THE THEORETICAL BEHAVIOR

OF A COMPLEX INELASTIC MATERIAL

by

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The notation used in this thesis is summarized below:

Symbol	Definition	Units
F	Unit of force	
L	Unit of length	
T	Unit of time	
K_1, K_2, K_o, K_R	Spring constants	F/L
C_3, C_4	Damping coefficients	FT/L
x, x_1, x_2, x_3, x_4	Displacements	L
P, P_1, P_2, P_3, P_4	Loads	F
M_o, M_R	Moduli of the material	F/L^2
T_a, T_b, T_c	Relaxation times	T
τ	Stress	F/L^2
γ	Strain	L/L
t	Time	T
E	Constant of proportionality between initial stress rate and the strain rate	F/L^2
ω	Frequency	cycles/T
A	Amplitude of strain	L/L
$\dot{P}$	Time derivative of load	F/T
$\dot{x}$	Time derivative of displacement	L/T
$\dot{\tau}$	Time derivative of stress	F/L^2T
$\dot{\gamma}$	Time derivative of strain	L/LT
exp.(a)	The base of the Natural logarithum raised to the a power	

Symbol	Definition	Units
s	Laplace transform of t	
$\frac{1}{(s-a)}$	Laplace transform of e^{at}	
$\frac{1}{(s-a)^2}$	Laplace transform of te^{at}	

III - INTRODUCTION

The mechanical behavior of an inelastic material lies somewhere between the action of a perfectly elastic solid, and a viscous liquid. It has been a common practice among engineers and research investigators to represent and visualize the action of an inelastic material by mechanical models. From these models, mathematical expressions are derived, which define the laws of behavior of the material under specific loadings and deformations. These laws are derived specifically from the differential equation governing the action of the model.

The mechanical model is usually composed of two common elements, a spring and a viscous dashpot. Each of these elements is a model of an extreme in mechanical behavior. The spring, which is known as a Hooke model, Fig. (1-a), represents a perfectly elastic solid. The stress in the solid is directly proportional to the strain, while the load on the spring is directly proportional to its displacement. The viscous dashpot, commonly called a Newton model, Fig. (1-b), corresponds in behavior to a viscous liquid. The stress in the liquid is directly proportional to the time rate of strain, and similarly the load on the dashpot is directly proportional to the time rate of displacement of the porous piston. The investigator, with his experience and judgement, combines these elements in such a fashion that their action represents the behavior of the material being studied.

In practical applications, investigators in the field of textiles (4, 5, 6, 8, 9, 10, 11, 12, 13, 16, 17, 18) have found these models, in the more common forms, useful in analyzing the behavior of

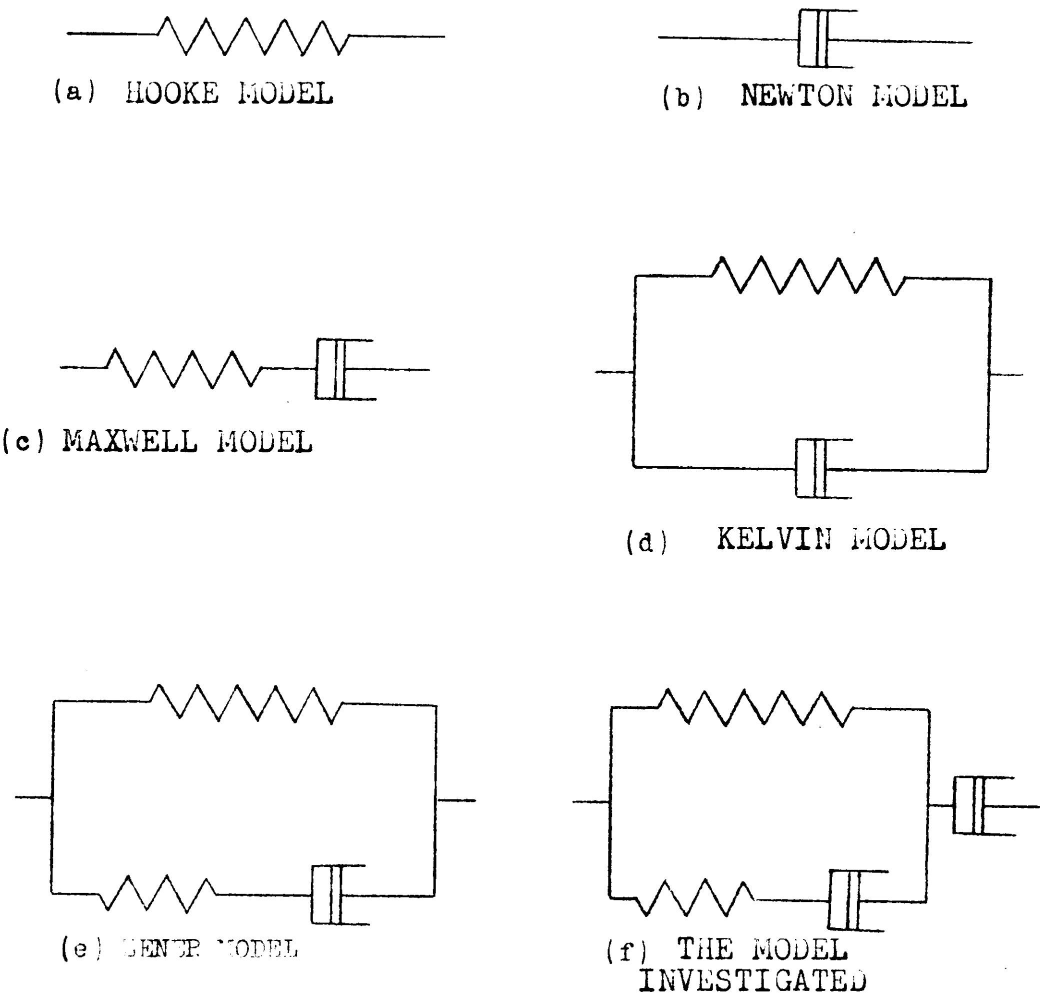


FIGURE 1.

TYPES OF MODELS

certain fibers. These models, in fact, are in common usage throughout the physics of high polymers. Not only are they used in analyzing the behavior of visco-elastic materials, but also as a method of specifying their properties (1).

With due respect to the research workers in the fields of polymeric materials, who pioneered the use of these models, the author feels that they will receive their real importance in practical engineering. Already Freudenthal, in his recent book on the inelastic behavior of engineering materials (7), has presented a scholarly treatise on the use of the more common models to represent the action of these materials, which cannot be done completely by Hooke's Law. In the field of Applied Mechanics, the models will serve as a means for establishing mathematical relations between stresses and strains, when a simple linear relation is no longer appropriate. The equations obtained from the model may be used as a general method of representing the great quantity of experimental data on materials, necessary to the practicing engineer. It is with the interest of applying these models to practical engineering that this thesis was written.

In Figs. (1-c, 1-d, 1-e), the more common models in use at the present time are shown. Each of these models will yield a stress-strain diagram that is applicable to an inelastic material, but they each fail, to a greater or lesser degree, in exhibiting other common characteristics.

Figure (1-c) shows the Maxwell Model, which has a stress-strain diagram that represents an inelastic material. This model exhibits a permanent set, hysteresis and stress relaxation, but does not show a Bauschinger effect, primary creep or history which is commonly called memory. Hence, this model does not completely describe an inelastic material.

In Fig. (1-d), we have the Kelvin model which has stress-strain characteristics of an inelastic solid, and also the properties of hysteresis and primary creep. Yet, this model exhibits neither a Bauschinger effect, stress relaxation, secondary creep nor a permanent set. Therefore, the Kelvin model does not represent the behavior of an inelastic solid as well as the Maxwell model.

The Zener model in Fig. (1-e) is the best representative of an inelastic material, among the more common models. Not only does it have the proper stress-strain characteristics, but also shows the following properties; a Bauschinger effect, hysteresis, primary creep, stress relaxation, and a memory. The properties it lacks are secondary creep, and a permanent set. The three models just described were discussed by P. F. Chenea (2).

Professor Dan Frederick of the Applied Mechanics Department at the Virginia Polytechnic Institute, suggested that a detailed study be made of the model shown in Fig. (1-f). This thesis concerns the behavior of this model and the properties that it exhibits. The method of investigation follows that proposed by Chenea (2).

IV - SYNOPSIS

A concise review of the investigation will be presented here, so the reader will have an overall picture of the study made, before entering into the details.

The differential equation governing the action of an inelastic material is derived from the load displacement relations existing in the model. The general solution of the differential equation is found, and the three forms of solution it may take are discussed.

Stress is determined as a function of time, assuming the material to be loaded at a constant strain rate. The effect on the stress of varying the strain rate is studied, and graphs are drawn to illustrate the results.

The equation of stress as a function of time is transformed into a stress-strain equation, and this equation is analyzed. Curves are drawn showing the results of varying the strain rate on the stress-strain relation, and also the effect of varying the constant of proportionality between the initial stress rate and the strain rate.

The material is found to exhibit a Bauschinger effect and also a permanent set when the unloading curve for stress as a function of strain is derived. The Bauschinger effect for various strain rates is illustrated in graphs.

The hysteresis loops for the material are drawn, and the energy loss in a strain cycle is determined. The energy loss is graphed as a function of the frequency.

Strain hardening characteristics of the material are investigated, and curves showing the effect of cold work at various strain rates are drawn. The effect of a change in the constant of proportionality between the initial stress rate and the strain rate as a result of cold work is shown.

The properties of creep and stress relaxation of the inelastic material and the laws describing these phenomena are derived as a function of time. Curves are drawn illustrating these properties, and the effect of more complicated loading.

Finally the material was investigated for a general memory effect, by allowing the strain to be an arbitrary function of time. The general solution, evaluated for the initial conditions, is found by Laplace Transforms, and the memory effect illustrated graphically.

V - INVESTIGATION AND DISCUSSION

In order to maintain an organized train of thought, a discussion of results will be integrated with each separate phase of the investigation.

A. The Differential Equation and its Solution

The differential equation governing the action of the model under loads and displacements will be derived from the relations existing between the components of load in each element and the total load, between the components of displacement in each element and the total displacement, and between the load in each element and the displacement of the element.

In Fig. (2) the model is shown, with the various spring constants, damping coefficients and displacement coordinates defined. The various model relations between the loads and displacements are also tabulated. Reference will be made to this figure throughout the derivation of the differential equation.

From relation (a) the total displacement is

$$x = x_2 + x_3 + x_4 \tag{A-1}$$

Differentiating this expression with respect to time, we get

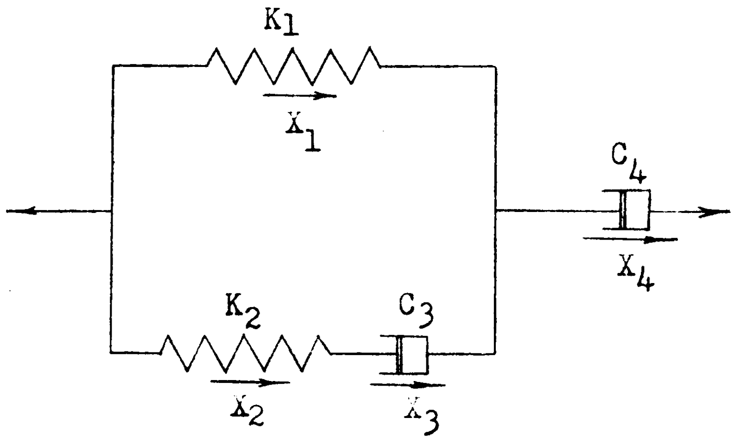
$$\dot{x} = \dot{x}_2 + \dot{x}_3 + \dot{x}_4, \tag{A-2}$$

and substituting (e), its derivative and (f) we arrive at

$$\dot{x} = \frac{\dot{P}}{K_2} - \frac{\dot{P}_1}{K_2} + \frac{P}{C_3} - \frac{P_1}{C_3} + \frac{P}{C_4} \quad . \tag{A-3}$$

Using the load relation (f) between P, P_1 and P_2 the previous equation becomes

$$\dot{x} = \frac{\dot{P}}{K_2} - \frac{K_1\dot{x}_1}{K_2} + \frac{P}{C_3} - \frac{K_1x_1}{C_3} + \frac{P}{C_4} \; . \tag{A-4}$$



THE MODEL

MODEL RELATIONS

DISPLACEMENTS

a) $X = X_2 + X_3 + X_4$

b) $X_1 = X_2 + X_3$

c) $X_1 = X - X_4$

LOADS

d) $P_1 = K_1 X_1$

e) $P_2 = K_2 X_2 = C_3 \dot{X}_3$

f) $P = C_4 \dot{X}_4 = P_1 + P_2$

WHERE

X = THE TOTAL DISPLACEMENT OF THE MODEL

P = THE TOTAL LOAD ON THE MODEL

FIGURE 2.

THE MODEL
AND
MODEL RELATIONS

Now substituting (b), the equation becomes

$$\dot{x} = \frac{\dot{P}}{K_2} - \frac{K_1}{K_2}(x - x_4) + \frac{P}{C_3} - \frac{K_1}{C_3}(x - x_4) + \frac{P}{C_4} \tag{A-5}$$

which, after differentiation and substitution of the derivatives of the relation (e), is

$$\ddot{x} = \frac{\ddot{P}}{K_2} - \frac{K_1\dot{x}}{K_2} + \frac{K_1\ \dot{P}}{K_2\ C_4} + \frac{\dot{P}}{C_3} - \frac{K_1\dot{x}}{C_3} + \frac{K_1\ P}{C_3\ C_4} + \frac{\dot{P}}{C_4}\ . \tag{A-6}$$

The previous equation may be written as

$$(K_2+K_1)\ \ddot{x} + \frac{K_1K_2}{C_3}\dot{x} = \ddot{P} + \dot{P}\left(\frac{K_1}{C_4} + \frac{K_2}{C_3} + \frac{K_2}{C_4}\right) + \frac{K_1K_2}{C_3C_4}P. \tag{A-7}$$

Substituting the following values

$$\frac{C_3}{K_1} = T_a,\ \frac{C_3C_4}{K_1K_2} = T_b^{\,2},\ \frac{C_3C_4}{C_3K_1 + K_2(C_4 + C_3)} = T_c,$$

$$(K_1 + K_2) = K_o \text{ and } K_2 = K_R$$

equation (A-7) becomes

$$K_o\ddot{x} + \frac{K_R\ \dot{x}}{T_a} = \ddot{P} + \frac{\dot{P}}{T_c} + \frac{P}{T_b{}^2} \tag{A-8}$$

Now dividing equation (A-8) by the area of the material, we get the differential equation in the form

$$M_o\ \ddot{\gamma} + \frac{M_R\ \dot{\gamma}}{T_a} = \ddot{\tau} + \frac{\dot{\tau}}{T_c} + \frac{\tau}{T_b{}^2} \tag{A-9}$$

This is the governing differential equation for the behavior of the inelastic material, where M_o and M_R are moduli of the material, and T_a. T_b and T_c its relaxation times.

