INVESTIGATION OF A MORE OPTIMUM TRAJECTORY THAN THE LOGARITHMIC SPIRAL IN SOLAR SAILING

by

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Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in candidacy for the degree of

MASTER OF SCIENCE

in

AEROSPACE ENGINEERING

July 1961

Blacksburg, Virginia

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LIST OF SYMBOLS

F_G	gravitational force
F_S	solar pressure force
θ	sail angle
m	total vehicle mass
$\bar{u}$	dimensional radial velocity (scalar component)
$\bar{v}$	dimensional tangential velocity (scalar component)
A	sail area
S_o	solar constant
r_o	reference radius
c	speed of light
r	radius (radial polar coordinate)
R	reflectivity factor
K^2	universal constant (gravitational)
M	sun's mass
A_T	tangential acceleration
A_R	radial acceleration
ϕ	angular polar coordinate
$(\dot{\ })$	differentiation with respect to dimensional time t
$(\ ')$	differentiation with respect to dimensionless time τ
α	pressure acceleration
a_o	gravitational acceleration
ρ	dimensionless radius

a	dimensionless parameter
u	dimensionless radial velocity (scalar component)
v	dimensionless tangential velocity (scalar component)
τ	dimensionless time

I. INTRODUCTION

Sailing through space in a vehicle propelled by solar pressure is a relatively new concept in propulsion. Needless to say, such a propulsive device has an inexhaustable supply of energy that can be utilized at a negligible cost. Though the thrust generated by such a device is necessarily small, thus restricting the use of such a system to travel through regions removed from planetary gravitational fields, the fact that the sail can be employed for the development of large speeds at little or no cost in energy more than compensates for its small propulsive power. The economy of operation in traveling large distances makes it competitive with more conventional thrusting systems.

It has been suggested by several investigators that a possible flight path for such a vehicle will be a logarithmic spiral. London[3], however, points out that the spiral is not the optimum time trajectory and suggested that other trajectories may be more efficient in the utilization of the solar sail. Assuming this to be the case, it is suggested that an investigation for a more efficient trajectory be made. Such is the study proposed in this thesis.

The logarithmic spiral path was first suggested by Garwin[1]. Proceeding from qualitative arguments, Garwin showed that the ratio of the radial velocity $(\bar{u})$ to the tangential velocity $(\bar{v})$

is independent of the path radius (r). Thus the path is an equiangular or logarithmic spiral. Tsu[2] assuming that the radial acceleration ($\dot{\bar{u}}$) was negligible, showed that the logarithmic spiral is a solution to the resulting approximate equations of motion. For values of the solar pressure acceleration (α) less than 0.25 cm/sec^2 the radial acceleration is less than 10 percent of the centrifugal acceleration, hence Tsu's assumption is a reasonable approximation within the above mentioned restriction on α. Bacon[8] in his studies showed that the logarithmic spiral represents a particular solution to the equations of motion; and London[3] extending Tsu's and Bacon's studies found that for values of $\alpha \leq 0.342$ cm/sec^2, the spiral is an exact solution for all sail angle settings. However, for values of $\alpha > 0.342$ cm/sec^2, there is a range of sail angle settings where the spiral solution is not valid. In another study, Hock, McMillan and Tanguay[5] also found that the logarithmic spiral is a solution to the problem but they failed to discuss its limitations.

The purpose of this thesis is to clearly define the limitations on the logarithmic spiral and to investigate the regions where the spiral fails. The limitation is obtained as a result of writing the equations of motion in dimensionless form and noting that one parameter, ''a'', which is the ratio of the radial to the tangential forces, plays a significant role on the shape of the trajectory. This parameter, as it appears in the dimensionless equations, is seen to imply that the logarithmic spiral is not a valid flight path when "a" $< 2\sqrt{2}$. In this range of values for "a" numerical solutions for the definition of the trajectory are required. These trajectories result in paths of motion which are more efficient than the logarithmic spiral.

II. THE EQUATIONS OF MOTION

The equations of motion are derived under the assumption of a vehicle moving in the sun's gravitational force field and free from the influence of all other celestial bodies. The regions of space are assumed to be so rarefied that the effects of lift and drag on the vehicle are negligible in comparison to the solar pressure force.

Figure 1 shows a vehicle propelled by a solar sail at some distance r from the center of the sun. This solar vehicle is propelled by a radiation pressure force (F_S) in the presence of the gravitational force (F_G). The angle θ is the sail angle taken as positive in the counterclockwise direction. Applying Newton's Second Law, the tangential and radial equations are found to be

$$F_S \sin \theta = \frac{d}{dt} (m \bar{v}) \qquad (1)$$

$$F_S \cos \theta - F_G = \frac{d}{dt} (m \bar{u}) \qquad (2)$$

respectively, where

$$F_S = \frac{A\, S_o\, r_o^2}{c\, r^2} (1 + R) \cos^2 \theta \qquad (3)$$

$$F_G = \frac{K^2\, M\, m}{r^2} \qquad (4)$$

$$F_S \sin \theta = m \, A_T \tag{5}$$

$$F_S \cos \theta - F_G = m \, A_R \tag{6}$$

where the tangential acceleration is

$$A_T = \dot{\bar{v}} + \frac{\bar{u}\,\bar{v}}{r} = r\,\ddot{\phi} + 2\,r\,\phi \tag{7}$$

and the radial acceleration is

$$A_R = \dot{\bar{u}} - \frac{\bar{v}^2}{r} = \ddot{r} - r\,\dot{\phi}^2 \tag{8}$$

Employing Tsu's notation, then

$$\frac{F_S}{m} = \alpha \cos^2 \theta \left(\frac{r_o}{r} \right)^2 \tag{9}$$

and

$$\frac{F_G}{m} = a_o \left(\frac{r_o}{r} \right)^2 \tag{10}$$

where

$$\alpha = \frac{A\,S_o\,(1 + R)}{c\,m} \tag{11}$$

$$a_o = \frac{K^2\,M}{r_o^{\,2}} = 0.592 \text{ cm/sec}^2 \text{ (at earth's orbit)} \tag{12}$$

On using Eqs. 7, 8, 9 and 10, Eqs. 5 and 6 become

$$\alpha \cos^2 \theta \sin \theta \left(\frac{r_o}{r} \right)^2 = \dot{\bar{v}} + \frac{\bar{u}\,\bar{v}}{r} = r\,\ddot{\phi} + 2\,\dot{r}\,\dot{\phi} \tag{13}$$

$$(\alpha \cos^3 \theta - a_o) \left(\frac{r_o}{r} \right)^2 = \dot{\bar{u}} - \frac{\bar{v}^2}{r} = \ddot{r} - r \dot{\phi}^2 \tag{14}$$

