Z_2\left[.3339 + \frac{.96}{10^6}\right] + Z_3\left[-.7340 + \frac{3.33 p^2}{10^6}\right] + Z_4\left[.4664 - \frac{1.69 p^2}{10^6}\right] + Z_5\left[-.0663 - \frac{1.54 p^2}{10^6}\right] = 0$$

Joint 4

$$Z_3\left[\frac{1}{h_{34}} + \frac{Q_3 p^2 h_{23} L_3'}{48 E I_c}\right] + Z_4\left[-\frac{1}{h_{34}} - \frac{1}{h_{45}} + \frac{Q_4 p^2 h_{34} L_3'}{48 E I_c} + \frac{Q_4 p^2 h_{23} L_3'}{48 E I_c} + \frac{Q_4 p^2 h_{34} h_{34}'}{24 E I_c}\right]$$

$$+ Z_5\left[\frac{1}{h_{45}} + \frac{Q_5 p^2 h_{56} L_5'}{48 E I_c} - \frac{Q_5 p^2 e_4}{24 E I_c}\right] = 0$$

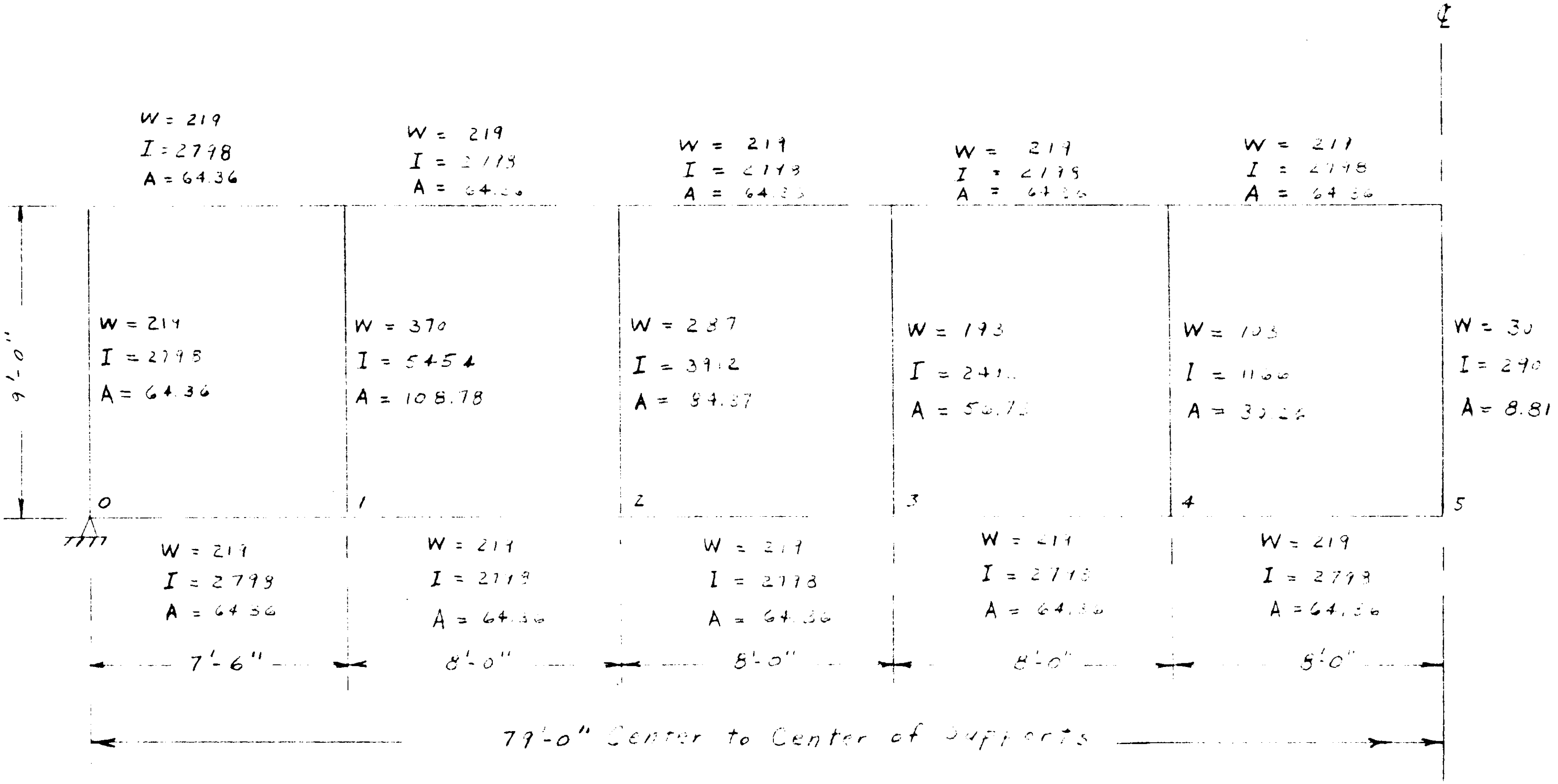
$$Z_3\left[1.0417 + \frac{4.06}{10^6} p^2\right] + Z_4\left[-2.0833 + \frac{12.12}{10^6} p^2\right] + Z_5\left[1.0417 + \frac{24.05}{10^6} p^2\right] = 0$$

Joint 5

$$Z_4\left[\frac{2}{h_{45}} + \frac{Q_4 p^2 h_{34} L_4'}{24 E I_c}\right] + Z_5\left[-\frac{2}{h_{45}} - \frac{2}{h_{56}} + \frac{Q_5 p^2 L_4'}{48 E I_c}(h_{34}+h_{45})\right.$$

$$\left. + \frac{Q_5 p^2 h_{45} h_{45}'}{24 E I_c} + \frac{Q_5 p^2 L_6'}{48 E I_c}(h_{56}+h_{67}) + \frac{Q_5 p^2 h_{56} h_{56}'}{24 E I_c}\right] + Z_6\left[\frac{2}{h_{56}} + \frac{Q_6 p^2 h_{67} L_6'}{24 E I_c}\right] = 0$$

$$Z_4\left[2.0833 + \frac{14.15}{10^6} p^2\right] + Z_5\left[-4.1667 + \frac{33.06}{10^6} p^2\right] + Z_6\left[2.0833 + \frac{14.15}{10^6} p^2\right] = 0$$



W = lbs per linear foot

$I = in^4$

$A = in^2$

Figure 16. Dimensions of Vierendeel Girder

Equations for Odd Mode Frequencies

$$
\left|\begin{array}{ccccc}
Z_1 & + \; Z_2 & + \; Z_3 & + \; Z_4 & + \; Z_5 \\
\hline
\left(-2.3807 + \frac{14.39}{10^6}p^2\right) & \left(1.0417 + \frac{2.90}{10^6}p^2\right) & & & \\
\left(1.0417 + \frac{2.16}{10^6}p^2\right) & \left(-2.4970 + \frac{10.08}{10^6}p^2\right) & \left(1.8690 - \frac{1.58}{10^6}p^2\right) & \left(-.4137 - \frac{2.75}{10^6}p^2\right) & \\
 & \left(.3339 + \frac{.96}{10^6}p^2\right) & \left(-.7340 + \frac{3.33}{10^6}p^2\right) & \left(.4664 - \frac{1.69}{10^6}p^2\right) & \left(-.0663 - \frac{1.54}{10^6}p^2\right) \\
 & & \left(1.0417 + \frac{4.06}{10^6}p^2\right) & \left(-2.0833 + \frac{12.12}{10^6}p^2\right) & \left(1.0417 + \frac{24.05}{10^6}p^2\right) \\
 & & & \left(4.1667 + \frac{28.30}{10^6}p^2\right) & \left(-4.1667 + \frac{33.06}{10^6}p^2\right)
\end{array}\right| = 0
$$

Equations for Even Mode Frequencies

$$
\left|\begin{array}{ccccccc}
Z_1 & + & Z_2 & + & Z_3 & + & Z_4 \\
\hline
\left(-2.3807 + \frac{14.39}{10^6} p^2\right) & & \left(1.0417 + \frac{2.90}{10^6} p^2\right) & & & & \\
\left(1.0417 + \frac{2.16}{10^6} p^2\right) & & \left(-2.4970 + \frac{10.08}{10^6} p^2\right) & & \left(1.8690 - \frac{1.580}{10^6} p^2\right) & & \left(-.4137 - \frac{2.75}{10^6} p^2\right) \\
 & & \left(.3339 + \frac{.96}{10^6} p^2\right) & & \left(-.7340 + \frac{3.33}{10^6} p^2\right) & & \left(.4664 - \frac{1.69}{10^6} p^2\right) \\
 & & & & \left(1.0417 + \frac{4.06}{10^6} p^2\right) & & \left(-2.0833 + \frac{12.12}{10^6} p^2\right)
\end{array}\right| = 0
$$

2. Computation Data - Without Shear Terms

Joint 1

$$Z_1\left[-\frac{1}{h_{01}}-\frac{1}{h_{12}}+\frac{Q_1p^2L_2'}{48EI_c}(h_{12}+h_{23})+\frac{Q_1p^2h_{12}h_{12}'}{24EI_c}\right]+Z_2\left[\frac{1}{h_{12}}+\frac{Q_2p^2h_{23}L_2'}{48EI_c}\right]=0$$

$$Z_1\left(-2.1528+\frac{14.39}{10^6}p^2\right)+Z_2\left(1.0417+\frac{2.90}{10^6}p^2\right)=0$$

Joint 2

$$Z_1\left[\frac{1}{h_{12}}+\frac{Q_1p^2h_{01}L_1'}{48EI_c}\right]+Z_2\left[-\frac{1}{h_{12}}-\frac{1}{h_{23}}+\frac{Q_2p^2L_1'}{48EI_c}(h_{01}+h_{12})+\frac{Q_2p^2h_{12}h_{12}'}{24EI_c}\right]$$

$$+Z_3\left[\frac{1}{h_{23}}+\frac{Q_3p^2h_{34}L_3'}{48EI_c}\right]=0$$

$$Z_1\left(1.0417+\frac{2.16}{10^6}p^2\right)+Z_2\left(-2.0833+\frac{11.23}{10^6}p^2\right)+Z_3\left(1.0417+\frac{4.07}{10^6}p^2\right)=0$$

