INFLUENCE OF LAYER WAVINESS
ON THE HYDROSTATIC RESPONSE
OF THICK COMPOSITE CYLINDERS

by

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INFLUENCE OF LAYER WAVINESS ON THE HYDROSTATIC RESPONSE OF THICK COMPOSITE CYLINDERS
(ABSTRACT)

by

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The influence of layer waviness in thick cross-ply composite cylinders subjected to hydrostatic pressure is investigated. The cylinders considered are graphite-epoxy with a 2:1 ratio of circumferential to axial layers. All cylinders considered contain 104 total layers with a layup of \([90/(90/0/90)_{17}]_S\), where a '0'' layer is taken to be in the axial direction. The influence of a single isolated group of wavy layers in an otherwise perfect cylinder is evaluated. Layer waviness in only the circumferential direction is considered, and the analysis is assumed to be valid only away from the cylinder ends. A parametric investigation is performed to determine the combined influence of wave location, wave amplitude, and cylinder geometry on hydrostatic response of the cylinder, particularly the stresses generated in and around the wave. The wave is assumed to be located either at the inner or the outer radius of the cylinder. Three wave amplitudes, \(\delta\), are considered: 1/2, 1, and 2 layer thicknesses. Only waves with a half wave length of 10 layer thicknesses are considered. Three cylinder geometries are considered, specifically ones with radius to thickness ratios of 5, 10, and 20.

Finite element analysis is used to determine the stress state within the imperfect, i.e., wave included, cylinders. Based on a maximum stress failure criterion, failure pressures are determined for each of the various wave and cylinder geometries. Failure pressures for the imperfect cylinders are compared with those for a perfect cylinder to determine the failure pressure reduction ratios due to fiber waviness. It is shown that
pressure capacity reductions of approximately 50% are possible for the range of parameters studied. Failure is primarily due to fiber compression, though interlaminar shear and interlaminar tension are a factor. Finite element analysis is also used to determine the failure pressure of the perfect cylinder due to buckling. This is done to determine whether failure due to buckling may overshadow material failure due to fiber waviness. It is shown that buckling is a factor in only one of the cylinder geometries considered, and only in the cases of mild layer waviness.

In addition to results, details about the finite element model are presented. These details include geometry of the wave, changes in material properties due to local fiber rotation and local volume fraction changes, boundary conditions, and justifications for modeling simplifications that were made in an effort to reduce computational costs and analysis times.
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CHAPTER 1

INTRODUCTION

The quality and performance of today's composite structures are largely dependent on the method by which these structures are manufactured. In particular, manufacturing defects, or anomalies, present in the structure can greatly affect its strength. If these anomalies cannot be prevented during the manufacturing process by means of improving the process, they must be allowed for in the initial design of the composite structure. Thus, the need arises for quantifying the effect of representative anomalies.

One anomaly which is present in thick composite cylinders is layer waviness, i.e., a local undulation of layers. Wavy layers are a direct result of the filament winding process by which many of today's composite cylinders are manufactured. This procedure involves winding tows, or bundles, of fibers around a mandrel of desired geometry. The process continues, varying the orientation of the tows to achieve the desired layup, until the desired cylinder wall thickness is obtained. As the cylinder is wound, varying winding tension may account for the initiation of waviness in underlying plies. For cylinders made from thermoset resins, this is of particular significance since the entire cylinder must be wound prior to consolidation and curing. Layer waviness may therefore be present through the entire thickness of the cylinder.

It has been speculated that this layer waviness has a detrimental effect on the response of the hydrostatically-loaded cylinder. To study this, the effect of an isolated group of wavy layers on the response of an otherwise perfect thick-walled graphite-epoxy composite cylinder subjected to hydrostatic loading is investigated. By analyzing an isolated group of wavy layers, herein collectively referred to as a wave, with
identifiable parameters such as wave amplitude and half wave length, results can be obtained which relate cylinder response, particularly the stresses, to variations in these wave parameters. The effects of radial location of the wave within the cylinder wall and cylinder radius to thickness ratios are also investigated. Thermal effects, although they do exist, have been shown in previous investigations to be small and are therefore not considered in this investigation.

Finite element techniques are used to model the wave on a ply-by-ply basis. To simplify the problem, previous wavy layer investigations have neglected curvature of the cylinder and modeled a representative wave geometry in rectangular coordinates. The present study utilizes cylindrical coordinates to represent the true cylindrical geometry. Curvature effects may be negligible, but by modeling in cylindrical coordinates, no assumptions need be made in this regard. So that the modeling techniques incorporated in this study may be applied to other problems, commercially available general purpose software is used for many aspects of the finite element model creation and analysis.

The effect of wavy layers on the hydrostatic response of cylinders is evaluated in terms of failure pressure. The maximum stress failure criterion is used to evaluate the failure pressure, the failure mode, and the location of failure within the cylinder wall. Results are presented which demonstrate the significant reduction in failure pressure due to the inclusion of the wave. For example, for the most severe wave case studied, failure pressure is seen to be approximately 50% of the failure pressure of the perfect graphite-epoxy cylinder. Although this reduction in failure pressure due to layer waviness is significant, it is conceivable that such reductions could be overshadowed by failure of the cylinder due to buckling.

To address the buckling issue, analyses are performed on perfect cylinders, i.e., no layer waviness present, to determine buckling pressures. The geometries of the cylinders
considered correspond to those used in the wavy layer analysis. Results are presented which show that for one of the geometries considered in this investigation, buckling of the perfect cylinder does occur prior to material failure due to layer waviness. This, however, is true only for the less severe cases of layer waviness considered in this investigation. For the more severe cases of layer waviness, material failure occurs prior to buckling of the perfect cylinder.

Overall then, in the context of hydrostatically-loaded thick-walled composite cylinders, this study shows that layer waviness has a detrimental effect on the pressure capacity of the cylinder. To follow in the next chapter is a brief review of the literature on the subject of layer waviness. The details of the specific problem addressed here are presented in Chapter 3. These details include: the geometry of the cylinder, the specific lamination sequence of the cylinder and wave, the nomenclature specific to this problem, the type of analysis used and the assumptions made in the analysis, the criterion used for failure, and other important information which establish the context of the specific problem being investigated. Details of the analytical models which are used in both the wavy layer analysis and the buckling analysis of the perfect cylinder are presented in Chapter 4. Verifications for these models are then presented in Chapter 5. Justification for the simplifications in the analysis is provided there together with accuracy checks for the analysis. Chapter 6 presents stress distributions resulting from the presence of layer waviness. These stress distributions are used to calculate failure pressure reduction ratios. Chapter 7 presents these failure pressure reduction ratios due to layer waviness. The discussion in Chapter 7 emphasizes the significant reduction in pressure capacity of the cylinder due to inclusion of wavy layers, and elaborates on the failure modes responsible for this decrease in strength. Results are also presented in Chapter 7 which compare the failure of the perfect cylinder due to buckling to the failure of the imperfect

Chapter 1: Introduction
cylinder due to layer waviness. Chapter 8 presents a comparison of results obtained in the current investigation with results from a previous investigation. Comparisons are made in terms of stress distributions due to layer waviness as well as failure pressure ratios due to layer waviness. Finally, a summary of important conclusions is presented in Chapter 9 to reemphasize the important issues addressed in this investigation. Recommendations for possible further study are also presented in Chapter 9.
CHAPTER 2

LITERATURE REVIEW

This chapter provides a brief review of previous research investigating the influence of layer waviness on the compression response of fiber-reinforced composite materials. The majority of work done to date has investigated the effect of layer waviness in the context of flat laminates. More recently, work has been done to determine the effect of layer waviness in context of a hydrostatic stress state within a cylindrical structure. This review will look at these two different types of analyses separately.

2.1 FLAT LAMINATES

Camoneschi [1] provides a thorough review of compression test methods, failure theories and mechanisms, and compression related experimental investigations. He experimentally investigated the effect of laminate thickness on the compression response of carbon/epoxy and glass/epoxy composite materials. Some level of layer waviness was present in his specimens and was quantified. Camoneschi witnessed a 20% reduction in compression strength of his $[0_{2/90}]_{hs}$ specimens when going from 0.25 inch thick specimens to 1.0 inch thick specimens. He attributes this to the combination of initial layer waviness present in the specimens and the fixture-induced layer curvature which developed in the specimen near the grips.

Adams and Hyer [2] provide a review of previous investigations regarding compression response in conjunction with fiber as well as layer waviness. The investigations cited include both in-plane as well as out-of-plane waviness.
considerations. Adams and Hyer investigated the compression response of flat composite laminates with intentionally fabricated layer waviness. The specimens had a cross-ply stacking sequence and were made from a carbon/thermoplastic material system. They experimentally investigated the effect of layer waviness on the static and fatigue compression strength of the laminates and used moiré interferometry to determine the localized strain response near the wavy region. They also performed finite element analysis on a representation of the fabricated specimens by modeling the actual geometry of the layer waviness. Material nonlinearities witnessed in the mechanical testing phase were also included in the finite element analysis. The mechanical testing demonstrated reductions in compression strength ranging from 1 to 36 percent, despite the fact that the wavy layer accounted for only 20 percent of the in-plane carrying capacity of the laminate. Layer waviness was characterized by wave amplitude to wave length ratios, \( \delta/\lambda \), and maximum layer rotation angle, \( \theta_{\text{max}} \). The reductions in compression strength were seen to increase with increasing \( \delta/\lambda \) and increasing \( \theta_{\text{max}} \). Fatigue specimens exhibited a one and a half decade loss of compression fatigue life as compared to the specimens without layer waviness. Moiré interferometric analysis indicated the presence of significant interlaminar shear and normal strains at interfaces in the vicinity of the wave. In a similar fashion, finite element results indicated areas of high interlaminar shear and normal stresses in the vicinity of the wave. Fiber compression failure was seen to be the dominant failure mode for less severe wave geometries with only slight reductions in strength. Interlaminar tension failure dominated the more severe wave geometries, resulting in strength reductions as high as 48 percent. The inclusion of material nonlinearities was insignificant, resulting in only slight reduction in predicted strength of the wavy layer laminates, and no changes in the failure modes.
Some of the investigations reviewed by Adams and Hyer [2] regarding compression response of flat laminates with initial fiber waviness are of special mention, but have not been reviewed here. These include Davis [3], Highsmith et al. [4], and Guynn [5]. It is felt that the models of fiber waviness used in these investigations could be used to model layer waviness.

2.2 CYLINDRICAL LAMINATES

Garala [6] experimentally tested cylinders under hydrostatic loading conditions. The carbon/epoxy cylinders were 0.6 inches thick and had an inner diameter of 7.0 inches. These cylinders were known to contain varying degrees layer waviness, with wave amplitudes ranging from 0.02 inches to 0.06 inches. The cylinders were observed to fail at pressures well below expected levels, and it was thought that layer waviness was at least partly responsible. Due to the experimental procedure whereby the cylinders were submerged in fluid, the failure process could not be observed. Therefore, the hypothesis that layer waviness was responsible for premature failure could not be confirmed.

Abdallah et al. [7] performed tests on ring specimens cut from cylinders with 7.0 inch inner diameters and 0.6 inch wall thicknesses. The cylinders were made from a variety of carbon fiber composite materials and manufactured using a variety of techniques. Varying degrees of circumferential layer waviness were present in the specimens, however, no quantitative measure of waviness was given. The lowest failure pressures occurred in the specimens with the most severe level of layer waviness, and failure occurred in regions of severe waviness. Photoelasticity and moiré interferometry results clearly indicated localized regions of high shear strains associated with the regions

Chapter 2: Literature Review
of layer waviness. No quantitative results in terms of shear strain were presented, but the authors indicated that this is planned for future work.

In a series of four papers, Hyer et al. [8] and Telegadas and Hyer [9,10,11] investigated the effect of isolated circumferential layer waviness on the response of a hydrostatically loaded graphite/epoxy cross-ply cylinder. The modeling was assumed to take place away from the ends of the cylinder, where a generalized plane deformation analysis is valid. In all cases, the cylinders considered were thick, i.e., greater than 100 layers, and had a 2:1 ratio of circumferential to axial layers. Hyer et al. [8] assumed a cosinusoidal geometry to be a suitable representation of layer waviness. They modeled an isolated group of wavy layers on a layer-by-layer basis using finite element techniques. Neglecting curvature effects of the cylinder, they modeled the group of layers in a rectangular coordinate system. The group of layers was effectively removed from the surrounding cylinder and modeled separately by applying necessary boundary conditions. These boundary conditions were calculated from an elasticity solution for the perfect cylinder also developed by Hyer [12]. For the single wave geometry analyzed, i.e., a wave amplitude of one layer thickness and a half wave length of ten layer thicknesses, significant interlaminar shear stresses were calculated in the vicinity of the wave.

Telegadas and Hyer [9] extended the previous work of Hyer et al. [8]. They investigated the effect of modeling the problem using a variety of finite element types, ranging from an element which assumed no fiber curvature whatsoever, to one which followed the curvilinear geometry of the layer waviness and included volume fraction changes due to the resin-rich and resin-depleted areas resulting from the layer waviness. The effect of the finite element type was investigated in terms of interlaminar normal and interlaminar shear stresses. Both pressure-induced stresses and thermally-induced
stresses were considered. Finally, they investigated the effect of radial location of the layer waviness within the cylinder wall by considering the layer waviness to be located at the inner radius, the outer radius, or the midradius of the cylinder. They found that in terms of the pressure-induced stresses, a relatively simple element produced results comparable to those produced by the more complicated element types. Their simple rotation element, which used a piece-wise linear approximation to model the curvilinear geometry of the waviness, produced results very similar to the most refined element incorporating curvilinear geometry and volume fraction changes. For the thermally-induced stresses, they found that the stress state was very sensitive to volume fraction change. Including volume fraction effects resulted in differences as great as 40 percent for the predicted thermally-induced interlaminar normal stresses. Even greater differences were predicted for the interlaminar shear stresses. Despite the sensitivity of the thermal stress predictions to volume fraction, the thermal stresses were seen to be quite small compared to the stresses generated by the pressure. The influence of the radial location of the layer waviness was shown to be significant. The worst case appeared to be for the wavy layers located at the inner radius of the cylinder. At this location, interlaminar tension was predicted, despite the overall compressive nature of the problem. The interlaminar shear stress was also shown to be greatest at this location.

Telegadas and Hyer [10] continued their previous work [9] by examining failure for varying degrees of severity of layer waviness. They again considered both pressure-induced and thermal-induced stresses. They employed a maximum stress failure criterion to evaluate three possible failure modes: failure in the fiber direction, interlaminar failure perpendicular to the fiber direction, and interlaminar shear failure. A total of nine wave geometries were considered representing three different levels of wave amplitude and half wave length. The effect of wave location was again considered by

Chapter 2: Literature Review
performing analyses with a wave located at the inner radius, the outer radius, or the midradius of the cylinder. For the most severe cases of layer waviness studied, 70 percent reductions in pressure capacity were predicted for the cylinder. With one exception, the failure was attributed to an interlaminar shear stress concentration located near the inflection point of the wave. The one exception occurred for the most severe wave considered. For this case, failure resulted from an interlaminar tensile stress concentration located beneath and near the location of maximum amplitude of the wave. In general, a wave located at the inner radius of the cylinder was seen to be most detrimental and inclusion of thermally-induced stresses was seen to further reduce to the strength of the cylinder, but not by much.

Some of the most recent work involving cylindrical laminates has been performed by Bogetti et al. [13] and Gillespie et al. [14]. This is a pair of reports which utilize the same two-dimensional ply waviness model for their analyses. In both cases, the influence of axial layer waviness is studied using a rectangular coordinate system. The first report develops the ply waviness model which predicts elastic properties and thermal expansion coefficients of sublaminates containing wavy layers, ply stresses for prescribed mechanical and thermal load cases, and strength reduction associated with ply waviness. Parametric studies are then performed on cross-ply thermoplastic composite cylinders to determine the effect of ply waviness on the stiffness and strength of the cylinder. Bogetti et al. found that stiffness reduction due to layer waviness is most significant in the direction of ply waviness, the axial direction in this case, and that the magnitude of stiffness reduction is highly sensitive to the level of ply waviness. They also found that the most significant strength reduction results from interlaminar shear failure of the wavy ply when loaded in the direction of undulation, and that strength reduction is also highly sensitive to the level of ply waviness.

Chapter 2: Literature Review
Gillespie et al. [14] present modified biaxial failure envelopes due to residual thermal stresses and layer waviness. They demonstrate that significant compressive strength reduction is only present in the axial direction. This is due to the fact that only axial layer waviness is considered. The strength reduction is attributed to interlaminar shear failure within the sublamine containing the wavy axial layer. Results are also presented which demonstrate the influence of layer undulation amplitude on the cylinder failure pressures. In general, Gillespie et al. find that for less severe wave amplitudes, fiber compression is the dominant failure mode, but that no significant reduction in failure pressure results. For more severe wave amplitudes, interlaminar shear is the dominant failure mode and significant reductions in failure pressure result. They also note that decreasing the interlaminar shear strength of the cylinder would result in shear failure due to waviness being triggered at much less severe wave amplitudes.

The review provided here of previous investigations regarding layer waviness demonstrates that some work has already been done to determine the effect of this type of anomaly. Although this brief review does not cover them all, the majority of such work has been performed in the context of flat laminates. The few investigations that have been performed on layer waviness in cylindrical laminates leave many questions unanswered. The effect of cylindrical geometry is one of primary interest for the current investigation. The investigations performed by Hyer et al. [8] and by Telegadas and Hyer [9,10,11] provide an excellent basis from which to proceed. The following chapters provide details of the analytical investigation which will characterize many of the combined effects of layer waviness and cylindrical structure geometry.
CHAPTER 3

PROBLEM STATEMENT

The effect of layer waviness on the material failure response of a hydrostatically-loaded thick composite cylinder is investigated. Thermal effects, although present, have been shown by Telegadas and Hyer [9,10,11] to be small compared to the pressure-induced effects and are therefore not considered. This chapter presents the details of the wavy layer problem, including the geometry and nomenclature of the cylinder and wave, and the material properties and the material failure criterion used. The buckling problem, though not studied as comprehensively, is also described in this chapter.

3.1 CYLINDER GEOMETRY

The cylinders being studied here are cross-ply graphite-epoxy having close to a 2:1 ratio of circumferential to axial plies. A 2:1 construction is ideal for hydrostatic loading, and considerable experimental, numerical, and analytical work has been done with this construction, both with graphite-epoxy and with glass-epoxy. The specific material stacking sequence is given as [90/(90/0/90)17]5, where 0° represents the axial direction. This stacking sequence, totaling 104 layers, is used for all cylinders in this investigation. Though not exactly 2:1, this layup is very close to 2:1. The extra 90° layer on the inner and outer radius represent the hoop windings generally added in actual cylinders. The various cylinder geometry parameters, together with the cylindrical r-θ-z coordinate system used throughout the discussion, are shown in Figure 3.1. A ply, or layer, thickness, h, in the perfect cylinder is taken to be 0.006 inches. The midradius of the cylinder is represented by R and the wall thickness by H. Three cylinder radius to
Figure 3.1. Cylindrical geometry and coordinate system.
thickness ratios, R/H, are considered in this investigation. They are 5, 10, and 20. All represent thick-walled cylinders. To achieve the various R/H ratios, the midradius of the cylinder is varied. In all cases, the cylinder wall thickness remains a constant 0.624 inches (104 layers at 0.006 in. per layer).

3.2 MATERIAL PROPERTIES

The material properties used for these cylinders coincide with those of previous investigations by Hyer, et al. [8] and Telegadas and Hyer [9, 10] for an assumed fiber volume fraction of 65%. Specifically, they are given as:

\[
\begin{align*}
E_1 &= 19.1 \times 10^6 \text{ psi} & E_2 &= 1.36 \times 10^6 \text{ psi} & E_3 &= 1.34 \times 10^6 \text{ psi} \\
G_{12} &= 0.80 \times 10^6 \text{ psi} & G_{23} &= 0.50 \times 10^6 \text{ psi} & G_{13} &= 0.80 \times 10^6 \text{ psi} \\
\nu_{12} &= 0.279 & \nu_{23} &= 0.340 & \nu_{13} &= 0.279
\end{align*}
\]

where \( E_1 \) = Young's modulus in the fiber direction
\( E_2, E_3 \) = Young's modulus transverse to fiber direction
\( G_{ij} \) = Shear modulus in the i-j plane
\( \nu_{ij} \) = Poisson's ratio in the i-j plane

3.3 WAVY LAYER ANALYSIS

The wavy layer geometry considered in this investigation is representative of those seen in many of the filament wound composite cylinders. For a cross-ply cylinder, wavy layers are commonly seen in the form illustrated in Figure 3.2. The lighter areas within the cylinder wall represent the circumferential layers, whereas the darker areas represent the axial layers. It is incorrect to assume that all wavy layers are alike, but certain identifiable characteristics are present in many of them which are included in the wavy layer geometry considered for this investigation.
Figure 3.2. Typical circumferential wavy layers in a composite cylinder.
First of all, the waviness is defined to be a result of circumferential layer undulation. This undulation is a result of variations in thickness of certain key circumferential layers. The axial layers, on the other hand, are assumed to remain straight and constant in thickness. Thus, the wave is assumed to span the entire axial length of the cylinder, with no variation in that direction. Layer waviness in the axial direction, though possibly as important, as demonstrated by Bogetti et al. [13] and Gillespie et al. [14], is beyond the scope of this investigation.

An isolated group of circumferential wavy layers is studied for all wavy layer analyses. Combined effects due to interaction of multiple circumferential waves are not considered. Such effects are, again, beyond the scope of this investigation. A schematic of the circumferential wavy layer problem is shown in Figure 3.3. Although the wave is shown located near the midradius of the cylinder, this investigation considers the wave to be either near the inner radius or near the outer radius of the cylinder. Due to the Lamé effect [15] associated with thick-walled cylinders, this approach is felt to represent extremes in response.

Since axial waviness is not considered in this investigation, the analysis is performed on a representative axial cross-section of the cylinder, i.e., an r-θ plane. As with several of the previous investigations, e.g., References 9-12, the cross-section is assumed to be far removed from any cylinder end effects. A generalized plane deformation analysis is thus used, whereby a ring, or slice, from the cylinder, as shown in Figure 3.4, is loaded externally by a pressure, P, at its outer radius, and with a uniform axial deformation resulting from the endcap forces due to P. The front and rear faces of the slice are required to remain planar and orthogonal to the axial direction.
Figure 3.3. Schematic of an isolated group of wavy layers.

Figure 3.4. Axial cross-section used for wavy layer analysis.
3.3.1 Wave Geometry

To be consistent with previous work by Hyer, et al. [8] and Telegadas and Hyer [9,10,11], layer waviness is characterized by a 14 layer thick sublamine with a stacking sequence of [90/(90/0/90)$_2$]$_S$. The sublamine has identifiable parameters of wave amplitude, $\delta$, and half wave length, $\lambda$. Equivalently, non-dimensional wave amplitude, $\bar{\delta}$, and non-dimensional half wave length, $\bar{\lambda}$, defined as

$$\bar{\delta} = \frac{\delta}{h}, \quad \bar{\lambda} = \frac{\lambda}{h}$$

(2)

are used to characterize the sublamine. These quantities are illustrated for a rectangular geometry in Figure 3.5a. For this investigation, however, these parameters must be adapted to a cylindrical geometry. The wave and its associated parameters are again shown in Figure 3.5b, this time in terms of a cylindrical geometry. Note that the circumferential and axial layers are identified in Figure 3.5. Both the rectangular and cylindrical geometries are shown because it is important to note that when considering the cylindrical geometry, it is often convenient to describe the arc subtended by the wavy layers, $\theta^*$. More will be said of this shortly.

