MISSILE AUTOPILOT DESIGN USING MU-SYNTHESIS

by

John Eugene Bibel

Thesis submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Master of Science
in
Aerospace Engineering

APPROVED:

[Signature]
Dr. Frederick H. Lutze, Chairman

[Signature]
Dr. Eugene M. Cliff

[Signature]
Dr. Mark R. Anderson

June 1998
Blacksburg, Virginia

Keywords: Missile Autopilot, H-Infinity Control, Mu-Synthesis, Structured Singular Value
c.2

LD
5655
V855
1998
B544
C.2
MISSILE AUTOPILOT DESIGN USING MU-SYNTHESIS

by

John Eugene Bibel

Dr. Frederick H. Lutze, Chairman

Department of Aerospace and Ocean Engineering

(ABSTRACT)

Due to increasingly difficult threats, current air defense missile systems are pushed to the limits of their performance capabilities. In order to defend against these more stressing threats, interceptor missiles require greater maneuverability, faster response time, and increased robustness to more severe environmental conditions. One of the most critical missile system elements is the flight control system, since its time constant is typically half of the total missile system time constant. Conventional autopilot design techniques have worked well in the past, but in order to satisfy future and more stringent design specifications, new design methods are necessary. Robust control techniques (in particular, H-Infinity Control and Mu-Synthesis) and their application to the design of missile autopilots are addressed in this thesis. In addition, conventional autopilot designs are performed as comparative benchmarks. This paper reviews the missile autopilot design problem and presents descriptions of the classical and H-Infinity/Mu design methods. Missile autopilot designs considering both rigid-body dynamics and elastic-body dynamics are presented. Comparisons of the design approaches and results are also discussed. The results show that the application of robust control techniques to the design of missile autopilots can improve the performance and stability robustness characteristics of the flight control system.
Dedication

This thesis is dedicated to the memory of my grandfather, John Bibel, who passed away during this work.
Acknowledgements

I would like to thank the many people who helped me throughout the development of this thesis. First, I would like to thank my chairman, Dr. Frederick Lutze, for his patience, his reviews of the thesis, and his assistance in getting the thesis to fruition. I also thank Dr. E. M. Cliff and Dr. M. R. Anderson for their suggestions and for serving on my committee. I would also like to acknowledge the guidance of Dr. Harold Stalford during the majority of the work.

I could not have completed this work without the support of the Naval Surface Warfare Center Dahlgren Division. First, I thank Sam Hardy, who gave me the chance to examine the robust control techniques under his Technology Block during the 1989 - 1991 timeframe and provided me with the aerodynamic data base. Dr. Tom Rice provided me with time to work on the effort and also with challenges to persevere. I thank Steve Malyevac for his help with the aerodynamic stability derivatives and his consultation with the designs - allowing me to bounce ideas off of him. Tom Hymer and Dr. Leroy Devan provided me with the bending mode data and the dynamic derivative data, respectively. David Hanger, Mark S. Jones and Mike Libeau also helped with various items during the effort. However, at NSWC, I am most indebted to Ernie Ohlmeyer, for his support under an Independent Research project, his encouragement, his guidance, and his inspiration to bring the work to a final product.

Most importantly, I would like to thank my wife, Denise, for her love, for her encouragement, for helping me with the final draft, and for giving me the time to work on
the thesis. I also thank my parents, John and Bert Bibel, for their support, and my grandparents for their wisdom and their recommendations to continue with higher learning.

Finally, I would like to thank God for giving me the strength, the knowledge, and the talents to carry out this endeavor. Surely, God has blessed me dearly.
# Table of Contents

Abstract ........................................................................................................... ii

Dedication ......................................................................................................... iii

Acknowledgements .......................................................................................... iv

Table of Contents ............................................................................................ vi

List of Figures ................................................................................................... ix

List of Tables ................................................................................................... xiv

Chapter 1.  Introduction .................................................................................... 1

Chapter 2.  Airframe Equations of Motion ......................................................... 5
  2.1 Rigid Body Dynamics .............................................................................. 5
  2.2 Flexible Body Dynamics ......................................................................... 23

Chapter 3.  Missile System Data ....................................................................... 30
  3.1 Airframe Data .......................................................................................... 30
  3.2 Aerodynamic Data ................................................................................... 30
  3.3 Flexible-Body Data .................................................................................. 42
  3.4 Instrumentation Models and Data ............................................................... 42

Chapter 4.  Autopilot Design Requirements ...................................................... 49

Chapter 5.  Autopilot Design Constraints and Problems ..................................... 55

Chapter 6.  Missile Autopilot Design Models .................................................... 63
  6.1 Pitch (Yaw) Planar Models: Small Angle of Attack .................................. 63
    6.1.1 Rigid Body Model with Accelerometer Offset .................................... 69
    6.1.2 Rigid Body Model with Instrumentation Dynamics .......................... 71
    6.1.3 Flexible Body Model .......................................................................... 71
    6.1.4 Model with Constraints ..................................................................... 74
  6.2 Pitch (Yaw) Planar Models: Large Angle of Attack .................................... 74
    6.2.1 High $\alpha$ Pitch Plane Model: Accelerometer Offset ....................... 80
    6.2.2 High $\alpha$ Pitch Plane Model: Instruments and Constraints ............. 81
  6.3 Roll Channel Model .................................................................................. 81
  6.4 Multivariable Coupled Pitch-Yaw-Roll Model ......................................... 85
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.4.1 Coupled Linear Design Model with Instrumentation</td>
<td>90</td>
</tr>
<tr>
<td>6.4.2 Coupled Linear Design Model: Accelerometer Offset</td>
<td>90</td>
</tr>
<tr>
<td>6.4.3 Coupled Linear Model: Output</td>
<td>91</td>
</tr>
<tr>
<td>Chapter 7. Classical Control Autopilot Design</td>
<td>94</td>
</tr>
<tr>
<td>7.1 Pitch (Yaw) Planar Autopilot Design</td>
<td>94</td>
</tr>
<tr>
<td>7.1.1 Gain Selection Procedure</td>
<td>96</td>
</tr>
<tr>
<td>7.1.2 Effects of Actuator and Sensor Dynamics</td>
<td>106</td>
</tr>
<tr>
<td>7.2 Flexible-Body Compensation</td>
<td>106</td>
</tr>
<tr>
<td>7.2.1 Gain Stabilization</td>
<td>107</td>
</tr>
<tr>
<td>7.2.2 Phase Stabilization</td>
<td>109</td>
</tr>
<tr>
<td>7.3 The Roll Autopilot</td>
<td>110</td>
</tr>
<tr>
<td>7.4 Pitch-Yaw-Roll Decoupling Compensation</td>
<td>117</td>
</tr>
<tr>
<td>Chapter 8. Introduction to H-Infinity Control, the Structured</td>
<td>125</td>
</tr>
<tr>
<td>8.1 Introduction</td>
<td>125</td>
</tr>
<tr>
<td>8.2 Historical Review</td>
<td>126</td>
</tr>
<tr>
<td>8.3 Mathematical Preliminaries</td>
<td>128</td>
</tr>
<tr>
<td>8.3.1 Complex Variables</td>
<td>128</td>
</tr>
<tr>
<td>8.3.2 Relevant Topics from Linear Algebra</td>
<td>130</td>
</tr>
<tr>
<td>8.3.3 Vector and Matrix Norms</td>
<td>132</td>
</tr>
<tr>
<td>8.3.3.1 Properties of Norms</td>
<td>132</td>
</tr>
<tr>
<td>8.3.3.2 Vector Norms</td>
<td>132</td>
</tr>
<tr>
<td>8.3.3.3 Matrix Norms</td>
<td>135</td>
</tr>
<tr>
<td>8.3.3.4 Induced Norms</td>
<td>136</td>
</tr>
<tr>
<td>8.3.4 Singular Values</td>
<td>140</td>
</tr>
<tr>
<td>8.3.4.1 Definition</td>
<td>140</td>
</tr>
<tr>
<td>8.3.4.2 Properties of Singular Values</td>
<td>141</td>
</tr>
<tr>
<td>8.3.4.3 Singular Value Decomposition</td>
<td>142</td>
</tr>
<tr>
<td>8.3.4.4 Importance of Singular Values</td>
<td>147</td>
</tr>
<tr>
<td>8.3.5 Signal and Function Norms</td>
<td>150</td>
</tr>
<tr>
<td>8.3.5.1 Signal Norms</td>
<td>150</td>
</tr>
<tr>
<td>8.3.5.2 Transfer Function Norms</td>
<td>151</td>
</tr>
<tr>
<td>8.3.6 Signal and Function Spaces</td>
<td>152</td>
</tr>
<tr>
<td>8.3.6.1 Time-Domain Spaces</td>
<td>153</td>
</tr>
<tr>
<td>8.3.6.2 Frequency-Domain Spaces</td>
<td>154</td>
</tr>
<tr>
<td>8.3.7 Engineering Relevances of Norms and Spaces</td>
<td>155</td>
</tr>
<tr>
<td>8.4 Robustness</td>
<td>159</td>
</tr>
<tr>
<td>8.4.1 Uncertainties</td>
<td>161</td>
</tr>
<tr>
<td>8.5 Modern Robust Control Techniques</td>
<td>162</td>
</tr>
<tr>
<td>8.6 $H_\infty$ and $\mu$ Methods of Robust Control</td>
<td>165</td>
</tr>
<tr>
<td>8.6.1 Design and Analysis Framework for Robust Control</td>
<td>165</td>
</tr>
<tr>
<td>8.6.2 Robust Compensator Synthesis and $H_\infty$ Control</td>
<td>167</td>
</tr>
</tbody>
</table>
8.6.3 Analysis and the Structured Singular Value ........................................ 170
  8.6.3.1 Nominal Stability ........................................................................ 172
  8.6.3.2 Nominal Performance .................................................................. 172
  8.6.3.3 Robust Stability ......................................................................... 172
  8.6.3.4 Uncertainty Descriptions .......................................................... 172
  8.6.3.5 Robust Stability for Unstructured Uncertainty ................. 173
  8.6.3.6 Robust Stability for Structured Uncertainty .................... 173
  8.6.3.7 Robust Performance ................................................................... 176
  8.6.4 Mu-Synthesis ............................................................................... 177

Chapter 9. Rigid-Body Missile Autopilot Design .............................................. 179
  9.1 Design Set 1 .................................................................................. 179
  9.1.1 Classical Missile Autopilot Design ............................................ 179
  9.1.2 Mu-Synthesis Autopilot Design ................................................... 180
  9.1.3 Autopilot Design Comparisons .................................................... 190
  9.2 Design Set 2 .................................................................................. 198
  9.2.1 Classical Missile Autopilot Design ............................................ 198
  9.2.2 Mu-Synthesis Autopilot Design ................................................... 203
  9.2.3 Autopilot Design Comparisons .................................................... 216

Chapter 10. Elastic Body Missile Autopilot Design ......................................... 223
  10.1 Conventional Autopilot Design and Analysis .................................. 225
  10.2 Mu-Synthesis Autopilot Design #1 .................................................. 228
  10.3 Mu-Synthesis Autopilot Design #2 .................................................. 241
  10.3.1 Controller Design Approach ....................................................... 241
  10.3.2 Analysis Results ....................................................................... 245
  10.4 Reduced-Order Mu Controller ......................................................... 257
  10.5 Autopilot Design Comparisons ....................................................... 258

Chapter 11. Summary and Conclusions ......................................................... 270

Chapter 12. Further Topics for Research ....................................................... 273

References ............................................................................................... 275

Bibliography for Chapter 8 .......................................................................... 282

Appendix A. Development of the Linearized Airframe Rigid Body Dynamics ...... 282

Vita ............................................................................................................ 323
# List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Missile Velocity and Rotational Rate Components</td>
<td>8</td>
</tr>
<tr>
<td>2.2</td>
<td>Missile Aerodynamic Forces and Moments</td>
<td>9</td>
</tr>
<tr>
<td>2.3</td>
<td>Aerodynamic Angle Definitions</td>
<td>10</td>
</tr>
<tr>
<td>2.4</td>
<td>Example Illustration of Aeroelastic Airframe</td>
<td>24</td>
</tr>
<tr>
<td>2.5</td>
<td>Typical Missile Bending Modes</td>
<td>25</td>
</tr>
<tr>
<td>2.6</td>
<td>Missile Control System with Flexible-Body Dynamics</td>
<td>27</td>
</tr>
<tr>
<td>3.1</td>
<td>Short-Range High-Performance Missile Configuration</td>
<td>31</td>
</tr>
<tr>
<td>3.2</td>
<td>Normal Force Coefficient vs. Angle of Attack</td>
<td>34</td>
</tr>
<tr>
<td>3.3</td>
<td>Pitching Moment Coefficient vs. Angle of Attack</td>
<td>35</td>
</tr>
<tr>
<td>3.4</td>
<td>Trim Normal Force</td>
<td>36</td>
</tr>
<tr>
<td>3.5</td>
<td>Pitch-Yaw-Roll Coupling</td>
<td>39</td>
</tr>
<tr>
<td>3.6</td>
<td>Causes of Pitch-Yaw-Roll Coupling</td>
<td>41</td>
</tr>
<tr>
<td>3.7</td>
<td>Missile Bending Mode Shapes for Gliding Flight</td>
<td>44</td>
</tr>
<tr>
<td>3.8</td>
<td>Moving IRU Coordinate Frame</td>
<td>47</td>
</tr>
<tr>
<td>4.1</td>
<td>Missile Homing Loop</td>
<td>50</td>
</tr>
<tr>
<td>5.1</td>
<td>Sources of Error in Guided Missile Systems</td>
<td>56</td>
</tr>
<tr>
<td>6.1</td>
<td>Missile Pitch Plane Linear Dynamics for Small Angle of Attack</td>
<td>66</td>
</tr>
<tr>
<td>6.2</td>
<td>Missile Pitch Plane Model with Actuator Dynamics</td>
<td>72</td>
</tr>
<tr>
<td>6.3</td>
<td>Missile Pitch Plane Model with Instrumentation Dynamics</td>
<td>73</td>
</tr>
<tr>
<td>Section</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>9.4</td>
<td>Unmodeled Dynamics Weighting Function</td>
<td>185</td>
</tr>
<tr>
<td>9.5</td>
<td>Mu-Synthesis Autopilot Design Model #1</td>
<td>187</td>
</tr>
<tr>
<td>9.6</td>
<td>Mu Design #1 Sensitivity &amp; Complementary Sensitivity Weights</td>
<td>188</td>
</tr>
<tr>
<td>9.7</td>
<td>Robust Performance Analysis of Mu Design #1</td>
<td>189</td>
</tr>
<tr>
<td>9.8</td>
<td>Acceleration Step Response, Mu Design #1</td>
<td>191</td>
</tr>
<tr>
<td>9.9</td>
<td>Frequency Response at Actuator Input, Mu Design #1</td>
<td>192</td>
</tr>
<tr>
<td>9.10</td>
<td>Frequency Response at Gyro Output, Mu Design #1</td>
<td>193</td>
</tr>
<tr>
<td>9.11</td>
<td>Comparison $\eta$ Step Response at Min &amp; Max $\alpha$, Classical Design #1</td>
<td>195</td>
</tr>
<tr>
<td>9.12</td>
<td>Comparison $\eta$ Step Response at Min &amp; Max $\alpha$, Mu Design #1</td>
<td>196</td>
</tr>
<tr>
<td>9.13</td>
<td>Robust Performance Analysis of Classical Autopilot #1</td>
<td>197</td>
</tr>
<tr>
<td>9.14</td>
<td>Acceleration Step Response, Classical Autopilot #2</td>
<td>199</td>
</tr>
<tr>
<td>9.15</td>
<td>Missile Angle of Attack and Pitch Rate Response to Step Command</td>
<td>200</td>
</tr>
<tr>
<td>9.16</td>
<td>Bode Plot for Loop Open at Actuator, Classical Autopilot Design</td>
<td>201</td>
</tr>
<tr>
<td>9.17</td>
<td>Bode Plot for Loop Open at Gyro, Classical Autopilot Design</td>
<td>202</td>
</tr>
<tr>
<td>9.18</td>
<td>Nyquist Plot for Loop Open at Actuator, Classical Autopilot Design</td>
<td>204</td>
</tr>
<tr>
<td>9.19</td>
<td>Unmodeled Dynamics Weight</td>
<td>206</td>
</tr>
<tr>
<td>9.20</td>
<td>Mu-Synthesis Design Model</td>
<td>208</td>
</tr>
<tr>
<td>9.21</td>
<td>General Interconnection Structure of the Design Model</td>
<td>209</td>
</tr>
<tr>
<td>9.22</td>
<td>S and T Weights for the Mu-Synthesis Design</td>
<td>211</td>
</tr>
<tr>
<td>9.23</td>
<td>Achieved Acceleration of Mu-Synthesis Design</td>
<td>212</td>
</tr>
<tr>
<td>9.24</td>
<td>Time Response of State Variables to Step Command, Mu-Synthesis Design</td>
<td>213</td>
</tr>
<tr>
<td>9.25</td>
<td>Bode Plot, Loop Open at Actuator, Mu-Synthesis Design</td>
<td>214</td>
</tr>
<tr>
<td>Section</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>----------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>9.26</td>
<td>Bode Plot, Loop Open at Gyro, Mu-Synthesis Design</td>
<td>215</td>
</tr>
<tr>
<td>9.27</td>
<td>Acceleration Step Response with Angle of Attack Variations,</td>
<td>217</td>
</tr>
<tr>
<td></td>
<td>Classical #2</td>
<td></td>
</tr>
<tr>
<td>9.28</td>
<td>Acceleration Step Response with Angle of Attack Variation,</td>
<td>218</td>
</tr>
<tr>
<td></td>
<td>Mu Design #2</td>
<td></td>
</tr>
<tr>
<td>9.29</td>
<td>Mu-Analysis of Classical Autopilot to Model Uncertainties</td>
<td>219</td>
</tr>
<tr>
<td>9.30</td>
<td>Mu-Analysis of Mu-Controller</td>
<td>220</td>
</tr>
<tr>
<td>10.1</td>
<td>Bending Mode Frequency Response</td>
<td>224</td>
</tr>
<tr>
<td>10.2</td>
<td>Bode Plot: Classical Design 1 at Actuator</td>
<td>226</td>
</tr>
<tr>
<td>10.3</td>
<td>Bode Plot: Classical Design 1 at Gyro</td>
<td>227</td>
</tr>
<tr>
<td>10.4</td>
<td>Acceleration: Classical Notch Filter Design</td>
<td>229</td>
</tr>
<tr>
<td>10.5</td>
<td>$\alpha$ and $q$: Classical Notch Filter Design</td>
<td>230</td>
</tr>
<tr>
<td>10.6</td>
<td>Bode Plot, Loop Open at Actuator Input: Classical Notch Filter Design</td>
<td>231</td>
</tr>
<tr>
<td>10.7</td>
<td>Bode Plot, Loop Open at Gyro Output: Classical Notch Filter Design</td>
<td>232</td>
</tr>
<tr>
<td>10.8</td>
<td>Unmodeled Dynamics Weight and Elastic Modes</td>
<td>234</td>
</tr>
<tr>
<td>10.9</td>
<td>S and T Weights for the Mu-Synthesis Design #1</td>
<td>236</td>
</tr>
<tr>
<td>10.10</td>
<td>Acceleration of $\mu$-synthesis Design #1</td>
<td>237</td>
</tr>
<tr>
<td>10.11</td>
<td>Time Response of Rigid-Body States: $\mu$-Synthesis Design</td>
<td>238</td>
</tr>
<tr>
<td>10.12</td>
<td>Bode Plot of $\mu$-synthesis Design: Loop Opened at Actuator</td>
<td>239</td>
</tr>
<tr>
<td>10.13</td>
<td>Bode Plot of $\mu$-synthesis Design: Loop Opened at Gyro</td>
<td>240</td>
</tr>
<tr>
<td>10.14</td>
<td>Output Performance Weights</td>
<td>243</td>
</tr>
<tr>
<td>10.15</td>
<td>Actuator Feedforward Uncertainty Weight</td>
<td>246</td>
</tr>
<tr>
<td>10.16</td>
<td>Model for $\mu$ Design #2</td>
<td>247</td>
</tr>
<tr>
<td>Section</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>-----------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>10.17</td>
<td>Interconnection Structure for $\mu$ Design #2</td>
<td>248</td>
</tr>
<tr>
<td>10.18</td>
<td>$\mu$ Controller #2 Loop Gains</td>
<td>250</td>
</tr>
<tr>
<td>10.19</td>
<td>Achieved Normal Accelerations</td>
<td>251</td>
</tr>
<tr>
<td>10.20</td>
<td>Angle of Attack and Body Pitch Rates</td>
<td>252</td>
</tr>
<tr>
<td>10.21</td>
<td>Bode Plot, Loop Open at Actuator Input</td>
<td>253</td>
</tr>
<tr>
<td>10.22</td>
<td>Nyquist Plot, Loop Open at Actuator</td>
<td>254</td>
</tr>
<tr>
<td>10.23</td>
<td>Bode Plot, Loop Open at Gyro Output</td>
<td>255</td>
</tr>
<tr>
<td>10.24</td>
<td>Bode Plot, Loop Open at Acceleration</td>
<td>256</td>
</tr>
<tr>
<td>10.25</td>
<td>Reduced-Order $\mu$-Controller Loop Gains</td>
<td>259</td>
</tr>
<tr>
<td>10.26</td>
<td>Acceleration Response Using Reduced-Order $\mu$-Controller</td>
<td>260</td>
</tr>
<tr>
<td>10.27</td>
<td>Bode Plot, Loop Open at Actuator, Reduced-Order $\mu$-Controller</td>
<td>261</td>
</tr>
<tr>
<td>10.28</td>
<td>Robustness Analysis of Classical Autopilot Design</td>
<td>264</td>
</tr>
<tr>
<td>10.29</td>
<td>Robustness Analysis of $\mu$-Synthesis Autopilot #1</td>
<td>265</td>
</tr>
<tr>
<td>10.30</td>
<td>Robustness Analysis of $\mu$-Synthesis Autopilot Design #2</td>
<td>266</td>
</tr>
<tr>
<td>10.31</td>
<td>Robustness Analysis of Reduced Order $\mu$-Controller</td>
<td>267</td>
</tr>
<tr>
<td>A.1</td>
<td>Definitions of Fin Deflections</td>
<td>299</td>
</tr>
</tbody>
</table>
List of Tables

3.1 Missile Data ................................................................. 32
3.2 Missile Aerodynamic Derivatives ........................................... 37
3.3 Flexible Body Data .......................................................... 43
7.1 Pitch Plane Autopilot Gain Calculation .................................. 103
7.2 Roll Autopilot Design Approach ........................................... 116
7.3 Autopilot Cross-Channel Decoupling Compensation Algorithm .... 123
A.1 Control Fin Deflection Relations ......................................... 300
Chapter 1. Introduction

As enemy threats evolve and improve, a homing missile's performance boundaries are pushed to the limits. The missile is simply required to fly faster and further to maintain its defensive effectiveness against the possible threats which may be encountered. As a result, the design of the total missile system becomes more difficult to accomplish.

One of the most important missile subsystems which must be designed to meet these threats and achieve the desired performance is that of the missile's flight control system. This system must interpret the command signals from the missile guidance system and steer the vehicle in the commanded direction. The primary objective of a missile autopilot are to control the missile’s flight path while meeting the specified performance requirements, (e.g., fast tracking response to guidance commands), and to maintain the specified stability margins of the airframe over the entire flight envelope, all in the presence of parameter variations, unmodeled dynamics and disturbances. That is, the feedback control design of the missile flight control system is to satisfy specified robust performance and robust stability requirements.

Many techniques of designing the missile flight control system have been studied\textsuperscript{1-37}. Historically, the classical control approach\textsuperscript{1-23, 28-30} has been applied the most often, and currently remains the most popular autopilot design technique. The reasons for the popularity of classical control application to missile autopilot design include: (1) that vast amounts of practical experience have been gained over the years with classical autopilot design methods, and (2) that the designs have proven successful in flight. The conventional
design approach is to design separate, uncoupled pitch, yaw, and roll channel autopilots using single-input single-output (SISO) classical control techniques. These decoupled designs are then incorporated, usually in some ad hoc method, to form the three-dimensional autopilot, considering the cross-coupling between channels.

Modern control methods\textsuperscript{22-32}, such as the linear quadratic regulator (LQR), linear quadratic gaussian (LQG) theory, and loop transfer recovery (LTR), have also been examined for missile autopilot design. These methods have been successfully applied and, because of their state-space nature, are better suited towards handling the multi-input, multi-output nature of the airframe. However, the direct application of these techniques is not without limitations also. The LQR has guaranteed stability margins, but assumes full-state feedback. The use of full-state feedback may lead to excessive instrumentation and complex missile systems. In the LQG method, an observer, in this case a Kalman filter, is designed to estimate the states of the plant to be controlled, but this approach does not have any guaranteed stability margins\textsuperscript{38}. In addition, when high-frequency effects are considered, the LQR/LQG controllers are overly sensitive to parameter variations, noise on the control inputs, and disturbances at the output\textsuperscript{23-25}. To regain the stability margins, the loop transfer recovery technique was devised as a modification to the LQG approach. Applications of the LQG/LTR technique to the design of missile autopilots\textsuperscript{26-32} have demonstrated this to be a feasible approach. However, the amount of recovery may be limited in some cases, such as for nonminimum phase systems (where the LTR attempts to design a compensator by asymptotically inverting the modeled plant dynamics). A more serious limitation of the LQG/LTR approach is that it cannot adequately treat uncertainties at more than one
location in the feedback loop. It is not uncommon that uncertainties occur, for example, at the input and at the output. Doyle\textsuperscript{33} presents a simple example with two uncertainties which cannot be adequately treated with all uncertainties placed at a single location. Another disadvantage of these modern control techniques is that they design a compensator in the time domain, and do not directly consider the frequency domain stability aspects. Thus, classical methods of stability analysis, in addition to singular value analysis, must be performed to verify desired stability requirements.

For the reasons described above, the application of $H_\infty/\mu$-synthesis control theory to missile autopilot design may provide potential benefits. $H_\infty/\mu$-synthesis control is a robust, multivariable (Multi-Input Multi-Output, or MIMO) design technique which integrates $H_\infty$ optimal control theory with the structured singular value ($\mu$) stability analysis. This approach is a frequency domain tool with a state-space realization of the compensator, and is an extension of both classical and modern control techniques. References 33, 34 and 35 present $H_\infty/\mu$-synthesis control as a viable missile autopilot design technique. The $H_\infty/\mu$-synthesis approach accounts for frequency domain performance and stability requirements, and is suited for the multivariable, coupled airframe control problem.

In this study, we investigate the design of missile flight control systems using the $H_\infty/\mu$-synthesis technique. We focus on the control of a short-range, surface-to-air, tail-controlled, tactical homing missile utilizing skid-to-turn (STT) steering. Two missile control problems will be treated herein. First, the problem of designing a planar (pitch) autopilot assuming a rigid-body airframe will be studied. Second, an autopilot will be designed for the pitch-plane case considering the missile's flexible body dynamics. Both designs will be
approached using classical and $H_\infty/\mu$-synthesis methods. A comparison, then, of the two
design techniques will be presented.

This report is organized as follows. Chapters 2 and 3 present the missile and
airframe models used in the design, and Chapter 4 describes the autopilot design
requirements. Autopilot design constraints, disturbances and parameter uncertainties are
discussed in the next chapter. In Chapter 6, the models of the plant are presented in block
diagram representations for the autopilot designs. Following this, we discuss the two design
approaches. A classical approach of missile autopilot design is described in Chapter 7, and
Chapter 8 presents a discussion of the $H_\infty/\mu$-synthesis control theory and background.
Designs of the pitch-plane rigid-body autopilot using both design approaches are presented
in Chapter 9, and the subsequent chapter describes the conventional and $\mu$-synthesis
autopilot designs with elastic mode compensation. Also, both latter chapters discuss the
results and compare the design approaches. Subsequent chapters summarize and address the
results from a broader perspective and also suggest further research.
Chapter 2. Airframe Equations of Motion

As the first step in the design of our control system we formulate a model of the plant, i.e., the dynamical system to be controlled, which in this case is the missile airframe. Our airframe model considers both rigid body and flexible body dynamics.

2.1 RIGID BODY DYNAMICS

We proceed with the missile autopilot design study by first defining a linear model of the airframe dynamics. The complete development of the linearized equations of motion of the rigid body airframe is presented in Appendix A. A summary of the model formulation follows.

The general force and moment equations of motion for a vehicle moving through some medium were derived in Reference 39. The derivation assumed a rigid body expressed in a body fixed axis system. To simplify the governing equations of motion, we assume:

1. The earth is flat and non-rotating in inertial space
2. The gravity vector is constant and is oriented positive in the inertial $z$ direction (down)
3. Missile buoyancy effects are neglected
4. Thrust acts along the missile’s centerline
5. The jet damping effects due to internal mass flow and momentum changes can be neglected
6. Gyroscopic contributions due to rotating machinery are neglected
7. The body coordinate frame's reference point coincides with the center of mass, with the body x-axis aligned with missile centerline.

8. The atmosphere is at rest relative to the earth.

9. The missile has a cruciform shape with a circular cross-section and symmetric mass and geometric properties.

The resulting force and moment equations of motion become

\[
\begin{align*}
F_x &= X_a - Mg \sin \theta + T = M(\dot{u} + qw - rv) \quad (2.1) \\
F_y &= Y_a + Mg \sin \phi \cos \theta = M(\dot{\psi} + ru - pw) \quad (2.2) \\
F_z &= Z_a + Mg \cos \phi \cos \theta = M(\dot{\psi} + pv - qu) \quad (2.3) \\
l &= I_x \dot{\phi} \\
m &= I_y \dot{\phi} + pr(I_z - I_x) \quad (2.5) \\
n &= I_z \dot{\phi} + pq(I_y - I_z) \quad (2.6)
\end{align*}
\]

where

1. missile velocity vector, \( \mathbf{v}_b = (u \ v \ w)^T \)

2. missile rotational rate vector, \( \omega_b = (p \ q \ r)^T \)

3. external body forces, \( \mathbf{F}_b = (F_x \ F_y \ F_z)^T \)

4. moments acting on the body, \( \mathbf{M}_b = (l \ m \ n)^T \)

5. inertia matrix

\[
\bar{I} = \begin{bmatrix}
I_x & 0 & 0 \\
0 & I_y & 0 \\
0 & 0 & I_z
\end{bmatrix}
\]

where \( I_x, I_y, \) and \( I_z \) are the moment-of-inertia terms (with \( I_y = I_z \) due to symmetry), and the product-of-inertia elements \( I_{xy}, I_{xz}, \) and \( I_{yz} \) are zero due to missile symmetry.

6. aerodynamic forces, \( \mathbf{F}_{a,b} = (X_a \ Y_a \ Z_a)^T. \)
7. T denotes the missile thrust

8. M is the mass of the missile

The orientation of the body axes system and the definition of the velocity and rotational rate components are given in Figure 2.1. A standard definition of aerodynamic forces and moments acting on a missile is depicted in Figure 2.2. The force in the x-direction, A, is called the axial force, and is defined positive in the aft direction. The y-direction force is called the side force, and the force acting normal to the body in the negative z-direction is known as the normal force, N. For a cruciform missile, the pitch and yaw aerodynamics are identical, assuming no aerodynamic coupling. The aerodynamic moments (l, m and n) are defined positive according to the right-hand-rule about the x, y and z axes, respectively.

However, for the autopilot design of symmetric missiles, it is customary to transform the states u, v, and w into the states: angle of attack, \( \alpha \), angle of sideslip, \( \beta \), andairspeed V. In addition, an aerodynamic roll angle \( \phi_A \) is derived for \( \alpha \) and \( \beta \). These aerodynamic angles are defined in Figure 2.3. The force equations are written in this form essentially because the aerodynamic coefficients are usually specified as functions of \( \alpha \) and \( \beta \), (or \( \phi_A \)), as well as Mach number, M, and tail deflection, \( \delta \). Using Figure 2.3, these aerodynamic angles can be related to the body velocity components

\[
\tan \alpha = \frac{w}{u} \tag{2.7}
\]

\[
\tan \beta = \frac{v}{u} \tag{2.8}
\]

\[
\tan \phi_A = \frac{\tan \beta}{\tan \alpha} = \frac{v}{w} \tag{2.9}
\]
Figure 2.1. Missile Velocity and Rotational Rate Components
Figure 2.2. Missile Aerodynamic Forces and Moments
Figure 2.3. Aerodynamic Angle Definitions
We point out that the definition of \( \beta \) given in Equation (2.8) is not the usual (Euler) definition of \( \beta \) found in aerodynamic models of aircraft. However, for symmetric missiles, this form retains the similarity between the normal and side, and pitch and yaw aerodynamics. We assume that the airspeed is approximated as constant and that the corresponding \( \dot{V} \) equation can be neglected since our interest is in the dynamics and the two acceleration directions transverse to the longitudinal body axis. Utilizing Equations (2.7) and (2.8), the following two equations are derived from the force equations,

\[
\dot{\alpha} = \cos^2 \alpha \left[ \frac{1}{M_u} (F_x - F_x \tan \alpha) - p \tan \beta + q \sec^2 \alpha - r \tan \alpha \tan \beta \right]
\] (2.10)

\[
\dot{\beta} = \cos^2 \beta \left[ \frac{1}{M_u} (F_y - F_y \tan \beta) + p \tan \alpha + q \tan \alpha \tan \beta - r \sec^2 \alpha \right]
\] (2.11)

Equations (2.4), (2.5), (2.6), (2.10) and (2.11) comprise the set of nonlinear missile equations of motion in the body reference frame to be used in our study.

Consider a general nonlinear set of dynamical equations expressed in the form

\[
f_i(x, \dot{x}, U, t) = 0, \quad i = 1,2, \ldots, n \text{ equations}
\] (2.12)

where \( x \) is an \( n \)-dimensional state vector and \( U \) is an \( m \)-dimensional control vector. We assume that the functions \( f_i \) in Equation (2.12) satisfy the hypothesis of the Implicit Function Theorem of mathematical analysis\(^4\) so that the linearized approximations to these functions are well defined about a reference condition. We can now proceed to form a linear model of the airframe. It is this linear model that will be utilized in the autopilot design process. The linearization is performed by neglecting the high order terms in a Taylor series expansion. This approach to linearizing the equations of motion considers some small perturbed state of the
system away from a specified reference condition. The resulting set of linear equations can be written in the forms

\[ \Delta \dot{x} = A \Delta x + B \Delta U \]  

(2.13)

where

\[ A = -\left( \frac{\partial f}{\partial \Delta x} \right)^{-1} \left( \frac{\partial f}{\partial x} \right)_{\text{ref}} \]  

(2.14-a)

\[ B = -\left( \frac{\partial f}{\partial \Delta x} \right)^{-1} \left( \frac{\partial f}{\partial U} \right)_{\text{ref}} \]  

(2.14-b)

Before linearization, the nonlinear set of airframe dynamical equations are expressed as

\[ f_1 = \cos^2 \alpha \left[ \frac{F_z}{M_o} - pt \tan \beta \right] - \sin \alpha \cos \alpha \left[ \frac{F_x}{M_o} + rt \tan \beta \right] + q - \dot{\alpha} = 0 \]  

(2.15)

\[ f_2 = \cos^2 \beta \left[ \frac{F_y}{M_o} + pt \alpha \right] - \sin \beta \cos \beta \left[ \frac{F_x}{M_o} - qt \alpha \right] - r - \dot{\beta} = 0 \]  

(2.16)

\[ f_3 = \frac{l}{I_x} - \dot{\rho} = 0 \]  

(2.17)

\[ f_4 = \frac{m}{I_y} - \dot{q} - pr \left[ \frac{I_x - I_z}{I_y} \right] = 0 \]  

(2.18)

\[ f_5 = \frac{n}{I_z} - \dot{r} - pq \left[ \frac{I_y - I_x}{I_z} \right] = 0 \]  

(2.19)

The state variables are selected to be the angle of attack, angle of sideslip, and the body rotational rates. Thus, the state vector is \( \mathbf{x} = (\alpha \ \beta \ p \ q \ r)^T \).

In addition, the control variables are chosen to be the effective pitch, yaw, and roll tail control surface deflections. The control vector, \( \mathbf{U} \), is thus defined as \( \mathbf{U} = (\delta P \ \delta Y \ \delta R)^T \). A
discussion of the missile control deflections is given in Appendix A.

The reference condition is assumed to be steady-state flight, with the missile flying at trim condition. Trimmed flight refers to the condition where the total net moments acting on the missile are zero. Additional assumptions at this equilibrium point include:

1. the mass properties of the missile are constant (this assumption implies that the rocket is at burnout)

2. the thrust terms can be neglected (this assumption draws upon the above assumption)

3. the effects of gravity can be neglected (since gravity can be compensated for in the missile guidance system)

4. the effects of the state variable derivatives (\(\dot{\alpha}, \dot{\beta}, \dot{\phi}, \dot{\psi}, \dot{\iota}\)) in aerodynamic the terms are usually relatively small and can be neglected.

The force and moment expressions then have the following functional form

\[
F_x = f[Mach \ No, \alpha, \beta, p, q, r, \delta P, \delta Y, \delta R]
\]

\[
M_x = f[Mach \ No, \alpha, \beta, p, q, r, \delta P, \delta Y, \delta R]
\]

The aerodynamic forces and moments can be related to dimensionless coefficients by the following expressions:

\[
X_A = F_x = QSC_x = -QSC_A = -A
\]  \hspace{1cm} (2.20)

\[
Y_A = F_y = Y = QSC_y
\]  \hspace{1cm} (2.21)

\[
Z_A = F_z = QSC_z = -QSC_N = -N
\]  \hspace{1cm} (2.22)

\[
l = QSDC_l
\]  \hspace{1cm} (2.23)

\[
m = QSDC_m
\]  \hspace{1cm} (2.24)

\[
n = QSDC_n
\]  \hspace{1cm} (2.25)

where \(Q\) is the dynamic pressure (\(= \frac{1}{2} \rho V^2\)), \(S\) is the reference area (for missiles, it is the body
cross-sectional area), and \(D\) is the reference length (the missile diameter). The aerodynamic forces and moments are also linearized about the reference condition by deleting the higher order terms in a Taylor series expansion

\[
\begin{align*}
X_A &= F_x = -A = -QS \left[ C_{\alpha x} \alpha + C_{\beta x} \beta + \left( \frac{D}{2V} \right) \left( C_{\alpha p} + C_{\alpha q} + C_{\alpha r} \right) \right] + C_{\alpha x} \delta P + C_{\alpha x} \delta Y + C_{\alpha x} \delta R \\
Y_A &= F_y = Y = QS \left[ C_{\alpha y} \alpha + C_{\beta y} \beta + \left( \frac{D}{2V} \right) \left( C_{\beta p} + C_{\beta q} + C_{\beta r} \right) \right] + C_{\beta y} \delta P + C_{\beta y} \delta Y + C_{\beta y} \delta R \\
Z_A &= F_z = -N = -QS \left[ C_{\alpha z} \alpha + C_{\beta z} \beta + \left( \frac{D}{2V} \right) \left( C_{\alpha p} + C_{\alpha q} + C_{\alpha r} \right) \right] + C_{\alpha z} \delta P + C_{\alpha z} \delta Y + C_{\alpha z} \delta R \\
l &= QSD \left[ C_{i x} \alpha + C_{i y} \beta + \left( \frac{D}{2V} \right) \left( C_{i p} + C_{i q} + C_{i r} \right) \right] + C_{i x} \delta P + C_{i x} \delta Y + C_{i x} \delta R \\
m &= QSD \left[ C_{m x} \alpha + C_{m y} \beta + \left( \frac{D}{2V} \right) \left( C_{m p} + C_{m q} + C_{m r} \right) \right] + C_{m x} \delta P + C_{m x} \delta Y + C_{m x} \delta R \\
n &= QSD \left[ C_{n x} \alpha + C_{n y} \beta + \left( \frac{D}{2V} \right) \left( C_{n p} + C_{n q} + C_{n r} \right) \right] + C_{n x} \delta P + C_{n x} \delta Y + C_{n x} \delta R
\end{align*}
\]

The derivatives of the aerodynamic coefficients are called the aerodynamic stability derivatives and are evaluated at the reference flight condition.