Joint 3

$$Z_2\left[\frac{1}{h_{23}}+\frac{Q_2p^2h_{12}L_2'}{48EI_c}\right]+Z_3\left[-\frac{1}{h_{23}}-\frac{1}{h_{34}}+\frac{Q_3p^2L_2'}{48EI_c}(h_{12}+h_{23})+\frac{Q_3p^2h_{23}h_{23}'}{24EI_c}\right]$$

$$+Z_4\left[\frac{1}{h_{34}}+\frac{Q_4p^2h_{45}L_4'}{48EI_c}\right]=0$$

$$Z_2\left(1.0417+\frac{2.90}{10^6}p^2\right)+Z_3\left(-2.0833+\frac{11.61}{10^6}p^2\right)+Z_4\left(1.0417+\frac{7.07}{10^6}p^2\right)=0$$

Joint 4

$$Z_3\left[\frac{1}{h_{34}}+\frac{Q_3p^2h_{23}L_3'}{48EI_c}\right]+Z_4\left[-\frac{1}{h_{34}}-\frac{1}{h_{45}}+\frac{Q_4p^2L_3'}{48EI_c}(h_{23}+h_{34})+\frac{Q_4p^2h_{34}h_{34}'}{24EI_c}\right]$$

$$+Z_5\left[\frac{1}{h_{45}}+\frac{Q_5p^2h_{56}L_5'}{48EI_c}\right]=0$$

$$Z_3\left(1.0417+\frac{4.07}{10^6}p^2\right)+Z_4\left(-2.0833+\frac{12.12}{10^6}p^2\right)+Z_5\left(1.0417+\frac{24.26}{10^6}p^2\right)=0$$

Joint 5

$$Z_4\left[\frac{1}{h_{45}}+\frac{Q_4p^2h_{34}L_4'}{48EI_c}\right]+Z_5\left[-\frac{1}{h_{45}}-\frac{1}{h_{56}}+\frac{Q_5p^2L_4'}{48EI_c}(h_{34}+h_{45})+\frac{Q_5p^2h_{45}h_{45}'}{24EI_c}\right]$$

$$+Z_6\left[\frac{1}{h_{56}}+\frac{Q_6p^2h_{67}L_6'}{48EI_c}\right]=0$$

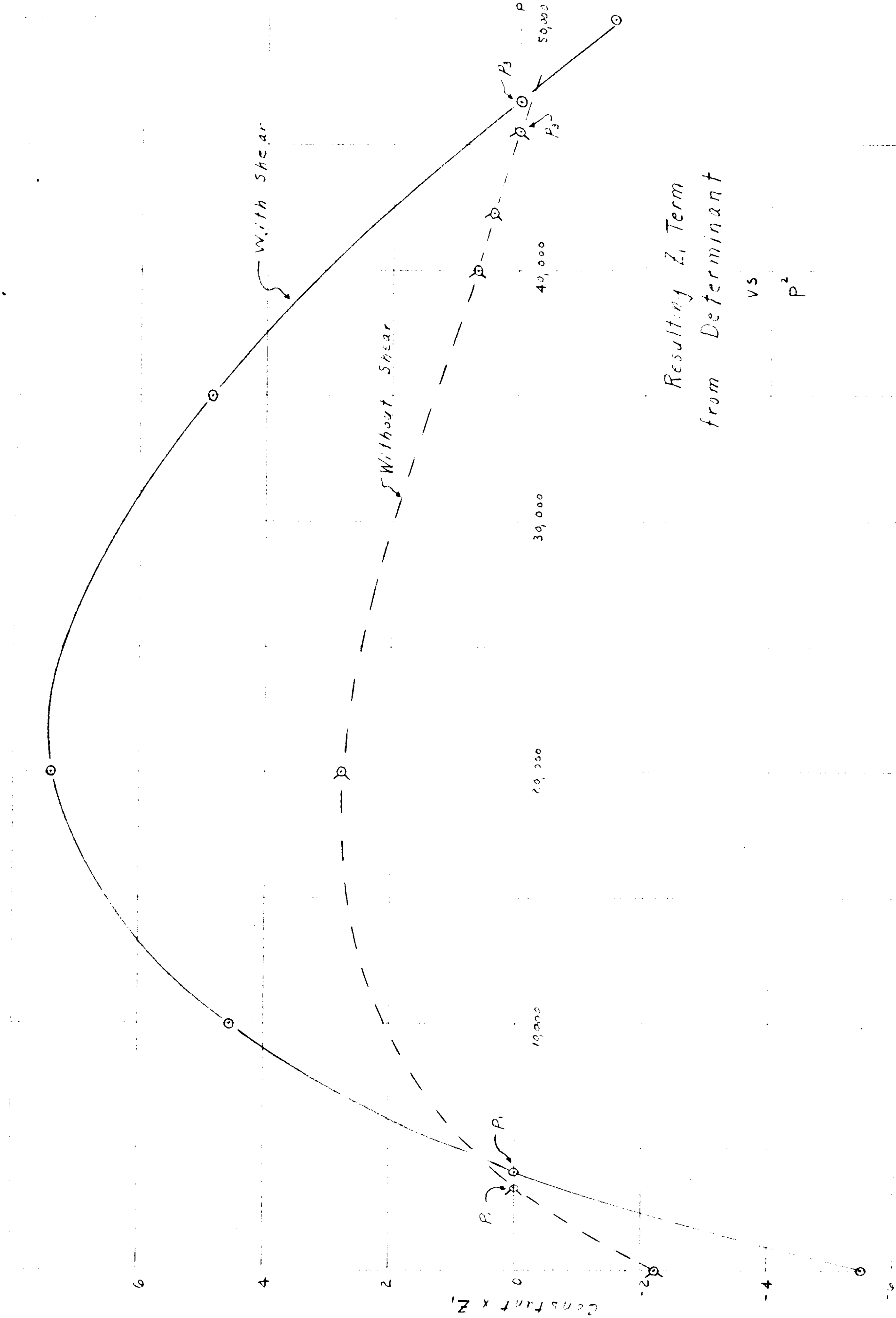
$$Z_4\left(1.0417+\frac{7.07}{10^6}p^2\right)+Z_5\left(-2.0833+\frac{16.54}{10^6}p^2\right)+Z_6\left(1.0417+\frac{7.07}{10^6}p^2\right)=0$$

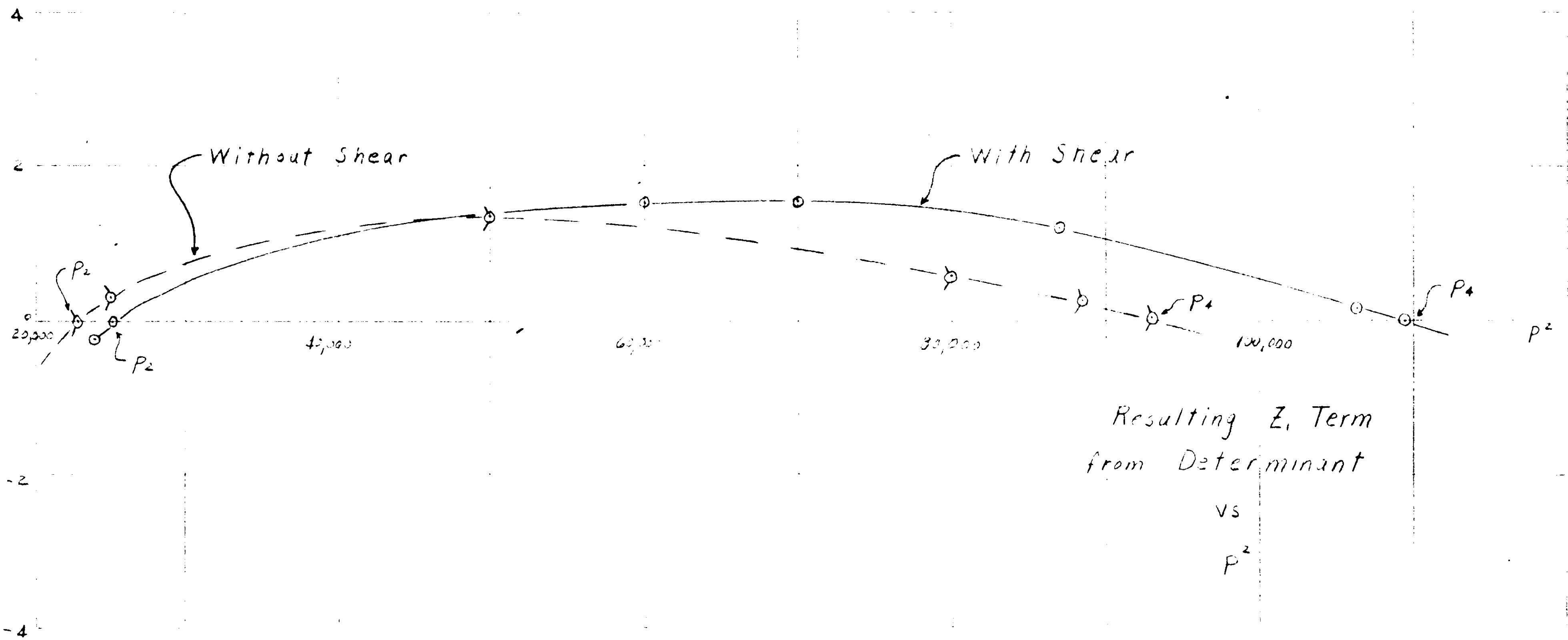
Equations for Odd Mode Frequencies

$$
\left|\begin{array}{ccccc}
Z_1 & + \; Z_2 & + \; Z_3 & + \; Z_4 & + \; Z_5 \\
\hline
\left(-2.1528 + \frac{14.39}{10^6}p^2\right) & \left(1.0417 + \frac{2.90}{10^6}p^2\right) & & & \\
\left(1.0417 + \frac{2.16}{10^6}p^2\right) & \left(-2.0833 + \frac{11.23}{10^6}p^2\right) & \left(1.0417 + \frac{4.07}{10^6}p^2\right) & & \\
 & \left(1.0417 + \frac{2.90}{10^6}p^2\right) & \left(-2.0833 + \frac{11.61}{10^6}p^2\right) & \left(1.0417 + \frac{7.07}{10^6}p^2\right) & \\
 & & \left(1.0417 + \frac{4.07}{10^6}p^2\right) & \left(-2.0833 + \frac{12.12}{10^6}p^2\right) & \left(1.0417 + \frac{24.26}{10^6}p^2\right) \\
 & & & \left(2.0834 + \frac{14.14}{10^6}p^2\right) & \left(-2.0833 + \frac{16.54}{10^6}p^2\right)
\end{array}\right| = 0
$$