For values of $\alpha \leqslant a_o$, the quantity $(a_o - \alpha \cos^3 \theta)$ is seen to be positive. Now defining the parameters

$$\lambda = (a_o - \alpha \cos^3 \theta) \, (r_o)^2 \tag{15}$$

$$\mu = (\alpha \sin \theta \cos^2 \theta) \, (r_o)^2 \tag{16}$$

One can show that Eqs. 13 and 14 take on the form

$$\frac{\mu}{r^2} = \frac{1}{r} \frac{d}{dt} (r^2 \dot{\phi}) \tag{17}$$

and

$$\frac{-\lambda}{r^2} = \ddot{r} - r \dot{\phi}^2 \tag{18}$$

Making the transformation

$$b = \frac{1}{r} \tag{19}$$

and letting

$$h = r^2 \dot{\phi} \tag{20}$$

using Eqs. 19 and 20 and eliminating t, i.e.

$$\frac{d}{dt} = \frac{h}{r^2} \frac{d}{d\phi} = h \, b^2 \frac{d}{d\phi} \tag{21}$$

one finds that Eqs. 17 and 18 reduce to

$$h \frac{dh}{d\phi} = \frac{\mu}{b} \tag{22}$$

$$h^2 \left(\frac{d^2b}{d\phi^2} + b \right) + h \frac{dh}{d\phi} \frac{db}{d\phi} = \lambda \tag{23}$$

respectively. On eliminating h, one obtains the result

$$\frac{-2}{b} = \frac{d}{d\phi} \left[\frac{\frac{-\lambda}{\mu} + \frac{1}{b} \frac{db}{d\phi}}{\frac{d^2b}{d\phi^2} + b} \right] \tag{24}$$

If the following change of variable is introduced

$$z = \int \frac{d\phi}{b} \tag{25}$$

Equation 24 can be reduced to a form given by Ince[6], which, however, is not integrable in terms of classical transcendants. Thus it is necessary to resort to a numerical solution of this equation. However, it can be seen by inspection that if b varies exponentially, Eq. 24 is satisfied. By writing

$$b = b_o \, e^{\beta\phi}$$

it is found that the assumed solution satisfies the equation, provided that

$$\beta^2 + \frac{\lambda}{\mu}\beta + 2 = 0 \tag{27}$$

or

$$\beta = \frac{-\lambda}{2\mu} \pm \frac{1}{2}\sqrt{\left(\frac{\lambda}{\mu}\right)^2 - 8} \tag{28}$$

Thus the logarithmic spiral is a solution if, and only if

$$\left(\frac{\lambda}{\mu}\right)^2 \geqslant 8 \tag{29}$$

III. THE DIMENSIONLESS FORM OF THE GOVERNING EQUATIONS

The parameters affecting the motion are best obtained by considering the equations of motion in dimensionless form. Also since the quantities dealt with are relative quantities, then the dimensionless form of the governing equation is highly suited to numerical analysis.

Choosing a representative distance as the initial or final distance of the vehicle from the sun, a characteristic velocity and time defined as

$$v_i = \left(\frac{\lambda}{r_i} \right)^{1/2} \tag{30}$$

and

$$\tau = \frac{v_i \, t}{r_i} \tag{31}$$

respectively can be introduced. Finally, introducing the dimensionless quantities

$$a = \frac{\lambda}{\mu} \; , \; \rho = \frac{r}{r_i} \; , \; u = \frac{\bar{u}}{v_i} \; , \; v = \frac{\bar{v}}{v_i} , \tag{32}$$

It is found that Eqs. 17 and 18 reduce to

$$\frac{d\,v}{d\,t} = \frac{1}{a\,\rho^2} - \frac{u\,v}{\rho} \tag{33}$$

and

$$\frac{du}{dt} = \frac{v^2}{\rho} - \frac{1}{\rho^2} \tag{34}$$

respectively. Now eliminating τ, by using the relation $\frac{d\rho}{d\tau} = u$, it is found that Eqs. 33 and 34 reduce to

$$\frac{dv}{d\rho} = \frac{1}{a\ \rho^2\ u} - \frac{v}{\rho} \tag{35}$$

and

$$\frac{du}{d\rho} = \frac{v^2}{\rho\ u} - \frac{1}{\rho^2\ u} \tag{36}$$

it is seen that as a result of writing the equations in dimensionless form, the performance of the solar sail is dependent solely on a single parameter ''a'', which is simply the ratio of the radial to the tangential force. Recalling that

$$a = \frac{\lambda}{\mu} = \frac{a_o - \alpha \cos^2 \theta}{\alpha \sin \theta \cos^2 \theta} \tag{37}$$

It is noted that ''a'' depends on the sail setting (θ) and the solar pressure acceleration (α), since the gravitational acceleration (a_o) remains constant. The parameter α is then dependent upon the payload and the sail characteristics. For fixed sail settings the larger values of α represent larger thrusts being produced by the solar sail.

IV. NUMERICAL SOLUTION

As mentioned previously values of "a" $< 2\sqrt{2}$ do not admit the spiral as a solution for the trajectory. Since the governing equations are found to have no known general solution, the region where the spiral fails was investigated numerically.

It is to be noted that if a is replaced by - a, and if u is replaced by - u, the equations will not change form. Thus it is sufficient to restrict "a" to positive values. Since r_i can be chosen as either an initial or a final distance from the sun then, without loss of generality, ρ can be restricted to values equal to or greater than unity.

Now Eqs. 35 and 36 are seen to be a system of first order non-linear differential equations. In this form they may be readily solved by the Runge-Kutta method of numerical integration. Recalling that

$$u = \frac{d\rho}{d\tau} \tag{38}$$

and

$$v = \rho \frac{d\phi}{d\tau} \tag{39}$$

then the following auxiliary relations may be employed to calculate the angular traverse and the time

$$\phi = \int_{\rho_o}^{\rho} \frac{v \, d\rho}{\rho \, u} \tag{40}$$

$$\tau = \int_{\rho_o}^{\rho} \frac{d\rho}{u} \tag{41}$$

respectively. Equations 40 and 41 lend themselves readily to Weddle's rule for numerical integration. The interpretive language of ''Runcible'' was used for programming and all computations were carried out on the IBM 650 digital computer.