Applying the linearization method and utilizing the definition of the Taylor series expansion for the aerodynamic forces and moments, the system matrix, \(A\), and the control matrix, \(B\), of Equation (2.13) become
\[ a_{11} = \frac{\partial f_1}{\partial \alpha} = \cos^2 \alpha \left[ -\frac{QS}{Mu} C_{N_x} - \frac{F_x}{Mu} - r \tan \beta \right] + \sin^2 \alpha \left[ \frac{F_x}{Mu} + r \tan \beta \right] + \sin \alpha \cos \alpha \left[ 2p \tan \beta - \frac{2F_x}{Mu} + \frac{QS}{Mu} C_{A_y} \right] \] (2.32)

\[ a_{12} = \frac{\partial f_1}{\partial \beta} = \cos^2 \alpha \left[ -\frac{QS}{Mu} C_{N_x} - \frac{P}{\cos^2 \beta} \right] - \sin \alpha \cos \alpha \left[ \frac{r}{\cos^2 \beta} - \frac{QS}{Mu} C_{A_y} \right] \] (2.33)

\[ a_{13} = \frac{\partial f_1}{\partial p} = \cos^2 \alpha \left[ -\frac{QS}{Mu} \left( \frac{D}{2V} \right) C_{N_x} - \tan \beta \right] - \sin \alpha \cos \alpha \left[ -\frac{QS}{Mu} \left( \frac{D}{2V} \right) C_{A_y} \right] \] (2.34)

\[ a_{14} = \frac{\partial f_1}{\partial q} = \cos^2 \alpha \left[ -\frac{QS}{Mu} \left( \frac{D}{2V} \right) C_{N_x} \right] - \sin \alpha \cos \alpha \left[ -\frac{QS}{Mu} \left( \frac{D}{2V} \right) C_{A_y} + 1 \right] \] (2.35)

\[ a_{15} = \frac{\partial f_1}{\partial r} = \cos^2 \alpha \left[ -\frac{QS}{Mu} \left( \frac{D}{2V} \right) C_{N_x} \right] - \sin \alpha \cos \alpha \left[ -\frac{QS}{Mu} \left( \frac{D}{2V} \right) C_{A_y} + \tan \beta \right] \] (2.36)

\[ a_{21} = \frac{\partial f_2}{\partial \alpha} = \cos^2 \beta \left[ \frac{QS}{Mu} C_{Y_x} + \frac{P}{\cos^2 \alpha} \right] - \sin \beta \cos \beta \left[ -\frac{q}{\cos^2 \alpha} - \frac{QS}{Mu} C_{A_y} \right] \] (2.37)

\[ a_{22} = \frac{\partial f_2}{\partial \beta} = \cos^2 \beta \left[ \frac{QS}{Mu} C_{Y_x} - \frac{F_x}{Mu} + q \tan \alpha \right] + \sin^2 \beta \left[ \frac{F_x}{Mu} - q \tan \alpha \right. \right. \]
\[ \left. \left. - \sin \beta \cos \beta \left[ 2p \tan \alpha + \frac{2F_x}{Mu} - \frac{QS}{Mu} C_{A_y} \right] \right] \] (2.38)

\[ a_{23} = \frac{\partial f_2}{\partial p} = \cos^2 \beta \left[ \frac{QS}{Mu} \left( \frac{D}{2V} \right) C_{Y_x} + \tan \alpha \right] - \sin \beta \cos \beta \left[ -\frac{QS}{Mu} \left( \frac{D}{2V} \right) C_{A_y} \right] \] (2.39)
\[ a_{24} = \frac{\partial f_2}{\partial q} = \cos^2\beta \left[ \frac{Q\Sigma}{M u} \left( \frac{D}{2V} \right) C_{r_s} \right] - \sin\beta \cos\beta \left[ - \frac{Q\Sigma}{M u} \left( \frac{D}{2V} \right) C_{A_r} - \tan\alpha \right] \bigg|_{\text{ref}} \] (2.40)

\[ a_{25} = \frac{\partial f_2}{\partial r} = \cos^2\beta \left[ \frac{Q\Sigma}{M u} \left( \frac{D}{2V} \right) C_{r_s} \right] - \sin\beta \cos\beta \left[ - \frac{Q\Sigma}{M u} \left( \frac{D}{2V} \right) C_{A_r} - 1 \right] \bigg|_{\text{ref}} \] (2.41)

\[ a_{31} = \frac{\partial f_3}{\partial \alpha} = \frac{Q\Sigma D}{I_x} C_{r_s} \bigg|_{\text{ref}} \] (2.42)

\[ a_{32} = \frac{\partial f_3}{\partial \beta} = \frac{Q\Sigma D}{I_x} C_{r_s} \bigg|_{\text{ref}} \] (2.43)

\[ a_{33} = \frac{\partial f_3}{\partial p} = \frac{Q\Sigma D}{I_x} \left( \frac{D}{2V} \right) C_{r_s} \bigg|_{\text{ref}} \] (2.44)

\[ a_{34} = \frac{\partial f_3}{\partial q} = \frac{Q\Sigma D}{I_x} \left( \frac{D}{2V} \right) C_{r_s} \bigg|_{\text{ref}} \] (2.45)

\[ a_{35} = \frac{\partial f_3}{\partial r} = \frac{Q\Sigma D}{I_x} \left( \frac{D}{2V} \right) C_{r_s} \bigg|_{\text{ref}} \] (2.46)

\[ a_{41} = \frac{\partial f_4}{\partial \alpha} = \frac{Q\Sigma D}{I_y} C_{r_s} \bigg|_{\text{ref}} \] (2.47)

\[ a_{42} = \frac{\partial f_4}{\partial \beta} = \frac{Q\Sigma D}{I_y} C_{r_s} \bigg|_{\text{ref}} \] (2.48)

\[ a_{43} = \frac{\partial f_4}{\partial p} = \frac{Q\Sigma D}{I_y} \left( \frac{D}{2V} \right) C_{r_s} - r \left( \frac{I_x - I_z}{I_y} \right) \bigg|_{\text{ref}} \] (2.49)

\[ a_{44} = \frac{\partial f_4}{\partial q} = \frac{Q\Sigma D}{I_y} \left( \frac{D}{2V} \right) C_{r_s} \bigg|_{\text{ref}} \] (2.50)
\[ a_{45} = \frac{\partial f_4}{\partial r} = \frac{QSD}{I_y} \left( \frac{D}{2V} \right) C_{\alpha r} + \rho \left( \frac{I_z - I_y}{I_z} \right) \bigg|_{\text{ref}} \]  
(2.51)

\[ a_{s1} = \frac{\partial f_5}{\partial \alpha} = \frac{QSD}{I_z} C_{\alpha s} \bigg|_{\text{ref}} \]  
(2.52)

\[ a_{s2} = \frac{\partial f_5}{\partial \beta} = \frac{QSD}{I_z} C_{\alpha s} \bigg|_{\text{ref}} \]  
(2.53)

\[ a_{s3} = \frac{\partial f_5}{\partial p} = \frac{QSD}{I_z} \left( \frac{D}{2V} \right) C_{\alpha r} + q \left( \frac{I_z - I_y}{I_z} \right) \bigg|_{\text{ref}} \]  
(2.54)

\[ a_{s4} = \frac{\partial f_5}{\partial q} = \frac{QSD}{I_z} \left( \frac{D}{2V} \right) C_{\alpha r} + \rho \left( \frac{I_z - I_y}{I_z} \right) \bigg|_{\text{ref}} \]  
(2.55)

\[ a_{55} = \frac{\partial f_5}{\partial r} = \frac{QSD}{I_z} \left( \frac{D}{2V} \right) C_{\alpha r} \bigg|_{\text{ref}} \]  
(2.56)

\[ b_{11} = \frac{\partial f_1}{\partial \delta P} = \cos^2 \alpha \left[ - \frac{QS}{Mu} C_{\alpha P} \right] - \sin \alpha \cos \alpha \left[ - \frac{QS}{Mu} C_{\alpha \rho} \right] \bigg|_{\text{ref}} \]  
(2.57)

\[ b_{12} = \frac{\partial f_1}{\partial \delta Y} = \cos^2 \alpha \left[ - \frac{QS}{Mu} C_{\alpha r} \right] - \sin \alpha \cos \alpha \left[ - \frac{QS}{Mu} C_{\alpha s} \right] \bigg|_{\text{ref}} \]  
(2.58)

\[ b_{13} = \frac{\partial f_1}{\partial \delta R} = \cos^2 \alpha \left[ - \frac{QS}{Mu} C_{\alpha r} \right] - \sin \alpha \cos \alpha \left[ - \frac{QS}{Mu} C_{\alpha s} \right] \bigg|_{\text{ref}} \]  
(2.59)

\[ b_{21} = \frac{\partial f_2}{\partial \delta \rho} = \cos^2 \beta \left[ \frac{QS}{Mu} C_{\gamma P} \right] - \sin \beta \cos \beta \left[ \frac{QS}{Mu} C_{\gamma \rho} \right] \bigg|_{\text{ref}} \]  
(2.60)

\[ b_{22} = \frac{\partial f_2}{\partial \delta Y} = \cos^2 \beta \left[ \frac{QS}{Mu} C_{\gamma r} \right] - \sin \beta \cos \beta \left[ \frac{QS}{Mu} C_{\gamma s} \right] \bigg|_{\text{ref}} \]  
(2.61)
\[ b_{23} = \frac{\partial f_2}{\partial \delta R} = \cos^2 \beta \left[ \frac{QS}{Mu} C_{\alpha x} \right] - \sin \beta \cos \beta \left[ -\frac{QS}{Mu} C_{\alpha x} \right] \] 

(2.62)

\[ b_{31} = \frac{\partial f_3}{\partial \delta P} = \left( \frac{QSD}{I_x} \right) C_{\alpha p} \] 

(2.63)

\[ b_{32} = \frac{\partial f_3}{\partial \delta Y} = \left( \frac{QSD}{I_x} \right) C_{\alpha y} \] 

(2.64)

\[ b_{33} = \frac{\partial f_3}{\partial \delta R} = \left( \frac{QSD}{I_x} \right) C_{\alpha x} \] 

(2.65)

\[ b_{41} = \frac{\partial f_4}{\partial \delta P} = \left( \frac{QSD}{I_y} \right) C_{\alpha p} \] 

(2.66)

\[ b_{42} = \frac{\partial f_4}{\partial \delta Y} = \left( \frac{QSD}{I_y} \right) C_{\alpha y} \] 

(2.67)

\[ b_{43} = \frac{\partial f_4}{\partial \delta R} = \left( \frac{QSD}{I_y} \right) C_{\alpha x} \] 

(2.68)

\[ b_{51} = \frac{\partial f_5}{\partial \delta P} = \left( \frac{QSD}{I_z} \right) C_{\alpha p} \] 

(2.69)

\[ b_{52} = \frac{\partial f_5}{\partial \delta Y} = \left( \frac{QSD}{I_z} \right) C_{\alpha y} \] 

(2.70)

\[ b_{53} = \frac{\partial f_5}{\partial \delta R} = \left( \frac{QSD}{I_z} \right) C_{\alpha x} \] 

(2.71)

The system output vector, \( Y \), can be related to the state and control vectors by

\[ Y = Cx + DU \] 

(2.72)
For our missile system, we will assume that the outputs are the lateral body aerodynamic accelerations, $A_Y$ and $A_Z$, and the body rotation rates, $p$, $q$, and $r$. The output vector is then $Y = (A_Y \ A_Z \ p \ q \ r)^T$. The missile accelerations are defined as

$$\begin{align*}
A_Y &= \frac{F_Y}{M} = \frac{Y}{M} = \frac{QS}{M} \left[ C_{Y_s} \alpha + C_{Y_b} \beta + \left( \frac{D}{2V} \right) \left( C_T P + C_T q + C_T r \right) \\
&\quad + C_{Y_{\delta P}} \delta P + C_{Y_{\delta \beta}} \delta \beta + C_{Y_{\delta \gamma}} \delta \gamma + C_{Y_{\delta \delta}} \delta \delta \right] \\
A_Z &= \frac{F_Z}{M} = -\frac{N}{M} = -\frac{QS}{M} \left[ C_{N_s} \alpha + C_{N_b} \beta + \left( \frac{D}{2V} \right) \left( C_N P + C_N q + C_N r \right) \\
&\quad + C_{N_{\delta P}} \delta P + C_{N_{\delta \beta}} \delta \beta + C_{N_{\delta \gamma}} \delta \gamma + C_{N_{\delta \delta}} \delta \delta \right]
\end{align*}$$

(2.73)  

In matrix form, the elements of the $C$ and $D$ matrices of Equation (2.72) can be defined as

$$\begin{align*}
c_{11} &= \frac{QS}{M} C_{Y_s} \\
c_{12} &= \frac{QS}{M} C_{Y_b} \\
c_{13} &= \frac{QS}{M} \left( \frac{D}{2V} \right) C_{Y_T} \\
c_{14} &= \frac{QS}{M} \left( \frac{D}{2V} \right) C_{Y_e} \\
c_{15} &= \frac{QS}{M} \left( \frac{D}{2V} \right) C_{Y_r} \\
c_{21} &= -\frac{QS}{M} C_{N_s} \\
c_{22} &= -\frac{QS}{M} C_{N_b}
\end{align*}$$

(2.75)  

(2.76)  

(2.77)  

(2.78)  

(2.79)  

(2.80)  

(2.81)
\[ c_{23} = -\frac{QS}{M} \left( \frac{D}{2V} \right) C_{N_r} \] (2.82)

\[ c_{24} = -\frac{QS}{M} \left( \frac{D}{2V} \right) C_{N_x} \] (2.83)

\[ c_{25} = -\frac{QS}{M} \left( \frac{D}{2V} \right) C_{N_r} \] (2.84)

\[ c_{31} = c_{32} = c_{33} = c_{34} = c_{35} = 0 \] (2.85-a)

\[ c_{41} = c_{42} = c_{43} = c_{45} = 0 \] (2.85-b)

\[ c_{51} = c_{52} = c_{53} = c_{54} = 0 \] (2.85-c)

\[ c_{33} = c_{44} = c_{55} = 1 \] (2.86)

\[ d_{11} = \frac{QS}{M} C_{Y_{e x}} \] (2.87)

\[ d_{12} = \frac{QS}{M} C_{Y_{e r}} \] (2.88)

\[ d_{13} = \frac{QS}{M} C_{Y_{e r}} \] (2.89)

\[ d_{21} = -\frac{QS}{M} C_{N_{e r}} \] (2.90)

\[ d_{22} = -\frac{QS}{M} C_{N_{e r}} \] (2.91)

\[ d_{23} = -\frac{QS}{M} C_{N_{e x}} \] (2.92)

\[ d_{31} = d_{32} = d_{33} = d_{41} = d_{42} = d_{43} = d_{51} = d_{52} = d_{53} = 0 \] (2.93)

Equations (2.32) - (2.93) describe our linear, coupled state-space model of the missile airframe at the specified reference condition.
For preliminary analysis and design, it is convenient to examine the totally decoupled planar dynamics of the pitch, yaw and roll channels. (Although the autopilot designs considered in this study are for the pitch/yaw plane, the roll dynamics are carried along for completeness.)

The decoupled state-space model is obtained by nulling the cross-coupling terms defined in the dynamical model described above. This decoupled model is described by the following equations:

\[
\dot{\alpha} = \left\{ \cos^2 \alpha \left[ -\frac{Q_S}{M_u} C_{N^*} + \frac{Q_S}{M_u} \left( C_{A^*} \alpha + \left( \frac{D}{2V} \right) C_{A^*} \delta P \right) \right] \\
+ \sin^2 \alpha \left[ -\frac{Q_S}{M_u} C_{A^*} \alpha + \left( \frac{D}{2V} \right) C_{A^*} \delta P \right] \\
+ \sin \alpha \cos \alpha \left[ 2\frac{Q_S}{M_u} \left( C_{N^*} + \frac{D}{2V} \right) C_{A^*} \delta P \right] + \frac{Q_S}{M_u} C_{A^*} \right\} \alpha \\
\] (2.94)

\[
\dot{q} = \left\{ \cos^2 \alpha \left[ -\frac{Q_S}{M_u} C_{N^*} \right] + \sin \alpha \cos \alpha \left[ \frac{Q_S}{M_u} \left( \frac{D}{2V} \right) C_{A^*} \right] + 1 \right\} q \\
+ \frac{Q_S}{M_u} C_{N^*} \\
\] (2.95)

\[
\dot{r} = \left\{ \cos^2 \beta \left[ \frac{Q_S}{M_u} C_{y^*} + \frac{Q_S}{M_u} \left( C_{A^*} \beta + \left( \frac{D}{2V} \right) C_{A^*} \right) \right] \\
+ \sin^2 \beta \left[ -\frac{Q_S}{M_u} C_{A^*} \beta + \left( \frac{D}{2V} \right) C_{A^*} \right] \\
- \sin \beta \cos \beta \left[ 2\frac{Q_S}{M_u} \left( C_{y^*} + \frac{D}{2V} \right) C_{A^*} \delta Y \right] - \frac{Q_S}{M_u} C_{A^*} \right\} r \\
+ \left\{ \cos^2 \beta \left[ \frac{Q_S}{M_u} \left( \frac{D}{2V} \right) C_{y^*} \right] + \sin \beta \cos \beta \left[ \frac{Q_S}{M_u} \left( \frac{D}{2V} \right) C_{A^*} \right] \right\} \delta Y \\
\] (2.96)

\[
\dot{\phi} = \left[ \frac{Q_S D}{I_x} \left( \frac{D}{2V} \right) C_{y^*} \right] p + \left[ \frac{Q_S D}{I_x} C_{A^*} \right] \delta R \\
\] (2.96)
\[ q = \left[ \frac{QSD}{I_y} c_{m_s} \right] \alpha + \left[ \frac{QSD}{I_y} \left( \frac{D}{2V} \right) c_{m_s} \right] q + \left[ \frac{QSD}{I_y} C_{n_s} \right] \delta P \quad (2.97) \]

\[ \dot{r} = \left[ \frac{QSD}{I_z} c_{n_s} \right] \beta + \left[ \frac{QSD}{I_z} \left( \frac{D}{2V} \right) c_{n_s} \right] r + \left[ \frac{QSD}{I_z} C_{n_s} \right] \delta Y \quad (2.98) \]

The acceleration output equations are

\[ A_y = \left[ \frac{QS}{M} C_{r_s} \right] \beta + \left[ \frac{QS}{m} \left( \frac{D}{2V} \right) C_{r_s} \right] r + \left[ \frac{QS}{M} C_{r_s} \right] \delta Y \quad (2.99) \]

\[ A_x = \left[ \frac{QS}{m} C_{N_s} \right] \alpha + \left[ \frac{QS}{m} \left( \frac{D}{2V} \right) C_{N_s} \right] q + \left[ \frac{QS}{M} C_{N_s} \right] \delta P \quad (2.100) \]

The decoupled equations can be further reduced by assuming (1) small angles of attack and sideslip, (2) small fin deflections at the reference condition, and (3) relatively small stability derivatives \( C_{N_s} \) and \( C_{r_s} \). With the small angle approximation in the decoupled equations above, the axial force terms may be neglected and the axial component of the velocity, \( u \), is approximately equivalent to the total velocity, \( V \). These assumptions are applied often in the conventional autopilot design approach. With these assumptions, the \( \dot{\alpha} \), \( \dot{\beta} \), and acceleration equations become

\[ \dot{\alpha} = \left[ \frac{QS}{MV} C_{N_s} \right] \alpha + q + \left[ \frac{QS}{MV} C_{N_s} \right] \delta P \quad (2.101) \]

\[ \dot{\beta} = \left[ \frac{QS}{MV} C_{r_s} \right] \beta - r + \left[ \frac{QS}{MV} C_{r_s} \right] \delta Y \quad (2.102) \]

\[ A_y = \left[ \frac{QS}{M} C_{r_s} \right] \beta + \left[ \frac{QS}{M} C_{r_s} \right] \delta Y \quad (2.103) \]
\[ A_Z = \left[ -\frac{Q_S}{M} C_{N_x} \right] \alpha + \left[ -\frac{Q_S}{M} C_{N_{\alpha \rho}} \right] \delta P \] (2.104)

Examination of the decoupled airframe equations above show that the pitch and yaw planes are symmetric as expected. The resulting symmetry between the pitch and yaw planes can be utilized in the design of the pitch/yaw autopilot.

2.2 FLEXIBLE BODY MODEL

The previous section addressed the dynamics of the rigid body airframe. In this section, we model the elasticity of the airframe. The bending dynamics of the missile are coupled with the rigid-body dynamics and therefore: may limit the autopilot bandwidth, may cause actuator heating and saturation problems, and may even destabilize the flight control system. For the long and slender airframe configuration considered here, the missile body bending effects are expected to have a strong impact on the missile's stability and performance characteristics. Thus, a model describing the bending dynamics of the airframe is formulated.

The tail deflection which causes the missile to maneuver also produces forces and moments which deform the airframe. For example, an airframe may assume various deformed shapes, with one such (exaggerated) shape illustrated in Figure 2.4. The airframe may flex through several bending modes which occur at various frequencies. Figure 2.5 shows what the first three bending modes may look like for a typical missile airframe.

The flexible body dynamics affects the missile lateral accelerations, Equations (2.103) and (2.104), and missile's body angular rates, as defined in Equations (2.97) and (2.98). Herein, we will treat the elastic body dynamics as being superimposed upon the rigid body dynamics. Thus, transfer functions for \((A_Z/\delta)_m\) and \((q/\delta)_m\), which describe the effects of bending can be
Figure 2.4 Example Illustration of Aeroelastic Airframe
$\phi_1$ = FIRST MODE SHAPE
$\phi_2$ = SECOND MODE SHAPE
$\phi_3$ = THIRD MODE SHAPE

Figure 2.5 Typical Missile Bending Modes
formulated. Since the flexible body dynamics alter the accelerations and body rates at any and all points along the missile airframe, we will define and utilize the elastic mode characteristics at the accelerometer and rate gyroscope location. (Here we assume that the accelerometer and gyro are packaged together in an inertial reference unit (IRU).) We consider the body bending characteristics at the IRU location because the acceleration and body rate will be utilized as feedback signals to the controller and the sensor measurements will be affected by the elastic body dynamics. Figure 2.6 depicts a typical missile control system that includes the effects of aeroelasticity.

Several approaches can be taken to obtain the missile bending dynamics\(^17,18,28,42,48,69\). For example, a lumped parameter model\(^17,18\), where the missile is broken down into a finite number of mass-spring systems, can be used to find body deflections due to input forces at various node points. The results can be used to form transfer functions of the bending dynamics.

Each bending mode is treated separately from the others, as indicated in Figure 2.6. For a particular bending mode, the vibrational dynamics can be written in the following form\(^17,18\) for the elastic dynamics at the accelerometer and gyro, respectively,

\[
\left( \begin{array}{c}
A_n \\
F_T/n
\end{array} \right) = \left( \begin{array}{c}
s^2 \Phi_n(x_T) \Phi_n(x_T) \\
(s^2 + 2 \zeta_n \omega_n s + \omega_n^2)M_{gn}
\end{array} \right)
\]

\[
\left( \begin{array}{c}
\dot{\Phi}_n \\
\dot{F}_T/n
\end{array} \right) = \left( \begin{array}{c}
s \Phi_n(x_T) \frac{d\Phi_n}{dx} \bigg|_{x=x_0} \\
(s^2 + 2 \zeta_n \omega_n s + \omega_n^2)M_{gn}
\end{array} \right)
\]

where

\(A_{n,fb}\) = lateral acceleration at accelerometer due to bending mode \(n\)

\(\dot{\theta}\) = \(q_n\) = rotational (pitch) rate of bending at gyro for mode \(n\)
Figure 2.6 Missile Control System Diagram with Flexible-Body Dynamics
\( F_T \) = force due to tail deflection (force input at tail hinge line)

\( \phi_n(x) \) = \( n^{th} \) mode shape

\( \phi_n(x_t) \) = displacement at tail

\( \phi_n(x_a) \) = displacement at accelerometer

\( \frac{d\phi}{dx}_{x = x_g} \) = modal slope at \( x_g \) (gyro)

\( M_{gn} \) = generalized mass of mode

\( \zeta_{fb} \) = structural damping ratio of mode

\( \omega_n \) = natural frequency of bending mode

Defining \( \frac{F_T}{\delta} \) as

\[
\frac{F_T}{\delta} = \frac{\text{force at tail fin}}{\text{tail deflection}} = QSC_{N_b} \tag{2.107}
\]

then the transfer functions describing the flexible dynamics become

\[
\left( \frac{A_n}{\delta} \right)_{fb} = \frac{s^2 \phi_n(x_t) \phi_n(x_a) (QSC_{N_b})}{(s^2 + 2\zeta_{fb} \omega_n s + \omega_n^2) M_{gn}} \tag{2.108}
\]

\[
\left( \frac{q}{\delta} \right)_{fb} = \frac{s \phi_n(x_t) \frac{d\phi}{dx}_{x = x_g} (QSC_{N_b})}{(s^2 + 2\zeta_{fb} \omega_n s + \omega_n^2) M_{gn}} \tag{2.109}
\]

Equations (2.108) and (2.109) provide linear models of acceleration and body rate due to aeroelastic effects of the airframe. In these transfer functions, the aerodynamic stability derivative of normal force due to fin deflection is based upon rigid body airframe aerodynamics.

However, another effect which may be significant in some cases is the change in the aerodynamic characteristics of an airframe due to bending deformations. Because the missile will deform slightly, the aerodynamic forces and moments acting on the missile may vary
from nominal values. This effect can be included in the $H_m/\mu$-synthesis design approach implicitly or explicitly as one of the sources of aerodynamic coefficient uncertainties. Coefficient or parameter uncertainties are discussed in Chapter 8, and missile aerodynamic coefficient uncertainties are addressed in Chapters 9 and 10.
Chapter 3. Missile System Data

A complete description of the airframe considered in the autopilot designs is now specified by the model in Chapter 2. Next, missile data, including aerodynamics, are presented. This information is then followed by a description of the instrumentation models used in this study.

3.1 AIRFRAME DATA

The missile considered in this study is a variant of the Advanced Point Defense Missile (APODS). The missile is a surface-launched, short-range, air-interceptor and is defined in Reference 50. The APODS has a slender, cruciform, body-tail configuration, designed to satisfy constraints imposed by the launch canister. However, a variant of this configuration is considered here. The airframe was lengthened and supplemented with dorsals to provide enhanced maneuverability, and is shown in Figure 3.1. A brief discussion on the dorsal sizing and placement is given in Reference 51. As mentioned previously, the tail-controlled missile is assumed to utilize skid-to-turn (STT) steering and is assumed to fly in the cross ("X") configuration. Specific airframe data is presented in Table 3.1. For this design study, the reference flight condition was selected to be a low altitude engagement, with the missile gliding at Mach 3 after burnout.

3.2 AERODYNAMIC DATA

For the pitch plane, plots of the normal force coefficient and pitching moment
Figure 3.1 Short-Range High-Performance Missile Configuration

*All dimensions in Calibers*
Table 3.1 Missile Data.

<table>
<thead>
<tr>
<th>Flight Condition</th>
<th>Boost</th>
<th>End-of-Boost (Sustain)</th>
<th>Burn-Out (Glide)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight (lb)</td>
<td>543.84</td>
<td>283.84</td>
<td>262.84</td>
</tr>
<tr>
<td>$x_{ru}$ (inches from nose)</td>
<td>118.32</td>
<td>87.52</td>
<td>78.72</td>
</tr>
<tr>
<td>$I_x$ (slug ft$^2$)</td>
<td>1.30</td>
<td>0.92</td>
<td>0.82</td>
</tr>
<tr>
<td>$I_y = I_z$ (slug ft$^2$)</td>
<td>266.45</td>
<td>154.00</td>
<td>121.78</td>
</tr>
</tbody>
</table>

d = 8. inches
S = 0.349 ft$^2$
$x_{iru} = 59.8$ (old value: 39.0) inches (from nose)
$x_{hinge-line} = 158.$ inches (from nose)
coefficient at Mach 3 are presented in Figures 3.2 and 3.3, respectively. The aerodynamic coefficients are plotted against the angle of attack for various pitch control deflections. The normal force is close to a quadratic function of angle of attack for fixed control deflection. For angles of attack less than 15 degrees, the normal force is fairly linear with respect to the control deflection for fixed angle of attack. The pitching moment is fairly linear with respect to angle of attack and control deflection for angles of attack below 10 degrees. The pitching moment has a nonlinearity with respect to angle of attack appearing between 15 and 25 degrees. The plot of $C_n$ vs. $\alpha$ yields information regarding the vehicle's static stability (statically stable if $C_{m_s} < 0$) and also whether the airframe can be trimmed for a given fin deflection. Figure 3.3 shows that the missile does have enough control authority to trim the missile for angles of attack greater than 30 degrees. A plot of the trim normal force data as a function of the trim angle of attack is displayed in Figure 3.4. The nonlinear aerodynamic nature of the airframe illustrated in these figures show why the autopilot design must be analyzed globally to ensure stable flight over the entire range of operational angles of attack.

The stability derivatives required for the autopilot design were determined from the aerodynamic force and moment data. The stability derivatives include the state and control derivatives, such as $C_{N_s}$, $C_{m_s}$, $C_{m_{sl}}$, $C_{Y_{sk}}$, $C_{l_{sk}}$, etc. The particular stability derivative data that were used in this pitch-plane autopilot design study for the given flight condition are specified in Table 3.2. The derivatives were determined by averaging their particular values over the expected angle of attack range, where the maximum angle of attack, (7 degrees),
Figure 3.2 Normal Force Coefficient vs. Angle of Attack
Mach = 3
X Configuration
Glide Condition

Figure 3.3 Pitching Moment Coefficient vs. Angle of Attack
Mach = 3
X Configuration
Glide Condition

Figure 3.4 Trim Normal Force
<table>
<thead>
<tr>
<th>Stability Derivative</th>
<th>Average value at Mach Number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>$C_{N_x}$</td>
<td></td>
</tr>
<tr>
<td>$C_{N_y}$</td>
<td></td>
</tr>
<tr>
<td>$C_{m_x}$</td>
<td></td>
</tr>
<tr>
<td>$C_{m_y}$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Dynamic Derivative</th>
<th>Mach Number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td>$C_{l_y}$</td>
<td>-28.66</td>
</tr>
<tr>
<td>$C_{m_y}$</td>
<td>-3130.4</td>
</tr>
<tr>
<td>$C_{n_y}$</td>
<td>-3130.4</td>
</tr>
</tbody>
</table>

37
was determined at the flight condition corresponding to the maximum structural acceleration limit of the airframe, specified in Chapter 5. The dynamic derivatives \( C_{y}, C_{q}, C_{n} \) were computed separately from the stability derivative data, and are also listed in Table 3.2. The remaining stability derivatives, such as \( C_{q}, C_{q}, C_{n}, \) etc., are components of aerodynamic coupling (to be discussed next) and are neglected in this study.

In addition to inplanar effects, the autopilot designer must consider out-of-plane effects such as pitch-yaw-roll coupling. These coupling phenomena include induced forces and moments (i.e., aerodynamic cross-coupling) and kinematic coupling. References 4,6,15,17,18,21,41,46,47, and 52-56 describe pitch-yaw-roll coupling phenomena in detail. The induced effects occur when the missile is pulling a symmetrical maneuver in some control plane (e.g., pitch), and a desired out-of-plane control deflection (e.g., yaw) also induces aerodynamic forces and moments in the remaining plane (e.g., roll) where no control action is desired. For example, consider a missile in a pitch maneuver, where the angle of attack is nonzero. If yaw controls are applied, a rolling moment is induced with the yaw maneuver, and similarly, an application of roll control deflections yield a side force and a yawing moment. Aerodynamic coupling refers to the forces and moments that result due to an asymmetric configuration relative to the airstream. In this case, consider a missile perturbed in yaw away from some pitch orientation. A sideslip results, yielding a side force and yaw moment, and creating an asymmetric configuration with respect to the flight condition. A rolling moment is then produced. Examples of both of these coupling effects are illustrated in Figure 3.5.
(a) Induced Control Coupling

(b) Aerodynamic Cross Coupling

Figure 3.5 Pitch-Yaw-Roll Coupling
These pitch-yaw-roll coupling phenomena are dependent on the missile configuration or shape, flight attitude, and flight condition. The coupling becomes more significant at low dynamic pressure and high angle of attack flight conditions. One reason for coupling is just pure kinematics. Coupling is also created by several complicated flow phenomena. One of the more important causes in this case is the presence of a shock on the windward side, causing an increased pressure acting on the windward surfaces and a decreased pressure field about the leeside surfaces. As a result, the effectiveness of the leeward surfaces is reduced while the windward surfaces experience an increase in effectiveness. This is called fin shading or fin blanking. Another pertinent reason for pitch-yaw-roll aerodynamic coupling is the effect of vortices. Vortices are shed from the nose, body, and forward surfaces (such as wings) and impact on the leeward tail control surfaces. These vortices change the local pressure distribution and produce moments which act on the missile. Other causes of aerodynamic coupling effects include wing tip effects, wing root effects, separation effects on the body and wing surfaces, sweepback effects, downwash effects, and interference effects (such as fin-on-fin interference). Some of these causes are depicted in Figure 3.6\textsuperscript{17,21,47}.

Because of the shape of the missile configuration considered in this study, the aerodynamic coupling effects described above are present at the high angles of attack. However, since we are constrained to relatively low angles of attack due to structural limitations, we will neglect the effects of aerodynamic coupling in this study.
Figure 3.6 Causes of Pitch-Yaw-Roll Coupling
3.3 FLEXIBLE-BODY DATA

In addition to the aerodynamics, the flexible body data must also be specified. References 57 and 58 discuss the bending properties for the missile configuration considered herein. We consider only the first three bending modes here. Table 3.3 gives this data for gliding flight. Figure 3.7 shows the first three bending mode shapes. The data presented in Table 3.3 can be substituted directly into the bending dynamic transfer functions presented in Equations (2.108) and (2.109). These equations provide a model of the flexible missile dynamics for which compensation is required in the autopilot design.

3.4 INSTRUMENTATION MODELS AND DATA

The effect of instruments on the design of the missile flight control system may also be significant and, therefore, cannot be neglected. We now define models of the fin servos (actuators), body rate gyros, and accelerometers. There are several ways of modeling these instruments, such as those described in References 7, 12-14, 17-20, 23-25, 28-35, 36-37, 41-42, 48-50, and 59. We utilize a quadratic model for these components.

For the fin actuator model, a second-order transfer function of the form

\[ \frac{\delta}{\delta_c} = \frac{1}{1 + \frac{2\zeta_{act}}{\omega_{act}}s + \frac{s^2}{\omega_{act}^2}} \]  \hspace{1cm} (3.1)

is used, where \( \zeta_{act} \) is the actuator damping ratio (= 0.7) and \( \omega_{act} \) is the actuator bandwidth (= 30 Hz = 188.5 rad/sec). Actuator constraints include a rate limit of 300 deg/sec and a 40 deg fin deflection limit.
Table 3.3. Flexible Body Data.

For Gliding (Burn-Out) Flight Condition

$\zeta_{fb} = \text{damping ratio} = 0.02$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Bending Frequency (rad/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>368</td>
</tr>
<tr>
<td>$\phi_n(x_r)$ (ft)</td>
<td>0.04114</td>
</tr>
<tr>
<td>$\phi_n(x_A)$ (ft)</td>
<td>-0.01976</td>
</tr>
<tr>
<td>$(d\phi/dx)_{inst}$</td>
<td>0.013657</td>
</tr>
<tr>
<td>$M_{in}$ (slug ft$^2$)</td>
<td>0.009356</td>
</tr>
</tbody>
</table>
Figure 3.7 Missile Bending Mode Shapes for Gliding Flight
Both pitch rate and acceleration are measured and fed back to the control system. The rate gyro is modeled as

\[
\frac{\dot{q}}{\dot{\theta}} = \frac{q}{q} = \frac{1}{1 + \frac{2\zeta_{rg} s}{\omega_{rg}} + \frac{s^2}{\omega_{rg}^2}}
\]  

(3.2)

with a natural frequency \(\omega_{rg} = 80\) Hz (500 rad/sec) and damping, \(\zeta_{rg}\), of 0.70. A maximum rate of ±400 deg/sec is assumed. Similarly, the accelerometer can be described by

\[
\frac{A_{meas}}{A} = \frac{1}{1 + \frac{2\zeta_{acc} s}{\omega_{acc}} + \frac{s^2}{\omega_{acc}^2}}
\]  

(3.3)

where \(\zeta_{acc} = 0.70\) and \(\omega_{acc} = 60\) Hz (377 rad/sec). A maximum acceleration of ±75 g’s is assumed to be the accelerometer limit. All the values assumed here for the instrumentation are within the capability of current technology.

The gyro and accelerometer sensors are assumed to be included as components of the missile’s inertial reference unit (IRU). Usually, the IRU is not situated at the center of mass of the missile, but along the missile axial centerline (x-axis) some distance, \(d\), offset from the center of mass. The distance of the IRU offset from the center of mass in the y and z directions is assumed to be relatively small. Thus, the acceleration measured at the IRU station may not be the missile acceleration at the center of mass, which is usually assumed to be the output of the plant and input to the controller. We now develop the equations that account for the accelerometer offset. We show that the measured accelerations at the IRU are functions of the angular accelerations \(\dot{\theta}\) and \(\ddot{\theta}\).

To account for this offset, consider the dynamics of a moving body at some point other
than at the origin of the object's body frame, O. This is depicted in Figure 3.9. The acceleration at the point P (IRU location) is given by

$$A = A_0 + \dot{d} + \omega \times \dot{d} + 2\omega \times \ddot{d} + \omega \times (\omega \times d)$$  \hspace{1cm} (3.4)

If the total flight trajectory of the missile were considered, the center of mass would shift in time as the rocket burned, until it reached the burn-out condition. After burnout, the offset distance, d, remains constant. Since a burnout condition is assumed for this autopilot design study, then \( \dot{d} = \ddot{d} = 0 \). Defining

$$d = \begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix}, \quad \omega = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

then, assuming a rigid body,

$$\begin{bmatrix} A_X \\ A_Y \\ A_Z \end{bmatrix}_{IRU} = \begin{bmatrix} A_{X_0} \\ A_{Y_0} \\ A_{Z_0} \end{bmatrix}_{CG} + \begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} \times \begin{bmatrix} d \\ 0 \end{bmatrix} + \begin{bmatrix} p \\ q \end{bmatrix} \times \begin{bmatrix} p \\ q \end{bmatrix} \times \begin{bmatrix} d \\ 0 \end{bmatrix}$$  \hspace{1cm} (3.5)

or

$$\begin{bmatrix} A_X \\ A_Y \\ A_Z \end{bmatrix}_{IRU} = \begin{bmatrix} A_{X_0} \\ A_{Y_0} \\ A_{Z_0} \end{bmatrix}_{CG} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \dot{d} + \begin{bmatrix} -d(q^2 + r^2) \\ dpq \\ dpr \end{bmatrix}$$  \hspace{1cm} (3.6)

Equation (3.6) quantifies the effect that the IRU offset position has on the measured acceleration. The flexible body effects at the accelerometer would then be summed with the above equation. In summary, the two key assumptions are that (1) the missile is flying at the burnout condition and (2) the only effective offset distance of the IRU is in the missile x-direction.
Figure 3.8 Moving IRU Coordinate Frame
The measured yaw and pitch accelerations are, respectively,

\[ A_{\gamma_{\text{meas}}} = A_{\gamma_q} + d\dot{\gamma} + dpq \]  
(3.7)

\[ A_{z_{\text{meas}}} = A_{z_q} - dq + dpr \]  
(3.8)

Decoupling the measured accelerations into planar expressions (assuming that the missile is roll stabilized such that the roll rate, \( \dot{\gamma} \), is negligibly small),

\[ A_{\gamma_{\text{meas}}} = A_{\gamma_q} + d\dot{\gamma} \]  
(3.9)

\[ A_{z_{\text{meas}}} = A_{z_q} - dq \]  
(3.10)

The effect of the accelerometer offset from the missile center of gravity can then be accounted for in the autopilot design by using Equations (3.7) and (3.8) for the MIMO autopilot design and Equations (3.9) and (3.10) for the planar autopilot design. The aeroelastic dynamics at the accelerometer would then be added to the accelerations as described earlier.
Chapter 4. Autopilot Design Requirements

To develop the primary goal of the autopilot, we summarize the role of the flight control system in the missile homing loop. A diagram of the missile homing loop is presented in Figure 4.1. Guidance (acceleration) commands are generated from target tracking information with the purpose of "guiding" the interceptor missile towards the target. The guidance commands are then sent to the autopilot. The autopilot operates on those commands to determine appropriate control surface deflection commands to "steer" the missile in the direction desired by the guidance system. The control surface deflection command signals drive the control-surface actuators to produce fin deflections, which, in turn, create aerodynamic forces and moments that cause the missile to maneuver. Rate gyro and accelerometer measurements of the maneuver are processed by the inertial reference unit to yield feedback control signals for the autopilot. The guidance system commands are compared to like quantities formed from accelerometer measurements, to form an acceleration differences or errors. These acceleration errors and rate gyro measurements are then used by the autopilot to generate new control surface deflection commands. A mission requirement for the missile is that it homes in close enough to the target so that its warhead blast may damage or "kill" the target. This mission requirement establishes the primary goal of the missile flight control system. That is, the autopilot should track guidance commands and control the missile such that the intercept miss distance between the interceptor missile and target is minimal.
Figure 4.1 Missile Homing Loop
Recall that the missile treated herein is a skid-to-turn airframe. This constraint means that the airframe is roll-attitude stabilized and that it maneuvers in the two planes of pitch and yaw. The roll attitude of the missile is held constant due to: guidance limitations (for example, rolling motion may cause the guidance commands to be in the incorrect plane due to system lags); a semi-active radar missile with polarized antennas must maintain roll orientation for proper alignment with illuminator polarization; directional fuzing; simplified guidance signals (such as for gravity bias and trajectory shaping supplements); and excessive rolling that may lead to loss of aerodynamic control. The guidance system commands maneuver accelerations in the pitch and yaw planes, while the autopilot must achieve the commanded accelerations and maintain a constant roll orientation. In this regard, the autopilot will serve as a "tracker" to minimize both the acceleration error \( (A_C - A) \) and roll angle error \( (\phi_C - \phi) \). These requirements provide another set of basic autopilot objectives.

To achieve the goals established above, the autopilot must perform several important functions. These include\(^{15,39,32,36}\):

1. provide airframe response characteristics that satisfy the specifications
2. maintain stability of the airframe
3. provide tolerance to plant variations, uncertainties and disturbances; in other words, must provide robustness
4. limit normal accelerations (or angles of attack or body rates)
5. limit control surface deflection angles and deflection rates
6. maintain certain bandwidths to respond to the guidance commands and not to the noise content in the acceleration commands
7. maintain appropriate frequency responses to avoid resonance with other missile components (e.g., actuators and sensors) and airframe bending modes
When we speak of the airframe response, we refer to several response characteristics of a missile system’s transient time behavior to a step guidance command input. These response characteristics are the effective first-order autopilot time constant, \( \tau_{AP} \), the response overshoot, the settling time, \( T_s \), and the steady-state acceleration tracking error. The time constant is defined as the time for the response to reach 63\% (1 - e\(^{-1}\)) of the commanded value, and is essentially a measure of response speed. The autopilot time constant must be fast enough to satisfy guidance and homing requirements. Overshoot should be minimized so that the response is acceptable for guidance and that the maneuver or structural limits are not exceeded. The settling time is the response time to achieve and maintain 95\% of the commanded value. A quick time to settle and minimum acceleration errors are desired for effective guidance.

Stability and robustness measures have classically been defined in the frequency domain as gain and phase margins. Designing large stability margins into the controller ensures that the system will be stable and will be tolerant to disturbances, uncertainties, and plant variations. However, good stability margins usually come at the expense of good response characteristics. This phenomena presents an interesting tradeoff for the autopilot designer.

The autopilot should also provide high frequency attenuation. Adequate attenuation would prevent the autopilot from responding to high frequency effects, such as flexible body dynamics, guidance noise, actuators, measurement instruments, and other high frequency noise disturbances.

Another function of the autopilot is to limit the normal (or lateral) accelerations. In effect, by limiting the normal accelerations, the achieved angle of attack becomes limited. These acceleration limits may be needed if the designer does not have high angle of attack data, the
missile structural limits are reached, or if the airframe cannot be stabilized or controlled at high angles of attack. Body rates may also need to be limited if large body rates lead to instability. In addition, tail fin angles and rates are constrained to account for limitations of the servos.

These functions then lead to autopilot design requirements. The requirements are defined by the engagement scenario (e.g., threat attributes, flight condition, etc.) and missile system characteristics, such as airframe parameters, guidance law, etc. The nominal design goals for this study are specified as follows for the pitch plane in classical SISO terms. To yield effective guidance at the reference condition, an autopilot time constant less than or equal to 0.2 seconds is desired. Other time response requirements are near zero (less than one-tenth percent) steady-state acceleration tracking error and a maximum overshoot of 5%. For stability, it is desirable to have the classical stability margins of at least 6 dB gain margin and 30 degrees of phase margin. For the rigid body autopilot design, the high frequency attenuation (loop gain at the actuator input) should be at least 20-30 dB at 300 rad/sec. To add further robustness to elastic mode uncertainties, we require at least 20-30 dB of attenuation at the bending mode frequencies with a gain stabilization approach, or at least ± 50 degrees of phase margin at the gain crossover points near the bending modes with a phase-stabilization approach. The stability margins specified above should be met when the signal flow is opened at the input to the actuator (with all the other loops closed) and when the loop is broken at the rate gyro output. In addition, when the flexible body dynamics are considered on the acceleration signal, the stability margins given above should also hold with the loop opened at the output of the accelerometer to ensure that the aeroelastic effects there do not cause stability problems.

Note that the requirements specified above are given in terms of classical design requirements. The current design practice is to use classical design requirements. However,
new multivariable requirements will also arise with the newer design technique, and will be left to a subsequent chapter for discussion.
Chapter 5. Autopilot Design Constraints and Problems

The missile flight control system is to be designed such that the goals described in the previous chapter are attained. However, the autopilot design process is not simple and straightforward. Many limitations and problems constrain the design and performance of the missile autopilot. An example of some of the sources of error in a guided missile system are illustrated in Figure 5.1\textsuperscript{41}. The constraints which affect the autopilot are to be considered necessarily in the design and/or analysis. Otherwise, the design requirements presented in the previous chapter may fail to be met.

This chapter introduces several constraints and problems that affect the performance and stability of missile autopilots. These problems were compiled from numerous references, the primary references being 4, 17, 18, 20, 33-35, 46-47 and 61-64. Qualitative descriptions of the limitations are presented and particular numerical values are specified where necessary. Some of the constraints presented here are not used in our current autopilot design work. Such constraints neglected here will be considered in future investigations.

First, recall that the missile airframe considered here is tail-controlled. To execute a pitch up maneuver, the leading edge of the tail control surfaces are initially deflected downward. The resulting normal missile acceleration (at the center of gravity) will then briefly respond in the opposite direction of the command before tracking the command. This opposite response is due to a tail-controlled missile having a transfer function with a right half-plane zero. That is, tail controlled missiles are nonminimum phase systems. Consequently, the autopilot's bandwidth is limited.
Figure 5.1 Sources of Error in Guided Missile Systems
The airframe also has several other limitations. Structurally, the airframe can only withstand a maximum amount of loading due to maneuver. For our airframe, 60 g's of acceleration is the structural limit, but 20 g's will be assumed as the guidance command limit as a margin of safety. Also related to the airframe structure, the autopilot is limited by body-bending dynamics or structural vibrations, as was discussed in a previous chapter. The first few structural modes may occur at frequencies within the bandwidth of the accelerometer and rate gyro sensors. The measurements being fed back to the autopilot are corrupted by the signals coming from the bending modes. In such a case, the corruption due to structural bending in the output of these sensors is interpreted as rigid body motions. Feeding back such corrupted signals to the autopilot could potentially cause performance degradation, saturation, actuator heating and instability. As given earlier, the bending effects can be linearly modeled with a second-order transfer function. This problem is further complicated by uncertainty in the bending mode frequency, modal slope and damping. Reference 35 suggests some reasons for this uncertainty, which include: the use of linear models for nonlinear airframe structural behavior; structural properties, such as stiffness, may vary from missile to missile as well as during flight, perhaps due to heating or change in the mass properties; and that the effective damping may be a function of vibration amplitude.

The autopilot design is also faced with three problems associated with the aerodynamics of the missile configuration: (1) variation over the flight envelope, (2) aerodynamic nonlinearities at a given flight condition, and (3) uncertainties of the given aerodynamic data. We treat these problems here. One prominent problem is the change in airframe response with flight condition. An ideal missile autopilot should provide an
identical response from one flight condition to another over the range of flight conditions in the missile's performance envelope. Some of the varying flight conditions include Mach number (or speed), altitude, mass and inertia characteristics, center of gravity location, angles of attack and sideslip, and aerodynamic roll angle. Current design practice uses adaptive autopilot gain scheduling to maintain the desired airframe response over the flight envelope. In this study, we will only investigate the autopilot design for one set of flight conditions (M, h, cg location, wt, and inertia properties). However, the missile uses angle of attack to maneuver. As the aerodynamic plots of Chapter 3 reveal, the aerodynamic force and moment coefficients are nonlinear functions of the aerodynamic angles (\( \alpha \) and \( \phi_a \)) for each given flight condition. This nonlinearity at a given flight condition presents the second aerodynamic problem - not only does the aerodynamics vary with angle of attack (for example), but it does so nonlinearly. In addition, the aerodynamic coefficient data used in the autopilot design may be highly uncertain. Some reasons for this uncertainty may include errors in the wind-tunnel data or aerodynamic prediction codes, slight airframe variations from one missile round to another (such as a slight asymmetry of the tail fins or dorsals) due to manufacturing tolerances, or changes in the missile airframe during flight due to heating or bending of the airframe. The amount of uncertainty associated with the aerodynamic data is difficult to obtain and varies in each separate design case because of the type and quality of the aerodynamic data. As a result, it is common in nonlinear Monte Carlo stability analysis to let the aerodynamic uncertainty vary as much as 100 percent.

A related limitation is the nonlinear coupling of the missile pitch, yaw, and roll channels during flight. The dynamics of the missile include kinematic, inertial, and
aerodynamic cross-coupling. The kinematic couplings are the products of the translational and rotational rates (such as $qw$, or its approximate equivalent $qa$). The products of the angular rates ($qr$, for example) are the inertial couplings. Both the kinematic and inertial cross-couplings are apparent from the equations of motion presented earlier. The aerodynamic cross-coupling (including the induced force and moment coupling) effects discussed in Chapter 3 have a greater impact at higher angles of attack. The above coupling problems, if not taken into account, may produce instability. Consequently, the cross-coupling effects need to be considered in the overall autopilot design.

The instrumentation included in the missile flight control system also induce errors. The effects that the actuator and sensors have on the time and frequency response of the flight control system are significant and need to be examined.

For example, consider the tail fin actuator system. Since the actuator is a nonlinear device, our use of a linear quadratic model introduces modeling errors. Some of the nonlinearities include servo angular position limit, angular rate limit, backlash, and hysteresis. (The position and rate limits were specified in a previous chapter.) In addition, the tail angle may have a bias due to fin misalignment. This angular bias may be as large as $\frac{1}{2}$ of a degree. (This error may be included with those mentioned above with regard to airframe asymmetries and aerodynamic uncertainties.) Also, the actuator dynamics, in particular, the phase characteristics, are highly uncertain near the bending mode frequencies.

The measurement devices are also imperfect and cause errors in the body rate and acceleration terms. As noted before, the sensors contain nonlinearities such as dead space, hysteresis, and limits. Values for the sensor limits were given in Chapter 3, but may be
assumed large enough such that saturation is not a problem. Also, internal component noise adds to the errors.