Equations for Even Mode Frequencies

$$
\begin{array}{cccc|l}
Z_1 & + \quad Z_2 & + \quad Z_3 & + \quad Z_4 & = 0 \\
\hline
\left(-2.1528 + \frac{14.39}{10^6}p^2\right) & \left(1.0417 + \frac{2.90}{10^6}p^2\right) & & & \\
\left(1.0417 + \frac{2.16}{10^6}p^2\right) & \left(-2.0833 + \frac{11.23}{10^6}p^2\right) & \left(1.0417 + \frac{4.07}{10^6}p^2\right) & & \\
 & \left(1.0417 + \frac{2.90}{10^6}p^2\right) & \left(-2.0833 + \frac{11.61}{10^6}p^2\right) & \left(1.0417 + \frac{7.07}{10^6}p^2\right) & \\
 & & \left(1.0417 + \frac{4.07}{10^6}p^2\right) & \left(-2.0833 + \frac{12.12}{10^6}p^2\right) &
\end{array}
$$


Resulting Z, Term
from Determinant
vs
P²
With Shear
Without Shear
P₁
P₁
P₃
P₃
Constant x Z,
6
4
2
0
-2
-4
10,000
20,000
30,000
40,000
50,000
P²


Without Shear
With Shear
P_2
P_2
P_4
P_4
4
2
0
-2
-4
20,000
100,000
P^2
Resulting Z_1 Term
from Determinant
vs
P^2

B. Calculation of Fundamental Frequency Using Raithel's Method

1. Computation Data - Fourth Approximation

Assume $k_o L_o = 0.75$

Member $A_o A_1$, $B_o B_1$

$h = .938$ $\qquad$ $hk_o L_o = .7035$

$\varphi = 1.0045$ $\qquad$ $\alpha = .3141$

$\psi = 1.0087$ $\qquad$ $\beta = .1577$

$cf = .5021$

$$K = \frac{.3141}{.09866 - .02487} = 4.2567$$

$\psi_1 = 1.0026$ $\qquad$ $\varphi_1 = .9923$ $\qquad$ $\psi_2 = 1.0013$ $\qquad$ $\varphi_2 = .9979$

$$fS\ (0) = \frac{12\ (1.0026)}{(.938)^3} = 14.578 \qquad fS\ (L) = \frac{12\ (.9923)}{(.938)^3} = 14.428$$

$$fM\ (0) = \frac{6\ (1.0013)}{(.938)^2} = 6.828 \qquad fM\ (L) = \frac{6\ (.9979)}{(.938)^2} = 6.805$$