The wave is assumed to be symmetric about the location $\theta = 0^\circ$. Referring to Figure 3.5b, the amplitude of the wave, $\delta$, is defined at this line of symmetry as the radial displacement, or radial offset, of the midinterface of the sublamine from its original circular geometry. The wave amplitudes considered in this investigation are $\delta = h/2$, $h$, and $2h$, or equivalently, from Equation 2, normalized wave amplitudes $\bar{\delta} = 1/2$, 1, and 2.

The arc length of the wave in the circumferential direction is characterized by its half wave length, $\lambda$, or, from Equation 2, normalized half wave length $\bar{\lambda}$. This is a measure of the distance from the line of symmetry at $\theta = 0^\circ$ to the point $\theta = \theta^*$, the point
a) Rectangular geometry.

b) Cylindrical geometry.

Figure 3.5. Layer waviness parameters in rectangular and cylindrical geometries.
at which layer waviness has subsided. For a given half wave length, \( \Theta^* \) is calculated from

\[
\Theta^* = \frac{\lambda}{R^*},
\]

(3)

where \( R^* \) represents the radius of the midinterface in the wavy sublaminate. A half wave length of \( \lambda = 10h \), or \( \bar{\lambda} = 10 \), is used for all analyses in this investigation. The value of \( \Theta^* \), however, changes with \( R^* \), and is therefore dependent on the radial location of the wave within the cylinder wall and on the R/H ratio of the cylinder.

Due to waviness, the radial location of the layer interfaces change relative to the perfectly circular geometry. The change of the interface locations within the wavy sublaminate, and hence within the wave, is assumed to be governed by the following cosinusoidal function of \( \Theta \):

\[
OFFSET = \Delta_r \cdot \frac{1}{2} \left[ 1 + \cos \left( \frac{\pi \Theta}{\Theta^*} \right) \right],
\]

(4)

where
- OFFSET = radial displacement of the interface from its original circular geometry. OFFSET varies as a function of interface location as well as circumferential location, \( \Theta \).
- \( \Delta_r \) = the radial displacement of the interface in question at the centerline of the wave, \( \Theta = 0^\circ \). The value of \( \Delta_r \) for each interface is illustrated in Figure 3.6.
- \( \Theta \) = circumferential coordinate
- \( \Theta^* \) = circumferential equivalent to a half wave length, \( \lambda \).

The function in Equation 4 is valid for the range \( 0^\circ \leq \Theta \leq \Theta^* \), at which point the local radial displacement of the interface due to waviness ceases and the circular shape of the interface is resumed. For example, at the interface located between the second and third layer of the wavy sublaminate, since there are two circumferential layers located
Figure 3.6. Radial displacement, $\Delta_r$, of each interface in the wavy sublaminate.
radially inward from this interface, the value of $\Delta r$ is equal to $\delta/2$. Each of these two inward layers has increased in thickness by $\delta/4$, resulting in a cumulative radial displacement of $\delta/2$. The offset from the original circular interface between the second and third layers is thus

$$\text{OFFSET} = \frac{\delta}{2} \cdot \frac{1}{2} \left[ 1 + \cos \left( \frac{\pi}{\theta_\ast} \right) \right].$$

To account for the change in the location of the layer interfaces due to layer waviness, some of the layers within the wavy sublamine must change thickness. This change in thickness is assumed to occur only in the circumferential layers, excluding the two center-most circumferential layers of the 14 layer sublamine. At $\theta = 0^\circ$, the four inner-most circumferential layers of the sublamine are assumed to increase in thickness an amount $\delta/4$, resulting in a net displacement of $\delta$ at the midinterface. Similarly, the four outermost circumferential layers within the wavy sublamine are assumed to decrease in thickness an amount $\delta/4$, so that the nonwavy interface location is resumed at the outer radius of the sublamine. The increase in thickness of the layers is assumed to be due to resin richness, while the decrease in thickness is assumed to be due to resin depletion. In practice, the non-uniform distribution of resin is, in part, responsible for waviness and is thus considered to be the primary mechanism here.

3.3.2 Material Properties

The material properties for the wave are related to those presented earlier in this chapter for the perfect cylinder, namely, Equation 1. These properties must be modified, however, due to the local fiber rotations and volume fraction changes due to the waviness of the layers. Fiber volume fractions range from 44% to 90% for the most severe case of waviness studied. The upper bound of 90% is an artificial limit placed on the model.
Calculating volume fractions strictly on the geometry of the wave results in fiber volume fractions greater than 100%. Since this is not possible, the artificial limit of 90% has been imposed. This same technique was used previously by Telegadas and Hyer [9,10,11].

A modified rule of mixtures approach is taken to determine the material properties in the wavy region. The details of the calculations are presented in Chapter 4.

3.3.3 Failure Pressure Calculations

A maximum stress failure criterion is used to determine the failure pressure. The maximum stress criterion is chosen because it is relatively simple to implement and clearly demonstrates the influence of various failure modes. The maximum stress failure criterion states that failure occurs when any of the following equalities holds:

\[
\frac{X_c}{\sigma_{s,c}} = 1 \\
\frac{X_t}{\sigma_{s,t}} = 1 \\
\frac{Y_c}{\sigma_{n,c}} = 1 \\
\frac{Y_t}{\sigma_{n,t}} = 1 \\
\frac{S}{|\tau_{ns}|} = 1
\]

(5)

where

\(\sigma_{s,c}\) = fiber direction compression stress  \\
\(\sigma_{s,t}\) = fiber direction tensile stress  \\
\(\sigma_{n,c}\) = interlaminar compression stress transverse to fiber direction  \\
\(\sigma_{n,t}\) = interlaminar tensile stress transverse to fiber direction  \\
\(\tau_{ns}\) = interlaminar shear stress
Strength parameters:

\[ X_c = \text{fiber direction compression strength} \]
\[ X_t = \text{fiber direction tensile strength} \]
\[ Y_c = \text{interlaminar compression strength transverse to fiber direction} \]
\[ Y_t = \text{interlaminar tensile strength transverse to fiber direction} \]
\[ S = \text{interlaminar shear strength} \]

The subscripts 'n' and 's' indicate directions normal and tangential to the interface, as illustrated in Figure 3.7. Within the wavy layer region, the 'n' and 's' directions are continuously changing with circumferential location, \( \theta \). It should be noted that for the nonwavy layer regions, the normal and tangential directions coincide with the radial, \( \varepsilon_r \), and circumferential, \( \varepsilon_\theta \), directions respectively. This is also illustrated in Figure 3.7.

The failure pressure due to layer waviness can be conveniently calculated from the expressions of Equation 5 by evaluating the ratios on the left side of the expressions using the stresses produced by a unit hydrostatic pressure. Using this method, five ratios are calculated, each corresponding to the five failure modes: fiber direction compression and tension, interlaminar compression and tension, and interlaminar shear. The lowest of the five ratios indicates the mode of failure. By using a unit pressure, the value of the lowest ratio is the failure pressure.

The strength parameters for this material are the same as those used previously by Telegadas and Hyer [10,11]:

\[ X_c = 180,000 \text{ psi} \]
\[ Y_c = 30,000 \text{ psi} \]
\[ Y_t = 7,500 \text{ psi} \]
\[ S = 13,500 \text{ psi} \]
The "n" direction is everywhere perpendicular to the interface, while the "s" direction is everywhere tangential to the interface.

Figure 3.7. Interlaminar "n-s" coordinate system.
It should be noted that these strength parameters are not modified to account for changes in volume fraction due to layer waviness. This is a limitation of the previous investigation by Telegadas and Hyer [10,11] and the current investigation.

3.3.4 Failure Pressure Ratios

To determine the failure pressure ratio, $P_{cr}/P_0$, the failure pressure due to layer waviness, $P_{cr}$, is divided by $P_0$, the pressure required to produce material failure in the perfect cylinder. Three values of $P_0$, given as 21.389 ksi, 11.445 ksi, and 5.988 ksi for the three cylinder geometries $R/H=5$, 10, and 20, respectively, are calculated from a maximum stress failure criterion based on the stresses in the perfect cylinder produced by a unit hydrostatic pressure. The stresses for the perfect cylinder are calculated from an elasticity-based solution by Hyer [12] which solves for the axisymmetric stress state in a multilayered cylinder. The stresses for the perfect cylinder are also solved for by a finite element analysis of a wave-free cylinder. This latter approach serves as a check on the accuracy of the finite element analysis and will be discussed in Chapter 5. Failure of the perfect cylinder occurs due to fiber direction compression failure at the inner radius of the cylinder. More will be said of this in the chapter on failure, Chapter 7.

3.4 BUCKLING ANALYSIS OF A PERFECT CYLINDER

The issue of cylinder buckling is very complicated. The analysis here provides a superficial analysis to determine critical failure pressures of very specific cylinders. This analysis is performed to address the issue of whether or not failure of the perfect cylinder due to buckling occurs prior to material failure due to wavy layer imperfections. Buckling analyses are performed on the three cylinder geometries being considered in this investigation, i.e., $R/H = 5$, 10, and 20. Buckling failure pressure, $P_{cr,buckling}$, is
calculated for each of these geometries. Because the interest lies in calculating the minimum failure pressure due to buckling, only the first buckling mode is considered.

3.4.1 Cylinder Geometry

Since the buckling mode shape is unknown, no symmetry assumptions can be made with respect to the circumferential direction of the cylinder. The entire circumference of the cylinder must therefore be included in the analysis. It is assumed that in practice, ring stiffeners will be placed along the length of the cylinder at intervals ranging from \( R/2 \) to \( R \). This essentially subdivides the cylinder into a number of equivalent axial sections which are identically loaded. The buckling analysis therefore needs to be performed only on a single axial section of the cylinder with length, \( L \), which corresponds to the distance between the centers of adjacent ring stiffeners, i.e., \( R/2 \) to \( R \), or alternatively, length to radius ratios, \( L/R \), of \( 1/2 \) to \( 1 \). Only two lengths of cylinder, \( L/R = 1/2 \) and \( 1 \), are analyzed in this investigation. The failure pressure due to buckling for \( L/R \) ratios between \( 1/2 \) and \( 1 \) will fall somewhere in between the failure pressures at these values.

To account for effects from adjoining sections of cylinder, clamped boundary conditions are imposed on either end of the section of cylinder being analyzed. This requires that either end of the cylindrical section remain perpendicular to the axial direction. A schematic of the analyzed section of the cylinder is shown in Figure 3.8.

3.4.2 Material Properties

The orthotropic material properties used in the buckling analysis reflect the true material properties of the full thickness graphite-epoxy cylinders used throughout this
a) Axial section used for buckling analysis.

b) Geometry parameters.

Figure 3.8. Cylinder geometry used in buckling analyses.
investigation. A scheme developed by Hyer [16] is used to calculate smeared three-dimensional properties based on the layer stacking sequence of the graphite-epoxy cylinders, i.e., [90/(90/0/90)17]S. The smeared properties are as follows:

\[ E_r = 0.14692 \times 10^7 \text{ psi} \quad E_\theta = 0.13354 \times 10^4 \text{ psi} \quad E_z = 0.71890 \times 10^7 \text{ psi} \]
\[ G_{rr} = 0.6681 \times 10^6 \text{ psi} \quad G_{\theta\theta} = 0.80 \times 10^6 \text{ psi} \quad G_{rz} = 0.56987 \times 10^6 \text{ psi} \quad (7) \]
\[ \nu_{rr} = 0.038256 \quad \nu_{\theta\theta} = 0.0530 \quad \nu_{rz} = 0.07078 \]

3.4.3 Buckling Failure Pressure Ratios

For comparison with failure pressure ratios due to layer waviness, the buckling pressures are divided by \( P_0 \), the pressure required to produce material failure in the perfect cylinder.
CHAPTER 4

ANALYTICAL MODEL

Details of the finite element models used for this investigation are presented in this chapter. The following modeling issues are addressed: geometry of the models, boundary conditions, material properties, and calculation of failure pressures. The specifics of each of the analyses, i.e., the layer waviness analysis and the buckling analysis of the perfect cylinder, are addressed separately.

For both analyses, commercially available general purpose programs are used whenever possible. PATRAN, a product of PDA Engineering [17], is used to create the geometries and apply boundary conditions for all of the models analyzed in this investigation. All processing of the models is performed with ABAQUS, a product of Hibbitt, Karlsson, and Sorenson, Inc. [18]. Material property calculations and implementation of a failure criterion are performed with FORTRAN programs developed in previous work and ones developed specifically for this investigation.

4.1 WAVY LAYER ANALYSIS

As stated earlier in Chapter 3, a generalized plane deformation analysis is used for the wavy layer analysis. This is a three-dimensional analysis which models a ring, or axial slice, from the cylinder of unit thickness in the axial direction. The ring is presumed to be located in a region removed from any boundary effects due to the ends of the cylinder. This type of isolated analysis is commonly referred to as a membrane analysis, in contrast to a boundary layer analysis which focuses on the ends of the cylinder. The modeled ring is presumed to remain planar, but axial loads due to the
presence of endcaps are permitted. Hence a generalized plane deformation condition is prescribed, rather than a plane strain condition. The latter would not allow for axial deformation due to the endcap loads.

The ring from the cylinder, including both wavy and nonwavy regions, is modeled on a layer by layer basis using three-dimensional triquadratic finite elements. This type of analysis is very computationally intensive. In an effort to minimize computation time and cost, simplifications are made to the model wherever possible.

Use of reduced rather than full integration finite elements for all the wavy layer analyses is the first of the modeling simplifications. This first simplification deals with finite element methodology rather than the specific layer waviness problem. The 'reduced' refers to the reduced Gaussian integration scheme used to evaluate the element stiffnesses [19]. The other two simplifications deal with geometric simplifications specific to the layer waviness problem being investigated here. The first of these geometric simplifications involves modeling a circumferential segment of the cylindrical ring, rather than the complete circumference (360°), and applying appropriate conditions at the boundaries of the segment to represent the unmodeled circumferential portion of the ring. The other geometric simplification involves modeling just half the thickness of the cylinder wall and applying the appropriate boundary conditions to represent the unmodeled thickness of the cylinder wall. Justifications and further details for each of the three simplifications mentioned here are presented later in Chapter 5.

The simplified layer waviness model is created and analyzed in three stages: preprocessing, processing, and postprocessing. Preprocessing includes such considerations as modeling the geometry of the wavy and nonwavy regions, application of boundary conditions to the ring segment, and calculation of material properties for the wavy and nonwavy regions. Processing involves calculating the stress state in the model.
for the given loading and boundary conditions. Postprocessing includes application of a maximum stress failure criterion, and calculation of the failure pressures and failure modes due to the hydrostatically-induced stress state. These three steps are discussed below.

4.1.1 Preprocessing

4.1.1.1 Wavy Layer Geometry

The geometry of the wavy and nonwavy regions of the cylinder is created using Patran Command Language (PCL), a programming feature of PATRAN. Separate PCL programs were written to generate a submodel representing the wavy layer region and several submodels to represent the surrounding nonwavy layer regions of cylinder ring segment. Each of these PCL programs is designed to offer the user an interactive capability within PATRAN. Although these programs are written specifically for this investigation, they are general enough to accommodate a variety of problems, and certainly extensions of the current problem. The source listings for the PCL programs to generate the wavy layer and the nonwavy layer regions, WAVYREGN and CYLINDER, are included in Appendix A.

The PCL programs give the user the ability to modify many of the geometric parameters of the problem at model creation time. In the case of the program to generate the wavy layer region, parameters which can be modified include the following: wave amplitude, half wave length, radial and circumferential wave location, ply thickness, and the number and type of three-dimensional finite elements used. In the case of the program to generate the nonwavy layer region of cylinder segment, the following parameters can be modified: inner and outer radius of the cylinder, arc length of the
segment, the number of layers in the cylinder, and the number and type of three-dimensional finite elements used. The PCL programs for both the wavy and nonwavy layer regions of the cylinder offer the flexibility of specifying axial model length. In this manner, finite element aspect ratios can be easily varied to suit a specific model. Although the ability to specify the axial model length is included, unit lengths in the axial direction are specified for all models included in this investigation.

The complete model is created by first producing a wavy layer submodel and then superimposing submodels of the nonwavy layer region around the wavy layer submodel. Joining these submodels together results in the complete model of the wavy and nonwavy portions of the cylinder segment. The combination of the two PCL programs therefore gives the user the ability to model any combination of cylinder and wavy layer geometry within the context of a 14 layer wavy sublamine. Although not used for this purpose in this investigation, the PCL programs give the user the ability to model multiple waves located at any radial and circumferential location throughout the cylinder wall. A two-dimensional view of a typical finite element mesh used for the wavy layer analysis is shown in Figure 4.1. As mentioned previously, and as will be discussed later, only a circumferential segment of the cylinder is modeled. In addition, Figure 4.1 shows only half of the cylinder wall thickness. The half-thickness segment modeled will also be discussed shortly. The wavy layer and nonwavy layer regions are modeled with two elements radially through the thickness of each layer. A minimum of two elements is required to obtain convergence of through-the-thickness stresses. One element is used in the axial direction. In the circumferential direction, eight elements are used between $\theta=0^\circ$ and $\theta=\theta^*$ and six elements are used between $\theta=\theta^*$ and the circumferential extent of the model, $\theta=\theta_{\text{wedge}}$. A total of 224 elements is thus used to model the 14 layer wavy region. A total of 1456 elements is used for the entire half-thickness segment model.
Figure 4.1. Finite element discretization for half-thickness model of a wave with $\delta=2$ located at the inner radius of a cylinder with $R/H=20$. 

Chapter 4: Analytical Model
(wavy and nonwavy layer regions). A typical three-dimensional triquadratic reduced integration element used for all wavy layer analyses is illustrated in Figure 4.2. The nodes, numbered with Arabic numerals, and the Gauss points, numbered with Roman numerals, are shown in the figure.

4.1.1.2 Boundary conditions

The boundary conditions applied are a result of three characteristics of the model: 1) the simplifications made in modeling a segment rather than a complete ring of the cylinder, 2) a generalized plane deformation analysis using three-dimensional elements, and 3) modeling only half the thickness of the cylinder wall.

The ability to model a segment of the cylinder rather than having to model an entire ring is due to the assumed symmetry of the wave and the localized effect it has on the stress state of the cylinder. A typical segment model and the accompanying required boundary conditions are shown in Figure 4.3. The assumed cosinusoidal geometry of the wave requires symmetry about line AA, i.e., about \( \theta=0^\circ \). This symmetry is represented by a classic roller-type boundary condition, restricting motion at AA in the circumferential direction. A similar boundary condition is applied along BB, or at \( \theta=\theta_{\text{wedge}} \), restricting motion in the circumferential direction. The exact circumferential location of line BB is determined only after preliminary analysis to determine the distance from the wave at which the stress state of the cylinder is no longer affected. Ideally, BB would be placed far enough away from the wavy region such that stress distributions along BB would match those for a cylinder with no imperfections. In practice, the boundary \( \theta=\theta_{\text{wedge}} \) is placed far enough from the wavy region so that the stress state in the wavy region is not influenced significantly by the presence of the boundary. More is said about this later in Chapter 5. Formally, the boundary conditions
(local node and Gauss point numbering indicated by Arabic and Roman numerals, respectively)

3-dimensional triquadratic reduced integration finite element

Figure 4.2. Finite element type used in wavy layer analyses.
Figure 4.3. Wavy layer segment model and boundary conditions.
at AA and BB are:

\[
\begin{align*}
\nu &= 0 \\
u, \omega &= 0 \\
\{ \theta = 0^\circ, \theta^{\text{edge}} \}
\end{align*}
\]  

(8)

where \(u, \nu, \) and \(w\) represent displacement in the radial, circumferential, and axial directions, respectively. Strictly speaking, with the boundary conditions being used at BB, symmetry about BB is assumed. This assumes that another wavy layer region is located a distance \(\theta^{\text{edge}}\) from BB. Essentially, with the modeling here, it is assumed that the two wavy regions do not interact.

In the early stages of this investigation, axial deformations resulting from endcap effects were applied to the segment using an axial force. Specifically, a simple area-based percentage of the axial force acting on the complete cross-section of the cylinder was applied to the segment. Since it is known a priori to be the hydrostatic pressure times the endcap area, axial force, in contrast to axial strain, is a much simpler condition to impose. Since the two-dimensional generalized plane strain elements available in ABAQUS allow for axial displacement deformation only, use of an axial force required the use of three-dimensional elements. To simulate a generalized plane deformation analysis using three-dimensional elements requires the application of special boundary conditions to the model. These conditions are illustrated in Figure 4.4. Specifically, axial motion of the entire back face (\(z=1\), i.e., unit axial thickness) of the segment is not allowed, thus ensuring that face remains planar. The front face (\(z=0\)) is allowed to move in the axial direction, but through the use of a multipoint constraint, is required to remain planar and orthogonal to the axial direction. The combination of these two restraints prescribe a generalized plane deformation condition.

In the later stages of the investigation, questions arose concerning the validity of applying a uniform axial force to represent endcap effects. The concern arose from the
Figure 4.4. Generalized plane deformation condition.
fact that the axial stresses in the vicinity of the isolated wavy layers were not the same as the axial stresses in the remaining circumferential locations of the cylinder, away from the wavy layers. Hence the axial force acting on the segment of the cylinder with the wavy layers would not be a simple percentage, based on area, of the axial force acting on the complete cross-section. To alleviate concerns, the analyses performed in later stages of this investigation utilized an elasticity solution to calculate the axial strain resulting from the hydrostatic loading for the given cylinder geometry. The axial strain due to endcap effects, rather than the axial force, was used for all subsequent analyses. There was a degree of confidence that the axial strain was the same at all circumferential locations, independent of the presence or absence of layer waviness. At this point, the two-dimensional generalized plane strain finite elements available in ABAQUS could have been used. However, the time and effort required to change to the generalized plane strain element resulted in the continued use of the three-dimensional element.

The final boundary conditions imposed on the model are due to hydrostatic loading. A unit pressure, \( P \), is applied externally to the cylinder. For a full-thickness cylinder model, this is the only pressure boundary condition necessary. However, for the half cylinder wall thickness models used in this investigation, the influence of this hydrostatic pressure is modeled in conjunction with the effect of the unmodeled portion of the cylinder wall.

As stated previously in Chapter 3, analyses are performed for waves located either near the inner radius or near the outer radius of the cylinder wall. Two characteristic half-thickness models are thus needed, one for the wave near the inner radius, and one for the wave near the outer radius of the cylinder wall. These two models are illustrated in Figure 4.5a. In both cases, the influence of the unmodeled cylinder wall is accounted for by applying a radial stress (pressure) boundary condition, \( P_{\text{midradius}} \), to the half-
a) Modeled and unmodeled portions of the cylinder wall. (solid lines represent modeled portion of cylinder wall)

b) Radial stress boundary conditions.

Figure 4.5. Half-thickness wavy layer models.
thickness models. The radial stress boundary condition is applied either at the inner radius or the outer radius of the half-thickness model, depending on whether the model represents the outer half-thickness or the inner half-thickness model, respectively. Specifically referring to Figure 4.5b, the boundary conditions are:

model representing the inner half of the cylinder wall (wave located near the inner radius of cylinder wall),

\[
\begin{align*}
\sigma_r &= 0 \quad \text{at} \quad r = R_{in} \\
\sigma_r &= -P_{midradius} \quad \text{at} \quad r = R
\end{align*}
\]  \hspace{1cm} (9)

model representing the outer half of the cylinder wall (wave located near the outer radius of the cylinder wall),

\[
\begin{align*}
\sigma_r &= -P_{midradius} \quad \text{at} \quad r = R \\
\sigma_r &= -P \quad \text{at} \quad r = R_{out}
\end{align*}
\]  \hspace{1cm} (10)

where \(P_{midradius}\) is calculated from an elasticity-based solution of the hydrostatically loaded perfect cylinder, and where \(R_{in}, R, \) and \(R_{out}\) are the inner radius, midradius, and outer radius of the cylinder, respectively. The midradial stresses, \(P_{midradius}\), are shown in Table 4.1 for each of the cylinder geometries considered in this investigation.