The rate gyro's also may contain drifts (including acceleration sensitive drift rates), cross-axis coupling due to misalignments, and random noise components. In equation form, the body rate measurements assume the form

$$\overline{\omega_{\text{meas}}} = \overline{\omega_b} + \overline{\omega_{\text{bias}}} + [g - \text{sens}] \overline{\eta_b} + [g^2 - \text{sens}] \overline{\eta_b}^2 + [MA] \overline{\omega_b} + \overline{\omega_{\text{noise}}}$$  \hspace{1cm} (5.1)

where $\omega_b$ is the nominal body rate vector, $\omega_{\text{bias}}$ denotes the drift rate (or bias) vector, $[g - \text{sens}]$ represents the acceleration sensitive drift, $[g^2 - \text{sens}]$ is the squared acceleration drift sensitivity, $[MA]$ denotes the misalignment of the gyro's, and $\omega_{\text{noise}}$ represents the random noise effects on the gyro's. In this study, the cross-channel gyro couplings will be assumed to be negligible. Gyro drift rates are random over the set of missiles and the noise varies randomly during flight. Typical drift values for a rate gyro are listed below:

- g Insensitive drift rate (or bias) \hspace{1cm} 5 deg/hr, max
- g Sensitive drift rate \hspace{1cm} 5 deg/hr/g, max
- $g^2$ Sensitive drift rate \hspace{1cm} 0.1 deg/hr/$g^2$, max
- randomness (or noise) \hspace{1cm} 0.2 deg/hr, RMS, 1σ

Accelerometer measurements also have other errors in addition to those mentioned above. Previously, we pointed out that the accelerometer measured values are slightly different than the accelerations at the missile center of gravity because of accelerometer location offsets. In addition, there are also errors such as accelerometer bias, and misalignment errors. The measured rigid body acceleration can thus be expressed as
\[ \bar{A}_{\text{meas}} = \bar{A}_{\text{acc}} + \bar{A}_{\text{bias}} + [MA]\bar{A}_{\text{acc}} \]  

(5.2)

where \( \bar{A}_{\text{acc}} \) is the nominal acceleration vector at the accelerometer, \( \bar{A}_{\text{bias}} \) denotes the accelerometer bias, and \([MA]\) represents misalignment effects. The accelerometer bias and misalignment errors are assumed to be \( \pm 0.02 \) g's and 0.003 radians, respectively. To account for the missile's elasticity, the flexible body acceleration would be added to the \( \bar{A}_{\text{acc}} \) term in Equation (5.2).

There are also other errors which act upon the missile system that may constrain the autopilot. The guidance system, which generates the acceleration/steering commands, is also contaminated by noise and error sources. Some of these errors are depicted in Figure 5.1. We do not desire that the autopilot respond to the noise (which is generally at the higher frequencies) in the guidance signals. The radome distorts the reflected radar return from the target and causes a radome aberration. This radome error, in particular the radome error slope, can lead to poor missile performance and even instability. There are several approaches of compensating for radome error, one of which is to adjust the autopilot gains to ensure missile stability. Another potential problem is thrust misalignment. However, we have previously assumed that there was no misalignment and that the rocket was at burnout. Thus, we will not examine the effects of thrust misalignment. Another problem is that of atmospheric turbulence and wind gust effects. These effects cause perturbations on the missile's flight path. Typical wind gusts may be on the order of 20 ft/sec, and may vary (uncorrelated) randomly or as time-correlated random processes. These additional error sources should be accounted for in the complete autopilot and missile design.

Two final constraints should also be mentioned. The first is unmodeled high-
frequency dynamics (such as the fourth and higher bending modes) and other modeling errors. These type of uncertainties are significant and are one reason why the loop transfer gain as indicated in a Bode plot is decreased ("rolled off") at high-frequencies. The other constraint is the cost of the flight control system and its components. Usually a cost-quality tradeoff must be made, where the lower cost components may contain larger noise values. However, this constraint is often handled by the systems engineer - the end result being the sensors and devices we have to work with.
Chapter 6. Missile Autopilot Design Models

A previous chapter presented the development of the governing dynamic equations of the missile airframe under various simplifying assumptions. Now, models of the airframe need to be formulated from those equations for application in the synthesis of a missile autopilot. Starting with the most simple model (pitch plane), models of varying degree of detail will be discussed. This discussion will include separate pitch (or yaw, because of missile symmetry) and roll models for both small and large angles of attack, as well as a description of the coupled pitch-yaw-roll, multivariable airframe.

6.1 PITCH (YAW) PLANAR MODELS: SMALL ANGLE OF ATTACK

We begin by formulating pitch-plane models of the airframe dynamics. First, a model of the airframe under a small angle-of-attack assumption will be presented. The state, output, and control vectors are, respectively,

\[
\begin{align*}
x &= \begin{bmatrix} \alpha \\ q \end{bmatrix} , \quad Y &= \begin{bmatrix} A_z \\ q \end{bmatrix} , \quad U = \{ \delta P \}
\end{align*}
\]

The linearized equations of motion from Chapter 2 are

\[
\dot{\alpha} = -\frac{QS}{MV} C_{\alpha \alpha_{\text{ref}}} \alpha + q + \left[ -\frac{QS}{MV} C_{\alpha \delta P_{\text{ref}}} \right] \delta P \tag{6.1}
\]

\[
\dot{q} = \left[ \frac{QSD}{I_y} C_{\alpha \alpha_{\text{qref}}} \right] \alpha + \left[ \frac{QSD}{I_y} \left( \frac{D}{2V} \right) C_{\alpha \alpha_{\text{qref}}} \right] q + \left[ \frac{QSD}{I_y} C_{\alpha \delta P_{\text{ref}}} \right] \delta P \tag{6.2}
\]

The normal acceleration output is given by
Next, we define $\eta_Z$ as the normal acceleration to the missile in G's, and defined positive upward (opposite direction of the missile-z axis). Reasons for this definition are: the normal force is defined positive up; the guidance system commanded accelerations are in G's; the accelerometer measures in units of G's; and because the acceleration is traditionally defined in this manner for missiles. Thus,

$$\eta_Z = -\frac{A_Z}{G}$$

(6.4)

and

$$\eta_Z = \left[ \frac{QS}{W} C_{Nz} \right]_{ref} \alpha + \left[ \frac{QS}{W} C_{Nz} \right]_{ref} \delta P$$

(6.5)

In addition, if the rate gyros measure the body pitch rate in units of degrees per second, then we will assume from here on that all angles will be specified in degrees. Treating the angular quantities in units of degrees, the acceleration equation above becomes

$$\eta_Z = \frac{1}{k} \left( \left[ \frac{QS}{W} C_{Nz} \right]_{ref} \alpha + \left[ \frac{QS}{W} C_{Nz} \right]_{ref} \delta P \right)$$

(6.6)

where $k$ is the conversion constant from radians to degrees ($k = 57.3$ degrees/radian).

Further, we can define

$$Z_a = \left[ \frac{QS}{MV} C_{Na} \right]_{ref}$$

(6.7)

$$Z_\delta = \left[ \frac{QS}{MV} C_{N \delta} \right]_{ref}$$

(6.8)
\[ M_a = \left[ \frac{QSD}{I_y} C_{m_a} \right]_{ref} \]  
\[ M_\delta = \left[ \frac{QSD}{I_y} C_{m_\delta} \right]_{ref} \]  
\[ M_q = \left[ \frac{QSD}{I_y} \left( \frac{D}{2V} \right) C_{m_q} \right]_{ref} \]  

(6.9)  
(6.10)  
(6.11)

where the stability derivatives \((C_{N_a}, C_{N_\delta}, C_{m_a}, C_{m_\delta}, \text{ and } C_{m_q})\) are all nondimensional. The \(\dot{\alpha}, \dot{q}, \) and \(\eta_z\) equations then become

\[ \dot{\alpha} = Z_a \alpha + q + Z_\delta \delta P \]  
\[ \dot{q} = M_a \alpha + M_q q + M_\delta \delta P \]  
\[ \eta_z = -\frac{V}{kG} [Z_a \alpha + Z_\delta \delta P] \]  

(6.12)  
(6.13)  
(6.14)

Note that \(\eta_z\) is the missile's normal acceleration in \(G\)'s at the center of gravity. The missile acceleration at the accelerometer location (in \(G\)'s) can be obtained from Equation (3.10)

\[ \eta_{z_{ac}} = \eta_z + \frac{d\dot{q}}{kG} \]  

(6.15)

where \(d\) is the distance from the cg to the accelerometer (positive in missile x-direction).

Equations (6.12) through (6.15) are illustrated in block diagram form in Figure 6.1.

Since we are considering a linear, time-invariant system, transfer functions of the airframe dynamics can be derived. These transfer functions are linear input-output operators which give the output of the system from a specified input function. The transfer functions that are of interest here those from the control input (the pitch control deflection) to the normal acceleration output, to the acceleration output at the accelerometer, and to the pitch.
Figure 6.1 Missile Pitch Plane Linear Dynamics for Small Angle of Attack
rate output, or \( \eta_Z(s)/\delta P(s) \), \( \eta_{Z_{\infty}}(s)/\delta P(s) \), and \( q(s)/\delta P(s) \), respectively.

Consider the linear, time-invariant system,

\[
\dot{x} = Ax + BU
\]  \hspace{1cm} (6.16)

Assuming that all the initial conditions are zero, then we can write an expression for the transfer function matrix, \( G(s) \), by applying the Laplace transform,

\[
G(s) = (sI - A)^{-1}B
\]  \hspace{1cm} (6.17)

Writing Equations (6.12) and (6.13) in matrix form

\[
\begin{pmatrix}
\dot{a} \\
\dot{q}
\end{pmatrix}
= 
\begin{bmatrix}
Z_a & 1 \\
M_a & M_q
\end{bmatrix}
\begin{pmatrix}
a \\
q
\end{pmatrix}
+ 
\begin{bmatrix}
Z_b \\
M_b
\end{bmatrix}
\delta P
\]  \hspace{1cm} (6.18)

Substituting the \( A \) and \( B \) matrices from the above equation into Equation (6.17), we find that the \( q(s)/\delta P(s) \) transfer function is given by

\[
\frac{q(s)}{\delta P(s)} = \frac{M_o s + (M_a Z_b - M_b Z_a)}{s^2 - (Z_a + M_q)s + (Z_a M_q - M_a)}
\]  \hspace{1cm} (6.19)

This transfer function can also be written in the form

\[
\frac{q(s)}{\delta P(s)} = \frac{K_3(1 + A_{31}s)}{1 + B_{11}s + B_{12}s^2}
\]  \hspace{1cm} (6.20)

where

\[
K_3 = \frac{(M_a Z_b - M_b Z_a)}{(Z_a M_q - M_a)}
\]  \hspace{1cm} (6.21)

\[
A_{31} = \frac{M_b}{(M_a Z_b - M_b Z_a)}
\]  \hspace{1cm} (6.22)
\[ B_{11} = \frac{-(Z_a + M_q)}{(Z_a M_q - M_a)} \]  
\[ B_{12} = \frac{1}{(Z_a M_q - M_a)} \]  

(6.23)  

(6.24)

For the output, the equations can be put in the form

\[ Y = C \alpha + D U \]  

(6.25)

The output vector is assumed to be \( Y = [\eta_z \ q]^T \). From Equation (6.14), the above equation becomes

\[
\begin{bmatrix}
\eta_z \\
q
\end{bmatrix} = \begin{bmatrix}
-(V Z_a/kG) & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
\alpha \\
q
\end{bmatrix} + \begin{bmatrix}
-(V Z_a/kG) \\
0
\end{bmatrix} \delta P
\]

(6.26)

To find the transfer functions of the outputs, the Laplace transform of Equation (6.25) leads to

\[ W(s) = Y(s) U^{-1}(s) = C(sI - A)^{-1} B + D \]  

(6.27)

Substituting the C and D matrices from Equation (6.26) into the above transfer function matrix relation, we find that the acceleration/control deflection transfer function is given by

\[ \frac{\eta_z(s)}{\delta P(s)} = \frac{\left(\frac{V}{kG}\right)\left[-Z_0 s^2 + Z_4 M_q s + (M_a Z_0 - M_q Z_a)\right]}{s^2 - (Z_a + M_q) s + (Z_a M_q - M_a)} \]  

(6.28)

This equation can be rearranged into the form

\[ \frac{\eta_z(s)}{\delta P(s)} = \frac{K_1 (1+A_{11} s + A_{12} s^2)}{1 + B_{11} s + B_{12} s^2} \]  

(6.29)

where

68
\[ K_1 = \left( \frac{V}{kG} \right) \left[ \frac{(M_a Z_a - M_e Z_e)}{(Z_a M_e - M_a)} \right] \]  
(6.30)

\[ A_{11} = \frac{Z_b}{(M_a Z_a - M_e Z_e)} \]  
(6.31)

\[ A_{12} = \frac{-Z_b}{(M_a Z_a - M_e Z_e)} \]  
(6.32)

and \( B_{11} \) and \( B_{12} \) were defined in Equations (6.23) and (6.24), respectively.

### 6.1.1 Rigid Body Model with Accelerometer Offset

Next, we find the effect of the accelerometer offset on the acceleration transfer function. Recall that the acceleration at the accelerometer location is given by Equation (6.15). Again, assuming zero initial conditions, applying the Laplace transform, and dividing through all terms by \( \delta P(s) \),

\[ \frac{\eta_{z_{\text{acc}}}(s)}{\delta P(s)} = \frac{\eta_z(s)}{\delta P(s)} + \left( \frac{d}{kG} \right) \frac{s q(s)}{\delta P(s)} \]  
(6.33)

Substituting in the two transfer functions defined above yields

\[ \frac{\eta_{z_{\text{acc}}}(s)}{\delta P(s)} = \frac{K_1 (1 + A_{11} s + A_{12} s^2)}{1 + B_{11} s + B_{12} s^2} + \left( \frac{d}{kG} \right) \frac{K_3 s (1 + A_{31} s)}{1 + B_{11} s + B_{12} s^2} \]  
(6.34)

Since \( K_1 \) and \( K_3 \) are related by the simple relationship

\[ K_3 = \left( \frac{kG}{V} \right) K_1 \]  
(6.35)

the \( \eta_{z_{\text{acc}}}(s) / \delta P(s) \) transfer function can be rewritten as
\[
\frac{\eta_{Z_{acc}}(s)}{\delta P(s)} = \frac{K_1 [1 + (A_{11} + \frac{d}{V})s + (A_{12} + \frac{d}{V}A_{31})s^2]}{1 + B_{11}s + B_{12}s^2}
\] (6.36)

This relation can be expressed in the form of
\[
\frac{\eta_{Z_{acc}}(s)}{\delta P(s)} = \frac{K_2 (1 + A_{21}s + A_{22}s^2)}{1 + B_{11}s + B_{12}s^2}
\] (6.37)

where
\[
K_2 = K_1
\] (6.38)
\[
A_{21} = A_{11} + \frac{d}{V}
\] (6.39)
\[
A_{22} = A_{12} + \left(\frac{d}{V}\right)A_{31}
\] (6.40)

and \(B_{11}\) and \(B_{12}\) are as specified earlier.

The measured acceleration at the accelerometer, \(\eta_{Z_{acc}}\), could also be considered as an output of the system. The new "modified" output vector, \(Y^*\), would be defined as
\[
Y^* = \begin{bmatrix} \eta_Z \\ \eta_{Z_{acc}} \\ q \end{bmatrix}
\]

This output vector contains the missile acceleration at the cg for performance analysis, and the acceleration at the accelerometer and the pitch rate for feedback to the control law. We obtain the linear expression for the acceleration at the accelerometer by substituting Equations (6.13) and (6.14) into Equation (6.15),
\[ \eta_{Z_{\infty}} = \left[ \frac{dM_a - VZ_a}{kG} \right] \alpha + \left[ \frac{dM_q}{kG} \right] q + \left[ \frac{dM_\delta - VZ_\delta}{kG} \right] \delta P \] \hspace{1cm} (6.41)

Then, the modified output equation of the form

\[ Y^* = C^* x + D^* U \] \hspace{1cm} (6.42)

becomes

\[
\begin{bmatrix}
\eta_z \\
\eta_{Z_{\infty}} \\
q
\end{bmatrix} =
\begin{bmatrix}
(-VZ_\phi/kG) & 0 \\
(dM_a - VZ_a)/kG & dM_q/kG \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\alpha \\
q
\end{bmatrix} +
\begin{bmatrix}
(-VZ_\phi/kG) \\
(dM_\delta - VZ_\delta)/kG \\
0
\end{bmatrix} \delta P \] \hspace{1cm} (6.43)

This equation provides us with an alternative state-space output equation for use in the autopilot design and analysis.

### 6.1.2 RIGID BODY MODEL WITH INSTRUMENTATION DYNAMICS

The above model can be further expanded by considering the instrumentation dynamics. An actuator model is given as a second-order transfer function in Equation (3.1). The pitch-plane airframe block diagram can then be drawn as shown in Figure 6.2. The measurement sensor models can also be included. Equations (3.2) and (3.3) give the rate gyro and accelerometer models, respectively. Figure 6.3 illustrates the airframe with instrumentation dynamics included.

### 6.1.3 FLEXIBLE BODY MODEL

Next we add the body-bending dynamics to the airframe model. Figure 2.6 illustrates
Figure 6.3 Missile Pitch Plane Model with Instrumentation Dynamics
how the flexible body dynamics are added to the rigid body model. The flexible-body
transfer functions are given in Equations (2.108) and (2.109). Including these effects in the
airframe model, the pitch-plane model becomes as shown in Figure 6.4.

6.1.4 MODEL WITH CONSTRAINTS

We can also include the constraints and the other problems that are discussed in
Chapter 5. Some of the more important effects to be included in an autopilot design are
actuator angular position and rate limits, and sensor errors and noises. The pitch plane
model can then be modified, as shown in Figure 6.5. Note that a structural G-limit is also
included to remind us that the this limit should not be exceeded.

6.2 PITCH (YAW) PLANAR MODELS: LARGE ANGLE OF ATTACK

The second model that we formulate is a linearized airframe high angle-of-attack
pitch dynamics model. The decoupled angle-of-attack and pitch rate dynamical equations
are given by (assuming that $C_{N_\alpha}$ and $C_{A_\alpha}$ are negligible),

$$\dot{\alpha} = Z_\alpha^* \alpha + q + Z_\delta^* \delta P$$  \hspace{1cm} (6.44)

where

$$Z_\alpha^* = \left\{ \cos^2 \alpha \left[ -\frac{QS}{Mu} C_{N_\alpha} + \frac{Qs}{Mu} (C_{A_\alpha} \alpha + C_{A_{\alpha r}} \delta P) \right] \right. + \sin^2 \alpha \left[ \frac{-Qs}{Mu} (C_{A_\alpha} \alpha + C_{A_{\alpha r}} \delta P) \right]
\left. + \sin \alpha \cos \alpha \left[ \frac{2QS}{Mu} (C_{N_\alpha} \alpha + C_{N_{\alpha r}} \delta P) + \frac{Qs}{Mu} C_{A_\alpha} \right] \right\}_{ref}$$  \hspace{1cm} (6.45)
Figure 6.4 Missile Pitch Plane Model with Instrumentation & Elastic-Body Dynamics
Figure 6.5 Missile Pitch Plane Model with Constraints and Limits
\[ Z_\delta^* = \left( \cos^2 \alpha \left[ -\frac{Qs}{Mu} C_{M_\alpha} \right] - \sin \alpha \cos \alpha \left[ -\frac{Qs}{Mu} C_{A_{\alpha \delta}} \right] \right)_{ref} \]  

(6.46)

and

\[ \dot{q} = M_\alpha \dot{\alpha} + M_q q + M_\delta \delta P \]

(6.47)

where \( M_\alpha, M_q, \) and \( M_\delta \) are as defined earlier. The acceleration output equation is also the same as before as given by Equation (6.14). The acceleration at the accelerometer is given by Equation (6.15). A block diagram of this high angle-of-attack pitch plane model is shown in Figure 6.6.

Transfer functions of the pitch rate and accelerations can be found similarly as for the small angle-of-attack dynamics. Choosing the state vector to be \( x = [\alpha \quad q]^T \), the high angle-of-attack pitch dynamics are given by the following matrix equation

\[
\begin{bmatrix}
\dot{\alpha} \\
\dot{q}
\end{bmatrix} =
\begin{bmatrix}
Z_\alpha^* & 1 \\
M_\alpha & M_q
\end{bmatrix}
\begin{bmatrix}
\alpha \\
q
\end{bmatrix}
+ \begin{bmatrix}
Z_\delta^* \\
M_\delta
\end{bmatrix} \delta P
\]

(6.48)

Substituting the A and B matrices into Equation (6.17) and solving for the pitch rate transfer function,

\[
\frac{q(s)}{\delta P(s)} = \frac{M_\delta s + (M_\alpha Z_\alpha^* - M_\delta Z_\delta^*)}{s^2 - (Z_\alpha^* + M_q s) + (Z_\alpha^* M_q - M_\alpha)}
\]

(6.49)

This transfer function can be put in the form

\[
\frac{q(s)}{\delta P(s)} = \frac{K_3 s}{1 + A_{31} s}
\]

(6.50)

where
Figure 6.6 Missile Pitch Plane Dynamics for High Angle of Attack
\[ K_3^* = \frac{(M_a Z_0^* - M_\delta Z_a^*)}{(Z_a^* M_q - M_\delta)} \]  
(6.51)

\[ A_{31}^* = \frac{M_\delta}{(M_a Z_0^* - M_\delta Z_a^*)} \]  
(6.52)

\[ B_{11}^* = \frac{-(Z_a^* + M_\delta)}{(Z_a^* M_q - M_\delta)} \]  
(6.53)

\[ B_{12}^* = \frac{1}{(Z_a^* M_q - M_\delta)} \]  
(6.54)

Next, we find the missile normal acceleration transfer function. Selecting the outputs to be the missile normal acceleration and the body pitch rate, the output matrix equation is

\[
\begin{bmatrix}
\eta_z \\
\dot{q}
\end{bmatrix} = \begin{bmatrix}
-(VZ_a^*/kG) & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
\alpha \\
\dot{\alpha}
\end{bmatrix} + \begin{bmatrix}
-(VZ_\delta^*/kG) \\
0
\end{bmatrix} \delta P
\]  
(6.55)

Note that this equation is the same as that given previously for the small angle-of-attack case, Equation (6.26). The output transfer function matrix is given by Equation (6.27).

Substituting the C and D matrices from Equation (6.55) into that equation, the \( \eta_z(s) / \delta P(s) \) transfer function is determined to be

\[
\eta_z(s) = \left( \frac{V}{kG} \right) \frac{Z_a s^2 + (Z_a Z_0^* - Z_a Z_a^* - Z_0^* M_a) s + (M_a Z_a^* - M_\delta Z_a^* - M_\delta Z_0^* Z_a^*)}{s^2 - (Z_a^* M_q + M_\delta)^2 + (Z_a^* M_q - M_\delta)}
\]  
(6.56)

This equation is similar to Equation (6.28), but contains extra terms in the numerator since the high angle of attack aerodynamic terms \( Z_a^* \) and \( Z_\delta^* \) do not lead to the simplification of the numerator coefficients. Equation (6.56) is also expressed as
\[
\frac{\eta_z(s)}{\delta P(s)} = \frac{K_1^* (1 + A_{11}^* s + A_{12}^* s^2)}{1 + B_{11}^* s + B_{12}^* s^2}
\]

(6.57)

where

\[
K_1^* = \left( \frac{V}{kG} \right) \left[ \frac{M_a Z_0 - M_b Z_u - M_q (Z_u Z_a^* - Z_a Z_b^*)}{(Z_a^* M_q - M_a)} \right]
\]

(6.58)

\[
A_{11}^* = \frac{Z_0 M_q + Z_0 Z_a^* - Z_a Z_b^*}{(M_a Z_0 - M_b Z_u - M_q (Z_u Z_a^* - Z_a Z_b^*))}
\]

(6.59)

\[
A_{12}^* = \frac{-Z_b}{(M_a Z_0 - M_b Z_u - M_q (Z_u Z_a^* - Z_a Z_b^*))}
\]

(6.60)

and \(B_{11}^*\) and \(B_{12}^*\) are given by Equations (6.53) and (6.54), respectively.

### 6.2.1 HIGH \(\alpha\) PITCH PLANE MODEL: ACCELEROMETER OFFSET

The transfer function for the acceleration at the accelerometer is found using Equation (6.15). Working through the algebra as before, we find

\[
\frac{\eta_{z_{acc}}(s)}{\delta P(s)} = \frac{K_2^* (1 + A_{21}^* s + A_{22}^* s^2)}{1 + B_{11}^* s + B_{12}^* s^2}
\]

(6.61)

where

\[
K_2^* = K_1^*
\]

(6.62)

\[
A_{21}^* = A_{11}^* + \frac{d}{V}
\]

(6.63)

\[
A_{22}^* = A_{12}^* + \left( \frac{d}{V} \right) A_{31}^*
\]

(6.64)

and \(B_{11}^*\) and \(B_{12}^*\) are as defined for the previous transfer functions. Examining the
corresponding state-space $Y'$ output equation in this case, we find that it is equivalent to that given by Equation (6.43).

6.2.2 HIGH $\alpha$ PITCH PLANE MODEL: INSTRUMENTS & CONSTRAINTS

This high angle-of-attack pitch plane model can also be adapted to include such effects as actuator, sensor, and flexible-body dynamics and various errors and noises as considered in the small angle of attack case. The final model can then be formulated similar to that for the small angle-of-attack case.

6.3 ROLL CHANNEL MODEL

The decoupled roll dynamics are given by Equation (2.96). If we assume that the roll system describes only the roll motion about the missile centerline, then the roll rate, $p$, is equivalent to $\dot{\phi}$. Equation (2.96) can then be expressed as

$$\ddot{\phi} = \left[ \frac{QSD}{I_x} \left( \frac{D}{2V} \right) C_{i_r} \right] \dot{\phi} + \left[ \frac{QSD}{I_x} C_{i_{u_r}} \right] \delta R \tag{6.65}$$

Defining

$$L_\phi = \left[ \frac{QSD}{I_x} \left( \frac{D}{2V} \right) C_{i_r} \right] \tag{6.66}$$

and

$$L_\theta = \left[ \frac{QSD}{I_x} C_{i_{u_r}} \right] \tag{6.67}$$

then the roll dynamics can be written as
\[
\dot{\phi} = L_\theta \dot{\phi} + L_\delta \delta R \tag{6.68}
\]

Taking the Laplace transform of the above equation, which is assumed to be linear and time-invariant with zero initial conditions

\[
s\phi - L_\theta \dot{\phi} = L_\delta \delta R \tag{6.69}
\]

If we define

\[
\omega_R = -L_\theta \dot{\phi} \tag{6.70}
\]

then the roll rate equation becomes

\[
s\phi + \omega_R \dot{\phi} = (s + \omega_R) \dot{\phi} = L_\delta \delta R \tag{6.71}
\]

The roll rate transfer function is then

\[
\frac{\phi(s)}{\delta R(s)} = \frac{L_\delta}{s + \omega_R} \tag{6.72}
\]

Similarly, the roll position transfer function is

\[
\frac{\phi(s)}{\delta R(s)} = \frac{L_\delta}{s(s + \omega_R)} \tag{6.73}
\]

A block diagram of the roll dynamics described above is illustrated in Figure 6.7. As for the pitch-plane models, the roll model can be supplemented with linear models of the actuator and rate gyro. Figure 6.8 shows this higher-order roll model, and also includes a couple of the primary error sources, such as actuator limits and gyro drifts. We remark that the roll-torsional flexible dynamics is absent from the model, but should be considered in the design of the roll autopilot.
Figure 6.7  Missile Roll Dynamics
6.4 MULTIVARIABLE, COUPLED PITCH-YAW-ROLL MODEL

Now we consider the coupled pitch-yaw-roll airframe dynamics to form a design model for the multivariable autopilot design. This design model is formulated only for the coupled, high angle-of-attack case. The linearized dynamics of the form of Equation (6.16) are presented in Equations (2.32) - (2.71), where the state and control vectors are

\[ x = \begin{pmatrix} \alpha \\ \beta \\ p \\ q \\ r \end{pmatrix}, \quad U = \begin{pmatrix} \delta P \\ \delta Y \end{pmatrix}, \]

The output takes the form given in Equation (6.25), where the output vector, \( Y \), is

\[ Y = \begin{pmatrix} \eta_Y \\ \eta_Z \\ p \\ q \\ r \end{pmatrix} \]

and the \( C \) and \( D \) matrices are specified by Equations (2.75) - (2.93).

To simplify these equations further, we assume that various stability derivatives are small and can be neglected. These terms are those derivatives with respect to the angular rotational rates, and include the following: \( C_{N\alpha}, C_{N\beta}, C_{Np}, C_{Ny}, C_{Yp}, C_{Yr}, C_{A\alpha}, C_{A\beta}, C_{A\gamma}, C_{I\alpha}, C_{I\beta}, C_{mp}, C_{m\gamma}, C_{n\alpha}, \text{ and } C_{n\beta} \). Neglecting these derivatives, the dynamical equations of motion (the \( A \) and \( B \) matrices, in particular) then become
\[
\begin{align*}
\alpha_{11} &= \left\{ \cos^2 \alpha \left[ -\frac{QS}{Mu} C_{N_\alpha} - r \tan \beta \right] + \sin^2 \alpha [r \tan \beta] \\
&+ \sin \alpha \cos \alpha \left[ 2p \tan \beta + \frac{QS}{Mu} C_{A_\beta} \right] \right\}_{\text{ref}} \\
&- \frac{QS}{Mu} (\sin^2 \alpha - \cos^2 \alpha) \left[ C_{A_\alpha} \alpha + C_{A_\beta} \beta + C_{A_\alpha} \delta P + C_{A_{\alpha \pi}} + C_{A_{\alpha \kappa}} \delta R \right] \\
&+ \frac{QS}{Mu} (\sin \alpha \cos \alpha) \left[ C_{N_\alpha} \alpha + C_{N_\beta} \beta + C_{N_{\alpha \pi}} \delta P + C_{N_{\alpha \kappa}} \delta Y + C_{N_{\alpha \kappa}} \delta R \right] \\
\alpha_{12} &= \left\{ \cos^2 \alpha \left[ -\frac{QS}{Mu} C_{N_\beta} - \frac{p}{\cos^2 \beta} \right] - \sin \alpha \cos \alpha \left[ \frac{r}{\cos^2 \beta} - \frac{QS}{Mu} C_{A_\beta} \right] \right\}_{\text{ref}} \\
\alpha_{13} &= \left\{ -\cos^2 \alpha \tan \beta \right\}_{\text{ref}} \\
\alpha_{14} &= 1 \\
\alpha_{15} &= \left\{ -\sin \alpha \cos \alpha \tan \beta \right\}_{\text{ref}} \\
\alpha_{21} &= \left\{ \cos^2 \beta \left[ \frac{QS}{Mu} C_{A_\beta} + \frac{p}{\cos^2 \alpha} \right] + \sin \beta \cos \beta \left[ \frac{q}{\cos^2 \alpha} + \frac{QS}{Mu} C_{A_\beta} \right] \right\}_{\text{ref}} \\
\alpha_{22} &= \left\{ \cos^2 \beta \left[ \frac{QS}{Mu} C_{A_\beta} + q \tan \alpha \right] - \sin^2 \beta [q \tan \alpha] \\
&- \sin \beta \cos \beta \left[ 2p \tan \alpha - \frac{QS}{Mu} C_{A_\beta} \right] \right\}_{\text{ref}} \\
&- \frac{QS}{Mu} (\sin^2 \beta - \cos^2 \beta) \left[ C_{A_\alpha} \alpha + C_{A_\beta} \beta + C_{A_{\alpha \pi}} + C_{A_{\alpha \kappa}} \delta R \right] \\
&- \frac{2QS}{Mu} (\sin \beta \cos \beta) \left[ C_{N_\alpha} \alpha + C_{N_\beta} \beta + C_{N_{\alpha \pi}} \delta P + C_{N_{\alpha \kappa}} \delta Y + C_{N_{\alpha \kappa}} \delta R \right]} \\
\end{align*}
\]
\[ a_{23} = \{ \cos^2 \beta \tan \alpha \}_{ref} \]  

(6.81)

\[ a_{24} = \{ \sin \beta \cos \beta \tan \varepsilon \}_{ref} \]  

(6.82)

\[ a_{25} = -1 \]  

(6.83)

\[ a_{31} = \left\{ \frac{QSD}{I_x} C_{i_x} \right\}_{ref} \]  

(6.84)

\[ a_{32} = \left\{ \frac{QSD}{I_x} C_{i_y} \right\}_{ref} \]  

(6.85)

\[ a_{33} = \left\{ \frac{QSD}{I_x} \left( \frac{D}{2V} \right) C_{i_y} \right\}_{ref} \]  

(6.86)

\[ a_{34} = a_{35} = 0 \]  

(6.87)

\[ a_{41} = \left\{ \frac{QSD}{I_y} C_{i_x} \right\}_{ref} \]  

(6.88)

\[ a_{42} = \left\{ \frac{QSD}{I_y} C_{i_y} \right\}_{ref} \]  

(6.89)

\[ a_{43} = \left\{ - \frac{I_x - I_t}{I_y} \right\}_{ref} \]  

(6.90)

\[ a_{44} = \left\{ \frac{QSD}{I_y} \left( \frac{D}{2V} \right) C_{i_y} \right\}_{ref} \]  

(6.91)
\[ a_{45} = \left\{ p \left( \frac{I_z - I_y}{I_{ref}} \right) \right\}_{ref} \]  
\[ a_{51} = \left\{ \frac{QSD}{I_z} C_{n_a} \right\}_{ref} \]  
\[ a_{52} = \left\{ \frac{QSD}{I_z} C_{n_b} \right\}_{ref} \]  
\[ a_{53} = \left\{ q \left( \frac{I_z - I_y}{I_z} \right) \right\}_{ref} \]  
\[ a_{54} = \left\{ p \left( \frac{I_z - I_y}{I_z} \right) \right\}_{ref} \]  
\[ a_{55} = \left\{ \frac{QSD}{I_z} \left( \frac{D}{2V} \right) C_{n_f} \right\}_{ref} \]  

and the B matrix elements are the same as given in Equations (2.57) - (2.71).

Similarly, the output C matrix elements become

\[ C_{11} = \left\{ \frac{1}{kG} \right\} \left\{ \frac{Q}{M} C_{Y_a} \right\}_{ref} \]  
\[ c_{12} = \left\{ \frac{1}{kG} \right\} \left\{ \frac{Q}{M} C_{Y_b} \right\}_{ref} \]  
\[ c_{13} = c_{14} = c_{15} = 0 \]  
\[ c_{21} = \left\{ \frac{1}{kG} \right\} \left\{ \frac{Q}{M} C_{n_a} \right\}_{ref} \]
\begin{align*}
c_{22} &= \left( \frac{1}{kG} \right) \left[ \frac{QS}{M} C_{N_t} \right]_{ref} \tag{6.102} \\
c_{23} &= c_{24} = c_{25} = 0 \tag{6.103} \\
c_{31} &= c_{32} = c_{34} = c_{35} = 0 \tag{6.104} \\
c_{41} &= c_{42} = c_{43} = c_{45} = 0 \tag{6.105} \\
c_{51} &= c_{52} = c_{53} = c_{54} = 0 \tag{6.106} \\
c_{33} &= c_{44} = c_{55} = 1 \tag{6.107}
\end{align*}

The D matrix entries are specified in Equations (2.87) - (2.93) and become

\begin{align*}
d_{11} &= \left( \frac{1}{kG} \right) \left[ \frac{QS}{M} C_{y_{sr}} \right] \tag{6.108} \\
d_{12} &= \left( \frac{1}{kG} \right) \left[ \frac{QS}{M} C_{y_{sr}} \right] \tag{6.109} \\
d_{13} &= \left( \frac{1}{kG} \right) \left[ \frac{QS}{M} C_{y_{ta}} \right] \tag{6.110} \\
d_{21} &= \left( \frac{1}{kG} \right) \left[ \frac{QS}{M} C_{n_{sr}} \right] \tag{6.111} \\
d_{22} &= \left( \frac{1}{kG} \right) \left[ \frac{QS}{M} C_{n_{sr}} \right] \tag{6.112} \\
d_{23} &= \left( \frac{1}{kG} \right) \left[ \frac{QS}{M} C_{n_{ta}} \right] \tag{6.113} \\
d_{31} &= d_{32} = d_{33} = d_{41} = d_{42} = d_{43} = d_{51} = d_{52} = d_{53} = 0 \tag{6.114}
\end{align*}

We have thus described the linearized, coupled pitch-yaw-roll airframe dynamics for high angle-of-attack analysis.
6.4.1 COUPLED LINEAR DESIGN MODEL WITH INSTRUMENTATION

In addition, as for the planar models discussed earlier, several other effects may be included in the coupled autopilot design model. Actuator models for the three channels may be added. As before, the second-order lag transfer function model can be implemented to characterize the linear actuator dynamics. Also, quadratic lag models of the rate gyros and accelerometers may contribute to the design model.

6.4.2 COUPLED LINEAR DESIGN MODEL: ACCELEROMETER OFFSET

To account for the accelerometer offsets from the center of gravity, the following relations were obtained in Chapter 3:

\[
A_{x_w} = A_{x_a} - d(q^2 + r^2) \quad (6.115)
\]

\[
A_{y_w} = A_{y_a} + \dot{d}r + dpq \quad (6.116)
\]

\[
A_{z_w} = A_{z_a} - dq + dpr \quad (6.117)
\]

Unfortunately, these equations are nonlinear. We now linearize Equations (6.115) - (6.117) to obtain a linear model of the accelerometer offset effects. To linearize the acceleration equations above, we apply the Taylor series expansion method discussed in Chapter 2. Instead of solving for the state rate of change \((dx/dt)\), we solve for the \(U\) term, (which we will substitute with the output \(Y\) in this case),

\[
\Delta Y = - \left( \frac{\partial Y}{\partial Y_{ref}} \right)_{ref} \left( \frac{\partial Y}{\partial \Delta x} \right)_{ref} \Delta x - \left( \frac{\partial Y}{\partial Y_{ref}} \right)_{ref} \left( \frac{\partial Y}{\partial \Delta x} \right)_{ref} \Delta x \quad (6.118)
\]

Recall that this equation describes a model of the system at some perturbed condition away
from some specified reference condition. Letting \( x = [p \ q \ r]^T \) and \( Y = [(A_{Mr} - A_{r})_x (A_{Mr} - A_{r})_y (A_{Mr} - A_{r})_z]^T \) and applying Equation (6.118) to Equations (6.115) - (6.117), we obtain the linearized equations for the accelerometer offset effect,

\[
\eta_{x_{sec}} = \left( \frac{\Delta A_{x_{sec}}}{G} \right) = \left( \frac{1}{G} \right) \left[ \Delta A_{x_{sec}} - 2d(q_{rf} \Delta q + r_{rf} \Delta r) \right] \tag{6.119}
\]

\[
\eta_{y_{sec}} = \left( \frac{\Delta A_{y_{sec}}}{G} \right) = \left( \frac{1}{G} \right) \left[ \Delta A_{y_{sec}} + d \Delta \tau + d(q_{rf} \Delta p + p_{rf} \Delta q) \right] \tag{6.120}
\]

\[
\eta_{z_{sec}} = \left( \frac{\Delta A_{z_{sec}}}{G} \right) = \left( -\frac{1}{G} \right) \left[ \Delta A_{z_{sec}} - d \Delta \dot{q} + d(r_{rf} \Delta p + p_{rf} \Delta r) \right] \tag{6.121}
\]

### 6.4.3 COUPLED LINEAR MODEL: OUTPUT

In addition, we may desire to find the multivariable \( Y^* \) output equation. This modified output vector now contains the \( Y \)- and \( Z \)-plane accelerations at their respective accelerometers, \( Y^* = [\eta_Y \ \eta_Z \ \eta_{y_{sec}} \ \eta_{z_{sec}} \ p \ q \ r]^T \). Rewriting Equations (6.120) and (6.121)

\[
\eta_{y_{sec}} = \eta_Y + \frac{d}{G}(\dot{\tau} + q_{rf} \hat{p} + p_{rf} \hat{q}) \tag{6.122}
\]

\[
\eta_{z_{sec}} = \eta_Z + \frac{d}{G}(\dot{\hat{q}} - r_{rf} \hat{p} - p_{rf} \hat{r}) \tag{6.123}
\]

Substituting in the relations for \( \eta_Y, \eta_Z, \hat{q}, \dot{\tau} \) defined previously with the A, B, C, and D matrix elements, then the equations for the acceleration at the accelerometers are
\[ \eta_{Y_{ac}} = \left[ c_{11} + \frac{da_{51}}{G} \right] \alpha + \left[ c_{12} + \frac{da_{52}}{G} \right] \beta + \left[ \frac{d}{G}(a_{13} + q_{ref}) \right] p + \left[ \frac{d}{G}(a_{34} + p_{ref}) \right] q \]
\[ + \left[ \frac{da_{53}}{G} \right] r + \left[ d_{11} + \frac{db_{21}}{G} \right] \delta P + \left[ d_{12} + \frac{db_{31}}{G} \right] \delta Y + \left[ d_{13} + \frac{db_{32}}{G} \right] \delta R \] (6.124)

\[ \eta_{Z_{ac}} = \left[ c_{21} + \frac{da_{41}}{G} \right] \alpha + \left[ c_{22} + \frac{da_{42}}{G} \right] \beta \]
\[ + \left[ \frac{d}{G}(a_{43} - r_{ref}) \right] p + \left[ \frac{d}{G}a_{44} \right] q + \left[ \frac{d}{G}(a_{45} - p_{ref}) \right] r \]
\[ + \left[ d_{21} + \frac{db_{41}}{G} \right] \delta P + \left[ d_{22} + \frac{db_{42}}{G} \right] \delta Y + \left[ d_{23} + \frac{db_{43}}{G} \right] \delta R \] (6.125)

In the matrix form of Equation (6.42), the modified output equation is

\[
\begin{bmatrix}
\eta_Y \\
\eta_Z \\
\eta_{Y_{ac}} \\
\eta_{Z_{ac}} \\
p \\
q \\
r
\end{bmatrix} =
\begin{bmatrix}
c_{11} & c_{12} & 0 & 0 & 0 \\
c_{21} & c_{22} & 0 & 0 & 0 \\
c_{31} & c_{32} & c_{33} & c_{34} & c_{35} \\
c_{41} & c_{42} & c_{43} & c_{44} & c_{45} \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta \\
p \\
q \\
r
\end{bmatrix}
+
\begin{bmatrix}
d_{11} & d_{12} & d_{13} \\
d_{21} & d_{22} & d_{23} \\
d_{31} & d_{32} & d_{33} \\
d_{41} & d_{42} & d_{43} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\delta P \\
\delta Y \\
\delta R
\end{bmatrix}
\] (6.126)

where the new modified matrix elements are

\[ c_{31}^* = c_{11} + \frac{da_{51}}{G} \]
\[ c_{32}^* = c_{12} + \frac{da_{52}}{G} \]
\[ c_{33}^* = d(a_{53} + q_{ref})/G \]
\[ c_{34}^* = d(a_{54} + p_{ref})/G \]
\[ e_{35}^* = d a_{35} / G \]
\[ c_{41}^* = c_{21} + d a_{41} / G \]
\[ c_{42}^* = c_{22} + d a_{42} / G \]
\[ c_{43}^* = d (a_{43} - r_{ref}) / G \]
\[ c_{44}^* = d a_{44} / G \]
\[ c_{45}^* = d (a_{45} - p_{ref}) / G \]
\[ d_{31}^* = d_{11} + d b_{31} / G \]
\[ d_{32}^* = d_{12} + d b_{32} / G \]
\[ d_{33}^* = d_{13} + d b_{33} / G \]
\[ d_{41}^* = d_{21} + d b_{41} / G \]
\[ d_{42}^* = d_{22} + d b_{42} / G \]
\[ d_{43}^* = d_{23} + d b_{43} / G \]

In the multivariable autopilot design, we may then use the \( Y^* \) output equation as defined above since it contains the acceleration at the accelerometer, which would be utilized in the feedback control law.

Finally, errors and limitations may then be added to the multivariable design model. We may want to account for such things as actuator angular position and rate limits, rate gyro drifts and noise, accelerometer biases, and aeroelastic dynamics.
Chapter 7. Classical Control Autopilot Design

The conventional approach in designing missile autopilots has been to use classical control theory. Several control loop topologies have been developed for various practical applications. We select one of these autopilot topologies for our design.

We take the following approach. First, the pitch (or yaw, i.e., symmetry) autopilot is designed for rigid body dynamics only. Next, the effects of the actuator and the sensor dynamics are then included, and the performance and stability requirements are checked. Third, flexible-body effects are accounted for and appropriate feedback control compensation is added as necessary. As before, the effects of the actuator and the sensors are included in the analysis. The details of the various stages of the conventional autopilot design approach are now discussed.

7.1 PITCH (YAW) PLANAR AUTOPilot DESIGN

The design approach for the rigid-body, decoupled pitch autopilot is based on the three-loop accelerometer and rate gyro feedback flight control system\(^{28}\). A diagram of this flight control system is illustrated in Figure 7.1. The flight control system shown in this figure reveals that the acceleration, body rate, and the integral of body rate are utilized for feedback in the control law. This method is well documented in References 9 - 13, and 28. We now present a synopsis of this design methodology.

Since the definition of the control law is already set (as given in Figure 7.1), the major task of the autopilot design is to determine the autopilot gains \(K_A\), \(K_I\), \(K_R\) and \(K_o\).
Figure 7.1 Pitch (Yaw) Plane Three-Loop Autopilot
These gains are set so that the design requirements specified in a previous chapter are satisfied. In this section, we discuss the gain selection procedure.

### 7.1.1 GAIN SELECTION PROCEDURE

First, we assume that we are controlling only the rigid-body airframe (thereby neglecting initially flexible modes and structural filter compensation), and that the actuator, rate gyro and accelerometer have faster time constants than the airframe. In this first stage of the design, then, $G_{ac}(s)$, $G_{ac}(s)$ and $G_{ang}(s)$ are all equal to one. Also, we will assume perfect fin mixing, and that the small angle-of-attack dynamical model serves as a good approximation to the vehicle dynamics.

The pitch rate and acceleration transfer functions from Chapter 6 are

$$G_1(s) = \frac{\eta_2(s)}{\delta P(s)} = \frac{\eta_1(s)}{\delta(s)} = \frac{K_1(1 + A_{11}s + A_{12}s^2)}{1 + B_{11}s + B_{12}s^2}$$  \hfill (7.1)

$$G_2(s) = \frac{\eta_{ac}(s)}{\delta P(s)} = \frac{\eta_A(s)}{\delta(s)} = \frac{K_2(1 + A_{21}s + A_{22}s^2)}{1 + B_{11}s + B_{12}s^2}$$  \hfill (7.2)

$$G_3(s) = \frac{q(s)}{\delta P(s)} = \frac{\hat{\delta}(s)}{\delta(s)} = \frac{K_3(1 + A_{31}s)}{1 + B_{11}s + B_{12}s^2}$$  \hfill (7.3)

Substituting these transfer functions into the autopilot system diagram of Figure 7.1 and solving for the normal acceleration output, the input-output transfer function, $\eta(s)/\eta_c(s)$, is

$$\frac{\eta(s)}{\eta_c(s)} = \frac{K_0K_4K_1}{K_3 + K_4K_2} \left[ \frac{1 + A_{11}s + A_{12}s^2}{1 + D_1s + D_2s^2 + D_3s^3} \right]$$  \hfill (7.4)

where
\[
D_1 = \frac{1 - K_R K_A K_I A_{31} - K_R K_I K_A A_{21}}{-K_R K_A K_I - K_R K_I K_A K_2}
\]

(7.5)

\[
D_2 = \frac{B_{11} - K_R K_A A_{31} - K_R K_I K_A A_{22}}{-K_R K_A K_I - K_R K_I K_A K_2}
\]

(7.6)

\[
D_3 = \frac{B_{12}}{-K_R K_A K_I - K_R K_I K_A K_2}
\]

(7.7)

The cubic denominator \(D(s)\) of the above transfer function is typical of tail-controlled missiles. This denominator can be written as a desired characteristic equation with the form

\[
D'(s) = (1 + \tau s) \left(1 + \frac{2\zeta}{\omega} s + \frac{s^2}{\omega^2}\right)
\]

(7.8)

where the closed-loop real pole, \(1/\tau\), the closed-loop frequency of the quadratic pair, \(\omega\), and the closed-loop damping of the quadratic pair, \(\zeta\), are specified to place the poles at some desired location. Actually, we initially specify \(\tau, \zeta,\) and \(\omega_{cr}\) (the gain crossover frequency of the open-loop transfer function, opened at the actuator, \(\delta\)). The quadratic pair frequency, \(\omega\), is then obtained from those parameters. Further relations then lead to the autopilot gains \(K_A, K_I,\) and \(K_R\). These equations will now be developed.