If we assume that the meterial is to be loaded at a constant strain rate, which is the usual method with modern testing machines, the equation becomes

$$\frac{M_R\ \dot{\gamma}}{T_a} = \ddot{\tau} + \frac{\dot{\tau}}{T_c} + \frac{\tau}{T_b{}^2}\ . \tag{A-10}$$

The general solution (14) for this differential equation is

$$\tau = C_1 \exp.\left\{\left[-1/2T_c + (1/4T_c^2 - 1/T_b^2)^{1/2}\right]t\right\} + C_2 \exp.\left\{\left[-1/2T_c - (1/4T_c^2 - 1/T_b^2)^{1/2}\right]t\right\} + T_b^2 \frac{M_R \dot{\gamma}}{T_a} \quad \text{(A-11)}$$

B. Forms of Solution

Case I: The coefficients of the time exponent are complex and unequal ($\frac{1}{4T_c^2} < \frac{1}{T_b^2}$). This indicates that the stress oscillates with time about some constant value of stress. As the time increases, the amplitude of oscillation decays and the stress approaches the constant value of stress as a limit.

Case II: The coefficients of the time exponent are real and negative ($\frac{1}{4T_c^2} > \frac{1}{T_b^2}$). For this case the stress increases with time, and then approaches a constant value from either above or below.

Case III: The coefficients of the time exponent are real and equal ($\frac{1}{4T_c^2} = \frac{1}{T_b^2}$), and the stress increases with time. The general solution for this case is

$$\tau = (C_1 + C_2 t) \exp. (-t/2T_c) + T_b^2 M_R \dot{\gamma}/T_a \quad \text{(B-1)}$$

In the following investigation, Case I will be neglected, since the author knows of no real material, which has the stress oscillating with time when loaded at a constant strain rate.

Of the remaining two cases, only Case III will be investigated in detail. The reason is that the stress-strain curves of Case III are common to many of our present industrial materials, whereas some of those of Case II are not. All the curves of Case II show a reversal of curvature which is not exhibited by many materials. In any event, Case III will show all the forms of mechanical behavior that are required.

For the benefit of completeness, the stress-strain curve for Case II will be drawn and analyzed in the next phase of the investigation. If the reader should be interested in this case and wish to determine the other characteristics, the mathematical investigation would follow that used for Case III exactly.

C. Stress-Strain Curve for Case II

The general solution of the differential equation is

$$\tau = C_1 \exp.\left\{\left[-1/2T_c + (1/4T_c^2 - 1/T_b^2)^{1/2}\right]t\right\} + C_2 \exp.\left\{\left[1/2T_c - (1/4T_c^2 - 1/T_b^2)^{1/2}\right]t\right\} + T_b^2 \frac{M_R\dot{\gamma}}{T_a}$$

where $\frac{1}{T_b^2} < \frac{1}{4T_c^2}$. If we let

$$m_1 = \frac{1}{2T_c} - \left(\frac{1}{4T_c^2} - \frac{1}{T_b^2}\right)^{\frac{1}{2}} \tag{C-1}$$

$$m_2 = \frac{1}{2T_c} + \left(\frac{1}{4T_c^2} - \frac{1}{T_b^2}\right)^{\frac{1}{2}} \tag{C-2}$$

the general solution becomes

$$\tau = C_1 \exp.(-m_1 t) + C_2 \exp.(-m_2 t) + T_b^2 \frac{M_R\dot{\gamma}}{T_a} \tag{C-3}$$

Loading the material at a constant strain rate and assuming that the initial stress rate is directly proportional to the strain rate, where E is the constant of proportionality, the initial conditions are $t = 0,\ \tau = 0,\ \gamma = 0,\ \dot{\tau} = E\dot{\gamma}$.

Using these initial conditions to evaluate the arbitrary constants, the solution for the stress as a function of time is

$$\tau = T_b^{2}\ \frac{M_R\dot{\gamma}}{T_a}\left[1-\frac{m_2\ \exp.\ (-m_1 t)}{(m_2 - m_1)}+\frac{m_1 \exp.(-m_2 t)}{(m_2 - m_1)}\right]+$$

$$+\frac{E\ \dot{\gamma}}{(m_2-m_1)}\left[\exp.\ (-m_1 t)\ -\ \exp.\ (-m_2 t)\right]. \qquad \text{(C-4)}$$

This may be transformed directly into a stress-strain equation, since at any instant the strain is

$$\gamma = t\ \dot{\gamma}\,. \qquad \text{(C-5)}$$

Solving for t in equation (C-5) and substituting its value in equation (C-4), we get the stress equal to this function of strain

$$\tau = T_b^{2}\ \frac{M_R\dot{\gamma}}{T_a}\left[1-\frac{m_2\ \exp.(-m_1\frac{\gamma}{\dot{\gamma}})}{(m_2 - m_1)}-\frac{m_1\ \exp.(-m_2\frac{\gamma}{\dot{\gamma}})}{(m_2 - m_1)}\right]+$$

$$+\frac{E\ \dot{\gamma}}{(m_2-m_1)}\left[\exp.\ (-m_1\ \frac{\gamma}{\dot{\gamma}})\ -\ \exp.(-m_2\ \frac{\gamma}{\dot{\gamma}})\right]. \qquad \text{(C-6)}$$

From this equation, the limiting values of the stress and its derivatives are

$$\lim_{\gamma \to 0} \tau = 0 \qquad\qquad \lim_{\gamma \to 0} \frac{d\tau}{d\gamma} = E$$

$$\lim_{\gamma \to \infty} \tau = T_b^{2}\ \frac{M_R\dot{\gamma}}{T_a} \qquad\qquad \lim_{\gamma \to \infty} \frac{d\tau}{d\gamma} = 0$$

Figure (3) shows the stress-strain curves for different values of strain rate. The curves increase and then become asymptotic to a limiting value of stress, from above or below. The value of the limiting stress is directly proportional to the strain rate. If a faster strain rate is used in the test, the limiting stress will increase.

The initial slopes of the curves are all equal and depend solely on the constant of proportionality between the initial stress rate and the strain rate.

Of particular interest is the curve for the lowest value of strain rate (γ_2). With a judicious choice of constants, this curve could be made to represent the stress-strain curve for steel up to and through the yielding region.

In Figure (4), the stress-strain curve for $\frac{1}{T_b^2} = \frac{1}{4T_c^2}$ is compared to curves for which $\frac{1}{T_b^2}$ is less than $\frac{1}{4T_c^2}$. When $\frac{1}{T_b^2}$ is only slightly less, the curves are similar and the limiting value of stress is still approached asymptotically from below. The curve of Case III ($\frac{1}{T_b^2} = \frac{1}{4T_c^2}$) may be used with proper constants to represent all curves of this form. When $\frac{1}{T_b^2}$ becomes much less than $\frac{1}{4T_c^2}$ the curve changes radically. A maximum stress is reached, which is greater than the limiting value of stress, and the limit stress is approached asympotically from above.

All curves of Case II have a reverse curvature, whereas those of Case III do not.

D. Stress as a Function of Time for Case III

We will assume that at the instant of loading, the initial stress rate is directly proportional to the strain rate. The symbol E will be used as the constant of proportionality, and depends upon the material being investigated. Using these initial conditions

$$t = 0, \quad \tau = 0, \quad \gamma = 0 \quad \text{and} \quad \dot{\tau} = E\,\dot{\gamma}$$

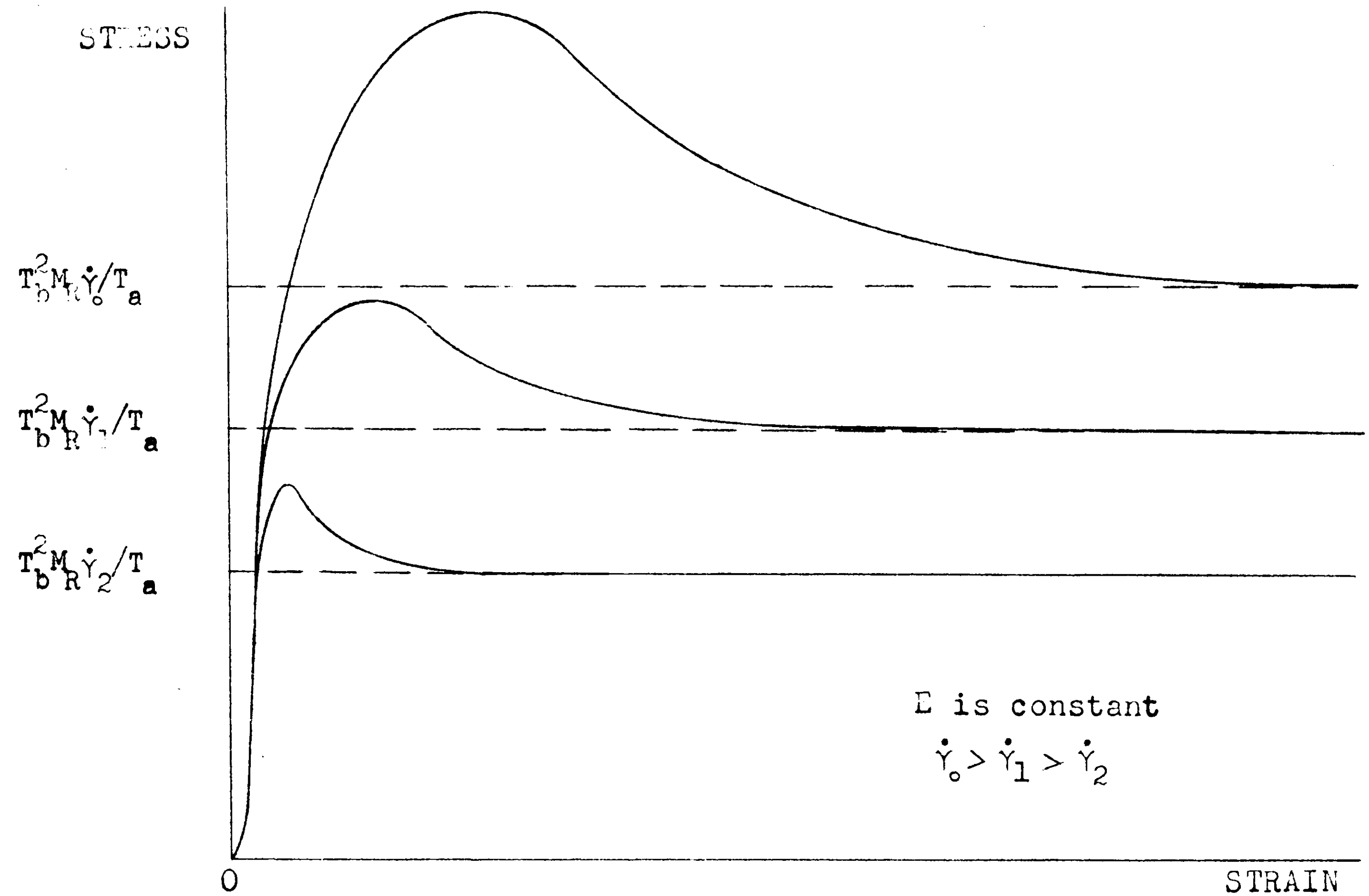


FIGURE 3.

STRESS-STRAIN CURVES FOR CASE II

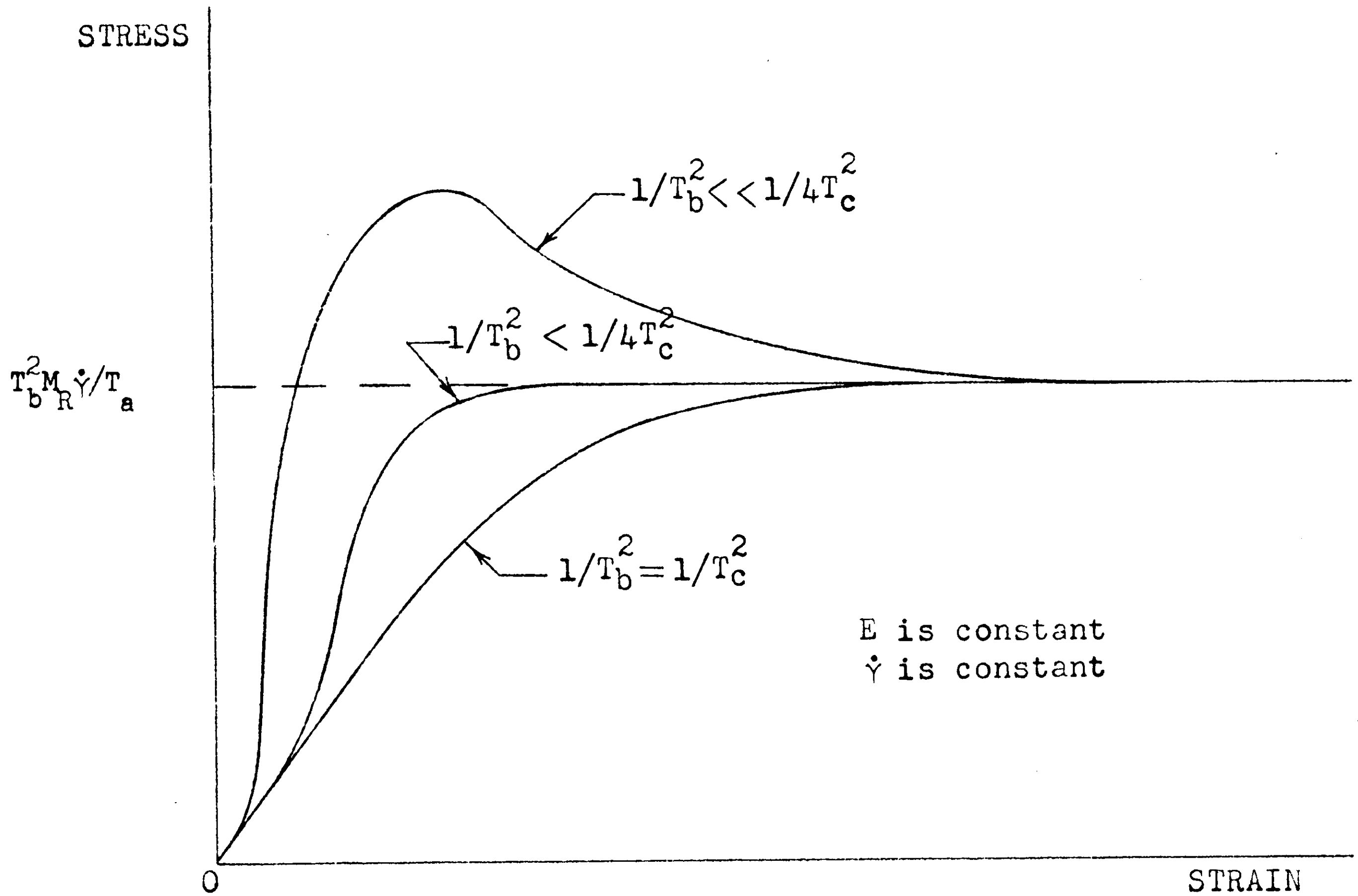


FIGURE 4c

STRESS-STRAIN CURVES FOR CASE II AND CASE III in the general solution (B-1)

$$\tau = (C_1 + C_2 t)\ \exp.(-t/2T_c) + T_b^{\,2}\ \frac{M_R \dot{\gamma}}{T_a}$$

the arbitrary constants are

$$C_1 = -\ T_b^{\,2}\ \frac{M_R \dot{\gamma}}{T_a} \tag{D-1}$$

$$C_2 = E\dot{\gamma} - T_b^{\,2}\ \frac{M_R \dot{\gamma}}{2T_c T_a} \tag{D-2}$$

The equation of stress as a function of time is

$$\tau = T_b^{\,2}\ \frac{M_R \dot{\gamma}}{T_a}\left[1 - \exp.(-t/2T_c\ ) - t\ \exp.(-t/2T_c)\right] + \\ + E\dot{\gamma}\ t\ \exp.\ (-t/2T_c). \tag{D-3}$$

Evaluating the equation at its limiting values, we get the following results,

$$\lim_{t \to 0} \tau = 0 \qquad\qquad \lim_{t \to 0} \dot{\tau} = E\dot{\gamma}$$

$$\lim_{t \to \infty} \tau = T_b^{\,2}\ \frac{M_R \dot{\gamma}}{T_a} \qquad\qquad \lim_{t \to \infty} \dot{\tau} = 0$$

In Fig. (5) and Fig. (6) are graphs of stress versus time. As the time approaches infinity, the curve becomes asymptotic to a limiting value of stress and indicates a truly plastic state of stress. The graph of Fig. (5) shows that the limiting value of stress in the inelastic material will increase with the strain rate, as will the stress rate. In Fig. (6), it is noted that the limiting value of stress is independent of the constant of proportionality between the initial stress rate and the strain rate, but that the stress rate will increase with an increase of this constant.