$$r = \frac{1.0048}{.938} = 1.0712 \qquad r^1 = \frac{.9945}{.938} = 1.0602$$

Members A_iA_{i+1} , B_iB_{i+1} $i = 1, 2, 3, 4$

$h = 1$ $hk_oL_o = .75$

$\varphi = 1.0048$ $\alpha = \dfrac{1.0048}{3} = .3349$

$\psi = 1.0093$ $\beta = \dfrac{1.0093}{6} = .1682$

$cf = .5022$ $K = \dfrac{.3349}{.11216 - .02829} = 3.9931$

$\psi_1 = 1.0034$ $\varphi_1 = .9898$ $\psi_2 = 1.0018$ $\varphi_2 = .9972$

$fS\ (0) = 12\ (1.0034) = 12.041$ $fS\ (L) = 12\ (.9898) = 11.878$

$fM\ (0) = 6\ (1.0018) = 6.011$ $fM\ (L) = 6\ (.9972) = 5.983$

$\mu = 1.0064$ $\mu^1 = .9927$

Member A_oB_o

$h = 1.125$ $hk_oL_o = .84375$

$\varphi = 1.0054$ $\alpha = \dfrac{1.0054\ (1.125)}{3} = .3370$

$\psi = 1.0105$ $\beta = \dfrac{1.125\ (1.0105)}{6} = .1895$

$cf = .5027$ $K = \dfrac{.3770}{.1421 - .0359} = 3.5499$

Member A_1B_1

$h = 1.085$ $\qquad hk_oL_o = .81375$

$\varphi = 1.0052$ $\qquad \alpha = \dfrac{1.125\ (1.0052)}{3\ (1.95)} = .1933$

$\psi = 1.0101$ $\qquad \beta = \dfrac{1.125\ (1.0101)}{6\ (1.95)} = .0971$

$cf = .5023$ $\qquad K = \dfrac{.1933}{.03736 - .00943} = 6.9209$

Member A_2B_2

$h = 1.1081$ $\qquad hk_oL_o = .8311$

$\varphi = 1.0053$ $\qquad \alpha = \dfrac{1.125\ (1.0053)}{3\ (1.40)} = .2693$

$\psi = 1.0103$ $\qquad \beta = \dfrac{1.125\ (1.0103)}{6\ (1.40)} = .1353$

$cf = .5024$ $\qquad K = \dfrac{.2693}{.0725 - .0183} = 4.9686$

Member A_3B_3

$h = 1.1475$ $\qquad hk_oL_o = .8606$

$\varphi = 1.0055$ $\qquad \alpha = \dfrac{1.125\ (1.0055)}{3\ (.86)} = .4384$

$\psi = 1.0107$ $\qquad \beta = \dfrac{1.125\ (1.0107)}{6\ (.86)} = .2204$

$cf = .5027$ $\qquad K = \dfrac{.4384}{.1922 - .0486} = 3.0529$

Member A_4B_4

$h = 1.1531$ $\qquad hk_oL_o = .8648$

$\varphi = 1.0055$ $\qquad \alpha = \dfrac{1.125\ (1.0055)}{3\ (.42)} = .8978$

$\psi = 1.0107$ $\qquad \beta = \dfrac{1.125\ (1.0107)}{6\ (.42)} = .4512$

$cf = .5026$ $\qquad K = \dfrac{.8978}{.8060 - .2036} = 1.4904$

Member A_5B_5

$h = 1.2251$ $\qquad hk_oL_o = .9188$

$\varphi = 1.0059$ $\qquad \alpha = \dfrac{1.125\ (1.0059)}{3\ (.10)} = 3.7721$

$\psi = 1.0114$ $\qquad \beta = \dfrac{1.125\ (1.0114)}{6\ (.10)} = 1.8964$

$cf = .5027$ $\qquad K = \dfrac{3.7721}{14.2287 - 3.5963} = .3548$

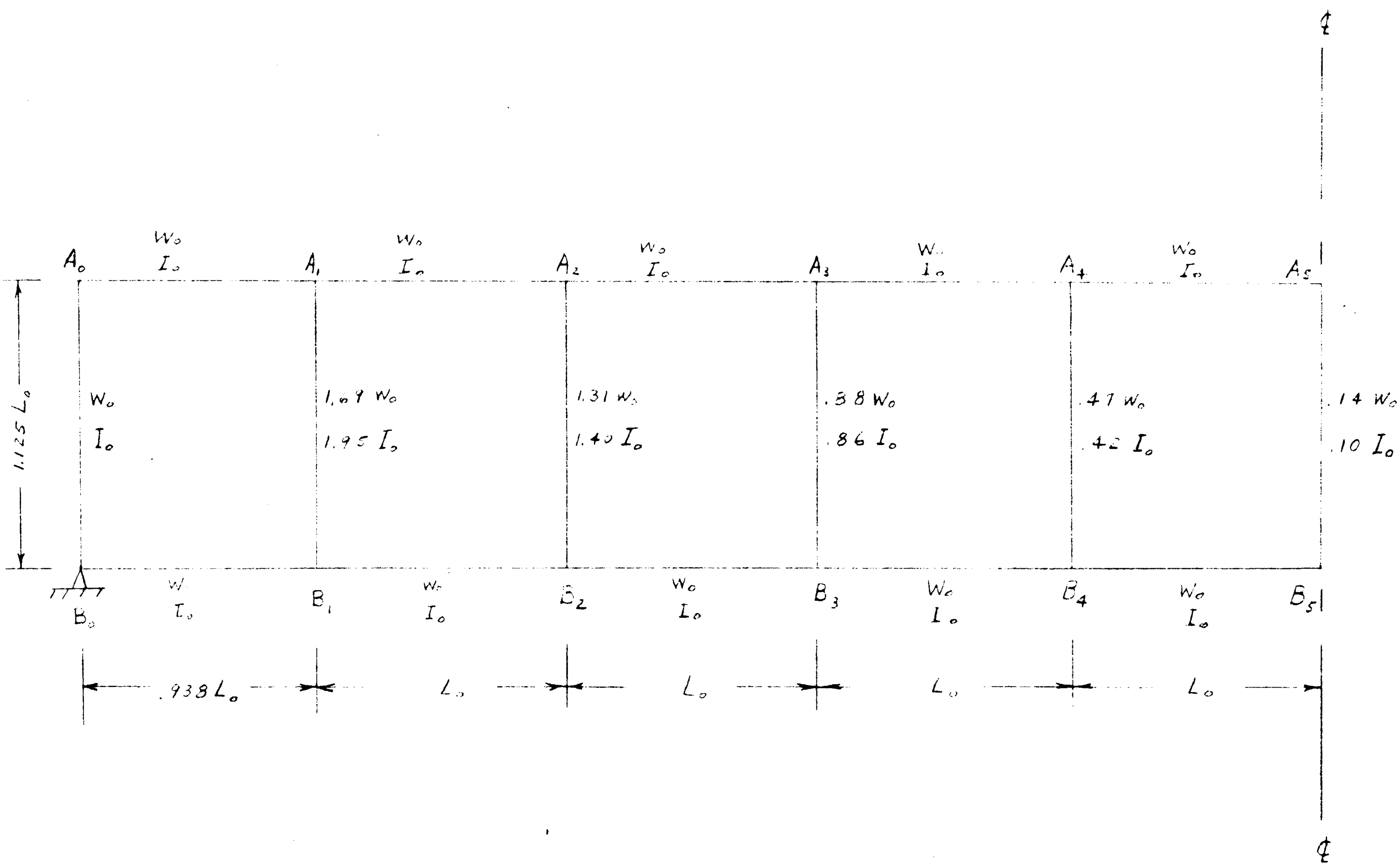
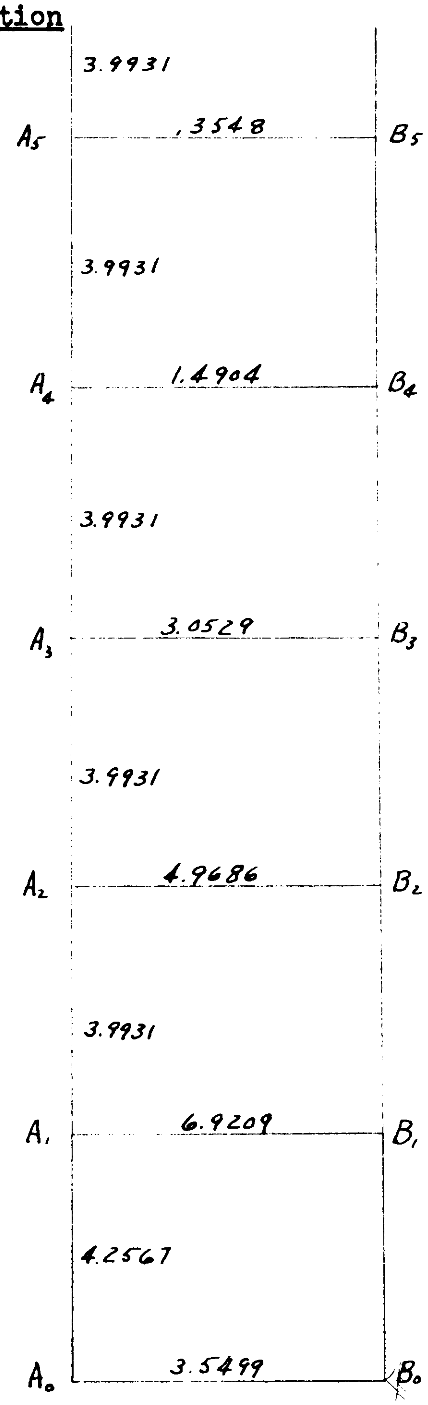


Figure 17. Equivalent Dimensions of Vierendeel Girder

Distribution Factors - Fourth Approximation



Joint 5	K	K/ΣK
$A_5 A_6$	3.9931	.4787
$A_5 B_5$	.3548	.0426
$A_5 A_4$	3.9931	.4787
	8.3410	1.0000

Joint 4		
$A_4 A_5$	3.9931	.4214
$A_4 B_4$	1.4904	.1572
$A_4 A_3$	3.9931	.4214
	9.4766	1.0000

Joint 3		
$A_3 A_4$	3.9931	.3617
$A_3 B_3$	3.0529	.2766
$A_3 A_2$	3.9931	.3617
	11.0391	1.0000

Joint 2		
$A_2 A_3$	3.9931	.3082
$A_2 B_2$	4.9686	.3836
$A_2 A_1$	3.9931	.3082
	12.9548	1.0000

Joint 1		
$A_1 A_2$	3.9931	.2632
$A_1 B_1$	6.9209	.4562
$A_1 A_0$	4.2567	.2806
	15.1707	1.0000

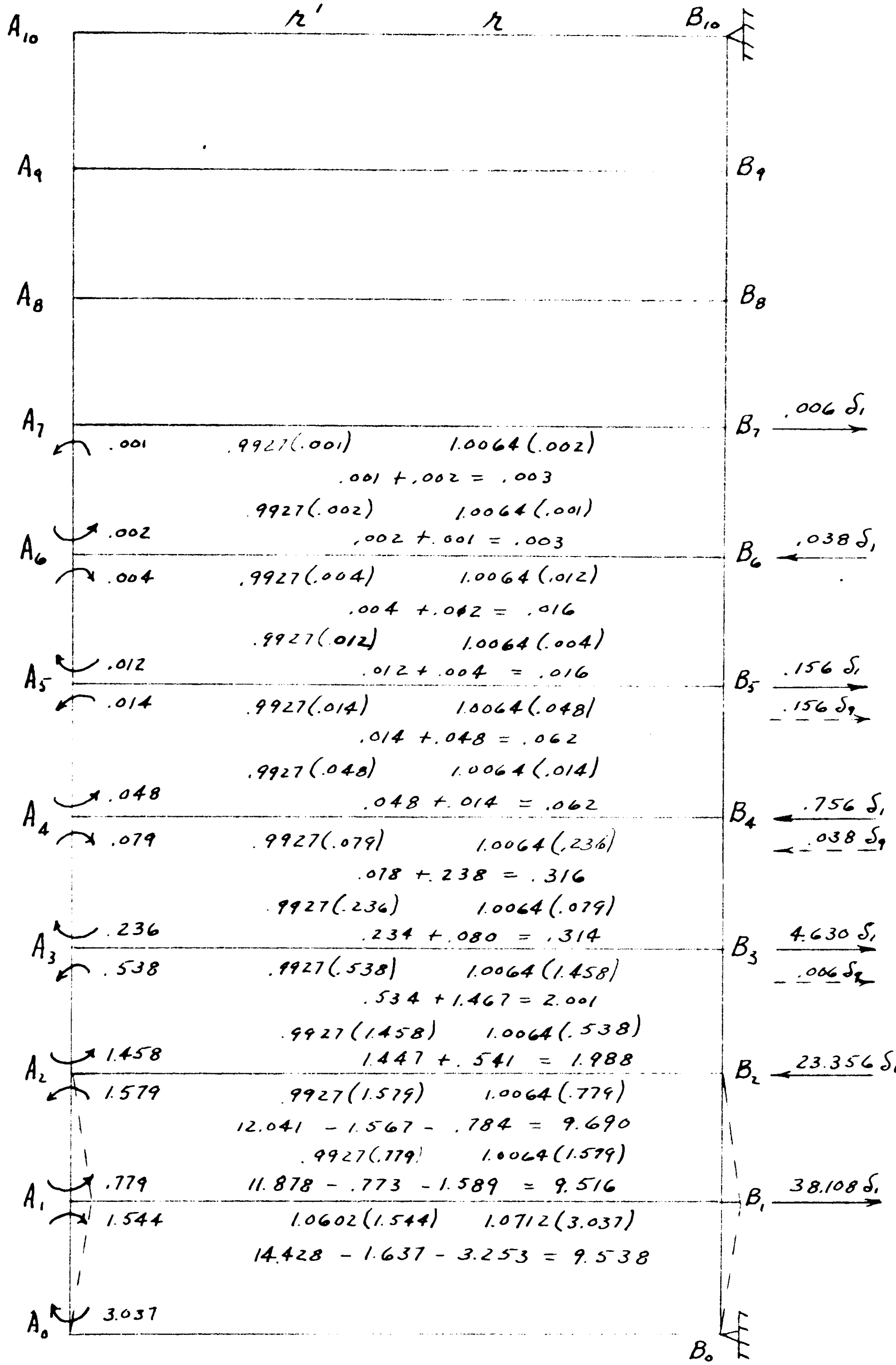
Joint 0		
$A_0 A_1$	4.2567	.5453
$A_0 B_0$	3.5499	.4547
	7.8066	1.0000

Fourth Approximation - Moments Due To δ_1

Joint	A_0		A_1			A_2			A_3			A_4			A_5		
Member	A_0B_0	A_0A_1	A_1A_0	A_1B_1	A_1A_2	A_2A_1	A_2B_2	A_2A_3	A_3A_2	A_3B_3	A_3A_4	A_4A_3	A_4B_4	A_4A_5	A_5A_4	A_5B_5	A_5A_6
K/ΣK	.4547	.5453	.2806	.4562	.2632	.3082	.3836	.3082	.3617	.2766	.3617	.4214	.1572	.4214	.4787	.0426	.4787
cf K/ΣK	.2286	.2738	.1409	.2291	.1322	.1548	.1927	.1548	.1816	.1390	.1816	.2116	.0790	.2116	.2404	.0214	.2404
		6.828	6.805		-5.983	-6.011											
			-1.870	.027	.931	.015	1.158		.931								
	-1.561	.016	.423	-.039	-.182	-.023	-.226		-.182								
	.353	-.024	-.097	-.010	.039	-.006	.68	-.143	.039								
	-.081	-.006	.029	.004	.015	.003	.018	.017	.015	-.110							
	.024	.003	-.005	-.001	-.004	-.001	-.006	-.007	-.004	.013	.027	-.143					
	-.004	-.001			.002		.002		.002	-.005		.017	.010	-.006	+.027	-.001	.001
												-.007					
	-1.269	6.816	5.285	-.019	-5.182	-6.023	.994	-.133	.801	-.102	.027	-.133	.010	-.006	.027	-.001	.001
	-2.522	-3.025	-.024	-.038	-.022	1.591	1.980	1.591	-.263	-.200	-.263	.054	.020	.054	-.013	-.001	-.013
	-3.791	3.791	5.261	-.057	-5.204	-4.432	2.974	1.458	.538	-.302	-.236	-.079	.030	.048	.014	-.002	-.012
		-6.828	-6.805		5.983	6.011											
		-3.037	-1.544		.779	1.579											