An expression for the crossover frequency, \(\omega_{cr}\), can be found by examining the open-loop characteristics. Figure 7.2 shows the flight control system with the loop opened at the actuator input. The open-loop transfer function, \(HG(s)\), can be expressed (for \(\eta_c = 0\)) as

\[
HG(s) = -K_{OL} \left[\frac{1 + AF_1 s + AF_2 s^2}{s(1 + B_{11}s + B_{12}s^2)}\right]
\]

(7.9)
Figure 7.2 Open-Loop Diagram of Flight Control System
where

\[ K_{OL} = K_R (K_A K_2 + K_1 K_3) \]  
\[ AF_1 = \frac{K_1 K_3 A_{31} + K_1 K_A K_2 A_{21}}{K_A K_2 + K_1 K_3} \]  
\[ AF_2 = \frac{K_A A_{31} + K_1 K_A K_2 A_{22}}{K_A K_2 + K_1 K_3} \]

To get \( \omega_{CR} \), we find the magnitude of the open-loop transfer function. At the crossover frequency, the magnitude of the open-loop transfer function is unity, and assuming that the crossover frequency occurs at high frequency (\( \omega = \omega_{CR} \rightarrow \infty \)), then an approximate relation for the crossover frequency is given by

\[ \omega_{CR} = -\frac{K_{OL} AF_2}{B_{12}} \]  

Substituting the equations for \( K_{OL} \), \( AF_2 \) and \( B_{12} \) into the above equation and assuming that the aerodynamic term \( A_{22} \) is small enough to be neglected, then,

\[ \omega_{CR} = K_R M_6 \]

The crossover frequency is thus a function of the \( K_R \) autopilot gain.

Next, we return back to the closed-loop to find the remaining relations to calculate the gains. In terms of the open-loop parameters, the closed-loop transfer function is

\[ \frac{\eta_c(s)}{\eta_c(s)} = \frac{K_R K_A K_2 K_3 K_1}{K_{OL}} \left[ \frac{1 + A_{11}s + A_{12}s^2}{1 + \left(AF_1 - \frac{1}{K_{OL}}\right)s + \left(AF_2 - \frac{B_{11}}{K_{OL}}\right)s^2 - \left(\frac{B_{12}}{K_{OL}}\right)s^3} \right] \]

Expanding the desired characteristic equation (7.8) and relating like terms with the denominator of Equation (7.15),

99
\[ \frac{2\xi}{\omega} + \tau = AF_1 - \frac{1}{K_{OL}} \quad (7.16) \]
\[ \frac{1}{\omega^2} + \frac{2\xi\tau}{\omega} = AF_2 - \frac{B_{11}}{K_{OL}} \quad (7.17) \]
\[ \frac{\tau}{\omega^2} = -\frac{B_{12}}{K_{OL}} \quad (7.18) \]

Solving Equation (7.13) for \( AF_2 \) and Equation (7.18) for \( K_{OL} \) and substituting the results into Equation (7.17), the frequency of the closed-loop quadratic pair is determined,
\[ \omega = \left[ \omega_{cr} + \frac{B_{11}}{B_{12}} \right] - 1 \quad (7.19) \]

If we represent the numerator of the open-loop transfer function, \( HG(s) \), as an open-loop quadratic pair,
\[ \text{NUM}(HG(s)) = 1 + \frac{2\xi_{OL}}{\omega_{OL}} s + \frac{s^2}{\omega^2_{OL}} \quad (7.20) \]

then by equating coefficients, we find
\[ AF_1 = \frac{2\xi_{OL}}{\omega_{OL}} \quad (7.21) \]
\[ AF_2 = \frac{1}{\omega^2_{OL}} \quad (7.22) \]

Solving Equation (7.22) for the natural frequency of the net open-loop feedback zeros, \( \omega_{OL} \), then utilizing Equations (7.13) and (7.18), \( \omega_{OL} \) can be expressed as
\[ \omega_{OL} = \frac{\omega}{\sqrt{\omega_{CR} \tau}} \]  

(7.23)

Substituting Equation (7.21) into Equation (7.16) and using Equations (7.13) and (7.22), we can solve for \( \zeta_{OL} \),

\[ \zeta_{OL} = \frac{\omega_{OL}}{2} \left[ \frac{2 \zeta}{\omega} + \tau - \frac{1}{\omega_{CR} B_{12} \omega_{OL}^2} \right] \]  

(7.24)

We have thus re-written the open-loop numerator as a classical quadratic expression.

The three autopilot gains that we are after \( K_A, K_o, \text{ and } K_R \) can then be found by simultaneously solving Equations (7.10), (7.11) and (7.12). If Equations (7.13), (7.21) and (7.22) are utilized, the resulting autopilot gains can be written as

\[ K_I = \frac{AF_2 - A_{32} - A_{31} (AF_1 - A_{21})}{(A_{22} - AF_2) (A_{31} - AF_1) - AF_2 (AF_1 - A_{21})} \]  

(7.25)

\[ K_A = \frac{K_3 \left( AF_2 - A_{31} \right)}{K_2 (A_{22} - AF_2)} \]  

(7.26)

and

\[ K_R = \frac{-\omega_{CR} B_{12} \omega_{OL}^2}{K_I (K_A K_2 + K_3)} \]  

(7.27)

The final gain to be computed is the steady-state (or dc) gain, \( K_o \). This gain is set such that in the limit as time approaches infinity, we desire the unit step response of \( \eta(t) \) to approach unity. Using the final value theorem,

\[ \lim_{s \to 0} \left[ \frac{\eta(s)}{\eta_c(s)} \right] = 1 \]  

(7.28)
The steady-state gain is then determined by setting $s$ equal to zero in the closed-loop transfer function, $\eta(s)/\eta_c(s)$, and solving

$$\frac{\eta_L(s)}{\eta_c(s)} = 1 = \frac{K_0 K_A K_1}{K_3 + K_A K_2}$$

for $K_0$. Then, we find

$$K_0 = \frac{K_3 + K_A K_2}{K_A K_1}$$ (7.29)

Utilizing the $K_2$ and $K_3$ relations from Chapter 6, the above equation can also be written as

$$K_0 = 1 + \left( \frac{kG}{K_A V} \right)$$ (7.30)

This then completes the sequence of autopilot gain determination. An algorithm can be developed to calculate these gains. This algorithm is presented in Table 7.1. Initial design values for $\zeta$ and $\tau$ are usually prescribed from prior miss distance analysis. Also, as a good rule of thumb, the crossover frequency, $\omega_{cr}$, should be chosen to be about one-third to one-fourth of the actuator bandwidth\textsuperscript{12, 13}.

The three autopilot gains $K_A$, $K_1$ and $K_3$ control the airframe response and influences, to first order, the three response parameters $\omega_{cr}$, $\zeta$, and $\tau$. The accelerometer gain $K_A$ influences the airframe time constant. Increasing $K_A$ tends to make the airframe response faster, without changing the frequency response characteristics very much. Changing the integrator gain $K_1$ influences the damping of the airframe response and also the gain of the frequency response. And, the rate loop gain $K_3$ mainly determines the open-loop crossover frequency, $\omega_{cr}$, and does not have much influence on the airframe transient time response characteristics. We may take advantage of these influences during the design process if gain
Table 7.1. Pitch Plane Autopilot Gain Calculation

1. Specify closed-loop real pole, $(1/\tau)$, the damping ratio of the closed-loop quadratic pair, $\zeta$, and the crossover frequency of the inner loop, $\omega_{cr}$

2. Compute the closed-loop quadratic pair natural frequency,
   \[
   \omega = \frac{\tau (\omega_{cr} + B_{11}/B_{12}) - 1}{2\zeta \tau}
   \]

3. Find the natural frequency of the net open-loop feedback zeros,
   \[
   \omega_{ol} = \frac{\omega}{\sqrt{\omega_{cr} \tau}}
   \]

4. Calculate the damping of the net open-loop feedback zeros,
   \[
   \zeta_{ol} = \frac{\omega_{ol} \left[ 2\zeta + \tau - \frac{1}{\omega_{cr} B_{12} \omega_{ol}^2} \right]}{2\omega_{ol} + \tau}
   \]

5. Compute the open-loop numerator coefficients, $AF_1$ and $AF_2$
   \[
   AF_1 = 2\zeta_{ol}/\omega_{ol}
   \]
   \[
   AF_2 = 1/\omega_{ol}^2
   \]

6. Determine the autopilot gains $K_a$, $K_p$, and $K_r$
   \[
   K_f = \frac{AF_2 - A_{22} - A_{31}(AF_1 - A_{21})}{(A_{22} - AF_1)(A_{31} - AF_1) - AF_2(AF_1 - A_{21})}
   \]
   \[
   K_A = \frac{AF_2 - A_{31}}{K_f (A_{22} - AF_2)}
   \]
   \[
   K_R = \frac{-\omega_{cr} B_{12} \omega_{ol}^2}{K_f (K_A K_2 + K_3)}
   \]

103
7. Find the steady-state gain, $K_0$

$$K_0 = 1 + \left( \frac{kG}{K_A V} \right)$$
modifications are necessary.

After the gains are calculated, classical performance and stability analyses, such as step response and frequency response (Bode, Nyquist) characteristics, are performed on the flight control system. The purpose of this analysis is to check that the requirements are satisfied. If the requirements are not met, then the gains are changed until the specifications are satisfied.

The gain calculation procedure is performed at the specified flight condition. One problem mentioned in a previous chapter is that the autopilot must control the airframe over the whole angle-of-attack range at that flight condition. So, the autopilot response is examined at various angles-of-attack. If the performance or stability characteristics are not satisfactory over the entire angle-of-attack range, then as a corrective procedure the autopilot gains can be scheduled over the angle-of-attack range to help achieve the desired response characteristics. Or, if constant gains are desired at that flight condition, various aerodynamic values (such as maximum $M_s$ and an intermediate $M_{a}$, see Reference 12) could be possibly used to obtain the desired response.

However, if we still do not get the airframe to respond as desired, there is a further correction we can employ. Recall that the autopilot is based on the low angle-of-attack, rigid-body transfer functions - Equations (7.1) - (7.3). To account for high angle-of-attack dynamics, we replace Equations (7.1) - (7.3) with their high angle-of-attack equivalents from Chapter 6, Equations (6.47), (6.54) and (6.58). Then, we obtain expressions for the autopilot gains in terms of the high angle-of-attack terms. This provides the autopilot design with a more accurate description of the airframe, and helps the controller design to meet the
desired response characteristics and stability margins. However, in this study, the use of the high angle of attack terms were not required to achieve an adequate classical autopilot design.

7.1.2 EFFECTS OF ACTUATOR AND SENSOR DYNAMICS

Once the autopilot gains are computed, the next stage in the design is to add the instrumentation models of the flight control system and recheck the time and frequency response characteristics. In particular, the quadratic lag models described in Chapter 3 are included in the analysis. If necessary, the autopilot gains are then changed to satisfy the design requirements. The design of the rigid-body airframe is then complete, and the next step is to examine the elastic effects of the airframe.

7.2 FLEXIBLE-BODY COMPENSATION

The flexible bending modes of the missile body may cause instability and should therefore be considered in the autopilot design and analysis. This instability could result in two ways. First, if the autopilot bandwidths are high enough compared to the structural mode bandwidths, coupling between the structural dynamics and the autopilot-airframe dynamics could lead to instability. Second, if the structural oscillations are sensed by the gyros and accelerometers and are feedback to the control system as sensed missile motion, then instability could result. The missile autopilot designer can compensate for these effects by proper location of the instruments, and/or by design techniques. These design techniques include phase stabilization, gain stabilization, or a combination of these methods\(^{14,17,18,20,28,34,35}\). For example, the first bending mode may be phase stabilized and the second flexible mode
compensated using gain stabilization. This section will describe the phase and gain stabilization approaches of structural bending-mode compensation.

Recall that above the flight control system analysis allowed for the effects of the instrumentation. The next step is then to add the linear dynamic models of the aeroelastic effects on the acceleration and body pitch rate feedback paths as discussed in Chapter 2. A good rule of thumb is to account for the bending effects of at least the first two or three bending modes. The primary effect of the flexible modes is to cause modal peaking of the magnitude of the frequency response (for the autopilot loop broken at the actuator) at the structural vibrational frequencies, leading to poor noise rejection, poor stability margins and even instability. Gain and phase stabilization techniques are standard classical control approaches used in compensating for the structural flexible dynamics.

7.2.1 GAIN STABILIZATION

Gain stabilization is probably the most common approach of suppressing the flexible modes for tactical missile autopilot design. Two of the more common gain stabilization approaches include autopilot gain adjustment and structural filtering. In the first approach, the autopilot gains are usually reduced to drop the loop transfer and its corresponding crossover frequency, thereby gaining the required attenuation at the flexible-body mode frequencies. However, by dropping the gains and reducing the crossover frequency, the time response of the autopilot-airframe becomes more sluggish. As a result, structural filtering is applied more often. This method attenuates the amplitude peaking at the bending frequencies by adding a structural filter to the flight control loop at the actuator input signal,
as shown in Figure 7.1. The structural filter is designed such that it has a deep notch at the structural mode frequency, and is given by the transfer function

\[
\frac{\delta_{c_2}(s)}{\delta_{c_1}(s)} = \frac{1 + \frac{2\zeta_{\text{num}}}{\omega_{N_i}} s + \frac{s^2}{\omega_{N_i}^2}}{1 + \frac{2\zeta_{\text{den}}}{\omega_{N_i}} s + \frac{s^2}{\omega_{N_i}^2}} \tag{7.31}
\]

where

- \(\omega_{N_i}\) = \(i^{th}\) mode bending frequency for notch
- \(\zeta_{\text{num}}\) = assumed bending mode damping ratio
- \(\zeta_{\text{den}}\) = notch filter damping

Values for \(\zeta_{\text{num}}\) are usually small (around 0.01 or 0.02) and values for the notch filter damping ratio, \(\zeta_{\text{den}}\), are usually greater than or equal to 0.5. The resulting magnitude of the loop gain should then be below unity gain (or zero db) at the flexible mode frequencies. Thus, the closed-loop system will be stable without even considering the system's phase lag at the bending frequencies.

In addition to the design goals set earlier, additional design measures are usually required. At the bending mode frequencies, the loop gain should typically have about 20-30 db of attenuation, for the loop opened at the actuator. Another problem to be considered is that the bending mode characteristics (damping, frequency, modal slope) may not be known exactly or may change during flight. To account for this, the gain margin should be at least 3 db (for the loop broken at the actuator) for a variation (say, \(\pm 5-10\%\)) of the nominal bending frequency.

The gain stabilization method has some important attributes. Since the loop gain is
kept below 0 db at the higher structural mode frequencies, the effects of noise on the system are minimized. In addition to being robust to noise, a gain stabilized design is also robust with respect to phase uncertainty. However, to obtain the desired attenuation at the bending frequency, the open-loop gain crossover frequency, $\omega_{cr}$, may need to be reduced to maintain the low-frequency phase and gain margins. Again, a reduction in $\omega_{cr}$ implies a slower time response and reduced performance.

7.2.2 PHASE STABILIZATION

The second approach to flexible mode compensation is phase stabilization. As with gain stabilization, there are several ways of accomplishing a phase-stabilized design. In this approach, the dynamics of the actuator and rate gyro components and structural filters (if necessary) are used to place the phase at the bending frequencies near 360 degrees. Thus, the phase at the flexible modes is far enough away from 180 degrees. By designing the controller to have large phase margins, the system can be stabilized, even though the amplitude of the structural mode peaks may exceed unity gain (or 0 db).

The phase stabilization approach may require a structural filter, which may take the form

$$\frac{\delta_{c_s}(s)}{\delta_{c_i}(s)} = \frac{1}{1 + \tau s}$$

where $\tau$ may be selected to yield a high break frequency. This type of filter may be necessary to tune the phase at the structural mode frequencies. Note that a notch filter is not utilized to attenuate the structural gain peaks in this approach.

Due to the uncertainty of the bending modes, another design goal for this technique
can be defined. For phase stabilization, it is also desired to have at least 50-70 degrees of phase margin at the high frequency gain crossover points. This requirement provides an extra measure of robustness with respect to the phase uncertainties of the flexible modes.

An advantage of phase stabilization is that it allows for higher crossover frequencies and thus improved response speeds. However, because the structural amplitude peaks may exceed 0 dB, noise may cause actuator heating or fin rate saturation problems. Since the noise occurs at the higher frequencies and also because of bending mode uncertainties, phase stabilization is not recommended for flexible body compensation of the higher structural modes.

7.3 THE ROLL AUTOPILOT

The missile under consideration in this study is assumed to be roll stabilized. Thus, a roll autopilot must be designed to maintain a constant roll attitude. Either a roll rate autopilot (commanding zero roll rate) or a roll position autopilot could be utilized. This study will consider a roll position autopilot as described in the Addendum to Reference 19. A diagram of this roll autopilot is presented in Figure 7.3. The roll autopilot utilized herein is similar to the pitch (yaw) autopilot discussed previously, and as a result, the design approach is similar.

For the roll autopilot design approach discussed here, we assume a perfect rigid-body (i.e., no torsional mode coupling). This assumption is made to keep the description of the design simple here, but the torsional modes need to be examined in a complete roll autopilot design.
Figure 7.3 Roll Channel Autopilot
The dynamics of the rigid-body roll channel are given by the transfer functions of Equations (6.69) and (6.70),

\[
\frac{\phi(s)}{\delta R(s)} = \frac{L_\delta}{s + \omega_R} \quad (7.33)
\]

\[
\frac{\phi(s)}{\delta R(s)} = \frac{L_\delta}{s(s + \omega_R)} \quad (7.34)
\]

The roll autopilot design procedure is to first compute the gains \(K_{AR}, K_{IR}\) and \(K_{RR}\) based upon desired loop characteristics. Then, classical control theory stability tests (Bode and Nyquist analysis) are utilized to check the requirements of the roll flight control system which includes the actuator and rate gyro models.

We begin by examining the open-loop characteristics, with the roll autopilot loop opened at the input to the actuator (\(\delta_c\) signal). The open-loop transfer function, \(G_{HOL}(s)\), is

\[
G_{HOL}(s) = \frac{K_0 \left[ 1 + \frac{2\zeta_\omega}{\omega_\omega} s + \frac{s^2}{\omega_\omega^2} \right]}{s^2(1 + s/\omega_R)} \quad (7.35)
\]

where

\[
K_0 = \frac{K_{RR}K_{IR}K_{AR}L_\delta}{\omega_R} \quad (7.36)
\]

\[
\frac{2\zeta_\omega}{\omega_\omega} = \frac{1}{K_{AR}} \quad (7.37)
\]

\[
\omega_\omega^2 = K_{IR}K_{AR} \quad (7.38)
\]

A desired open-loop crossover frequency can be utilized to specify one of the autopilot gains. At the crossover, the magnitude of the open-loop transfer function is unity. Finding the
expression for the magnitude and assuming that the crossover frequency, \( \omega_{CR} \), is relatively large, and also that the aerodynamic roll damping is relatively small, or

$$\omega_{CR} > \omega_o > \omega_R$$

the crossover frequency is approximately

$$\omega_{CR} = K_{RR} I_0$$ \hspace{1cm} (7.39)

Thus, the gain \( K_{RR} \) can be selected to obtain approximately the desired open-loop crossover frequency. The roll rate-loop gain, \( K_{RR} \), is chosen such that the crossover frequency is high enough to yield a fast autopilot response but low enough to prevent stability problems due to instrumentation and cross-channel coupling effects. A rule of thumb for the selection of the roll rate loop gain is that the roll crossover frequency should be at least double or triple of the pitch/yaw crossover frequency, and about one-third of the roll actuator bandwidth\(^{13}\).

The next step is to set the gain \( K_{IR} \) based on phase margin considerations. To obtain a more accurate value for the phase margin, we add the actuator and rate gyro models. The open-loop transfer function becomes

$$G(s) = \frac{K_{a} G_a(s) G_r(s)}{s^2(s + \omega_R)} \hspace{1cm} (7.40)$$

where \( G_a(s) \) and \( G_r(s) \) are the actuator and rate gyro quadratic lag models discussed in an earlier chapter. The phase angle, \( \phi(\omega) \) can be determined by considering the separate components of the open-loop transfer function,

$$\phi(\omega) = -\tan^{-1}\left[ \frac{2 \zeta_{a}(\omega/\omega_{act})}{1 - \left(\frac{\omega}{\omega_{act}}\right)^2} \right] - \tan^{-1}\left[ \frac{2 \zeta_{r}(\omega/\omega_{r})}{1 - \left(\frac{\omega}{\omega_{r}}\right)^2} \right] + \tan^{-1}\left[ \frac{2 \zeta_{o}(\omega/\omega_{o})}{1 - \left(\frac{\omega}{\omega_{o}}\right)^2} \right] - 2\left(\frac{\pi}{2}\right) - \tan^{-1}\left(\frac{\omega}{\omega_{R}}\right)$$

113
The phase margin, PM, is found at the gain crossover frequency, \( \omega_{CR} \), and is the phase angle away from -180 degrees, or

\[
PM = \phi(\omega_{CR}) + \pi
\]

(7.41)

If we assume that

\[
\frac{(\omega_{CR})^2}{\omega_o} > 1
\]

then the phase margin is

\[
PM = -\tan^{-1} \left[ \frac{2 \zeta_{act} \left( \frac{\omega_{CR}}{\omega_{act}} \right)}{1 - \left( \frac{\omega_{CR}}{\omega_{act}} \right)^2} \right] - \tan^{-1} \left[ \frac{2 \zeta_{rg} \left( \frac{\omega_{CR}}{\omega_{rg}} \right)}{1 - \left( \frac{\omega_{CR}}{\omega_{rg}} \right)^2} \right] + \tan^{-1} \left[ \frac{-K_{IR}}{\omega_{CR}} \right] - \tan^{-1} \left[ \frac{\omega_{CR}}{\omega_R} \right]
\]

(7.42)

Once the crossover frequency has been chosen, Equation (7.42) can be rearranged to obtain the integrator gain, \( K_{IR} \),

\[
K_{IR} = -\omega_{CR} \left\{ \tan(\text{PM}) + \left[ \frac{2 \zeta_{act} \left( \frac{\omega_{CR}}{\omega_{act}} \right)}{1 - \left( \frac{\omega_{CR}}{\omega_{act}} \right)^2} \right] + \left[ \frac{2 \zeta_{rg} \left( \frac{\omega_{CR}}{\omega_{rg}} \right)}{1 - \left( \frac{\omega_{CR}}{\omega_{rg}} \right)^2} \right] \right\}
\]

(7.43)

Thus, we have used the open-loop crossover frequency and phase margin to set two of the roll autopilot gains.

To select the third autopilot gain, we turn our attention to the closed-loop characteristics, in particular, the autopilot response speed. Again, we assume that the actuator and rate gyro bandwidths are relatively larger than the roll system bandwidth, i.e., \( G_{act}(s) = G_{rg}(s) = 1 \). Then, the closed-loop transfer function is
\[
\frac{\phi(s)}{\phi_{c}(s)} = \frac{K_{IR}K_{AR}K_{RR}L_{o}}{s^3 + s^2(L_0K_{RR} + \omega_R) + s(K_{IR}K_{RR}L_o) + K_{IR}K_{AR}K_{RR}L_{o}}
\]  
(7.44)

The effective first-order time constant, \(\tau_{eff}\), of the closed-loop roll autopilot is approximately,

\[
\tau_{eff} = \frac{1}{K_{AR}}
\]  
(7.45)

By specifying the desired effective time constant, we set the final roll autopilot gain, \(K_{AR}\). The values of allowable time constant (and thus, \(K_{AR}\)) are limited by stability conditions. Applying the Routh-Hurwitz criteria to Equation (7.44), stability is guaranteed if: 
1. the denominator coefficients of Equation (7.44) are all positive, 
2. the condition below holds

\[
(L_oK_{RR} + \omega_R) > K_{AR} \approx \frac{1}{\tau_{eff}}
\]  
(7.46)

Using this criteria as a constraint on the desired effective time constant, \(\tau_{eff}\), we can set the gain \(K_{AR}\) to yield the desired response speed while maintaining stability of the basic rigid-body airframe. If the desired gain margin is not achieved, the gain \(K_{AR}\) could be modified to yield the required margin. The procedure for the roll autopilot design is presented in Table 7.2.

To check the design, the actuator and rate gyro models are then added to the system and classical control stability and time response tests are then performed. If the specifications aren't satisfied, the control gains could be modified to yield the desired performance.

We have now covered the design approach of the separate pitch, yaw, and roll autopilot channels. The next step is to combine these single channel designs into a multivariable autopilot design that accounts for the cross-channel aerodynamic coupling.
Table 7.2. Roll Autopilot Design Approach.

1. Specify desired open-loop crossover frequency, $\omega_{CR}$, phase margin, PM, and effective time constant, $\tau_{eff}$

2. Compute roll rate-loop autopilot gain, $K_{RR}$,

$$K_{RR} = \frac{\omega_{CR}}{L_0}$$

3. Use phase margin to find roll integrator gain, $K_{IR}$,

$$K_{IR} = -\omega_{CR} \left\{ \tan(\text{PM}) + \frac{2 \zeta_{act} \left( \frac{\omega_{CR}}{\omega_{act}} \right)}{1 - \left( \frac{\omega_{CR}}{\omega_{act}} \right)^2} + \frac{2 \zeta_{rg} \left( \frac{\omega_{CR}}{\omega_{rg}} \right)}{1 - \left( \frac{\omega_{CR}}{\omega_{rg}} \right)^2} \right\}$$

4. Calculate autopilot gain $K_{AR}$, from effective time constant

$$K_{AR} = \frac{1}{\tau_{eff}}$$

5. Perform classical control analysis. If necessary, modify the three roll autopilot gains to obtain the desired performance and stability,
7.4 PITCH-YAW-ROLL DECOUPLING COMPENSATION

The final stage of the conventional autopilot design approach is to minimize the effect of the pitch-yaw-roll coupling effects. The basic idea is essentially decouple the pitch, yaw and roll channels. The purpose of the decoupling is to maintain the skid-to-turn system with two normal (pitch and yaw) steering channels and a roll stabilized channel, especially since these autopilots are designed using single-input single-output classical control theory.

The approach to decoupling the flight control channels is usually an ad hoc one. The conventional classical technique is to make the roll channel faster than the pitch and yaw channels to remove roll disturbances caused by pitch-yaw-roll coupling. With this approach, the roll autopilot bandwidth should be about twice that of the pitch (yaw) autopilot. Schelke designed a "stabilizer" to account for the induced roll moment. References 17 and 18 describe the addition of "$H_D$" and "$N_D$" decoupling paths between the roll and yaw channels. State space approaches have also been proposed to decouple multivariable flight control systems.

In this report, we will take a similar state-space approach. Recall the linear, time-invariant state-space description of the airframe dynamics presented in Chapter 2,

\[
\begin{align*}
\dot{x} &= Ax + BU \\
Y &= Cx + DU
\end{align*}
\]  

(7.47) (7.48)

where

\[
\begin{align*}
x &= \begin{bmatrix} \alpha \\ \beta \\ p \\ q \\ r \end{bmatrix}, & Y &= \begin{bmatrix} A_y \\ A_z \\ p \\ q \\ r \end{bmatrix}, & U &= \begin{bmatrix} \delta P \\ \delta Y \\ \delta R \end{bmatrix}
\end{align*}
\]
and the A, B, C, and D matrices were defined in Equations (2.30) - (2.69) and (2.73) - (2.91). At this point in the design, we have developed SISO pitch (yaw) and roll channel autopilots which produce the uncoupled control signals $\delta P$, $\delta Y$, and $\delta R$. The objective is to obtain a control vector, $U_C$, that compensates for the cross-channel coupling (or decouples the channels) exemplified in the fully coupled state-space description of the system. The compensated control signals can then be employed to control the airframe.

To begin, we form a reduced-order model from the state-space description given above by concentrating on the moment equations and the cross-coupling that is evident in these equations. The reduced-order model becomes

$$\dot{w} = A_R x_R + B_R U_C$$  \hspace{1cm} (7.49)

where $w$ is the body angular rate vector, $U_C$ is the compensated control (fin deflection) command vector, and $x_R$ is an input vector composed of the angle-of-attack and angle-of-sideslip. We assume that $B_R$ has an inverse $B_R^{-1}$. From the moment equations, then the reduced order model is

$$\begin{bmatrix}
\dot{\alpha} \\
\dot{\beta} \\
\dot{\gamma} \\
\end{bmatrix} = \begin{bmatrix}
a_{31} & a_{32} \\
a_{41} & a_{42} \\
a_{51} & a_{52} \\
\end{bmatrix} \begin{bmatrix}
\alpha \\
\beta \\
\gamma \\
\end{bmatrix} + \begin{bmatrix}
b_{31} & b_{32} & b_{33} \\
b_{41} & b_{42} & b_{43} \\
b_{51} & b_{52} & b_{53} \\
\end{bmatrix} \begin{bmatrix}
\delta P \\
\delta Y \\
\delta R \\
\end{bmatrix}$$  \hspace{1cm} (7.50)

The goal is to cancel (or minimize) the cross-coupling aerodynamic terms which influence the body rates. These cross-coupling terms include:

1. $\alpha$ and $\beta$ coupling into the roll rate ($a_{31}$ and $a_{32}$).
2. $\alpha$ causing yaw rate and $\beta$ causing pitch rate ($a_{42}$ and $a_{51}$, respectively)
3. $\delta P$ and $\delta Y$ coupling into the roll rate ($b_{31}$ and $b_{32}$)
4. $\delta P$ causing yaw rate and $\delta Y$ causing pitch rate ($b_{s1}$ and $b_{s2}$, respectively)

5. $\delta R$ coupling into the pitch and yaw rates ($b_{s1}$ and $b_{s3}$)

The desired, decoupled system would cancel these terms, leaving

\[
\begin{bmatrix}
\dot{\rho} \\
\dot{\alpha} \\
\dot{\beta}
\end{bmatrix} =
\begin{bmatrix}
0 & 0 \\
a_{41} & 0 \\
0 & a_{52}
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix} +
\begin{bmatrix}
0 & 0 & b_{s3} \\
b_{41} & 0 & 0 \\
0 & b_{s2} & 0
\end{bmatrix}
\begin{bmatrix}
\delta P \\
\delta Y \\
\delta R
\end{bmatrix}
\]

(7.51)

which has the form

\[
\dot{\mathbf{x}} = A \mathbf{x} + B \mathbf{u}
\]

(7.52)

We assume that $B_{D}$ has an inverse $B_{D}^{-1}$. To achieve this decoupled system, a linear feedback control law (similar to that given in References 65-67) is applied,

\[
U_{C} = FX_{R} + GU
\]

(7.53)

Substituting this equation into Equation (7.49) and equating like terms with those in Equation (7.52), we find

\[
F = B_{R}^{-1}(A_{D} - A_{R})
\]

(7.54)

\[
G = B_{R}^{-1}B_{D}
\]

(7.55)

Or, factoring out the G matrix, the compensated control commands become

\[
U_{C} = G(U + G^{-1}FX_{R})
\]

(7.56)

Letting

\[
M = G^{-1} = B_{D}^{-1}B_{R}
\]

(7.57)

and

\[
N = -G^{-1}F = B_{D}^{-1}(A_{R} - A_{D})
\]

(7.58)

the compensated control law is
\[ U_C = M^{-1}(U - Nx_x) \] (7.59)

The above equation serves as the decoupling compensation equation.

First, we find the \( M \) matrix. Inverting the \( B_D \) matrix from Equation (7.51) and substituting into Equation (7.57) along with \( B_R \) (from Equation (7.50)),

\[
M = \begin{bmatrix}
1 & \frac{b_{42}}{b_{41}} & \frac{b_{43}}{b_{41}} \\
\frac{b_{51}}{b_{52}} & 1 & \frac{b_{53}}{b_{52}} \\
\frac{b_{31}}{b_{33}} & \frac{b_{32}}{b_{33}} & 1
\end{bmatrix}
\] (7.60)

Using the elements of the \( B \) matrix from Chapter 2, the off-diagonal elements of this \( M \) matrix are

\[
M_{12} = \frac{b_{42}}{b_{41}} = \frac{C_{m_y}}{C_{m_x}}
\] (7.61-a)

\[
M_{13} = \frac{b_{43}}{b_{41}} = \frac{C_{m_x}}{C_{m_y}}
\] (7.61-b)

\[
M_{21} = \frac{b_{52}}{b_{51}} = \frac{C_{m_x}}{C_{m_y}}
\] (7.61-c)

\[
M_{23} = \frac{b_{53}}{b_{52}} = \frac{C_{m_y}}{C_{m_x}}
\] (7.61-d)

\[
M_{31} = \frac{b_{32}}{b_{31}} = \frac{C_{l_x}}{C_{l_y}}
\] (7.61-e)

\[
M_{32} = \frac{b_{33}}{b_{32}} = \frac{C_{l_y}}{C_{l_x}}
\] (7.61-f)

The next step is to calculate the \( N \) matrix using Equation (7.58) and \( B_D^{-1} \). Making the substitutions and performing the linear algebra, we get

\[
N = \begin{bmatrix}
0 & \frac{a_{42}}{b_{41}} \\
\frac{a_{51}}{b_{52}} & 0 \\
\frac{a_{31}}{b_{33}} & \frac{a_{32}}{b_{33}}
\end{bmatrix}
\] (7.62)

Again, the elements of the \( A \) and \( B \) matrices from Chapter 2 are then substituted into the above nonzero terms to define
\[ N_{12} = a_{41}/b_{41} = C_{m_{1}}/C_{m_{42}} \]  \hspace{1cm} (7.63-a)
\[ N_{21} = a_{51}/b_{52} = C_{\phi_{1}}/C_{\phi_{42}} \]  \hspace{1cm} (7.63-b)
\[ N_{31} = a_{53}/b_{53} = C_{\phi_{1}}/C_{\phi_{43}} \]  \hspace{1cm} (7.63-c)
\[ N_{32} = a_{52}/b_{53} = C_{\phi_{4}}/C_{\phi_{43}} \]  \hspace{1cm} (7.63-d)

In addition, the inverse of the M matrix must be determined to utilize Equation (7.59). The inverse of M (M\(^{-1}\) = G) is

\[
G = \begin{bmatrix}
(1-M_{23}M_{32})/\Delta M & (M_{13}M_{32}-M_{12})/\Delta M & (M_{12}M_{23}-M_{13})/\Delta M \\
(M_{23}M_{31}-M_{21})/\Delta M & (1-M_{23}M_{31})/\Delta M & (M_{13}M_{21}-M_{12})/\Delta M \\
(M_{21}M_{32}-M_{22})/\Delta M & (M_{12}M_{32}-M_{31})/\Delta M & (1-M_{12}M_{21})/\Delta M
\end{bmatrix} \quad (7.64)
\]

where

\[
\Delta M = (1-M_{23}M_{32}) + M_{12}(M_{23}M_{31}-M_{21}) + M_{13}(M_{21}M_{32}-M_{31}) \quad (7.65)
\]

Multiplying out the matrices M\(^{-1}\)N (= -F) yields

\[
F = \begin{bmatrix}
-(G_{12}N_{21} + G_{13}N_{31}) & -(G_{11}N_{12} + G_{13}N_{32}) \\
-(G_{22}N_{21} + G_{23}N_{31}) & -(G_{21}N_{12} + G_{23}N_{32}) \\
-(G_{32}N_{21} + G_{33}N_{31}) & -(G_{31}N_{12} + G_{33}N_{32})
\end{bmatrix} \quad (7.66)
\]

The coupling compensation is then obtained from Equation (7.53). Substituting in the appropriate matrices and vectors,

\[
\delta P_{C} = G_{11}\delta P + G_{12}\delta Y + G_{13}\delta R + F_{11}\alpha + F_{12}\beta \quad (7.67)
\]
\[
\delta Y_{C} = G_{21}\delta P + G_{22}\delta Y + G_{23}\delta R + F_{21}\alpha + F_{22}\beta \quad (7.68)
\]
\[
\delta R_{C} = G_{31}\delta P + G_{32}\delta Y + G_{33}\delta R + F_{31}\alpha + F_{32}\beta \quad (7.69)
\]

The uncoupled fin deflection commands (\(\delta P, \delta Y, \delta R\)) are produced by the separate channel pitch, yaw and roll autopilots, and the inputs \(\alpha\) and \(\beta\) are obtained from the Inertial
Reference Unit (IRU). Note that in this decoupling method, the vector $x_R$ is not really a state vector, but is an exogenous input vector composed of the angles of attack and sideslip. The angle of attack components $\alpha$ and $\beta$ are calculated in the IRU based on integration of the accelerometer values, and are not really measured states.

We have thus developed a state-space method for decoupling the pitch-yaw-roll autopilot channels. This decoupling attempts to cancel the cross-channel coupling effects on the missile body rates, thus preventing induced aerodynamic moments from acting on the missile. Essentially, this approach takes advantage of apriori knowledge of cross-channel aerodynamic terms (obtained from wind tunnel tests or aerodynamic prediction codes). This technique accounts for the expected coupling by compensating the fin deflection commands from the individual pitch, yaw and roll autopilots. An algorithm for implementation of this compensation is provided in Table 7.3.

The discussion of the conventional autopilot design approach using classical control theory is now complete. We are now ready to examine the background and theory of our next autopilot design approach: that is, $H_\infty$ control and $\mu$-synthesis.
Table 7.3. Autopilot Cross-Channel Decoupling Compensation Algorithm

1. Input current flight condition from IRU (M, α, β, φ)

2. Input fin deflection commands from pitch/yaw/roll autopilots (δP, δY, δR)

3. Estimate the following aerodynamic derivatives:
   - $C_{iα}$, $C_{iβ}$, $C_{iγ}$, $C_{iγγ}$
   - $C_{mβ}$, $C_{mγ}$, $C_{mγγ}$, $C_{mγγγ}$
   - $C_{nα}$, $C_{nαγ}$, $C_{nαγγ}$, $C_{nαγγγ}$

4. Compute $M$ matrix elements
   
   \[ M_{12} = C_{mα}/C_{mα} \]
   \[ M_{13} = C_{mγ}/C_{mγ} \]
   \[ M_{21} = C_{nβ}/C_{nβ} \]
   \[ M_{23} = C_{nγ}/C_{nγ} \]
   \[ M_{31} = C_{iα}/C_{iα} \]
   \[ M_{32} = C_{iβ}/C_{iβ} \]

5. Calculate $N$ matrix elements
   
   \[ N_{12} = C_{mα}/C_{mα} \]
   \[ N_{21} = C_{nβ}/C_{nβ} \]
   \[ N_{31} = C_{iα}/C_{iα} \]
   \[ N_{32} = C_{iβ}/C_{iβ} \]

6. Find determinant of $M$ matrix
   \[ ΔM = (1 - M_{23}M_{32}) + M_{12}(M_{23}M_{31} - M_{21}) + M_{13}(M_{21}M_{32} - M_{31}) \]

123
7. Determine inverse of M matrix (\(\equiv\) G matrix)

\[
\begin{align*}
G_{11} & = (1 - M_{22}M_{33})/\Delta M \\
G_{12} & = (M_{13}M_{32} - M_{13}M_{23})/\Delta M \\
G_{13} & = (M_{12}M_{33} - M_{13}M_{23})/\Delta M \\
G_{21} & = (M_{23}M_{31} - M_{23}M_{13})/\Delta M \\
G_{22} & = (1 - M_{13}M_{31})/\Delta M \\
G_{23} & = (M_{13}M_{21} - M_{23}M_{12})/\Delta M \\
G_{31} & = (M_{21}M_{32} - M_{23}M_{31})/\Delta M \\
G_{32} & = (M_{12}M_{31} - M_{23}M_{12})/\Delta M \\
G_{33} & = (1 - M_{12}M_{21})/\Delta M
\end{align*}
\]

8. Find components of F matrix (where \(F = -M^{-1}N\))

\[
\begin{align*}
F_{11} & = -(G_{12}N_{21} + G_{13}N_{31}) \\
F_{12} & = -(G_{11}N_{12} + G_{13}N_{32}) \\
F_{21} & = -(G_{22}N_{21} + G_{23}N_{31}) \\
F_{22} & = -(G_{21}N_{12} + G_{23}N_{32}) \\
F_{31} & = -(G_{32}N_{21} + G_{33}N_{31}) \\
F_{32} & = -(G_{31}N_{12} + G_{33}N_{32})
\end{align*}
\]

9. Calculate compensated fin deflection commands

\[
\begin{align*}
\delta P_C & = G_{11} \delta P + G_{12} \delta Y + G_{13} \delta R + F_{11} \alpha + F_{12} \beta \\
\delta Y_C & = G_{21} \delta P + G_{22} \delta Y + G_{23} \delta R + F_{21} \alpha + F_{22} \beta \\
\delta R_C & = G_{31} \delta P + G_{32} \delta Y + G_{33} \delta R + F_{31} \alpha + F_{32} \beta
\end{align*}
\]
Chapter 8. Introduction to H-Infinity Control, the Structured Singular Value and Mu-Synthesis

8.1 INTRODUCTION

The concept of robustness and its importance in control system design is not new. Classical control designers build robustness into their control system designs by ensuring satisfactory stability (gain and phase) margins at critical points in the control system loop. Modern control methods such as optimal and linear quadratic control can treat multiple-input multiple-output (MIMO or multivariable) systems; however, (until recently), modern control designs do not specifically address modeling errors or other model uncertainties. Over the past 15 years, substantial research efforts have concentrated on adding robustness to the multivariable control design techniques. One of the emerging techniques for (nonconservatively) building in robustness to specific modeling uncertainties is Mu-Synthesis.

Mu-Synthesis is a (post-)modern robust, multivariable control design technique which combines H-Infinity (H∞) control with Structured Singular Value (SSV or μ) analysis. But what do the concepts of robustness, H∞ control, μ and μ-Synthesis really mean, and what are their importance in control system design? There are literally hundreds of reports that address these subjects. However, to control system designers starting out in H∞ and μ methods, the sheer number of reports can be overwhelming and the highly mathematical nature of many of these reports can make them seem foreign. This chapter is meant to fill that gap and to serve as a tutorial regarding H∞ and μ robust control synthesis and analysis methods. As such, this chapter is not intended as a rigorous mathematical exposition full
of theory development and derivations, but is written as an introduction to the \( H_\infty \) and \( \mu \) methods of robust control.

We begin this chapter with a historical review of the development of the \( H_\infty \) and \( \mu \) methods. In section 8.3, we present some mathematical background that form the basis of the \( H_\infty \) and \( \mu \) methods. Following this, the concept of robustness is introduced. A description of the general analyses and synthesis framework is then given in Section 8.5. A summary of linear systems theory and several stability measures are then discussed. Section 8.7 presents uncertainty descriptions and defines degrees of stability, performance and robustness. These concepts are then applied to single-input single-output (SISO) systems and multiple-input multiple-output (MIMO) systems in Sections 8.8 and 8.9, respectively. The final section then discusses \( H_\infty \) control and \( \mu \)-Synthesis. A bibliography comprised of the reports from which this chapter was drawn is provided at the conclusion of this report.

8.2 HISTORICAL REVIEW

Modern Control Theory, the dominant control theory paradigm in the theoretical community during the 1960's and 1970's, has its basis in Stochastic Optimal Control and Estimation (LQG) Theory. That theory essentially restricts model uncertainty to additive noise. It provides a methodology for analyzing the impact of noise on system performance and synthesizing a controller to reduce that impact. The inadequacies of that view of uncertainty became widely accepted in the late 1970's, as robustness to plant uncertainty became a major theme in the Modern Control Theory community. Ironically, modern robust control development involved a renewed interest in the Classical Control paradigm of Bode\textsuperscript{68}
and Horowitz, which Modern Control Theory displaced within the theoretical community if not among practicing engineers. Initially, the new direction provided useful design tools, including Singular Value Analysis and Multivariable Loop Shaping. While providing an important perspective, as well as practical techniques, the methods based on singular values still require rather restrictive assumptions about uncertainty. In particular, plant uncertainty must be essentially modelled as a single unstructured perturbation. The Structured Singular Value, also called \( \mu \), was developed at the beginning of the 1980’s to correct that deficiency.

The synthesis method of H-Infinity control was introduced to the control community by Zames and Helton and further developed by many other researchers. Several approaches to the solution of the \( H_\infty \) control problem were formulated, but the computational aspects of these methods were unwieldy. The work of Doyle, Glover, Khargonekar and Francis, however, resulted in a more computationally practical solution. This approach solves the H-Infinity control problem in the time-domain as the stabilizing solutions of two algebraic Riccati equations.

Doyle introduced and defined the structured singular value \( \mu \) as the answer to the control problem, wherein he derived an upper and lower bound to \( \mu \). Doyle’s upper bound is computable using frequency dependent D scales and H-Infinity Synthesis. Consequently, Doyle’s upper bound to the complex \( \mu \) synthesis control problem is computable as a sequence of solutions of algebraic Riccati equations. Computations of the lower bound provides an estimate on how close the upper bound is to the defined answer \( \mu \).

Structured Singular Value theory together with the theory of H-Infinity control has
provided a very powerful mathematical framework for the analysis, synthesis and control
design of complex multivariable control systems. They have formed the basis for a new
paradigm for control theory broader in scope and content than that provided by Classical or
Modern Control Theory. In addition, these methods have already been successfully
demonstrated, e.g., experimental flexible space structures work at California Institute of
Technology through the work of Balas and Doyle.79

8.3 MATHEMATICAL PRELIMINARIES

In this section, we lay the mathematical groundwork for the remainder of the chapter.
We begin with a review of basic topics such as complex variables and linear algebra. Then,
we discuss vector and matrix norms, singular values, function spaces, and signal and system
(transfer function) norms. Examples are presented to illustrate the concepts.