<table>
<thead>
<tr>
<th>Cylinder Geometry (R/H)</th>
<th>Midradius pressure (P_{midradius}) (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.56138</td>
</tr>
<tr>
<td>10</td>
<td>0.53615</td>
</tr>
<tr>
<td>20</td>
<td>0.51937</td>
</tr>
</tbody>
</table>

Chapter 4: Analytical Model
This concludes the discussion of the geometry and boundary conditions for the wavy layer models. To this stage in the preprocessing, the model has been created entirely within PATRAN, with the single exception of the multipoint constraint used to prescribe the generalized plane deformation condition. This preliminary PATRAN model is then translated, via PATABA, into a preliminary ABAQUS input file which includes the prescribed geometry and boundary conditions. The nodal coordinates and element connectivity included in this preliminary input file are used in the calculation of material properties for the model.

4.1.1.3 Material Properties

A program was written to calculate the material properties for each element using geometric information included in the preliminary ABAQUS input file. These material properties are dependent on local fiber orientation and volume fraction change for each of the elements representing the wavy layers. The source listing for the program, ORSOLMTL, is included in Appendix B.

Fiber orientation is calculated for each of the wavy elements based on the average fiber angle within element. This average angle is illustrated in Figure 4.6. The node numbers in Figure 4.6 represent the local node numbering for the element. The average fiber angle, $\alpha$, for the element is taken to be the angle of a line joining local node numbers 9 and 11 relative to the global $X$ direction. This method of calculating average fiber angles for a discrete number of elements in the circumferential direction establishes a piece-wise linear approximation to the true curvilinear geometry of the wavy fibers. As a result, error is introduced into the model. It is felt that this error is quite small, since it was shown by Telegadas and Hyer [9] that piece-wise geometry approximations produce results very similar to those using a more complicated curvilinear layer geometry. This
Figure 4.6. Average fiber angle of a wavy element.
error can also be minimized through finite element mesh refinement. More is said about this later in Chapter 6 (see Section 6.2).

The local fiber orientation can be defined in the ABAQUS input file in two ways. The first uses the local fiber orientation to define a separate material, and hence a separate *MATERIAL card, for each of the elements representing the wavy layers. This involves simple transformations of a three-dimensional stiffness matrix based on the calculated fiber orientation. The second method, and the one used here, uses the *ORIENTATION keyword available in ABAQUS. The *ORIENTATION keyword defines local material orientation, and requires fewer material definitions than the first approach. In the case of a cross-ply laminate with constant volume fraction, only two material definitions are required: a material definition representing circumferential layers, and a material definition representing axial layers. The *ORIENTATION feature defines a local user-defined coordinate system and is referenced to specified elements through the *SOLID SECTION keyword. The logical choice is to define local coordinate systems with axes that are everywhere normal and tangential to the fiber directions. These continuously changing local 'n-s' coordinates were illustrated in Figure 3.7. This means defining a separate local coordinate system for each of the elements in the wavy layer region. For the surrounding elements in the nonwavy layer regions, this means defining a just single r-Ө-z coordinate system. This method has an added benefit. Within ABAQUS, defining a local coordinate system for an element dictates that the stress components are evaluated in that local coordinate system. Therefore, stresses are calculated in directions normal and tangential to the fiber directions, i.e., σ_n, τ_{ns}, and σ_s as illustrated in Figure 4.7. Since failure strength parameters are also defined in terms of directions normal and tangential to the fiber directions, implementation of a failure criterion becomes a relatively simple task.
The "n" direction is everywhere perpendicular to the fiber direction, while the "s" direction is everywhere tangential to the fiber direction.

Figure 4.7. Stress components in the n-s coordinate system.
Changes in fiber volume fractions are calculated for each of the elements making up the wavy region. Finite element nodal geometry included in the preliminary ABAQUS input file is used to calculate the change in volume fractions due to fiber waviness. The fiber volume fraction calculations are based on the following equations:

\[ \eta_f = 0.65, \eta_m = 0.35 \quad (11) \]

\[ \eta_{f, w} = \frac{\eta_f \cdot V_p}{V_w} \quad (12) \]

\[ \eta_{m, w} = 1 - \eta_{f, w} \quad (13) \]

where

- \( \eta_f \) = fiber volume fraction of a nonwavy element
- \( \eta_m \) = matrix volume fraction of a nonwavy element
- \( \eta_{f, w} \) = fiber volume fraction of a wavy element
- \( \eta_{m, w} \) = matrix volume fraction of a wavy element
- \( V_p \) = volume of a nonwavy element
- \( V_w \) = volume of a wavy element.

It is necessary to calculate the volume of a wavy element and the volume of that same element, prior to introduction of the wave. In the latter case, the element is nonwavy. The volumes of the wavy and nonwavy elements are dependent on the position of the element within the wave as well as the element's radial location within the cylindrical segment. Because the model is a unit thickness in the axial direction, the volume of the element is simply its axial cross-sectional area. To determine axial cross-sectional area, the element is analyzed as if the element were two-dimensional. The two-dimensional element area is calculated from the determinant of the Jacobian matrix [20]. The Jacobian matrix is defined as
\[ [J] = \left[ \sum_{i=1}^{n} X_i \frac{\partial \psi_i}{\partial \xi} \sum_{i=1}^{n} Y_i \frac{\partial \psi_i}{\partial \eta} \right] \left[ \sum_{i=1}^{n} X_i \frac{\partial \psi_i}{\partial \eta} \sum_{i=1}^{n} Y_i \frac{\partial \psi_i}{\partial \xi} \right], \quad (14) \]

where \( X_i \) = global X coordinate of the ith node
\( Y_i \) = global Y coordinate of the ith node
\( \psi_i = f(\xi, \eta) \) = shape function for the ith degree of freedom.

The shape functions corresponding to a master finite element are shown in Figure 4.8.

The Jacobian, \( J \), is then

\[ J = \det [J] = |J|. \quad (15) \]

Area is calculated by integrating the Jacobian over the domain of the finite element, namely,

\[ A = \int_{\eta=-1}^{1} \int_{\xi=-1}^{1} J \text{d}\xi \text{d}\eta. \quad (16) \]

Results from sample element volume calculations, area times unit axial thickness, are included in Appendix C for the case of elements with no waviness. The volume calculations using the Jacobian method are compared with exact volume calculations. The results show the accuracy for the Jacobian method of calculating element volumes.

Fiber and matrix volume fractions are calculated directly using Equations 11-13. A modified rule of mixtures approach is used to calculate changes in material properties due to these volume fraction changes. The modified rule of mixtures approach incorporates stress partitioning parameters [21] to overcome the inability of the fundamental rule of mixtures equations to accurately calculate shear and transverse...
Master Finite Element

\[ -1 \leq \eta \leq +1, \quad -1 \leq \xi \leq +1 \]

\[
\psi_1 = \frac{1}{4}(1 - \xi)(1 - \eta)(-1 - \xi - \eta)
\]

\[
\psi_2 = \frac{1}{4}(1 + \xi)(1 - \eta)(-1 + \xi - \eta)
\]

\[
\psi_3 = \frac{1}{4}(1 + \xi)(1 + \eta)(-1 + \xi + \eta)
\]

\[
\psi_4 = \frac{1}{4}(1 - \xi)(1 + \eta)(-1 - \xi + \eta)
\]

\[
\psi_9 = \frac{1}{2}(1 - \xi)(1 + \xi)(1 - \eta)
\]

\[
\psi_{10} = \frac{1}{2}(1 + \xi)(1 - \eta)(1 + \eta)
\]

\[
\psi_{11} = \frac{1}{2}(1 - \xi)(1 + \xi)(1 + \eta)
\]

\[
\psi_{12} = \frac{1}{2}(1 - \xi)(1 - \eta)(1 + \eta)
\]

Figure 4.8. Shape functions for a two-dimensional quadratic serendipity finite element.
material properties. The equations for calculating the in-plane material properties of a layer based on the modified rule of mixtures are as follows:

\[ E_1 = \nu_y E_{1f} + \nu_m E_m \]  \hspace{1cm} (17)

\[ E_2 = \frac{(1 + \nu_y^*)}{\left(\frac{1}{E_{1f}} + \frac{\nu_y^*}{E_m}\right)} \]  \hspace{1cm} (18)

\[ E_3 = \frac{(1 + \nu_s^*)}{\left(\frac{1}{E_{1f}} + \frac{\nu_s^*}{E_m}\right)} \]  \hspace{1cm} (19)

\[ \nu_{12} = \nu_y \nu_{12f} + \nu_m \nu_m \]  \hspace{1cm} (20)

\[ G_{12} = \frac{(1 + \nu_s^*)}{\left(\frac{1}{G_{1f}} + \frac{\nu_s^*}{G_m}\right)} \]  \hspace{1cm} (21)

where the subscript '1' indicates the longitudinal direction, i.e. along the fiber direction, and the '2' and '3' indicate the two directions transverse to the fiber direction. The quantity \( E_1 \) is the longitudinal Young's modulus and \( E_2 \) and \( E_3 \) are the two transverse Young's moduli. The quantities \( \nu_{12} \) and \( G_{12} \) are Poisson's ratio and shear modulus, respectively. Based on typical epoxy materials, the following are assumed for all calculations:

\[ E_m = 0.5 \times 10^6 \text{ psi} \]
\[ \nu_m = 0.35 \]

The reduced matrix/fiber volume ratios, \( \nu_y^* \) and \( \nu_s^* \), are defined as:

\[ \nu_y^* = \frac{\eta_y \cdot \nu_m}{\nu_f}, \quad \nu_s^* = \frac{\eta_s \cdot \nu_m}{\nu_f} \]  \hspace{1cm} (22)

where \( \eta_y \) and \( \eta_s \) are the transverse and shear stress partitioning parameters respectively. Values of \( \eta_y = 0.516 \) and \( \eta_s = 0.316 \), as derived by Tsai [22], are used for all models. The values for the longitudinal Young's modulus of the fiber, \( E_{1f} \), and the transverse
Young's moduli of the fiber, $E_{2f}$ and $E_{3f}$, are back-calculated from Equations 17, 18, and 19 respectively, using a fiber volume fraction of $\nu_f = 0.65$ and material properties given by Equation 1. Likewise, the Poisson's ratio of the fiber, $\nu_{12f}$, and the shear modulus of the fiber, $G_{fx}$, are back calculated using Equations 20 and 21, respectively. Equation 18 has been rearranged to demonstrate a typical back-calculation:

$$E_{2f} = \left[ \frac{(1+\nu_y^*)}{E_1} - \frac{\nu_y^*}{E_m} \right]^{-1}.$$  \hspace{1cm} (23)

As with the material properties for the nonwavy region given in Equation 1, the following relations are assumed for materials in the wavy region:

$$G_{13} = G_{12}$$ \hspace{1cm} (24)

$$\nu_{13} = \nu_{12}.$$ \hspace{1cm} (25)

The material properties $G_{23}$ and $\nu_{23}$, as with previous work by Telegadas and Hyer [9] and Adams and Hyer [2], are based on scaling of the material properties in the nonwavy region:

$$G_{23} = \left( \frac{G_{23}^*}{G_{12}^*} \right) G_{12}$$ \hspace{1cm} (26)

$$\nu_{23} = \left( \frac{\nu_{23}^*}{\nu_{12}^*} \right) \nu_{12},$$ \hspace{1cm} (27)

where $G_{23}^*$, $G_{12}^*$, $\nu_{23}^*$, and $\nu_{12}^*$ represent the material properties in the nonwavy region. The calculated material properties are assembled into a compliance matrix which is inverted to yield a stiffness matrix. The calculated stiffness matrix is used in an ABAQUS *MATERIAL card to define material properties. Since the volume fractions vary for each of the wavy elements, a separate material definition is required for each of these elements.

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Material properties due to local fiber orientation and volume fraction changes are calculated in the program ORSOLMTL for each of the wavy elements. The program ORSOLMTL generates material definitions for the wavy elements in the form of *SOLID SECTION, *MATERIAL, and *ORIENTATION cards which are then manually inserted into the ABAQUS input file. The *MATERIAL and *ORIENTATION cards are also inserted into the input file to define material properties for the nonwavy portion of the cylindrical segment that surrounds the wave. This concludes the material definitions necessary for the finite element model.

4.1.1.4 Output

The final aspect of the model deals with the calculation of stresses and deformations. The PATABA translator gives the user the flexibility to specify the types of results to be calculated, and in the case of stress calculations, the location at which these stresses are to be reported.

For all analyses included in this investigation, all six components of stress are calculated and reported by ABAQUS at the Gauss points of each element. Eight Gauss points are present in a three-dimensional reduced integration element, with two Gauss points in each dimension. The location of these Gauss points is given by the two-point formula [23].

In addition to stresses, deformations are calculated in this investigation. This is done to provide a rough check of the boundary conditions applied to the model. ABAQUS calculates and reports the deformations at each of the nodes. However, in order for these calculations to be performed in the cylindrical r-θ-z coordinate system, a parameter in the ABAQUS input file must be correctly defined. By default, PATABA creates an input file specifying that deformations be calculated in the global x-y-z
coordinate system. The GLOBAL parameter in the *NODE FILE and *NODE PRINT cards, set to YES by PATABA, must be changed to NO in order for deformations to be calculated in the r-θ-z coordinate system.

A sample ABAQUS input file is included in Appendix D. A summary of pertinent ABAQUS keywords and their significance is included in Table 4.2.

### Table 4.2
**AB AQUS Keyword Definitions**

<table>
<thead>
<tr>
<th>Effect</th>
<th>ABAQUS Keywords</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nodal Coordinate Definition</td>
<td>*NODE</td>
<td>1</td>
</tr>
<tr>
<td>Cylindrical Coordinate System</td>
<td>*TRANSFORM</td>
<td>2</td>
</tr>
<tr>
<td>Node Set Creation</td>
<td>*NSET</td>
<td>3</td>
</tr>
<tr>
<td>Element Property Assignment</td>
<td>*SOLID SECTION</td>
<td>4</td>
</tr>
<tr>
<td>Material Definition</td>
<td>*MATERIAL</td>
<td>5</td>
</tr>
<tr>
<td>Element Coordinate System</td>
<td>*ORIENTATION</td>
<td>6</td>
</tr>
<tr>
<td>Multipoint Constraint</td>
<td>*EQUATION</td>
<td>7</td>
</tr>
<tr>
<td>Specifying Calculated Quantities</td>
<td>*NODE FILE, *NODE PRINT</td>
<td></td>
</tr>
</tbody>
</table>

1. Automatically defined by PATABA translator.
2. Transfers nodal degrees of freedom from x,y,z to r,θ,z. Necessary for defining cylindrical boundary conditions. Automatically defined by PATABA as a result of r,θ,z coordinate system used in PATRAN.
3. Groups nodes together into a set (referenced by *EQUATION multipoint constraint). Automatically defined by PATABA if a PATRAN Named Component is created which contains the desired nodes to be included in the node set.
4. References *MATERIAL and *ORIENTATION names. Created by ORSOLMTL.
5. Created by ORSOLMTL.
6. Defines normal and tangent to local fiber orientations. Created by ORSOLMTL.
7. References node set containing nodes in front face of model.
4.1.2 Postprocessing

Two postprocessing programs were written to transform the raw output data from the ABAQUS analysis into data to which a failure criterion could be applied, and from which failure pressures due to the hydrostatically-induced stress state could be calculated.

The first of these programs, REDUCE_ABAQUSDAT, extracts the stress information from the ABAQUS output file and calculates the r-θ-z coordinates of the Gauss points at which these stresses were calculated. Pressures to produce failure for the fiber, interlaminar normal, and interlaminar shear modes are computed from the $\sigma_S$, $\sigma_n$, and $\tau_{ns}$ stress components, respectively. These failure pressures are then arranged in numerical order to locate the minimum failure pressure for each of the three failure modes. The location within the model at which the minimum failure pressures occur are recorded as well.

The second program, LYRBYLYR, reads the stress data and the r-θ-z coordinates of the Gauss points at which the stresses are calculated and organizes the information on a layer-by-layer basis. Interlaminar stress values are calculated for the continuous through-the-thickness components of stress $\sigma_n$ and $\tau_{ns}$. These interlaminar stresses are calculated as an average between the calculated values at the two Gauss points adjacent to and on either side of a layer interface, as illustrated in Figure 4.9. The program then writes stress values to files arranged in the following ways: by layer (at Gauss points), by interface ($\sigma_n$ and $\tau_{ns}$ only), and by circumferential location (at Gauss points with the same $\theta$ coordinate). These formats for viewing the data were instrumental in validating the finite element model results, the assumptions made in simplifying the model, and after these steps, for providing insight into the influence of layer waviness.
Figure 4.9. Interlaminar stress calculation.
4.2 BUCKLING ANALYSIS OF A PERFECT CYLINDER

The buckling analysis is performed to address the issue of whether or not failure due to buckling occurs in a perfect cylinder at pressures lower than those required to produce material failure due to wavy layer effects. In a manner similar to the wavy layer analysis, this analysis can be broken down into preprocessing, processing, and postprocessing. Preprocessing again includes geometry, boundary conditions, and material property considerations. Processing involves eigenvalue extraction for the first buckling mode of the cylinder. Postprocessing involves calculation of a failure pressure based on the calculated eigenvalue, though this final step is trivial.

4.2.1 Preprocessing

4.2.1.1 Cylinder Buckling Geometry

Buckling analyses are performed on all three of the cylinder geometries considered in this investigation, i.e., R/H=5, 10, and 20. Two lengths, in terms of length to radius ratios, are modeled in this investigation. They are L/R=1/2 and 1, and both are considered for each of the three R/H values. Thus, a total of six buckling models are considered. Due to the fact that the buckled mode shape is unknown, symmetry or asymmetry with respect to the circumferential direction, θ, cannot be assumed. The complete cylinder circumference must therefore be modeled. The PCL program, CYLINDER, used previously in conjunction with the wavy layer model, is used here to generate the geometry for each of the buckling models.

As in the wavy layer models, the cylinders here are modeled with three-dimensional elements, thus allowing the axial loading condition resulting from endcap effects. Again, triquadratic reduced integration elements are used, with justification for
their use being provided in the next chapter. Each of the six models contains 36 elements in the circumferential direction, 2 elements in the radial direction (through the cylinder wall thickness), and 6 elements in the axial direction. These numbers are used for all six models to impose consistency between models, while at the same time producing good aspect ratios for all elements. Typical cylinder models are shown in Figure 4.10 for the two L/R ratios studied.

4.2.1.2 Boundary conditions

The cylinders are loaded externally with hydrostatic pressure, \( P \), and an axial load equivalent to the pressure times the area of the endcap. The axial load is applied to one end of the cylinder which is free to move in the axial direction. The axial motion of the other end of the cylinder is fixed. Clamped boundary conditions are enforced on each end of the cylinder. The clamped conditions consist of fixing the radial and circumferential displacements on both ends of the cylinder. The loaded end of the cylinder, although it can move in the axial direction, is required to remain planar and perpendicular to the axial direction. This is achieved with a multipoint constraint similar to that used in the wavy layer analysis. These boundary conditions are illustrated in Figure 4.11 for the axial section of cylinder being modeled.

4.2.1.3 Material Properties

The smeared properties defined in Chapter 3 are used for all six of the buckling models. These properties are repeated here for convenience as

\[
\begin{align*}
E_r &= 0.14692 \times 10^7 \text{ psi} & E_\theta &= 0.13354 \times 10^8 \text{ psi} & E_z &= 0.71890 \times 10^7 \text{ psi} \\
G_{rs} &= 0.6681 \times 10^6 \text{ psi} & G_{rz} &= 0.80 \times 10^6 \text{ psi} & G_{rz} &= 0.56987 \times 10^6 \text{ psi} & (7, \text{ repeated}) \\
\nu_{rs} &= 0.038256 & \nu_{rz} &= 0.0530 & \nu_{rz} &= 0.07078
\end{align*}
\]
Figure 4.10. Finite element discretization used for each of the buckling analyses.
Figure 4.11. Boundary conditions for cylinder buckling analysis.
4.2.1.4 Output

The output for the buckling analysis is a series of eigenvalues corresponding to the buckling pressures of the cylinder. Since the lowest buckling pressure is the only one of interest, only the first eigenvalue is required to converge. The steps involved in calculating the eigenvalue consist of:

1) initially loading the model statically to approximately 10% of its critical buckling load
2) storing the stiffness matrix
3) incrementing the static loading
4) extracting the eigenvalues

These four steps are performed respectively by ABAQUS by defining the following keyword cards:

1) **STATIC, NLGEOM** (initial loading; ABAQUS refers to this as 'dead' loading)
2) **BUCKLE, DEAD** (storage of stiffness matrix)
3) **STATIC, NLGEOM** (incremental loading; ABAQUS refers to this as 'live' loading)
4) **BUCKLE, LIVE** (eigenvalue extraction)

A sample ABAQUS buckling input file has been included in Appendix D.

4.2.2 Postprocessing

The eigenvalue calculated by ABAQUS is used to calculate the critical buckling pressure of the cylinder. The following equation is used to do this:

\[ P_{cr, buckling} = (\text{eigenvalue} \times "\text{live" loads}) + "\text{dead" loads} \]  \hspace{1cm} (28)
This completes the discussion of modeling issues for the wavy layer and buckling analyses. The necessary tools for postprocessing the finite element results have also been developed or defined. Some important questions regarding accuracy of the models arises at this point. Are the stresses predicted by ABAQUS reasonable for the wavy layer models, and likewise, are the buckling failure pressures predicted by ABAQUS reasonable for the buckling models? To address the first question regarding stresses, an analysis is performed using ABAQUS to determine how well it predicts stresses on a layer-by-layer basis. This first analyses is performed on a perfect cylinder to which a solution is known. Specifically, an analysis is performed which compares the results of a layer-by-layer finite element analysis for a perfect cylinder with results from elasticity for a perfect cylinder. In terms of a wavy layer analysis, stress convergence is checked to ensure that stress results are accurate within the context of the wavy model being analyzed. Finally, in terms of the buckling analysis for a perfect cylinder, ABAQUS finite element analyses are performed for thin-walled cylinders and compared with results predicted by thin-shell cylinder theory. While this comparison for thin-walled cylinder buckling pressures does not prove that the buckling pressures predicted by ABAQUS for thick-walled cylinders are accurate, the comparisons do determine whether the ABAQUS analysis is free of major errors.

A second question regarding the wavy layer model can also be asked. Are the simplifications made in reducing model size and calculation times justified? To address this question, the three simplifications made to the model are each investigated. Results are presented in the following chapter which support the use of reduced integration finite elements to model the layer waviness. Also, results are presented from a study conducted to determine the minimum arc length of segment that can be used and still correctly model the waviness as being an isolated effect in an otherwise perfect cylinder.

Chapter 4: Analytical Model
Finally, results are presented which validate modeling a segment of the cylinder using only half the cylinder wall thickness rather than the full cylinder wall thickness.
CHAPTER 5

ANALYTICAL MODEL VERIFICATION

Results are presented in this chapter which address the questions raised at the end of the previous chapter, namely, the questions concerning the validity of the finite element analyses performed in this investigation. To reiterate, the first question asks whether or not such an analysis is accurate, even for the general case of a cylinder with no imperfections included. In regards to the wavy layer analysis, two additional questions are addressed. The first additional question asks whether or not using a simplified model for the wavy layer analysis, i.e., a half-thickness cylindrical segment made up of reduced integration elements, is justified in modeling this problem. The second additional question asks whether or not the results calculated by ABAQUS for this simplified wavy layer model are accurate. Finally, the question of accuracy must also be addressed in the context of the buckling analysis for the perfect cylinder.