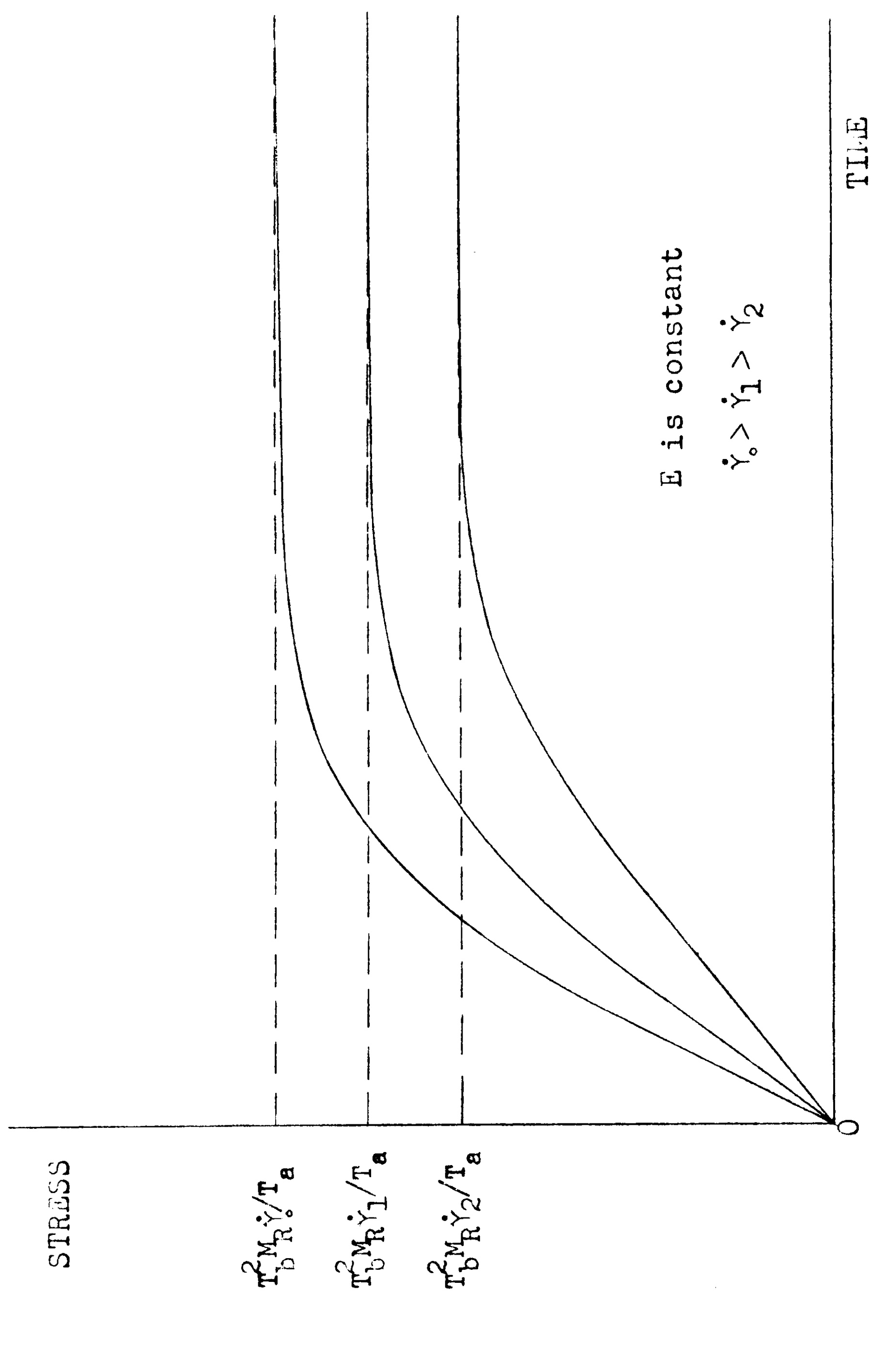


FIGURE 5.

STRESS VERSUS TIME CURVES

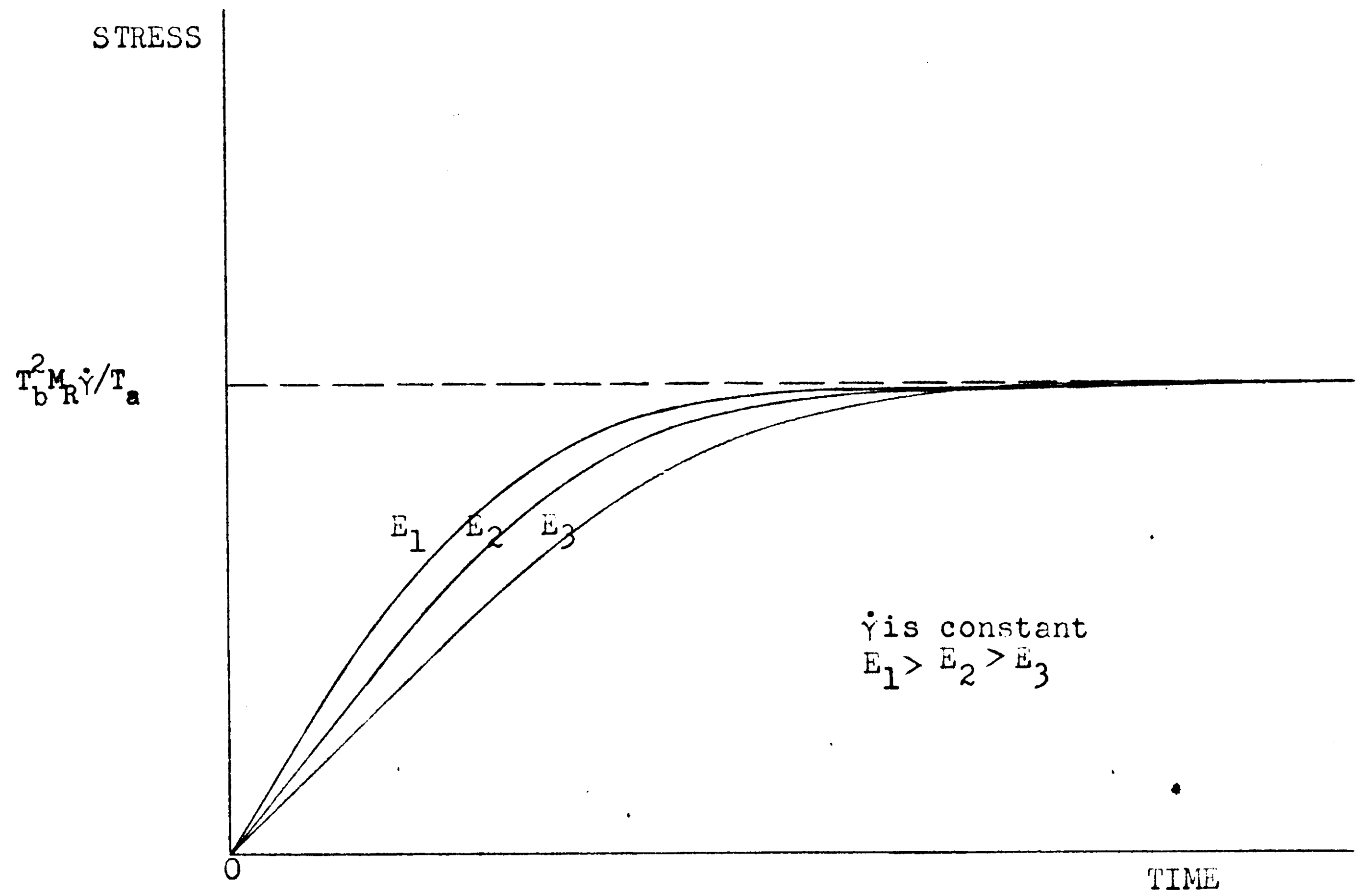


FIGURE 6.

STRESS VERSUS TIME CURVES

E. Stress as a Function of Strain for Case III

The equation for stress as a function of time is developed directly from equation (D-3) in the previous phase of the investigation. Since the strain rate is a constant, the strain at any instant is

$$\gamma = t\,\dot{\gamma} \tag{E-1}$$

therefore

$$t = \frac{\gamma}{\dot{\gamma}} \tag{E-2}$$

substituting this value of t in equation (D-3) we get the equation of stress as a function of strain

$$\tau = T_b^2\,\frac{M_R\dot{\gamma}}{T_a}\left[1 - \exp.(-\gamma/2T_c\dot{\gamma}) - \frac{\gamma\,\exp.(-\gamma/2T_c\dot{\gamma})}{2T_c\dot{\gamma}}\right] +$$
$$+\,E\gamma\,\exp.\,(-\gamma/2T_c\dot{\gamma}) \tag{E-3}$$

The limiting values of the stress and its derivatives are

$$\lim_{\gamma\to 0} \tau = 0 \qquad\qquad \lim_{\gamma\to 0} \frac{d\tau}{d\gamma} = E$$

$$\lim_{\gamma\to\infty} \tau = T_b^2\,\frac{M_R\dot{\gamma}}{T_a} \qquad\qquad \lim_{\gamma\to\infty} \frac{d\tau}{d\gamma} = 0$$

The stress in the inelastic material increases with the strain and becomes asymptotic to a limiting value. As the curve of stress versus strain becomes asymptotic, the material approaches a purely plastic state since there is no appreciable increase in the stress with continued yielding. The value of the limiting stress is directly proportional to the strain rate, while the initial slope of the stress-strain curve is entirely independent of it. The initial slope of the stress-strain curve will vary only with the constant of proportionality between the initial stress rate and the strain rate. The value of this constant does not affect the limiting value of stress.

These conditions are illustrated in the graphs of stress versus strain shown in Fig. (7) and Fig. (8).

F. Bauschinger Effect for Case III

To show a Bauschinger effect in the material, we load to a stress $\tau = \tau_o$ and a corresponding strain $\gamma = \gamma_o$, and then reverse the direction of loading. Time will be measured from the position $\tau = \tau_o$ and $\gamma = \gamma_o$. This will yield an unloading curve of stress as a function of strain.

Applying these initial conditions

$$t = 0, \quad \tau = \tau_o, \quad \gamma = \gamma_o \quad \text{and} \quad \dot{\tau} = -E\dot{\gamma}$$

to the general solution of the differential equation (B-1), now with a negative strain rate

$$\tau = (C_1 + C_2 t)\ \exp.(-t/2T_c) - T_b^2 \frac{M_R \dot{\gamma}}{T_a}$$

we get for the two unknown constants

$$C_1 = \tau_o + T_b^2 \frac{M_R \dot{\gamma}}{T_a} \tag{F-1}$$

$$C_2 = \frac{\tau_o}{2T_c} + T_b^2 \frac{M_R \dot{\gamma}}{2T_c T_a} - E\dot{\gamma} \tag{F-2}$$

The equation for the stress as a function of time is therefore

$$\tau = -T_b^2 \frac{M_R \dot{\gamma}}{T_a} \left[1 - \exp.(-t/2T_c) - t\ \exp.\ (-t/2T_c)\right] +$$

$$+ \tau_o \left(1 + \frac{t}{2T_c}\right) \exp.\ (-t/2T_c) - E\dot{\gamma}\ t\ \exp.(-t/2T_c) \tag{F-3}$$

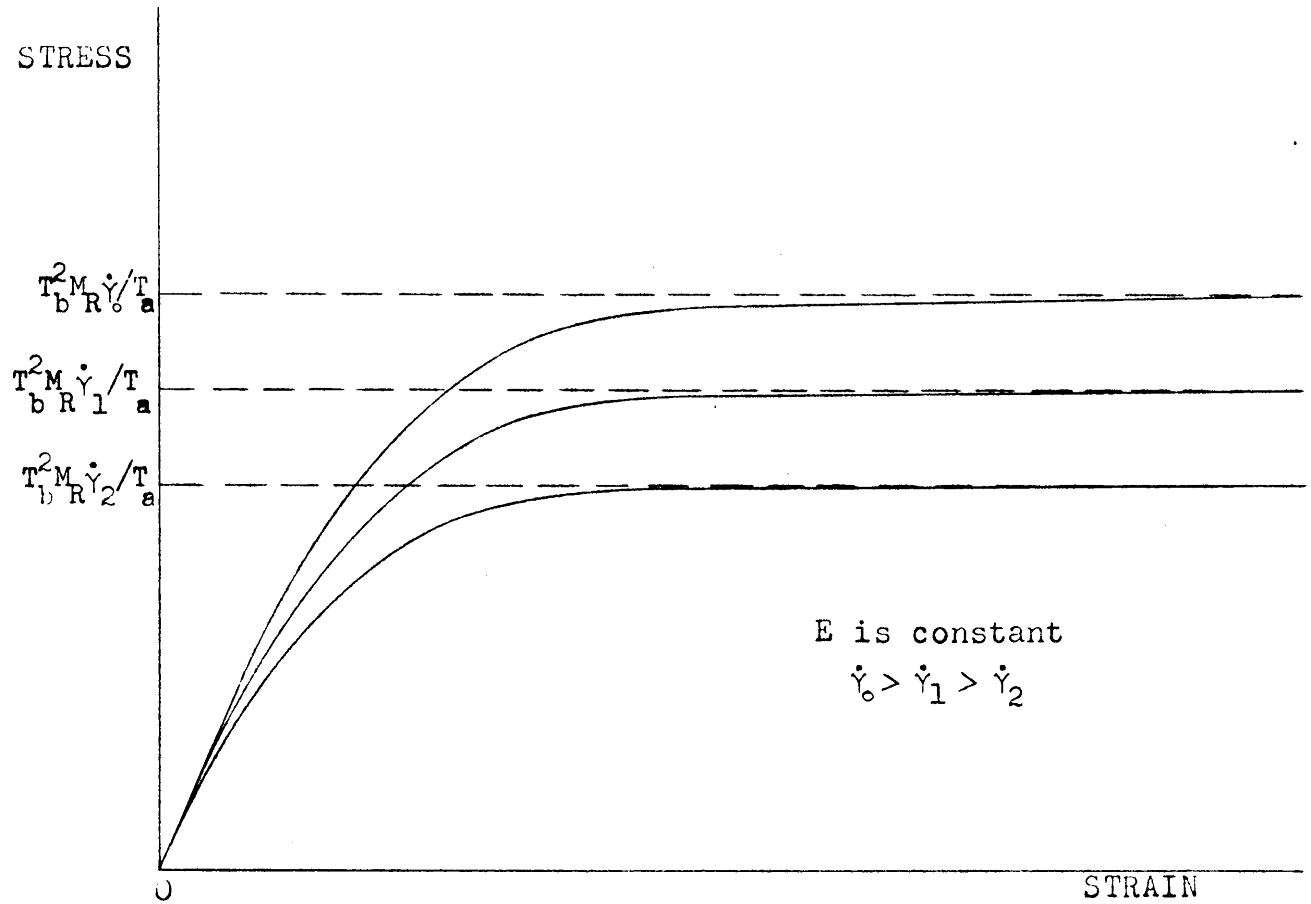


FIGURE 7.