Joint	A_6			A_7			A_8			A_9			A_{10}	
Member	A_6A_5	A_6B_6	A_6A_7	A_7A_6	A_7B_7	A_7A_8	A_8A_7	A_8B_8	A_8A_9	A_9A_8	A_9B_9	A_9A_{10}	$A_{10}A_9$	$A_{10}B_{10}$
K/ΣK	.4214	.1572	.4214	.3617	.2766	.3617								
cf K/ΣK	.2116	.0790	.2116	.1816	.1390	.1816								
	-.006													
				.001										
	-.006			.001										
	.002	.002	.002											
	-.004	.002	.002	.001										

Shear Due To Translation of Column # 1



Inertia Force of Column # 1

$$b_1 c_1 = 1.125 \frac{L_o}{L_o} \times 1.69 \frac{\omega_1}{\omega_o} = 1.901$$

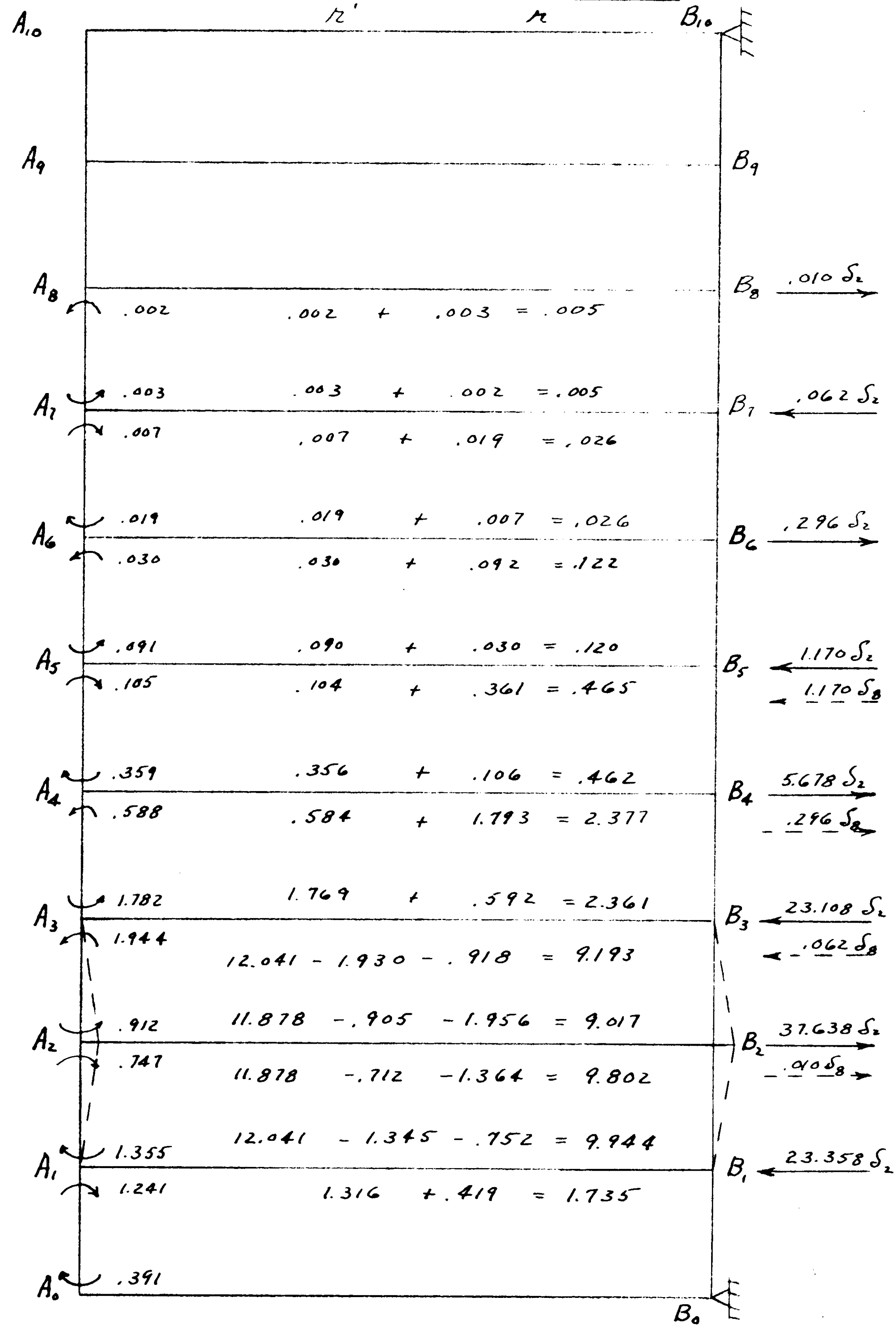
$$F_1 = 1.901 (k_o L_o)^4 \frac{E I_o}{L_o^3} \delta_1$$

Fourth Approximation - Moments Due To δ_2

Joint	A_0		A_1			A_2			A_3			A_4			A_5		
Member	A_0B_0	A_0A_1	A_1A_0	A_1B_1	A_1A_2	A_2A_1	A_2B_2	A_2A_3	A_3A_2	A_3B_3	A_3A_4	A_4A_3	A_4B_4	A_4A_5	A_5A_4	A_5B_5	A_5A_6
$K/\Sigma K$	.4547	.5453	.2806	.4562	.2632	.3082	.3836	.3082	.3617	.2766	.3617	.4214	.1572	.4214	.4787	.426	.4787
cf $K/\Sigma K$	.2286	.2738	.1409	.2291	.1322	.1548	.1927	.1548	.1816	.1390	.1816	.2116	.0790	.2116	.2404	.0214	.2404
					6.011	5.983		-5.983	-6.011								
		-.847		-1.377		-.795		1.092		.836		1.092					
		.194	.179	.315		.182		-.152		-.116		-.152					
	.149																
		-.070		-.113	-.044	-.065	-.055	.021	-.044	.016	-.203	.021					
	-.023		-.028										-.076		-.203		
		.022		.036	-.001	.021	-.001	.042	-.001	.032	-.003	.042		.049		.004	
			.006												-.003		-.010
	.005	-.002		-.003	.001	-.002	.001	-.005	.001	-.004		-.005	-.001	.002			
	.131	-.763	.157	-1.142	5.967	5.324	-.055	-4.983	-6.055	.764	-.206	.998	-.077	.051	-.206	.004	-.010
	.260	.312	-1.398	-2.273	-1.311	-.088	-.110	-.088	1.988	1.520	1.988	-.410	-.153	-.410	.101	.009	.101
	.391	-.391	-1.241	-3.415	4.656	5.236	-.165	-5.071	-4.067	2.284	1.784	.588	-.230	-.359	-.105	.013	.091
					-6.011	-5.983		5.983	6.011								
					-1.355	-.747		.912	1.944								

Joint		A_6			A_7			A_8			A_9		A_{10}	
Member	A_6A_5	A_6B_6	A_6A_7	A_7A_6	A_7B_7	A_7A_8	A_8A_7	A_8B_8	A_8A_9	A_9A_8	A_9B_9	A_9A_{10}	$A_{10}A_9$	$A_{10}B_{10}$
$K/\Sigma K$	.4214	.1572	.4214	.3617	.2766	.3617								
cf $K/\Sigma K$	.2116	.0790	.2116	.1816	.1390	.1816								
	.049	-.004		-.010			.002							
	.002		.002		.001									
	.051	-.004	.002	-.010	.001		.002							
	-.021	-.008	-.021	.003	.003	.003								
		-.012	-.019	-.007	.004	.003	.002							
	.030													

Shear Due To Translation of Column # 2



Inertia Force of Column # 2

$$b_2 c_2 = 1.125 \frac{L_o}{L_o} \times 1.31 \frac{\omega_o}{\omega_o} = 1.474$$