8.3.1 COMPLEX VARIABLES

The elements of a vector $\bar{X}$ or a matrix $A$ may be real or complex numbers. The field
of real numbers as denoted as $\mathbb{R}$ and the field of complex numbers is represented by $\mathbb{C}$. The
notation $\mathbb{C}^{\times}$ thus symbolizes the field of complex matrices of dimension $n \times m$. For
example, the terminology $\bar{x} \in \mathbb{C}^n$ represents a vector $\bar{x} = [x_1 \ x_2 \ x_3 \ldots x_n]^T$, where all the
elements $x_i$ are members of the field of complex numbers (i.e., $\bar{x}$ is an $n$-dimensional
complex-valued column vector). This can also be written as $x_i \in \mathbb{C}, \forall \ i$, where $\forall$ symbolizes
"for all".

Recall that a complex number contains both real and imaginary parts, $x = a + jb$,
where \( j \) denotes the imaginary part (\( i \) also commonly denotes the imaginary part). The complex conjugate of \( x \) is given by \( x^* \), and as defined as \( x^* = a - jb \). The magnitude of a complex number can be found by using
\[
| x | = \sqrt{x^*x} = \sqrt{a^2 + b^2}
\]
(8.1)

An \( n \)-dimensional complex vector \( \bar{x} \) can be written in the form
\[
\bar{x} = \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix} = \begin{bmatrix}
a_1 + jb_1 \\
a_2 + jb_2 \\
\vdots \\
a_n + jb_n
\end{bmatrix}
\]
(8.2)

The complex conjugate transpose of the vector \( \bar{x} \), \( \bar{x}^H \), is given by \( \bar{x}^H = (x_1^* \ x_2^* \ldots x_n^*) \). For example, if
\[
\bar{x} = \begin{bmatrix}
1 + j2 \\
3 - j4
\end{bmatrix}
\]
then its complex conjugate transpose is
\[
\bar{x}^H = \begin{bmatrix}
1 - j2 & 3 + j4
\end{bmatrix}
\]

Similarly, we can define an \( n \times m \) complex valued matrix \( A \), or \( A \in \mathbb{C}^{n \times m} \). Then, \( A^H \) represents the \( m \times n \) dimensional, complex conjugate transpose of the matrix \( A \). As an example, let
\[
A = \begin{bmatrix}
-j & 1 + j2 \\
2 - j3 & 3
\end{bmatrix}
\]
then

129
\[ A^R = \begin{bmatrix} j & 2 + j3 \\ 1 - j2 & 3 \end{bmatrix} \]

### 8.3.2 RELEVANT TOPICS FROM LINEAR ALGEBRA

We begin with some definitions. A real matrix \( R \) is called orthogonal if the transpose of the matrix is equivalent to the matrix inverse, or \( R^T = R^{-1} \). A complex matrix \( A \) is termed Hermitian if \( A = A^H \) is satisfied. Also, a complex matrix \( U \) is unitary if the complex conjugate transpose is equivalent to the inverse, \( U^H = U^{-1} \).

Next, let \( A \) be some \( n \times n \) matrix. The eigenvalues, \( \lambda_i \), of the matrix \( A \) are determined from

\[
(\lambda I - A)v = 0
\]

where \( v \) is the eigenvector. The eigenvalues are obtained by solving

\[
\det(\lambda I - A) = 0
\]

The set of eigenvalues \( \lambda_i, i = 1, 2, 3, ..., n \), comprise the spectrum of \( A \), \( \Omega(A) \). The eigenvectors \( v_i \) can be found by solving Equation (8.3) or

\[
Av = \lambda v
\]

for a given eigenvalue. The spectral radius of the matrix \( A \), \( \rho(A) \), is defined as the eigenvalue with largest absolute magnitude,

\[
\rho(A) = \max_i |\lambda_i(A)|
\]

As an example, let \( A \) be

\[
A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}
\]
The eigenvalues are determined using Equation (8.4) to be $\lambda_1 = 1$ and $\lambda_2 = 5$. Equation (8.3) is utilized to find the corresponding eigenvectors,

for $\lambda_1 = 1$, $\quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

for $\lambda_2 = 5$, $\quad v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

The spectral radius of the matrix $A$ is then $\rho(A) = 5$.

For some cases, the eigenvalues take on special properties:

1. If the matrix $A$ is Hermitian, then all the eigenvalues are real

2. For a unitary matrix, all the eigenvalues have unit magnitude

3. The matrix $A$ is said to be positive definite (and positive semi-definite) if all the real parts of the eigenvalues of $A$ are $> 0$ (and $\geq 0$). Alternately, we can define a square matrix $A$ to be positive definite (and positive semi-definite) if $x^H Ax > 0$ (and $x^H Ax \geq 0$) for all $x \neq 0$

4. The matrix $A$ is said to be positive definite (and positive semi-definite) if all the real parts of the eigenvalues of $A$ are $> 0$ (and $\geq 0$), and,

5. If some of the real parts of the eigenvalues of the matrix $A$ are positive and some negative, $A$ is called indefinite.

Going back to the last example, $A$ is positive definite since all the eigenvalues are real and positive. This result can be verified by choosing any vector $\bar{x}$ where $\bar{x} \neq 0$ and showing that $\bar{x}^H A \bar{x}$ is positive.
8.3.3 VECTOR AND MATRIX NORMS

In addition to such measures as length and magnitude, various other types of measures can be defined for vectors and matrices. In particular, we refer to norms, which are essentially measures of "size" that satisfy certain properties.

8.3.3.1 Properties of Norms

Norms are real valued functions \(|x|\) defined over the field \(F\), where \(F\) is the field of real \(\mathbb{R}\) or complex \(\mathbb{C}\) numbers \((x \in F^n)\), that satisfy the following four properties:

1. \(|x| > 0, x \neq 0\)  
   Positivity

2. \(|x| = 0, x = 0\)  
   Positive Indefiniteness

3. \(|\alpha x| = |\alpha| |x|, \alpha \in \mathbb{C}\)  
   Homogeneity

4. \(|x + y| \leq |x| + |y|, x, y \in \mathbb{C}^n\)  
   Triangle Inequality

Many types of norms can be defined. We will examine in the following sections a few of the most common vector and matrix norms.

8.3.3.2 Vector Norms

Consider the vector \(x \in F^n\). The vector \(p\)-norm of \(x\) is defined as

\[
|x|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \quad 1 \leq p < \infty
\]

(8.7)

Here, we define three of the most common \(p\) norms \((p = 1, 2, \infty)\).
\[ |x|_1 = \sum_{i=1}^{n} |x_i| \]  \hspace{1cm} (8.8)

\[ |x|_2 = \left( \sum_{i=1}^{n} |x_i|^2 \right)^{1/2} = (x^H x)^{1/2} \]  \hspace{1cm} (8.9)

where the vector 2-norm is referred to as the Euclidean norm and gives the length of the vector \( x \), and

\[ |x|_\infty = \max_{1 \leq i \leq n} |x_i| \]  \hspace{1cm} (8.10)

For example, consider some two-dimensional vector, \( x \in \mathbb{C}^2 \). The three norms defined above for this example are

\[ |x|_1 = |x_1|_1 = |x_1| + |x_2| \]

\[ |x|_2 = \sqrt{|x_1|^2 + |x_2|^2} \]

\[ |x|_\infty = \max\{|x_1|, |x_2|\} \]

These norms are depicted in Figure 8.1.

A second example illustrates the Euclidean norm for a complex vector. Let

\[ x = \begin{bmatrix} 1 + j4 \\ 2 - j2 \end{bmatrix} \]

Then
Figure 8.1. Example of Two-Dimensional Vector Norm
\[ x^H = [1 - j4 \ 2 + j2] \]

and

\[ x^H x = [(1 - j4)(1 + j4) + (2 + j2)(2 - j2)] = 25 \]

Thus,

\[ \|x\|_2 = \sqrt{x^H x} = 5 \]

8.3.3.3 Matrix Norms

Next, we examine matrix norms. Consider the \( m \times n \) dimensional matrix \( A \) to have real and/or complex elements, \( A \in F^{m \times n} \), where \( F \) is the field of real and complex numbers defined above in Section 8.3.3.1. The matrix 1, 2, and \( \infty \) norms are defined as

\[ \|A\|_1 = \max_j \sum_i |a_{ij}| \quad (8.11) \]

\[ \|A\|_2 = \max_i \left[ \lambda_i(A^H A) \right]^{1/2} \quad (8.12) \]

\[ \|A\|_\infty = \max_i \sum_j |a_{ij}| \quad (8.13) \]

The matrix 1-norm is the maximum column sum, the matrix 2-norm is the square root of the largest eigenvalues of \( A^H A \) and is known as the Spectral norm, and the matrix \( \infty \)-norm is the maximum row sum. These matrix norms are specific cases of the matrix p-norm, defined by

\[ \|A\|_p = \max_{x \neq 0} \frac{|Ax|_p}{\|x\|_p} \quad (8.14) \]

The matrix p-norm \( \|A\|_p \) bounds the amplifying power of the matrix \( A \),
\[ |Ax| \leq |A| |x| \]  
(8.15)

In addition to the matrix norms defined above, there is another matrix norm that is important - the Frobenius norm. The Frobenius (or Euclidean) matrix norm is the root sum squares of the magnitudes of the matrix elements,

\[ |A|_F = \left[ \sum_i \sum_j |a_{ij}|^2 \right]^{1/2} = \left[ \text{Trace}(A^HA) \right]^{1/2} \]  
(8.16)

As an example, consider the complex matrix \( A \) from Section 8.3.1,

\[ A = \begin{bmatrix} -j & 1 + j2 \\ 2 - j3 & 3 \end{bmatrix} \]

The matrix norm \( |A|_1 \) is the maximum column sum, in this case column 2, which is equal to 5.24 (for column 2). The norm \( |A|_\infty \) is equal to 6.61, where row 2 contains the maximum sum. For the matrix 2-norm, \( A^HA \) is

\[ A^HA = \begin{bmatrix} 14 & 4 + 10j \\ 4 - 10j & 14 \end{bmatrix} \]

The maximum eigenvalue is equal to 24.77, thus \( |A|_2 = 4.98 \). The Frobenius norm is evaluated by either taking the square root of the sum of the squares of the \( A \) matrix element magnitudes or by taking the square root of the trace of \( A^HA \). Either way, \( |A|_F = 5.29 \).

8.3.3.4 Induced Norms

An induced norm is a special case of a norm. An induced norm is induced by a vector norm on the input space into the output space. Consider the equation \( y = Ax \), where \( x \) is an input vector \( (x \in \mathbb{F}^n) \) to \( A \) (a linear map), and \( y \) is the output vector \( (y \in \mathbb{F}^n) \). The
induced norm of the linear map \( A, \| A \|_i \) is defined by

\[
\| A \|_i = \sup_{x \neq 0} \frac{\| Ax \|}{\| x \|} = \sup_{\| x \| = 1} \| Ax \| \tag{8.17}
\]

where "sup" represents supremum. [The supremum is similar to the maximum, but also takes into account possible bounds on the function. As such, the supremum is equivalent to the "least upper bound" of the function. For example, consider the function \( f(t) = 1 - e^t \), shown in Figure 8.2(b). This function is bounded by 1, which is the least upper bound, thus the supremum of \( f(t) = 1 \).] From the definition given in Equation (8.17), matrix p-norms are induced norms because they are induced by vector norms. However, the Frobenius norm is not an induced norm. In addition, induced norms satisfy the following properties:

1. \( \| Ax \| \leq \| A \| \| x \| \)
2. \( \| \alpha A \| = |\alpha| \| A \| \)
3. \( \| A+B \| \leq \| A \| + \| B \| \)
4. \( \| AB \| \leq \| A \| \| B \| \)

The induced norm gives a measure of the maximum gain of a matrix. To illustrate this, consider the input vector \( x \) to be the set of all two-dimensional unit vectors centered at the origin \( X \in \mathbb{R}^2 \), as depicted in Figure 8.3(a). Thus, the input is simply the unit disk. If the input is operated on by \( A \), the elements of the output are magnified as shown in figure 8.3(b). The induced norm \( \| A \|_i \) is then the upper bound or maximum magnification (Figure 8.3(b)). Why is the induced norm so important? The induced norm quantifies the amount of magnification (for the worst case) due to unit input signals. Thus, induced norms will be important later when we minimize the worst case output of a system due to unit input disturbances.

137
Figure 8.2. Example Functions of Time

(a) \( f_1(t) = e^{t/\tau} \sin t \)

(b) \( f_2(t) = 1 - e^{t/\tau} \)
Figure 8.3. Example of an Induced Norm
8.3.4 SINGULAR VALUES

The use of singular values in the design and analysis of multivariable control system
design has been accepted and applied since the late 1970's. In this section, we define the
singular value, give some of the properties of singular values, summarize how we might
determine the singular values using singular value decomposition, and examine why singular
values are useful in multivariable control system design.

8.3.4.1 Definition

Let $A$ be an $m \times n$ real or complex matrix, $A \in \mathbb{F}^{m \times n}$, and $k = \min \{m, n\}$. The
singular values of the matrix $A$, $\sigma_i(A)$, are the non-negative square roots of the eigenvalues
of $A^H A$ (or equivalently $AA^H$),

$$\sigma_i(A) = \sqrt{\lambda_i(A^H A)} = \sqrt{\lambda_i(AA^H)} \geq 0, \quad i = 1, 2, \ldots, k$$  \hspace{1cm} (8.18)

The singular values are non-negative since $A^H A$ (and $AA^H$) are Hermitian (i.e., the
eigenvalues are real). The maximum singular value is denoted by $\sigma(A)$ and the minimum
singular value is represented by $\sigma(A)$. From the prior definition of the matrix 2-norm (or
spectral norm), it turns out that the spectral norm of the A matrix is equivalent to the
maximum singular value of the matrix $A$, $\| A \|_2 = \sigma(A)$. (Alternatively, we can say from
the definition of the matrix p-norm or induced norm that the matrix norm induced by the
vector 2-norm, $\| x \|_2$, is the maximum singular value, $\sigma(A)$. Thus, the maximum and
minimum singular values of the matrix $A$ can be written as

$$\sigma(A) = \max_{x \neq 0} \frac{\| Ax \|_2}{\| x \|_2} = \max_{\| x \|_2 = 1} \| Ax \|_2 = \| A \|_2$$  \hspace{1cm} (8.19)
\[ \sigma(A) = \min_{x \neq 0} \frac{|Ax|_2}{|x|_2} = \min_{|x|_2 = 1} |Ax|_2 = \frac{1}{|A^{-1}|_2} \quad (8.20) \]

if the inverse of \( A \) exists. Thus, the inverse of the spectral norm of \( A^{-1} \) is the minimum singular value of \( A \), \( \sigma \). Assuming that \( A \) operates on some input vector \( x \) such that \( y = Ax \), then the maximum and minimum singular values of \( A \) describe the maximum and minimum magnification of the input vector by \( A \).

The condition number of the matrix \( A \) is given by

\[ \kappa(A) = \frac{\sigma(A)}{\sigma(A)} \quad (8.21) \]

and is a measure of the difficulty of inverting the \( A \) matrix. If \( \kappa(A) \) is near zero, then the matrix \( A \) is "ill-conditioned" and an inversion of the \( A \) matrix could be numerically unreliable.

### 8.3.4.2 Properties of Singular Values

There are several properties which singular values can be shown to satisfy. Here, we will present some of the more useful identities. First, the maximum and minimum singular values provide upper and lower bounds, respectively, of the eigenvalues of \( A \),

\[ \sigma(A) \leq |\lambda_1(A)| \leq \sigma(A) \quad (8.22) \]

(In the special case when \( A \) is Hermitian, then the eigenvalues of \( A \), \( \lambda_i(A) \), and the singular values of \( A \), \( \sigma_i(A) \), are the same.) Second, from the definitions of the maximum and minimum singular values, we can also say

\[ \sigma(A) = \frac{1}{\sigma(A^{-1})} \quad (8.23) \]
\[ \sigma(A) = \frac{1}{\sigma(A^{-1})} \] (8.24)

if the matrix A is nonsingular. If A is singular, then \( \sigma(A) = 0 \). Third, consider unitary matrices. If U and V are unitary matrices, then

\[ \sigma_1(UA) = \sigma_1(A) \] (8.25)
\[ \sigma_1(AV) = \sigma_1(A) \] (8.26)

This result shows that unitary matrices are norm invariant. Also, if the A matrix is unitary, then \( \lambda_i(A^H A) = 1 \) for all i, and

\[ \sigma(A) = \sigma_1(A) = \sigma(A) = 1 \] (8.27)

Finally, some other useful properties are:

\[ \sigma_i(\alpha A) = |\alpha| \sigma_i(A) \quad \text{where } \alpha = \text{scalar} \]
\[ \bar{\sigma}(A+B) \leq \bar{\sigma}(A) + \bar{\sigma}(B) \quad \text{(Triangle Inequality)} \]
\[ \bar{\sigma}(AB) \leq \bar{\sigma}(A) \bar{\sigma}(B) \]
\[ \sigma(AB) \geq \sigma(A) \sigma(B) \]

8.3.4.3 Singular Value Decomposition

Given the definition and properties of singular values, the question of how they can be efficiently computed remains. Singular value decomposition not only provides a method by which singular values can be calculated, but also provides some geometric insights into the directionality that the operator A has on the input vector x (given \( y = Ax \)).

Let A be any \( m \times n \) dimensional matrix (\( A \in \mathbb{F}^{m \times n} \)). The factorization of the A matrix in the following manner
\[ A = U \Sigma V^H \] (8.28)

is called Singular Value Decomposition (SVD). In the above equation, the singular values are the diagonal elements of the matrix \( \Sigma \) (\( \Sigma \in \mathbb{R}^{n \times n} \)), are non-negative and real, and are arranged in descending order,

\[ \Sigma = \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix}, \quad n \geq m \] (8.29a)

or

\[ \Sigma = \begin{bmatrix} \Sigma_1 & 0 \end{bmatrix}, \quad n \leq m \] (8.29b)

where

\[ \Sigma_1 = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_k \end{bmatrix}, \quad k = \min(m, n) \] (8.29c)

The \( U \) and \( V \) matrices of Equation (8.28) are unitary (i.e., \( U^H U = I \) and \( V^H V = I \)) with the column vectors

\[ U = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix} \in \mathbb{F}^{m \times m} \] (8.30a)

\[ V = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix} \in \mathbb{F}^{n \times m} \] (8.30b)

If \( A \) is real, then \( U \) and \( V \) are also real. The columns of \( U, u_i \), are the eigenvectors of \( AA^H \) and the columns of \( V, v_i \), are the eigenvectors of \( A^H A \), and are thus the left and right singular vectors, respectively, of the matrix \( A \).

Singular value decomposition displays the directionality of the linear transformation, \( y = A x = U \Sigma V^H x \). Since the spectral norm of a unitary matrix is one, then the singular
value decomposition can be thought of as a rotation \((V^H)\) followed by a scaling \((\Sigma)\), in turn followed by another rotation \((U)\) on the input \(x\). If the input \(x\) is assumed to lie in a unit (hyper)sphere (or \(|x|_2 = 1\)), then \(A\) can be thought of, geometrically, as a transformation from the unit (hyper)sphere to a (hyper)ellipsoid, where the lengths of the semi-axes of the (hyper)ellipsoid are the singular values of \(A\).

As an example, consider a 2-dimensional example (from Reference 80), where

\[
A = \begin{bmatrix}
0.8712 & -1.3195 \\
1.5783 & -0.0947
\end{bmatrix}
\]

The singular value decomposition of the \(A\) matrix yields

\[
U = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}, \quad \Sigma = \begin{bmatrix}
2 & 0 \\
0 & 1
\end{bmatrix}, \quad V = \frac{1}{2} \begin{bmatrix}
\sqrt{3} & 1 \\
-1 & \sqrt{3}
\end{bmatrix}
\]

The rotation and scaling for this example is illustrated in Figure 8.4\(^8\). Since we are dealing with a 2-dimensional space \((A \in \mathbb{R}^{2\times2})\), we can visualize the input as a unit circle and the output as an ellipsis, where the semi-major axis has a length of 2 and the semi-minor axis has unit length, corresponding to the maximum \((\sigma = 2)\) and minimum \((\sigma = 1)\) singular values, respectively.

A similar view of singular value decomposition can be obtained from an input-output perspective. Figure 8.5 shows the singular value decomposition of the operator \(A\) in block diagram form. The vector \(v_1\) (\(v_n\)) corresponds to the highest (lowest) gain input direction (i.e., the input direction with the most amplification of the input), and \(u_1\) (\(u_n\)) are the largest (smallest) gain output directions (or, the output directions in which the inputs are most (least) effective.

144
Figure 8.4 Example Geometric Visualization of SVD
Figure 8.5 Input-Output View of SVD
8.3.4.4 Importance of Singular Values

Singular values are useful for several reasons. First, singular values can be viewed from a geometric perspective of how the matrix $A$ amplifies the input and in what directions. The maximum and minimum singular values provide an upper and lower bound, respectively, on the mapping of the input. Thus, $\sigma(A)$ gives the worst case output to bounded inputs. Second, the minimum singular value $\sigma(A)$ provides a measure of how near the matrix $A$ is to being singular or rank deficient. One interpretation of the minimum singular value of the matrix $A$ is that it represents the distance between $A$ and its nearest singular matrix.

Singular values are a better measure of the near singularity of a matrix than the eigenvalues or the determinant of the matrix. Consider the simple illustration with a non-singular matrix $M$, whose inverse is given by

$$M^{-1} = \begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix}$$

The determinant of $M^{-1}$ is equal to one, and both eigenvalues are also unity. The singular values of $M^{-1}$ as well as $M$ are approximately $\tilde{\sigma} \approx 100$ and $\sigma \approx .01$. However, a small perturbation $\Delta$ could make this matrix singular. This result is indicated by the relatively small minimum singular value. In addition, the large difference in value between the maximum and minimum singular values indicate that this matrix is also ill-conditioned.

Consider the singularity of the sum

$$M^{-1} + \Delta = \begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix} + \Delta$$

This sum could be made singular with the small perturbation
\[ \Delta = \begin{bmatrix} 0 & 0 \\ .01 & 0 \end{bmatrix} \]

The maximum singular value of \( \Delta \) is \( \tilde{\sigma}(\Delta) = .01 \). We see that the sum of \( M^4 \) and \( \Delta \) goes singular for a perturbation whose maximum singular value is near the minimum singular value of \( M^4 \). The maximum singular value of a non-singular matrix is always equal to the inverse of the minimum singular value of its inverse (Equation 8.23). Thus, we see that the sum \( M^4 + \Delta \) goes singular for a perturbation whose maximum singular value is near the inverse of the maximum singular value of \( M \). Third, singular value decomposition can be utilized to determine the rank of the matrix \( A \). The rank of matrix \( A \) is equal to the non-zero singular values along the diagonal of the \( \Sigma \) matrix. Fourth, the singular values describe the "size" or "gain" of the matrix. Assuming that the \( A \) matrix transforms the input \( x \) via the operation \( y = Ax \), and that the input vector has unit length, \( \| x \|_2 = 1 \), then we define \( A \) to be large if \( \sigma(A) \gg 1 \), and small if \( \tilde{\sigma} \ll 1 \). In addition, efficient algorithms (in particular, using singular value decomposition) are available for the computation of the singular values.

Singular values have also become an important tool in the design and analysis of multivariable control systems. For single-input single-output (SISO or scalar) systems, the concept of "gain" is synonymous with the classical Bode magnitude plot. This Bode magnitude is equal to the amount of amplification of an input sinusoidal signal by the system. For multiple-input multiple-output systems with the transfer function matrix \( M \), the extension of the Bode gain is the singular values of \( M \) evaluated at a frequency on the imaginary axis, \( M(j\omega) \). (When the singular values are determined for the transfer matrix
$M(j\omega)$ as a function of frequency, the singular values are also sometimes referred to as principal gains.) These singular values of the system are given by $\sigma[M(j\omega)]$. Thus, we replace the SISO Bode gain with a range of singular values, bounded between $\bar{\sigma}[M(j\omega)]$ and $\underline{\sigma}[M(j\omega)]$, for the MIMO case. The use of singular values provides a measure of the amplification of the input signal space. Consider the transfer function matrix $M$ evaluated at some frequency $M(j\omega)$. The largest amplification that a sinusoidal signal of frequency $\omega$ will receive as it passes through the system $M$ is equal to the maximum singular value of $M$ at $s = j\omega$. Thus, the singular values give upper and lower limits of the magnification that the system $M(s)$ imparts on the input space (i.e., maximum and minimum magnification bounds of the system output space). A SISO system has only one singular value $\sigma[M(j\omega)]$ since $M(s)$ is a scalar transfer function. For a SISO system then, the maximum singular value, $\bar{\sigma}[M(j\omega)]$, is equivalent to the Bode gain $|M(j\omega)|$. The maximum gain that a stable linear system $M(s)$ can deliver is then the supremum of the maximum singular values taken over all frequencies.

The relevances of singular values to MIMO control system design are summarized as follows. First, since singular values are the generalization of the Bode magnitude plot to multivariable systems, the singular values of $M(j\omega)$, in particular $\bar{\sigma}(j\omega)$ and $\underline{\sigma}(j\omega)$, are useful for "loopshaping". Loopshaping can be defined as a compensator design technique for shaping the loop gain such that properties such as performance (e.g., command following), disturbance rejection, noise attenuation and robustness to high frequency unmodeled dynamics are achieved. Second, since singular values provide a better measure of the near singularity of a matrix and also provide upper and lower bounds on the "gain" amplification
of the input signal space, singular values can provide guarantees of stability robustness for a system subject to perturbations (i.e., uncertainties). And third, since the maximum singular value can be interpreted as the maximum magnification of the input, it is useful in assessing the worst case response of a system to bounded inputs. These topics will be discussed in further detail in later sections.

8.3.5 SIGNAL AND FUNCTION NORMS

As described in the previous section, singular values give the transforming amplification that a matrix (or a system evaluated at some frequency) has on its input. Similarly, norms basically are a measure of the "size" of some element. These elements could be vectors, matrices, scalar or vector signals and even transfer functions of a system. Also, several norms can be defined for each of these elements, such as the 2-norm or the ∞-norm. Specifically, the 2-norm and the ∞-norm are of relevance here. In this section, we examine the norms on signals and transfer functions.

8.3.5.1 Signal Norms

Consider a scalar function f(t) of some real variable t. This function could represent some time signal. In general, we can define a p-norm of that signal, ∥f(t)∥_p, as

\[ ∥f(t)∥_p = \left\{ \int_{-\infty}^{\infty} |f(t)|^p \, dt \right\}^{1/p} \]  \hspace{1cm} (8.31)

According to this definition, then the 2-norm of f(t), ∥f(t)∥_2, is

\[ ∥f(t)∥_2 = \left\{ \int_{-\infty}^{\infty} |f(t)|^2 \, dt \right\}^{1/2} \]  \hspace{1cm} (8.32)
Thus, the 2-norm of $f(t)$ describes square-integrable functions of time, and characterizes the root-mean-square (RMS) values of the signal. The square of the signal 2-norm is

$$
\| f(t) \|_2^2 = \int_{-\infty}^{\infty} |f(t)|^2 \, dt
$$

(8.33)

and is defined as the "energy" contained in $f(t)$. Thus, the 2-norm of a signal can also be interpreted as a function (in particular, the square root) of the amount of energy in $f(t)$.

The $\infty$-norm of $f(t)$, $\| f(t) \|_{\infty}$, turns out to be the maximum of $f(t)$ over $t$, or

$$
\| f(t) \|_{\infty} = \text{ess sup}_t f(t)
$$

(8.34)

where ess sup denotes the "essential supremum", which basically means the maximum value of the signal. This norm specifies the maximum (or peak) magnitude of the signal $f(t)$.

If we let $x(t)$ represent a vector of signals that are functions of $t$, the 2-norm of $x(t)$ is defined as

$$
\| x(t) \|_2 = \left( \int_{-\infty}^{\infty} \text{trace} [x(t)x^T(t)] \, dt \right)^{1/2} = \left( \int_{-\infty}^{\infty} x^T(t)x(t) \, dt \right)^{1/2}
$$

(8.35)

Thus, the 2-norm of $x(t)$ is found by integrating the sum of the squared elements of the time response, and can be interpreted as the square root of the energy of the vector-valued signal $x(t)$.

8.3.5.2 Transfer Function Norms

Next, we let $G(s)$ represent the transfer function (matrix) of some system $G$. The 2-norm of $G(s)$ can be found by integrating the sum of the squared elements’ frequency response magnitudes,

where $G^H(j\omega)$ is the complex conjugate transpose of the frequency response matrix. The $\infty$-
\[ |G(j\omega)|_2 = \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace} [G^H(j\omega)G(j\omega)] d\omega \right\}^{1/2} \] (8.36)

norm of \( G(s) \) is essentially the peak maximum singular value of \( G(j\omega) \) over frequency,

\[ |G(j\omega)|_\infty = \text{ess sup} \sigma[G(j\omega)] \] (8.37)

Thus, \( |G(j\omega)|_\infty \) can be interpreted as the maximum "gain" of the transfer function evaluated along the imaginary axis. The \( \infty \)-norm is also an induced norm that is induced by the 2-norms of the input and output signal spaces. Let \( G \) be the (linear) operator that transforms \( u \) into \( y \) via the output relation \( y(s) = G(s)u(s) \) in the frequency domain, with the counterpart via convolution \( y = G * u \) in the time domain. Then, the \( \infty \)-norm of \( G \) is

\[ |G|_\infty = \sup_{u \neq 0} \frac{|y|_2}{|u|_2} = \sup_{u \neq 0} \frac{|Gu|_2}{|u|_2} \] (8.38)

Or, another way, the \( \infty \)-norm of the system \( G \) is the worst case \( |y|_2 \) taken over all the input signals \( u \) satisfying \( |u|_2 \leq 1 \),

\[ |G|_\infty = \sup_{|u|_2 \leq 1} |y|_2 = \sup_{|u|_2 \leq 1} |Gu|_2 \] (8.39)

Thus, \( |G|_\infty \) can be interpreted as the worst case square root energy of the output for unit energy input to the system.

8.3.6 SIGNAL AND FUNCTION SPACES

We assume that signals and transfer functions can be placed in certain function spaces, which essentially are defined as having certain properties of the signals or transfer functions. These function spaces can be defined in the time or frequency domain.

The spaces that are of relevance in this report are the Lebesque (L) space and the
Hardy (H) space. The Lebesgue spaces contain functions that can be defined on a line; for example, time \( t \) in the time domain, and on the imaginary axis \( j\omega \) in the frequency domain. Hardy spaces contain functions that are defined and bounded in a half plane in the frequency domain or whose functions are zero for either \( t > 0 \) or \( t < 0 \) (e.g., causal systems) in the time domain. These spaces are usually denoted with subscripts \((L_p \text{ or } H_p)\), where the subscript \( p \) gives the type of norm defined on that space. For example, the subscript 2 refers to spaces containing square-integrable functions and the \( \infty \) subscript refers to those spaces containing bounded functions. (We say that the function \( X(t) \) of the real variable \( t \), \( -\infty \leq t \leq \infty \), is square-integrable if

\[
\int_{-\infty}^{\infty} \text{Trace}[X^*(t)X(t)] \, dt < \infty
\]

(8.40)

where \( X \) maps the input space or domain set of all real numbers, \( \mathbb{R} \), (here, all real values of \( t \)), into the output space or range of the function, i.e., \( X(t) \) can be a complex matrix-valued function. Or, \( X : \mathbb{R} \to \mathbb{C}^{m \times n} \).) Some pertinent spaces defined in the time and frequency domains are now given.

8.3.6.1 Time-Domain Spaces

Let \( \mathbb{R} \) denote the set of all real numbers \( t \), \( -\infty \leq t \leq \infty \), i.e., \( \mathbb{R} = (-\infty, \infty) \). The Lebesgue space \( L_2(-\infty, \infty) \) is a linear space of matrix-valued functions defined on \( \mathbb{R} \) with the inner product

\[
\langle X, Y \rangle = \int_{-\infty}^{\infty} \text{Trace}[X^*(t)Y(t)] \, dt
\]

(8.41)

Thus, \( L_2(\mathbb{R}) \) defines the space of square-integrable (in the sense of Lebesque) functions on
For scalar functions of time, $f(t)$, $L_2(\mathbb{R})$ contains square-integrable functions defined on $\mathbb{R}$ with the (bounded) norm

$$
\| f(t) \|_2 = \left( \int_{-\infty}^{\infty} |f(t)|^2 \, dt \right)^{1/2} < \infty
$$

(8.42)

Thus, this space contains all time signals which have bounded energy. Or, another interpretation, functions in this space can be characterized basically as those that decay in time. The space $H_2(\mathbb{R})$ is a subspace of $L_2$ which has functions that are zero for $t < 0$ (i.e., the $H_2$ space contains causal functions of $t$). This space is also denoted as $L_2[0, \infty)$. The space $L_{\infty}(\mathbb{R})$ contains the set of all (essentially) bounded functions of time, with the norm

$$
\| f(t) \|_{L_{\infty}} = \text{ess sup}_{t} |f(t)| < \infty
$$

(8.43)

This definition can easily be extended to vector spaces.

As an example, consider the two scalar signals $f_1(t) = e^{\pi t} \sin t$, and $f_2(t) = 1 - e^{\pi t}$, with $t \geq 0$ in both cases. These signals are illustrated in Figure 8.2. The signal $f_1(t)$ belongs to both $L_2(\mathbb{R})$ and $L_{\infty}(\mathbb{R})$ since the function is square-integrable and since its maximum magnitude is bounded. The function $f_2(t)$ belongs to $L_{\infty}(\mathbb{R})$ since it is bounded in time, but not to $L_2(\mathbb{R})$ since it is not square-integrable (i.e., the function does not decay to zero as time increases). If the function $f_1(t)$ is further defined to be causal (i.e., $f_1(t) = 0$ for $t < 0$), then it would also belong to the $H_2(\mathbb{R})$ space.

### 8.3.6.2 Frequency-Domain Spaces

Let $j\mathbb{R}$ denote the imaginary axis, where $\mathbb{R}$ represents the set of frequencies $-\infty < \omega < \infty$. A point on the imaginary axis is then given by $j\omega$. The Lebesque space $L_2(j\mathbb{R})$
contains matrix-valued functions which are square-integrable (in the sense of Lebesgue) on the imaginary axis, with the inner product

\[
<F, G> = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}[F^*(j\omega) G(j\omega)] d\omega
\]  

(8.44)

The space \( H_2(C, C^{\infty}) \) is a subspace of \( L_{\infty}(jR, C^{\infty}) \) with functions \( F(s) \) that are analytic in \( \text{Re} s > 0 \) (i.e., in the right half plane) and are square-integrable,

\[
\sup_{\sigma > 0} \int_{-\infty}^{\infty} \text{Trace}[F^*(\sigma + j\omega) F(\sigma + j\omega)] d\omega < \infty
\]  

(8.45)

These properties imply that functions \( F(s) \) in \( H_2 \) have no poles in the right half plane (RHP) (stable) and whose frequency response satisfies \( \lim_{\omega \to \infty} F(j\omega) = 0 \), i.e., rolls off at high frequencies. Therefore, the rational transfer functions are strictly proper (the order of the numerator of \( F(s) \) is less than the order of denominator of \( F(s) \)). The space \( L_{\infty}(jR, C^{\infty}) \) contains functions \( F(s) \), for \( s = j\omega \), that are (essentially) bounded on the imaginary axis \( jR \), with

\[
\|F\|_\infty = \text{ess sup}_{\omega} \sigma[F(j\omega)] < \infty
\]  

(8.46)

The space \( H_{\infty}(C, C^{\infty}) \) is a subspace of \( L_{\infty}(jR, C^{\infty}) \) which contains functions \( F(s) \) that are analytic and bounded in the open RHP, i.e., \( F(s) \) is stable, and

\[
\sigma[F(s)] < \infty
\]  

(8.47)

in the open RHP (\( \text{Re}(s) > 0 \)). Thus, the \( H_{\infty} \) space contains all proper and stable transfer functions.

### 8.3.7 Engineering Relevances of Norms and Spaces

There are several engineering relevances that we can draw from the previous
mathematical concepts. These relevances pertain to singular values, spaces and norms.

First, singular values are useful in multivariable system analysis because they show the magnitude and direction of transformation the system has on the input space. In addition, the maximum singular value of a system evaluated at a frequency \( s = j\omega \) is equivalent to the \( \infty \)-norm of the system.

Regarding spaces, both \( \mathbf{RH}_2 \) and \( \mathbf{RH}_\infty \) are spaces of stable transfer functions, with \( \mathbf{RH}_2 \) being a subspace of \( \mathbf{RH}_\infty \). (The prefix \( \mathbf{R} \) denotes real rational functions.) From a frequency domain perspective, functions in \( \mathbf{RH}_2 \) roll off at high frequency because of the square-integrable property, whereas functions in \( \mathbf{RH}_\infty \) could have some nonzero gain at high frequency. This property implies that transfer functions in \( \mathbf{RH}_2 \) are stable and strictly proper, and those in \( \mathbf{RH}_\infty \) are stable and proper. For a state-space interpretation, a transfer function in \( \mathbf{RH}_2 \) assumes the realization

\[
\dot{x} = Ax + Bu \\
y = Cx
\]  

(8.48)

where \( A \) is stable and the system is strictly proper (\( D = 0 \)), and a transfer function (matrix) in \( \mathbf{RH}_\infty \) takes the form

\[
\dot{x} = Ax + Bu \\
y = Cx + Du
\]  

(8.49)

where the system is stable and proper.

A couple of relevances can be made for the 2-norm. First, let \( y = Gu \), where the input \( u \) is assumed to be a vector white noise process with the autocovariance of unit intensity \( \mathbb{E}[uu^T] = I \), \( y \) is the output and \( G \) represents the system. The RMS output power of \( y \) is \( \{\mathbb{E}[y^Ty]\}^{1/2} \), which is equivalent to the 2-norm of \( G \), \( \|G\|_2 \). Thus, the 2-norm gives
a measure of the RMS output to white noise. For example, the Kalman-Bucy filter minimizes the mean square estimation error, which is equivalent to minimizing the expected values of the squares of the 2-norm of the estimation error,

$$E\{\|e(t)\|_2^2\} = E\{\int_0^1 [e^T(t)e(t)] \, dt\}$$  \hspace{1cm} (8.50)

which is the same as minimizing the square of the closed-loop system transfer function 2-norm, for the same filter formulation. In this case, the $H_2$ optimal estimator is identical to the steady-state Kalman filter.

We can also draw two engineering relevances for the $\infty$-norm. Again, let $y = Gu$, where $u$ is the input signal, $G$ is the system and $y$ is the output. The square root of energy in a signal $u$ is the 2-norm of the signal, $\|u\|_2$. The infinity norm of the system $G$, $\|G\|_\infty$, is equal to the worst case square root of the energy out for unit energy in. Similarly, the infinity norm is also induced by the semi-norm of power and in addition by the semi-norm of power spectral density. (Recall that bounded power signals may be persistent signals in time such as sines and cosines. And recall that power spectral density signals represent signals with fixed or bounded spectral characteristics; white noise is the limit of such signals. Such signals can be passed through shaping filters to produce signals with any desired frequency content.) Thus, $\|G\|_\infty$ is equal to the worst case power out for unit power in, and, likewise, is equal to the worst case "spectral density" out for unit "spectral density" in. That is, the (square root of the) worst case energy out, the worst case power out and the worst case "spectral density" out are one and the same number when measured in their own norm or semi-norm. The first engineering relevance of the infinity norm is then this: If a control designer seeks to design a controller for minimizing the transfer of energy (i.e., the worst case
output energy), or the transfer of power or the transfer of white noise through a closed-loop system $G$ which depends on the controller, then the controller should be designed so that the infinity norm of $G$, $\|G\|_\infty$, is minimized over all controllers that stabilize the closed-loop system $G$.

The second engineering relevance of the infinity norm is that the infinity norm possesses the submultiplicative property which is essential in establishing robustness of uncertain systems. Consider the cascade of two systems $M_1$ and $M_2$ as given by $y(s) = M_2(s)M_1(s)u(s)$. The infinity norm satisfies the inequality:

$$\|M_2M_1\|_\infty \leq \|M_2\|_\infty \|M_1\|_\infty \tag{8.51}$$

This inequality says, in particular, that the energy gain through both systems $M_1$ and $M_2$ is less than or equal to the product of the energy gain through each individual system. In contrast, the 2-norm does not satisfy the submultiplicative property of Equation (8.51). It does not satisfy this property for the simplest of systems. For example, let $M_1$ and $M_2$ be the same SISO system defined by the transfer function

$$M_1(s) = M_2(s) = \frac{1}{\tau s + 1}, \quad \tau > 0$$

The 2-norms are given by

$$\|M_2M_1\|_2 = \frac{1}{2\sqrt{\tau}}$$

$$\|M_2\|_2 \|M_1\|_2 = \frac{1}{\sqrt{2\tau}} \cdot \frac{1}{\sqrt{2\tau}} = \frac{1}{2\tau}$$

which violates the submultiplicative property for $\tau > 1$, that is,
\[ |M_2M_1|_2 = \frac{1}{2\sqrt{\tau}} > |M_2|_2 |M_1|_2 = \frac{1}{2\tau}, \quad \tau > 1 \]

This inequality can be made as large as desired.

Robust stability of the closed-loop system, where \( M_i \) is a nominally stable closed-loop system, depends on the stability of the closed-loop system depicted in Figure 8.6 for all uncertainties \( \Delta \) satisfying the known bound on the 2-norm \( |\Delta|_2 \). The stability of the closed-loop system presented above cannot be established from knowledge of the 2-norm \( |M_i|_2 \) and the bound on the 2-norm \( |\Delta|_2 \) of the uncertainty \( \Delta \). What is needed is the 2-norm bound \( |\Delta M_i|_2 \) on the product \( \Delta M_i \) for all uncertainties \( \Delta \); but this quantity cannot be computed since it depends on the particular dynamics of \( \Delta \) and since it cannot be bounded by the product of \( |M_i|_2 \) and \( |\Delta|_2 \), (the submultiplicative property is invalid for 2-norms). Stability of the closed-loop system shown in Figure 8.6 depends on how much gain cycles through the loop from \( u \) going in to \( y \) coming out. That gain can be bounded by using infinity norms \( |\cdot|_\infty \) but not by using 2-norms \( |\cdot|_2 \). Consequently, the infinity norm is a tool for establishing robust stability of uncertain systems but the 2-norm is not applicable.

### 8.4 ROBUSTNESS

Feedback controllers are necessary in the design of most dynamic physical systems. The control system ensures stability of the closed-loop system, rejects the influence of disturbances, and allows the system to achieve certain performance objectives. The controller must accomplish these goals in the presence of modeling errors and uncertainties.
Figure 8.6. Closed-Loop System with Uncertainties
The design of feedback controllers is based on some nominal model of the physical system (or plant). However, it is impossible to perfectly model the dynamics of most physical systems mathematically. Reasons for modeling errors and plant uncertainties include: (1) partial or imperfect knowledge of the system, (2) simplifying assumptions in the mathematical description, such as neglected high frequency dynamics, (3) linearization of a nonlinear dynamical model, and (4) time-varying plant behavior. As a result, the model is an approximation and will not exactly match the true system's behavior. If the controller is designed to work with the nominal model and does not account for the possible deviations of the true system from the nominal model, the system may go unstable or not be able to meet the performance goals. Tolerance to such model uncertainties and disturbances is termed robustness.

8.4.1 UNCERTAINTIES

In general, there are two main classes of uncertainties: structured and unstructured. Structured uncertainties affect the plant in some known way. There is no interaction or coupling between structured uncertainty elements. Structured uncertainties can arise as parameter variations from inaccurate knowledge of the coefficients in the differential equations that describe the physical system. For example, consider a system modeled as a second-order differential equation

\[
\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = u
\]  

(8.52)

If the the coefficients \(a\) and \(b\) of the above equation are subject to variations \(\delta_a\) and \(\delta_b\) relative to their nominal values, then Equation (1) becomes
\[
\frac{d^2x}{dt^2} + (a + \delta_a) \frac{dx}{dt} + (b + \delta_b)x = u
\] (8.53)

The nominal and perturbed systems are illustrated in block diagram form in Figures 8.7(a) and 8.7(b), respectively. The uncertainties \(\delta_a\) and \(\delta_b\) can be represented in the block diagonal (i.e., structured) form as depicted in Figure 8.7(c).

Unstructured uncertainties have no particular structure in the plant model, and are used to represent unspecified dynamics; they also allow for interactions (or coupling) between the assumed uncertainty elements. Unstructured uncertainties arise from many sources, such as unmodeled high frequency dynamics, nonlinearities, simplifying assumptions and other neglected phenomena. Since uncertainties are characterized mathematically with expected bounds, unstructured uncertainties can be represented as the set of transfer functions bounded in a disk on the imaginary plane (Nyquist perspective) or as those with bounded gain and unknown phase characteristics (Bode viewpoint). An illustration of unstructured uncertainties is conceptualized in Figure 8.8. In almost all applications, the uncertainty model of the physical system will be a mixture of structured and unstructured uncertainties (i.e., mixed model uncertainty). In general, the uncertainty elements may be real or complex, scalars or matrices, fixed or functions of frequency.

8.5 MODERN ROBUST CONTROL TECHNIQUES

Classical control design practice accounts for model uncertainty by requiring adequate gain and phase margins based on Bode and Nyquist analysis. Such an approach treats uncertainties in only a general sense without specifying the form of individual uncertainty.
Figure 8.7. Example of Structured Uncertainty
Figure 8.8. Unstructured Uncertainty Descriptions
models. As a result, an accurate measure of robustness is not produced.

Since the 1970's, many modern robust control methods have evolved to handle the effects of model uncertainties in the design of controllers, as discussed in References 81 and 82. These include such methods as the Linear Quadratic Gaussian/Loop Transfer Recovery (LQG/LTR) approach, loop shaping techniques, $H_\infty$ and $\mu$ methods, game theoretic or minimax methods, Lyapunov robust stabilization, the Quantitative Feedback Theory (QFT) of Horowitz, and robust eigenstructure assignment approaches, to name a few. Mu-Synthesis is a multiple-input multiple-output (MIMO) robust control design technique that combines $H_\infty$ Control with $\mu$-Analysis in an iterative, two-step optimization process. In the next section, the $H_\infty$ and $\mu$ methods of robust control will be reviewed.