STRESS VERSUS STRAIN CURVES

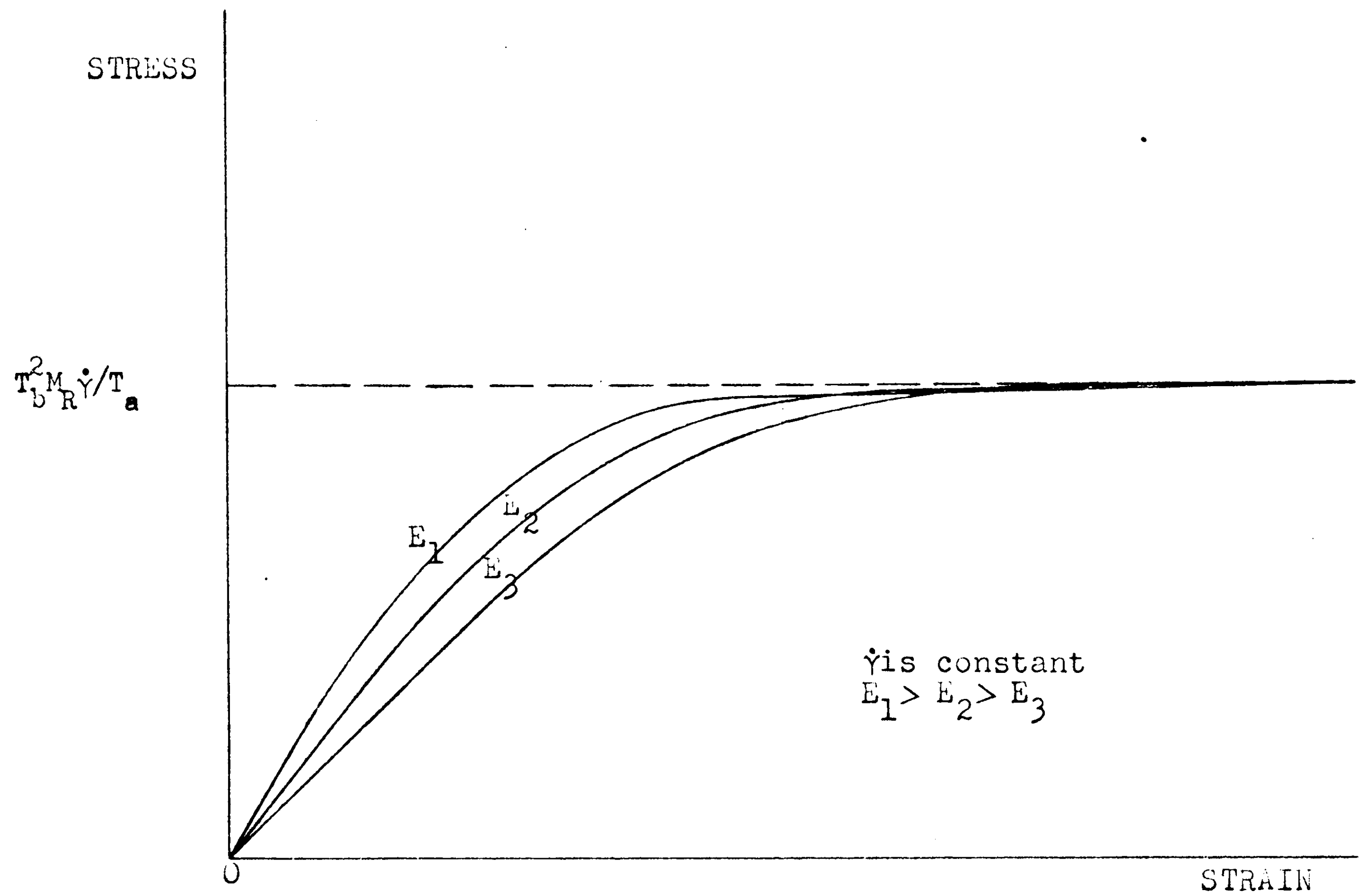


FIGURE 8.

STRESS VERSUS STRAIN CURVES

With the strain rate held constant throughout the unloading test, the strain at any instant may be found from the equation

$$\gamma = \gamma_0 - t\,\dot{\gamma} \tag{F-4}$$

With this relation, time can be found as a function of γ, γ_0 and $\dot{\gamma}$

$$t = \frac{\gamma_0 - \gamma}{\dot{\gamma}} \tag{F-5}$$

Substituting this expression for t in equation (F-3), we get the stress on unloading as a function of the strain

$$\tau = -\frac{T_b^2 M_R \dot{\gamma}}{T_a}\left\{1 - \exp.\left[-(\gamma_0-\gamma)/2T_c\dot{\gamma}\right] - \frac{(\gamma_0-\gamma)}{2T_c\,\dot{\gamma}}\exp.\left[-(\gamma_0-\gamma)/2T_c\dot{\gamma}\right]\right\} +$$

$$+\,\tau_0\left[1 + \frac{(\gamma_0-\gamma)}{2T_c\dot{\gamma}}\right]\exp.\left[-(\gamma_0-\gamma)/2T_c\dot{\gamma}\right] - E(\gamma_0-\gamma)\exp.\left[-(\gamma_0-\gamma)/2T_c\dot{\gamma}\right]. \tag{F-6}$$

The limiting values for the stress and its derivatives are given below

$$\lim_{\gamma \to \gamma_0} \tau = \tau_0 \qquad\qquad \lim_{\gamma \to \gamma_0} \frac{d\tau}{d\gamma} = E$$

$$\lim_{\gamma \to \infty} \tau = -T_b^2\,\frac{M_R\dot{\gamma}}{T_a} \qquad\qquad \lim_{\gamma \to \infty} \frac{d\tau}{d\gamma} = 0$$

As the strain decreases and becomes negative, the stress decreases and becomes negative, approaching a limiting value asymptotically. If the strain rate used in unloading is equal to the strain rate used in loading, the value of this asymptote will be numerically the same as that of the loading curve. An increase in the strain rate will numerically increase the asymptote, while a decrease in the strain rate will decrease the asymptote. The initial slope of the unloading curve is independent of the strain rate, and is a function only of the constant of proportionality between the initial stress rate and the strain rate. This constant has no effect on the value of the asymptote.

The slope of the unloading curve is dependent upon the maximum value of stress reached during loading, and the slope increases with an increase in this stress. The slope of the unloading curve is greater than the slope of the loading curve by a term including the maximum stress reached in loading. As a result of this, the unloading curve does not pass through the origin when the stress is zero, but crosses the strain axis to the right of the origin. This indicates the establishment of a permanent set in the material.

The Bauschinger effect in the inelastic material is obvious for a strain rate on unloading that is unequal to the strain rate used in loading. It is interesting to note that the material will show a negative Bauschinger effect, i.e., the yield strength is greater when the direction of loading is reversed, if the strain rate used in unloading is greater than that used in loading.

For the case where the strain rate used in loading and unloading are equal, the difference between the loading equation and the unloading equation indicates a Bauschinger effect. A graph of these two equations establishes this fact.

These phenomena, which have just been discussed, are shown graphically in Fig. (9) and Fig. (10).

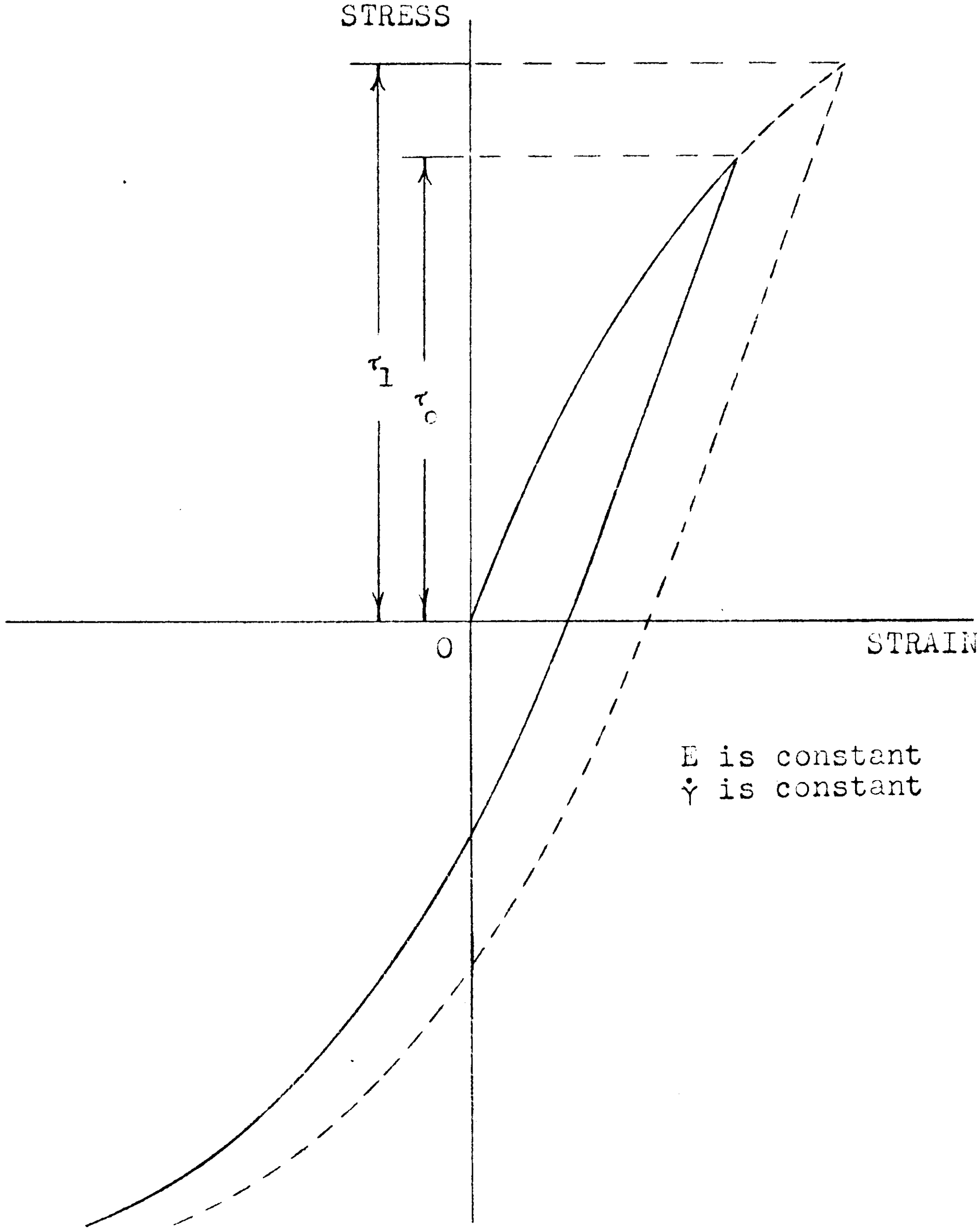


FIGURE 9.

LOAD AND UNLOADING CURVES

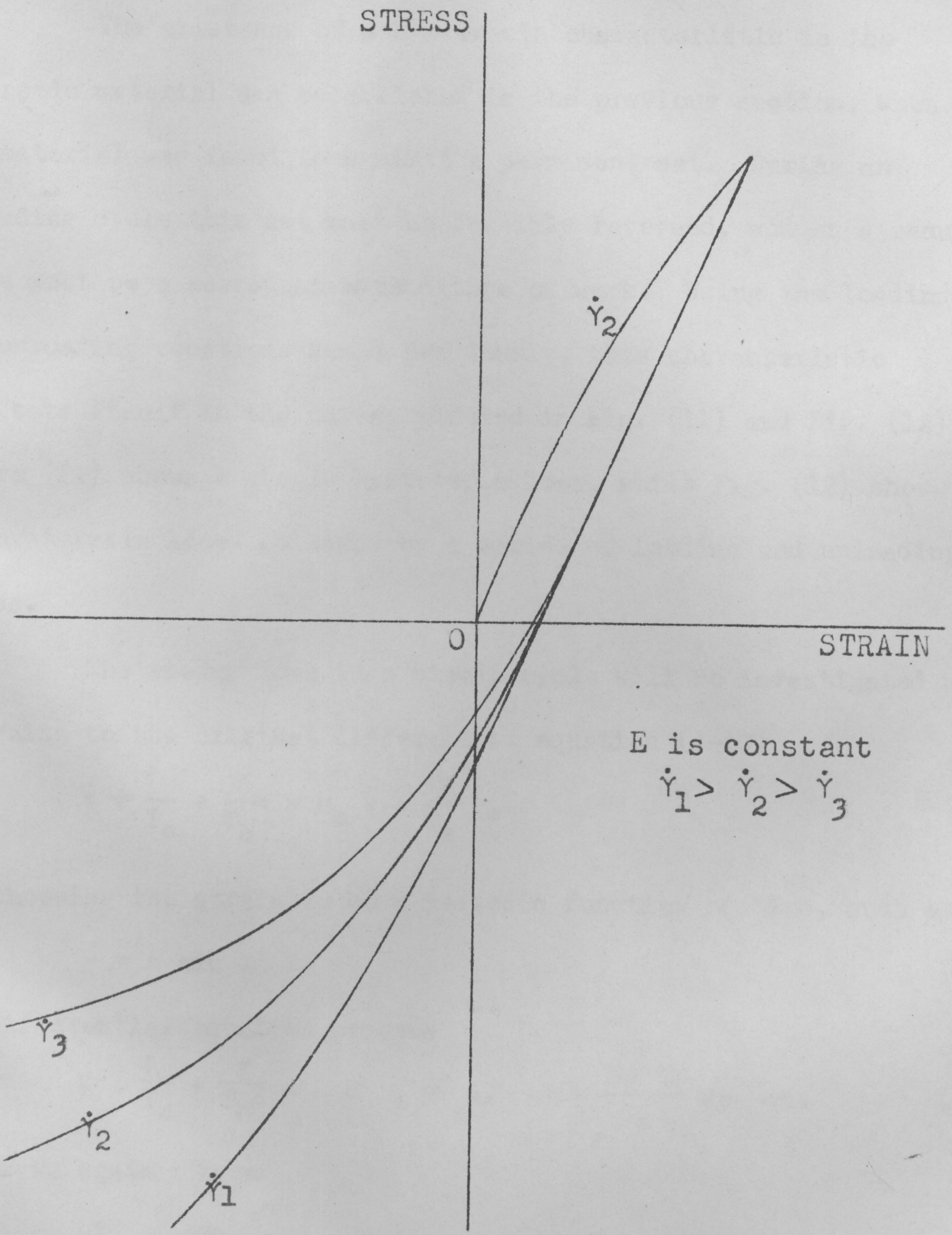


FIGURE 10.