Three sets of initial conditions were applied to Eqs. 35 and 36. For each set of initial conditions the parameter ''a'' was varied from a = 0.5 to a = 2.5 in increments of 0.5. A tabulation of the calculations made is given in Table 1, and the results are plotted in Figures 3 through 8 respectively.

V. THE LOGARITHMIC SPIRAL IN DIMENSIONLESS FORM

In order to compare the numerical solutions obtained with that of a spiral it was necessary to express the results of the spiral trajectory in dimensionless form. The equation of the logarithmic spiral in dimensionless form is written as

$$\rho = \rho_0 \, e^{\beta \phi} \tag{42}$$

where β is the tangent of the spiral angle. Now, differentiating Eq. 42, it is found that

$$\rho' = \rho \, \beta \, \phi' \tag{43}$$

where the prime denotes differentiation with respect to τ. Since

$$u = \rho' \quad \text{and} \quad v = \rho\phi' \tag{44}$$

Eq. 43 reduces to

$$u = \beta \, v \tag{45}$$

and on substituting Eq. 45 into Eqs. 35 and 36 it is found that

$$u = \pm \frac{\beta \sqrt{2}}{\rho^{1/2}} \tag{46}$$

and

$$v = \pm \frac{\sqrt{2}}{\rho^{1/2}} \tag{47}$$

and Eq. 41 becomes

$$\tau = \frac{\sqrt{2}}{3\beta} \left(\rho^{3/2} - \rho_o^{3/2}\right) \quad \text{for } \rho_o < \rho \tag{48}$$

For a logarithmic spiral Eqs. 46 and 47 must be satisfied for all values of time.

Since a comparison of the numerical solution of Eqs. 35 and 36 with the spiral, for the same value of ''a'' is impossible, the comparison will have to be made for the same value of α. On this basis of comparison one can compare the resulting trajectories with the same initial conditions or for trajectories which connect two given end points.

The comparison of the spiral and numerical solution was accomplished in the following two ways: first, identical initial conditions were given to both the spiral and numerical solution and the trajectories allowed to progress an equal radial distance; this result is denoted as run 2 (see Table 1); secondly, the trajectories for the spiral and the numerical solution were made to pass through the same end points in each case. It is noted that here the initial velocities, dictating the passage of the two different trajectories through the same two points, were different; specifically the initial velocities of the spiral trajectories were higher than their corresponding counterparts in the numerical solutions.

The spiral trajectories, and their time to transfer along these trajectories, are compared with the computed results in Figures 3 through 8.

VI. DISCUSSION

In this thesis all the parameters affecting the performance of the solar sail have been reduced to a single parameter, namely, ''a'', which is simply the ratio of the radial to tangential force. As ''a'' is reduced in value, a more efficient performance results. Efficiency, here, being the criteria for time to transfer radially. For values of ''a'' equal to or above $2\sqrt{2}$ a logarithmic spiral trajectory will satisfy the governing equations; however, below a value of $2\sqrt{2}$ this trajectory is not a solution. The trajectory described by the sailing vehicle in this area was generated numerically and the efficiency of the trajectory compared with that of the spiral. Two spirals were considered for purposes of comparison. The spiral generated from identical initial conditions, and a spiral passing through the end points of the numerical solution. The time to transfer along each is compared with the time required to transfer along the numerically generated solution.

The results obtained showed the numerically generated solution to be far superior to the spiral with identical initial conditions based on the transfer time criteria. The spiral passing through the extremities of the numerical solution compares more favorably with the numerical solution but still requires a greater time to effect the transfer. Hence, the numerical solution is clearly found to be more efficient than either of the two spiral paths considered.

VII. CONCLUSIONS

As a result of this study it has been found that all parameters affecting the vehicle's performance could be reduced to a single parameter, namely, the parameter ''a'', and that as the value of ''a'' is reduced the performance capability of the vehicle is increased.

Figure 2 is a plot of ''a'' versus θ for different values of α. It is seen that the numerical solution is superior to the spiral since no limit exists on the minimum value of ''a''. Essentially α is fixed by the sail's characteristics and the payload. Tsu[2] sets a practical limit on α at 0.3 cm/sec^2 consistent with technological advances at that time. However, technological advancements in this field will increase Tsu's limit, allowing the more efficient numerical solution to be utilized.

Comparing the numerical solution with the logarithmic spiral solution, for the same initial conditions (see Figure 5), it appears that the trajectories characterized by values of $a < 2\sqrt{2}$ would be quite useful a means of providing escape speed.

Finally, the calculations presented here indicate that for values of $a < 2\sqrt{2}$ considerable saving in time required to transfer between radial distances could be achieved. For example, Figure 6 shows the numerical solution for ''a'' = 2.5 to require only 27 percent of the total time to transfer as its spiral counterpart.

VIII. SUMMARY

It was pointed out by London[3] that the logarithmic spiral trajectory is not the optimum for a solar sail and suggested that other types of trajectories are necessary for the most efficient utilization of the sail. The investigation presented herein indicated that the performance of a sail is governed by a parameter ''a'', which is the ratio of the radial to tangential force. For values of ''a'' less than $2\sqrt{2}$ the logarithmic spiral was not a possible solution to the governing equations of motion.

It has been found that for a given initial condition the smaller the value of ''a'' the more efficient is the trajectory. Those paths of motion characterized by values of ''a'' less than $2\sqrt{2}$ were generated numerically and found to require less time for transfer between radial distances than any logarithmic spiral considered for purposes of comparison.