8.6 $H_\infty$ AND $\mu$ METHODS OF ROBUST CONTROL

8.6.1 DESIGN AND ANALYSIS FRAMEWORK FOR ROBUST CONTROL

For the application of the $H_\infty$ and $\mu$ methods, the control design and analysis problem is cast into the following framework. The physical system is modeled as a finite-dimensional, linear, time-invariant system. This model includes a description of the uncertainties and their structure, and is illustrated in Figure 8.9. The uncertainties are represented by $\Delta$, $P$ is the generalized plant, and $K$ denotes the controller. Any linear combination of inputs, outputs, perturbations, and controller can be arranged to fit this interconnection framework. The input vector, $d$, denotes all external disturbances acting on the plant, such as commands, physical disturbances and sensor noise. The output vector, $e$, contains, in particular, the regulated performance variables. Feedback measurements are denoted by the vector $y$, and

165
Figure 8.9. Framework for Robust Control
\( u \) represents the control vector that is input to the plant actuators. The input disturbances, output errors and uncertainties are normalized to unity bounds (\( \|d\|_2 \leq 1 \), \( \|e\|_2 \leq 1 \), and \( \hat{\phi}(\Delta) < 1 \)) by frequency dependent weights, which are incorporated into the generalized plant model, \( P \).

### 8.6.2 ROBUST COMPENSATOR SYNTHESIS AND \( H_\infty \) CONTROL

For robust controller synthesis, the design problem is to find a controller that stabilizes the closed-loop system and provides tolerance to model uncertainties. The control problem's performance objective can be stated in terms of norms. For example, the application of the 2-norm to the control problem leads to \( H_2 \) Control, which is commonly known as LQG control. However, LQG does not guarantee robust stability. Attempts to modify the LQG approach to recover robustness (LQG/LTR) have had difficulties (in particular, with non-minimum phase systems and multiple uncertainties). On the other hand, the \( \infty \)-norm possesses attractive properties for application to robust control. First, the \( \infty \)-norm possesses the submultiplicative property, which was given in Equation (8.51). In contrast, the 2-norm does not satisfy the submultiplicative property. To establish robust stability of an uncertain system, the gain or amplification that occurs through the loop is important (using either MIMO Nyquist theory or the Small Gain Theorem). That gain can be bounded by using infinity norms, but not by using 2-norms. Another useful property of the \( \infty \)-norm for control is derived from its definition. The infinity norm of the system \( M \), \( \|M\|_\infty \), can be interpreted as the worst case square root energy of the output for unit energy input to \( M \). Thus, the infinity norm is useful if the performance objective is to minimize the transfer of disturbance energy through the system. For these reasons, the \( \infty \)-norm is a
powerful tool for robust control, and led to the initial development of $H_\infty$ Control theory (Zames\textsuperscript{75}).

To formalize the $H_\infty$ control problem, consider the control framework in Figure 8.10. In this diagram, $P(s)$ represents the generalized plant transfer matrix and $K(s)$ is a linear transfer matrix description of the controller. The plant $P$ has two inputs ($W$ and $u$) and two outputs ($z$ and $y$), and absorbs weighting functions on the (disturbance) inputs, $W$, and the (performance) outputs, $Z$. Weighting functions on the uncertainties are also absorbed into the generalized plant $P$. Closing the feedback control loop, the transformation (or mapping) of the input $W$ to the output $Z$, $T_{zw}$, is called the lower linear fractional transformation, $F_1(P,K)$,

$$ Z = T_{zw} W = F_1(P,K) W $$  \hspace{1cm} (8.54)

where

$$ F_1(P,K) = P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21} $$ \hspace{1cm} (8.55)

Here, $P_{ij}$ are the four submatrices formed by partitioning the matrix $P$.

The $H_\infty$ control problem is then to find a real-rational, proper controller $K$ that makes the closed-loop system internally stable and that minimizes the infinity-norm of the input/output map $T_{zw}$, i.e.,

$$ \min_K \| T_{zw} \|_\infty = \min_K \| F_1(P,K) \|_\infty $$ \hspace{1cm} (8.56)

Thus, $H_\infty$ control yields a closed-loop system $M$ whose $\infty$-norm is minimized over all controllers that stabilize the closed-loop system. The $H_\infty$ controller can have several interpretations. In one view, $H_\infty$ control minimizes the energy gain between the exogenous inputs $W$ and the generalized error outputs $Z$. Another interpretation is that $H_\infty$ control
\begin{align*}
\begin{bmatrix} z_e \end{bmatrix} &= Z \\
W &= \begin{bmatrix} v' \\
\nu' 
\end{bmatrix}
\end{align*}

Figure 8.10. Controller Synthesis Diagram
minimizes the largest of the maximum singular values of the transformation $T_{zw}$ found over all frequencies. As such, the solution minimizes the maximum possible amplification of the (disturbance) inputs through the system. So, in general, $H_\infty$ Control may be thought of as a "worst-case" minimization design approach.

The $H_\infty$ optimal control (central) solution is obtained via the solutions of two algebraic Riccati equations$^{77-78}$. The controller $K$ has the same order as the generalized plant, and also has a separation structure similar to that for LQG ($H_2$) controllers. That is, the $H_\infty$ controller contains both controller gains and observer (filter) gains. Several solution methods to the $H_\infty$ problem have been developed; however, the state-space solution given in References 77-78 is the most computationally practical.

8.6.3 ANALYSIS AND THE STRUCTURED SINGULAR VALUE

For analysis purposes, the control loop is closed and the controller is incorporated in with the generalized plant to form the closed-loop system transfer matrix, $M$. The analysis framework is shown in Figure 8.11, where $\Delta$ represents the model uncertainty. The system $M$ has two inputs ($v, d$) and two outputs ($z, e$). The input/output map from $d$ to $e$ can be expressed as the upper linear fractional transformation, $F_u(M, \Delta)$,

$$e = F_u(M, \Delta)d$$

where

$$F_u(M, \Delta) = M_{22} + M_{21} \Delta (I - M_{11} \Delta)^{-1} M_{12}$$

and $M_i$ are matrix partitions of $M$. Tests for stability and performance analysis are now presented.
Figure 8.11. Analysis Framework
8.6.3.1 Nominal Stability (NS)

The nominal closed-loop system (i.e., no uncertainties) is said to be nominally stable if $M$ is stable. This property can be checked by calculating the eigenvalues of the closed-loop system $M$. If all eigenvalues have real parts less than zero, the system is nominally stable.

8.6.3.2 Nominal Performance (NP)

Let the inputs $d$ and the outputs $e$ be bounded: $|d|_2 \leq 1$ and $|e|_2 \leq 1$, and take $\Delta$ to be zero. The system satisfies nominal performance if

$$|M_{22}|_\omega = \sup_\omega |M_{22}(j\omega)| \leq 1 \quad (8.59)$$

In essence, the system is said to satisfy nominal performance if $M$ is stable and satisfies the performance requirements.

8.6.3.3 Robust Stability (RS)

Robust stability refers to the stability of the perturbed closed-loop system. In general, the system is robustly stable if the closed-loop feedback system is stable in the presence of parameter variations and other model uncertainties, $\Delta$. A discussion of the uncertainty, or $\Delta$, block is now given.

8.6.3.4 Uncertainty Descriptions

It is simplest to consider $\Delta$ to be complex with a block diagonal structure

$$\Delta = \text{diag}\{\Delta_1, \Delta_2, \ldots, \Delta_n\} \quad (8.60)$$

since this is convenient for analysis and synthesis. The individual uncertainties, $\Delta_i$, that make
up the $\Delta$ block may have one (or a combination) of several representations: (1) scalar block - uncertainty appears only once in the real system, such as a parameter uncertainty, (2) repeated scalar blocks - where the uncertainty appears multiple times in the real system, such as dynamic pressure, and (3) full blocks - examples of which include multivariable neglected dynamics and the performance block. Robust stability tests can be defined for various uncertainty descriptions, such as unstructured or structured uncertainties.

8.6.3.5 Robust Stability for Unstructured Uncertainty

In this case, $\Delta$ is a full uncertainty block. For bounded uncertainty, $\tilde{\sigma}(\Delta) < 1$, the closed-loop system satisfies robust stability if

$$\|M_{11}\|_\infty = \sup_{\omega} \tilde{\sigma}[M_{11}(j\omega)] < 1 \quad (8.61)$$

8.6.3.6 Robust Stability for Structured Uncertainty

In most practical circumstances, at least some of the structure of the uncertainty is known. If the analysis test for unstructured uncertainty given in Equation (8.61) were used for systems with structured uncertainty, the estimate for robust stability would be conservative. To reduce the conservatism, the structure of the uncertainty must be accounted for. In one case, assume the uncertainties are all repeated complex scalars ($\Delta = \delta_j$). The system is robustly stable if $\rho(M_{11}) \leq 1$, where $\rho(M_{11})$ is the spectral radius of $M_{11}$ (i.e., the magnitude of the maximum eigenvalue of $M_{11}$, $\max |\lambda(M_{11})|$).

Allowing for more generality in the uncertainty structure, assume that $\Delta$ takes the form
\[ \Delta = \left\{ \text{diag}[\delta_1 I_{n_1}, \ldots, \delta_i I_{n_i}, \ldots, \delta_N I_{n_N}, \Delta_j, \ldots, \Delta_j, \ldots, \Delta_m] : \delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j \times m_j} \right\} \]  

(8.62)

In other words, \( \Delta \) takes the form of complex-valued, block diagonal perturbations. Each \( \delta_i \) scalar block is repeated \( r_i \) times, and each \( \Delta_j \) full block has dimension \( m_j \times m_j \). Further, the uncertainties may be members of a bounded subset of those given above,

\[ B \Delta = \{ \Delta \in \Delta : \bar{\sigma}(\Delta) \leq 1 \} \]  

(8.63)

Then, the structured singular value \( \mu(M(j\omega)) \) of a complex matrix \( M(j\omega) \) with respect to the block structure \( \Delta \) is defined as\(^3\)

\[
\mu(M(j\omega)) = \left\{ \begin{array}{c}
\frac{1}{\min_{\Delta \in \Delta} \{ \bar{\sigma}(\Delta); \det(I - M(j\omega)\Delta) = 0 \}} \\
\text{or } 0, \det(I - M(j\omega)\Delta) = 0
\end{array} \right\}.
\]  

(8.64)

This says that \( \mu \) is the inverse of the smallest magnitude of a destabilizing perturbation of \( M \). The structured singular value is then a measure of robustness of the model to structured perturbations. In practice, the supremum of \( \mu \) over frequency gives the inverse of the minimum sized destabilizing perturbation. Thus the structured singular value provides a quantitative measure of how much uncertainty can be tolerated before the system becomes unstable. The system is said to be robustly stable if

\[
\sup_{\omega} \mu(M_{ij}(j\omega)) \leq 1
\]  

(8.65)

The optimization problem given in Equation (8.65) is not globally convex and the computation of \( \mu \) is difficult to perform. However, useful upper and lower bounds can be found for \( \mu \). For any perturbation set \( \Delta, \mu \) is bounded by

174
\[ \rho(M(j\omega)) \leq \mu(M(j\omega)) \leq \sigma(M(j\omega)) \quad (8.66) \]

This equation says that the structured singular value of \( M(j\omega) \) is bounded above by the maximum singular value of \( M(j\omega) \) and is bounded below by the spectral radius of \( M(j\omega) \). These bounds, however, are not tight, and are therefore not very useful for computation.

To improve the bounds, we introduce the transformation matrices \( U \) and \( D \), where \( U \) is a diagonal, unitary \( (U'U = I) \) matrix that belongs to the set \( U \) \( (U \in U) \), and the \( D \) matrix is diagonal and invertible and belongs to the set \( D \) \( (D^{-1} \exists D \in D) \). The sets \( U \) and \( D \) have the same structure as \( \Delta \), with the properties that \( \mu \) is \( U \)-invariant and is also invariant under a similarity transformation with \( D \), i.e.,

\[
\begin{align*}
\mu(UM) &= \mu(MU) = \mu(M) : \quad U \in U \\
\mu(DMD^{-1}) &= \mu(M) : \quad D \in D
\end{align*} \quad (8.67)
\]

These properties allow for a tighter set of bounds on \( \mu \),

\[
\max_{U \in U} \rho[\left[U(j\omega)M(j\omega)\right]] \leq \mu[M(j\omega)] \leq \inf_{D \in D} \sigma[\left[D(j\omega)M(j\omega)D^{-1}(j\omega)\right]] \quad (8.68)
\]

The equality on the lower bound is true for all \( M(j\omega) \) and all structures \( \Delta \), but the computational optimization problem of finding this lower bound has many local maxima and is therefore difficult to compute. The equality on the upper bound holds in some cases, but is not true in general. However, the optimization problem for the upper bound calculation is convex, and the D-scales can be utilized to obtain an upper bound value that is fairly close to the \( \mu \) value in most cases. So, an approximate answer to the \( \mu \) problem is

\[
\sup_{\omega} \mu[M(j\omega)] \leq \sup_{\omega} \inf_{D \in D} \|D(j\omega)M(j\omega)D^{-1}(j\omega)\|_\infty \quad (8.69)
\]

In practice, robust stability is checked using
\[ \sup_{\omega} \inf_{D \in \mathcal{D}} \tilde{\sigma}(D(\omega) M_{11}(j\omega) D^{-1}(\omega)) \leq 1 \quad (8.70) \]

**8.6.3.7 Robust Performance (RP)**

In general, robust performance means that the closed-loop system is stable and achieves the performance specifications in the presence of modeling uncertainties. The performance requirements are specified by means of weighting functions that are applied to the regulated output variables. The performance block, \( \Delta_{s+1} \), is the transfer matrix between the disturbance inputs, \( d \), and the error outputs, \( e \). The performance block is treated as a full uncertainty block, assuming \( \tilde{\sigma}(\Delta_{s+1}) \leq 1 \). The robust performance problem is converted into an equivalent robust stability problem by augmenting the (robust stability) uncertainty block,

\[ \Delta_{aug} = \{ \text{diag}(\Delta, \Delta_{s+1}); \Delta \in \Delta \} \quad (8.71) \]

The system exhibits robust performance if

\[ \sup_{\omega} \mu_{\Delta_{aug}}(M(j\omega)) < 1 \quad (8.72) \]

provided that \( d \) and \( e \) have unity bounded 2-norms and that \( \Delta \in B\Delta \). The upper bound for \( \mu \) is utilized in practice to check for robust performance,

\[ \sup_{\omega} \mu_{\Delta_{aug}}[M(j\omega)] \leq \sup_{\omega} \inf_{D \in \mathcal{D}} \tilde{\sigma}(D(\omega) M(j\omega) D^{-1}(\omega)) \leq 1 \quad (8.73) \]

If the upper bound with respect to the augmented uncertainty block is less than unity over all frequencies, then the closed-loop system is guaranteed to remain stable and satisfy the performance requirements in the presence of all the specified structured uncertainties in the set \( \Delta \). Note that this upper bound test has become an \( \infty \)-norm test.

176
8.6.4 MU-SYNTHESIS

The ultimate goal of designing a controller is to satisfy the robust performance criterion. This goal is met by minimizing the sufficient condition for robust performance given in Equation (24) over all stabilizing controllers, that is,

\[
\inf_{K} \sup_{\omega} \inf_{D(\omega)} \sigma[D(\omega)M(P,K)(j\omega)D(\omega)^{-1}] \tag{8.74}
\]

The computation above is done iteratively by solving for the controller \(K\) and the D-scales \(D(\omega)\), with the goal

\[
|DM(P,K)D^{-1}|_\infty \leq 1 \tag{8.75}
\]

The successive iterations are known as "DK Iterations".

The DK iteration process is summarized as follows. Consider some stabilizing controller \(K\). Closing the control loop yields the lower linear fractional transformation, \(F_L(P,K) = M(P,K)\). The D-scales yield the transformation \(DM(P,K)D^{-1}\). The upper bound is the infimum of the infinity norm of this latter transformation taken over all stabilizing controllers \(K\) and all admissible D-scales. This is an optimization problem in the two variables \(K\) and \(D\). For a given \(D\), we have an \(H_\infty\) control problem. For a fixed controller \(K\), an optimal D-scale can be computed. The D-scales are computed to minimize the upper bound approximation. That is, the largest singular value of the transformation is minimized at each frequency as a convex optimization problem in \(D\). The resulting \(D(\omega)\) is then fit with a stable, rational transfer function with a stable inverse. This iterative process is performed until the \(\infty\)-norm of \(DF_L(P,K)D^{-1}\) is close to its \(\mu\) value. This iterative optimization problem is convex when either variable is fixed, but is not globally convex and is not guaranteed to converge to the global optimal \(K\) and \(D\). However, the DK approximation has been shown
to be quite good in a large number of engineering applications, making the technique very useful for the design of robust controllers.

The engineering process for control design using $\mu$-Synthesis can be described in the following steps:

1. **Formulate the model of the plant,**
2. **Set up the uncertainty description for the model, including the uncertainty weights,**
3. **Design the weighting functions for the inputs and performance outputs,**
4. **Perform the DK iteration until convergence,**
5. **Analyze the resulting design for Nominal Stability, Nominal Performance, Robust Stability and Robust Performance,**
6. **If necessary, repeat steps 2 - 5 until the design requirements are satisfied.**

The first three steps above are the most critical steps in the design process. In the formulation of the plant and uncertainty models, there is a tradeoff between how much detail from the real system to include in the design model versus the complexity of the controller that is synthesized. A controller of too large dimension may be too cumbersome to implement in a practical flight computer. Also, a design model that is overly complex in terms of its uncertainty description may result in a compensator that is extremely robust to perturbations, but at the cost of performance. In addition, the selection of the input/output weights are important because they act like "tuning knobs" in the $H_\infty$ and $\mu$-Synthesis methods. There are many problem-dependent issues regarding the design of these weights that must be considered, as discussed in Reference 83.
Chapter 9. Rigid-Body Missile Autopilot Designs

With the background behind us, we are now ready to proceed with the autopilot designs. In this chapter, we examine two sets of classical and Mu-Synthesis autopilot designs for the missile described earlier. In this chapter, the airframe is assumed to be perfectly rigid. The classical design approach for both these sets was outlined in Chapter 7. The \( \mu \)-Synthesis design model differed between the two design sets as our design experience with the technique matured. These two sets of designs are now discussed and compared.

9.1 DESIGN SET 1

The first set of classical and \( \mu \)-Synthesis designs was performed for a preliminary case; with the missile's IRU package located 39 inches from the nose, forward of the location specified in Chapter 3. (The IRU location was shifted after the first set of autopilot designs was completed to reduce the effects of the structural vibrations on the sensors.) We discuss the classical design and results, the \( \mu \)-Synthesis design and results, then compare the resulting designs.

9.1.1 CLASSICAL MISSILE AUTOPILOT DESIGN

Recall that the design procedure from Chapter 7 is to set several autopilot gains and then perform classical control analysis to examine the open- and closed-loop system characteristics. The autopilot gains were first set by specifying values of \( \tau \) and \( \zeta \) to be 0.08 and 0.7, respectively, and a open-loop crossover frequency of one-third to one-fourth the
actuator bandwidth \((\omega_c = 50 \text{ rad/sec})\). Performance and stability analyses were then performed on the flight control system. The gain and phase margin specifications were satisfied at the low frequencies and the step time response was very quick, but the high frequency attenuation goal of \(20 \text{ db} \) attenuation at a frequency of \(300 \text{ rad/sec} \) was not met. The crossover frequency was decreased to \(25 \text{ rad/sec} \) to satisfy that requirement. The autopilot gains that were found were: \(K_A = 0.4614, K_I = 13.4623, K_R = 0.0220, \) and \(K_0 = 2.1939\). Figures 9.1 - 9.3 show the results of the performance and stability analysis at the nominal design condition. The acceleration response to a unit step acceleration command displayed in Figure 9.1 shows that the steady-state error and overshoot criteria are satisfied, and that the autopilot has a time constant of about \(0.186 \text{ sec} \). The gain and phase Bode plots of Figures 9.2 (loop opened at the actuator input) and 9.3 (open-loop at gyro output) verify stability margins that exceed the requirements at the low frequencies, and also attains the attenuation desired at high frequency.

### 9.1.2 MU-SYNTHESIS AUTOPILOT DESIGN

The \(\mu\)-synthesis design is performed by formulating a design model and uncertainties acting on that system, designing performance weights, and then successively iterating on the \(H_\infty\) controller and the optimal D-scales until \(\mu\) and/or the D-scales converge to the optimal solution. The \(\mu\)-synthesis technique provides a very useful tool to design a controller, however, the "real engineering" of this method is the development of an appropriate design model and weight selection.

We begin the design by forming the basic model of the airframe from the linear rigid
Figure 9.1 Acceleration Step Response, Classical Autopilot #1
Figure 9.2 Frequency Response at Actuator Input, Classical Autopilot #1
Figure 9.3 Frequency Response at Gyro Output, Classical Autopilot #1
body equations of motion discussed in a previous section. This model is a second-order system, where the angle of attack, $\alpha$, and the pitch rate, $q$, are the state variables. A second-order actuator model is also included in the design model to account for actuator dynamics.

Next, uncertainties acting on the system are defined. From an examination of the aerodynamic stability derivative variations over the range of angles of attack, the largest variations are on the pitching moment (270%) and normal force (15%) coefficients due to angle of attack. These variations are translated into weights (or scale factors) for the uncertainties acting on the $M_\alpha$ and $Z_\alpha$ terms. A third uncertainty is added from the pitch deflection to the pitch rate to account for high frequency unmodeled dynamics. We are essentially accounting for the flexible mode dynamics of the missile. The weight presented in Figure 9.4 represents that for the unmodeled dynamics. At low frequency, the unmodeled dynamic weight is relatively small since the dynamics are well described by the linear rigid body model, and ramped up at the higher frequencies where the flexible modes are assumed to act. To attain the high frequency gain attenuation requirement, the unmodeled dynamics weight is designed to have at least 20 db of gain at a frequency of 300 rad/sec. This type of weight is conservative since we cover to some extent the high frequency range instead of weighting just the specific flexible modes. We utilize this conservative unmodeled dynamics weight when the information regarding the high frequency modes are not available or the bending modes are uncertain.

Disturbances acting on the missile system are also considered. These may also include exogenous inputs, such as reference command signals. For example, the autopilot is forced by a guidance acceleration command. So, this command is treated as a disturbance.
Figure 9.4 Unmodeled Dynamics Weighting Function
input. Other disturbance inputs are assumed to be accelerometer bias and rate gyro drift. Nominal values are obtained from hardware specifications: \( A_{\text{bias}} = 0.001 \) g's and \( q_{\text{drift}} = 5 \) deg/hr.

Completing the Mu-synthesis design model, performance variables (i.e., the output error vector, \( e \)) are selected. These included: (1) the error between the commanded acceleration and the achieved acceleration, or the sensitivity function, (2) the achieved acceleration, or the complementary sensitivity function, (3) actuator fin rate, (4) actuator fin angle, and (5) actuator input signal. Weights for the actuator fin rate and fin deflection angle are taken to be the inverse of the actuator saturation limits. (Actuator rate limits are assumed to be 300 rad/sec and fin deflection limits are considered to be 40 degrees.) The weight on the input to the actuator is assumed to be fairly small (1E-5). The Mu-synthesis design model is illustrated in Figure 9.5.

The design process consists of selecting various sensitivity function and complementary sensitivity function weights, proceeding with the mu-synthesis, testing for robust performance, and then examining the time response and frequency response (Bode) characteristics. The final set of performance weights are presented in Figure 9.6. First order D-scale weights are chosen to keep the order of the controller as small as possible without giving up too much accuracy in the fits of the optimal D-scales.

The bandwidth of the closed loop design is pushed higher to achieve the maximum performance while satisfying the stability requirements. The upper bound of \( \mu \) for the robust performance analysis is given in Figure 9.7, and demonstrates robust performance. Analysis of the mu-synthesis flight control system, which includes both the gyro and accelerometer
Figure 9.5. Mu-Synthesis Autopilot Design Model #1
Figure 9.6  Mu Design #1 Sensitivity & Complementary Sensitivity Weights
Figure 9.7 Robust Performance Analysis of Mu Design #1
models, is performed next. Figure 9.8 shows the closed-loop system acceleration response to a unit step acceleration command. A time constant of 0.16 seconds is obtained. The open-loop frequency characteristics (at the actuator input and gyro output) are presented in Figures 9.9 and 9.10. The low frequency gain and phase margins just satisfy the design requirements, and the loop attenuation property for the high frequency requirement is satisfied due to roll-off. The order of the mu-synthesis controller is 14. The order of the weighted plant is also 14. It's states originate from: rigid body - 2, actuator dynamics - 2, unmodeled dynamics weight - 2, performance weights - 2, and 6 states from the D-scale fits (3 each for the left and right D-scales). We expect that the order of the compensator can be reduced by means of model reduction techniques without sacrificing a loss in performance and or stability.

9.1.3 AUTOPILOT DESIGN COMPARISONS

To highlight the merits of both techniques, we compare the design methodologies and analysis results of both the classical controller and mu-synthesis controller. Both approaches are fairly straightforward, given the classical autopilot loop structure and the mu-synthesis tools. The design approaches are similar, both comprised of selecting weights or gains until the requirements are satisfied. The mu-synthesis technique yields a controller structure that is robust in performance and stability. The classical controller design is based on years of experience.

Utilizing classical control analyses, both compensators are found to have similar performance and stability margins. The classical autopilot has higher phase margins at the
Figure 9.8 Acceleration Step Response, Mu Design #1
Figure 9.9 Frequency Response at Actuator Input, Mu Design #1
Figure 9.10 Frequency Response at Gyro Output, Mu Design #1
low frequency crossover points and the mu-synthesis autopilot has better gain tolerance at
the high frequency specification. Also, the mu-synthesis controller shows slight improved
time response performance.

However, recall that the objectives of the designs are to control the missile over the
full operational angle of attack range for the given Mach number and altitude flight
condition. Since the designs are based on intermediate values of the stability derivatives,
time response characteristics are also checked at the extreme angle of attack conditions. The
acceleration responses at the minimum and maximum angles of attack are plotted with the
nominal response in Figure 9.11 for the classical autopilot. The results show that the
autopilot stabilizes the airframe at the extreme angle of attack conditions. But, the response
is not very robust to the aerodynamic variations, as exemplified by the overshoot in the
minimum angle of attack case. A similar analysis for the mu-synthesis designed autopilot is
presented in Figure 9.12. In this case, the autopilot is robust to aerodynamic parameter
variation (as expected since it is accounted for in the design process), with not much
variation of the acceleration response at either extreme angle of attack flight condition
relative to that at the design condition. As a further check on the classical design, a mu-
analysis is performed to examine robust performance qualities of that controller. The upper
bound of mu is shown in Figure 9.13, and shows that the classical flight control system is not
robust to the aerodynamic and unmodeled dynamics for the same level of performance
specifications as utilized by the mu-synthesis designed autopilot.
Figure 9.11 Comparison $\eta$ Step Response at Min & Max $\alpha$, Classical Design #1
Figure 9.12 Comparison $\eta$ Step Response at Min & Max $\alpha$, Mu Design #1
Figure 9.13 Robust Performance Analysis of Classical Autopilot #1
9.2 DESIGN CASE 2

In this case, the missile IRU is assumed to be at the nominal location given in Table 3.1. The classical autopilot design approach is the same as in the previous design. However, since the previous $\mu$-Synthesis design just satisfied the stability margin requirements, additional uncertainties were inserted in the design model to increase the stability margins and give the autopilot designer more flexibility in achieving the design requirements. Both of these designs are discussed below.

9.2.1 CLASSICAL MISSILE AUTOPILOT DESIGN

The autopilot gains were initially set by specifying $\tau = 0.08$, $\zeta = 0.7$ and an open-loop crossover frequency of one-third to one-fourth the actuator bandwidth, $\omega_{cr} = 50$ rad/sec. From this input data, the autopilot gains were calculated. However, the Nyquist plot loop transfer requirement failed to be satisfied, i.e., the loop transfer entered the 0.5 radius ball about the Nyquist critical point. By decreasing the crossover frequency to 45 rad/sec, the resulting design satisfied all the design requirements. In this case, the autopilot gains were: $K_A = 0.512$, $K_t = 25.289$, $K_R = 0.04$ and $K_g = 2.0764$. The time response of the closed-loop system's acceleration, angle of attack and pitch rate to a unit step acceleration command are given in Figures 9.14 and 9.15. Figure 9.14 shows that the achieved normal acceleration satisfies the overshoot and steady-state tracking criteria, and the effective time constant for this design is 0.124 seconds. The angle of attack and pitch rate state variables are shown to be well behaved in Figure 9.15. The frequency response characteristics are given by the Bode plots in Figures 9.16 and 9.17, for the loop opened at the actuator input and the gyro
Figure 9.14 Acceleration Step Response, Classical Autopilot #2
Figure 9.15. Missile Angle of Attack and Pitch Rate Response to Step Command
Figure 9.16. Bode Plot for Loop Open at Actuator, Classical Autopilot Design
Figure 9.17. Bode Plot for Loop Open at Gyro, Classical Autopilot Design
output, respectively. For the loop broken at the actuator, the gain margin is 9.6 db, the phase margin is 37.7 degrees and there is 25.5 db of high frequency attenuation. Similarly for the open-loop characteristics at the gyro, the gain margins are 9.6 db and -11.2 db, the phase margin is 36.3 degrees, and the attenuation at 300 rad/sec is 25.4 db. The Nyquist plot for the loop broken at the actuator is shown in Figure 9.18 and exemplifies the loop transfer characteristics about the 0.5 radius ball around the critical point. Thus, the classical autopilot design satisfies all the time and frequency domain design specifications.

9.2.2 MU-SYNTHESIS AUTOPILOT DESIGN

The mu-synthesis design is performed in the following steps: formulate the design model along with a description of the uncertainties associated with that model, design the performance weights, and then successively iterate on the $H_\infty$ controller and the D-scales until the $\mu$ upper bound and/or the D-scales converge to a solution. (The formulation of the system design model and the specification of the performance weights are the critical steps in producing a good controller design.) After the solution converges, the robust stability and robust performance criteria and design requirements are evaluated. Recall that in this study the mu-synthesis design also must satisfy classical type of specifications. If the robustness properties or the design requirements are not attained, the weights and/or design model are modified in order to meet the specifications.

The current Mu-Synthesis design model is built around the rigid body airframe equations of motion, given in Chapters 2 and 6. Instrumentation models of the actuator, gyro and accelerometer are also added to ensure precise modeling of the true system.
Figure 9.18. Nyquist Plot for Loop Open at Actuator, Classical Autopilot Design
dynamics. In addition, the accelerometer offset effect on the acceleration is included.

Next, the uncertainty models are added to the design model. Five uncertainties are included here: high frequency unmodeled dynamics, two aerodynamic coefficient variation uncertainties, and input and output uncertainties. The first uncertainty, a high frequency unmodeled dynamics uncertainty, was considered to account for any high frequency effects, such as noise and the flexible-body modes, that are not modeled. This uncertainty was weighted by a second-order transfer function with small gain at low frequency and sufficiently large gain at high frequency. This weight is used as a "control knob" to meet the high frequency attenuation requirements. This weight is illustrated in Figure 9.19.

The next two perturbations included are aerodynamic parameter variations. The pitching moment and normal force derivatives with respect to angle of attack, \( C_{m_a} \) and \( C_{n_a} \), have the largest variations of the stability derivatives over the operating range. (The operating range is considered to be the angle of attack operating range, and the aerodynamic stability derivative variations are due to the nonlinear characteristics of the pitching moment and the normal force with angle of attack.) Considering an average value for the aerodynamic stability derivatives at the design flight condition, the uncertainties are 270% and 15%, respectively. These variations translate into uncertainty weights (or scale factors) acting on the \( M_a \) and \( Z_a \) terms. These uncertainties are considered as structured uncertainties since they affect the plant model in a specific, structured manner. The remaining two uncertainties are added at the loop breaking points (the actuator input and the gyro output) to build in more stability margin (robustness) at those particular points in the loop. By adjusting the weight (gain) of these uncertainties, explicit control of the
Figure 9.19. Unmodeled Dynamics Weight
tradeoff between the system performance and the stability margins at these points could be controlled. In this design, constant values of 0.05 over frequency are utilized, and produce adequate stability margins. These perturbations are included in the design model primarily because of the stability margin design specifications.

Next, disturbance inputs are considered. Three inputs are included in the design model: (1) acceleration command, (2) gyro drift, and (3) accelerometer bias. The first disturbance input is the normal acceleration command from the guidance system, which the autopilot is to track. Measurement sensor disturbances account for the other two inputs. These are the gyro drift (or noise) and the acceleration bias (or noise). Values for these terms are obtained from hardware specifications: $A_{\text{bias}} = 0.0001$ g's and $q_{\text{drift}} = 5$ deg/hr.

Completing the mu-synthesis design model, performance variables (i.e., the output error vector, e) are selected. These include: (1) the tracking error between the commanded normal acceleration and the achieved missile normal acceleration, (i.e., the sensitivity function, S), (2) the achieved normal acceleration (i.e., the complementary sensitivity function, T), (3) the actuator fin rate, and (4) the actuator fin angle. The mu-synthesis design model considered here is presented in Figure 9.20. The linear interconnection is rearranged as shown in Figure 9.21 to fit the form of the general analysis and synthesis framework given in Figure 8.9. These error outputs are normalized with weights that are set initially according to the performance and stability requirements. The sensitivity function weight is initially set based upon performance requirements. The performance specifications translate to the $W_s(s)$ weight as follows: (1) a time constant of approximately 0.1 seconds - cross the S and T weights at 11 rad/sec; (2) 0.1% steady-state tracking error - low frequency
Figure 9.20. Mu-Synthesis Design Model
Figure 9.21. General Interconnection Structure of the Design Model
magnitude of \( S \) weight = 1/0.001 = 1000; (3) limit overshoot - high frequency gain of \( S \) weight = 0.01. The complementary sensitivity function weight, \( W_4(s) \), is set by: (1) low frequency gain = 1/20 g's structural acceleration limit; and (2) high frequency gain set at 100 to aid stability margins. The control servo weights on the fin deflection angle and the fin deflection rate are set as the inverse of their respective saturation limits. Figure 9.22 gives the sensitivity function and complementary sensitivity function weights.

The Mu-Synthesis design converged after three iterations, with a maximum value of Mu of 0.75; thus robust performance is guaranteed for the system in the presence of the perturbations considered in the design. Figures 9.23 and 9.24 show the time response of the missile achieved normal acceleration and the model states to a unit step acceleration command. The missile responds to the guidance command with an effective first-order time constant of 0.108 seconds. This corresponds to a 13% improvement in the time response over the classical design. The Bode plots for the Mu-Synthesis controller are given in Figures 9.25 and 9.26. The gain margin, phase margin, and high frequency attenuation characteristics are: (1) for the loop broken at the actuator, 12.5 db of gain margin, 32.3 degrees of phase margin, and 22 db of attenuation; (2) for the loop opened at the gyro, 13.5 db and -8.2 gain margins, 32.1 and -75.6 degrees phase margins, and 21.8 db of attenuation at 300 rad/sec. In addition, although not given here, the Nyquist plots show that the loop transfer remains outside the prescribed ball about the critical point.

Examining the control structure, the mu-synthesis controller has 26 states. These states come from the generalized weighted plant, whose states originate from: rigid-body dynamics - 2, instrumentation dynamics - 6, unmodeled dynamics weight - 2, performance
Figure 9.22. S and T Weights for the Mu-Synthesis Design
Figure 9.23. Achieved Acceleration of Mu-Synthesis Design
Figure 9.24. Time Response of State Variables to Step Command, Mu-Synthesis Design
Figure 9.25. Bode Plot, Loop Open at Actuator, Mu-Synthesis Design
Figure 9.26. Bode Plot, Loop Open at Gyro, Mu-Synthesis Design
weights - 2, and 14 states from the D-scales fits (5 curve fits that lead to 7 states each for the left and right D-scales). For implementation purposes, model reduction techniques could probably reduce the order without sacrificing much performance or stability robustness.

9.2.3 AUTOPILOT DESIGN COMPARISONS

With regards to the design specifications, both designs satisfy all the design requirements. The Mu-Controller has a slightly faster response and has comparable stability margins, and the classical design yields slightly better attenuation at 300 rad/sec. In addition, both designs should be robust to angle of attack variation. That is, a constant controller over the angle of attack operating range is desired. Figures 9.27 and 9.28 show the performance of the classical and Mu-synthesis controllers, respectively, to a unit step acceleration command. The response is given at the minimum, nominal, and maximum angles of attack. As these figures demonstrate, both controllers yield the desired robustness with respect to angle of attack. It is noted, however, that for this design condition, the angle of attack range is relatively small (seven degrees) and that at other design conditions at which high angles of attack (20 - 30 degrees) are necessary, it is likely that the classical controller would fail to produce this performance robustness with respect to angle of attack variation, as demonstrated by the classical controller in Reference 35. However, it is more likely that the classical controller would be easier to gain schedule than the 26 state mu controller.

Mu-Analysis is also performed on both the classical and the Mu-Synthesis designs. Plots showing the nominal performance, robust stability, and robust performance curves for both designs are presented in Figures 9.29 and 9.30. Figure 9.29 shows that the classical
Figure 9.27. Acceleration Step Response with Angle of Attack Variation, Classical #2
Figure 9.28. Acceleration Step Response with Angle of Attack Variation, Mu Design #2
Figure 9.29. Mu-Analysis of Classical Autopilot to Model Uncertainties
Figure 9.30. Mu-Analysis of Mu-Controlier
design satisfies the nominal performance and robust stability criteria, but does not guarantee robust performance. The Mu-Synthesis design analysis results presented in Figure 9.30 show that nominal performance, robust stability, and robust performance are achieved. Thus, the Mu-Controller is more robust to the model uncertainties described in the previous section. Equivalently, the Mu-Synthesis autopilot design can tolerate more uncertainty and still satisfy the performance requirements.

Comparing the controllers, the classical autopilot design is a relatively simple first-order controller, while the Mu-Synthesis design has 26 states. This is why the design model and the details included therein are important for the Mu-Synthesis approach. Again, model reduction could likely reduce the Mu-Controller to a more manageable size.

The conventional autopilot design approach utilizes classical proportional and integral control to design a feedback loop structure. The design from that point consists of finding gains to achieve the performance and stability requirements. Classical SISO stability analysis tests (Bode, Nyquist diagrams) are used to check the frequency response for gain and phase margins and stability. Therefore, robustness can be checked at only one place in the loop at a time. Time response characteristics are examined to check the performance requirements and closed-loop behavior.

In the $\mu$-Synthesis approach, a design model that includes the system dynamics, input and performance weights, and a specific (structured) model of the uncertainties associated with that model is used with $H_\infty$ control and $\mu$-Analysis tools to compute a linear feedback controller. Instead of gains, the weighting functions are tailored to ensure that the design requirements are met. Nominal closed-loop stability is checked by calculating the system's
eigenvalues. Robust stability and robust performance are examined with the structured singular value. Mu-Analysis allows us to check for closed-loop stability and performance with regards to structured uncertainty for the whole loop in a MIMO setting.

Also, it is beneficial to perform mu-analysis on classical autopilot designs and classical SISO frequency and time response analyses on $\mu$-Synthesis designs. In this manner, both methods complement each other - the result being an improved autopilot design.
Chapter 10. Elastic Body Missile Autopilot Designs

In this chapter, we examine the effect of the airframe bending dynamics on the design of the flight control system. During flight, aeroelastic forces also act on the airframe, causing high-frequency vibratory modes. The vibrations affect both the acceleration and the pitch rate. In this chapter, designs neglecting the elastic effect on the acceleration and considering the elastic behavior on both acceleration and pitch rate are examined. For analysis, it is important to consider at least the first two or three elastic modes of the airframe. The bending dynamics are discussed in Chapter 2, specifically Equations (2.108 and 2.109) describe the bending effect on the body acceleration and pitch rate, respectively. Here, we consider the first three modes. The data for the given transfer function above for the first three modes is presented in Table 3.3 and the magnitude of the frequency response for these modes is illustrated in Figure 10.1. These bending mode dynamics are then added to the rigid body dynamics.

This chapter is divided into five sections. The first section examines some classical design approaches, the second section describes a conservative \( \mu \)-Synthesis design, the third section explores a non-conservative \( \mu \) design, a third-order controller reduced from the final \( \mu \)-autopilot is discussed in section 4, and a comparison of the design approaches is given in the final section. The only change in the design requirements as used in Chapter 9 is that the 20 db attenuation specification at high frequency should be satisfied at the modal peak frequencies instead of only at 300 rad/sec.
Figure 10.1. Bending Mode Frequency Response
10.1. CONVENTIONAL AUTOPILOT DESIGN AND ANALYSIS

The design process is to design the pitch autopilot for the rigid-body airframe first (by setting the autopilot gains $K_A$, $K_i$ and $K_R$), and then to add compensation for the flexible modes. Some of the methods that can be utilized to eliminate structural mode instabilities and to achieve the design requirements are: (1) the reduction of the autopilot gain(s), in particular, $K_R$; (2) proper choice of the gyro location, hinge line location, or the control surface center of gravity; (3) gain stabilization, i.e., notch filtering; and (4) phase stabilization\(^{14}\). For this study, methods (1) and (3) are used to satisfy the attenuation requirement at the flexible mode frequencies.

The autopilot gains were initially set for the rigid body autopilot design as discussed in the previous chapter. When the flexible dynamics are added into the analysis, the attenuation requirement at the first bending mode is not met. Using the first design approach of reducing the autopilot gains, the inner rate loop crossover frequency, $\omega_{CR}$, is reduced from 45 rad/sec to 10 rad/sec. As a result, the rate loop gain, $K_r$, and the integrator gain, $K_i$, are reduced from 0.04 and 25.3 to 0.009 and 2.6, respectively. The acceleration loop gain, $K_A$, is 1.5 and the dc gain, $K_0$, is 1.4. Figures 10.2 and 10.3 show the Bode plots for this design for the loop broken at the actuator input and the gyro output, respectively. Even though this design satisfied the stability margin and flexible mode attenuation goals, the time constant of 0.35 seconds did not meet the performance requirement.

The second classical approach of flexible mode compensation is the application of a notch filter tuned to the first bending mode. The notch filter is given by
Figure 10.2. Bode Plot: Classical Design 1 at Actuator
Figure 10.3. Bode Plot: Classical Design 1 at Gyro
\[ G_{\text{notch}}(s) = \frac{s^2 + \left( \frac{2\zeta_{\text{notchD}}}{\omega_{\text{notch}}} \right)s + 1}{\frac{s^2}{\omega_{\text{notch}}} + \left( \frac{2\zeta_{\text{notchN}}}{\omega_{\text{notch}}} \right)s + 1} \] (10.1)

where, \( \zeta_{\text{notchD}} \) is set to 0.5, and \( \omega_{\text{notch}} \) and \( \zeta_{\text{notchN}} \) are tuned to the respective first mode values. This structural notch filter is then added to the flight control loop as shown in Figure 7.1. Starting with \( \omega_{\text{cr}} = 45 \) rad/sec, the notch filter design satisfied the performance and flexible mode attenuation goals, but the stability margins were shy of the requirements. Dropping the inner rate loop crossover to 35 rad/sec allowed us to satisfy all the design specifications. The time responses of the achieved normal acceleration, the angle of attack, and the pitch rate to a unit step commanded acceleration are shown in Figures 10.4 and 10.5, respectively. For this autopilot design, the effective first-order time constant is 0.143 seconds. Figures 10.6 and 10.7 give the Bode plots for the loop opened at the actuator input and the gyro output, respectively. These Bode diagrams show adequate stability margins and high frequency attenuation.

10.2. MU-SYNTHESIS AUTOPILOT DESIGN #1

Recall that the objective is to design a \( \mu \) controller that handles the effects of the airframe's elastic dynamics. With the \( \mu \)-Synthesis technique, the approach is to account for the flexible body modes in the design model in some way. This may include modeling the first elastic body mode or several flexible-body modes, utilizing some form of uncertainty description associated with the elastic body dynamics, and/or employing some performance
Figure 10.4. Acceleration: Classical Notch Filter Design
Figure 10.5. $\alpha$ and $q$: Classical Notch Filter Design
Figure 10.6. Bode Plot, Loop Open At Actuator Input: Classical Notch Filter Design
Figure 10.7. Bode Plot, Loop Open at Gyro Output:
Classical Notch Filter Design
weight(s) to help suppress the loop gain for adequate flexible-body mode attenuation. In this chapter, gain-stabilized techniques are examined to achieve satisfactory flexible-body mode attenuation.

The Mu-Synthesis design approach here is identical to that formulated in Section 9.2.2., i.e., the μ-Synthesis design model is the same. Only the weighting functions are changed here. Thus, for this μ-Synthesis autopilot design, the flexible body dynamics are not included in the design model. Instead, high frequency unmodeled dynamics weights that cover all the flexible modes are utilized to account for the elastic behavior of the missile. This is a conservative mu-synthesis approach since the unmodeled dynamics weighting function covers the known flexible mode dynamics with sufficient margin to account for unknown magnitude and frequency uncertainties that may be associated with the bending model dynamics and/or data. This design approach is considered as a gain-stabilization approach since the effect of this weighting will be to force the controller gain to have the characteristics of the inverse of the uncertainty weighting, thereby causing a suppression of the loop transfer gain at the frequencies covered by the uncertainty weighting. A second-order transfer function with small gain at low frequency and sufficiently large gain at high frequency is used as a "control knob" to meet the high frequency attenuation requirements.

The unmodeled dynamics weight is given by

\[
W_{\text{UD}}(s) = \frac{K_{\text{UD}} (s/0.0025 + 1)(s/25 + 1)}{(s/250 + 1)^2}
\]  

(10.2)

where \(K_{\text{UD}} = 1.75 \times 10^4\). Figure 10.8 illustrates how this weight covers the flexible body dynamics (only the higher magnitude pitch rate flexible dynamics are given).
Figure 10.8. Unmodeled Dynamics Weight and Elastic Modes
Again, this approach is conservative because we do not model particular variations of the flexible dynamics, but instead, by amply covering all the modes, allow essentially for unstructured perturbations associated with the bending modes, as long as they remain bounded by the weighting function. This type of weighting approach would be appropriate when information regarding the flexible dynamics of the missile is not available or is not too dependable (the flexible mode data is in general more uncertain as the frequency gets higher).