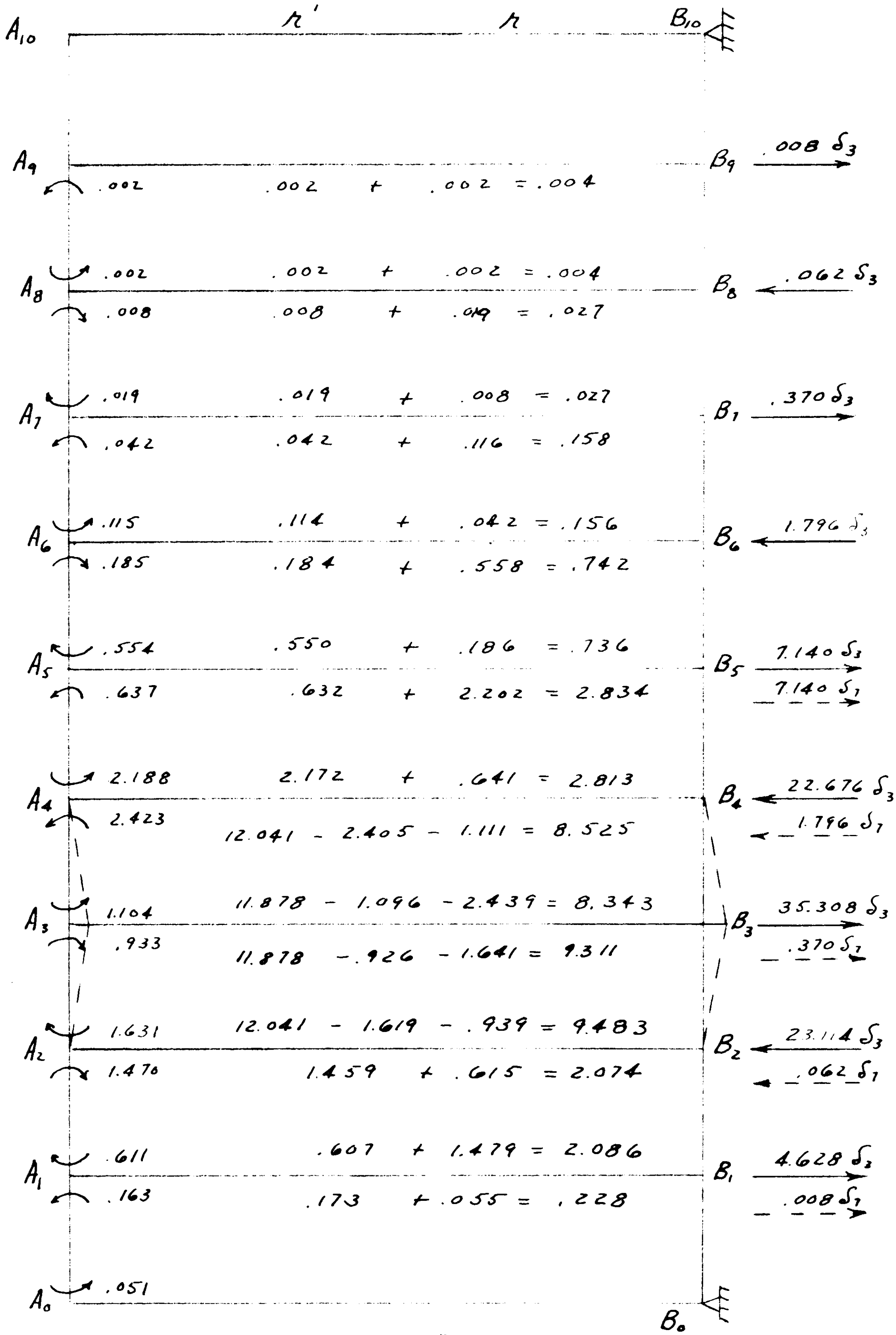
$$F_2 = 1.474\,(k_o L_o)^4 \frac{EI_o}{L_o^3}\,\delta_2$$

Fourth Approximation - Moments Due To δ_3

Joint	Ao		A1			A2			A3			A4			A5		
Member	AoBo	AoA1	A1Ao	A1B1	A1A2	A2A1	A2B2	A2A3	A3A2	A3B3	A3A4	A4A3	A4B4	A4A5	A5A4	A5B5	A5A6
K/ΣK	.4541	.5453	.2806	.4562	.2632	.3082	.3836	.3082	.3617	.2766	.3617	.4214	.1572	.4214	.4787	.0426	.4787
cf K/ΣK	.2286	.2738	.1409	.2291	.1322	.1548	.1927	.1548	.1816	.1390	.1816	.2116	.0790	.2116	.2404	.0214	.2404
								6.011	5.983		-5.983	-6.011					
					-.931		-1.158		-.931		1.272		.475	-.306	1.272		
					.179		.223		.179				-.013				
		.106		.172		.009				-.067	-.036				-.036		
	-.024		-.029				-.045					-.088	.008	-.005		-.027	
		-.015			-.036			-.088	-.036		.021				.021		.065
				-.025		-.014	.011										
	.009		.011		.009			.013	.009	.010	-.003	.013	-.001	.001	-.003	-.001	-.002
		.001				.001											
	-.002		-.003	.002	-.004		-.005		-.004								
	-.017	.092	-.021	.149	-.783	.086	-.974	5.936	5.200	-.057	-4.729	-6.086			1.254	-.028	.063
	-.034	-.041	.184	.299	.172	-1.556	-1.936	-1.556	-.150	-.115	-.150	2.498	.469	-.310	-.617	-.055	-.617
													.932	2.498			
	-.051	.051	.163	.448	-.611	-1.470	-2.910	4.380	5.050	-.172		-3.588	1.401	2.188	.637	-.083	-.554
								-6.011	-5.983		-4.879	6.011					
								-1.631	-.933		5.983	2.423					
											1.104						

Joint	A6			A7			A8			A9			A10	
Member	A6A5	A6B6	A6A7	A7A6	A7B7	A7A8	A8A7	A8B8	A8A9	A9A8	A9B9	A9A10	A10A9	A10B10
K/ΣK	.4214	.1572	.4214	.3617	.2766	.3617	.3082	.3836	.3082					
cf K/ΣK	.2116	.0790	.2116	.1816	.1390	.1816	.1548	.1927	.1548					
	-.306													
		.021		.065										
	-.005		-.012											
					-.009		-.012							
	.001	-.001	.002	-.002										
					.001	.002	.002	.002		.002				
	-.310	.023	-.010	.063	-.008	.002	-.010	.002		.002				
	.125	.047	.125	-.021	-.016	-.021	.002	.003	.002					
	-.185	.070	.115	.042	-.024	-.019	-.008	.005	.002	.002				

Shear Due To Translation of Column # 3



Inertia Force of Column # 3

$$b_3 c_3 = 1.125 \frac{L_o}{L_o} \times .88 \frac{\omega_o}{\omega_o} = .990$$

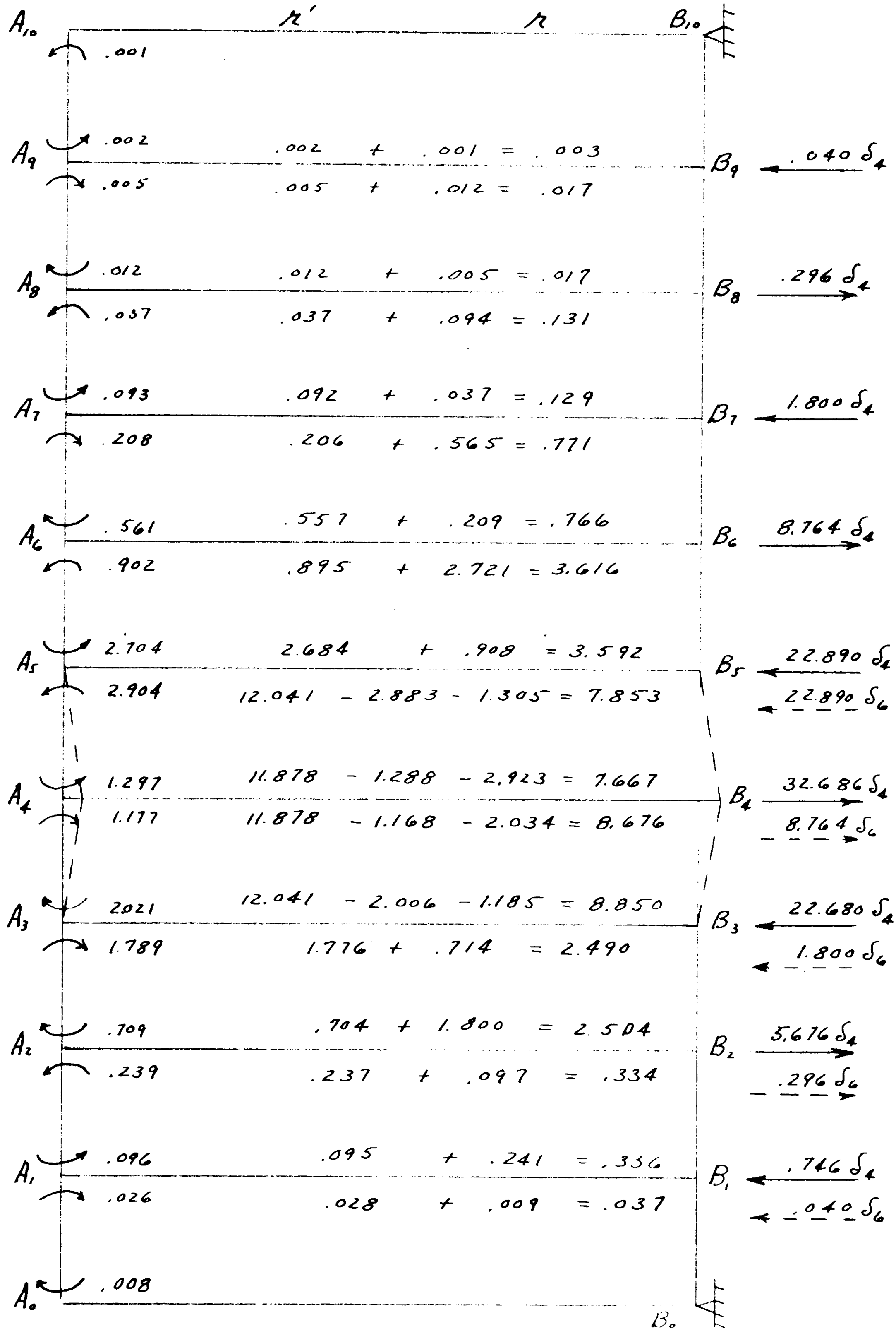
$$F_3 = .990 (k_o L_o)^4 \frac{E I_o}{L_o^3} \delta_3$$