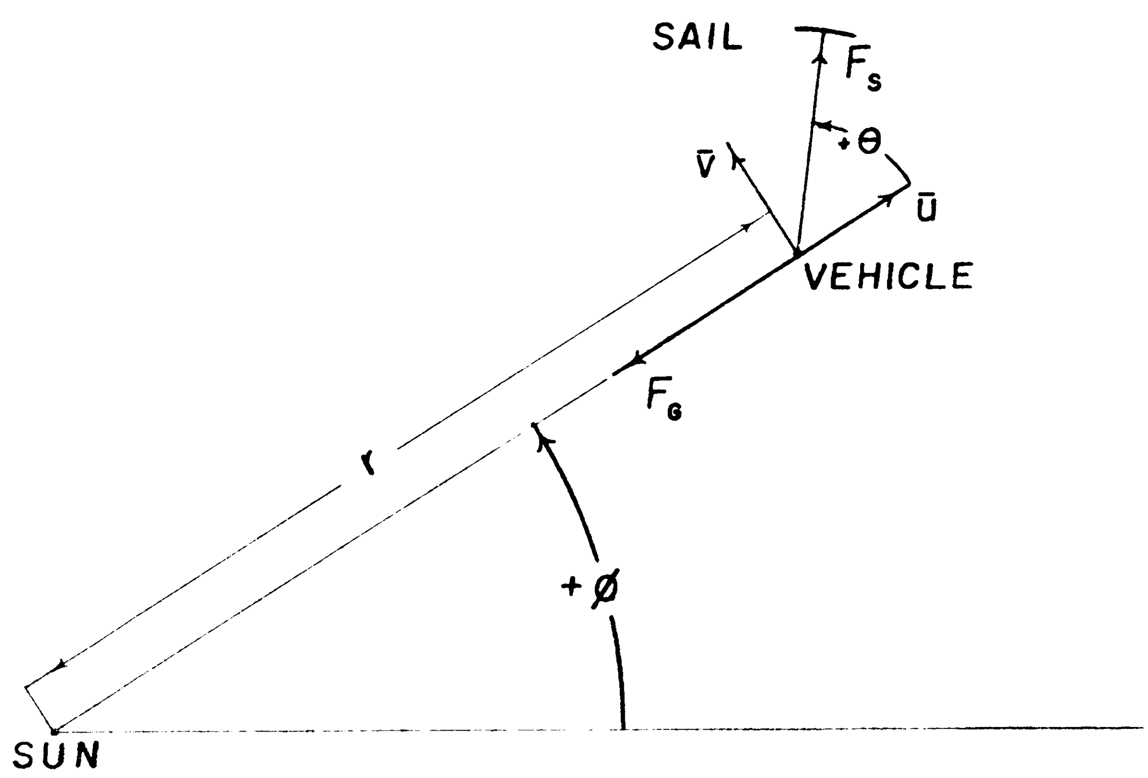

FIGURE I SCHEMATIC OF THE COORDINATE SYSTEM

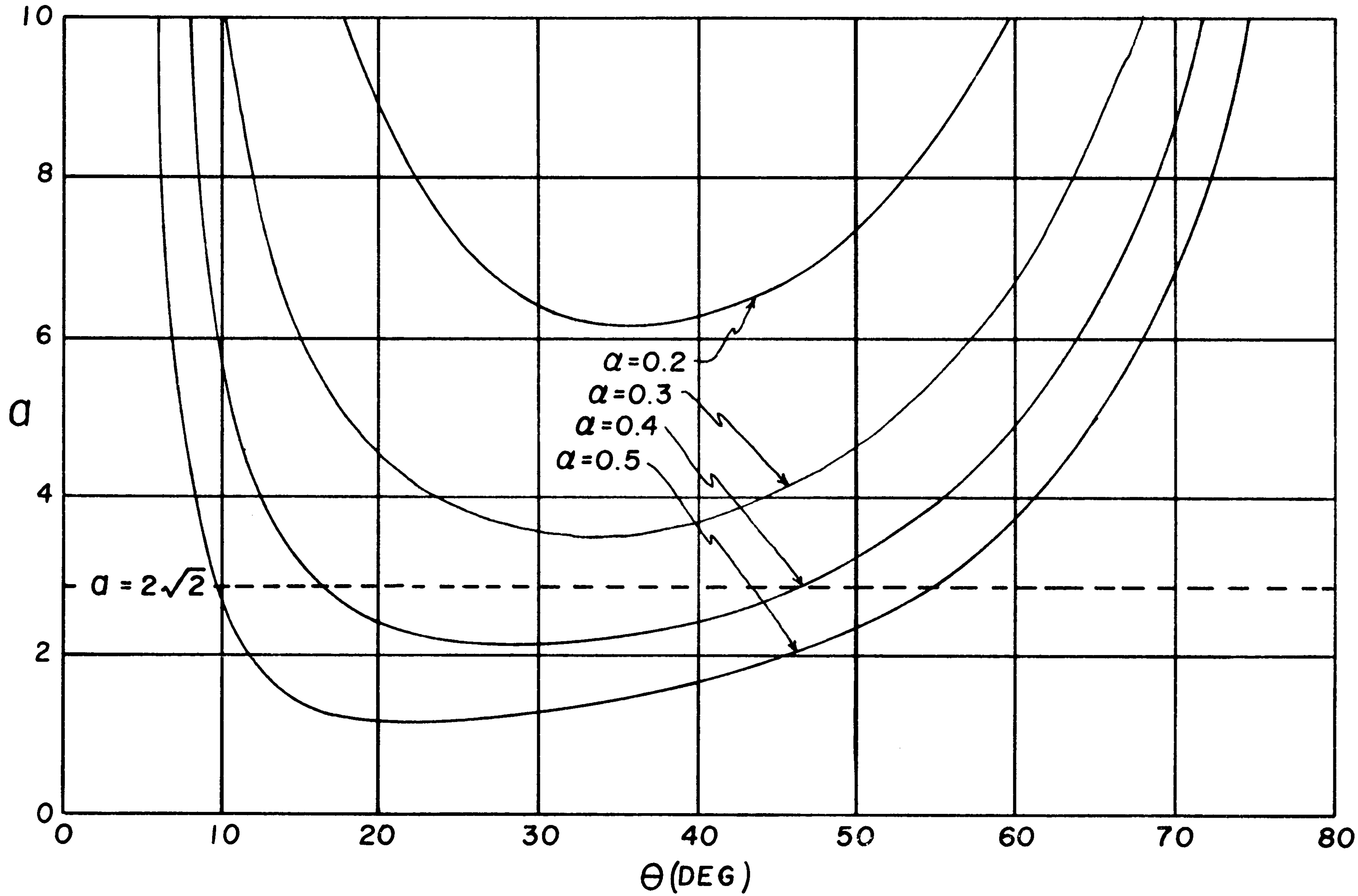

FIGURE 2 a VS θ