The sensitivity and complementary sensitivity weights utilized in this design are plotted in Figure 10.9. In pushing the performance bandwidth higher, we found that the input/output balls of uncertainty could be dropped to a small radius (0.05) and still meet the gain and phase margin requirements at the loop breaking points. After three DK iterations, the solution converged to a maximum \( \text{mu} \) of 0.9; thus robust performance is guaranteed for the system in the presence of the perturbations considered in the design. Figures 10.10 and 10.11 show the time response of the missile achieved normal acceleration and the model states to a unit step acceleration command. The responses behave as expected, except that the pitch rate has some oscillations due to the suppressed first elastic mode. The missile responds to the guidance command with an effective first-order time constant of 0.136 seconds. The stability margins and body-bending modal attenuation requirements were satisfied, as can be seen from the Bode plots given in Figures 10.12 and 10.13.

The resulting \( \text{mu} \)-synthesis controller has 26 states. These states come from the generalized weighted plant, whose states originate from: rigid-body dynamics - 2, instrumentation dynamics - 6, unmodeled dynamics weight - 2, performance weights - 2, and
Figure 10.9 S and T Weights for the Mu-Synthesis Design #1
Figure 10.10. Acceleration of $\mu$-synthesis Design #1
Figure 10.11. Time Response of Rigid-Body States:
\( \mu \)-Synthesis Design
Figure 10.12. Bode Plot of $\mu$-synthesis Design:
Loop Opened at Actuator
Figure 10.13. Bode Plot of $\mu$-synthesis Design:
Loop Opened at Gyro
14 states from the D-scales fits (5 curve fits that lead to 7 states each for the left and right D-scales). For implementation purposes, model reduction techniques could probably reduce the order without sacrificing much performance or stability robustness.

There was one drawback to this design - the "ringing" of the transient body pitch rate time response. Although this oscillation does not appear on the acceleration response, the transient pitch rate behavior could potentially cause actuator problems (such as heating or wearing out the gears) or airframe structural concerns, and is, in general, undesirable. Within the design framework of this approach, the pitch rate oscillations could possibly be eliminated by changing the weighting functions in the design model (in particular, by dropping the crossover frequency of the pitch rate uncertainty weighting and/or increasing the magnitude at higher frequencies (where the bending modes occur). However, this would come at the expense, most likely, of a slightly slower time response.

10.3. MU-SYNTHESIS AUTOPILOT DESIGN #2

In the previous section, only the body elastic dynamics on the missile pitch rate were considered in the design and analysis. Here, we also examine the elastic dynamics that occur on the acceleration signal. This mu design is not as conservative as the previous mu autopilot since the actual flexible body dynamics are considered in the mu design model and in the analysis.

10.3.1. CONTROLLER DESIGN APPROACH

The $H_\infty/\mu$ control design model is built around both the rigid-body and elastic-body
airframe dynamics discussed in Chapters 2 and 6. Disturbance inputs to the design model include the acceleration command signal, rate gyro drift and accelerometer bias. A gyro drift value of 5 deg/hr and an accelerometer bias value of 0.0001 g's are utilized in the design model to weight those respective inputs.

The regulated output performance variables considered here are the error between the acceleration command and the achieved normal acceleration, the fin deflection angle, the actuator fin rate, and the body pitch rate. All of these outputs are normalized using frequency dependent weighting functions, which are prescribed by the performance specifications. The weight on the acceleration error (which is also the sensitivity function, S), $W_s(s)$, is selected such that its inverse reflects the desired shape of the sensitivity function. (A good sensitivity function has low gain at low frequencies for good command following, low gain in the appropriate frequency ranges for good disturbance rejection properties, and high gain at high frequencies to limit overshoot.) The weight on the sensitivity function is generally selected as a low pass transfer function with its low frequency gain set equal to the inverse of the desired steady-state tracking error, its crossover frequency set approximately equal to the inverse of desired closed-loop time constant, and its high frequency magnitude selected to limit overshoot (the larger this high frequency gain, the more effectively the overshoot is limited). Using a first-order low pass weight for $W_s(s)$ and inverting the time response design specifications, the sensitivity function weight is chosen as

$$W_s(s) = \frac{100(0.0258s + 1)}{10.327s + 1} \quad (10.3)$$

This weight is shown in Figure 10.14. The weights on the fin angle position and the actuator fin rate are selected as constants over frequency, assumed as the inverses of their respective
Figure 10.14. Output Performance Weights
saturation limits. Weighting these actuator states helps to prevent these variables from saturating during flight. And finally, the body pitch rated was weighted to help achieve high frequency control loop gain roll-off and also to help meet the high frequency attenuation goals. As opposed to the sensitivity function weight, \( W_q(s) \) is high pass with a small gain at low frequency and large gain at higher frequencies to suppress the effects of high frequency unmodeled dynamics and sensor noise. A second order transfer function was utilized for this weight, with its low frequency gain set equal to the inverse of the assumed maximum rate of the rate gyro (400 deg/sec) and its crossover frequency and high frequency gain tuned to satisfy the design specifications. This \( q \) output performance weight is given by

\[
W_q(s) = \frac{(1/400)(0.025s + 1)^2}{(1.78e^{-s} + 1)^2}
\]  

(10.4)

This weight is also shown in Figure 10.14 with the sensitivity function weight.

The two remaining outputs of the design model include the feedback signals to the controller. These are the error between the commanded acceleration and the measured normal acceleration and also the measured body pitch rate.

Also considered in the design model are four model perturbations or uncertainties. These include two aerodynamic stability coefficient variation uncertainties and two actuator perturbations. One of the desired controller attributes is to remain insensitive to angle of attack changes during flight. However, the pitching moment and normal force aerodynamic characteristics are nonlinear over the angle of attack range of the airframe. To account for this in the design model, multiplicative parameter uncertainties are considered upon the pitching moment and normal force aerodynamic stability derivatives with respect to angle of
attack. The weights on these uncertainties are selected as the percentage variation of those derivatives about their respective average values over the angle of attack range (\( M_x \) variation = 270\% and \( N_x \) variation: 15\%). To satisfy the specified high frequency attenuation and stability margin/robustness goals, feedforward (multiplicative) and feedback (divisive) uncertainties are added at the actuator input signal. A constant value weighting of 0.2 was found to be adequate for the actuator feedback uncertainty because of contributions from other sources. (A value of 0.5 would guarantee 6 db gain margins and 30 degree phase margins.) For the actuator feedforward uncertainty weight, a second order high pass weight is used to help tune the controller. This uncertainty weight is given by

\[
W_{\omega a}(s) = \frac{(1/100)(0.049s + 1)^2}{(9.805e^{-4}s + 1)^2}
\]  

(10.5)

and is presented in Figure 10.15.

All of these elements are then incorporated into the \( \mathcal{H}_\omega/\mu \) control design model. A block diagram of the design model is displayed in Figure 10.16. Figure 10.17 shows the general interconnection structure diagram for this design model.

10.3.2. \textbf{ANALYSIS RESULTS}

Several types of analyses are required to verify that the design specifications are satisfied. These include examination of the controller and the closed-loop system poles, time response, frequency response and \( \mu \)-analysis. (The \( \mu \)-analysis is discussed in section 10.5.)

The resulting \( \mu \) controller, obtained after two DK Iterations, stabilizes the closed-loop system and is of 27th order. These states come from the generalized plant, which includes
Figure 10.15. Actuator Feedforward Uncertainty Weight
Figure 10.16. Model for μ Design #2
Figure 10.17. Interconnection Structure for $\mu$ Design #2
the performance and uncertainty weights and the D-scale weights. For example, four poles of the compensator are the same as those of the performance and uncertainty weights. The $H_\infty$ controller synthesis builds in notches at the three body bending mode frequencies in the pitch rate channel, as shown in the plot of the control loop transfer, Figure 10.18. Note that this differs from the conventional notch filter approach, which are added at the actuator input (see Chapter 7). The acceleration error control loop gain, however, does not have any notches, but rolls off to obtain high frequency attenuation. The poles of the closed-loop system are comprised largely of those of the controller and of the open-loop plant, and are all stable as expected.

The time response of the closed-loop system to a unit step acceleration command exhibits fast performance. The achieved acceleration is shown in Figure 10.19. The time constant is 0.086 seconds, and the time response displays approximately 2% overshoot. The missile’s angle of attack and body pitch rate, plotted in Figure 10.20, demonstrate that the flexible body dynamics do not disturb these states.

Frequency response analysis was performed at the plant input and measurement output locations. In particular, the gain margin, phase margin, body bending mode attenuation and Nyquist loop transfer requirements were checked. The Bode and Nyquist plots for the loop opened at the actuator input are shown in Figures 10.21 and 10.22, respectively. The Nyquist plot of Figure 10.22 is the close-up view about the critical point and shows that the loop transfer remains outside the required disk. Figures 10.23 and 10.24 give the Bode plots for the loop broken at the gyro and accelerometer output locations, respectively, and demonstrate adequate frequency response requirements.
Figure 10.18. $\mu$ Controller #2 Loop Gains
Figure 10.19. Achieved Normal Acceleration
Figure 10.20. Angle of Attack and Body Pitch Rate
Figure 10.21. Bode Plot, Loop Open at Actuator Input
Figure 10.22. Nyquist Plot, Loop Open at Actuator
Figure 10.23. Bode Plot, Loop Open at Gyro Output
Figure 10.24. Bode Plot, Loop Open at Acceleration
10.4. REDUCED-ORDER MU CONTROLLER

The 27 state controller described above would make for an unnecessarily large controller to actually implement onboard a missile guidance and control computer. To make implementation more amenable, it is desired to reduce the order of the controller as low as possible. At the same time, the requirements given in Chapter 4 must also be satisfied. The only exception to this is the requirement on the loop transfer robustness specification on the Nyquist plot. This requirement was included for the full-order $\mu$-controller design to enforce good gain and phase margins, especially when considering its reduced-order variant.

The controller order was reduced by inspection of the controller poles and zeros of the separate channels. This included neglecting the controller's high frequency dynamics and canceling stable pole/zero combinations located close to each other. The initial simplification led to a 7th order controller in the acceleration error channel and a 12th order one in the pitch rate channel. By further examination of the remaining poles and zeros, decoupled third-order controllers were obtained for the acceleration error and pitch rate channels. The resulting controller transfer functions are

\[
K_{\text{acc}}(s) = \frac{-5.3}{s + 0.1} \left[ \frac{s + 5.5}{s + 6.4} \right] \left[ \frac{s + 80}{s + 200} \right]
\]

\[
K_\phi(s) = 0.2 \left[ \frac{s + 28}{s + 6.4} \right] \left[ \frac{s^2 + 2 \zeta_N \omega_N + \omega_N^2}{s^2 + 2 \zeta_D \omega_D + \omega_D^2} \right]
\]

where $\zeta_N = 0.01$, $\omega_N = 378$ rad/sec, $\zeta_D = 0.06$ and $\omega_D = 754$ rad/sec. The controller in the acceleration error channel has a pole near zero to act as an integrator for zero steady-state error, a lead-lag element near crossover and a lead-lag before the first flexible body mode.
The controller for the pitch rate channel includes a lead-lag combination near the crossover frequency and a notch filter element for the first flexible body mode. Note that only the first notch filter was retained, as compared to the three notch filters for the full-order controller, since the attenuation at the higher frequencies is adequate enough to satisfy the design requirements. Also, the two channels share only one common pole.

The analyses of the controller and of the open- and closed-loop systems verify that the design requirements are satisfied. The reduced-order controller's loop gains are displayed in Figure 10.25, and the notch is evident in the pitch rate channel gain. Figure 10.26 shows the time response of the achieved normal acceleration to a unit step acceleration command. The time response characteristics are similar to those for the full-order controller, except now we achieve an even slightly faster time constant of 0.078 seconds. (Though not shown, the time responses of the angle-of-attack and the pitch rate do not display any undesirable attributes.) In addition, the frequency response specifications, such as stability margins and high frequency attenuation, are satisfied. Figure 10.27 gives the Bode plot for the loop opened at the actuator, for example. The μ-analysis of this design, given in the next section, also show robust stability and performance conditions are satisfied. Thus, these third-order controllers for the two channels are more amenable to onboard missile computer implementation and provide robust stability and robust performance characteristics for the flight control system.

10.5. AUTOPILOT DESIGN COMPARISONS

This chapter examined five autopilot designs that accounted for body-bending effects:
Figure 10.25. Reduced-Order Mu-Controller Loop Gains
Figure 10.26. Acceleration Response Using Reduced-Order Mu-Controller
Figure 10.27. Bode Plot, Loop Open at Actuator, Reduced-Order Mu-Controller
two using conventional design practice and two using mu-synthesis. The fifth design was obtained by reducing the order of the best mu-controller candidate. The design philosophy that was carried throughout all the designs was to try to obtain the fastest (achieved acceleration) time response possible, while satisfying the overshoot and steady-state error requirements, as well as the frequency domain stability margins and bending mode attenuation specifications. Since the first classical design (in which the approach was to back down on the crossover frequency until the attenuation and stability margin requirements were met) did not satisfy the time constant goal, only the remaining four designs for the flexible vehicle will be compared here.

With regards to the time-domain specifications as set forth in Chapter 4, all four designs satisfy all the design requirements. With the classically designed autopilot serving as the comparison baseline, the three mu-based controllers each had faster time constants. The first mu autopilot was about five percent faster, while the second mu-controller allowed for a 40% time constant improvement. The reduced-order mu-controller did even slightly better, with a 45% improvement in time constant. This is the first notable advantage of the mu-based designs over the conventional autopilot.

We also compare the overshoot and the steady-state error characteristics of the resulting designs. The classical controller has zero overshoot, as does the first mu-synthesis autopilot. The second mu-controller had about 2% overshoot and the reduced-order mu-autopilot had about 4% overshoot. Thus, with the faster time response comes slight overshoot - but both within design specification. As for the steady-state error attributes, all designs lead to approximately zero error.
However, one time response aberration is noted. The pitch rate for the first mu-synthesis autopilot design had some slight "ringing" in the transient response. Though this design had satisfied all the design requirements, including the attenuation requirement at the bending mode frequencies, this response attribute would probably not be desirable for actual flight. This is why the second mu-synthesis design approach was examined.

Analysis using singular values and the structured singular value was also performed on the classical and the mu-synthesis designs. Specifically, the conditions for nominal performance, robust stability, and robust performance as described in Chapter 8 were checked. Plots depicting nominal performance, robust stability and robust performance are presented in Figures 10.28 - 10.31 for the four designs described in the preceding sections. Figure 10.28 shows that the classical (notch filter) design satisfies nominal performance, but does not guarantee robust stability nor robust performance. The mu-synthesis design #1 analysis results presented in Figure 10.29 show that nominal performance, robust stability, and robust performance are achieved. Figure 10.30 displays the nominal performance, robust stability, and robust performance for the second mu-controller, and Figure 10.31 is analogous for the reduced-order mu-autopilot. As in the first mu design case, all the conditions (NP, RS and RP) were satisfied. This means that the closed-loop system will exhibit robust performance in the presence of the uncertainties considered in the design model (aero variations and actuator uncertainties). Note that the robustness does not suggest that the stability margins will be satisfied at the output locations since output feedback uncertainties were not included in the design model. However, these uncertainties could have been included in the design model at the expense of a larger controller and slightly slower
Figure 10.28. Robustness Analysis of Classical Autopilot Design
Figure 10.29. Robustness Analysis of Mu-Synthesis Autopilot #1
Figure 10.30. Robustness Analysis of Mu-Synthesis Autopilot Design #2
Figure 10.31. Robustness Analysis of Reduced Order Mu-Controller
response. Summarizing these results, the mu-synthesis designs are more robust to model uncertainties (as modeled herein). Equivalently, the mu-synthesis autopilot design can tolerate more uncertainty and still satisfy the performance requirements. This is the second notable advantage of the mu design process over the conventional classical approach.

Comparing the size of the controllers, the classical notch filter design had three states (one for the rigid body and two for the notch filter), while the first mu-synthesis design had 26 states and the second mu autopilot had 27 states. This illustrates a drawback of the mu-synthesis design approach - unwieldy controllers. This also suggests the importance of the design model and the details included therein to the mu-synthesis approach. The designer has the flexibility to try to keep the model relatively low-order to keep the resulting controller manageable, or to add as much details to the design model as desired, and then apply order-reduction techniques to the resulting controller. In this work, engineering judgement and control experience was used to reduce the second mu-controller to a reduced form that only had three states per feedback channel. Thus, we demonstrated that controller order reduction can be performed with little sacrifice of performance or robustness.

The conventional autopilot design approach utilizes classical proportional and integral control to design a feedback loop structure and a quadratic notch filter to handle the first body-bending mode effects. The design from that point consists of finding gains to achieve the performance and stability requirements. Classical SISO stability analysis tests (Bode, Nyquist diagrams) are used to check the frequency response for gain and phase margins and stability at particular points in the loop. Therefore, robustness can be checked at only one place in the loop at a time. Time response characteristics are examined to check the
performance requirements and closed-loop behavior.

In the $\mu$-synthesis approach, a design model that includes the system dynamics, input and performance weights, and a specific (structured) model of the uncertainties associated with that model is used with $H_\infty$ control and $\mu$-analysis tools to compute a linear feedback controller. Instead of gains, the weighting functions are tailored to ensure that the design requirements are met. Nominal closed-loop stability is checked by calculating the system's eigenvalues. Robust stability and robust performance are examined with the structured singular value. Mu-analysis allows us to check for closed-loop stability and performance with regards to structured uncertainty for the whole loop in a MIMO setting.

Though both design approaches are vastly different, their synthesis and analysis tools complement each other. For example, by applying mu-analysis to a classical control design, weaknesses to (structured) uncertainties acting simultaneously at various points in the loop can be examined. This would be especially valuable to designs that have multiple inputs and multiple outputs, that in a classical design, would have to be broken into separate single-input single-output systems. For a mu-synthesis design, being able to perform classical time response analysis and Bode and/or Nyquist analysis would help to tie the resulting design performance and robustness characteristics back to concepts that conventional control design engineers are quite familiar with and that missile autopilot requirements are still specified as. Thus, regardless of the synthesis technique that is applied, the end result is an improved autopilot design because of the use of the complementary analysis tools from each approach.
Chapter 11. Summary and Conclusions

In this report, we compared the design of missile flight control systems using both classical control and $\mu$-Synthesis. After deriving the linear equations of motion for a missile airframe and specifying data on a high performance missile, autopilot design constraints and limitations were given. Following this, the autopilot design requirements were specified. Then, linear autopilot design models were developed. A detailed review of classical missile autopilot design was then given, followed by a review of the $\mu$-Synthesis design technique. Subsequent chapters addressed the design of missile autopilots for both the rigid-body and elastic-body cases.

We found that at the nominal design condition, both design techniques could be used to design an autopilot that satisfied the design requirements. (The exception was the first classical design approach for flexible-body compensation, reduction of the autopilot gains - the acceleration time response was not quick enough.) Even at off-nominal or perturbed design conditions, both design techniques could stabilize the airframe. However, only the $\mu$-Synthesis technique could guarantee robust performance (satisfy the performance specifications at off-nominal design conditions). This was one of the advantages that we found for the $\mu$-synthesis design approach.

For example, for the first rigid body case that was examined, the time response of the achieved acceleration to a step command for the $\mu$ design was robust to the extreme angle-of-attack variation, but for the same extreme alpha flight condition, the conventional autopilot design had an overshoot (of about 10%) that did not satisfy the design
specification. Thus, to achieve a satisfactory design for that flight condition, new gains would need to be calculated for the classical autopilot and gain scheduling employed as a function of alpha. The mu-synthesis design was then more robust to angle-of-attack variations (since we could build that attribute into the design), and would not require gain scheduling with angle-of-attack (in the case examined). This is an off-shoot advantage due to the robust performance advantage of the mu-synthesis technique.

The second notable advantage that we found for the mu-synthesis approach was that it could yield faster time responses than those using the conventional design approach, while still meeting the stability margin and attenuation specifications. For example, the second mu-synthesis design for the flexible airframe had a time constant improvement on the order of 40%. The significance of this result is as follows: for a short-range, ship self-defense missile homing against a highly maneuverable anti-ship threat, the miss distance rises almost exponentially as a function of missile (and subsequently autopilot) time constant. Thus, the chances of intercepting and negating the maneuverable threat can be significantly increased by reducing the autopilot time constant.

The only disadvantage found in this work was the order of the resulting mu-synthesis controller. The order can get quite large with the mu-synthesis technique, due to the states considered in the design plant, performance and uncertainty weights, and D-scale functions necessary in the mu-synthesis DK iterations. The main problem with high order controllers for missile flight control is computer implementation of the complex controllers, especially when considering that the mu-autopilots themselves will most likely require gain scheduling over a full flight envelope of a high performance tactical missile. However, order-reduction
techniques can be employed to reduce the controller order and complexity without sacrificing much performance or robustness. This was demonstrated in the final design of this research.

The structured singular value can be utilized to examine both stability and performance aspects in control design; treat uncertainties, disturbances and performance easily as tradeoffs versus stability; treat uncertainty at several points in the loop simultaneously; and treat stability and performance for MIMO systems without having to break the loop at several points to perform SISO analysis tests. In addition, even if another design approach was utilized, the H\(_\infty\)/\(\mu\)-synthesis technique could be applied to serve as a benchmark by which the other controller could be judged, from both robust stability and robust performance perspectives. Thus, H\(_\infty\)/\(\mu\)-synthesis and analysis techniques provide valuable tools in the design of missile flight control systems.
Chapter 12. Further Topics for Research

This research examined the application of H-Infinity/Mu-Synthesis techniques to the design of missile autopilots and found that they can provide valuable analysis and synthesis tools. However, this work just scratched the surface of the issue of how mu-based controllers could be designed for full-up, practical, missile flight controllers. Herein, we examined (pitch) planar designs only, for both the rigid-body and flexible-body cases. Though the final mu-design for the flexible case was very satisfactory, more improvements could be made to the mu synthesis approach utilized here to handle the elastic body dynamics, especially if one considers (structured and unstructured) uncertainties associated with the flexible body dynamics. Other extensions of this $H_\infty/\mu$-synthesis work could include:

- fully coupled pitch/yaw/roll dynamics
- the amount of gain scheduling required for full envelope autopilots
- gain scheduling approaches for the (relatively high-order) mu controllers over the flight envelope
- systematic controller order reduction techniques and their application to obtain manageable flight control algorithms
- further elaboration on the trade-offs between the complexity of the design model and uncertainty descriptions, the use of controller order techniques, and the resulting performance and robustness characteristics
- the use of $H_\infty/\mu$-synthesis with other nonlinear (such as feedback linearization) and/or other adaptive control techniques, with the goal of reducing or eliminating the gain-scheduling problem
- the use of $H_\infty/\mu$-synthesis for the design of (integrated) missile guidance and control systems

273
Work has already been started in the missile control community for some of these additional research areas. By thoroughly examining the above topics, the question of whether H-Infinity Control / Mu-Synthesis based techniques could be used for practical on-board autopilot implementations could perhaps be answered.
References


Doyle, John C., "Synthesis of Robust Controllers and Filters", Proc. of 22nd Conf. on Decision and Control, Dec 1983.


283


Notes from "Theory and Application of Robust Multivariable Control" short course, MUSYN, Arcadia, CA, Sept. 1989.


Appendix A. Development of the Linearized Airframe Rigid Body Dynamics

Several approaches can be taken to obtain the rigid body airframe dynamics, such as those documented in References 7, 15, 17, 18, 35, 36 and 39-44. This report will formulate the equations of motion for a rigid airframe by simplifying the dynamical equations presented in Reference 39, express them as a function of aerodynamic angles, then linearize the resulting equations for design and analysis purposes.

The general force and moment equations of motion for a vehicle moving through some medium were derived in Reference 39. These six equations were derived assuming a rigid body and expressed in a body fixed axis system. Under the following assumptions:

1. The earth is flat and non-rotating in inertial space
2. The gravity vector is constant and is oriented positive in the inertial z direction (down)
3. Missile buoyancy effects are neglected
4. Thrust acts along the missile's centerline
5. The jet damping effects due to internal mass flow and momentum changes can be neglected
6. Gyroscopic contributions due to rotating machinery are neglected
7. The body coordinate frame's reference point coincides with the center of mass, with the body x-axis aligned with missile centerline
8. The atmosphere is at rest relative to the earth,

the equations of motion become
\[ F_X = M(\dot{u} + qw - rv) \]  
\[ F_Y = M(\dot{v} + ru - pw) \]  
\[ F_Z = M(\dot{w} + pv - qu) \]  
\[ l = I_x \dot{\phi} - I_{xy} \dot{\theta} - I_{xz} \dot{\tau} + (I_x - I_y)qr + I_{xz}(r^2 - q^2) + I_{yy}pr - I_{xy}qr \]  
\[ m = I_y \dot{\theta} - I_{yz} \dot{\phi} - I_{xt} \dot{\tau} + (I_x - I_y)pr + I_{yz}(q^2 - r^2) + I_{yy}pq - I_{xy}qr \]  
\[ n = I_t \dot{\tau} - I_{zt} \dot{\phi} - I_{yt} \dot{\theta} + (I_y - I_z)pq + I_{yt}(q^2 - p^2) + I_{xy}qr - I_{yz}pr \]  

where

\[ F_X = X_a - Mg \sin \theta + T \]  
\[ F_Y = Y_a + Mg \sin \phi \cos \theta \]  
\[ F_Z = Z_a + Mg \cos \phi \cos \theta \]  

In the above equations, we define in the body frame

1. missile velocity vector, \( V_b = (u \ v \ w)^T \)
2. missile rotational rate vector, \( \omega_b = (p \ q \ r)^T \)
3. external body forces, \( F_b = (F_x \ F_y \ F_z)^T \)
4. moments acting on the body, \( M_b = (l \ m \ n)^T \)
5. inertia matrix

\[
\mathbf{\bar{I}} = \begin{bmatrix}
I_x & I_{xy} & I_{xz} \\
I_{yx} & I_y & I_{yz} \\
I_{zx} & I_{zy} & I_z
\end{bmatrix}
\]

where \( I_x, I_y, \) and \( I_z \) are the moment-of-inertia terms and \( I_{xy}, I_{xz}, \) and \( I_{yz} \) are the product-of-inertia elements

6. aerodynamic forces, \( F_{aa} = (X_A \ Y_A \ Z_A)^T. \)

Also, \( T \) represents the missile thrust and \( M \) denotes the mass of the missile.
Next, we consider the missile configuration. In the current work, the missile will be assumed to be cruciform with circular body cross-sections. As a result, the airframe has two planes of geometrical and mass symmetry. Because the body axes are defined to be aligned with the principal axes of inertia, the product-of-inertia terms \( I_{xy}, I_{xz}, I_{yz} \) disappear. In addition, the moment-of-inertia about the \( y \)-axis is identical to that about the \( z \)-axis, or \( I_y = I_z \). The moment equations then simplify to

\[ l = I_x \dot{\phi} \]  
\[ m = I_y \dot{\psi} + pr(I_x - I_y) \]  
\[ n = I_y \dot{\lambda} + pq(I_x - I_y) \]  

Equations (A.1)-(A.3) and (A.10)-(A.12) describe the three-dimensional, nonlinear motion of a cruciform airframe in the body axes system.

However, for the autopilot design of symmetric, skid-to-turn missiles, it is customary to transform the states \( u, v, \) and \( w \) into the states: angle of attack, \( \alpha \), angle of sideslip, \( \beta \), and airspeed \( V \). In addition, an aerodynamic roll angle \( \phi_A \) can also be defined that, combined with the total angle of attack, can determine \( \alpha \) and \( \beta \). These aerodynamic angles are defined in Figure 2.3. The force equations are written in terms of \( V, \alpha \) and \( \beta \) essentially because the aerodynamic coefficients are usually specified as functions of \( \alpha \) and \( \beta \), (or \( \alpha_T \) and \( \phi_A \)), as well as Mach number, \( M \), and tail deflection, \( \delta \). Using Figure 2.3, these aerodynamic angles can be related to the body velocity components

\[ \tan \alpha = \frac{w}{u} \]  
\[ \tan \beta = \frac{v}{u} \]
\[ \tan \phi_\alpha = \frac{v}{w} = \frac{\beta}{\alpha} \quad (A.15) \]

Note that \( \alpha \) and \( \beta \) as defined above are not equivalent to the standard Euler angles of attack and sideslip. Also, the square of the missile velocity is

\[ v^2 = u^2 + v^2 + w^2 \quad (A.16) \]

Taking the time derivatives of Equations (A.13), (A.14) and (A.16),

\[ \frac{d}{dt} [w = u \tan \alpha] \]

\[ \dot{w} = \dot{u} \tan \alpha + u \dot{\alpha} \sec^2 \alpha \quad (A.17) \]

\[ \frac{d}{dt} [v = u \tan \beta] \]

\[ \dot{v} = \dot{u} \tan \beta + u \dot{\beta} \sec^2 \beta \quad (A.18) \]

and

\[ 2V\ddot{\gamma} = 2u\dot{u} + 2v\dot{\gamma} + 2w\ddot{w} \quad (A.19) \]

Solving these expressions for \( \dot{\alpha}, \dot{\beta}, \) and \( \ddot{\gamma}, \)

\[ \dot{\alpha} = \frac{1}{u \sec^2 \alpha} [\dot{w} - \dot{u} \tan \alpha] \]

\[ \ddot{\alpha} = \frac{\cos^2 \alpha}{u} [\ddot{w} - \dot{u} \tan \alpha] \quad (A.20) \]

\[ \dot{\beta} = \frac{1}{u \sec^2 \beta} [\dot{v} - \dot{u} \tan \beta] \]

\[ \ddot{\beta} = \frac{\cos^2 \beta}{u} [\ddot{v} - \dot{u} \tan \beta] \quad (A.21) \]
\[ \dot{\gamma} = \frac{(u\dot{u} + v\dot{v} + w\dot{w})}{V} \]  

(A.22)

The force equations (A.1) - (A.3) are then solved for time rates of change of the body velocity components,

\[ \dot{u} = \frac{F_x}{M} - qw + rv \]  

(A.23)

\[ \dot{v} = \frac{F_y}{M} - ru + pw \]  

(A.24)

\[ \dot{w} = \frac{F_z}{M} - pv + qu \]  

(A.25)

Substituting the above relations into Equations (A.20) - (A.22),

\[ \dot{\alpha} = \frac{\cos^2\alpha}{u} \left[ \frac{F_z}{M} - pv + qu - \tan\alpha \left( \frac{F_x}{M} - qw + rv \right) \right] \]

\[ \dot{\beta} = \frac{\cos^2\beta}{u} \left[ \frac{F_y}{M} - ru + pw - \tan\beta \left( \frac{F_x}{M} - qw + rv \right) \right] \]

\[ \dot{\gamma} = \frac{u \left( \frac{F_x}{M} - qw + rv \right) + v \left( \frac{F_y}{M} - ru + pw \right) + w \left( \frac{F_z}{M} - pv + qu \right)}{V} \]

Utilizing the definition of \( \alpha, \beta \) and \( V \), these equations simplify to

\[ \dot{\alpha} = \cos^2\alpha \left[ \frac{F_z}{Mu} - p\tan\beta + q - \tan\alpha \left( \frac{F_x}{Mu} - q\tan\alpha + r\tan\beta \right) \right] \]

\[ \dot{\alpha} = \cos^2\alpha \left[ \frac{F_z}{Mu} - \frac{F_x}{Mu} \tan\alpha - p\tan\beta + q(1 + \tan^2\alpha) - r\tan\alpha \tan\beta \right] \]  

(A.26)
\[
\dot{\beta} = \cos^2 \beta \left[ \frac{F_y}{Mu} - r + p \tan \alpha - \tan \beta \left( \frac{F_x}{Mu} - q \tan \alpha + r \tan \beta \right) \right]
\]

\[
\dot{\beta} = \cos^2 \beta \left[ \frac{F_y}{Mu} \tan \beta + p \tan \alpha + q \tan \alpha \tan \beta - r(1 + \tan^2 \beta) \right]
\]  \hspace{1cm} (A.27)

\[\dot{v} = \left[ \frac{1}{Mx^{1/2}} (F_x + F_y \tan \beta + F_z \tan \alpha) \right]\]  \hspace{1cm} (A.28)

where

\[\kappa = 1 + \tan^2 \alpha + \tan^2 \beta\]  \hspace{1cm} (A.29)

Substituting the Pythagorean relation,

\[1 + \tan^2 x = \sec^2 x\]  \hspace{1cm} (A.30)

into Equations (A.26) and (A.27), the time rates of change of \(\alpha\) and \(\beta\) become

\[\dot{\alpha} = \cos^2 \alpha \left[ \frac{1}{Mu} (F_x - F_y \tan \alpha) - p \tan \beta + q \sec^2 \alpha - r \tan \alpha \tan \beta \right]\]  \hspace{1cm} (A.31)

\[\dot{\beta} = \cos^2 \beta \left[ \frac{1}{Mu} (F_y - F_x \tan \beta) + p \tan \alpha + q \tan \alpha \tan \beta - r \sec^2 \alpha \right]\]  \hspace{1cm} (A.32)

Equations (A.10) - (A.12), (A.28), (A.31) and (A.32) yield a three-dimensional, nonlinear description of the rigid airframe dynamics in the body reference frame. In general, the body forces and moments are functions of the state variables and control variables for a specified flight condition.

However, these equations of motion are not amenable for flight control design and analysis. To obtain an appropriate dynamical model, we can approximate the transient behavior of the system using small disturbance theory, which is also known as small perturbation theory. In general, we "linearize" the system and examine the response of the
system at some "perturbed" condition in the neighborhood of an assumed reference flight condition.

In this study, our interest is in the dynamics and two missile accelerations in the directions transverse to the longitudinal body axes. Consequently, we assume that the airspeed \( V \) is approximated as a constant and that the corresponding \( V \) equation can be neglected. Thus, we consider the dynamics of the missile as described by the five equations (A.10) - (A.12), (A.31) and (A.32).

Many references have demonstrated how the equations of motion can be linearized\(^{15,17,18,35,36,39-42} \). The classical approach to linearization is to perturb important variables away from the reference condition by some small quantity (for example, \( x = x_{\text{ref}} + \Delta x \)), substitute into the equations of motion, expand and simplify the equations, in terms of the perturbed quantities. However, the current work will perform the same task but in a slightly different manner using a Taylor Series expansion linearization method described in References 35 and 39.

The method of linearization based on a Taylor Series expansion can be described as follows. Any nonlinear set of dynamical equations can be written in the form

\[
\mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{U}, t) = 0, \quad i = 1, 2, \ldots, n \text{ equations} \tag{A.33}
\]

where \( \mathbf{x} \) is an \( n \)-dimensional state vector and \( \mathbf{U} \) is an \( m \)-dimensional control vector. We assume that the functions \( f_i \) in Equation (A.33) satisfy the hypothesis of the Implicit Function Theorem of mathematical analysis\(^{45} \). Thus, the linearized equations are well defined about a reference condition. For a specified reference condition,
\[ f_i(x, \dot{x}, U, t)_{\text{ref}} = 0, \quad i = 1, 2, \ldots, n \text{ equations} \] \hspace{1cm} (A.34)

If small perturbations are considered to the state and control vectors,

\[ x = x_{\text{ref}} + \Delta x \] \hspace{1cm} (A.35)

\[ \dot{x} = \dot{x}_{\text{ref}} + \Delta \dot{x} \] \hspace{1cm} (A.36)

\[ U = U_{\text{ref}} + \Delta U \] \hspace{1cm} (A.37)

the original nonlinear set of differential equations can then be expressed in the form

\[ f(x_{\text{ref}} + \Delta x, \dot{x}_{\text{ref}} + \Delta \dot{x}, U_{\text{ref}} + \Delta U, t) = 0 \] \hspace{1cm} (A.38)

Expanding this equation in a Taylor Series about the nominal condition,

\[ f(x_{\text{ref}} + \Delta x, \dot{x}_{\text{ref}} + \Delta \dot{x}, U_{\text{ref}} + \Delta U, t) = f(x, \dot{x}, U, t) + \left. \frac{\partial f}{\partial x} \right|_{\text{ref}} \Delta x \]

\[ + \left. \frac{\partial f}{\partial \dot{x}} \right|_{\text{ref}} \Delta \dot{x} + \left. \frac{\partial f}{\partial U} \right|_{\text{ref}} \Delta U + \ldots \] \hspace{1cm} (A.39)

From the definition of the reference condition given in Equation (A.34) and neglecting the higher order terms of the expansion,

\[ \left. \frac{\partial f}{\partial x} \right|_{\text{ref}} \Delta \dot{x} + \left. \frac{\partial f}{\partial \dot{x}} \right|_{\text{ref}} \Delta x + \left. \frac{\partial f}{\partial U} \right|_{\text{ref}} \Delta U = 0 \] \hspace{1cm} (A.40)

Rearranging this equation,

\[ \Delta \dot{x} = \left( \left. \frac{\partial f}{\partial \dot{x}} \right|_{\text{ref}} \right)^{-1} \left( \left. \frac{\partial f}{\partial x} \right|_{\text{ref}} \right) \Delta x - \left( \left. \frac{\partial f}{\partial \dot{x}} \right|_{\text{ref}} \right)^{-1} \left( \left. \frac{\partial f}{\partial U} \right|_{\text{ref}} \right) \Delta U \] \hspace{1cm} (A.41)

This can be written in the standard linear form

\[ \Delta \dot{x} = A \Delta x + B \Delta U \] \hspace{1cm} (A.42)

where

295
\[ A = - \left( \frac{\partial f}{\partial x} \right)^{-1} \left( \frac{\partial f}{\partial x} \right)_{\text{ref}} \]  
(A.43)

\[ B = - \left( \frac{\partial f}{\partial x} \right)^{-1} \left( \frac{\partial f}{\partial U} \right)_{\text{ref}} \]  
(A.44)

and

\[ \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \]  
(A.45)

\[ \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \]  
(A.46)
The A matrix is known as the system matrix and the B matrix is called the input matrix. If the reference condition is chosen to be in equilibrium, then the A and B matrices will be constant. The system in this case is called a linear, time-invariant system.

We begin the linearization of the airframe dynamical equations by writing Equations (A.10) - (A.12) and (A.31), (A.32) in the form of Equation (A.33),

\[ f_1 = \frac{F_z}{M u} \cos^2 \alpha - \frac{F_x}{M u} \sin \alpha \cos \alpha - p \tan \beta \cos^2 \alpha + q - r \sin \alpha \cos \alpha \tan \beta - \dot{\alpha} = 0 \]  

\[ f_1 = \cos^2 \alpha \left[ \frac{F_z}{M u} - p \tan \beta \right] - \sin \alpha \cos \alpha \left[ \frac{F_x}{M u} + r \tan \beta \right] + q - \dot{\alpha} = 0 \] \hspace{0.5cm} (A.48)

\[ f_2 = \frac{F_y}{M u} \cos^2 \beta - \frac{F_x}{M u} \sin \beta \cos \beta + p \tan \alpha \cos^2 \beta + q \tan \alpha \sin \beta \cos \beta - r - \dot{\beta} = 0 \]  

\[ f_2 = \cos^2 \beta \left[ \frac{F_y}{M u} + p \tan \alpha \right] - \sin \beta \cos \beta \left[ \frac{F_x}{M u} - q \tan \alpha \right] - r - \dot{\beta} = 0 \] \hspace{0.5cm} (A.49)

\[ f_3 = \frac{l}{l_x} - \ddot{p} = 0 \] \hspace{0.5cm} (A.50)
\[ f_4 = \frac{m}{I_y} \dot{\phi} - pr \left[ \frac{I_x - I_z}{I_y} \right] = 0 \]  
(A.51)

\[ f_5 = \frac{n}{I_z} \dot{\gamma} - pq \left[ \frac{I_y - I_x}{I_z} \right] = 0 \]  
(A.52)

The state variables are selected to be the angle of attack, angle of sideslip, and the body rotational rates. Thus, the state vector is \( \mathbf{x} = (\alpha \ \beta \ p \ q \ r)^T \).

A tail-controlled missile is steered aerodynamically by deflecting the tail-fins to achieve a desired maneuver. For a skid-to-turn (STT) control system, the roll channel is attitude-stabilized, and the maneuvers are performed in the pitch and yaw planes. To perform this maneuver, the autopilot must command the proper effective pitch, yaw, and roll tail deflections, \( \delta P, \delta Y, \) and \( \delta R \), respectively. Figure A.1 presents the positive definitions of the fin deflections for pitch, yaw, and roll maneuvers. If the missile is stabilized at a zero degree roll attitude, the positive deflections are given in Figure A.1(a), and Figure A.1(b) shows how the fins are used for control when the missile is in the 45 degree roll orientation. The positive arrows in Figure A.1 indicate the direction of the fin leading edges and the direction of the control forces. Note that there are three control commands (pitch, yaw and roll) and four fins. A squeeze mode command is introduced, then, that allows us to determine the four individual fin deflection commands. The squeeze mode, \( \delta SQ \), is usually defined in such a way so that there is no effective maneuver and minimizes the axial force. Relations between the control commands and the individual fin deflections can then be obtained. Table A.1 gives these solutions for two definitions of positive fin deflection.

For missile control, then, the control variables are selected to be the effective pitch, yaw, and roll fin deflections. The control vector, \( \mathbf{U} \), is defined as \( \mathbf{U} = (\delta P \ \delta Y \ \delta R)^T \). We
\(\phi_A = 0 \text{ DEG, PLUS ("+") CONFIGURATION}\)

<table>
<thead>
<tr>
<th>(\delta P, \text{ PITCH COMMAND})</th>
<th>(\delta Y, \text{ YAW COMMAND})</th>
<th>(\delta R, \text{ ROLL COMMAND})</th>
</tr>
</thead>
<tbody>
<tr>
<td>![Diagram 1]</td>
<td>![Diagram 2]</td>
<td>![Diagram 3]</td>
</tr>
</tbody>
</table>

\(\phi_A = 45 \text{ DEG, ("X") CONFIGURATION}\)

<table>
<thead>
<tr>
<th>(\delta P, \text{ PITCH COMMAND})</th>
<th>(\delta Y, \text{ YAW COMMAND})</th>
<th>(\delta R, \text{ ROLL COMMAND})</th>
</tr>
</thead>
<tbody>
<tr>
<td>![Diagram 4]</td>
<td>![Diagram 5]</td>
<td>![Diagram 6]</td>
</tr>
</tbody>
</table>

NOTE: ARROWS SHOW DIRECTION OF TAIL PANEL FORCES.
VIEW LOOKING FORWARD
POSITIVE COMMANDS SHOWN

Figure A.1 Definitions of Fin Deflections
Table A.1  Control Fin Deflection Relations

<table>
<thead>
<tr>
<th></th>
<th>Pitch ($\delta P$)</th>
<th>Yaw ($\delta Y$)</th>
<th>Roll ($\delta R$)</th>
<th>Squeeze ($\delta SQ$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_A = 0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Leading Edge</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Up</td>
<td>$(\delta_2 + \delta_4)/2$</td>
<td>$(\delta_1 + \delta_3)/2$</td>
<td>$(\delta_1 - \delta_2 + \delta_3 + \delta_4)/4$</td>
<td>$(-\delta_1 + \delta_2 + \delta_3 + \delta_4)/4$</td>
</tr>
<tr>
<td>Clockwise</td>
<td>$(-\delta_2 + \delta_4)/2$</td>
<td>$(\delta_1 - \delta_3)/2$</td>
<td>$(\delta_1 + \delta_2 + \delta_3 + \delta_4)/4$</td>
<td>$(\delta_1 - \delta_2 + \delta_3 - \delta_4)/4$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Pitch ($\delta P$)</th>
<th>Yaw ($\delta Y$)</th>
<th>Roll ($\delta R$)</th>
<th>Squeeze ($\delta SQ$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_A = 45$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Leading Edge</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Up</td>
<td>$(\delta_1 + \delta_2 + \delta_3 + \delta_4)/4$</td>
<td>$(-\delta_1 + \delta_2 - \delta_3 + \delta_4)/4$</td>
<td>$(\delta_1 - \delta_2 + \delta_3 + \delta_4)/4$</td>
<td>$(-\delta_1 + \delta_2 + \delta_3 - \delta_4)/4$</td>
</tr>
<tr>
<td>Clockwise</td>
<td>$(-\delta_1 - \delta_2 + \delta_3 + \delta_4)/4$</td>
<td>$(\delta_1 - \delta_2 - \delta_3 + \delta_4)/4$</td>
<td>$(\delta_1 + \delta_2 + \delta_3 + \delta_4)/4$</td>
<td>$(\delta_1 - \delta_2 + \delta_3 - \delta_4)/4$</td>
</tr>
</tbody>
</table>
assume that the forces and moments have the following functional form

\[ F_b = f[MachNo, \alpha, \beta, p, q, r, \delta P, \delta Y, \delta R] \]  \hspace{1cm} (A.53)

\[ M_b = f[MachNo, \alpha, \beta, p, q, r, \delta P, \delta Y, \delta R] \]  \hspace{1cm} (A.54)

Note that the force terms have been assumed not to be functions of thrust or gravity. These contributions are neglected because we are interested in the aerodynamic stability and control aspects of the airframe and also because the effects of these terms can be compensated for in the guidance system. The thrust effects may also be neglected if the rocket is assumed to be at the burnout condition. In addition, the dependence of \( F \) and \( M \) on the state derivatives (\( \dot{\alpha}, \dot{\beta}, \dot{p}, \dot{q}, \dot{r} \)) is negligible.

We will assume the reference condition to be steady-state flight, with the missile flying at the trim condition. Trimmed flight refers to the condition where the total moments acting on the missile are zero. Also, since the missile is assumed to be at burnout at the reference condition, the mass properties of the missile are assumed to be constant.