BAUSCHINGER EFFECT

G. Hysteresis for Case III

The existence of a hysteresis characteristic in the inelastic material was established in the previous section, when the material was found to exhibit a permanent set. During an unloading cycle this set must be forcibly reversed, and as a result, there must be a permanent expenditure of work. Using the loading and unloading equations found previously, this characteristic manifests itself in the curves plotted in Fig. (11) and Fig. (12). Figure (11) shows a single hysteresis loop, while Fig. (12) shows the hysteresis loops obtained by a series of loading and unloading cycles.

The energy loss in a strain cycle will be investigated now. Returning to the original differential equation (A-10)

$$\ddot{\tau} + \frac{\dot{\tau}}{T_c} + \frac{\tau}{T_b^2} = M_o \ddot{\gamma} + \frac{M_R \dot{\gamma}}{T_a} ,$$

and choosing the strain to be a periodic function of time, such as

$$\gamma = A \sin \omega t \tag{G-1}$$

the differential equation becomes

$$\ddot{\tau} + \frac{\dot{\tau}}{T_c} + \frac{\tau}{T_b^2} = - M_o A \omega^2 \sin \omega t + \frac{M_R A\omega}{T_a} \cos \omega t. \tag{G-2}$$

Now if we again choose

$$\frac{1}{4T_c^2} = \frac{1}{T_b^2} ,$$

the general solution of the differential equation is

$$\tau = (C_1 + C_2 t) \exp. (-t/2T_c) - M_1 \sin \omega t + M_2 \cos \omega t \tag{G-3}$$

where

$$M_1 = \frac{1}{(\omega^2 + \frac{1}{4T_c^2})^2} \left(A \frac{M_o \omega^2}{4T_c^2} - A M_o \omega^4 - A \frac{M_R \omega^2}{T_a T_c}\right) \tag{G-4}$$

$$M_2 = \frac{1}{(\omega^2+\frac{1}{4T_c^2})^2} \quad (A\,\frac{M_R\,\omega}{4T_c^2\,T_a} - A\,\frac{M_R\omega^3}{T_a} + A\,\frac{M_o\omega^3}{T_c}). \qquad \text{(G-5)}$$

With the following initial conditions

$t = 0,\ \tau = 0$ and $\dot{\tau} = E\dot{\gamma}_o$ where $\dot{\gamma} = \dot{\gamma}_o$ at $t = 0$

the arbitrary constants become

$$C_1 = -M_2 \qquad \text{(G-6)}$$

$$C_2 = E\dot{\gamma}_o - \frac{M_2}{2T_c} + M_1\,\omega \qquad \text{(G-7)}$$

The solution for the stress is

$$\tau = -M_2\,\exp.(-t/2T_c) + t\,\exp.(-t/2T_c)\quad (E\dot{\gamma}_o - \frac{M_2}{2T_c} + M_1\,\omega) - $$
$$- M_1\,\sin\omega t + M_2\cos\omega t. \qquad \text{(G-8)}$$

The energy loss in a strain cycle is

$$\text{E.L.} = \int_0^{\frac{2\pi}{\omega}} \tau\,\dot{\gamma}\,dt. \qquad \text{(G-9)}$$

Substituting the time derivative of the strain from equation (G-1), and the stress as a function of time given in equation (G-8) into equation (G-2), the energy loss is

$$\text{E.L.} = \pi A M_2. \qquad \text{(G-10)}$$

Replacing M_2 by its value, the energy loss is

$$\text{E.L.} = \frac{\pi A^2\,\omega}{(\omega^2+\frac{1}{4T_c^2})^2} \quad (\frac{M_R}{4T_c^2T_a} - \frac{M_R\omega^2}{T_a} + \frac{M_o\omega^2}{T_c}). \qquad \text{(G-11)}$$

The limiting values of the energy loss are given below

$$\lim_{\omega\to 0} \text{E.L.} = 0 \qquad\qquad \lim_{\omega\to\infty} \text{E.L.} = 0$$

As the frequency increases, the energy loss increases to a maximum value, then decreases with a further increase in frequency. In Fig. (13), a graph is shown of energy loss as a function of the frequency.

H. Strain Hardening for Case III

To investigate for the property of strain hardening in the inelastic material, we will load to $\tau = \tau_0$ and $\gamma = \gamma_0$, and then unload to $\tau = 0$ and $\gamma = \gamma_1$. Upon reaching this latter position, we will begin to reload immediately. Time will be measured from the start of the reloading test. The strain rate during reloading will still be a constant. The initial conditions for this test are

$$t = 0, \quad \tau = 0, \quad \gamma = \gamma_1, \text{ and } \dot{\tau} = E\dot{\gamma}.$$

Using these conditions in the general solution of the differential equation (B-1), and solving for the arbitrary constants, we obtain this equation of stress as a function of time (see phase D of the investigation)

$$\tau = T_b^2 \frac{M_R\dot{\gamma}}{T_a}\left[1 - \exp.\left(-t/2T_c\right) - t \exp.\left(-t/2T_c\right)/2T_c\right] + \\ + E\dot{\gamma}\, t \exp.\left(-t/2T_c\right) \qquad \text{(H-1)}$$

Using a constant strain rate for reloading, the strain at any instant is

$$\gamma = \gamma_1 + t\,\dot{\gamma} \qquad \text{(H-2)}$$

We can use this relation to transform the equation (H-1) so that the stress becomes a function of the strain. Solving for t in equation (H-2),

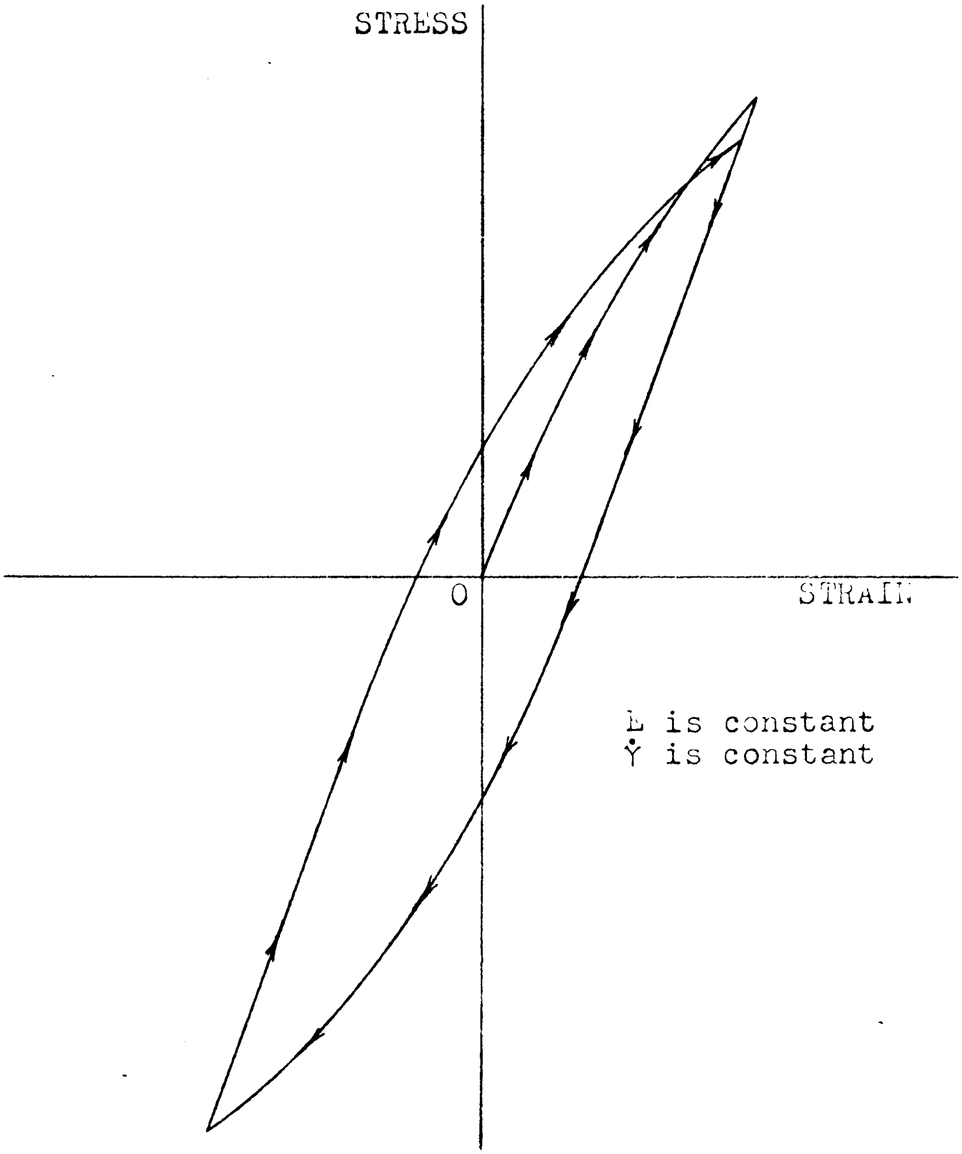


FIGURE 11.

HYSTERESIS LOOP FOR

LOADING CYCLE

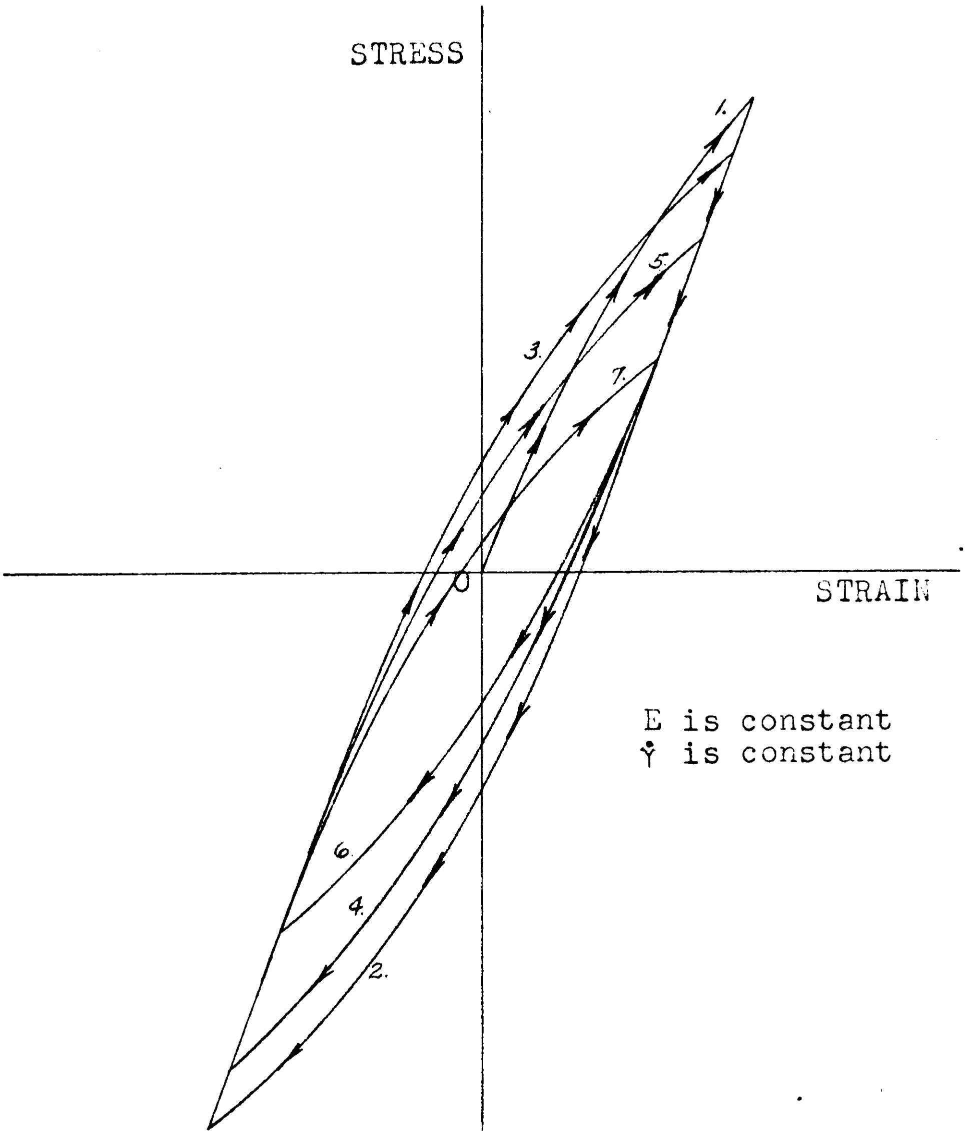


FIGURE 12.

HYSTERESIS LOOPS FOR

LOADING CYCLES

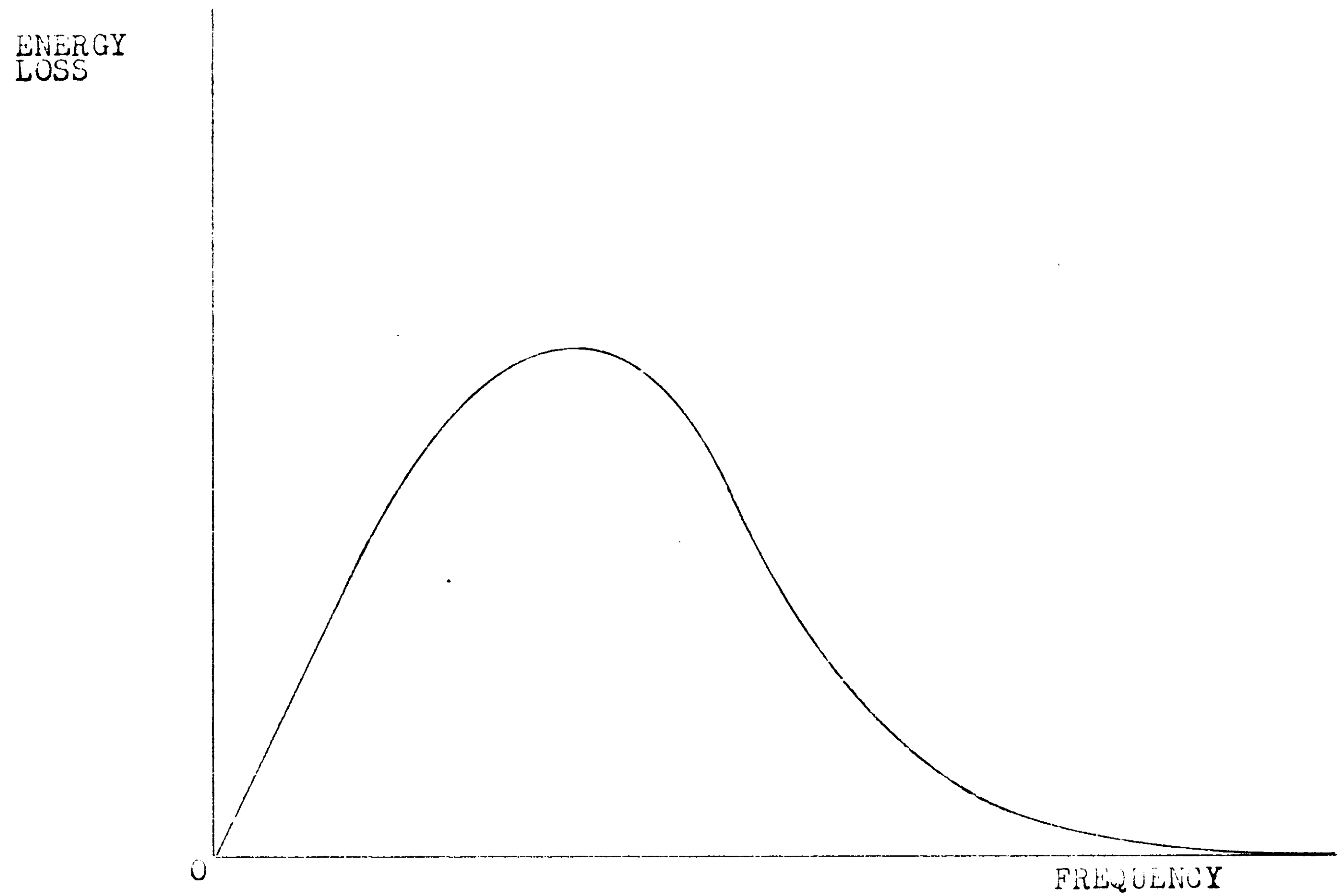


FIGURE 13.