5.1 GENERAL VERIFICATIONS

To provide a general verification of the finite element analysis, the capabilities of ABAQUS are evaluated in terms of calculating stresses on a layer-by-layer basis for the case of a hydrostatically-loaded perfect, i.e., no layer waviness present, composite cylinder. Results are presented in this section which demonstrate this capability by comparing results calculated by ABAQUS to those calculated by elasticity theory for the identical problem.

The geometry of the cylinder studied in this general verification phase is identical to that of one of the cylinders with included wave geometry, specifically R/H=5 and with
the layup of $[90/(90/0/90)]_S$. The only difference is that the cylinder does not include the wave. This is an axisymmetric problem, and as a result, simplifications can be made to the finite element model without sacrificing accuracy. Namely, a circular segment of the cylinder can be modeled, as was done for the wavy model described in Chapter 4. The segment has a unit thickness in the axial direction. Unlike the wavy layer models, the full-thickness 104 layer segment is modeled in this verification.

The boundary conditions for modeling the segment of the perfect cylinder are identical to those for the model of the segment including the wave, namely, classic roller-type displacement boundary conditions shown in Figure 4.2 are placed at both circumferential boundaries of the segment. Hence, circumferential motion is prevented at these boundaries. Referring to Figure 4.3, a generalized plane deformation condition is prescribed; axial motion along the back face ($z=1$) of the model is prohibited, and the front face ($z=0$) is allowed to move in the axial direction, but is required to remain planar and perpendicular to the axial direction. A unit pressure, $P$, is applied externally to the cylindrical segment. A simple area-based percentage of the axial force acting on the complete cross-section of the cylinder is applied to the segment to simulate endcap effects.

The radial, circumferential, and axial stresses are compared using values calculated from the finite element and elasticity analyses. For this axisymmetric problem, there are no shear stresses. Finite element results are obtained by modeling the segment on a layer-by-layer basis using two elements through the thickness of each layer. Although no justification for their use has yet been provided, reduced integration elements are used for this analysis to reduce computation time. In retrospect, justification for their use is provided by the fact that the stresses calculated using reduced integration elements are virtually identical to those calculated from elasticity theory. The elasticity theory results

Chapter 5: Analytical Model Verification
are obtained from Hyer's work [12]. The results shown here resemble those previously reported results for perfect cross-ply cylinders. The program was modified to calculate stresses at locations corresponding to the Gauss point locations in the finite element model. In this manner, comparisons are made between the finite element and elasticity results at identical radial locations within the cylinder wall. Comparisons are illustrated in Figures 5.1, 5.2, and 5.3 for the radial stress, $\sigma_r$, circumferential stress, $\sigma_\theta$, and axial stress, $\sigma_z$, respectively. In the figures the stresses have been normalized by the applied pressure, $P$. The radial location within the cylinder wall is normalized to range from $-1/2$ at the inner radius of the cylinder to $+1/2$ at the outer radius of the cylinder. This nondimensional radial position, $\rho$, is given by

$$\rho = \frac{r - R}{H}. \quad (29)$$

With all three stress components, results calculated by finite element analysis and elasticity theory are virtually identical, including the details of the zigzag character of the radial stress components in Figure 5.1. Note in Figure 5.2, that for the perfect cylinder, due to the Lamé effect [15], the layers near the inner radius of the cylinder, $\rho=-1/2$, experience higher circumferential stresses than the layers near the outer radius of the cylinder, $\rho=+1/2$. It is also seen in Figure 5.2 that the layers with circumferentially oriented fibers carry about ten times the circumferential stress of the axial layers. For the axial stress, Figure 5.3, the stress ratio between the axial and circumferential layers is also about 10:1, reflecting the high level of orthotropy of the graphite-epoxy. It is therefore concluded that results calculated by ABAQUS are accurate for the perfect cylinder geometry. It cannot necessarily be assumed that stress predictions for the wavy layer model will be as accurate, but evidence has been provided to justify using ABAQUS to model the wavy layer problem.
Figure 5.1. Comparison between finite element and elasticity solution to radial stress component, $\sigma_r$, for perfect cylinder with $R/H=5$. 

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Figure 5.2. Comparison between finite element and elasticity solution to circumferential stress component, \( \sigma_\theta \), for perfect cylinder with \( R/H = 5 \).
Figure 5.3. Comparison between finite element and elasticity solution to axial stress component, $\sigma_z$, for perfect cylinder with $R/H=5$. 

Chapter 5: Analytical Model Verification
5.2 WAVY LAYER MODEL VERIFICATIONS

In this section, two issues are addressed regarding the analysis of the cylinder with wavy layers. The first issue regards the use of a simplified model to accurately analyze the effect of the wave. Aspects of the simplified model include use of reduced rather than full Gaussian integration to compute the element stiffnesses, use of a circular segment rather than the complete ring of the cylinder, and use of a 52 layer half-thickness model rather than a 104 layer full-thickness model. Each of these three aspects are discussed and results are presented to justify their use.

The second issue addressed regards stress convergence in the context of the simplified wavy layer model. Stress convergence is checked to ensure the accuracy of the model. Results are presented which show good stress convergence for these simplified finite element models.

5.2.1 Justifications for a Simplified Wavy Layer Model

Motivation exists for including simplifications in the analysis of the cylinder with wavy layers. Achieving faster program execution times for parametric studies, and making use of limited computing capabilities are two prime considerations. Results are presented which justify use of three major simplifications to the wavy layer model. These three simplifications are:

1) Use of reduced integration rather than full integration elements;
2) Modeling a circular segment of the cylinder rather than the entire 360° ring;
3) Modeling a 52 layer half-thickness segment rather than the 104 layer full-thickness segment.

The first of these simplifications deals with finite element methodology, but in the context of a wavy layer analysis. The latter two simplifications are specific to the wave and cylinder geometry being studied.
Justification is provided for each of the simplifications by comparing the simplified case with the unsimplified case, and by showing that results from the two cases are virtually identical. An idealized justification would consider each of the simplifications separately, isolated from any other simplification. As an example, to justify the first simplification, use of reduced integration elements, ideally one would model the full-thickness 360° ring using both reduced and full integration elements. Comparison of the results would be independent of the second and third simplifications concerning circumferential model length and model thickness. However, limited computing capabilities require that aspects of each of the three simplifications be included in every model. Justification is therefore provided for each simplification by varying characteristics pertaining only to that specific simplification, while keeping all other variables constant. As an example, in the following justification for the use of reduced integration elements, the two models compared are cylindrical segments whose thickness is only a fraction of the total cylinder wall thickness. The only difference between the two models is the type of finite element used - reduced integration elements are used in one model, and full integration elements are used in the other model. In this manner, the effect of element type is evaluated.

5.2.1.1 Reduced vs. Full Integration Elements

The issue of reduced vs. full integration elements for a finite element analysis is very complicated. The brief justification presented here is not intended to address this finite element issue on a global scale. It does, however, show that for the specific problem investigated here, results predicted by reduced integration elements are very close to those predicted by full integration elements. So close, in fact, are the results, that reduced integration elements are justified for use throughout this investigation.
Due to computing requirements for full integration analyses, the issue of reduced vs. full integration elements is addressed within the context of a 14 layer cylinder containing a 14 layer wavy sublaminate. In effect, a cylinder is modeled which is 14 layers in thickness and contains a full-thickness wave. This is, in fact, a worst case scenario, where the hydrostatic pressure is distributed through only 14 layers, all of them being imperfect. In contrast, a less severe case is that of the 104 layer full-thickness cylinder in which the 14 imperfect layers are a small percentage of the total thickness. The boundary conditions for this model are the same as those used for the wavy layer models presented in Chapter 4. A unit external hydrostatic pressure is applied to a circular segment of a cylinder, in this case only 14 layers in thickness. Motion in the circumferential direction is again prevented by roller-type boundary conditions at the circumferential boundaries (θ=0° and θ=θwedge) of the segment. A generalized plane deformation condition is established through axial boundary conditions on the front face (z=0) and back face (z=1) of the model.

Results are presented in terms of interlaminar stress distributions at the most critical interfaces within the 14 layer cylinder segment. The layer and interface numbering scheme used to identify these interfaces is illustrated in Figure 5.4. The layers are numbered consecutively from 1 to 14 beginning with the inner-most layer and proceeding outward. The interface nomenclature, e.g., interface 1-2, indicates the interface between two adjacent layers, e.g., interface 1-2 refers to the interface located between layers 1 and 2.

Figures 5.5 and 5.6 show interlaminar normal, $\sigma_n$, and interlaminar shear, $\tau_{ns}$, components of stress, respectively, computed by both the full integration method and the reduced integration method. The stresses, normalized by the applied pressure, are plotted vs. circumferential coordinate, $\theta$, which is normalized by $\theta^*$, the extent of the wave, as

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Figure 5.4. Layer and interface numbering scheme for the 14 layer cylindrical segment used in reduced vs. full integration analysis.
a) Interface 8-9: location of maximum interlaminar compression.

b) Interface 3-4: location of maximum interlaminar tension.

Figure 5.5. Comparison between interlaminar normal stresses, $\sigma_n$, calculated by reduced integration elements and full integration elements.
Stresses are shown along interface 10-11 within the model corresponding to the location of maximum interlaminar shear stress.

Figure 5.6. Comparison between interlaminar shear stresses, $\tau_{ns}$, calculated by reduced integration elements and full integration elements.
previously illustrated in Figure 3.5. For the interlaminar normal component, stress distributions are shown at two interfaces within the model corresponding to the interfaces of maximum interlaminar compression, Figure 5.5a, and maximum interlaminar tension, Figure 5.5b. For the interlaminar shear component, Figure 5.6, the stress distribution is shown for the interface corresponding to the maximum absolute value of interlaminar shear stress, since, in terms of shear failure, only the absolute value is important.

From both Figures 5.5 and 5.6, it is seen that nearly the same trends are predicted by both integration schemes. The reduced integration analysis, in effect, provides an 'average' to the fluctuating results provided by the full integration analysis. The deviation of the full integration values from the 'average' reduced integration values are seen to be a small percentage of the total stress. The reduced integration and full integration analyses are therefore considered to produce equivalent results. Reduced integration, due to its reduced computation time and expense, is therefore used for all subsequent analyses.

It should be noted that the fluctuations in stress seen in Figures 5.5 and 5.6 are inherent to the piece-wise linear representation of the curvilinear geometry of layer waviness. More will be said about the fluctuations in Chapter 6.

5.2.1.2 Circular Segment vs. 360° Ring Model

To further reduce computation time and expense, a simplification is made in terms of reducing the circumferential length of the wavy layer model. This simplification is allowed as a result of the assumed symmetry of the wave, and the localized effect of the wave on the stress state of the cylinder. The combination of these two characteristics allows a circumferential segment of the cylinder to be modeled rather than the entire
360° ring. Details of the boundary conditions used to achieve this simplification were presented in Chapter 4. A brief summary of these conditions is presented here.

Referring to Figure 4.2, the assumed cosinusoidal geometry of the wave requires symmetry, and hence no circumferential motion, at θ=0° (AA in Figure 4.2). A similar boundary condition restricts motion in the circumferential direction at θ=θ_{wedge} (BB in Figure 4.2). As mentioned previously in Chapter 4, the boundary at θ_{wedge} would ideally be located beyond the extent of the influence of the wave, namely, at a circumferential location where the stress state in the imperfect cylinder (wave included) matches the stress state in the perfect cylinder (no wave included). In practice, however, modeling such a segment, suspected to be greater than ten half wave lengths in circumferential length, would exceed the computing limitations of this investigation. These computing limitations require that a much shorter circumferential segment be modeled, i.e., two to three half wave lengths in length. As a result, the boundary at θ_{wedge} is placed at a circumferential location at which the stress distribution does not match that of the perfect cylinder. To show this mismatch, the stress distribution along θ=θ_{wedge} for the worst case wavy layer model considered in this investigation, i.e., a wave with δ̅=2 and λ̅=10 located at the outer radius of a cylinder with R/H=20, is compared to the stress distribution for a perfect cylinder with R/H=20. For this worst case layer model, the total circumferential length of the model is approximately three half wave lengths (θ*=0.26962721° and θ_{wedge}=0.75°). A full-thickness model is used for the wavy layer analysis. The comparison of radial, circumferential, axial, and shear components of stress, as calculated from a full-thickness wavy layer model and an elasticity-based solution to the perfect cylinder, are shown in Figures 5.7, 5.8, 5.9, and 5.10, respectively. The fact that the radial, circumferential, and shear stress distributions for the wavy layer model and the perfect cylinder differ indicates that the boundary is
Figure 5.7. Radial stress distribution, $\sigma_r$, at $\theta=\theta_{\text{wedge}}$ for a wavy layer model and the radial stress distribution for the perfect cylinder.
Figure 5.8. Circumferential stress distribution, $\sigma_\theta$, at $\theta=\theta_{\text{wedge}}$ for a wavy layer model and the circumferential stress distribution for the perfect cylinder.
Figure 5.9. Axial stress distribution, $\sigma_z$, at $\theta=\theta_{\text{wedge}}$ for a wavy layer model and the axial stress distribution for the perfect cylinder.
Figure 5.10. Shear stress distribution, $\tau_{r\theta}$, at $\theta=\theta_{\text{wedge}}$ for a wavy layer model and the shear stress distribution for the perfect cylinder.
incorrectly applied at $\theta_{\text{wedge}}$ for the wavy layer model. The equivalence of the axial stress distributions for the wavy layer model and the perfect cylinder, Figure 5.9, indicates the absence of axial effects due to the wave geometry modeled in this investigation. Fortunately, the mismatch in the radial, circumferential, and shear stress distributions is not a problem. It will be shown that the boundary can be placed at circumferential locations within only a couple of half wave lengths of the end of the wave and still have little effect on the stress state in the immediate vicinity of the wave, and hence the predicted failure due to layer waviness. It should be emphasized that since failure is predicted to occur in this region, the stress state in the immediate vicinity of the wave is of primary concern. However, having a perfect match between the stress distributions for a wavy layer model and the stress distributions for the perfect cylinder at $\theta=\theta_{\text{wedge}}$ would be ideal.

The validity of using a circumferentially small model, i.e., three half wave lengths long, to analyze the stress state in the immediate vicinity of the wave is illustrated by comparing results from two models, each representing the worst case of layer waviness considered in this investigation, i.e., as just mentioned, a wave with $\delta=2$, $\lambda=10$ located at the outer radius of a cylinder with $R/H=20$. The first of these models has the $\theta_{\text{wedge}}$ boundary located at roughly two half wave lengths from the wave ($\theta^*=0.26962721^*$ and $\theta_{\text{wedge}}=0.75^*$) and the other has the $\theta_{\text{wedge}}$ boundary located at roughly four and a half half wave lengths from the wave ($\theta^*=0.26962721^*$ and $\theta_{\text{wedge}}=1.5^*$). It is shown in Table 5.1 that the most critical failure pressure ratios, calculated from the critical stress state in the immediate vicinity of the wave, are virtually identical (within 4%) for the two models. When examining Table 5.1, reference should be made to Figure 5.11, a figure showing the layer and interface numbering scheme for the 104 layer segment. This scheme was shown in Figure 4.9 in reference to calculation of interlaminar stresses and is

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Figure 5.11. Layer and interface numbering scheme for the 104 layer full-thickness cylindrical segment.
repeated here for convenience. As important, the failure mode and the location of failure are identical for the two models, indicating that failure prediction is not affected by the two model arc lengths.

**TABLE 5.1**
Comparison of Most Critical Failure Pressure Ratios for 0.75° and 1.5° Wedge Models

<table>
<thead>
<tr>
<th>P_cr/P_0</th>
<th>% change</th>
<th>Failure Mode*</th>
<th>Location within Model**</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.75°</td>
<td>1.5°</td>
<td>0.75°</td>
<td>1.5°</td>
</tr>
<tr>
<td>.521</td>
<td>.502</td>
<td>shear</td>
<td>shear</td>
</tr>
<tr>
<td></td>
<td></td>
<td>int 100-101</td>
<td>int 100-101</td>
</tr>
<tr>
<td></td>
<td></td>
<td>θ=0.10886°</td>
<td>θ=0.10886°</td>
</tr>
<tr>
<td>.549</td>
<td>.530</td>
<td>shear</td>
<td>shear</td>
</tr>
<tr>
<td></td>
<td></td>
<td>int 100-101</td>
<td>int 100-101</td>
</tr>
<tr>
<td></td>
<td></td>
<td>θ=0.07448°</td>
<td>θ=0.07448°</td>
</tr>
<tr>
<td>.563</td>
<td>.544</td>
<td>shear</td>
<td>shear</td>
</tr>
<tr>
<td></td>
<td></td>
<td>int 100-101</td>
<td>int 100-101</td>
</tr>
<tr>
<td></td>
<td></td>
<td>θ=0.12605°</td>
<td>θ=0.12605°</td>
</tr>
<tr>
<td>.589</td>
<td>.567</td>
<td>shear</td>
<td>shear</td>
</tr>
<tr>
<td></td>
<td></td>
<td>int 100-101</td>
<td>int 100-101</td>
</tr>
<tr>
<td></td>
<td></td>
<td>θ=0.10886°</td>
<td>θ=0.10886°</td>
</tr>
<tr>
<td>.599</td>
<td>.576</td>
<td>fiber</td>
<td>fiber</td>
</tr>
<tr>
<td></td>
<td></td>
<td>layer 100</td>
<td>layer 100</td>
</tr>
<tr>
<td></td>
<td></td>
<td>θ=0.00573°</td>
<td>θ=0.00573°</td>
</tr>
<tr>
<td>.604</td>
<td>.582</td>
<td>shear</td>
<td>shear</td>
</tr>
<tr>
<td></td>
<td></td>
<td>int 100-101</td>
<td>int 100-101</td>
</tr>
<tr>
<td></td>
<td></td>
<td>θ=0.14324°</td>
<td>θ=0.14324°</td>
</tr>
<tr>
<td>.614</td>
<td>.591</td>
<td>fiber</td>
<td>fiber</td>
</tr>
<tr>
<td></td>
<td></td>
<td>layer 102</td>
<td>layer 102</td>
</tr>
<tr>
<td></td>
<td></td>
<td>θ=0.00573°</td>
<td>θ=0.00573°</td>
</tr>
<tr>
<td>.617</td>
<td>.595</td>
<td>shear</td>
<td>shear</td>
</tr>
<tr>
<td></td>
<td></td>
<td>int 100-101</td>
<td>int 100-101</td>
</tr>
<tr>
<td></td>
<td></td>
<td>θ=0.12605°</td>
<td>θ=0.12605°</td>
</tr>
<tr>
<td>.621</td>
<td>.598</td>
<td>fiber</td>
<td>fiber</td>
</tr>
<tr>
<td></td>
<td></td>
<td>layer 100</td>
<td>layer 100</td>
</tr>
<tr>
<td></td>
<td></td>
<td>θ=0.02865°</td>
<td>θ=0.02865°</td>
</tr>
<tr>
<td>.622</td>
<td>.598</td>
<td>shear</td>
<td>shear</td>
</tr>
<tr>
<td></td>
<td></td>
<td>int 100-101</td>
<td>int 100-101</td>
</tr>
<tr>
<td></td>
<td></td>
<td>θ=0.16043°</td>
<td>θ=0.16043°</td>
</tr>
</tbody>
</table>

* 'shear' indicates interlaminar shear failure (τ_n); 'fiber' indicates fiber compression failure (σ_f).
** 'int' indicates the interface at which interlaminar shear failure occurs; 'layer' indicates the layer in which fiber compression failure occurs.

Because the results from the 0.75° and 1.5° models are virtually identical, the following important conclusion can be made: beyond some minimum circumferential location, the precise location of the boundary at θ=θ_wedge does not have much of an

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effect on the calculated stress state in the immediate vicinity of the wave. No attempt has been made to determine this minimum circumferential location, but it undoubtedly depends on the severity of the layer waviness and most likely also depends on the cylinder geometry and the location of the wave within the cylinder wall. It is suspected that it also depends on lamination sequence, though that is beyond the scope of this study. It has been shown here that for the worst case of waviness considered in this investigation, a total circumferential model length of approximately three half wave lengths ($\theta^{*}=0.26962721$ and $\theta^{\text{wedge}}=0.75^*$) produces results virtually equivalent to a model twice that long ($\theta^{*}=0.26962721$ and $\theta^{\text{wedge}}=1.5^*$). As a general rule of thumb, a circumferential model length of two to three half wave lengths is therefore used for all models in this investigation. The precise values of $\theta^{*}$ and $\theta^{\text{wedge}}$ are given in Table 5.2 for each of the models considered herein.

<table>
<thead>
<tr>
<th>Cylinder Geometry (R/H)</th>
<th>Wave Location</th>
<th>$\theta^{*}$ (degrees)</th>
<th>$\theta^{\text{wedge}}$ (degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>Inner</td>
<td>1.2062268</td>
<td>2.5</td>
</tr>
<tr>
<td></td>
<td>Outer</td>
<td>1.0140846</td>
<td>2.5</td>
</tr>
<tr>
<td>10</td>
<td>Inner</td>
<td>0.5758370</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td>Outer</td>
<td>0.5280718</td>
<td>1.5</td>
</tr>
<tr>
<td>20</td>
<td>Inner</td>
<td>0.2815518</td>
<td>0.75</td>
</tr>
<tr>
<td></td>
<td>Outer</td>
<td>0.2696272</td>
<td>0.75</td>
</tr>
</tbody>
</table>
Justification has been provided for modeling a segment, rather than the entire 360° ring, of the cylinder for the worst case of layer waviness considered in this investigation. It is therefore assumed that a cylindrical segment can be used for all wavy layer analyses. This simplification significantly reduces the computation time and expense for the wavy layer analyses. One further simplification is still possible, however. This final simplification reduces the size of the model by a factor of two by modeling only half of the total cylinder wall thickness.

5.2.1.3 Half-Thickness vs. Full-Thickness Model

The motivation for modeling only half the cylinder wall thickness is again to reduce computation time and expense. To justify this simplification, it is necessary to show that the results calculated for a half-thickness analysis do not differ significantly from those for a full-thickness analysis. Two cases are examined here, one for an inner-half model and one for an outer-half model. Each of the models represent a worst case wavy layer analysis, the first for a wave located at the inner radius of the cylinder, and the second for a wave located at the outer radius of the cylinder. The worst case wavy layer analysis corresponds to a cylinder geometry of $R/H=20$ and a wave geometry of $\bar{\delta}=2$ and $\bar{\lambda}=10$. The only difference between the two cases is the radial location of the wave within the cylinder wall. Two cases were chosen because the application of the pressure boundary condition is slightly different for the two cases, as previously described in Chapter 4 (see Figure 4.4). Recall, the pressures applied at the midradius of the half-thickness models were shown in Table 4.1.

Results are presented here which compare the failure pressure ratios calculated from the half-thickness models with the failure pressure ratios from the full-thickness models. The comparisons are made for each of the four failure modes, i.e., interlaminar
normal compression and tension, $\sigma_n$, interlaminar shear, $\tau_{ns}$, and fiber direction, $\sigma_f$, at locations within the model corresponding to the maximum level of stress for each respective stress component. Since this is where failure will first occur, the location of maximum stress is of primary interest. Comparisons are shown in Table 5.3 for both the inner wave and outer wave models.