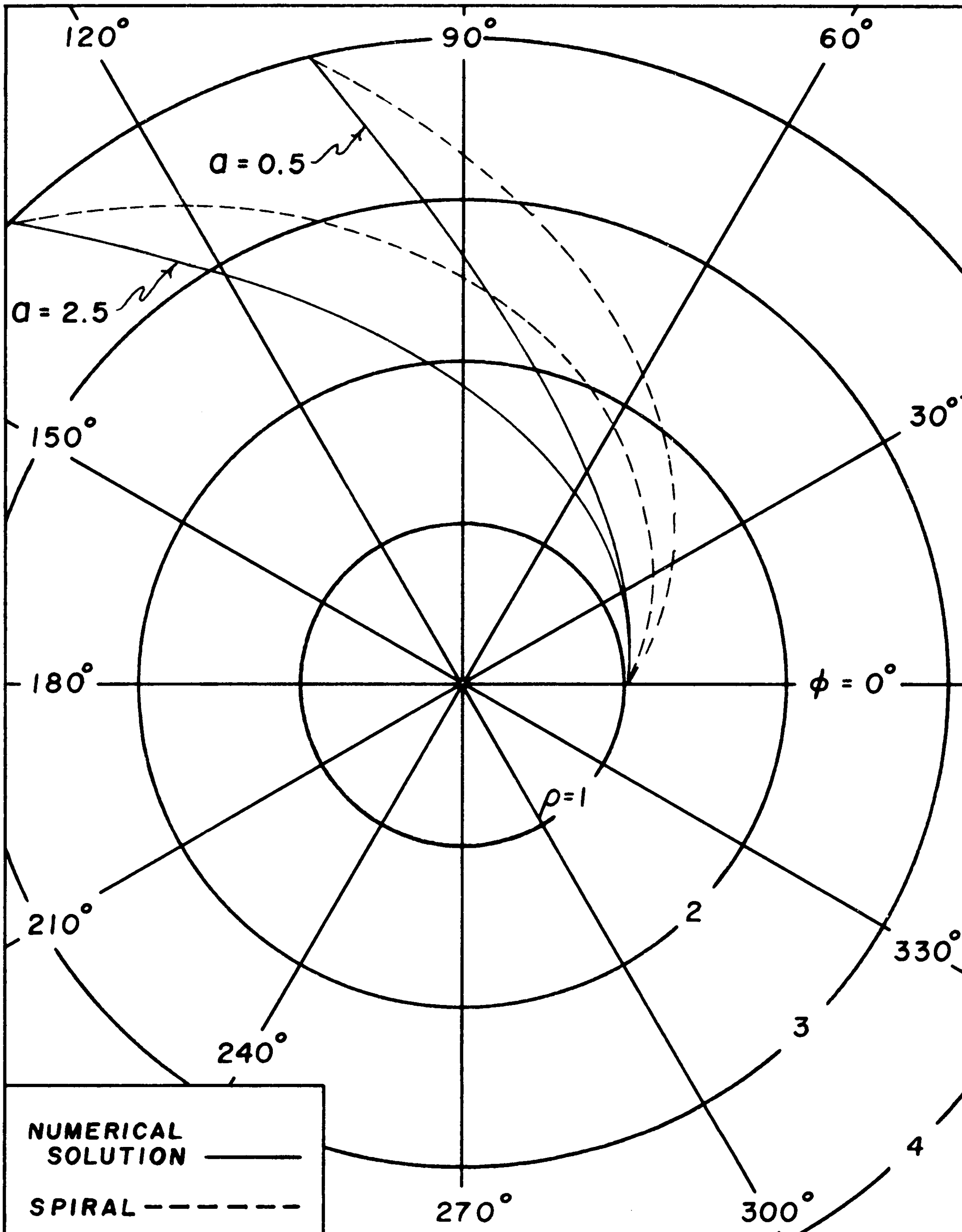

FIGURE 3 TRAJECTORIES (u=0.25, v=1.0)

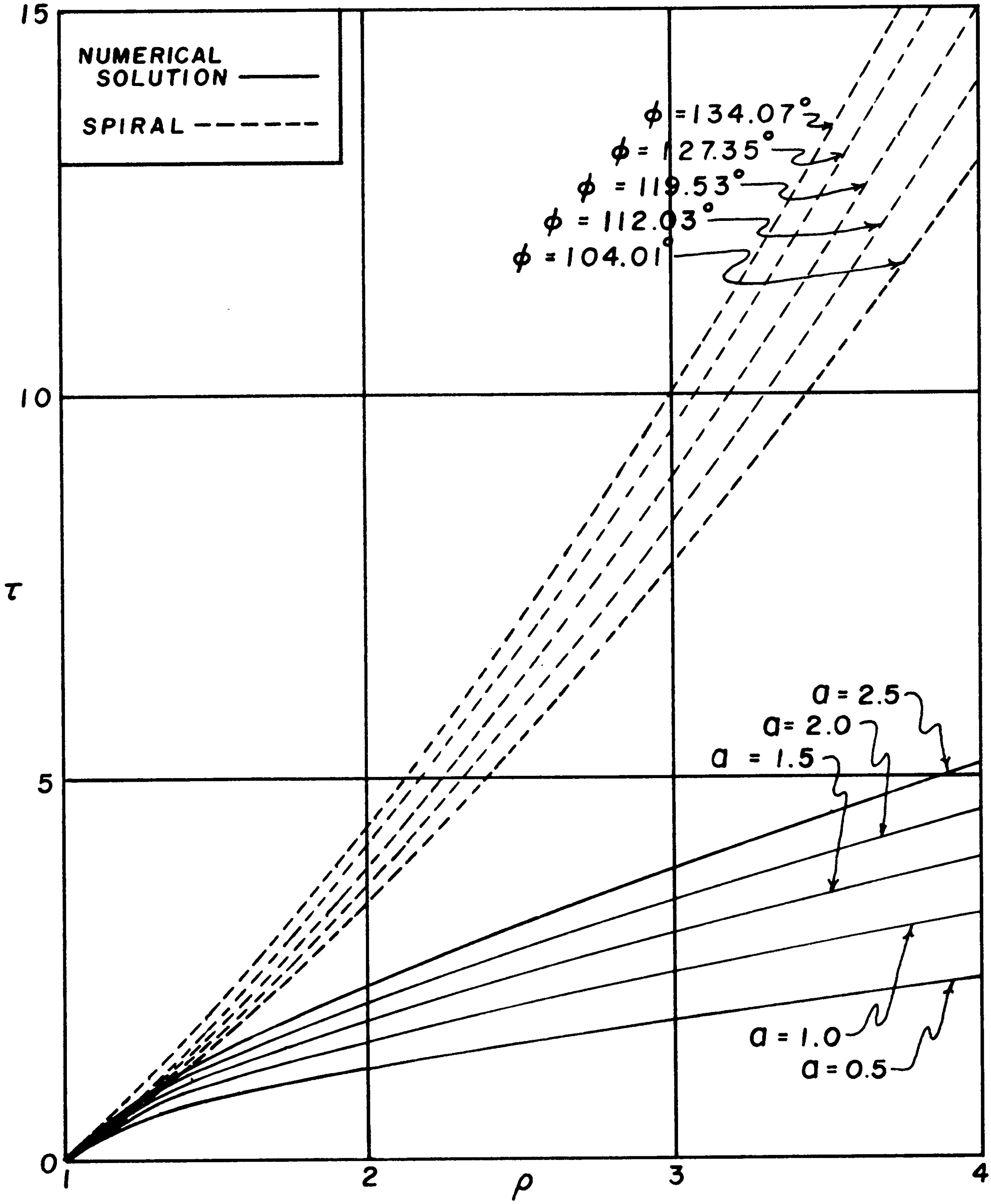

FIGURE 4 TIME TO TRANSFER ($u = 0.25$, $v = 1.0$)