ENERGY LOSS VERSUS FREQUENCY CURVE

we substitute its value in equation (H-1) and obtain

$$\tau = \frac{T_b{}^2 M_R \dot{\gamma}}{T_a}\left\{1 - \exp.\left[-(\gamma-\gamma_1)/2T_c\dot{\gamma}\right] - \frac{(\gamma-\gamma_1)}{2T_c\dot{\gamma}}\exp.\left[-(\gamma-\gamma_1)/2T_c\dot{\gamma}\right]\right\} +$$

$$+ E(\gamma - \gamma_1)\exp.\left[-(\gamma - \gamma_1)/2T_c\dot{\gamma}\right] \quad . \qquad \text{(H-3)}$$

If the material is reloaded at the same strain rate used in the initial loading test, and the constant of proportionality between the initial stress rate and the strain rate is the same value, the reload curve will be the same as the initial loading curve, but shifted to the right a value of strain γ_1. Both curves will approach the same limiting stress, and the yield point will not change from the value obtained in the initial loading. However, the common practice in cold working is to reload at a greater strain rate. This does not affect the initial slope of the stress strain curve on reloading, which is dependent upon the constant of proportionality between the initial stress rate and the strain rate, but will increase the asymptotic value of the reload curve. Hence, the material will show strain hardening if the cold working is done at a faster strain rate. This is illustrated in Fig. (14).

Should the constant of proportionality between the initial stress rate and strain rate be different on reloading, the slope of the reload curve would not be the same as that of the initial loading curve. The slope would increase with an increase in this constant and it would decrease should this constant decrease. There would be no effect on the limiting stress since the value of the asymptote is independent of this constant.

If a period lapses before reloading, the stress will remain zero, but the strain will relax, according to the law given in the section on creep, to some value. The reloading curves after this interval will still follow the laws determined above. In Fig. (15), the effects of the last two characteristics of the material are shown.

I. Stress Relaxation for Case III

The stress relaxation law, which the stress follows after the material has been subjected to a constant strain, will be determined by loading the material to a stress $\tau = \tau_o$ and a corresponding strain $\gamma = \gamma_o$. The strain will be held constant at a value of $\gamma = \gamma_o$. Time will be measured from the instant the strain becomes constant. The original differential equation (A-9), with the strain now constant, becomes

$$\ddot{\tau} + \frac{\dot{\tau}}{T_c} + \frac{\tau}{T_b{}^2} = 0 \qquad \text{(I-1)}$$

With

$$\frac{1}{T_b{}^2} = \frac{1}{4T_c{}^2}$$

the general solution of the differential equation is

$$\tau = (C_1 + C_2 t)\ \exp.\ (-t/2T_c) \qquad \text{(I-2)}$$

Applying these initial conditions to the general solution

$$t = 0, \quad \tau = \tau_o, \quad \dot{\tau} = -\dot{\tau}_o$$

we determine the following values for the arbitrary constants

$$C_1 = \tau_o \qquad \text{(I-3)}$$

$$C_2 = \frac{\tau_o}{2T_c} - \dot{\tau}_o \qquad \text{(I-4)}$$

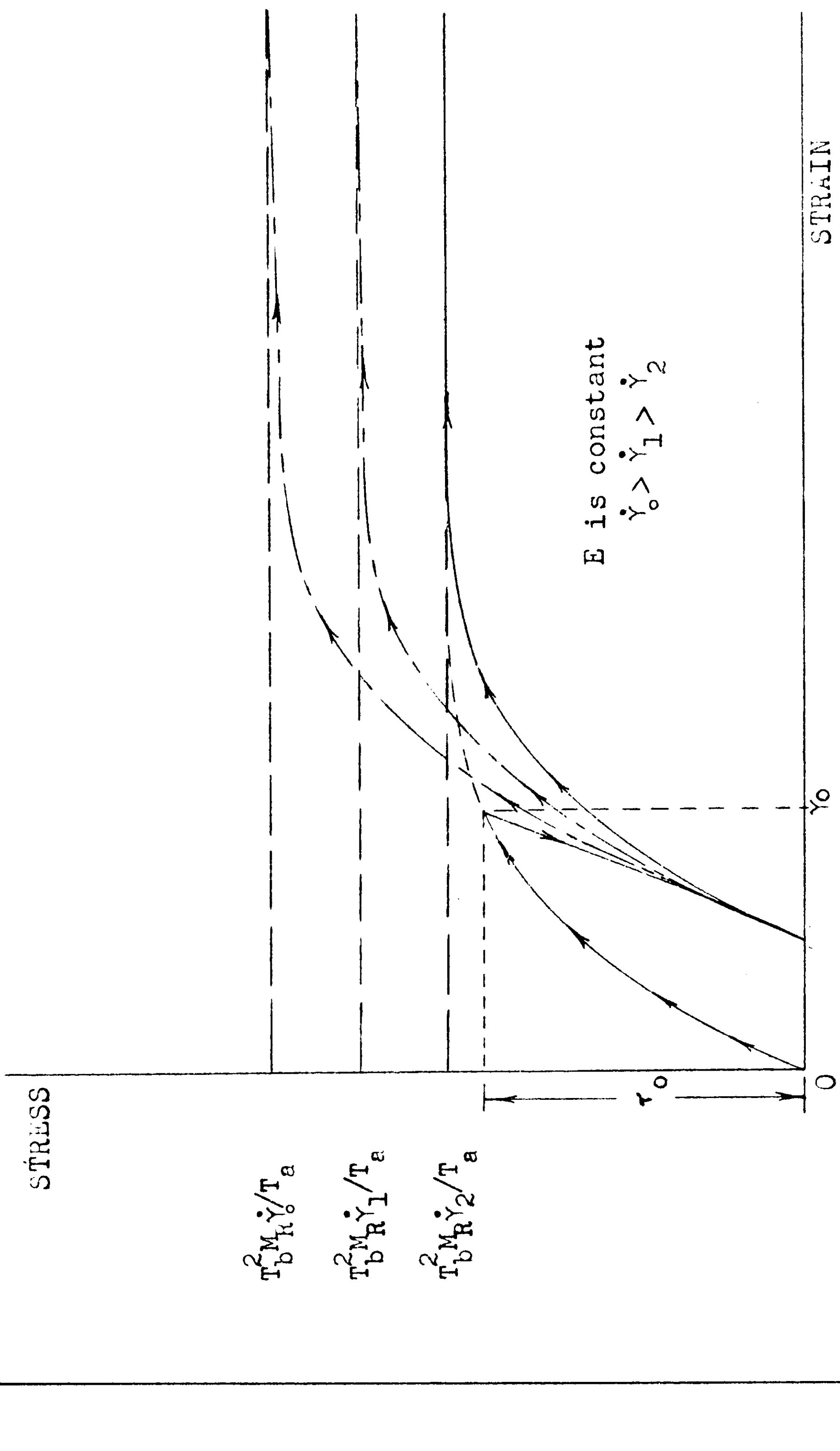


FIGURE 14.

STRAIN HARDENING CURVES

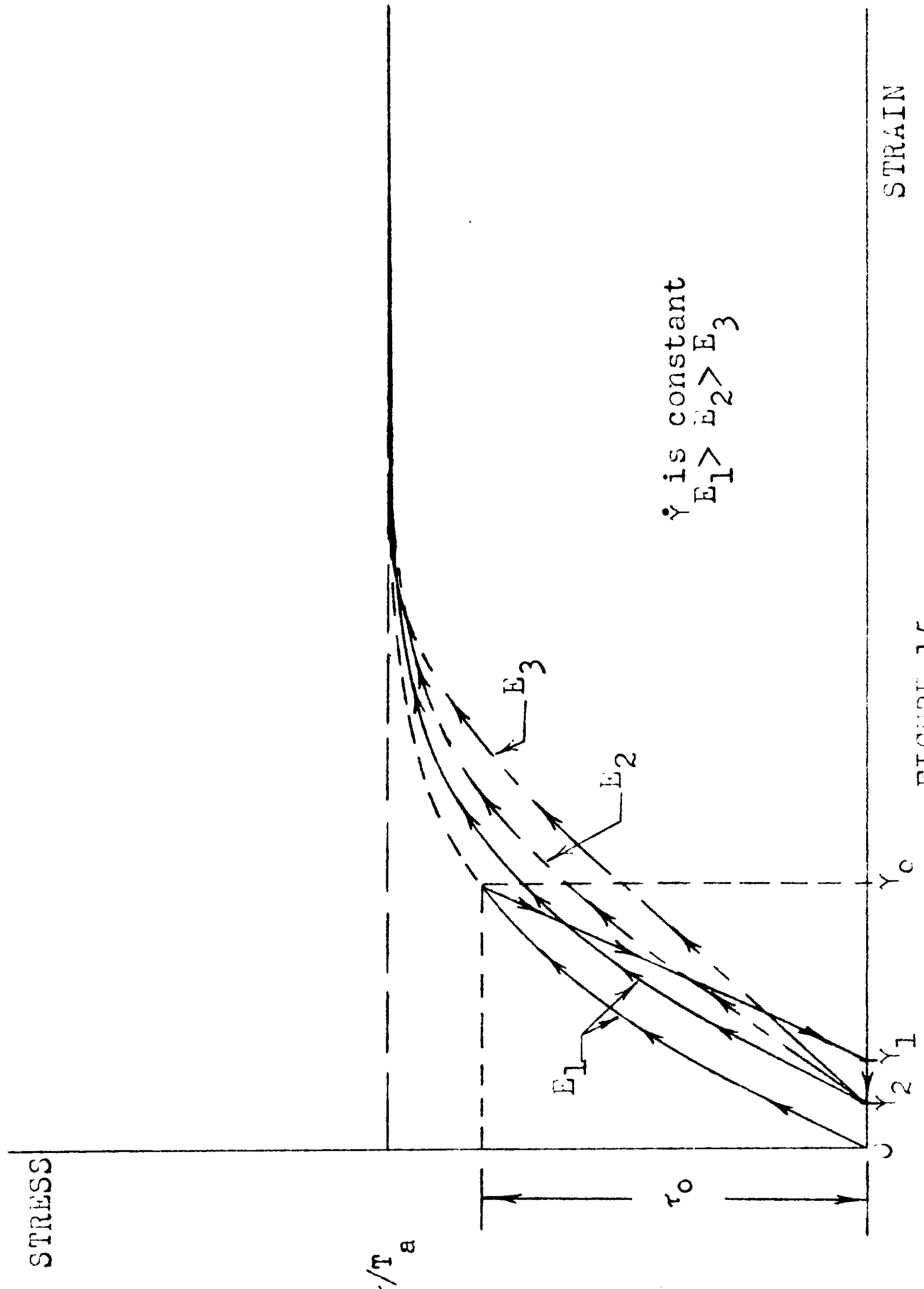


FIGURE 15.

STRAIN HARDENING CURVES

Using these constants in equation (I-2), the stress relaxation law is

$$\tau = \tau_o \left(1 + \frac{t}{2T_c}\right) \exp. (-t/2T_c) - \dot{\tau}_o \, t \exp. (-t/2T_c). \qquad \text{(I-5)}$$

The limiting values of the stress are

$$\lim_{t \to 0} \tau = \tau_o \qquad\qquad \lim_{t \to 0} \dot{\tau} = -\dot{\tau}_o$$

$$\lim_{t \to \infty} \tau = 0 \qquad\qquad \lim_{t \to \infty} \dot{\tau} = 0$$

With the strain held constant, the stress relaxes from its maximum value to zero. The rate at which the stress relaxes varies with the value of the maximum stress reached just before holding the strain constant. As this maximum stress is increased, the slope of the relaxation curve increases. The initial rate of stress relaxation is dependent upon the material, and must be evaluated experimentally. Figure (16) shows stress relaxation curves, with the stress relaxing from different maximum values. In Fig. (17), the loading process to obtain a stress relaxation curve is shown.

J. Creep for Case III

The creep characteristics of the material will be found by loading to some stress $\tau = \tau_o$ and corresponding strain $\gamma = \gamma_o$. Then the stress will be held constant. Time will be measured from the instant the stress becomes equal to τ_o. The strain will then be found as a function of time.