<table>
<thead>
<tr>
<th>Location of Wave</th>
<th>Stress Component</th>
<th>$P_{cr}/P_o$</th>
<th>Half-thickness Model</th>
<th>Full-thickness Model</th>
<th>% Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner Radius</td>
<td>interlaminar tension (+$\sigma_n$)</td>
<td>0.829 (1)</td>
<td>0.842 (1)</td>
<td>1.54</td>
<td></td>
</tr>
<tr>
<td></td>
<td>interlaminar compression (-$\sigma_n$)</td>
<td>1.687 (2)</td>
<td>1.710 (2)</td>
<td>1.35</td>
<td></td>
</tr>
<tr>
<td></td>
<td>interlaminar shear ($\tau_{ns}$)</td>
<td>0.590 (3)</td>
<td>0.598 (3)</td>
<td>1.34</td>
<td></td>
</tr>
<tr>
<td></td>
<td>fiber direction ($\sigma_f$)</td>
<td>0.562 (4)</td>
<td>0.571 (4)</td>
<td>1.58</td>
<td></td>
</tr>
<tr>
<td>Outer Radius</td>
<td>interlaminar tension (+$\sigma_n$)</td>
<td>0.836 (5)</td>
<td>0.857 (5)</td>
<td>2.45</td>
<td></td>
</tr>
<tr>
<td></td>
<td>interlaminar compression (-$\sigma_n$)</td>
<td>2.336 (6)</td>
<td>2.355 (6)</td>
<td>0.81</td>
<td></td>
</tr>
<tr>
<td></td>
<td>interlaminar shear ($\tau_{ns}$)</td>
<td>0.521 (7)</td>
<td>0.529 (7)</td>
<td>1.51</td>
<td></td>
</tr>
<tr>
<td></td>
<td>fiber direction ($\sigma_f$)</td>
<td>0.599 (8)</td>
<td>0.608 (8)</td>
<td>1.48</td>
<td></td>
</tr>
</tbody>
</table>

1 occurs at interface 11-12 (within the wavy sublamine).
2 occurs at interface 11-12 (within the wavy sublamine).
3 occurs at interface 10-11 (within the wavy sublamine).
4 occurs in layer 13 (within the wavy sublamine).
5 occurs in layer 13 (within the wavy sublamine).
6 occurs at interface 93-93 (within the wavy sublamine).
7 occurs at interface 93-94 (within the wavy sublamine).
8 occurs in layer 100 (within the wavy sublamine).
The percent change in Table 5.3 indicates the percentage difference in failure pressure ratios predicted by the two models. The location within the model at which these most critical failure pressure ratios occur is also noted. Very little variation is seen between failure pressure ratios predicted by the half-thickness and the full-thickness models (within 2% in most cases). This is true for both the inner and outer wave models. Since the models considered here represent the worst case of waviness for an inner and outer radius wave, it is reasonable to presume that agreement between half-thickness and full-thickness models for all other cases is as good or better. Half-thickness models are therefore justified for use in all of the layer waviness analyses conducted in this investigation.

5.2.1.4 Summary of Wavy Layer Modeling Simplifications

Evidence has been provided which supports the use of three simplifications to the wavy layer analytical model. The simplified model is a half-thickness cylindrical segment made up of three-dimensional reduced integration elements. One more issue needs to be addressed. That issue is the convergence of stresses within the simplified models.

5.2.2 Stress Convergence of the Simplified Wavy Layer Model

To ensure the accuracy of the stresses calculated by ABAQUS for the simplified wavy layer analysis, stress convergence is verified for each of the continuous stress components. For a perfect cylinder loaded with hydrostatic pressure, the radial, $\sigma_r$, component of stress and shear component of stress in the r-\(\theta\) plane, $\tau_{r\theta}$, should be continuous. In terms of the wavy layer problem, the interlaminar normal and interlaminar shear stress components, $\sigma_n$ and $\tau_{ns}$, must be continuous.
Since this is where failure is predicted to occur, convergence of the continuous stress components is most critical at the location of maximum stress. Results are presented here which verify stress convergence for the most severe stresses due to layer waviness encountered in this investigation, namely, stresses resulting from a wave with $\bar{\delta}=2$ and $\bar{\lambda}=10$ located at the outer radius of a cylinder with $R/H=20$. Stress convergence for all other models is as good or better than this worst-case model.

To verify stress convergence, continuity of stress is checked across the interface of interest, i.e., the interface where the maximum stress occurs. Stresses calculated by finite element methods are calculated at the Gauss integration points within each of the elements. For a three-dimensional reduced integration element, there are a total of eight Gauss points, two in each of the three natural coordinate directions. These locations were shown schematically in Figures 4.2 and 4.9 and are shown again here in Figure 5.12 for convenience. The two rows of Gauss points, one on each side of the interface and shown with a bold '+' in Figure 5.12, are the ones of interest. At any circumferential location, if the stresses calculated at the Gauss points on both sides of the interface are equivalent, or nearly equivalent, the stresses are considered to be continuous, or converge, at the interface. As an example, the values of stress at the two Gauss points enclosed by a dashed rectangle in Figure 5.12 should be nearly equivalent for stress convergence to exist along that interface at that circumferential location.

In all cases of waviness studied, the maximum stress occurs at interfaces within the 14 layer wavy sublaminate. The exact location within the sublaminate depends on the specific case of layer waviness being studied. For the worst case model presented here, i.e., a wave occupying layers 91 through 104, the maximum interlaminar normal stress

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Figure 5.12. Gauss point locations in reference to layer interfaces.
occurs at interface 98-99. The maximum interlaminar shear stress occurs at interface 100-101. The interlaminar stresses calculated at the Gauss points above and below these interfaces are shown in Figure 5.13 as open squares and circles, respectively, for both the interlaminar normal and interlaminar shear components. The closed triangles represent the values of interlaminar stress at the interface, taken as the average between the two Gauss point values. The solid line connecting the triangular data points is used for clarity, to identify the sequence of data points in the circumferential direction. Excellent convergence is seen for the interlaminar normal stress component as indicated by the near equivalence of the stresses at Gauss points on both sides of the interface, Figure 5.13a. The convergence of the interlaminar shear stress component, Figure 5.13b, is not as good, but is acceptable. It should be emphasized that the stresses shown in Figure 5.13 are calculated using two elements through the thickness of each layer. Calculations using one element per layer exhibited poor stress convergence. Two elements per layer is seen to be the minimum allowable to achieve good stress convergence. It should also be noted that the fluctuation in interlaminar stress values is again seen here as it was in the discussion pertaining to using reduced vs. full integration. The solid line representing the interlaminar stress at the interface, a simple connection of the data points, could be replaced by a curve fit through the data points. A curve fit through the data is thought to be a more accurate representation of the true stress state due to layer waviness. More will be said about this in Chapter 6 (see Section 6.2).

It should be noted that stress convergence, although good at the locations of maximum stress, is not good everywhere in the model. The most obvious example of this involves the shear stress component at interface 90-91, the interface separating the nonwavy and wavy regions. The shear stresses calculated at the Gauss points on both sides of this interface are shown in Figure 5.14. It is obvious that convergence is very
a) Interlaminar normal stress, $\sigma_n$, along interface 98-99.

b) Interlaminar shear stress, $\tau_{ns}$, along interface 100-101.

Figure 5.13. Convergence of interlaminar stresses at locations of maximum interlaminar stress within the model.
Interlaminar shear stress, $\tau_{ns}$, along interface 90-91.

Figure 5.14. Lack of convergence of interlaminar shear stress.
poor along this interface from $\theta/\theta^*=0$ to $\theta/\theta=1$, as evident from the large difference in calculated stress at Gauss points on the two sides of the interface. This phenomenon is not fully understood. However, it is felt that the convergence is poor because the stresses are changing most rapidly with radial location at this interface separating the nonwavy and wavy regions. The displacement-based finite element solution cannot accurately compute the stresses when the stress gradients,

$$\frac{\partial \sigma_r}{\partial r}, \frac{\partial \tau_{\theta \phi}}{\partial r}, \text{etc.},$$

are high. These gradients involve second derivatives of displacements with respect to the spatial coordinates, a difficult calculation for any displacement-based finite element scheme to perform accurately. Although convergence is poor at this location, the magnitude of the stress is small compared to that at the location of maximum stress. Since failure is not predicted in this region, the lack of convergence at this location is therefore not an issue. It should be noted that analyses were performed using four elements through the thickness of each layer to determine if convergence was improved. Very little improvement in stress convergence was witnessed for these four element per layer analyses.

In summary, convergence of stress is good at the locations of interest, the interfaces where maximum interlaminar stress occurs. The stresses calculated for this worst case of layer waviness are therefore assumed to be accurate.

5.2.3 Wavy Layer Model Summary

Evidence has been presented to verify the accuracy of the simplified wavy layer model for the worst case of layer waviness considered in this investigation. The model of a half-thickness cylinder segment modeled on a layer-by-layer basis with two three-
dimensional reduced integration elements through the thickness of each layer is therefore assumed to accurately calculate the stress state for all layer waviness geometries considered in this investigation. The failure predictions for these layer waviness geometries are also assumed to be accurate in the context of the applied failure criterion.

5.3 BUCKLING MODEL VERIFICATIONS

To validate the finite element solution to the buckling response of the perfect cylinder, comparisons are made between finite element results and those given by semi-closed-form solutions [24,25]. The finite element analysis models the cylinders using reduced integration three-dimensional elements, the same type used in the wavy layer analyses. The semi-closed-form solutions used in these comparisons are based on Donnell thin-shell theory. Hence, all analyses performed for this aspect of model verification are done for thin-walled cylinders, specifically, for cylinders with R/H=100.

Analyses are performed using both finite elements and semi-closed-form solutions to determine the buckling response of thin-walled cylinders to hydrostatic and pure axial loading. Although response to only hydrostatic loading is of interest in this investigation, the pure axial loading condition is analyzed to provide further validation of the finite element model. Simple support boundary conditions are prescribed on either end of the thin-walled cylinders used in this verification. Schematics of the loading and boundary conditions used here are shown in Figure 5.15.

Both isotropic and orthotropic materials are analyzed to again achieve a more thorough verification of the finite element model. Steel was considered for the isotropic material with the following assumed properties:

\[ E = 30 \times 10^6 \text{ psi} \quad v = 0.33. \]  

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Figure 5.15. Loading and boundary conditions used in buckling model verifications.
The orthotropic material properties used are identical to those of the thick-walled cylinders (see Chapter 3, Equation 7). They represent through-the-thickness smeared material properties for the graphite-epoxy cylinders used throughout this investigation.

Comparisons between the critical axial buckling load or the critical buckling pressure calculated by finite elements and thin-shell theory are shown in Table 5.4. The type of loading, material type, and percent error corresponding to each of the comparisons are shown as well.

**TABLE 5.4**
Buckling Model Verifications

<table>
<thead>
<tr>
<th>Loading Type</th>
<th>Material</th>
<th>Finite Element</th>
<th>Thin-Shell Theory</th>
<th>Percent Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial</td>
<td>Steel</td>
<td>45,184,500 lb.</td>
<td>44,889,683 lb. *</td>
<td>0.6</td>
</tr>
<tr>
<td>(F&lt;sub&gt;cr&lt;/sub&gt;)</td>
<td>Graphite-Epoxy</td>
<td>6,140,850 lb.</td>
<td>6,028,488.7 lb. **</td>
<td>1.9</td>
</tr>
<tr>
<td>Hydrostatic</td>
<td>Steel</td>
<td>128.07 psi</td>
<td>133.6 psi **</td>
<td>4.1</td>
</tr>
<tr>
<td>(P&lt;sub&gt;cr&lt;/sub&gt;)</td>
<td>Graphite-Epoxy</td>
<td>121.79 psi</td>
<td>124.23 psi **</td>
<td>2.0</td>
</tr>
</tbody>
</table>

* Reference 24.
** FORTRAN code based on Donnell thin-shell theory [25].

The percent errors seen in Table 5.4 are sufficiently low to justify modeling the cylinders for buckling analysis using finite elements. The three-dimensional reduced integration finite elements are seen to predict buckling at values very close to those predicted by thin-shell theory. The finite element modeling scheme presented in Chapter 4 is therefore assumed to be valid for the thick-walled cylinder analyses considered in this investigation.
Attention now turns to using the validated finite element models to study the stresses in and around the wave, and to make predictions of reductions in the cylinder's pressure capacity due to layer waviness. These topics are discussed in the next two chapters.
CHAPTER 6

ANALYTICAL RESULTS

This chapter provides an in depth study of the effects of layer waviness on the stresses in the vicinity of the wave. Insight into the general character of the influence of layer waviness on the stresses is gained by this examination. More importantly, by examining the stress distributions at various locations in and around the wave, insight is gained regarding the location(s) at which material failure is likely to occur and the mechanism(s) responsible for this failure.

6.1 STRESS DISTRIBUTIONS DUE TO LAYER WAVINESS

Distributions are presented in this section for the three stress components of primary interest: interlaminar normal stress, $\sigma_n$, interlaminar shear stress, $\tau_{ns}$, and fiber direction stress, $\sigma_S$. As previously shown in Chapter 5, the axial stress, $\sigma_z$, is not significantly affected by the circumferential layer waviness being modeled in this investigation. Results for the axial stress component are therefore not presented here.

The stress distributions are presented in four sections. Each section contains results which illustrate one aspect of the problem being studied. The first section contains stress distributions for a representative wave as a function of radial and circumferential location. This first section provides a view of the general character of the stress distributions in and around the wave. The second section contains results which illustrate the influence of the wave's radial location on the stress state. The third section contains results which illustrate the influence of wave amplitude, $\delta$, on the stress state. Finally,
the fourth section contains results which illustrate the effect of cylinder geometry, $R/H$, on the stress state due to layer waviness.

6.1.1 Stress Distributions as a Function of Position In and Around the Wave

Stress distributions are presented in this section for the most severe case of layer waviness considered in this investigation, i.e., a wave with $\bar{\delta}=2$ and $\bar{\lambda}=10$. The wave is located at the outer radius of a cylinder with $R/H=20$. Although the stresses presented here are specific to this cylinder and wave geometry, some characteristics of the distributions are similar for all cylinder and wave geometries considered in this investigation. It should be noted that the interlaminar stress distributions shown here are actually curves fit to the actual Gauss point stress values calculated by the finite element analysis. The fiber direction stresses, since they are discontinuous across the circumferential-axial layer interfaces, have been presented as scatter distributions through the thickness of the cylinder, and at discrete circumferential locations. Each of the data points for fiber direction stress represents an actual Gauss point value calculated by the finite element analysis.

The interlaminar normal stress, $\sigma_n$, interlaminar shear stress, $\tau_{ns}$, and fiber direction stress, $\sigma_s$, are shown in Figures 6.1, 6.2, and 6.3, respectively. The interlaminar stresses as a function of circumferential location, $\theta$, are shown in Figures 6.1 and 6.2 along various interface locations through the cylinder wall. The circumferential location is normalized by $\theta^*$, the extent of the actual wave, as was shown in Figure 3.5. The stresses are normalized by the applied pressure, $P$. The layer interfaces are numbered as shown previously in Figure 5.7. It is important to note that for this case of the wave being located at the outer radius of the cylinder, the midinterface of the wavy
Figure 6.1. Interlaminar normal stress distributions, $\sigma_{n}$, through the thickness of a cylinder containing a wave located at the outer radius: $R/H=20$, $\bar{\varepsilon}=2$, $\bar{\alpha}=10$. 

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g) Interface 92-93.

i) Interface 85-86.

h) Interface 90-91.

j) Interface 80-81.

Figure 6.1 (continued). Interlaminar normal stress distributions, $\sigma_n$, through the thickness of a cylinder containing a wave located at the outer radius: $R/H = 20$, $\bar{\delta} = 2$, $\bar{\kappa} = 10$. 

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Figure 6.2. Interlaminar shear stress distributions, $\tau_{ns}$, through the thickness of a cylinder containing a wave located at the outer radius: $R/H=20$, $\bar{\delta} = 2$, $\bar{\lambda} = 10$. 

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Figure 6.2 (continued). Interlaminar shear stress distributions, $\tau_{ns}$, through the thickness of a cylinder containing a wave located at the outer radius: $R/H=20$, $\bar{\delta}=2$, $\bar{\lambda}=10$. 
Figure 6.3. Fiber direction stress distributions, $\sigma_s$, through the thickness of a cylinder with a wave located at the outer radius: $R/H=20$, $\delta=2$, $\lambda=10$. 

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c) $\theta/\theta^*=0.4675$.

d) $\theta/\theta^*=0.7225$.

Figure 6.3 (continued). Fiber direction stress distributions, $\sigma_S$, through the thickness of a cylinder with a wave located at the outer radius: $R/H=20$, $\bar{s}=2$, $\bar{\lambda}=10$. 

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Figure 6.3 (continued). Fiber direction stress distributions, $\sigma_s$, through the thickness of a cylinder with a wave located at the outer radius: $R/H=20$, $\bar{\sigma}=2$, $\bar{\kappa}=10$. 

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g) $\theta/\theta^* = 1.84875$.

h) $\theta/\theta^* = 2.6775$.

Figure 6.3 (continued). Fiber direction stress distributions, $\sigma_s$, through the thickness of a cylinder with a wave located at the outer radius: $R/H=20$, $\delta=2$, $\lambda=10$. 

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sublamine is at interface 97-98. The fiber direction stress distributions are presented somewhat differently than the interlaminar stress distributions. The fiber direction stresses are shown as a function of nondimensional radial location, \( \rho_0 \), through the thickness of the cylinder wall at various circumferential locations. With \( \rho_0 \) as a measure of radial location, and for a wave located at the outer radius of the cylinder, the midinterface of the wavy sublamine varies from \( \rho_0 = 0.452 \) at \( \theta/\theta^* = 0.0 \) to \( \rho_0 = 0.433 \) for \( \theta/\theta^* \geq 1.0 \). However, for clarity, the midinterface of the wavy sublamine is shown at \( \rho_0 = 0.433 \) for all circumferential locations.

As can be seen in Figures 6.1, 6.2, and 6.3, the wave-induced perturbations in the three stress components, \( \sigma_n \), \( \tau_{ns} \), and \( \sigma_s \), are localized and confined to areas in close proximity to the wave. Specifically referring to Figures 6.1 and 6.2, localization in the circumferential direction is demonstrated by the interlaminar normal and interlaminar shear stress distributions which return from a perturbed distribution within the wave to a uniform distribution (zero in the case of \( \tau_{ns} \)) within two half wave lengths of the end of the wave, i.e., at circumferential locations \( 1.0 < \theta/\theta^* < 3.0 \). The fiber direction stress distribution, however, is not as circumferentially localized as the interlaminar stresses. Fiber direction stress, as seen in Figure 6.3h, is significantly influenced at nearly two half wave lengths from the end of the wave. This far reaching perturbation in the fiber direction stress, discussed earlier in the context of choosing a model arc length, is a result of the mismatch in circumferential stiffness of the circumferential and axial layers, i.e., the high circumferential stiffness of the circumferential layers and the low circumferential stiffness of the adjacent axial layers. Due to their low circumferential stiffness, the axial layers require a large circumferential length to help unload the wave-induced fiber direction stress perturbations present in the circumferential layers.
The perturbations in the three components of stress, in addition to being circumferentially localized, are localized in the radial direction. Referring to Figures 6.1 and 6.2, localization in the radial direction is demonstrated by the fact that near uniform interlaminar normal and interlaminar shear stress distributions are present at radial locations within ten layers from the wave, i.e., along interface 80-81. In a similar manner, the fiber direction stress shown in Figure 6.3 has returned to its unperturbed state within ten layers of the wave, i.e., along layer 81.

More evidence of the localized effect of the wave is the fact that the maximum values of stress always occur somewhere within the 14 layer wavy sublamine. These maxima are shown in Figures 6.1c and 6.1f, 6.2b, and 6.3a for the interlaminar normal, interlaminar shear, and fiber direction components of stress, respectively.

The midinterface of the wave, interface 97-98, and the inflection point of the wave, θ/θ*=0.5, appear to play a significant role in the distributions of all three stress components. The remainder of this section will describe how these two geometric characteristics of the wave affect each of the three stress distributions.

The distribution of σn along any particular interface within the wavy sublamine appears to be related to the location of that interface with respect to the midinterface of the sublaminate. For instance, referring to Figures 6.1a through 6.1c which represent interfaces located above the midinterface of the wavy sublamine, a maximum compressive stress develops between the peak of the wave, θ/θ*=0.0, and the inflection point of the wave, θ/θ*=0.5, while a minimum compressive stress develops near the end of the wave, θ/θ*=1.0. In contrast, referring to Figures 6.1d through 6.1h which represent interfaces located below the midinterface of the wavy sublamine, the maximum compressive stress develops near the end of the wave, θ/θ*=1.0, while the minimum compressive stress, which in many cases is not compressive, but rather is
tensile, develops at the peak of the wave, \( \theta/\theta^* = 0.0 \). A physical interpretation of this phenomenon is given by a 'bent beam in compression' analogy, illustrated in Figure 6.4. In the bent beam analogy, the central two circumferential layers of the wavy sublaminate, one on each side of the midinterface, are analogous to a bent beam. As illustrated in Figure 6.4a, a bent beam loaded in compression, with no surrounding material to restrict its deformation, bends upward at the line of symmetry, causing regions 2 and 3, indicated on the figure, to move upward. Also, regions 1 and 4, as indicated in the figure, are forced to move downward. When the bent beam is surrounded by material, as illustrated in Figure 6.4b, two distinct regions of compression and two distinct regions of tension are present between the beam and the surrounding material. The locations of these regions of compression and tension above and below the bent beam correlate fairly well with interlaminar normal stress distributions above and below the central layers of the wavy sublaminate. Namely, above the beam, the maximum compressive stress occurs at locations along the length of the beam between the line of symmetry and the inflection point of the beam (analogous to circumferential locations between the peak, \( \theta/\theta^* = 0.0 \), and the inflection point, \( \theta/\theta^* = 0.5 \), of the wave), and the maximum tensile stress, or minimum compressive stress, occurs at locations along the length of the beam between the inflection point and the end of the beam (analogous to circumferential locations between the inflection point, \( \theta/\theta^* = 0.5 \), and the end, \( \theta/\theta^* = 1.0 \), of the wave). In a similar manner, below the beam, the maximum compressive stress occurs at locations along the length of the beam between the inflection point and the end of the beam, while the maximum tensile stress, or minimum compressive stress, occurs at locations along the length of the beam between the line of symmetry and the inflection point of the beam. While this analogy has some shortcomings, it is a useful parallel to the wavy layer problem.
a) Bent beam in compression (no surrounding material present).

b) Bent beam in compression with surrounding material.

Figure 6.4. Bent beam analogy to interlaminar normal stress state, $\sigma_n$, in a cylinder containing layer waviness.
In terms of the interlaminar shear stress distributions, $\tau_{ns}$, the midinterface of the wavy sublamine does not appear to play a significant role, i.e., the dependence of the stress distribution on the circumferential coordinate appears to be independent of radial location within the sublamine. The inflection point of the wave, however, appears to be very significant. Referring to Figure 6.2, the maximum value of interlaminar shear stress always occurs very near the inflection point of the wave, $\theta/\theta^* = 0.5$. Within the wavy region, $0.0 < \theta/\theta^* < 1.0$, the stress distribution to either side of this maximum value is very near symmetric.

The midinterface of the wavy sublamine and the inflection point of the wave both appear to play a significant role in the fiber direction stresses, $\sigma_3$. Within the wavy sublamine, $0.0 < \theta/\theta^* < 1.0$, see Figures 6.3a through 6.3e, and $\rho > 0.36$, the maximum fiber direction compressive stress within the circumferential layers always occurs above the midinterface of the sublamine. Also, referring specifically to Figures 6.3a and 6.3b which represent circumferential locations between the peak and the inflection point of the wave, this maximum fiber direction stress always occurs at the inner-most radial position within the two adjacent circumferential layers (note the location of the 'Maximum fiber stress' indicated in Figure 6.3a). In contrast, referring to Figures 6.3d and 6.3e which represent circumferential locations between the inflection point and the end of the wave, for layers above the midinterface of the sublamine, the maximum fiber direction compressive stress occurs at the outer-most radial position within the two adjacent circumferential layers. For layers below the midinterface of the sublamine, the maximum fiber direction stress always occurs near the inner-most radial position within the two adjacent circumferential layers. A transition in the location of the maximum fiber direction stress within the two adjacent circumferential layers definitely occurs somewhere near the inflection point of the wave, as indicated by Figure 6.3c, in which

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there is really no apparent trend in the distribution within the two adjacent circumferential layers.