Fourth Approximation - Moments Due To δ_4

Joint	A_o		A_1			A_2			A_3			A_4			A_5		
Member	A_oB_o	A_oA_1	A_1A_o	A_1B_1	A_1A_2	A_2A_1	A_2B_2	A_2A_3	A_3A_2	A_3B_3	A_3A_4	A_4A_3	A_4B_4	A_4A_5	A_5A_4	A_5B_5	A_5A_6
$K/\Sigma K$	.4547	.5453	.2806	.4562	.2632	.3082	.3836	.3082	.3617	.2766	.3617	.4214	.1572	.4214	.4787	.0426	.4787
cf $K/\Sigma K$	.2286	.2738	.1409	.2291	.1322	.1548	.1927	.1548	.1816	.1390	.1816	.2116	.0790	.2116	.2404	.0214	.2404
											6.011	5.983		-5.983	-6.011		
								-1.092		-.836		-1.092		1.445			
					.169		.210	.121	.169	.093		.121		.043		.129	-.306
		-.024		-.039	-.048	-.022	-.060	-.009	-.048	-.007	-.107	-.009				.004	-.002
	.003	.012	.003	.020	.007	.012	.009	.021	.007	.016		.021	-.040	.025	-.107	.002	.001
		-.004		-.007	-.003	-.004	-.003	-.002	-.003	-.002		-.002					
		.001		.002		.001											
	.003	-.015	.003	-.024	.123	-.013	.156	-.961	.125	-.736	5.904	5.022	-.040	-4.470	-6.118	.135	-.307
	.005	.007	-.029	-.047	-.027	.252	.314	.252	-1.914	-1.464	-1.914	-.216	-.080	-.216	3.011	.268	3.011
	.008	-.008	-.026	-.071	.096	.239	.470	-.709	-1.789	-2.200	3.990	4.806	-.120	-4.68[illegible]	-3.107	.403	2.704
											-6.011	-5.983		5.983	6.011		
											-2.021	-1.177		1.297	2.904		

Joint		A_6		A_7			A_8			A_9			A_{10}	
Member	A_6A_5	A_6B_6	A_6A_7	A_7A_6	A_7B_7	A_7A_8	A_8A_7	A_8B_8	A_8A_9	A_9A_8	A_9B_9	A_9A_{10}	$A_{10}A_9$	$A_{10}B_{10}$
$K/\Sigma K$	.4214	.1572	.4214	.3617	.2766	.3617	.3082	.3836	.3082	.2632	.4562	.2806	.5453	.4547
cf $K/\Sigma K$	.2116	.0790	.2116	.1816	.1390	.1816	.1548	.1927	.1548	.1322	.2291	.1409	.2738	.2286
	1.445													
	.043	-.114	.056	-.306	.043		.056							
	.025	-.001		-.002		-.009		-.011		-.009				
		.001	-.006	.001	-.004	.003	-.006	.003	.001	.003	.001		.001	
	1.513	-.114	.050	-.307	.039	-.006	.050	-.008	.001	-.006	.001		.001	
	-.611	-.228	-.611	.099	.076	.099	-.013	-.016	-.013	.001	.002	.002		
	902	-.342	-.561	-.208	.115	.093	.037	-.024	-.012	-.005	.003	.002	.001	

Shear Due To Translation of Column # 4



Inertia Force of Column # 4

$$b_4 c_4 = 1.125 \frac{L_o}{L_o} \times .47 \frac{\omega_o}{\omega_o} = .529$$

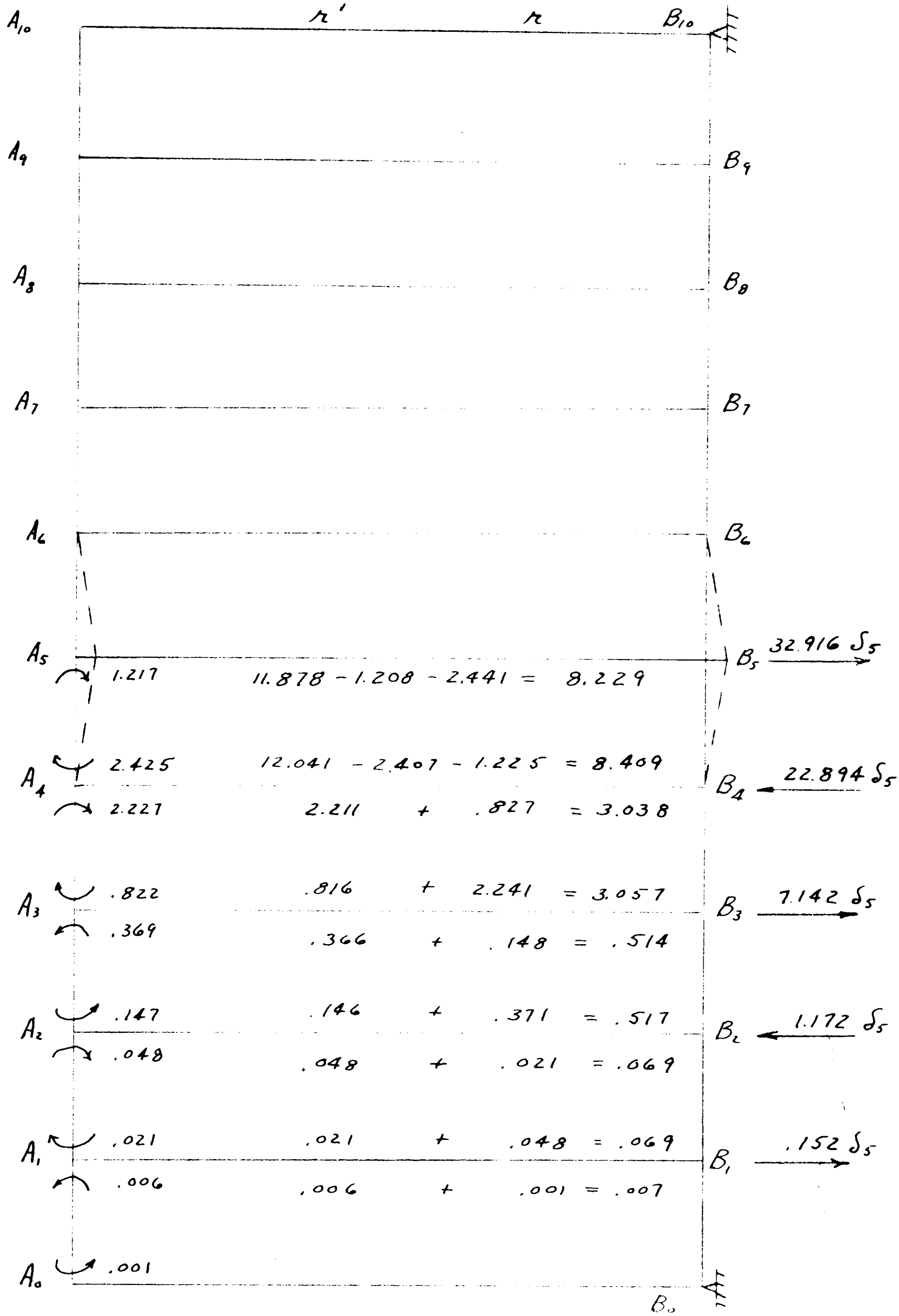
$$F_4 = .529 (k_o L_o)^4 \frac{E I_o}{L_o^3} \delta_4$$

Fourth Approximation - Moments Due To δ_5

Joint	A0		A1			A2			A3			A4			A5		
Member	A_0B_0	A_0A_1	A_1A_0	A_1B_1	A_1A_2	A_2A_1	A_2B_2	A_2A_3	A_3A_2	A_3B_3	A_3A_4	A_4A_3	A_4B_4	A_4A_5	A_5A_4	A_5B_5	A_5A_6
$K/\Sigma K$	.4547	.5453	.2806	.4562	.2632	.3082	.3836	.3082	.3617	.2766	.3617	.4214	.1572	.4214	.4787	.0426	.4787
cf $K/\Sigma K$	.2286	.2738	.1409	.2291	.1322	.1548	.1927	.1548	.1816	.1390	.1816	.2116	.0790	.2116	.2404	.0214	.2404
														6.011	5.983		
											-1.272		-.475		-1.272		
								.231		.177	.052	.231	.019		.052		
					-.036	.005	-.045		-.036	-.027	.003	-.035	.001		.003		
		.005		.008	.012	-.003	.014	-.035	.012	.002		.002					
		-.003		-.005	-.002	.001	-.003	.002	-.002								
		.001		.002													
		.003		.005	-.026	.003	-.034	.198	-.026	.152	-1.217	.198	-.455	6.011	4.766		
	-.001	-.002	.006	.010	.005	-.051	-.064	-.051	.395	.302	.395	-2.425	-.904	-2.425			
	-.001	.001	.006	.015	-.021	-.048	-.098	.147	.369	.454	-.822	-2.227	-1.359	3.586	4.766		
														-6.011	-5.983		
														-2.425	-1.217		

	A6			A7			A8			A9			A10	
	A_6A_5	A_6B_6	A_6A_7	A_7A_6	A_7B_7	A_7A_8	A_8A_7	A_8B_8	A_8A_9	A_9A_8	A_9B_9	A_9A_{10}	$A_{10}A_9$	$A_{10}B_{10}$

Shear Due To Translation Of Column # 5



Inertia Force of Column # 5

$$b_5 c_5 = 1.125 \frac{L_0}{L_0} \times .14 \frac{w_0}{w_0} = .158$$

$$F_5 = .158 (k_0 L_0)^4 \frac{E I_0}{L_0^3} \delta_5$$

Equations of Forces on the Columns

Fourth Approximation

$$1.901\,(k_oL_o)^4\frac{EI_o}{L_o^3}X_1 = \frac{EI_o}{L_o^3}\Big(38.108\,X_1 - 23.358\,X_2 + 4.636\,X_3 - .786\,X_4$$
$$+ .152\,X_5\Big)$$

$$1.474\,(k_oL_o)^4\frac{EI_o}{L_o^3}X_2 = \frac{EI_o}{L_o^3}\Big(-23.356\,X_1 + 37.648\,X_2 - 23.176\,X_3 + 5.972\,X_4$$
$$- 1.172\,X_5\Big)$$

$$.990\,(k_oL_o)^4\frac{EI_o}{L_o^3}X_3 = \frac{EI_o}{L_o^3}\Big(4.636\,X_1 - 23.170\,X_2 + 35.678\,X_3 - 24.480\,X_4$$
$$+ 7.142\,X_5\Big)$$

$$.529\,(k_oL_o)^4\frac{EI_o}{L_o^3}X_4 = \frac{EI_o}{L_o^3}\Big(-.794\,X_1 + 5.974\,X_2 - 24.472\,X_3 + 41.450\,X_4$$
$$- 22.894\,X_5\Big)$$

$$.158\,(k_oL_o)^4\frac{EI_o}{L_o^3}X_5 = \frac{EI_o}{L_o^3}\Big(.312\,X_1 - 2.340\,X_2 + 14.280\,X_3 - 45.780\,X_4$$
$$+ 32.916\,X_5\Big)$$