The original differential equation, with the stress held constant becomes

$$M_o \ddot{\gamma} + \frac{M_R \dot{\gamma}}{T_a} = \frac{\tau_o}{T_b^2} \qquad \text{(J-1)}$$

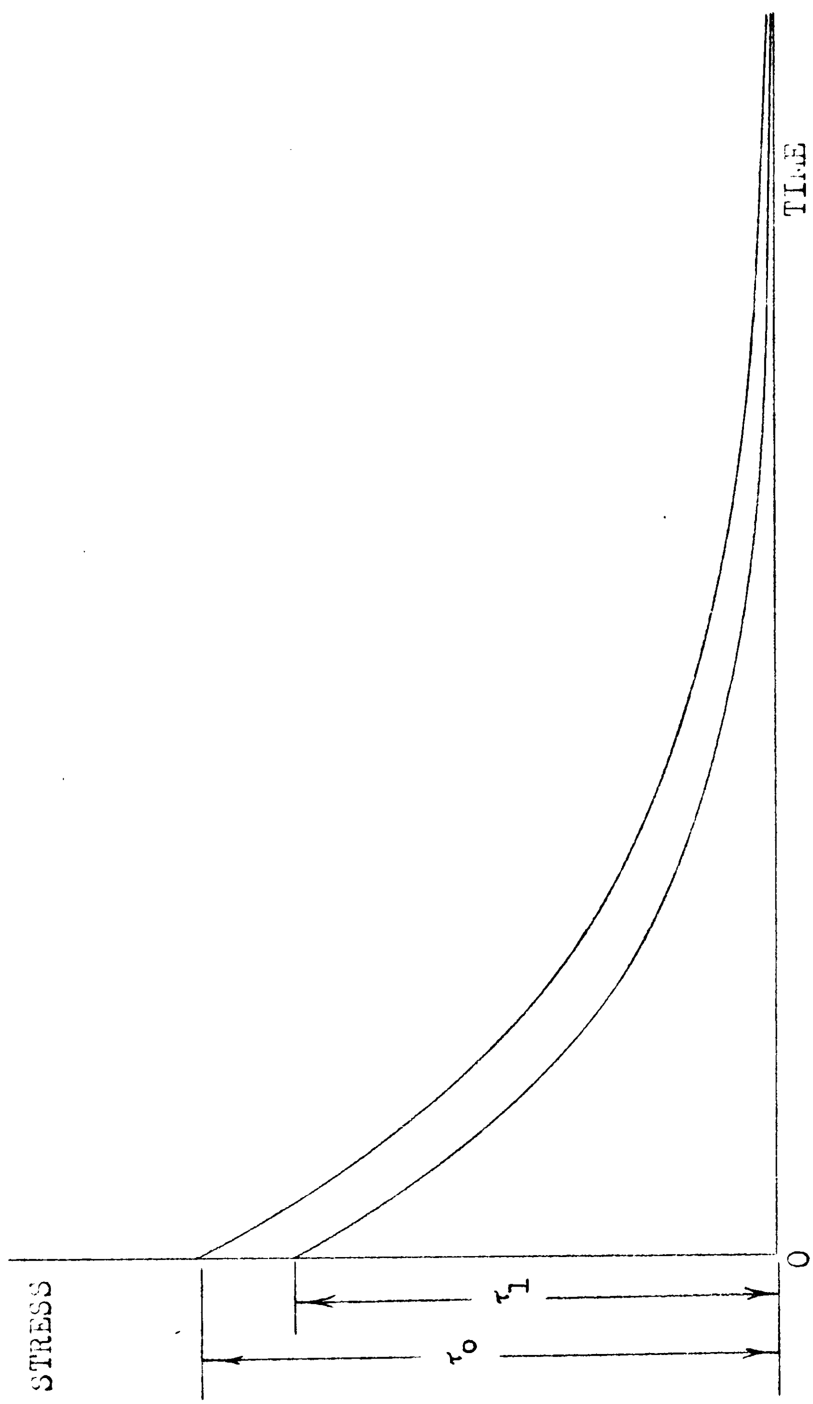


FIGURE 16.

STRESS RELAXATION CURVES

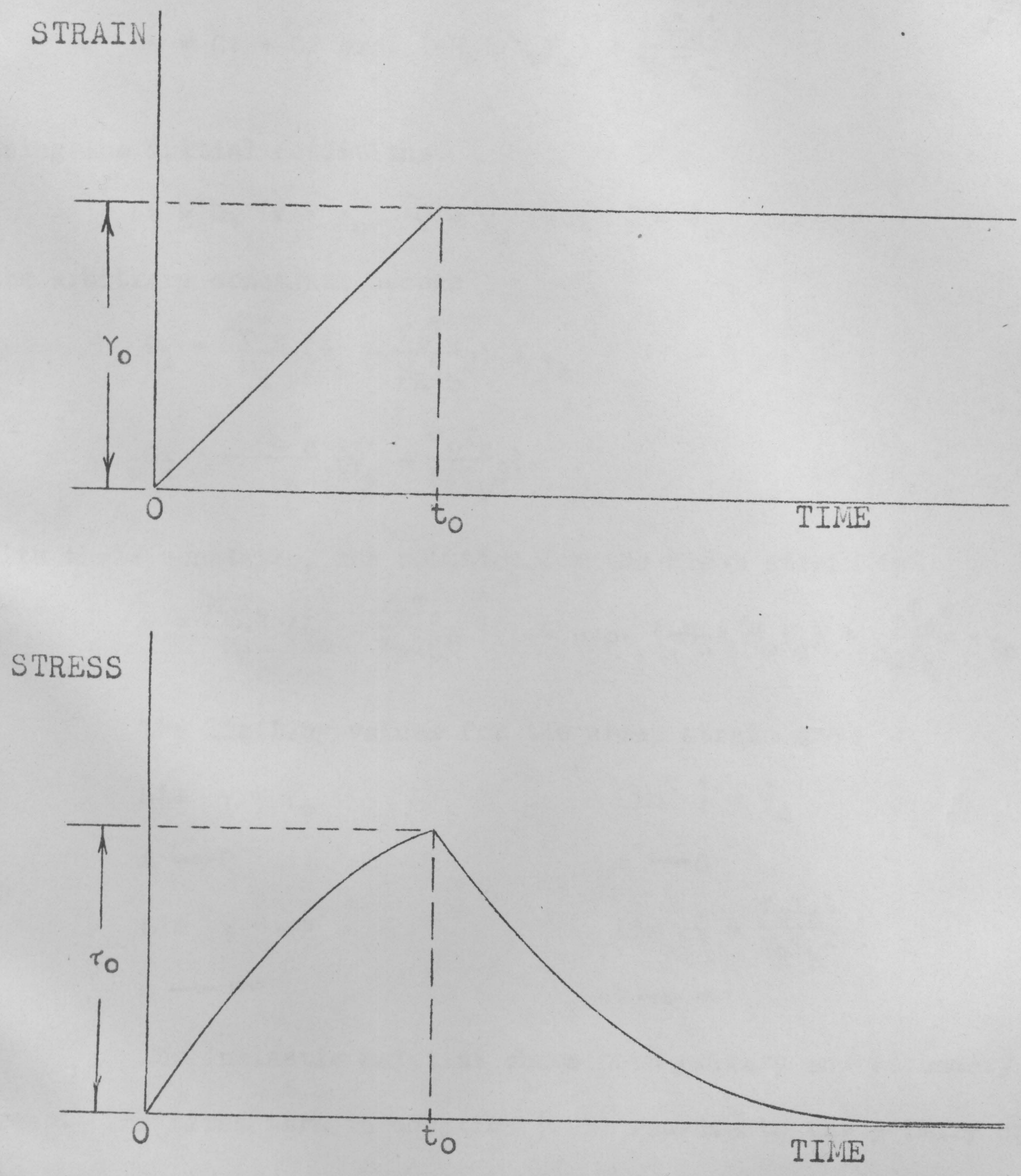


FIGURE 17.

STRAIN - TIME AND STRESS - TIME

DIAGRAMS

FOR STRESS RELAXATION

for which the general solution is

$$\gamma = C_1 + C_2 \exp.\ (-M_R t/M_o T_a) + \frac{\tau_o T_a t}{M_R T_b{}^2} \qquad \text{(J-2)}$$

Using the initial conditions

$$t = 0, \quad \tau = \tau_o, \quad \gamma = \gamma_o \quad \text{and} \quad \dot{\gamma} = \dot{\gamma}_o$$

the arbitrary constants become

$$C_1 = \frac{M_o T_a}{M_R} (\dot{\gamma}_o - \frac{\tau_o T_a}{M_R T_b{}^2}) + \gamma_o \qquad \text{(J-3)}$$

$$C_2 = - \frac{M_o T_a}{M_R} (\dot{\gamma}_o - \frac{\tau_o T_a}{M_R T_b{}^2}). \qquad \text{(J-4)}$$

With these constants, the solution for the creep strain is

$$\gamma = \frac{M_o T_a}{M_R} (\dot{\gamma}_o - \frac{\tau_o T_a}{M_R T_b{}^2})\left[1 - \exp.\ (-M_R t/M_o T_a)\right] + \frac{\tau_o T_a t}{M_R T_b{}^2} + \gamma_o \qquad \text{(J-5)}$$

The limiting values for the creep strain are

$\lim\limits_{t \to 0} \gamma = \gamma_o$	$\lim\limits_{t \to 0} \dot{\gamma} = \dot{\gamma}_o$
$\lim\limits_{t \to \infty} \gamma = \infty$	$\lim\limits_{t \to \infty} \dot{\gamma} = \frac{\tau_o T_a t}{M_R T_b{}^2}$

The inelastic material shows both primary and secondary creep. The first term in equation (J-5) represents the primary creep and the second term, the secondary creep. Both of these terms are shown independently, Fig. (18). The initial slope of the creep curve is dependent on the material, and must be determined in the laboratory. The initial strain and the slope of the creep curve are dependent on the value of the constant stress applied to the material. The higher the value of the constant stress, the greater is the initial strain, and the steeper the slope of the creep curve. In Fig. (19), creep curves are shown with different values for the constant stress.

If the constant stress at which the creep test is run is applied to the material instantly, in the form of a dead weight, the strain rate may be assumed infinite. As a result, the dashpots are considered rigid, and all the load is absorbed initially in the spring. This yields an initial strain in the model equal to the stress multiplied by a spring constant equivalent to the elastic action of the two model springs in parallel. In the inelastic material, there will be an initial strain due to an instantaneous elastic action of the material. This initial deformation is defined in terms of the moduli of the material and the stress as

$$\gamma_o = \frac{M_R\,(M_o - M_R)}{M_o}\,\tau_o\,, \tag{J-6}$$

where τ_o is the applied stress. After this initial strain, the material will creep according to the law given in equation (J-5).

If the applied constant stress is removed instantly, the value of strain will be reduced immediately by a value of strain given in equation (J-6). The reason for this action is the same as that for the existence of the initial strain when loading.

After the removal of the load τ_o and the instantaneous reduction in strain, the strain will reduce according to the law

$$\gamma = \gamma_2 - \frac{M_o T_a \dot{\gamma}_R}{M_R}\left[1 - \exp.\,(-M_R t/M_o T_a)\right], \tag{J-7}$$

where γ_2 is the value of the strain immediately after removing the load, and $\dot{\gamma}_R$ is the initial recovery strain rate. In equation (J-7) time is being measured after removal of the load. As can be seen from equation (J-7), complete recovery does not occur, but rather the strain approaches a limiting value asymptotically. This limiting value of strain is given below

$$\lim_{t \to \infty} \gamma = \gamma_2 - \frac{M_o T_a \dot{\gamma}_R}{M_R} \qquad \lim_{t \to \infty} \dot{\gamma} = 0$$

The previously discussed phenomena is illustrated in the stress-time and strain-time curves of Fig. (20).

K. Memory Effect for Case III

To show in general the memory characteristic of the material, the strain will be chosen as an arbitrary function of time. In the differential equation (A-10), the left side becomes an arbitrary function of time G(t). Equation (A-9) becomes

$$\ddot{\tau} + \frac{\dot{\tau}}{T_c} + \frac{\tau}{T_b^2} = M_o \ddot{\gamma} + \frac{M_R \dot{\gamma}}{T_a} = G(t). \qquad \text{(K-1)}$$

This equation will be solved by Laplace transforms (3) using these initial conditions

$$t = 0, \; \tau = \tau_o \text{ and } \dot{\tau} = \dot{\tau}_o.$$

Remembering that

$$T_b = 2T_c$$

the transformed equation is

$$s^2\tau(s) - s\,\tau(o) - \dot{\tau}(o) + \frac{s}{T_c}\tau(s) - \frac{\tau(o)}{T_c} + \frac{\tau(s)}{4T_c^2} = G(s). \qquad \text{(K-2)}$$

Substituting the initial conditions and rewriting, the equation (K-2) becomes

$$\tau(s)\left(s^2 + \frac{s}{T_c} + \frac{1}{4T_c^2}\right) = G(s) + s\,\tau_o + \frac{\tau_o}{T_c} + \dot{\tau}_o. \qquad \text{(K-3)}$$

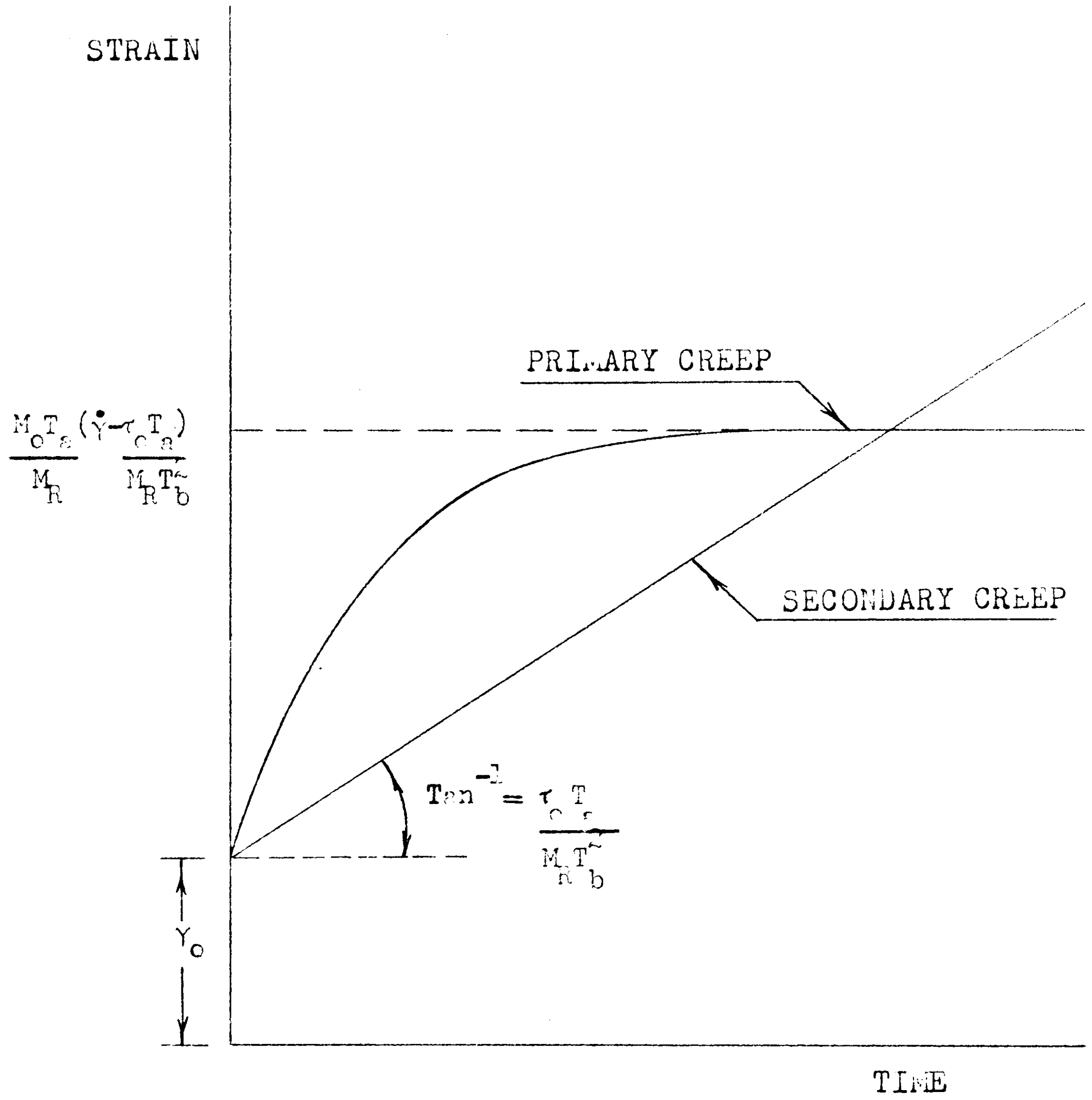


FIGURE 18.