The linearization is now performed, evaluating the partial derivative matrices defined in Equations (A.41) - (A.47) of the functions defined in Equations (A.48) - (A.52). First, the partial derivative of the functions with respect to the state derivatives (\( \partial f/\partial \dot{x} \)) will be evaluated.

\[ \frac{\partial f_1}{\partial \dot{\alpha}} = -1 \]  \hspace{1cm} (A.55)

\[ \frac{\partial f_1}{\partial \dot{\beta}} = \frac{\partial f_1}{\partial \dot{p}} = \frac{\partial f_1}{\partial \dot{q}} = \frac{\partial f_1}{\partial \dot{r}} = 0 \]  \hspace{1cm} (A.56)
\[
\frac{\partial f_2}{\partial \dot{\phi}} = -1 \\
\frac{\partial f_2}{\partial \dot{\alpha}} = \frac{\partial f_2}{\partial \dot{\beta}} = \frac{\partial f_2}{\partial \dot{q}} = \frac{\partial f_2}{\partial \dot{r}} = 0
\]

(A.57)

\[
\frac{\partial f_3}{\partial \dot{\alpha}} = \frac{\partial f_3}{\partial \dot{\beta}} = \frac{\partial f_3}{\partial \dot{q}} = \frac{\partial f_3}{\partial \dot{r}} = 0
\]

(A.58)

\[
\frac{\partial f_4}{\partial \dot{\alpha}} = \frac{\partial f_4}{\partial \dot{\beta}} = \frac{\partial f_4}{\partial \dot{q}} = \frac{\partial f_4}{\partial \dot{r}} = 0
\]

(A.59)

\[
\frac{\partial f_5}{\partial \dot{\alpha}} = \frac{\partial f_5}{\partial \dot{\beta}} = \frac{\partial f_5}{\partial \dot{q}} = \frac{\partial f_5}{\partial \dot{r}} = 0
\]

(A.60)

Thus,

\[
\frac{\partial f}{\partial \dot{x}} = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1
\end{bmatrix} = -I
\]

(A.61)

(A.62)

(A.63)

(A.64)

Using this result, Equations (A.43) and (A.44) simplify to

\[
A = \frac{\partial f'}{\partial \dot{x}_{ref}}
\]

(A.65)

(A.66)
Next, the partial derivative of $f$ with respect to the state variables, (i.e., the A matrix), is evaluated.

\[
B = \frac{\partial f}{\partial U}_{\text{ref}} \tag{A.67}
\]

\[
\frac{\partial f_1}{\partial \alpha} = -2 \cos \alpha \sin \alpha \left[ \frac{F_z}{Mu} - p \tan \beta \right] + \cos^2 \alpha \left[ \frac{1}{Mu} \frac{\partial F_z}{\partial \alpha} \right] - (\sin^2 \alpha - \cos^2 \alpha) \left[ \frac{F_x}{Mu} + r \tan \beta \right] - \sin \alpha \cos \alpha \left[ \frac{1}{Mu} \frac{\partial F_x}{\partial \alpha} \right]_{\text{ref}} \tag{A.68}
\]

\[
\frac{\partial f_1}{\partial \beta} = \cos^2 \alpha \left[ \frac{1}{Mu} \frac{\partial F_z}{\partial \beta} - F_x \frac{\partial \tan \beta}{\cos^2 \beta} \right] + \sin^2 \alpha \left[ \frac{F_x}{Mu} + r \tan \beta \right] + \sin \alpha \cos \alpha \left[ \frac{2F_z}{Mu} - \frac{1}{Mu} \frac{\partial F_z}{\partial \beta} \right]_{\text{ref}} \tag{A.69}
\]

\[
\frac{\partial f_1}{\partial p} = \cos^2 \alpha \left[ \frac{1}{Mu} \frac{\partial F_z}{\partial p} - \tan \beta \right] - \sin \alpha \cos \alpha \left[ \frac{1}{Mu} \frac{\partial F_x}{\partial p} \right]_{\text{ref}} \tag{A.70}
\]

\[
\frac{\partial f_1}{\partial q} = \cos^2 \alpha \left[ \frac{1}{Mu} \frac{\partial F_z}{\partial q} \right] - \sin \alpha \cos \alpha \left[ \frac{1}{Mu} \frac{\partial F_x}{\partial q} \right] + 1_{\text{ref}} \tag{A.71}
\]

\[
\frac{\partial f_1}{\partial r} = \cos^2 \alpha \left[ \frac{1}{Mu} \frac{\partial F_z}{\partial r} \right] - \sin \alpha \cos \alpha \left[ \frac{1}{Mu} \frac{\partial F_x}{\partial r} + \tan \beta \right]_{\text{ref}} \tag{A.72}
\]

\[
\frac{\partial f_2}{\partial \alpha} = \cos^2 \beta \left[ \frac{1}{Mu} \frac{\partial F_y}{\partial \alpha} + \frac{P}{\cos^2 \alpha} \right] - \sin \beta \cos \beta \left[ \frac{q}{\cos^2 \alpha} - \frac{1}{Mu} \frac{\partial F_x}{\partial \alpha} \right]_{\text{ref}} \tag{A.73}
\]
\[ \frac{\partial f_2}{\partial \beta} = -2 \cos \beta \sin \beta \left[ \frac{F_y}{Mu} + p \tan \alpha \right] + \cos^2 \beta \left[ \frac{1}{Mu} \left( \frac{\partial F_y}{\partial \beta} \right) \right] \\
+ (\sin^2 \beta - \cos^2 \beta) \left[ \frac{F_x}{Mu} - q \tan \alpha \right] - \sin \beta \cos \beta \left[ \frac{1}{Mu} \left( \frac{\partial F_x}{\partial \beta} \right) \right] \]

\[ \frac{\partial f_2}{\partial \beta} = \cos^2 \beta \left[ \frac{1}{Mu} \left( \frac{\partial F_y}{\partial \beta} \right) - \frac{F_x}{Mu} + q \tan \alpha \right] + \sin^2 \beta \left[ \frac{F_x}{Mu} - q \tan \alpha \right] \\
- \sin \beta \cos \beta \left[ 2p \tan \alpha + \frac{2F_y}{Mu} + \frac{1}{Mu} \left( \frac{\partial F_x}{\partial \beta} \right) \right] \]  

(A.74)

\[ \frac{\partial f_2}{\partial p} = \cos^2 \beta \left[ \frac{1}{Mu} \left( \frac{\partial F_y}{\partial p} \right) + \tan \alpha \right] - \sin \beta \cos \beta \left[ \frac{1}{Mu} \left( \frac{\partial F_x}{\partial p} \right) \right] \]  

(A.75)

\[ \frac{\partial f_2}{\partial q} = \cos^2 \beta \left[ \frac{1}{Mu} \left( \frac{\partial F_y}{\partial q} \right) \right] - \sin \beta \cos \beta \left[ \frac{1}{Mu} \left( \frac{\partial F_x}{\partial q} \right) - \tan \alpha \right] \]  

(A.76)

\[ \frac{\partial f_2}{\partial r} = \cos^2 \beta \left[ \frac{1}{Mu} \left( \frac{\partial F_y}{\partial r} \right) \right] - \sin \beta \cos \beta \left[ \frac{1}{Mu} \left( \frac{\partial F_x}{\partial r} \right) \right] - 1 \]  

(A.77)

\[ \frac{\partial f_3}{\partial \alpha} = \frac{1}{I_x} \left( \frac{\partial I}{\partial \alpha} \right) \]  

(A.78)

\[ \frac{\partial f_3}{\partial \beta} = \frac{1}{I_x} \left( \frac{\partial I}{\partial \beta} \right) \]  

(A.79)

\[ \frac{\partial f_3}{\partial p} = \frac{1}{I_x} \left( \frac{\partial I}{\partial p} \right) \]  

(A.80)
\[
\frac{\partial f_5}{\partial q} = \frac{1}{I_x} \left( \frac{\partial l}{\partial q} \right)_{ref}
\] (A.81)

\[
\frac{\partial f_5}{\partial r} = \frac{1}{I_x} \left( \frac{\partial l}{\partial r} \right)_{ref}
\] (A.82)

\[
\frac{\partial f_4}{\partial \alpha} = \frac{1}{I_y} \left( \frac{\partial m}{\partial \alpha} \right)_{ref}
\] (A.83)

\[
\frac{\partial f_4}{\partial \beta} = \frac{1}{I_y} \left( \frac{\partial m}{\partial \beta} \right)_{ref}
\] (A.84)

\[
\frac{\partial f_4}{\partial p} = \frac{1}{I_y} \left( \frac{\partial m}{\partial p} \right) - r \left( \frac{I_x - I_z}{I_y} \right)_{ref}
\] (A.85)

\[
\frac{\partial f_4}{\partial q} = \frac{1}{I_y} \left( \frac{\partial m}{\partial q} \right)_{ref}
\] (A.86)

\[
\frac{\partial f_4}{\partial r} = \frac{1}{I_y} \left( \frac{\partial m}{\partial r} \right) + p \left( \frac{I_x - I_z}{I_y} \right)_{ref}
\] (A.87)

\[
\frac{\partial f_5}{\partial \alpha} = \frac{1}{I_x} \left( \frac{\partial n}{\partial \alpha} \right)_{ref}
\] (A.88)

\[
\frac{\partial f_5}{\partial \beta} = \frac{1}{I_x} \left( \frac{\partial n}{\partial \beta} \right)_{ref}
\] (A.89)

\[
\frac{\partial f_5}{\partial p} = \frac{1}{I_x} \left( \frac{\partial n}{\partial p} \right) + q \left( \frac{I_x - I_y}{I_x} \right)_{ref}
\] (A.90)

\[
\frac{\partial f_5}{\partial q} = \frac{1}{I_x} \left( \frac{\partial n}{\partial q} \right) + p \left( \frac{I_x - I_y}{I_x} \right)_{ref}
\] (A.91)

\[
\frac{\partial f_5}{\partial r} = \frac{1}{I_x} \left( \frac{\partial n}{\partial r} \right)_{ref}
\] (A.92)
In addition, Equation (A.67) is applied to find the control matrix, B.

\[
\frac{\partial f_1}{\partial \delta P} = \cos^2 \alpha \left[ \frac{1}{M_u} \left( \frac{\partial F_x}{\partial \delta P} \right) \right] - \sin \alpha \cos \alpha \left[ \frac{1}{M_u} \left( \frac{\partial F_x}{\partial \delta P} \right) \right]_{\text{ref}} \tag{A.93}
\]

\[
\frac{\partial f_1}{\partial \delta Y} = \cos^2 \alpha \left[ \frac{1}{M_u} \left( \frac{\partial F_y}{\partial \delta Y} \right) \right] - \sin \alpha \cos \alpha \left[ \frac{1}{M_u} \left( \frac{\partial F_x}{\partial \delta Y} \right) \right]_{\text{ref}} \tag{A.94}
\]

\[
\frac{\partial f_1}{\partial \delta R} = \cos^2 \alpha \left[ \frac{1}{M_u} \left( \frac{\partial F_z}{\partial \delta R} \right) \right] - \sin \alpha \cos \alpha \left[ \frac{1}{M_u} \left( \frac{\partial F_x}{\partial \delta R} \right) \right]_{\text{ref}} \tag{A.95}
\]

\[
\frac{\partial f_2}{\partial \delta P} = \cos^2 \beta \left[ \frac{1}{M_u} \left( \frac{\partial F_y}{\partial \delta P} \right) \right] - \sin \beta \cos \beta \left[ \frac{1}{M_u} \left( \frac{\partial F_x}{\partial \delta P} \right) \right]_{\text{ref}} \tag{A.96}
\]

\[
\frac{\partial f_2}{\partial \delta Y} = \cos^2 \beta \left[ \frac{1}{M_u} \left( \frac{\partial F_y}{\partial \delta Y} \right) \right] - \sin \beta \cos \beta \left[ \frac{1}{M_u} \left( \frac{\partial F_x}{\partial \delta Y} \right) \right]_{\text{ref}} \tag{A.97}
\]

\[
\frac{\partial f_2}{\partial \delta R} = \cos^2 \beta \left[ \frac{1}{M_u} \left( \frac{\partial F_y}{\partial \delta R} \right) \right] - \sin \beta \cos \beta \left[ \frac{1}{M_u} \left( \frac{\partial F_x}{\partial \delta R} \right) \right]_{\text{ref}} \tag{A.98}
\]

\[
\frac{\partial f_3}{\partial \delta P} = \frac{1}{I_x} \left( \frac{\partial l}{\partial \delta P} \right)_{\text{ref}} \tag{A.99}
\]

\[
\frac{\partial f_3}{\partial \delta Y} = \frac{1}{I_x} \left( \frac{\partial l}{\partial \delta Y} \right)_{\text{ref}} \tag{A.100}
\]

\[
\frac{\partial f_3}{\partial \delta R} = \frac{1}{I_x} \left( \frac{\partial l}{\partial \delta R} \right)_{\text{ref}} \tag{A.101}
\]

\[
\frac{\partial f_4}{\partial \delta P} = \frac{1}{I_y} \left( \frac{\partial m}{\partial \delta P} \right)_{\text{ref}} \tag{A.102}
\]
\[
\frac{\partial f_i}{\partial \delta Y} = \frac{1}{I_y} \left( \frac{\partial m}{\partial \delta Y} \right)_{\text{ref}} \tag{A.103}
\]

\[
\frac{\partial f_i}{\partial \delta R} = \frac{1}{I_y} \left( \frac{\partial m}{\partial \delta R} \right)_{\text{ref}} \tag{A.104}
\]

\[
\frac{\partial f_s}{\partial \delta P} = \frac{1}{I_z} \left( \frac{\partial n}{\partial \delta P} \right)_{\text{ref}} \tag{A.105}
\]

\[
\frac{\partial f_s}{\partial \delta Y} = \frac{1}{I_z} \left( \frac{\partial n}{\partial \delta Y} \right)_{\text{ref}} \tag{A.106}
\]

\[
\frac{\partial f_s}{\partial \delta R} = \frac{1}{I_z} \left( \frac{\partial n}{\partial \delta R} \right)_{\text{ref}} \tag{A.107}
\]

Recall that the parameters defined in Equations (A.68) - (A.107) are evaluated at the specified reference condition. Thus, we have transformed the set of nonlinear differential equations to a set of coupled, linear equations of the state-space form

\[
i = Ax + BU \tag{A.108}
\]

where

\[
A = a_{ij} = \frac{\partial f_i}{\partial x_j} \tag{A.109}
\]

and

\[
B = b_{ik} = \frac{\partial f_i}{\partial U_k} \tag{A.110}
\]

This set of equations describes the system dynamics at some small deviation from the nominal operating point.

The system output vector, Y, can be related to the state and control vectors by
\[ Y = Cx + DU \]  

(A.111)

For our missile system, we will assume that the outputs are the lateral body accelerations, \( A_y \) and \( A_z \), and the body rotation rates, \( p \), \( q \), and \( r \). The output vector is then \( Y = (A_y \ A_z \ p \ q \ r)^T \). The missile accelerations are defined as

\[ A_y = \frac{F_y}{M} \]  

(A.112)

\[ A_z = \frac{F_z}{M} \]  

(A.113)

Expressions for \( F_y \) and \( F_z \) were given in Equations (A.8) and (A.9). But, since we assume that the thrust and gravity contributions can be neglected, only the aerodynamic forces are considered.

In accordance with the definition of the reference condition, the aerodynamic forces and moments are also linearized with a Taylor Series expansion about the trim point. For example,

\[ F_x = F_x|_{\text{ref}} + \frac{\partial F_x}{\partial \alpha}|_{\text{ref}} \Delta \alpha + \frac{\partial F_x}{\partial \beta}|_{\text{ref}} \Delta \beta + \ldots + \frac{\partial F_x}{\partial P}|_{\text{ref}} \Delta P \]

[\text{ref}]  

(A.114)

[\text{ref}]

\[ + \frac{\partial F_x}{\partial Y}|_{\text{ref}} \Delta Y + \ldots + \frac{\partial F_x}{\partial \dot\alpha}|_{\text{ref}} \Delta \dot\alpha + \ldots \]

The higher order terms and contributions due to state derivatives are neglected because they are relatively small compared to the other terms. Equation (A.114) simplifies to
\[
\Delta F_x = \frac{\partial F_x}{\partial \alpha} \Delta \alpha + \frac{\partial F_x}{\partial \beta} \Delta \beta + \frac{\partial F_x}{\partial p} \Delta p + \frac{\partial F_x}{\partial q} \Delta q
\]
\[
+ \frac{\partial F_x}{\partial r} \Delta r + \frac{\partial F_x}{\partial \delta P} \Delta \delta P + \frac{\partial F_x}{\partial \delta Y} \Delta \delta Y + \frac{\partial F_x}{\partial \delta R} \Delta \delta R
\]  
(A.115)

If we change the notation to reflect that the parameters are perturbations away from the reference point,

\[
F_x = \frac{\partial F_x}{\partial \alpha} \Delta \alpha + \frac{\partial F_x}{\partial \beta} \Delta \beta + \frac{\partial F_x}{\partial p} \Delta p + \frac{\partial F_x}{\partial q} \Delta q
\]
\[
+ \frac{\partial F_x}{\partial r} \Delta r + \frac{\partial F_x}{\partial \delta P} \Delta \delta P + \frac{\partial F_x}{\partial \delta Y} \Delta \delta Y + \frac{\partial F_x}{\partial \delta R} \Delta \delta R
\]  
(A.116)

This expansion can be performed for all the forces and moments. Note that \( \mathbf{M}_{\text{ref}} = 0 \) at the trim reference condition. Similarly, then

\[
F_y = \frac{\partial F_y}{\partial \alpha} \Delta \alpha + \frac{\partial F_y}{\partial \beta} \Delta \beta + \frac{\partial F_y}{\partial p} \Delta p + \frac{\partial F_y}{\partial q} \Delta q
\]
\[
+ \frac{\partial F_y}{\partial r} \Delta r + \frac{\partial F_y}{\partial \delta P} \Delta \delta P + \frac{\partial F_y}{\partial \delta Y} \Delta \delta Y + \frac{\partial F_y}{\partial \delta R} \Delta \delta R
\]  
(A.117)

\[
F_z = \frac{\partial F_z}{\partial \alpha} \Delta \alpha + \frac{\partial F_z}{\partial \beta} \Delta \beta + \frac{\partial F_z}{\partial p} \Delta p + \frac{\partial F_z}{\partial q} \Delta q
\]
\[
+ \frac{\partial F_z}{\partial r} \Delta r + \frac{\partial F_z}{\partial \delta P} \Delta \delta P + \frac{\partial F_z}{\partial \delta Y} \Delta \delta Y + \frac{\partial F_z}{\partial \delta R} \Delta \delta R
\]  
(A.118)

\[
l = \frac{\partial l}{\partial \alpha} \Delta \alpha + \frac{\partial l}{\partial \beta} \Delta \beta + \frac{\partial l}{\partial p} \Delta p + \frac{\partial l}{\partial q} \Delta q + \frac{\partial l}{\partial r} \Delta r
\]
\[
+ \frac{\partial l}{\partial \delta P} \Delta \delta P + \frac{\partial l}{\partial \delta Y} \Delta \delta Y + \frac{\partial l}{\partial \delta R} \Delta \delta R
\]  
(A.119)
\[ m = \frac{\partial m}{\partial \alpha_{\text{ref}}} \alpha + \frac{\partial m}{\partial \beta_{\text{ref}}} \beta + \frac{\partial m}{\partial \rho_{\text{ref}}} \rho + \frac{\partial m}{\partial q_{\text{ref}}} q + \frac{\partial m}{\partial r_{\text{ref}}} r 
 + \frac{\partial m}{\partial \delta P_{\text{ref}}} \delta P + \frac{\partial m}{\partial \delta Y_{\text{ref}}} \delta Y + \frac{\partial m}{\partial \delta R_{\text{ref}}} \delta R \] (A.120)

\[ n = \frac{\partial n}{\partial \alpha_{\text{ref}}} \alpha + \frac{\partial n}{\partial \beta_{\text{ref}}} \beta + \frac{\partial n}{\partial \rho_{\text{ref}}} \rho + \frac{\partial n}{\partial q_{\text{ref}}} q + \frac{\partial n}{\partial r_{\text{ref}}} r 
 + \frac{\partial n}{\partial \delta P_{\text{ref}}} \delta P + \frac{\partial n}{\partial \delta Y_{\text{ref}}} \delta Y + \frac{\partial n}{\partial \delta R_{\text{ref}}} \delta R \] (A.121)

The aerodynamic forces and moments can be related to dimensionless coefficients by the following expressions:

\[ X_A = F_X = QSC_X = -QSC_A = -A \] (A.122)

\[ Y_A = F_Y = Y = QSC_T \] (A.123)

\[ Z_A = F_Z = QSC_Z = -QSC_N = -N \] (A.124)

\[ l = QSDC_l \] (A.125)

\[ m = QSDC_m \] (A.126)

\[ n = QSDC_n \] (A.127)

where \( Q \) is the dynamic pressure (\( = \frac{1}{2} \rho V^2 \)), \( S \) is the reference area (for missiles, it is the body cross-sectional area), and \( D \) is the reference length (the missile diameter). Equations (A.122) - (A.127) can also be written as a function of the Taylor Series expansions in Equations (A.116) - (A.121),

\[ X_A = F_X = -A = -QS \left[ C_{A_x} \alpha + C_{A_y} \beta + \left( \frac{D}{2V} \right) \left( C_{A_P} P + C_{A_q} q + C_{A_r} r \right) + C_{A_{\delta P}} \delta P + C_{A_{\delta Y}} \delta Y + C_{A_{\delta R}} \delta R \right] \] (A.128)

310
\[ Y_A = F_Y = Y = QS \left[ C_{Y_a} \alpha + C_{Y_q} q + \left( \frac{D}{2V} \right) C_{Y_p} p + C_{Y_r} r \right] + C_{Y_a} \delta P + C_{Y_q} \delta Y + C_{Y_r} \delta R \] (A.129)

\[ Z_A = F_Z = -N = -QS \left[ C_{N_a} \alpha + C_{N_q} q + \left( \frac{D}{2V} \right) C_{N_p} p + C_{N_r} r \right] + C_{N_a} \delta P + C_{N_q} \delta Y + C_{N_r} \delta R \] (A.130)

\[ l = QSD \left[ C_{l_a} \alpha + C_{l_q} q + \left( \frac{D}{2V} \right) C_{l_p} p + C_{l_r} r \right] + C_{l_a} \delta P + C_{l_q} \delta Y + C_{l_r} \delta R \] (A.131)

\[ m = QSD \left[ C_{m_a} \alpha + C_{m_q} q + \left( \frac{D}{2V} \right) C_{m_p} p + C_{m_r} r \right] + C_{m_a} \delta P + C_{m_q} \delta Y + C_{m_r} \delta R \] (A.132)

\[ n = QSD \left[ C_{n_a} \alpha + C_{n_q} q + \left( \frac{D}{2V} \right) C_{n_p} p + C_{n_r} r \right] + C_{n_a} \delta P + C_{n_q} \delta Y + C_{n_r} \delta R \] (A.133)

The derivatives of the aerodynamic coefficients (for example, \( \left( \frac{1}{QS} \right) \frac{\partial F}{\partial \alpha} = \frac{\partial C_{X_a}}{\partial \alpha} = C_{X_a} \)) are called the aerodynamic stability derivatives. A description of the important stability derivatives can be found in References 7, 40, and 42.

The aerodynamic force expansions given above can be utilized to define the lateral accelerations, \( A_Y \) and \( A_Z \). From Equations (A.112) and (A.113), the yaw and pitch accelerations are

\[ A_Y = \frac{F_Y}{M} = \frac{Y}{M} = QS \left[ C_{Y_a} \alpha + C_{Y_q} q + \left( \frac{D}{2V} \right) C_{Y_p} p + C_{Y_r} r \right] + C_{Y_a} \delta P + C_{Y_q} \delta Y + C_{Y_r} \delta R \] (A.134)
\[ A_z = \frac{F_z}{M} = -\frac{N}{M} = -\frac{Q S}{M} \left[ C_{N_4} \alpha + C_{N_p} \beta + \left( \frac{D}{2V} \right) \left( C_{N_4} P + C_{N_q} Q + C_{N_r} r \right) + C_{N_p} \delta P + C_{N_2} \delta Y + C_{N_r} \delta R \right] \] (A.135)

Utilizing the definition of the stability derivatives, the linear airframe dynamical equations can be rewritten. The elements of the system matrix, \( A \), and the control matrix, \( B \), become

\[ a_{11} = \frac{\partial f_1}{\partial \alpha} = \cos^2 \alpha \left[ -\frac{Q S}{M u} C_{N_4} \alpha - \frac{F_X}{M u} - r \tan \beta \right] + \sin^2 \alpha \left[ \frac{F_X}{M u} + r \tan \beta \right] + \sin \alpha \cos \alpha \left[ 2 p \tan \beta - \frac{2 F_z}{M u} + \frac{Q S}{M u} C_{A_4} \right] \] (A.136)

\[ a_{12} = \frac{\partial f_1}{\partial \beta} = \cos^2 \alpha \left[ -\frac{Q S}{M u} C_{N_4} \alpha - \frac{P}{\cos^2 \beta} \right] \] (A.137)

\[ a_{13} = \frac{\partial f_1}{\partial p} = \cos^2 \alpha \left[ -\frac{Q S}{M u} \left( \frac{D}{2V} \right) C_{N_4} \alpha - \tan \beta \right] - \sin \alpha \cos \alpha \left[ -\frac{Q S}{M u} \left( \frac{D}{2V} \right) C_{A_4} \right] \] (A.138)

\[ a_{14} = \frac{\partial f_1}{\partial q} = \cos^2 \alpha \left[ -\frac{Q S}{M u} \left( \frac{D}{2V} \right) C_{N_4} \alpha \right] - \sin \alpha \cos \alpha \left[ -\frac{Q S}{M u} \left( \frac{D}{2V} \right) C_{A_4} + 1 \right] \] (A.139)

\[ a_{15} = \frac{\partial f_1}{\partial r} = \cos^2 \alpha \left[ -\frac{Q S}{M u} \left( \frac{D}{2V} \right) C_{N_4} \alpha \right] - \sin \alpha \cos \alpha \left[ -\frac{Q S}{M u} \left( \frac{D}{2V} \right) C_{A_4} + \tan \beta \right] \] (A.140)

\[ a_{21} = \frac{\partial f_2}{\partial \alpha} = \cos^2 \beta \left[ \frac{Q S}{M u} C_{N_4} \alpha + \frac{P}{\cos^2 \alpha} \right] - \sin \beta \cos \beta \left[ -\frac{q}{\cos^2 \alpha} - \frac{Q S}{M u} C_{A_4} \right] \] (A.141)

312
\[ a_{22} = \frac{\partial f_2}{\partial \rho} = \cos^2 \beta \left[ \frac{QS}{Mu} C_{y_b} \left( F_x + q \tan \alpha \right) \right] + \sin^2 \beta \left[ -\frac{F_x}{Mu} - q \tan \alpha \right] \]

\[ - \sin \beta \cos \beta \left[ 2 \rho \tan \alpha + 2 F_y \frac{Q}{Mu} - \frac{Q}{Mu} C_{A_b} \right] \text{ref} \tag{A.142} \]

\[ a_{23} = \frac{\partial f_2}{\partial \rho} = \cos^2 \beta \left[ \frac{Q}{Mu} \left( \frac{D}{2V} \right) C_{y_b} + \tan \alpha \right] - \sin \beta \cos \beta \left[ - \frac{Q}{Mu} \left( \frac{D}{2V} \right) C_{A_b} \right] \text{ref} \tag{A.143} \]

\[ a_{24} = \frac{\partial f_2}{\partial \rho} = \cos^2 \beta \left[ \frac{Q}{Mu} \left( \frac{D}{2V} \right) C_{y_b} \right] - \sin \beta \cos \beta \left[ - \frac{Q}{Mu} \left( \frac{D}{2V} \right) C_{A_b} - \tan \alpha \right] \text{ref} \tag{A.144} \]

\[ a_{25} = \frac{\partial f_2}{\partial r} = \cos^2 \beta \left[ \frac{Q}{Mu} \left( \frac{D}{2V} \right) C_{y_b} \right] - \sin \beta \cos \beta \left[ - \frac{Q}{Mu} \left( \frac{D}{2V} \right) C_{A_b} \right] - 1 \text{ref} \tag{A.145} \]

\[ a_{31} = \frac{\partial f_3}{\partial \alpha} = \frac{QSD}{I_x} C_{i_s} \text{ref} \tag{A.146} \]

\[ a_{32} = \frac{\partial f_3}{\partial \beta} = \frac{QSD}{I_x} C_{i_b} \text{ref} \tag{A.147} \]

\[ a_{33} = \frac{\partial f_3}{\partial \rho} = \frac{QSD}{I_x} \left( \frac{D}{2V} \right) C_{i_s} \text{ref} \tag{A.148} \]

\[ a_{34} = \frac{\partial f_3}{\partial \rho} = \frac{QSD}{I_x} \left( \frac{D}{2V} \right) C_{i_s} \text{ref} \tag{A.149} \]
\[ a_{35} = \frac{\partial f}{\partial r} = \frac{QSD}{I_x} \left( \frac{D}{2V} \right) C_{r_{\text{ref}}} \]  
(A.150)

\[ a_{41} = \frac{\partial f}{\partial \alpha} = \frac{QSD}{I_y} C_{\alpha_{\text{ref}}} \]  
(A.151)

\[ a_{42} = \frac{\partial f}{\partial \beta} = \frac{QSD}{I_y} C_{\beta_{\text{ref}}} \]  
(A.152)

\[ a_{43} = \frac{\partial f}{\partial \psi} = \frac{QSD}{I_y} \left( \frac{D}{2V} \right) C_{m_{\text{ref}}} - r \left( \frac{I_x - I_y}{I_y} \right) \]  
(A.153)

\[ a_{44} = \frac{\partial f}{\partial q} = \frac{QSD}{I_y} \left( \frac{D}{2V} \right) C_{m_{\text{ref}}} \]  
(A.154)

\[ a_{45} = \frac{\partial f}{\partial \tau} = \frac{QSD}{I_y} \left( \frac{D}{2V} \right) C_{m_{\text{ref}}} + p \left( \frac{I_x - I_y}{I_y} \right) \]  
(A.155)

\[ a_{51} = \frac{\partial f}{\partial \alpha} = \frac{QSD}{I_z} C_{\alpha_{\text{ref}}} \]  
(A.156)

\[ a_{52} = \frac{\partial f}{\partial \beta} = \frac{QSD}{I_z} C_{\beta_{\text{ref}}} \]  
(A.157)

\[ a_{53} = \frac{\partial f}{\partial \psi} = \frac{QSD}{I_z} \left( \frac{D}{2V} \right) C_{m_{\text{ref}}} + q \left( \frac{I_x - I_y}{I_z} \right) \]  
(A.158)

\[ a_{54} = \frac{\partial f}{\partial q} = \frac{QSD}{I_z} \left( \frac{D}{2V} \right) C_{m_{\text{ref}}} + p \left( \frac{I_x - I_y}{I_z} \right) \]  
(A.159)
\[ a_{ss} = \frac{\partial f_3}{\partial r} = \frac{QSD}{I_x} \left( \frac{D}{2V} \right) C_{ns} \bigg|_{ref} \] (A.160)

\[ b_{11} = \frac{\partial f_1}{\partial \delta P} = \cos^2 \alpha \left[ -\frac{QS}{M_u} C_{ntr} \right] - \sin \alpha \cos \alpha \left[ -\frac{QS}{M_u} C_{atr} \right] \bigg|_{ref} \] (A.161)

\[ b_{12} = \frac{\partial f_1}{\partial \delta Y} = \cos^2 \alpha \left[ -\frac{QS}{M_u} C_{ntr} \right] - \sin \alpha \cos \alpha \left[ -\frac{QS}{M_u} C_{atr} \right] \bigg|_{ref} \] (A.162)

\[ b_{13} = \frac{\partial f_1}{\partial \delta R} = \cos^2 \alpha \left[ -\frac{QS}{M_u} C_{ntr} \right] - \sin \alpha \cos \alpha \left[ -\frac{QS}{M_u} C_{atr} \right] \bigg|_{ref} \] (A.163)

\[ b_{21} = \frac{\partial f_2}{\partial \delta P} = \cos^2 \beta \left[ \frac{QS}{M_u} C_{rtr} \right] - \sin \beta \cos \beta \left[ -\frac{QS}{M_u} C_{atr} \right] \bigg|_{ref} \] (A.164)

\[ b_{22} = \frac{\partial f_2}{\partial \delta Y} = \cos^2 \beta \left[ \frac{QS}{M_u} C_{rtr} \right] - \sin \beta \cos \beta \left[ -\frac{QS}{M_u} C_{atr} \right] \bigg|_{ref} \] (A.165)

\[ b_{23} = \frac{\partial f_2}{\partial \delta R} = \cos^2 \beta \left[ \frac{QS}{M_u} C_{rtr} \right] - \sin \beta \cos \beta \left[ -\frac{QS}{M_u} C_{atr} \right] \bigg|_{ref} \] (A.166)

\[ b_{31} = \frac{\partial f_3}{\partial \delta P} = \left( \frac{QSD}{I_x} \right) C_{atr} \bigg|_{ref} \] (A.167)

\[ b_{32} = \frac{\partial f_3}{\partial \delta Y} = \left( \frac{QSD}{I_x} \right) C_{atr} \bigg|_{ref} \] (A.168)

\[ b_{33} = \frac{\partial f_3}{\partial \delta R} = \left( \frac{QSD}{I_x} \right) C_{atr} \bigg|_{ref} \] (A.169)

315
\[ b_{41} = \frac{\partial f_4}{\partial \delta P} = \left( \frac{QSD}{I_y} \right) C_{max}^{ref} \]  
(A.170)

\[ b_{42} = \frac{\partial f_4}{\partial \delta Y} = \left( \frac{QSD}{I_y} \right) C_{max}^{ref} \]  
(A.171)

\[ b_{43} = \frac{\partial f_4}{\partial \delta R} = \left( \frac{QSD}{I_y} \right) C_{max}^{ref} \]  
(A.172)

\[ b_{51} = \frac{\partial f_5}{\partial \delta P} = \left( \frac{QSD}{I_z} \right) C_{nap}^{ref} \]  
(A.173)

\[ b_{52} = \frac{\partial f_5}{\partial \delta Y} = \left( \frac{QSD}{I_z} \right) C_{nap}^{ref} \]  
(A.174)

\[ b_{53} = \frac{\partial f_5}{\partial \delta R} = \left( \frac{QSD}{I_z} \right) C_{nap}^{ref} \]  
(A.175)

Similarly, the C and D matrices of Equations (A.111) can be defined using the accelerations in Equations (A.134) and (A.135),

\[ c_{11} = \frac{Q \bar{S}}{M} C_{r_e} \]  
(A.176)

\[ c_{12} = \frac{Q \bar{S}}{M} C_{r_s} \]  
(A.177)

\[ c_{13} = \frac{Q \bar{S}}{M} \left( \frac{D}{2V} \right) C_{r_e} \]  
(A.178)
\[ c_{14} = \frac{QS}{M} \left( \frac{D}{2V} \right) c_{N_e} \]  
(A.179)

\[ c_{15} = \frac{QS}{M} \left( \frac{D}{2V} \right) c_{N_e} \]  
(A.180)

\[ c_{21} = -\frac{QS}{M} c_{N_e} \]  
(A.181)

\[ c_{22} = -\frac{QS}{M} c_{N_e} \]  
(A.182)

\[ c_{23} = -\frac{QS}{M} \left( \frac{D}{2V} \right) c_{N_e} \]  
(A.183)

\[ c_{24} = -\frac{QS}{M} \left( \frac{D}{2V} \right) c_{N_e} \]  
(A.184)

\[ c_{25} = -\frac{QS}{M} \left( \frac{D}{2V} \right) c_{N_e} \]  
(A.185)

\[ c_{31} = c_{32} = c_{34} = c_{35} = 0 \]  
(A.186)

\[ c_{33} = 1 \]  
(A.187)

\[ c_{41} = c_{42} = c_{43} = c_{45} = 0 \]  
(A.188)

\[ c_{44} = 0 \]  
(A.189)

\[ c_{51} = c_{52} = c_{53} = c_{54} = 0 \]  
(A.190)

\[ c_{55} = 0 \]  
(A.191)
\[ d_{11} = \frac{QS}{M} C_{\nu_{xy}} \]  
(A.192)

\[ d_{12} = \frac{QS}{M} C_{\nu_{sz}} \]  
(A.193)

\[ d_{13} = \frac{QS}{M} C_{\nu_{sx}} \]  
(A.194)

\[ d_{21} = -\frac{QS}{M} C_{N_{sy}} \]  
(A.195)

\[ d_{22} = -\frac{QS}{M} C_{N_{sx}} \]  
(A.196)

\[ d_{23} = -\frac{QS}{M} C_{N_{sz}} \]  
(A.197)

\[ d_{31} = d_{32} = d_{33} = 0 \]  
(A.198)

\[ d_{41} = d_{42} = d_{43} = 0 \]  
(A.199)

\[ d_{51} = d_{52} = d_{53} = 0 \]  
(A.200)

We have thus obtained our linear, coupled, state-space model of the missile airframe at a specified flight condition, with the form

\[ \dot{x} = Ax + Bu \]

\[ y =Cx + Du \]

where
\[
\begin{bmatrix}
\alpha \\
\beta \\
p \\
q \\
r
\end{bmatrix} = 
\begin{bmatrix}
A_Y \\
A_Z \\
p \\
q \\
r
\end{bmatrix}
\]

For preliminary analysis and design, it is convenient to examine the totally decoupled planar dynamics of the pitch, yaw and roll channels. The decoupled state-space model is obtained by nulling the cross-coupling terms defined in the dynamical model described previously. This decoupled model is given by

\[
\begin{bmatrix}
\dot{\alpha} \\
\dot{\beta} \\
p \\
q \\
\dot{r}_{dec}
\end{bmatrix} = \begin{bmatrix}
a_{11} & 0 & 0 & a_{14} & 0 \\
0 & a_{22} & 0 & 0 & a_{25} \\
0 & 0 & a_{33} & 0 & 0 \\
a_{41} & 0 & 0 & a_{44} & 0 \\
0 & a_{52} & 0 & 0 & a_{55}_{dec}
\end{bmatrix} \begin{bmatrix}
\alpha \\
\beta \\
p \\
q \\
r_{dec}
\end{bmatrix} + \begin{bmatrix}
b_{11} & 0 & 0 \\
0 & b_{22} & 0 \\
0 & 0 & b_{33} \\
b_{41} & 0 & 0 \\
0 & b_{52} & 0
\end{bmatrix} \begin{bmatrix}
\delta P \\
\delta Y \\
\delta Y_{dec}
\end{bmatrix} \tag{A.201}
\]

and

\[
\begin{bmatrix}
A_Z \\
A_Y \\
p \\
q \\
\dot{r}_{dec}
\end{bmatrix} = \begin{bmatrix}
0 & c_{12} & 0 & 0 & c_{15} \\
c_{21} & 0 & 0 & c_{24} & 0 \\
0 & 0 & c_{33} & 0 & 0 \\
0 & 0 & 0 & c_{44} & 0 \\
0 & 0 & 0 & 0 & c_{55}_{dec}
\end{bmatrix} \begin{bmatrix}
\alpha \\
\beta \\
p \\
q \\
r_{dec}
\end{bmatrix} + \begin{bmatrix}
0 & d_{12} & 0 \\
d_{21} & 0 & 0 \\
0 & 0 & d_{33} \\
d_{41} & 0 & 0 \\
0 & d_{52} & 0
\end{bmatrix} \begin{bmatrix}
\delta P \\
\delta Y \\
\delta Y_{dec}
\end{bmatrix} \tag{A.202}
\]

The totally decoupled linearized equations of motion are
\[ \dot{\alpha} = \left\{ \cos^2 \alpha \left[ -\frac{QS}{Mu} C_{N_e} + \frac{QS}{Mu} \left( C_{A_e} \alpha + \left( \frac{D}{2V} \right) C_{A_e} q + C_{A\delta} \delta P \right) \right] \\
+ \sin^2 \alpha \left[ -\frac{QS}{Mu} \left( C_{A_e} \alpha + \left( \frac{D}{2V} \right) C_{A_e} q + C_{A\delta} \delta P \right) \right] \\
+ \sin \alpha \cos \alpha \left[ 2\frac{QS}{Mu} C_{N_e} + \frac{QS}{Mu} \left( C_{A_e} \alpha + \left( \frac{D}{2V} \right) C_{A_e} q + C_{A\delta} \delta P \right) \right] \right\} \alpha \\
(A.203) \\
+ \left\{ \cos^2 \alpha \left[ -\frac{QS}{Mu} \left( C_{N_e} \alpha + \left( \frac{D}{2V} \right) C_{N_e} q + C_{N\delta} \delta P \right) \right] \\
+ \sin \alpha \cos \alpha \left[ \frac{QS}{Mu} \left( C_{A_e} \alpha + \left( \frac{D}{2V} \right) C_{A_e} q + C_{A\delta} \delta P \right) \right] \right\} q \\
+ \left\{ \cos^2 \alpha \left[ -\frac{QS}{Mu} C_{N_e} \right] - \sin \alpha \cos \alpha \left[ -\frac{QS}{Mu} C_{A_e} \right] \right\} \delta P \]

\[ \dot{\beta} = \left\{ \cos^2 \beta \left[ \frac{QS}{Mu} C_{Y_e} + \frac{QS}{Mu} \left( C_{A_e} \beta + \left( \frac{D}{2V} \right) C_{A_e} r + C_{A\delta} \delta Y \right) \right] \\
+ \sin^2 \beta \left[ -\frac{QS}{Mu} \left( C_{A_e} \beta + \left( \frac{D}{2V} \right) C_{A_e} r + C_{A\delta} \delta Y \right) \right] \\
- \sin \beta \cos \beta \left[ \frac{2QS}{Mu} \left( C_{Y_e} \beta + \left( \frac{D}{2V} \right) C_{Y_e} r + C_{Y\delta} \delta Y \right) - \frac{QS}{Mu} C_{A_e} \right] \right\} \beta \\
(A.204) \\
+ \left\{ \cos^2 \beta \left[ \frac{QS}{Mu} \left( C_{Y_e} \beta + \left( \frac{D}{2V} \right) C_{Y_e} r + C_{Y\delta} \delta Y \right) \right] \\
+ \sin \beta \cos \beta \left[ \frac{QS}{Mu} \left( C_{A_e} \beta + \left( \frac{D}{2V} \right) C_{A_e} r + C_{A\delta} \delta Y \right) \right] \right\} r \\
+ \left\{ \cos^2 \beta \left[ \frac{QS}{Mu} C_{Y_e} \right] + \sin \beta \cos \beta \left[ \frac{QS}{Mu} C_{A_e} \right] \right\} \delta Y \]

\[ \dot{p} = \left[ \frac{QSD}{I_x} \left( \frac{D}{2V} \right) C_{I_x} \right] \left[ \frac{QSD}{I_x} \right] \left[ C_{I_x} \right] \delta R \]

\[ \dot{q} = \left[ \frac{QSD}{I_y} \left( \frac{D}{2V} \right) C_{m_x} \right] \left[ \frac{QSD}{I_y} \right] \left[ C_{m_x} \right] \delta p \]

\[ \dot{r} = \left[ \frac{QSD}{I_z} \left( \frac{D}{2V} \right) C_{m_x} \right] \left[ \frac{QSD}{I_z} \right] \left[ C_{m_x} \right] \delta r \]

To simplify these expressions, we evaluate the linearized terms at the reference flight.
conditions. Recall that the reference condition was assumed to be trim flight. In addition, we will assume small angles of attack and sideslip, and that the fin deflections are small at the reference condition. With the small angle approximation and decoupled channel assumptions, the axial force terms may be neglected, and the forward velocity component, \(u\), is approximately equal to the total velocity, \(V\). The \(\dot{\alpha}\) and \(\dot{\beta}\) equations above then simplify to

\[
\dot{\alpha} = \left[ -\frac{QS}{MV} C_{N_{\alpha}} \right] \alpha + \left[ 1 - \frac{QS}{MV} \left( \frac{D}{2V} \right) C_{N_{\alpha}} \right] \dot{q} + \left[ -\frac{QS}{MV} C_{N_{\alpha r}} \right] \delta P
\]  
(A.208)

\[
\dot{\beta} = \left[ \frac{QS}{MV} C_{Y_{\beta}} \right] \beta + \left[ \frac{QS}{MV} \left( \frac{D}{2V} \right) C_{Y_{\beta}} \right] \dot{r} + \left[ \frac{QS}{MV} C_{Y_{\beta r}} \right] \delta Y
\]  
(A.209)

Further, the derivatives \(C_{N_{\alpha}}\) and \(C_{Y_{\beta}}\) are usually small and difficult to obtain. Neglecting these terms, \(\dot{\alpha}\) and \(\dot{\beta}\) become

\[
\dot{\alpha} = \left[ -\frac{QS}{MV} C_{N_{\alpha}} \right] \alpha + \dot{q} + \left[ -\frac{QS}{MV} C_{N_{\alpha r}} \right] \delta P
\]  
(A.210)

\[
\dot{\beta} = \left[ \frac{QS}{MV} C_{Y_{\beta}} \right] \beta - \dot{r} + \left[ \frac{QS}{MV} C_{Y_{\beta r}} \right] \delta Y
\]  
(A.211)

Examination of the \(\dot{\alpha}\), \(\dot{\beta}\), \(\dot{q}\), and \(\dot{r}\) equations reveals that the pitch and yaw dynamics are symmetric.

The linearized, decoupled roll dynamics are given by

\[
\dot{\phi} = \left[ \frac{QSD}{I_x} \left( \frac{D}{2V} \right) C_{\phi} \right] \dot{p} + \left[ \frac{QSD}{I_x} C_{\phi m} \right] \delta R
\]  
(A.212)

The above equation can be utilized in the design of the roll autopilot.

In addition to the dynamics presented above, the decoupled lateral accelerations of the missile are
\[ A_y = \left[ \frac{QS}{M} C_{Y_x} \right] \beta + \left[ \frac{QS}{M} \left( \frac{D}{2V} \right) C_{Y_r} \right] r + \left[ \frac{QS}{M} C_{Y_{\alpha r}} \right] \delta Y \] (A.213)

\[ A_z = \left[ -\frac{QS}{M} C_{N_x} \right] \alpha + \left[ -\frac{QS}{M} \left( \frac{D}{2V} \right) C_{N_r} \right] q + \left[ -\frac{QS}{M} C_{N_{\alpha r}} \right] \delta P \] (A.214)

As before, the derivatives \( C_{Y_x} \) and \( C_{N_x} \) are assumed to be negligible. With this assumption, the missile lateral acceleration equations become

\[ A_y = \left[ \frac{QS}{M} C_{Y_x} \right] \beta + \left[ \frac{QS}{M} C_{Y_{\alpha r}} \right] \delta Y \] (A.215)

\[ A_z = \left[ -\frac{QS}{M} C_{N_x} \right] \alpha + \left[ -\frac{QS}{M} C_{N_{\alpha r}} \right] \delta P \] (A.216)

In these equations, symmetry between the pitch and yaw lateral accelerations is evident. The resulting symmetry between the pitch and yaw planes can be utilized in the design of the pitch/yaw autopilot(s).

322
Vita

John Eugene Bibel was born on September 29, 1962, in Mount Pleasant, PA. He graduated from Yough Senior High School in 1980, and then attended the Pennsylvania State University. He received a Bachelor of Science in Aerospace Engineering in May, 1984. Since then, he has been employed at the Naval Surface Warfare Center, Dahlgren, VA. While at NSWC, the author has worked in the areas of simulation, performance analysis, guidance, estimation, navigation, control, flight testing, and systems engineering, with application to tactical missiles and guided projectiles. The author is a senior member of the American Institute of Aeronautics and Astronautics (AIAA). He married his wife, Denise, in June, 1994, and has a stepson, Bryan, and a son, John Elijah.