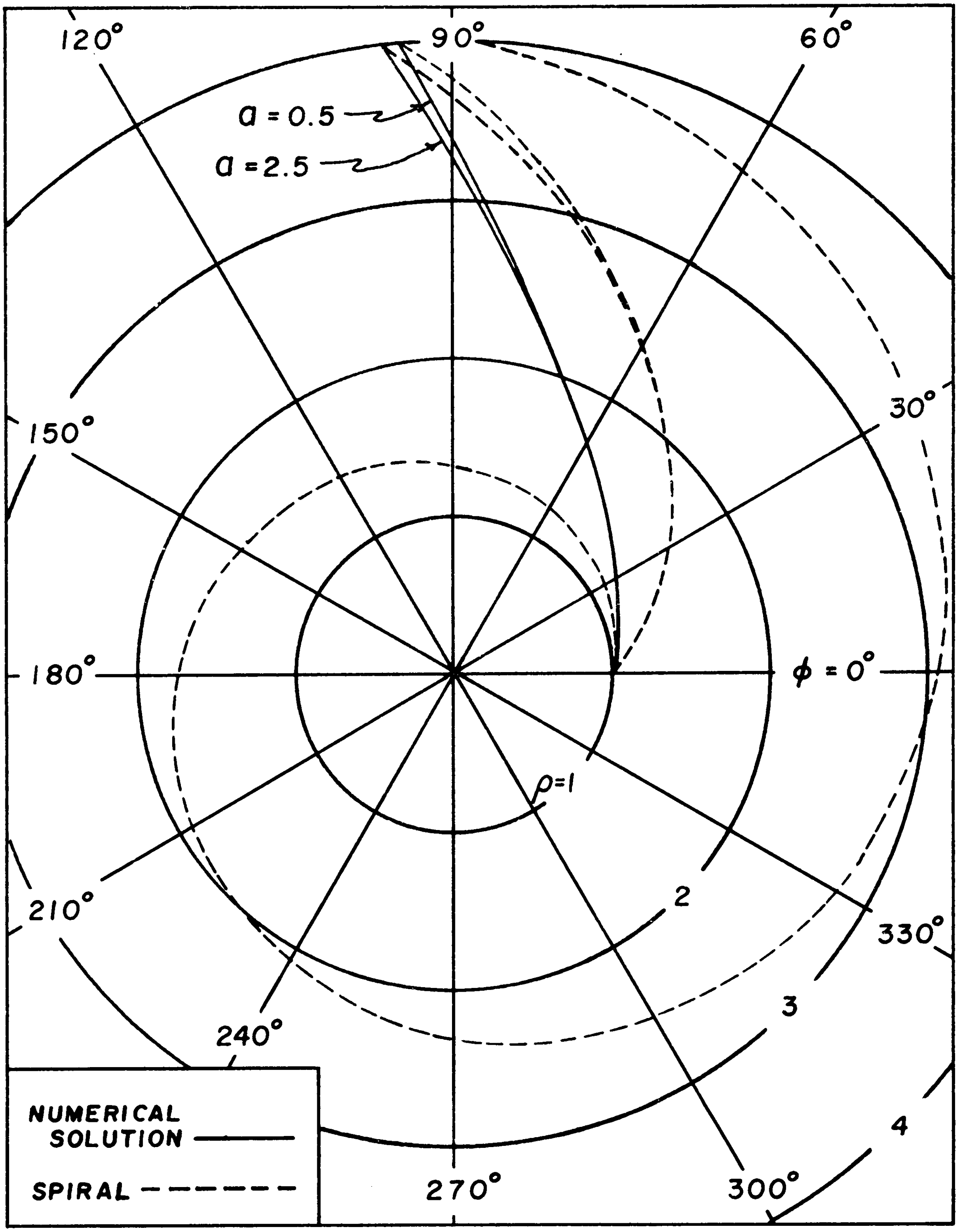

FIGURE 5 TRAJECTORIES($u = 0.25$, $v = \sqrt{2}$)