Results have been presented in this section which demonstrate the localized effect of the wave. The locations of maximum stress have also been shown to occur somewhere within the wavy sublaminate. Since these maximum stresses are of primary consideration for this investigation, the following sections will address the issues of the wave's radial location, wave geometry, and cylinder geometry by considering stress distributions only at locations within the wavy sublaminate.

6.1.2 Effect of the Wave's Radial Location on the Stress State

Results are presented in this section which compare stress distributions for the worst case of a wave located at the inner radius of the cylinder and for the worst case of a wave located at the outer radius of the cylinder. These worst case comparisons correspond to a cylinder with \( R/H = 20 \) containing a wave with \( \delta = 2 \) and \( \lambda = 10 \). Interlaminar stress distributions for the wave located at the inner radius are compared with interlaminar stress distributions for the wave located at the outer radius. This comparison of interlaminar stresses is made at interfaces which represent identical relative locations within the 14 layer wavy sublaminate. An interface numbering scheme has been shown in Figure 6.5 which identifies corresponding interfaces for the case of a wave located at the inner radius of the cylinder and for the case of a wave located at the outer radius of the cylinder. This interface numbering scheme, independent of the wave's radial location, is used in the following comparisons of interlaminar stress. With this numbering scheme, the midinterface of the wavy sublaminate is identified as interface 7*-8*. Fiber direction stress distributions are compared for the two wave locations along the same circumferential location, i.e., at identical \( \theta/\theta^* \) locations. The midinterface of
Figure 6.5. Layer and interface numbering scheme independent of wave's radial location.
the wavy sublamine is located at $\rho = -0.433$ for the wave located at the inner radius of the cylinder and at $\rho = +0.433$ for the wave located at the outer radius of the cylinder.

Interlaminar normal stress distributions, $\sigma_n$, are shown in Figure 6.6 as a function of circumferential location, and at various interfaces within the wavy sublamine, for the wave located at the inner radius of the cylinder and for the wave located at the outer radius of the cylinder. In general, the trends are similar for the two cases. Namely, in regions above the midinterface of the wavy sublamine, Figures 6.6a through 6.6c, a maximum compressive stress develops somewhere in the range $0.0 < \theta/\theta^* < 0.5$, between the peak and the inflection point of the wave, respectively. The maximum tensile stress, in the case of the wave located at the inner radius of the cylinder, or minimum compressive stress, in the case of the wave located at the outer radius of the cylinder, develops near the end of the wave, $\theta/\theta^* = 1.0$. It is felt that in the case of the wave located at the outer radius of the cylinder, a tensile stress does not develop above the midinterface of the wave due to the higher interlaminar compressive stress state near the outer radius of the cylinder due to the applied pressure, $P$. It should be noted that for locations above the midinterface of the wavy sublamine, the previously discussed bent beam analogy shows excellent correlation with interlaminar normal stress distributions for a wave located at the inner radius of the cylinder, while only moderate correlation for a wave located at the outer radius of the cylinder. In regions below the midinterface of the wavy sublamine, Figures 6.6d through 6.6f, a region of interlaminar tension develops near the peak of the wave, $\theta/\theta^* = 0.0$, for both the wave located at the inner radius of the cylinder and for the wave located at the outer radius of the cylinder. It should be noted that for these locations below the midinterface of the wavy sublamine, the bent beam analogy correlates better with a wave located at the outer radius of the cylinder than with a wave located at the inner radius of the cylinder. It is also interesting
Figure 6.6. Interlaminar normal stress distributions, $\sigma_n$, for cylinders with a wave located at the inner radius versus the outer radius of the cylinder: $R/H=20$, $\bar{\sigma}=2$, $\bar{\lambda}=10$. 

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Wave at inner radius.  
Wave at outer radius. 

(d) Interface 6*-7*.

Wave at inner radius.  
Wave at outer radius. 

(e) Interface 4*-5*.

Wave at inner radius.  
Wave at outer radius. 

(f) Interface 2*-3*.

Figure 6.6 (continued). Interlaminar normal stress distributions, $\sigma_n$, for cylinders with a wave located at the inner radius versus the outer radius of the cylinder: $R/H=20$, $\bar{\delta}=2$, $\bar{\lambda}=10$. 

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to note that in regions below the midinterface of the wavy sublaminate, the highest interlaminar tension develops for the wave located at the outer radius of the cylinder, despite the higher overall interlaminar normal compressive stress state, due to the applied hydrostatic pressure, and the lower circumferential stress state near the outer radius of the cylinder. The lower circumferential stress state is important in terms of the bent beam analogy discussed previously. A lower circumferential stress state results in less bending of the beam, and hence the interlaminar tension induced below the midinterface of the wave would be less for the wave located at the outer radius of the cylinder than for the wave located at the inner radius of the cylinder. Although the bent beam analogy represents, with some degree of accuracy, the trends for the wave located at the inner radius of the cylinder and for the wave located at the outer radius of the cylinder, the analogy is not accurate enough to represent the relative differences between the two cases.

The interlaminar shear stress distributions, $\tau_{ns}$, are shown in Figure 6.7 for the wave located at the inner radius of the cylinder and for the wave located at the outer radius of the cylinder. The stress distributions for the two cases are seen to be very similar, including the fact that the maximum interlaminar shear stress for a wave located at the inner radius of the cylinder and for a wave located at the outer radius of the cylinder occurs above the midinterface of the wavy sublaminate and near the inflection point of the wave. One characteristic which is noted to be different is that at locations above the midinterface of the wavy sublaminate, Figures 6.7a through 6.7c, the magnitude of the maximum shear stress is slightly greater for a wave located at the outer radius of the cylinder than for a wave located at the inner radius of the cylinder. The opposite is true for locations below the midinterface of the wavy sublaminate, Figures 6.7d through 6.7f. Also for locations below the midinterface of the sublaminate, a
Wave at *inner* radius.

Wave at *outer* radius.

a) Interface 12*-13*.

Wave at *inner* radius.

Wave at *outer* radius.

b) Interface 10*-11*.

Wave at *inner* radius.

Wave at *outer* radius.

c) Interface 8*-9*.

Figure 6.7. Interlaminar shear stress distributions, $\tau_{ns}$, for cylinders with a wave located at the *inner* radius versus the *outer* radius of the cylinder: $R/H=20$, $\bar{u} = 2$, $\bar{x} = 10$.

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Figure 6.7 (continued). Interlaminar shear stress distributions, $\tau_{ns}$, for cylinders with a wave located at the inner radius versus the outer radius of the cylinder: $R/H=20$, $\bar{\delta}=2$, $\bar{\kappa}=10$. 

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change in sign of the interlaminar shear stress is seen only for a wave located at the outer radius of the cylinder. This change in sign occurs at circumferential locations beyond the end of the wave, i.e., at $\theta/\theta^* > 1.0$.

The fiber direction stress distributions, $\sigma_3$, are shown in Figure 6.8 for a wave located at the inner radius of the cylinder and for a wave located at the outer radius of the cylinder. Very similar distributions are demonstrated for the two cases. In both cases, the maximum fiber direction compression stress occurs at Gauss points closest to the peak of the wave, $\theta/\theta^*=0.02125$, and above the midinterface of the wavy sublaminates. It is suspected that an extrapolation of the fiber direction stress distributions would show that the maximum fiber direction stress actually occurs at the peak of the wave, $\theta/\theta^*=0.0$.

As expected, the absolute maximum occurs for the wave located at the inner radius of the cylinder, due to the higher hydrostatically-induced circumferential stresses present near the inner radius of the cylinder.

Results have been presented in this section which demonstrate the large number of similarities and the few differences in the behavior of the stresses for a wave located at the inner radius of the cylinder and for a wave located at the outer radius of the cylinder. Because of the similarities in the stress distributions for the two wave locations, the following discussions of the influence of wave geometry and cylinder geometry on the stress state will consider only a wave located at the outer radius of the cylinder. These results will be representative of the response due to layer waviness.

An interesting interpretation of the results presented should be noted at this point. With the stresses normalized by the applied pressure, the figures just discussed can be interpreted as illustrative of stress concentration effects. Stress concentration effects are generally associated with geometric discontinuities such as holes and notches. However, a wavy layer is a geometric discontinuity and hence lends itself to the stress
Figure 6.8. Fiber direction stress distributions, $\sigma_\theta$, for cylinders with a wave located at the *inner* radius versus the *outer* radius of the cylinder: $R/H=20$, $\bar{\delta}=2$, $\bar{\lambda}=10$. 

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c) $\theta/\theta^* = 0.4675$.

d) $\theta/\theta^* = 0.7225$.

Figure 6.8 (continued). Fiber direction stress distributions, $\sigma_s$, for cylinders with a wave located at the inner radius versus the outer radius of the cylinder: $R/H=20$, $\delta = 2$, $\tilde{x} = 10$. 

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e) $\theta/\theta^* = 0.9775$.

f) $\theta/\theta^* = 1.3175$.

Figure 6.8 (continued). Fiber direction stress distributions, $\sigma_8$, for cylinders with a wave located at the inner radius versus the outer radius of the cylinder: $R/H = 20$, $\delta = 2$, $\lambda = 10$. 

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Figure 6.8 (continued). Fiber direction stress distributions, $\sigma_\theta$, for cylinders with a wave located at the inner radius versus the outer radius of the cylinder: $R/H=20$, $\delta=2$, $T=10$. 

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concentration factor interpretation. Thus, for example, referring to Figure 6.2b, it can be said that in a cylinder with $R/H=20$ and for a wave with $\delta=2$ and $\lambda=10$ located at the outer radius of the cylinder, the stress concentration factor for interlaminar shear stress is 4. Hence with an applied pressure of 1000 psi, the maximum interlaminar shear stress is 4000 psi. Similar interpretations can be developed for the other stress components.

6.1.3 Effect of Wave Amplitude on the Stress State

Results are presented in this section for a wave located at the outer radius of a cylinder with $R/H=20$. Interlaminar normal, interlaminar shear, and fiber direction stress distributions, shown in Figures 6.9, 6.10, and 6.11, respectively, are compared for three wave geometries considered in this investigation, i.e., wave amplitudes of $\delta=1/2$, 1, and 2 and a half wave length of $\lambda=10$.

As expected, referring to Figure 6.9, the peak tensile and peak compressive interlaminar normal stresses increase with increasing wave amplitude. However, the location of these peaks is not significantly affected by the wave amplitude. It is interesting to note that independent of the amplitude of the wave, the attenuation of the perturbations in the interlaminar normal stress appears to occur over the same small circumferential region between $\theta/\theta^*=1.0$ and $\theta/\theta^*=1.5$. In other words, the primary influence of layer waviness occurs within the wavy region, $0.0 < \theta/\theta^* < 1.0$.

Referring to Figure 6.10, similar trends in the interlaminar shear stress are seen for each of the wave amplitudes. Note the similar circumferential location of the peak shear stress, which in turn maintains the symmetry about the inflection point of the wave, $\theta/\theta^*=0.5$, regardless of the wave amplitude. As noted in the previous section, for locations below the midinterface of the wave, Figures 6.10d through 6.10f, the sign of the interlaminar shear stress changes from negative to positive at locations beyond the
Figure 6.9. Interlaminar normal stress distributions, $\sigma_n$, for a cylinder with a wave located at the outer radius: $R/H=20$, $\bar{x}=1/2$, 1, and 2, $x=10$. 

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Figure 6.10. Interlaminar shear stress distributions, $\tau_{ns}$, for a cylinder with a wave located at the outer radius: $R/H=20$, $\bar{x}=1/2$, 1, and 2, $\bar{x}=10$. 

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Figure 6.11. Fiber direction stress distributions, \( \sigma_\theta \), for a cylinder with a wave located at the outer radius: \( R/H = 20 \), \( \bar{\delta} = 1/2, 1, \) and \( 2 \), \( \bar{k} = 10 \).
c) $\theta/\theta^*=0.4675$.

d) $\theta/\theta^*=0.7225$.

Figure 6.11 (continued). Fiber direction stress distributions, $\sigma_S$, for a cylinder with a wave located at the outer radius: $R/H=20$, $\bar{\delta}=1/2$, 1, and 2, $\bar{\tau}=10$. Chapter 6: Analytical Results
Figure 6.11 (continued). Fiber direction stress distributions, $\sigma_S$, for a cylinder with a wave located at the outer radius: $R/H=20$, $\bar{\delta}=1/2$, 1, and 2, $\bar{\lambda}=10$. 

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g) $\theta/\theta^*=1.84875$.

h) $\theta/\theta^*=2.6775$.

Figure 6.11 (continued). Fiber direction stress distributions, $\sigma_s$, for a cylinder with a wave located at the outer radius: $R/H=20$, $\bar{\delta}=1/2$, 1, and 2, $\bar{\kappa}=10$. 

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end of the wave, $\theta/\theta^* > 1.0$. However, this sign change occurs only for the largest of the wave amplitudes.

Finally, referring to Figure 6.11, the perturbations in the fiber direction stress, as well as the stress gradients seen two adjacent circumferential layers, increase with increasing wave amplitude. As pointed out in the previous section, for radial locations above the midinterface of the wavy sublaminate, $\rho > 0.433$, the location of the maximum fiber direction stress changes from near the bottom of the two adjacent circumferential layers at circumferential locations between the peak of the wave and the inflection point of the wave, $0.0 < \theta/\theta^* < 0.5$, to near the top of the two adjacent circumferential layers at circumferential locations between the inflection point and the end of the wave, $0.5 < \theta/\theta^* < 1.0$. Essentially, the gradient of the circumferential fiber direction compressive stresses changes sign with circumferential location. This characteristic is independent of wave amplitude.

6.1.4 Effect of Cylinder Geometry on the Stress State

Results are presented in this section for three cylinder geometries and a single wave geometry. The three cylinder geometries are $R/H=5$, 10, and 20, and the wave geometry is $\bar{\delta}=2$ and $\bar{\lambda}=10$. The wave is located at the outer radius of each of the cylinders. By considering a single wave geometry and wave location within the cylinder wall, the effect of cylinder geometry is isolated.

The interlaminar normal and interlaminar shear stress distributions, shown in Figures 6.12 and 6.13, respectively, are remarkably similar to those presented in the previous section illustrating the effect of wave amplitude, Figures 6.9 and 6.10. It is therefore apparent that, in terms of the interlaminar stress components, decreasing the curvature of the cylinder, i.e., increasing the value of $R/H$, is comparable to increasing
Figure 6.12. Interlaminar normal stress distributions, $\sigma_n$, for cylinders with a wave located at the outer radius: $R/H=5$, 10, and 20, $\bar{\alpha}=2$, $\bar{\Gamma}=10$. 

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Figure 6.13. Interlaminar shear stress distributions, $\tau_{ns}$, for cylinders with a wave located at the outer radius: $R/H=5, 10$, and $20$, $\bar{\sigma}=2$, $\bar{\lambda}=10$. 

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the curvature of the wave, i.e., increasing the wave amplitude. From the interlaminar stress distributions, it is clear that increasing the radius of the cylinder results in a higher levels of interlaminar stress due to layer waviness.

As do the interlaminar stresses, the fiber direction stresses, shown in Figure 6.14 as a function of cylinder geometry, show a remarkable resemblance to the fiber direction stresses as a function of wave amplitude. An increased perturbation in the fiber direction stresses is seen as a result of increasing the radius of the cylinder. Figure 6.11 demonstrated that increasing the curvature of the wave, i.e., increasing the wave amplitude, had the same effect. The notable difference here though, is that the fiber direction stresses inward from the wavy region are quite different for the three cylinder geometries. This difference is due to the linear relation between the circumferential stress and the radius of curvature, $R/H$. Increasing the radius of curvature results in an increased circumferential stress in the cylinder, even for the case of a cylinder with no included wave. In a similar manner, increasing the radius of curvature for a cylinder including a wave results in an increased wave-induced perturbation in the fiber direction stresses.
Figure 6.14. Fiber direction stress distributions, $\sigma_S$, for cylinders with a wave located at the outer radius: $R/H=5$, 10, and 20, $\bar{s}=2$, $\bar{k}=10$. 

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Figure 6.14(continued). Fiber direction stress distributions, $\sigma_\theta$, for cylinders with a wave located at the outer radius: $R/H=5$, 10, and 20, $\delta=2$, $\kappa=10$. 

c) $\theta/\theta^* = 0.4675$.

d) $\theta/\theta^* = 0.7225$. 

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Figure 6.14(continued). Fiber direction stress distributions, $\sigma_\theta$, for cylinders with a wave located at the outer radius: $R/H=5$, 10, and 20, $\delta=2$, $\kappa=10$. 

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g) $\theta/\theta^*=1.84875$.

h) $\theta/\theta^*=2.6775$.

Figure 6.14(continued). Fiber direction stress distributions, $\sigma_5$, for cylinders with a wave located at the outer radius: $R/H=5$, 10, and 20, $\bar{\theta}=2$, $\bar{r}=10$. 

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6.2 COMMENTS ON THE CALCULATION OF INTERLAMINAR STRESSES

The interlaminar stress distributions just shown in Section 6.1 are curves fit to the actual data calculated by the finite element analysis. A Lowess (locally weighted regression) [26] curve fitting method is used whereby a piece-wise linear least square fit is made at each data point. It is felt that this curve fitting method gives a good representation of the trends in the actual data. A typical curve fit to actual data is shown as a solid line in Figure 6.15. The data points are connected by a dashed line in Figure 6.15 to show the oscillatory nature of the data in the circumferential direction. It is felt that this oscillatory behavior is a result of the fiber angle discontinuities at the circumferential boundaries of the elements resulting from the previously described piece-wise linear approximation to the curvilinear layer geometry (see Chapter 4). These fiber angle discontinuities are illustrated in Figure 6.16. It is proposed that refining the mesh (increasing the number of finite elements) in the circumferential direction will reduce the magnitude of the fiber angle discontinuities at the element boundaries, resulting in better stress convergence in the circumferential direction, and hence less oscillation in the calculated stresses. A brief analysis of this proposed method was performed by increasing the number of finite elements in the circumferential direction from 8 to 12 in the wavy region, $0.0 < \theta/\theta^* < 1.0$, and from 6 to 8 outside the wavy region, $\theta/\theta^* > 1.0$. These minor mesh refinements resulted in little improvement in the calculated stresses. It was concluded that substantial mesh refinement in the circumferential direction was necessary to produce any significant improvement in circumferential stress convergence. Due to the limited computing capabilities in this investigation, no further action was taken on this matter. It should also be noted that, since the true curvilinear fiber geometry is approximated by a piece-wise linear geometry, the calculated interlaminar stresses are actually perpendicular to the direction of the piece-wise approximation, not

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Interlaminar normal stress distribution, $\sigma_n$, along interface 100-101 in a cylinder with $R/H=20$ containing a wave with $\bar{s}=2$ and $\bar{x}=10$ at the outer radius of the cylinder.

Figure 6.15. Example curve fit to actual data values calculated by finite element analysis.
Discontinuities in fiber angle

Piecewise linear approximation to the true curvilinear path of a fiber within a wavy circumferential layer

Exaggerated curvature of a wavy circumferential layer
(element boundaries indicated by dashed lines)

Figure 6.16. Discontinuities in the fiber angle at circumferential element boundaries.
perpendicular to the true curvilinear fiber geometry. This is not really an issue due to the relatively small curvature of the actual wavy layers.

One final point to be made about the interlaminar stresses regards the calculation of failure pressure, $P_{Cr}$. Although it is felt that a curve fit to the interlaminar stress data gives a much better representation of the trends in the stresses, the maximum value of stress, as calculated directly from the Gauss points of the finite element analysis, is used to determine the failure pressure of the cylinder due to layer waviness. Use of the maximum value of stress is done primarily for convenience. As illustrated in Figure 6.17, a calculation based on the maximum value of stress can be different than one based on the maximum value of stress from a curve fit to the actual values of stress.

With typical results discussed which provide insight into the character of the stresses in and around the wavy layers, and which provide information regarding the influence of the various geometric parameters on these stresses, attention now turns to failure. The following chapter discusses this important aspect of the layer waviness investigation.
Figure 6.17. Value of stress used to calculate the failure pressure of the cylinder due to layer waviness.
CHAPTER 7

FAILURE PREDICTIONS

Results are presented in this chapter which are based on using the maximum stress failure criterion for analyzing the most critical stresses due to layer waviness. Failure pressure ratios are presented which compare the pressure necessary to cause material failure in the imperfect cylinder with the pressure necessary to cause material failure in the perfect cylinder. The ratios are important in demonstrating the reduction due to layer waviness of the cylinder's hydrostatic pressure capacity. Failure pressure ratios are also presented regarding buckling of the perfect cylinder. The buckling ratios compare the pressure necessary to cause buckling of the perfect cylinder with the pressure necessary to cause material failure in the perfect cylinder. Results are then presented which compare the failure pressure ratios from the wavy layer and buckling analyses. Comparing the failure pressure ratios from the two analyses for a hydrostatically loaded composite cylinder demonstrates the significance of failure of the cylinder due to layer waviness relative to failure of the cylinder due to buckling.

7.1 WAVY LAYER FAILURE PRESSURE RATIOS

Results are presented in this section which analyze the effect of layer waviness on the pressure capacity of the cylinder. More specifically, failure pressure ratios, $P_{cr}/P_0$, are used to evaluate the material failure response of a cylinder including a wave as compared to the material failure response of a cylinder without a wave, i.e., a perfect cylinder. These failure pressure ratios indicate the reduction in pressure capacity of the cylinder which contains wavy layers. The definition of the failure pressure ratio was
presented previously in Chapter 3, and is repeated here for convenience, namely, the pressure, $P_{cr}$, which causes material failure of the cylinder with wavy layers is normalized by the failure pressure of the perfect cylinder, $P_0$. Recall, as stated in Chapter 3, failure in the perfect cylinder is due to a fiber direction compression failure in a layer with circumferentially oriented fibers. This failure occurs near the inner radius of the cylinder due to the higher circumferential stresses at the inner radius of a thick-walled cylinder (the Lamé effect [15]), and the value of $P_0$ depends on the cylinder geometry. The three values of $P_0$ are 21.389 ksi, 11.445 ksi, and 5.988 ksi for the cylinder geometries with $R/H=5$, 10, and 20, respectively. Because of the four failure modes which are considered in this investigation, i.e., interlaminar normal tension, interlaminar normal compression, interlaminar shear, and fiber direction compression, four wave-induced failure pressures, $P_{cr}$, are studied for each combination of cylinder and wave geometry. Likewise, four failure pressure ratios are studied for each combination of cylinder and wave geometry. Obviously, for a given cylinder, only one pressure ratio is critical, but much can be learned, at least initially, by examining all four.

The failure pressure ratios are presented in three sections. The first of these sections considers the geometric effects of the problem by analyzing the effect of cylinder and wave geometry on each of the four possible failure modes, and hence the four failure pressure ratios. The second section looks at the interactions of each of the four failure modes by comparing the four failure pressure ratios for a specific cylinder geometry. The final section looks only at the most critical failure pressure ratios from those presented in the previous two sections. It is important to note that only a failure pressure ratio less than unity is interpreted as a reduction in the cylinder's pressure capacity. A failure pressure ratio greater than unity for a particular failure mode implies that layer waviness does not cause failure in that mode. If none of the four failure
pressure ratios are less than unity, fiber direction compression failure will occur at the inner radius of the cylinder, outside the wavy region, despite the presence of a wave.