Fifth Order Determinant **Let $(k_0 L_0)^4 = x$**

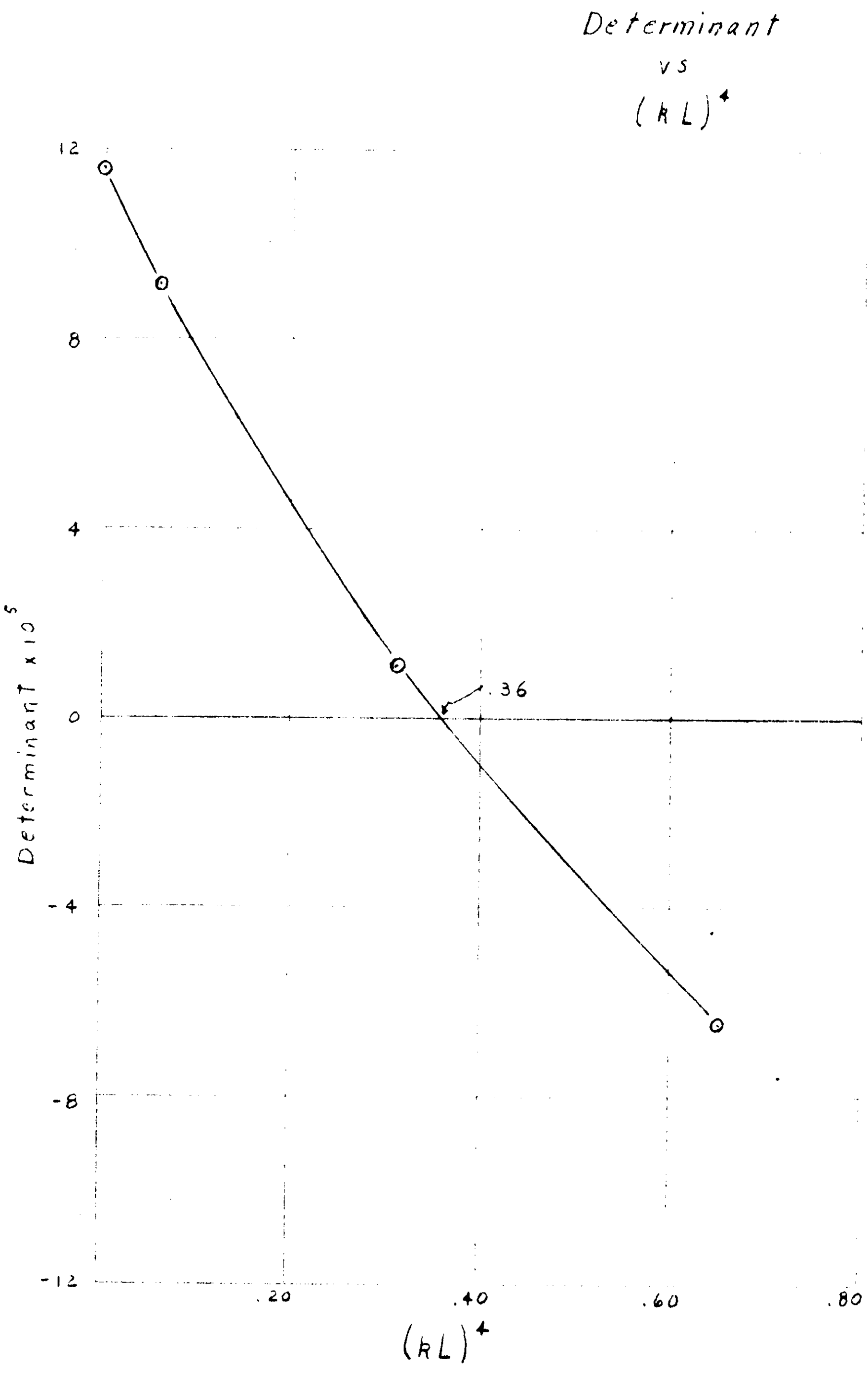
$$\begin{vmatrix} (38.108 - 1.901x) & -23.358 & 4.636 & -.786 & .152 \\ -23.356 & (37.648 - 1.474x) & -23.176 & 5.972 & -1.172 \\ 4.636 & -23.170 & (35.678 - .990x) & -24.480 & 7.142 \\ -.794 & 5.974 & -24.472 & (41.450 - .529x) & -22.894 \\ .312 & -2.340 & 14.280 & -45.780 & (32.916 - .158x) \end{vmatrix} = 0$$

Calculation For The Fundamental Frequency

$$p^2 = \frac{(k_0 L_0)^4 E I_0 g}{L_0^4 \, w_0}$$

$$p^2 = \frac{.36\,(30\times10^6)(2.798\times10^3)(386)}{(85\times10^6)\,(18.25)} = 7.52\times10^3$$

$$p = \frac{86.7}{2\pi} = 13.8 \text{ cps}$$


Determinant
vs
(kL)^4
12
8
4
0
-4
-8
-12
.36
.20
.40
.60
.80
Determinant x 10^5
(kL)^4

*
Table 1. Values of φ and ψ

kL	φ	ψ	kL	φ	ψ
0.00	1.0000	1.0000	2.62	1.5748	2.1305
1.00	1.0064	1.0124	2.64	1.6101	2.2004
1.10	1.0094	1.0183	2.66	1.6484	2,2658
1.20	1.0135	1.0261	2.68	1.6902	2.3592
1.30	1.0187	1.0302	2.70	1.7358	2.4497
1.40	1.0254	1.0492	2.72	1.7859	2.5493
1.50	1.0339	1.0657	2.74	1.8411	2.6589
1.60	1.0358	1.0865	2.76	1.9022	2.7806
1.70	1.0579	1.1125	2.78	1.9703	2.9160
1.80	1.0746	1.1449	2.80	2.0465	3.0676
1.90	1.0953	1.1854	2.82	2.1323	3.2384
2.00	1.1212	1.2360	2.84	2.2296	3.4323
2.05	1.1366	1.2661	2.86	2.3409	3.6542
2.10	1.1538	1.2997	2.88	2.4695	3.9104
2.15	1.1732	1.3377	2.90	2.6196	4.2096
2.20	1.1950	1.3807	2.92	2.7969	4.5635
2.25	1.2199	1.4294	2.94	3.0204	5.0059
2.30	1.2481	1.4849	2.96	3.2695	5.5666
2.35	1.2804	1.5486	2.98	3.5938	6.1546
2.40	1.3177	1.6219	3.00	4.0100	6.9863
2.45	1.3608	1.7071	3.02	4.5638	8.1214
2.50	1.4113	1.8068	3.04	5.3360	9.6366
2.52	1.4339	1.8582	3.06	6.4871	11.9372
2.54	1.4582	1.8995	3.08	8.3861	15.7351
2.56	1.4850	1.9509	3.10	9.8877	18.7414
2.58	1.5122	2.0063	3.12	22.7400	44.4508
2.60	1.5423	2.0659	π	∞	∞

* **Raithel, A., "La Dinamica Dei Sistemi Solidali**, Nota I, Giornale Genio Civile, V. 90, 1952, p 630

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Table 2. Values of φ_s and ψ_s

kL	φ_s	ψ_s	kL	φ_s	ψ_s
0.00	1.0000	1.0000	1.50	0.8818	1.1041
0.50	0.9986	1.0012	1.55	0.8642	1.1196
0.60	0.9971	1.0047	1.60	0.8446	1.1370
0.70	0.9946	1.0047	1.65	0.8227	1.1566
0.80	0.9908	1.0080	1.70	0.7982	1.1784
0.90	0.9853	1.0128	1.75	0.7710	1.2029
1.00	0.9776	1.0196	1.80	0.7409	1.2300
1.10	0.9670	1.0289	1.85	0.7067	1.2600
1.20	0.9530	1.0412	1.90	0.6689	1.2946
1.30	0.9283	1.0573	1.95	0.6267	1.3325
1.40	0.9114	1.0779	2.00	0.5797	1.3754

*Raithel, A., La Dinamica Dei Sistemi Soladali,
Nota II, Giornale Genio Civile, V. 90, 1952, p. 717-18

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Table 3. Values of φ_1, ψ_1, φ_2, and ψ_2

kL	φ_1	ψ_1	φ_2	ψ_2
0.00	1.0000	1.0000	1.0000	1.0000
0.50	0.9976	1.0002	0.9989	1.0001
0.60	0.9957	1.0011	0.9985	1.0004
0.70	0.9925	1.0025	0.9980	1.0013
0.80	0.9872	1.0042	0.9964	1.0021
0.90	0.9787	1.0070	0.9948	1.0034
1.00	0.9684	1.0103	0.9907	1.0050
1.10	0.9543	1.0154	0.9867	1.0070
1.20	0.9356	1.0222	0.9818	1.0107
1.30	0.9115	1.0309	0.9747	1.0145
1.40	0.8803	1.0411	0.9661	1.0198
1.50	0.8423	1.0547	0.9554	1.0264
1.55	0.8201	1.0625	0.9491	1.0301
1.60	0.7957	1.0713	0.9428	1.0341
1.65	0.7692	1.0814	0.9345	1.0388
1.70	0.7423	1.0959	0.9261	1.0438
1.75	0.7069	1.1029	0.9170	1.0497
1.80	0.6711	1.1157	0.9069	1.0565
1.85	0.6332	1.1294	0.8959	1.0621
1.90	0.5912	1.1442	0.8837	1.0694
1.95	0.5459	1.1608	0.8709	1.0771
2.00	0.4966	1.1785	0.8569	1.0856

*Raithel, A., La Dinamica Dei Sistemi Solidali, Nota II, Giornale Genio Civile, V. 90, 1952, p. 717-18