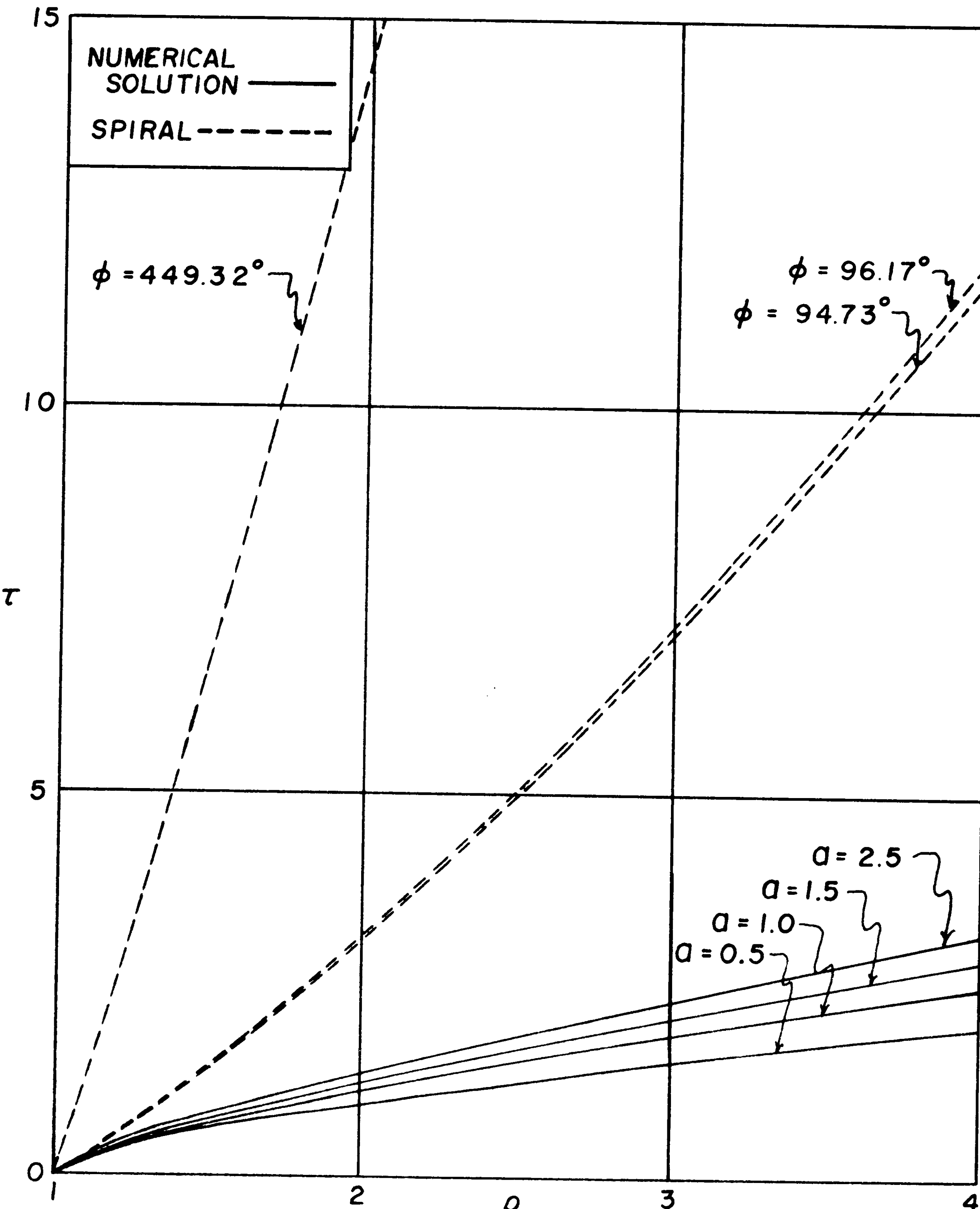

FIGURE 6 TIME TO TRANSFER (u=0.25, $v=\sqrt{2}$)

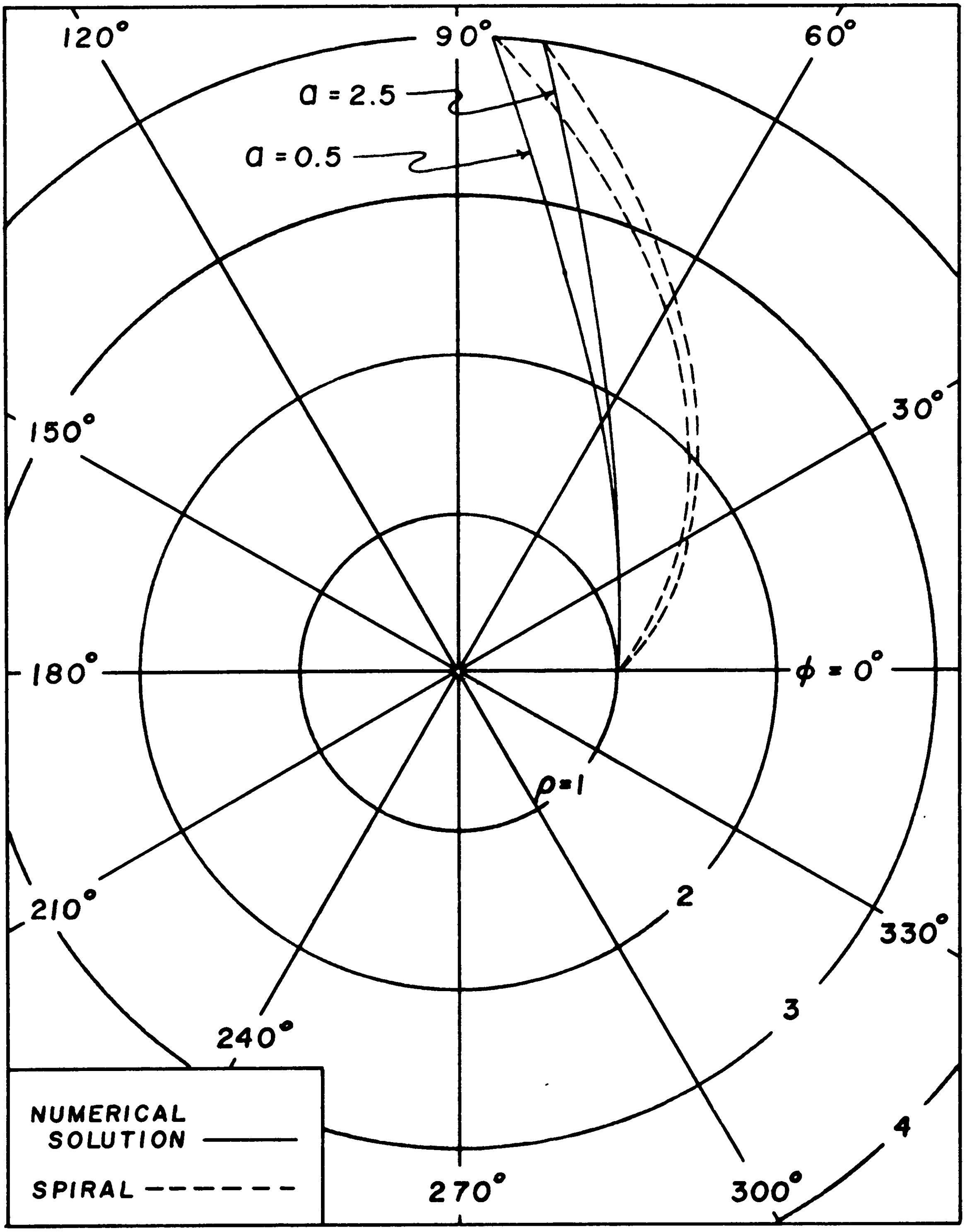

FIGURE 7 TRAJECTORIES (u=0.25, V_f=2.0)