7.1.1 Effect of Cylinder and Wave Geometry on Failure Pressure Ratios

Failure pressure ratios for the interlaminar normal tension, $+\sigma_n$, failure mode are shown in Table 7.1 for each of the three wave geometries located at the inner radius of each of the three cylinder geometries and those same three wave geometries located at the outer radius of each of the three cylinder geometries. The rows in the table are arranged so that for a given wave location, i.e., inner radius or outer radius, the radius of curvature of the cylinder increases as you move down the rows. For each wave location and cylinder geometry, the failure pressure ratios are arranged in three columns, each column representing a different wave geometry. The columns are arranged so that the wave amplitude, $\bar{\delta}$, increases from left to right. Similar table arrangements are used for the remainder of this section.

Since in the perfect cylinder interlaminar normal tension never develops, the failure pressure necessary to cause interlaminar tension failure in the perfect cylinder is infinite. The fact that any of the wavy layer failure pressure ratios, $P_{cr}/P_0$, due to interlaminar normal tension shown in Table 7.1 are finite indicates the significance of interlaminar tension caused by layer waviness.
Table 7.1
Failure Pressure Ratios for
Interlaminar Normal Tension, \( +\sigma_n \), Failure Mode

<table>
<thead>
<tr>
<th>Wave Location</th>
<th>Cylinder Geometry (R/H)</th>
<th>Wave Geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \delta = 1/2, \lambda = 10 )</td>
<td>( \delta = 1, \lambda = 10 )</td>
</tr>
<tr>
<td>Inner Radius</td>
<td>5</td>
<td>10.315</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>6.298</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>3.886</td>
</tr>
<tr>
<td>Outer Radius</td>
<td>5</td>
<td>N/A*</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>N/A*</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>N/A*</td>
</tr>
</tbody>
</table>

* N/A indicates that no interlaminar normal tensile stresses are induced in this particular cylinder and for this particular wave geometry, i.e., \( P_{cr}/P_o \) is infinite.

To provide additional insight into the character of failure, the failure pressure ratios shown in Table 7.1 are plotted in Figure 7.1. The figure shows failure pressure ratio, \( P_{cr}/P_o \), vs. wave amplitude, \( \delta \), with cylinder geometry, R/H, and wave location as parameters. One important thing to note in the figure is the dotted line at \( P_{cr}/P_o = 1 \). This dotted line indicates a threshold, below which, failure due to interlaminar normal tension will potentially occur at a pressure lower than the pressure, \( P_o \), to cause the standard fiber direction compression failure at the inner radius of the cylinder, outside the wavy region. The wording 'will potentially occur' is used because of the possibility of some other mode of failure having a failure pressure ratio less than that of the interlaminar normal tension failure pressure ratio. This point will be made clear shortly when other failure modes are studied. Similar interpretations of the dotted line at \( P_{cr}/P_o \) will be
Figure 7.1. Failure pressure ratios for interlaminar normal tension, $\sigma_n$, failure mode.

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made for the other three failure modes. It is seen from Figure 7.1 and from Table 7.1 that interlaminar normal tension failure will potentially occur, i.e., $P_{Cr}/P_o < 1.0$, for some of the cylinders containing a wave with $\delta = 2$ and $\lambda = 10$. The ordering of the failure pressure ratios for the various cylinder geometries, i.e., the larger value of $R/H$ resulting in a lower value of $P_{Cr}/P_o$, implies that decreasing the radius of the cylinder seems to assist slightly in suppressing interlaminar normal tension failure. This is true only for the smaller wave amplitudes, i.e., $\delta < 2$. As the wave amplitude increases, compared to the slope for other values of $R/H$, the greater slope of the relation between $P_{Cr}/P_o$ and $\delta$ for a wave located at the inner radius of a cylinder with $R/H=5$ implies that beyond $\delta = 2$, the smaller radius of the cylinder with may very well result in interlaminar normal tension failure at pressures below those required to cause interlaminar normal tension failure in a cylinder with $R/H=10$ or 20. Furthermore, it is anticipated that for these larger wave amplitudes, the wave located at the outer radius of the cylinder may cause interlaminar normal tension failure at pressures less than those caused by a wave located at the inner radius of the cylinder. This anticipated behavior is a result of the greater slope of the relation in Figure 7.1 between $P_{Cr}/P_o$ and $\delta$ for a wave located at the outer radius than for a wave located at the inner radius of a cylinder for values of $R/H$ equal to 10 and 20. Note that in Figure 7.1, the '∞' relate to the 'N/A' shown in Table 7.1, indicating an infinite failure pressure ratio.

Failure pressure ratios are shown in Table 7.2 for the interlaminar normal compression failure mode, $-\sigma_n$. The asterisks indicate that failure pressure ratios due to layer waviness at the inner radius of the cylinder are overshadowed by failure pressure ratios due to the higher interlaminar normal compressive stress state at the outer radius of the cylinder, outside the wavy region. The failure pressure ratios due to the interlaminar normal compressive stress state at the outer radius of the cylinder, independent of layer
waviness, are given as 1.403, 2.621, and 5.010 for the cylinder geometries R/H=5, 10, and 20, respectively. To arrive at these values, the failure pressure due to interlaminar normal compression is calculated to be 30 ksi, this from the 1 psi interlaminar compressive stress at the outer radius of the cylinder, and the 30 ksi interlaminar compressive strength of the material, as defined previously in Chapter 3. The three failure pressure ratios are thus calculated by dividing the 30 ksi failure pressure by the failure pressure of the perfect cylinder due to fiber direction compression, $P_o$, which was previously defined to be 21.389 ksi, 11.445 ksi, and 5.988 ksi for cylinders with R/H=5, 10, and 20, respectively. It should be noted that these failure pressure ratios marked by an asterisk imply that failure would be caused by the external pressure itself, crushing the material at the outer surface of the cylinder. Physically, this is an impossible situation.
Table 7.2
Failure Pressure Ratios for Interlaminar Compression, \(-\sigma_N\), Failure Mode

<table>
<thead>
<tr>
<th>Wave Location</th>
<th>Cylinder Geometry (R/H)</th>
<th>Wave Geometry</th>
<th>(\delta = 1/2, \lambda = 10)</th>
<th>(\delta = 1, \lambda = 10)</th>
<th>(\delta = 2, \lambda = 10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner Radius</td>
<td>5</td>
<td>1.403*</td>
<td>1.403*</td>
<td>1.403*</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>2.621*</td>
<td>2.621*</td>
<td>1.596</td>
<td></td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>5.010*</td>
<td>2.835</td>
<td>1.687</td>
<td></td>
</tr>
<tr>
<td>Outer Radius</td>
<td>5</td>
<td>1.343</td>
<td>1.256</td>
<td>1.104</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>2.377</td>
<td>2.045</td>
<td>1.688</td>
<td></td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>3.896</td>
<td>3.066</td>
<td>2.336</td>
<td></td>
</tr>
</tbody>
</table>

* Failure pressure ratio is independent of layer waviness. Failure pressure ratio is calculated from interlaminar normal compressive stresses at the outer radius of the cylinder, outside the wavy region. That is, \(P_{cr}/P_{o} = 1.403, 2.621,\) and \(5.010\) for the cylinders with \(R/H = 5, 10,\) and \(20,\) respectively, are calculated by dividing the interlaminar normal compressive failure pressure, \(P_{c} = 30\) ksi, by the fiber direction compressive failure pressure, \(P_{o} = 21.389\) ksi, \(11.445\) ksi, and \(5.988\) ksi for the perfect cylinders with \(R/H = 5, 10,\) and \(20,\) respectively.

The fact that the failure pressure ratios shown in the table for the wave at the outer radius of the cylinder are not marked with an asterisk implies that when located there, the wave does increase the interlaminar normal compressive stress state to levels above that present in the perfect cylinder. This increase in the interlaminar normal compressive stress relative to the perfect cylinder, and hence the decrease in failure pressure ratio, is seen to be much less significant for the smaller wave amplitudes, as indicated from Table 7.2 and from Figure 7.2, which shows the tabulated values as relations between \(P_{cr}/P_{o}\) and \(\overline{\delta}\) for the interlaminar normal compression failure mode. Also, for the smaller wave amplitudes, \(\overline{\delta} < 1,\) it is seen from the table and from the figure, that a wave located at the
Figure 7.2. Failure pressure ratios for interlaminar normal compression, $-\sigma_n$, failure mode.
outer radius of all cylinders results in lower failure pressure ratios than a wave located at the inner radius of the cylinders. With these smaller wave amplitudes, the wave has a limited effect on the interlaminar normal compressive stresses. Hence, the lower failure pressure for the wave at the outer radius is a direct result of the higher pressure-induced interlaminar normal compressive stress near the outer radius of the cylinder overshadowing any perturbation in the stress caused by a small amplitude wave at the inner radius of the cylinder. However, with large amplitude waves at the inner radius, this overshadowing effect by the high pressure-induced stresses at the outer radius is overcome by the wave-induced perturbations in stress at the inner radius of the cylinder. For example, referring to Figure 7.2, at wave amplitudes greater than \( \bar{\delta} = 1.0 \), the wave located at the inner radius of a cylinder with \( R/H = 20 \) is seen to result in lower failure pressure ratios than when the wave is located at the outer radius of the cylinder. Similar behavior is seen for a cylinder with \( R/H = 10 \) at wave amplitudes near \( \bar{\delta} = 2.0 \). It is suspected that for wave amplitudes greater than \( \bar{\delta} = 2.0 \), this type of behavior will also take place for a cylinder with \( R/H = 5 \). A possible explanation of this phenomena is given by the bent beam analogy presented earlier in this chapter. It is felt that for the larger wave amplitudes, the interlaminar normal compressive stresses developing in a wave located at the inner radius of the cylinder are higher than for a wave located at the outer radius of the cylinder due to the higher fiber direction compressive stresses at the inner radius of the cylinder, i.e., the Lamé effect [15]. These higher fiber direction compressive stresses at the inner radius of the cylinder induce a higher interlaminar normal compressive stress in the wave located at the inner radius of the cylinder because of the bent beam effect. The wave-induced interlaminar normal compressive stresses in the wave at the inner radius of the cylinder are large enough to overcome those induced in the wave at the outer radius plus the higher pressure-induced interlaminar normal

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compressive stresses at the outer radius of the cylinder. The result is a lower failure pressure ratio for the wave located at the inner radius of the cylinder. This behavior is seen in Figure 7.2 to be most prevalent for the cylinders with larger radii, as indicated by the fact that the slope of the relation between \( P_{Cr}/P_0 \) and \( \bar{\delta} \) for a cylinder with \( R/H=20 \) is greater than the slope for a cylinder with \( R/H=10 \). Note that although interlaminar normal compression failure pressure ratios are not less than unity for any of the wave amplitudes considered in this investigation, it is suspected that for slightly larger wave amplitudes, i.e., \( \bar{\delta} > 2 \), these ratios may become a factor in determining the failure due to layer waviness.

Failure pressure ratios for the interlaminar shear failure mode, \( \tau_{ns} \), are shown in Table 7.3 and are plotted in Figure 7.3. Again, as with interlaminar normal tension, the pressure required to produce interlaminar shear failure in the perfect cylinder is infinite, and the finite values of the tabulated interlaminar shear failure pressure ratios implies the significance of layer waviness in terms of the induced shear stress. The fact that the failure pressure ratios are significantly less than unity for the larger wave amplitudes indicates that interlaminar shear failure may be play a leading role in failure due to layer waviness. It is seen from Figure 7.3 by the virtual overlay of the three relations between \( P_{Cr}/P_0 \) and \( \bar{\delta} \), that for a wave located at the inner radius of the cylinder, cylinder geometry has virtually no effect on the interlaminar shear failure pressure ratios, and hence on the shear stress due to layer waviness. In contrast, for a wave located at the outer radius of the cylinder, a spread in the relations is seen, indicating the influence of cylinder geometry on the shear stress. It is suspected that the fiber direction compressive stresses are responsible for this phenomena. This will be explained further in the following discussion of the fiber direction compression failure mode.
### Table 7.3
Failure Pressure Ratios for Interlaminar Shear, $\tau_{\text{NS}}$, Failure Mode

<table>
<thead>
<tr>
<th>Wave Location</th>
<th>Cylinder Geometry (R/H)</th>
<th>Wave Geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\delta = 1/2$, $\bar{\lambda} = 10$</td>
<td>$\delta = 1$, $\bar{\lambda} = 10$</td>
</tr>
<tr>
<td>Inner Radius</td>
<td>5</td>
<td>1.943</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>1.919</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>1.895</td>
</tr>
<tr>
<td>Outer Radius</td>
<td>5</td>
<td>2.583</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>2.263</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>2.043</td>
</tr>
</tbody>
</table>
Figure 7.3. Failure pressure ratios for interlaminar shear, $\tau_{ns}$, failure mode.
Finally, failure pressure ratios are shown in Table 7.4 for the fiber direction compression failure mode, $-\sigma_s$. These fiber direction compression failure pressure ratios are by far, at least for the smaller wave amplitudes, the lowest ratios of the four failure modes. The failure pressure ratios are plotted in Figure 7.4. From the table and the figure, it is seen that a wave located at the inner radius of the cylinder results in fiber direction compression failure pressure ratios lower than those for a wave located at the outer radius of the cylinder. This physically makes sense due to the higher fiber direction compressive stresses near the inner radius of the cylinder. It is also obvious from Figure 7.4 that fiber direction compression failure pressure ratios decrease virtually in constant proportion to increasing wave amplitude.

<table>
<thead>
<tr>
<th>Wave Location</th>
<th>Cylinder Geometry (R/H)</th>
<th>Wave Geometry</th>
<th>Wave Geometry</th>
<th>Wave Geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\delta = 1 \div 2$, $\lambda = 10$</td>
<td>$\delta = 1$, $\lambda = 10$</td>
<td>$\delta = 2$, $\lambda = 10$</td>
<td></td>
</tr>
<tr>
<td>Inner Radius</td>
<td>5</td>
<td>0.897</td>
<td>0.770</td>
<td>0.584</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.883</td>
<td>0.751</td>
<td>0.562</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.877</td>
<td>0.750</td>
<td>0.562</td>
</tr>
<tr>
<td>Outer Radius</td>
<td>5</td>
<td>0.967</td>
<td>0.819</td>
<td>0.646</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.947</td>
<td>0.799</td>
<td>0.623</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.912</td>
<td>0.767</td>
<td>0.599</td>
</tr>
</tbody>
</table>
Figure 7.4. Failure pressure ratios for fiber direction compression, $-\sigma_S$, failure mode.
Of particular interest is the behavior of the fiber direction compression failure pressure ratios seen in Figure 7.4 which is very similar to the behavior seen for the interlaminar shear failure pressure ratios in Figure 7.3. Namely, for a wave located at the inner radius of the cylinder, cylinder geometry has very little effect on the fiber direction compression failure pressure ratios, as indicated by the near coincidence of the relations between $P_{cr}/P_o$ and $\delta$ for a wave at the inner radius. For a wave located at the outer radius, however, cylinder geometry seems to have a significant effect on the ratios, as indicated by the obvious spread in the relations for a wave at the outer radius. The similarity of this behavior for the two failure modes implies that the behavior is derived from the same source. It is felt that the source responsible for this behavior is related to the normalizing pressure, $P_o$, which is more closely related to the wave-induced failure pressure, $P_{cr}$, for a wave located at the inner radius of the cylinder than for a wave located at the outer radius of the cylinder. The closer relation between $P_o$ and $P_{cr}$ for a wave located at the inner radius of the cylinder exists because the fiber direction compressive stress at the inner radius of the perfect cylinder used to calculate $P_o$ acts as the far-field fiber direction stress, i.e., the circumferential stress at circumferential locations far from the wave, for a wave located at the inner radius of the cylinder. For a wave located at the outer radius of the cylinder, the far-field fiber direction stress acting on the wave is less than that for a wave at the inner radius, due to the often-mentioned Lamé effect [15] in thick cylinders. Hence, the far-field fiber direction stress acting on a wave at the outer radius is less than that used to calculate $P_o$. The Lamé effect is illustrated in Table 7.5 which shows the far-field fiber direction stresses at the inner radius and outer radius of each of the three cylinder geometries subjected to a unit hydrostatic pressure. The change in the fiber direction stress from the inner radius to the outer radius of each of the cylinders is clearly seen in the table as a percent change.
Also, each of the cylinder geometries exhibit a different percent change. Specifically, the cylinder with $R/H=20$ has a far-field fiber direction stress at the outer radius 5.69% less in magnitude than the far-field fiber direction stress at its inner radius. The cylinder with $R/H=5$ and 10 have variations of 9.99% and 9.09%, respectively. To form a failure pressure ratio for a wave located at the outer radius of a cylinder with $R/H=20$, the failure pressure for this geometry is normalized by a value of $P_0$ which is calculated based on a fiber direction stress at the inner radius of the perfect cylinder with $R/H=20$, i.e., a fiber direction stress which is roughly 5.69% greater in magnitude than the far-field fiber direction stress acting on the wave at the outer radius of the cylinder. Likewise, the failure pressures for waves located at the outer radius of cylinders with $R/H=5$ and 10 are normalized by values of $P_0$ calculated from fiber direction stresses at the inner radius of perfect cylinders with $R/H=5$ and 10, i.e., fiber direction stresses which are roughly 9.99% and 9.09% greater in magnitude than the far-field fiber direction stresses acting on the wave at the outer radius of these cylinders. It is felt that the variation of percentage difference in the normalizing pressures with respect to $R/H$ is directly responsible for the relations in Figure 7.4 between $P_{cr}/P_0$ and $\delta$ for waves located at the outer radius of the cylinders not being coincident. Examining the relations closely, one sees that the lack of coincidence from the inner to outer wave location for a given cylinder geometry is in close proportion to the percentage difference for each cylinder geometry, i.e., 9.99%, 9.09%, and 5.69%, for $R/H=5$, 10, and 20, respectively.

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Table 7.5
Fiber Direction Stresses for Cylinders with
R/H=5, 10, and 20 Subjected to Unit Hydrostatic Pressure

<table>
<thead>
<tr>
<th>Cylinder Geometry (R/H)</th>
<th>Far-field Fiber Direction Stresses $\sigma_s$ (psi)</th>
<th>% Change</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Inner Radius</td>
<td>Outer Radius</td>
</tr>
<tr>
<td>5</td>
<td>-8.41</td>
<td>-7.57</td>
</tr>
<tr>
<td>10</td>
<td>-15.73</td>
<td>-14.30</td>
</tr>
<tr>
<td>20</td>
<td>-30.06</td>
<td>-28.35</td>
</tr>
</tbody>
</table>

It was suggested in the previous discussion of interlaminar shear failure pressure ratios that the fiber direction compressive stresses are responsible for the behavior of the interlaminar shear failure pressure ratios. The evidence for this is provided by the fact that the behavior of the interlaminar shear failure pressure ratios is virtually identical to the behavior of the fiber direction compression failure pressure ratios. Namely, the relations between $P_{cr}/P_0$ and $\bar{\delta}$ for interlaminar shear failure shown in Figure 7.3 are independent of cylinder geometry for a wave located at the inner radius of the cylinder, while the relations are dependent of cylinder geometry for a wave located at the outer radius of the cylinder. The reasoning just presented regarding the variation in the normalizing failure pressure, $P_0$, provides an explanation for the behavior of the interlaminar shear failure pressure ratios, and implies that interlaminar shear stresses are greatly affected by the fiber direction stresses.

7.1.2 Failure Mode Interaction

Results are again presented in this section in terms of failure pressure ratios, but this time, additional insight into the response due to layer waviness is gained by focusing

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attention on the interaction of the failure modes. To investigate this interaction, all four failure modes, $+\sigma_n$, $-\sigma_n$, $\tau_{ns}$, and $-\sigma_s$, will be studied simultaneously for both wave locations and all three wave geometries, while independently considering each of the three cylinder geometries. As in the figures shown in the previous section, the dotted line at $P_{cr}/P_0=1$ indicates a threshold, below which failure will occur at a pressure lower than that required to produce standard fiber direction compression failure at the inner radius of the cylinder.

The failure pressure ratios for a cylinder with $R/H=5$ are shown in Table 7.6. The table is arranged similarly to those in the previous section, but instead of each row representing a different cylinder geometry, each row now represents a different failure mode. For a given wave location, i.e., inner radius or outer radius, the first row, representing the interlaminar normal tension failure mode, $+\sigma_n$, is followed by the second, third, and fourth rows representing the interlaminar normal compression failure mode, $-\sigma_n$, the interlaminar shear failure mode, $\tau_{ns}$, and the fiber direction compression failure mode, $-\sigma_s$, respectively. For each row, the wave geometry is again arranged in three columns with the wave amplitude increasing from left to right.
Table 7.6
Failure Pressure Ratios for a Cylinder with R/H=5

<table>
<thead>
<tr>
<th>Wave Location</th>
<th>Failure Mode</th>
<th>Wave Geometry</th>
<th>( \bar{\delta} = 1/2, \bar{\lambda} = 10 )</th>
<th>( \bar{\delta} = 1, \bar{\lambda} = 10 )</th>
<th>( \bar{\delta} = 2, \bar{\lambda} = 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner Radius</td>
<td>+( \sigma_n )</td>
<td>10.315</td>
<td>3.198</td>
<td>1.122</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-( \sigma_n )</td>
<td>1.403*</td>
<td>1.403*</td>
<td>1.403*</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \tau_{ns} )</td>
<td>1.943</td>
<td>1.052</td>
<td>0.623</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-( \sigma_s )</td>
<td>0.897</td>
<td>0.770</td>
<td>0.584</td>
<td></td>
</tr>
<tr>
<td>Outer Radius</td>
<td>+( \sigma_n )</td>
<td>N/A**</td>
<td>N/A**</td>
<td>N/A**</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-( \sigma_n )</td>
<td>1.343</td>
<td>1.256</td>
<td>1.104</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \tau_{ns} )</td>
<td>2.583</td>
<td>1.243</td>
<td>0.648</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-( \sigma_s )</td>
<td>0.967</td>
<td>0.819</td>
<td>0.646</td>
<td></td>
</tr>
</tbody>
</table>

* Failure pressure ratio is independent of layer waviness. Failure pressure ratio is calculated from interlaminar normal compressive stresses at the outer radius of the cylinder, outside the wavy region.

** Interlaminar normal tensile stresses do not occur for any of the wave geometries located at the outer radius of the cylinder with R/H=5.

The failure pressure ratios from Table 7.6 are plotted in Figure 7.5. Figure 7.5a and 7.5b represent the same data, but the range of values of \( P_{cr}/P_0 \) is reduced in Figure 7.5b to more closely examine the failure mode interaction taking place near \( P_{cr}/P_0=1.0 \). As indicated by the double asterisks in Table 7.6, interlaminar normal tensile stresses never occur for any of the wave geometries located at the outer radius of the cylinder with R/H=5. Those interlaminar normal tensile stresses which occur for waves located at the inner radius of the cylinder with R/H=5 all result in failure pressure ratios which are greater than unity. Hence, the interlaminar normal tension failure mode is not critical for this cylinder geometry.

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b) The range of failure pressure ratios in a) has been reduced in b) to show the details around the more critical failure pressure ratios.

Figure 7.5. Failure pressure ratios for all wave geometries in a cylinder with R/H=5.
As indicated by the single asterisk in Table 7.6 and by the horizontal line at \( P_{cr}/P_0 = 1.403 \) in Figure 7.5, interlaminar normal compressive stresses within the wavy region for a wave located at the inner radius of a cylinder with \( R/H = 5 \) never exceed the interlaminar normal compressive stresses which occur at the outer radius of the cylinder. Hence, the failure pressure ratio for all three wave geometries is shown to be 1.403, the ratio calculated from the applied pressure causing compression failure of the outer surface of the cylinder, the impossible situation alluded to earlier. As will be shown shortly, for cylinders with larger radii, i.e., cylinders with \( R/H = 10 \) and 20, the interlaminar normal compressive stresses within the wavy region of a wave located at the inner radius of the cylinder do exceed those at the outer radius of the cylinder for some values of \( \delta \). For a wave located at the outer radius of the cylinder with \( R/H = 5 \), interlaminar normal compressive failure pressure ratios are seen to decrease with increasing wave amplitude, but all ratios remain above unity. In summary, then, the interlaminar normal compressive failure mode is therefore noncritical for a cylinder with \( R/H = 5 \).