COMPONENTS OF CREEP

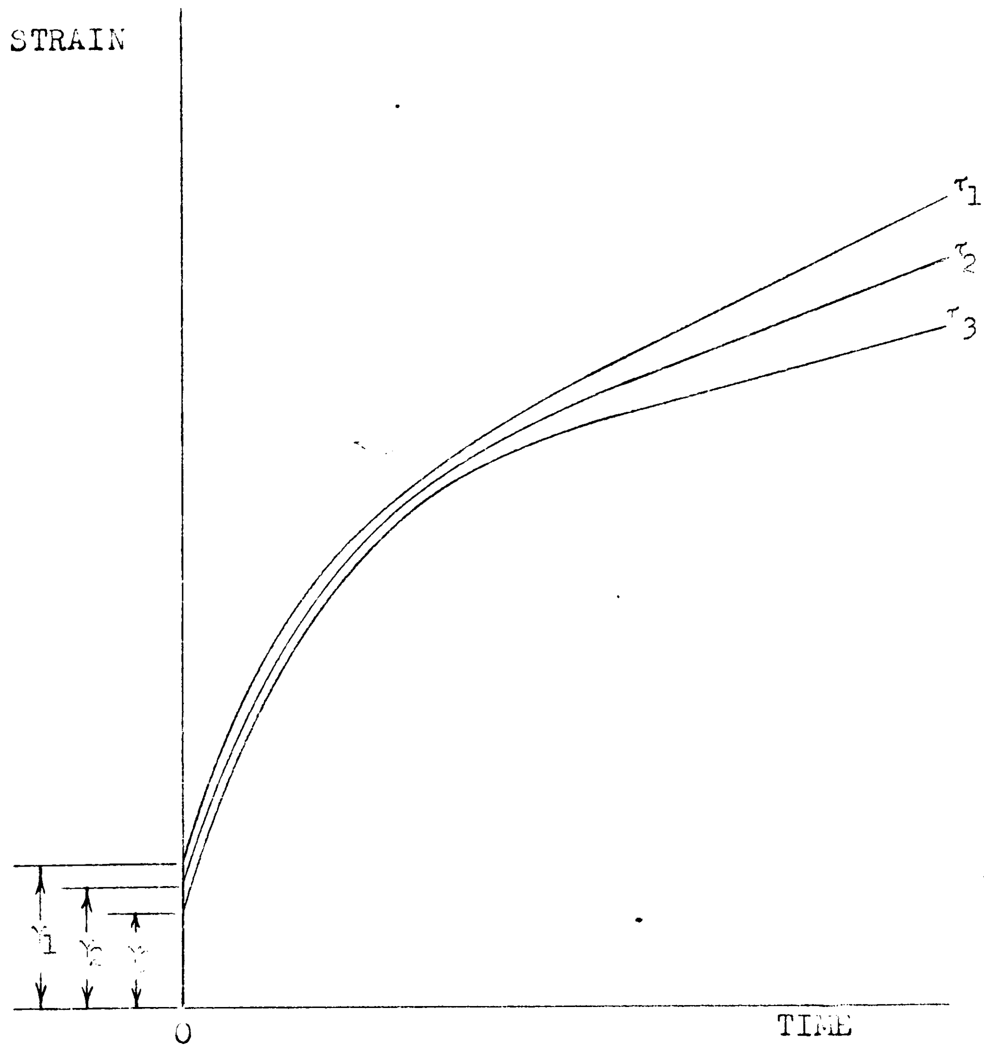


FIGURE 19.

CREEP CURVES

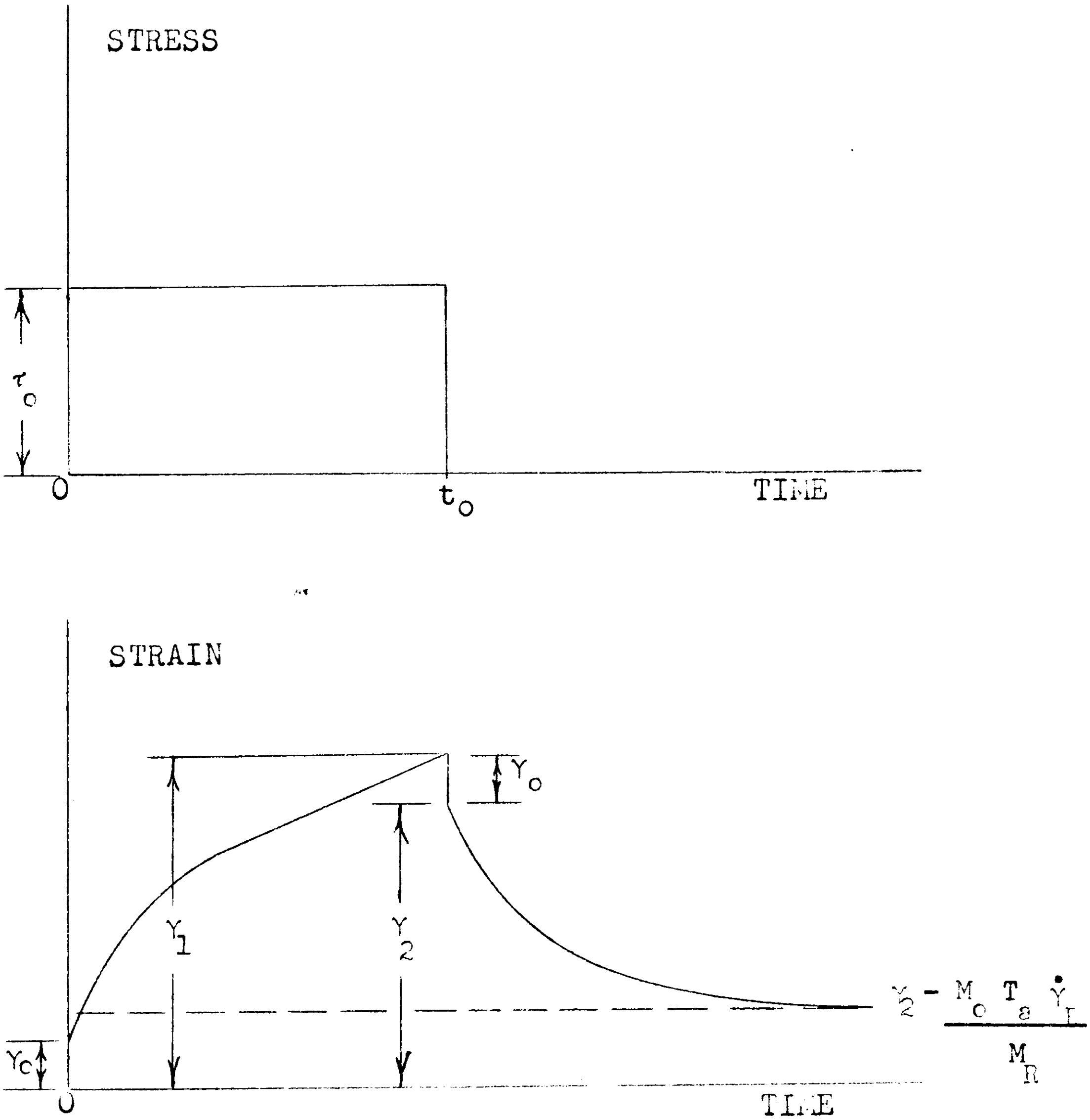


FIGURE 20.

STRESS - TIME AND STRAIN - TIME

DIAGRAMS

FOR CREEP CURVES

Solving (K-3) for $\tau(s)$ and collecting terms we have

$$\tau(s) = \frac{\tau_o}{(s+\frac{1}{2T_c})} + \frac{\tau_o}{2T_c(s+\frac{1}{2T_c})^2} + \frac{\dot{\tau}_o}{(s+\frac{1}{2T_c})^2} + \frac{G(s)}{(s+\frac{1}{2T_c})^2} . \qquad \text{(K-4)}$$

Applying the inverse transformation to (K-4), we obtain τ as this function of t.

$$\tau = \tau_o \exp. (-t/2T_c)(1 + \frac{t}{2T_c}) + \dot{\tau}_o t \exp. (-t/2T_c) + $$
$$+ \int_0^t G\,(t - \varepsilon)\ \varepsilon \exp. (-\varepsilon/2T_c)\ d\varepsilon, \qquad \text{(K-5)}$$

which contains the convolution integral.

The first two terms of equation (K-5) are instantaneous effects depending upon the value of time. The third term involves the memory function, and since it is an integral over all values of time from zero to the instant of investigation, it modifies the stress by the previous history of the material. Figure (21) shows the effect of the memory function.

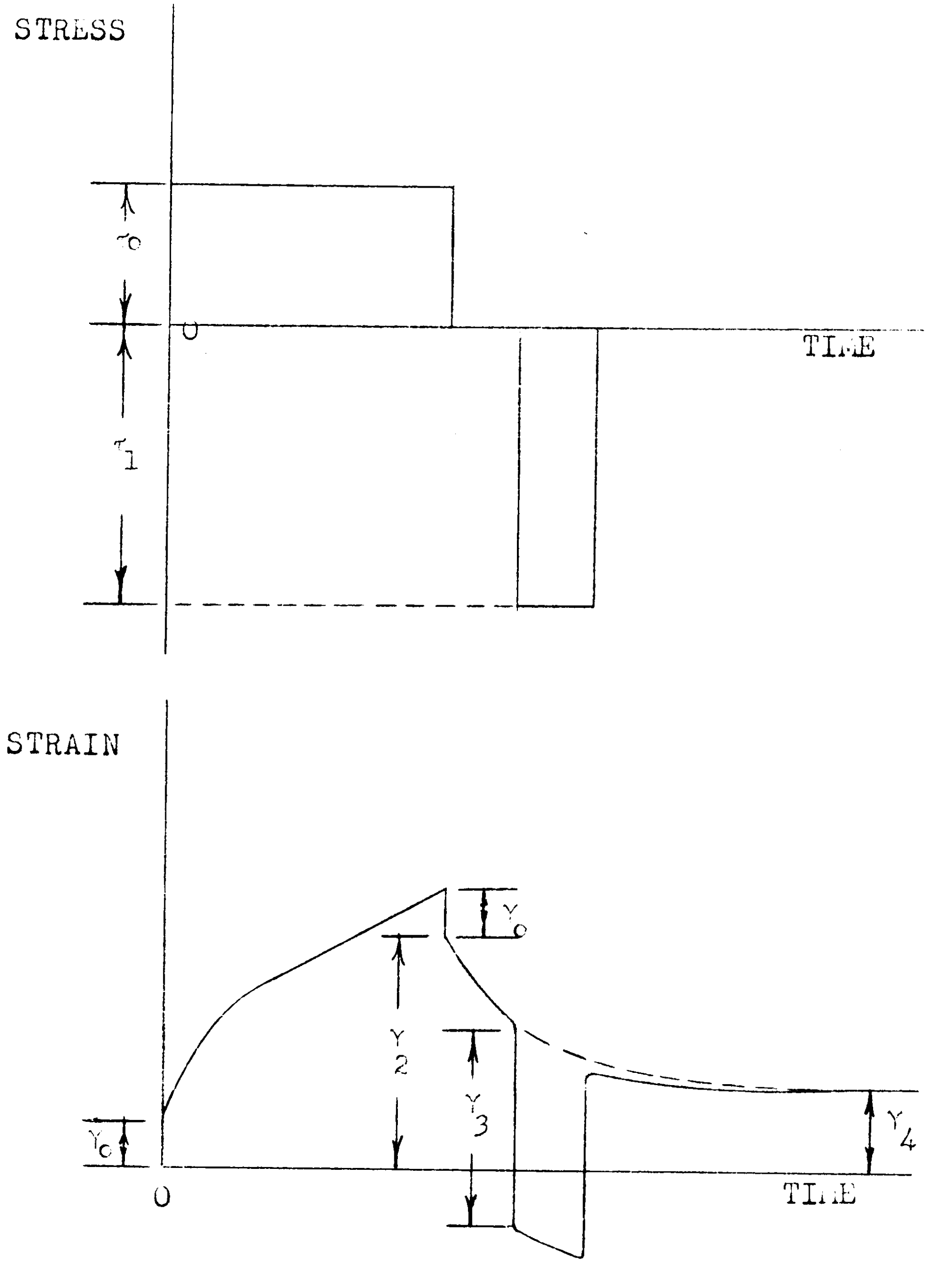


FIGURE 21.

MEMORY EFFECT OF THE MODEL

VI. PRACTICAL APPLICATION

Initially, the material being studied must satisfy three conditions; a) it must have a stress-strain curve with a form similar to that in Fig. (7), b) when subjected to a constant strain, the experimental stress relaxation curve must correspond to the curve in Fig. (16), and c) when loaded as shown in the stress-time curve of Fig. (20), it should yield a strain-time curve similar to that shown in the same figure.

To apply the equations determined in the investigation, it is necessary to establish the value of the six unknown constants. These six constants are E, T_b, T_c, T_a, M_o and M_R, which actually may be reduced to five since it is assumed that $T_b^2 = 4\ T_c^2$. The value of the constants will be determined from the aforementioned experimental curves. From the three curves, the values of E, T_c and T_b can be obtained directly, and three relations established containing the remaining unknown constants T_a, M_o and M_R from which their values may be obtained.

From the stress-strain curve, the value of E is determined by the initial slope. From the stress relaxation curve, Fig. (16), a value of stress and its corresponding time is chosen. The initial slope and stress are determined directly from the graph. Using these values in the stress relaxation law (I-5), the constant T_c can be determined. With the value of T_c known, T_b can be found immediately since

$$T_b = 2\ T_c.$$

The limiting value of stress in the stress-strain curve is equal to a function of T_b, M_R, $\dot{\gamma}$, and T_a, where T_b is now known. The value of $\dot{\gamma}$ was established during the experiment. This yields an equation where there are two unknowns T_a and M_R. The equation is

$$T_b{}^2 \frac{M_R \dot{\gamma}}{T_a} = \text{Limiting value of stress} \qquad \text{(A)}$$

Now, from the experimental curves corresponding to Fig. (20), two more relations are established, from which the unknown constants can be determined. Referring to the strain-time curve, the initial strain γ_o is a function of the moduli of the inelastic material, and the applied constant load. This function is

$$\frac{M_R \ (M_o - M_R)}{M_o} \tau_o = \text{Initial strain,} \qquad \text{(B)}$$

where τ_o, the constant applied stress is known, and the initial strain is obtained directly from the graph. The value of strain, which remains permanently in the material after suddenly removing the constant load, and allowing time for strain recovery is

$$\gamma_2 - \frac{M_o \ T_a \ \dot{\gamma}_R}{M_R} = \text{Permanent strain} \qquad \text{(C)}$$

The permanent strain, the initial slope of the recovery curve $\dot{\gamma}_R$ and the value of strain remaining in the material immediately after removal of the constant load γ_2 are determined from the strain-time curve.

The three equations A, B and C are functions of the three remaining unknown constants, and the value of teese constants may be determined.

With the constants in the equations governing the action of the material known, the equations may now be used to predict the action of the material under various loading conditions.

VII - CONCLUSIONS

The model investigated, exhibits and defines in mathematical expressions all the laws of common mechanical behavior of an inelastic solid. As a result, it excels the more common models discussed in the introduction not only in completeness of behavior, but also in possibilities of application.

In a practical sense, it is of interest to note the similarity between the stress-strain curves of the model and those of some aluminum alloys, Duralumin, magnesium and copper. The creep and strain recovery curve of Fig. (20) is in close agreement with that of real metals discussed by Nadai (15).

This seems to indicate that the model will be useful in giving mathematical equations for the behavior of these metals under different test conditions.

VIII - ACKNOWLEDGEMENT

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