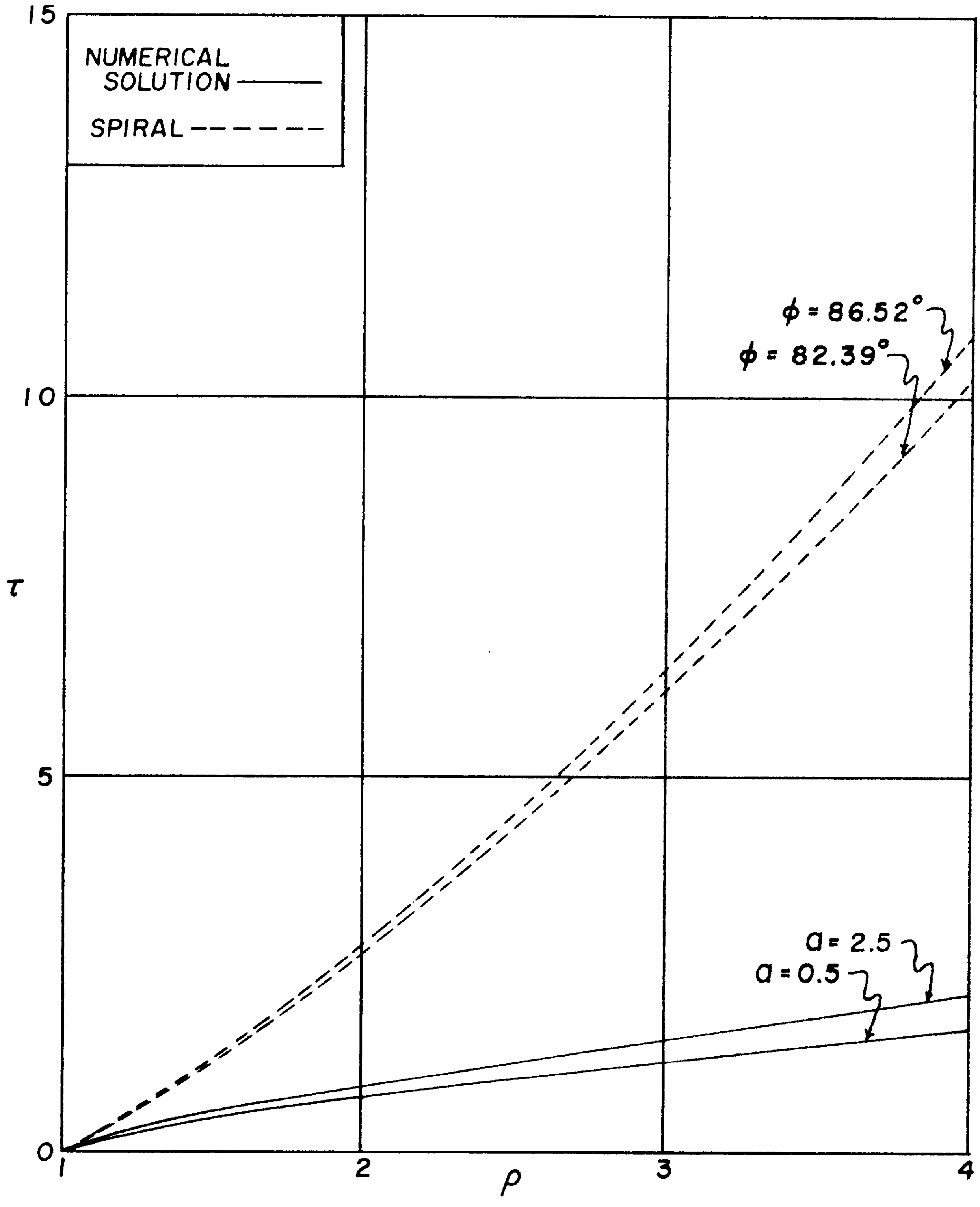

FIGURE 8 TIME TO TRANSFER $(u = 0.25,\ v = 2.0)$

Run	Numerical Solution: u	v	a	τ	Spiral Solution: u	v	β	τ	Ø (Final)
	0.25	1	2.5	5.15	$0.5924\sqrt{2}$	$\sqrt{2}$	0.5924	16.71	134.07°
	0.25	1	2.0	4.62	$0.6237\sqrt{2}$	$\sqrt{2}$	0.6237	15.87	127.35°
1	0.25	1	1.5	3.91	$0.6645\sqrt{2}$	$\sqrt{2}$	0.6645	14.90	119.53°
	0.25	1	1.0	3.21	$0.7090\sqrt{2}$	$\sqrt{2}$	0.7090	13.96	112.03°
	0.25	1	0.5	2.32	$0.7637\sqrt{2}$	$\sqrt{2}$	0.7637	12.96	104.01°
	0.25	$\sqrt{2}$	2.5	3.22	$0.8259\sqrt{2}$	$\sqrt{2}$	0.8259	11.99	96.17°
	0.25	$\sqrt{2}$	2.0	3.06	$0.8283\sqrt{2}$	$\sqrt{2}$	0.8283	11.95	95.90°
2	0.25	$\sqrt{2}$	1.5	2.85	$0.8312\sqrt{2}$	$\sqrt{2}$	0.8312	11.91	95.56°
	0.25	$\sqrt{2}$	1.0	2.53	$0.8347\sqrt{2}$	$\sqrt{2}$	0.8347	11.86	95.16°
	0.25	$\sqrt{2}$	0.5	2.01	$0.8384\sqrt{2}$	$\sqrt{2}$	0.8384	11.81	94.73°
3	0.25	2	2.5	2.08	$0.9640\sqrt{2}$	$\sqrt{2}$	0.9640	10.26	82.39°
	0.25	2	0.5	1.63	$0.9180\sqrt{2}$	$\sqrt{2}$	0.9180	10.78	86.52°

INITIAL CONDITIONS FOR THE NUMERICAL AND THE SPIRAL SOLUTIONS

TABLE 1

X. ACKNOWLEDGEMENTS

The author wishes to express his sincere appreciation to Dr. Hassan A. Hassan for his guidance during the development of this thesis. Without Dr. Hassan's encouragement and sound advice this thesis would have lacked the technical completeness gained only through experience.

The author would like to thank Dr. James B. Eades for his time and interest in contributing to the clarity of this thesis.

Finally, the author wishes to thank for her patience in typing this thesis.

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ABSTRACT

A logarithmic spiral trajectory is not the optimum trajectory for a solar sailing space vehicle. Investigation of the governing equations showed that the performance of the solar sail depends on one parameter ''a''. This parameter ''a'', defined simply as the ratio of radial to tangential forces, was seen to be significant for two reasons: first, the logarithmic spiral is restricted to values of $a = 2\sqrt{2}$; secondly, decreased values of ''a'' indicate better performance characteristics. For values of $a < 2\sqrt{2}$ a new trajectory was generated and found to be more efficient than any logarithmic spiral trajectory.