At least for the larger wave amplitudes, the interlaminar shear stresses induced by the wave located at either the inner radius or the outer radius of a cylinder with \( R/H = 5 \) result in failure pressure ratios significantly less than unity. From Figure 7.5b, it is seen that the slopes of the relations between \( P_{cr}/P_0 \) and \( \delta \) for the interlaminar shear failure pressure ratios are slightly different for the two wave locations, with the greater slope occurring for the wave located at the outer radius. In the previous section, it was stated that due to their similar behavior, the fiber direction stresses were responsible for the behavior of the shear stresses. It would appear from the variation in the slope of the relations between \( P_{cr}/P_0 \) and \( \delta \) for interlaminar shear shown in Figure 7.5b, that the interlaminar normal compressive stresses may also contribute to the behavior of the shear stresses.
stresses, i.e., the higher pressure-induced interlaminar normal compressive stress at the outer radius of the cylinder may account for the greater slope in the relation between $P_{cr}/P_o$ and $\bar{\delta}$ for the interlaminar shear failure pressure ratio for a wave located at the outer radius of the cylinder. Similar variations in slope of the relations for interlaminar shear failure pressure ratios will be seen for the other cylinder geometries.

Finally, all of the fiber direction compressive failure pressure ratios in Table 7.6 are seen to be less than unity. From Figure 7.5b, this fiber direction compression failure mode is seen to overshadow all other failure modes for all wave geometries and both wave locations in a cylinder with $R/H=5$. It is very interesting to note, however, that at large wave amplitudes, a change in failure mode has almost occurred from the fiber direction compression failure mode to the interlaminar shear failure mode. Namely, for a wave with $\bar{\delta}=2$ located at the outer radius of the cylinder, the failure pressure ratios for the interlaminar shear and fiber direction compression failure modes are virtually identical. Due to the greater slope of the relation between $P_{cr}/P_o$ and $\bar{\delta}$ for the interlaminar shear failure mode, it is suspected that for wave amplitudes $\bar{\delta} > 2$, the interlaminar shear failure mode will overshadow the fiber direction compression failure mode and become the predicted mode of failure for the cylinder.

Failure pressure ratios for a cylinder with $R/H=10$ are shown in Table 7.7 and are plotted in Figure 7.6. As done previously, the scale has been changed in Figure 7.6b to better show what is occurring near $P_{cr}/P_o=1.0$. Remarks similar to those stated for the cylinder with $R/H=5$ can be made about the cylinder with $R/H=10$. Some interesting points to note, however, are that the failure pressure ratios calculated from the interlaminar normal tensile stresses, are for the first time, less than unity, at least for a wave with $\bar{\delta}=2$ located at the inner radius of the cylinder. As seen in Figure 7.6a, for a wave located at the inner radius of the cylinder, the large slope of the relation between

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$P_{cr}/P_o$ and $\bar{\delta}$ for interlaminar normal tension implies that this mode of failure is rapidly becoming important for larger wave amplitudes. It is also evident by the presence of a failure pressure ratio in Table 7.7, that significant interlaminar normal tensile stresses are beginning to occur in a cylinder with a wave amplitude of $\bar{\delta}=2$ located at the outer radius of the cylinder. The combination of these two characteristics of the interlaminar normal tensile stress implies that increasing the radius of the cylinder, results in increased wave-induced interlaminar normal tensile stresses, especially for wave amplitudes between $\bar{\delta}=1/2$ and $\bar{\delta}=1$. It was stated in the previous section, however, that for wave amplitudes $\bar{\delta} > 2$, it is anticipated that an interlaminar normal tension failure will occur for the cylinder with the lower value of $R/H$, i.e., $R/H=5$, at a value of $P_{cr}/P_o$ smaller than for a cylinder with a higher value of $R/H$. 

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Table 7.7
Failure Pressure Ratios for a Cylinder with $R/H=10$

<table>
<thead>
<tr>
<th>Wave Location</th>
<th>Failure Mode</th>
<th>$\delta = 1/2, \bar{\lambda} = 10$</th>
<th>$\delta = 1, \bar{\lambda} = 10$</th>
<th>$\delta = 2, \bar{\lambda} = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner Radius</td>
<td>$+\sigma_n$</td>
<td>6.298</td>
<td>1.939</td>
<td>0.883</td>
</tr>
<tr>
<td></td>
<td>$-\sigma_n$</td>
<td>2.621*</td>
<td>2.621*</td>
<td>1.596</td>
</tr>
<tr>
<td></td>
<td>$\tau_{ns}$</td>
<td>1.919</td>
<td>1.021</td>
<td>0.591</td>
</tr>
<tr>
<td></td>
<td>$-\sigma_s$</td>
<td>0.883</td>
<td>0.751</td>
<td>0.562</td>
</tr>
<tr>
<td>Outer Radius</td>
<td>$+\sigma_n$</td>
<td>N/A**</td>
<td>N/A**</td>
<td>1.935</td>
</tr>
<tr>
<td></td>
<td>$-\sigma_n$</td>
<td>2.377</td>
<td>2.045</td>
<td>1.688</td>
</tr>
<tr>
<td></td>
<td>$\tau_{ns}$</td>
<td>2.263</td>
<td>1.107</td>
<td>0.568</td>
</tr>
<tr>
<td></td>
<td>$-\sigma_s$</td>
<td>0.947</td>
<td>0.799</td>
<td>0.623</td>
</tr>
</tbody>
</table>

* Failure pressure ratio is independent of layer waviness. Failure pressure ratio is calculated from interlaminar normal compressive stresses at the outer radius of the cylinder, outside the wavy region.

** Interlaminar normal tensile stresses do not occur for these wave geometries located at the outer radius of the cylinder with $R/H=10$. 

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b) The range of failure pressure ratios in a) has been reduced in b) to show the details around the more critical failure pressure ratios.

Figure 7.6. Failure pressure ratios for all wave geometries in a cylinder with R/H=10.
The wave-induced interlaminar normal compressive stresses from a wave located at the inner radius of a cylinder with \( R/H = 10 \) exceed the pressure-induced interlaminar normal compressive stresses at the outer radius of the cylinder. Evidence of this is provided in Table 7.7 by the value of \( P_{cr}/P_o \) for a wave with \( \delta = 2 \) which is less than the value of \( P_{cr}/P_o = 2.621 \), and in Figure 7.6a, which shows a deviation from the horizontal line at \( P_{cr}/P_o = 2.621 \) of the relation between \( P_{cr}/P_o \) and \( \delta \) for interlaminar normal compression for a wave located at the inner radius. This interlaminar normal compressive failure mode, however, remains insignificant at this point due to the failure pressure ratios which are greater than unity.

The interlaminar shear and fiber direction compression failure pressure ratios for a cylinder with \( R/H = 10 \) exhibit behavior, seen in Figure 7.6b, very similar to that for a cylinder with \( R/H = 5 \). For the cylinder with \( R/H = 10 \), however, a change in failure mode does occur at large wave amplitudes. This change takes place for a wave located at the outer radius of the cylinder with \( \delta \) between 1.75 and 2. It is suspected that for wave amplitudes just above \( \delta = 2 \), a similar change in failure mode will occur for a wave located at the inner radius of a cylinder with \( R/H = 10 \).

Failure pressure ratios for a cylinder with \( R/H = 20 \) are shown in Table 7.8 and plotted in Figure 7.7. As seen in Figure 7.7a, the relation between \( P_{cr}/P_o \) and \( \delta \) for the interlaminar normal tension failure pressure ratios for a wave located at the inner radius of a cylinder with \( R/H = 20 \) is not as steep as was seen for the cylinder with \( R/H = 10 \). The slope of the relation is, however, still significant and implies that the interlaminar normal tension failure mode may dominate for wave amplitudes \( \delta > 2 \). This is particularly the case for the wave located at the outer radius of the cylinder, a condition where the relation between \( P_{cr}/P_o \) and \( \delta \) has slope even greater than that for the wave at the inner radius of the cylinder.

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Table 7.8
Failure Pressure Ratios for a Cylinder with R/H=20

<table>
<thead>
<tr>
<th>Wave Location</th>
<th>Failure Mode</th>
<th>Wave Geometry</th>
<th>( \delta = 1/2, \lambda = 10 )</th>
<th>( \delta = 1, \lambda = 10 )</th>
<th>( \delta = 2, \lambda = 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner</td>
<td>+( \sigma_n )</td>
<td>3.886</td>
<td>1.626</td>
<td>0.829</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-( \sigma_n )</td>
<td>5.010*</td>
<td>2.835</td>
<td>1.687</td>
<td></td>
</tr>
<tr>
<td>Radius</td>
<td>( \tau_{ns} )</td>
<td>1.895</td>
<td>1.014</td>
<td>0.590</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-( \sigma_s )</td>
<td>0.877</td>
<td>0.750</td>
<td>0.562</td>
<td></td>
</tr>
<tr>
<td>Outer</td>
<td>+( \sigma_n )</td>
<td>N/A**</td>
<td>2.281</td>
<td>0.836</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-( \sigma_n )</td>
<td>3.896</td>
<td>3.066</td>
<td>2.336</td>
<td></td>
</tr>
<tr>
<td>Radius</td>
<td>( \tau_{ns} )</td>
<td>2.043</td>
<td>1.015</td>
<td>0.521</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-( \sigma_s )</td>
<td>0.912</td>
<td>0.767</td>
<td>0.599</td>
<td></td>
</tr>
</tbody>
</table>

* Failure pressure ratio is independent of layer waviness. Failure pressure ratio is calculated from interlaminar normal compressive stresses at the outer radius of the cylinder, outside the wavy region.
** Interlaminar normal tensile stresses do not occur for this wave geometry located at the outer radius of the cylinder with R/H=20.

The insignificance of the interlaminar normal compression failure mode is again noted for this cylinder geometry by the interlaminar normal compression failure pressure ratios in Table 7.8 which are greater than unity for all wave geometries considered.

Finally, very similar trends to those seen for the cylinders with R/H=5 and 10 are seen for the cylinder with R/H=20 in terms of the interlaminar shear failure pressure ratios and the fiber direction compression failure pressure ratios. They dominate! However, as important, a change in failure mode is again seen for the case of a wave located at the outer radius of the cylinder with R/H=20, whereby the interlaminar shear failure mode dominates over the fiber direction compression failure mode for wave

Chapter 7: Failure Predictions
b) The range of failure pressure ratios in a) has been reduced in b) to show the details around the more critical failure pressure ratios.

Figure 7.7. Failure pressure ratios for all wave geometries in a cylinder with R/H=20.
amplitudes greater than approximately $\bar{\delta}=1.75$. From the greater slope of its relation between $P_{cr}/P_o$ and $\bar{\delta}$ in Figure 7.7b, similar domination of the interlaminar shear failure mode over the fiber direction compression failure mode is anticipated for wave amplitudes greater than $\bar{\delta}=2$ for a wave located at the inner radius of a cylinder with $R/H=20$.

7.1.3 Most Critical Failure Pressure Ratios

A summary of the most critical failure pressure ratios is presented in this section for all of the cylinder geometries and wave geometries considered in this investigation. These most critical failure pressure ratios are shown in Table 7.9 along with the expected mode of failure, i.e., fiber direction compression failure, $-\sigma_s$, or interlaminar shear failure, $\tau_{ns}$. Interlaminar normal tension and compression failure modes are not a factor. As expected, the largest wave amplitude, $\bar{\delta}=2$, results in the lowest failure pressure ratios, indicating that the largest wave amplitudes are most detrimental to the pressure capacity of the cylinder. The worst of the cases, a wave with $\bar{\delta}=2$ located at the outer radius of a cylinder with $R/H=20$, is seen to reduce the pressure capacity of the cylinder to 52% of the original capacity, i.e., $P_{cr}/P_o=0.521$. The tabulated failure pressure ratios are shown as a function of $\bar{\delta}$ in Figure 7.8. From the slope of the relations between $P_{cr}/P_o$ and $\bar{\delta}$, it can be expected that further reductions in pressure capacity of the cylinders will result for even larger wave amplitudes.
Table 7.9
Most Critical Failure Pressure Ratios for all Cylinder and Wave Geometries

<table>
<thead>
<tr>
<th>Wave Location</th>
<th>Cylinder Geometry (R/H)</th>
<th>Wave Geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\bar{d} = 1/2$, $\bar{\lambda} = 10$</td>
<td>$\bar{d} = 1$, $\bar{\lambda} = 10$</td>
</tr>
<tr>
<td>Inner Radius</td>
<td>5</td>
<td>0.897 (-$\sigma_s$)</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.883 (-$\sigma_s$)</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.877 (-$\sigma_s$)</td>
</tr>
<tr>
<td>Outer Radius</td>
<td>5</td>
<td>0.967 (-$\sigma_s$)</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.947 (-$\sigma_s$)</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.912 (-$\sigma_s$)</td>
</tr>
</tbody>
</table>
Figure 7.8. Most critical failure pressure ratios.
This completes the rather detailed analysis of failure caused by layer waviness. Attention is now focused on the failure response of the perfect cylinder due to buckling to see how failure due to buckling compares with failure due to layer waviness.

7.2 BUCKLING FAILURE PRESSURE RATIOS

Results are presented in this section regarding the failure of the perfect cylinder, i.e., no wave included, due to buckling. It was stated in Chapter 3 that this analysis is performed to address the issue of whether or not failure of the perfect cylinder due to buckling occurs prior to material failure due to wavy layer imperfections. It was also stated that since the minimum critical failure pressure due to buckling is of interest, only the first buckling mode is considered.

A total of six perfect cylinder geometries were analyzed to determine their buckling response. These six geometries included each of the three cylinder geometries, i.e., \( R/H = 5, 10, \text{ and } 20 \), which were each analyzed for length to radius ratios, \( L/R \), of 1/2 and 1. The deformed cylinder geometries corresponding to the first buckling mode are shown in Figure 7.9 for each of the six geometries. The buckled mode shapes of each of the six geometries are characterized by a single half-wave in the axial direction and varying numbers of full waves in the circumferential direction, depending on the particular cylinder geometry. Specifically, the number of full waves seen in the circumferential direction are: 6 for \( R/H = 20, \ L/R = 1/2 \); 5 for \( R/H = 20, \ L/R = 1 \); 5 for \( R/H = 10, \ L/R = 1/2 \); 4 for \( R/H = 10, \ L/R = 1 \); and 4 for both \( R/H = 5, \ L/R = 1/2 \) and \( R/H = 5, \ L/R = 1 \). The number of waves are shown in the figure alongside the corresponding buckled cylinder geometry.
(m,n): m=number of axial half-waves, n=number of circumferential full waves

a) R/H=20, L/R = 1/2 (1,6)  
b) R/H=20, L/R = 1 (1,5)

c) R/H=10, L/R=1/2 (1,5)  
d) R/H=10, L/R=1 (1,4)

e) R/H=5, L/R=1/2 (1,4)  
f) R/H=5, L/R=1 (1,4)

Figure 7.9. Deformed cylinder geometries corresponding to first buckling mode.
Critical buckling failure pressures, \( P_{\text{cr, buckling}} \), were calculated from the eigenvalue solutions, and buckling failure pressure ratios, \( P_{\text{cr, buckling}}/P_0 \), were calculated by normalizing the buckling failure pressure by the failure pressure required to cause material failure in the perfect cylinder. These buckling failure pressure ratios are shown in Table 7.11 for each of the six cylinder geometries.

### Table 7.11
Buckling Failure Pressure Ratios

<table>
<thead>
<tr>
<th>Cylinder Geometry</th>
<th>Failure Pressure Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R/H )</td>
<td>( L/R )</td>
</tr>
<tr>
<td>5</td>
<td>1/2</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>1/2</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>20</td>
<td>1/2</td>
</tr>
<tr>
<td>20</td>
<td>1</td>
</tr>
</tbody>
</table>

It is seen from Table 7.11 that the only buckling failure pressure ratio less than unity, and hence of any concern, is that for the cylinder with \( R/H = 20 \) and \( L/R = 1 \). To compare this buckling failure pressure ratio with those calculated for material failure due to layer waviness, Figure 7.8 has been modified to include the buckling failure pressure ratio for a cylinder with \( R/H = 20 \) and \( L/R = 1 \). This buckling failure pressure ratio is seen as a dotted line in Figure 7.10. From Figure 7.10, it is seen that failure of the perfect cylinder with \( R/H = 20 \) due to buckling will occur at nearly the same pressure as material
Figure 7.10. Most critical failure pressure ratios, buckling considerations included.
failure of the same cylinder geometry due to a wave with \( \delta = 1.1 \) located at its inner radius. In a similar manner, failure of the perfect cylinder with \( R/H = 20 \) due to buckling will occur at nearly the same pressure as material failure of the cylinder due to a wave with \( \delta = 1.2 \) located at its outer radius.

From this superficial buckling analysis of the perfect cylinder, it is seen that buckling of the perfect cylinder should be considered as a possible failure mechanism which may occur prior to material failure of a cylinder due to layer waviness. Although the analysis is beyond the scope of this investigation, it is suspected that the incorporation of layer waviness into the perfect cylinder geometry may tend to decrease the buckling strength of the cylinder. For such a case, failure of the imperfect cylinder due to buckling will be even more critical than that seen here for the perfect cylinder. Hence, failure due to buckling needs to be considered as a possible mechanism of failure in a cylinder containing layer waviness.

This completes the discussion of failure response due to layer waviness. Before altogether concluding the discussion of layer waviness, it is important to compare the results obtained in the current study with available past work. This is done in the following chapter.
CHAPTER 8

COMPARISONS WITH PREVIOUS WORK

Comparisons are presented in this chapter which correlate the results obtained in the current investigation with results from previous work regarding cylinder response due to layer waviness. The predicted stress distributions in and around the wave as well as predicted failure due to layer waviness are compared.

8.1 INTERLAMINAR STRESS DISTRIBUTIONS

Results are presented in this section regarding comparisons of wavy layer analyses performed in the current investigation and in the previous investigation by Telegadas and Hyer [10,11]. The analyses in the current investigation and the previous investigation regarding a wave located at the inner radius of a cylinder with R/H=5 are very similar. However, three principal differences between the investigations do exist. The first difference is that the current investigation models the true cylindrical geometry, whereas the previous investigation modeled a rectangular geometry and applied special boundary conditions to the model in an attempt to simulate the response of the cylindrical geometry. The second difference is that the current investigation models a wavy sublamine which is 14 layers in thickness, whereas the previous investigation modeled a wavy sublamine which was 20 layers in thickness. The third and final difference is that the cylinder modeled in the current investigation is 104 layers thick, whereas the cylinder modeled in the previous investigation was only 102 layers thick. Despite these differences, it is felt that the two investigations are worthy of comparison, especially since they model identical wave parameters, $\bar{\delta}$ and $\bar{\lambda}$, and cylinder geometry parameters,
R/H, at least for the case of the wave located at the inner radius of a cylinder with R/H=5. To compare results, since the previous investigation was in rectangular coordinates, the stress distributions from that investigation are shown as a function of normalized distance, \(X/X^*\), where \(X^*\) represents the location at which the wavy region ends, analogous to \(\theta^*\) in the current investigation.

The first comparison demonstrates the similarity in the stress distributions at the midinterface of each of the wavy sublaminates. As seen in Figure 8.1a, there is a difference in the value of interlaminar normal stresses between the two investigations, with the previous investigation predicting stresses significantly greater in magnitude than those predicted by the current investigation. Although this difference in the magnitude of the stresses exists, the trends in the distributions are very similar. For example, for both investigations, the maximum interlaminar normal compressive stress is seen to occur between the peak of the wave, \(\theta/\theta^*=X/X^*=0.0\), and the inflection point of the wave, \(\theta/\theta^*=X/X^*=0.5\). Similarly, for both investigations, the minimum interlaminar normal compressive stress is seen to occur near the end of the wave, \(\theta/\theta^*=X/X^*=1.0\).

Two distinct differences are noted for the two investigations in terms of the trends in the interlaminar normal stress. The first of the differences occurs at the peak of the wave, \(\theta/\theta^*=X/X^*=0.0\). The current investigation implies that at the peak of the wave, interlaminar normal compressive stress is not independent of wave amplitude, as indicated by the varying level of stress for the three wave amplitudes. In contrast, the previous investigation implies that at the peak of the wave, interlaminar normal stress is independent of wave amplitude. The second of the differences occurs at the circumferential boundary of the model, i.e., \(\theta/\theta^*=2.0\) for the current investigation and \(X/X^*=3.0\) for the previous investigation. The current investigation shows that the value of interlaminar normal stress at \(\theta/\theta^*=2.0\) is independent of wave amplitude. Assuming

Chapter 8: Comparisons with Previous Work
a) Interlaminar normal stress, $\sigma_n$.

b) Interlaminar shear stress, $\tau_{ns}$.

Figure 8.1. Comparison with previous work for a wave located at the inner radius of a cylinder: $R/H=5$, $\bar{\delta}=1/2$, 1, and 2, $\bar{\lambda}=10$.

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that the effect of the wave attenuates by $\theta/\theta^* = 2.0$, this independence is correct. In contrast, the previous investigation shows that the value of interlaminar normal stress at $X/X^* = 3.0$ is not independent of wave amplitude.

It is known that the boundary conditions imposed on the rectangular geometry in the previous investigation did not accurately model the true response for the actual cylindrical geometry. Specifically, with a rectangular model, the decrease in the radial stress with radial distance from the outer radius of the cylinder, where the pressure boundary condition is applied, simply could not be represented. With the rectangular model, a 'radial' stress applied at the upper boundary would have to be counterbalanced by a 'radial' stress on the lower boundary of exactly the same magnitude. With a cylindrical geometry, the stress on the bottom boundary is less, owing to circumferential stresses contributing to equilibrium. With a rectangular geometry, there can be no circumferential stress. Thus in a rectangular model, the interlaminar normal stress is wrong by a factor proportional to radial distance through the wavy layer region. As can be seen in Figure 8.1a, by simply shifting the past results upward, they would agree fairly well with the current investigation.

The interlaminar shear stress distributions shown in Figure 8.1b indicate that in addition to the trends in the interlaminar shear stress, the values of maximum interlaminar shear stress predicted by the two investigations are very similar. Although the previous investigation predicts interlaminar shear stresses which are slightly greater than those predicted by the current investigation, the only appreciable difference in the value of maximum interlaminar shear stress occurs for a wave with $\bar{\delta} = 2$.

Although they are very similar, there are variations in the trends of the interlaminar shear stress distributions for the two investigations which should be pointed out. First of all, as was seen for the interlaminar normal stress distribution, the current investigation

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imply a dependence on wave amplitude of the value of interlaminar shear stress at the peak of the wave, \( \theta/\theta^* = X/X^* = 0.0 \). The previous investigation again shows no dependence on wave amplitude of the value of interlaminar shear stress at the peak of the wave. A second variation in the interlaminar shear stress distributions between the two investigations occurs near the end of the wave, \( \theta/\theta^* = X/X^* = 1.0 \). The current investigation shows no change in sign of the interlaminar shear stress near the end of the wave, whereas the previous investigation does. It is suspected that both of these variations in the interlaminar shear stress distributions occur as a result of the different application of boundary conditions for the two investigations. It is felt that the true cylindrical geometry incorporated in the current investigation more accurately predicts the interlaminar shear stresses due to layer waviness.

8.2 FAILURE PRESSURE RATIOS

The second comparison between the current and previous investigations looks at the predicted failure pressure ratios and the predicted mode of failure. The most critical failure pressure ratios predicted by each of the investigations are presented in Table 8.1. These ratios predict the material failure response for each of the three wave geometries for a wave located at the inner radius of a cylinder with \( R/H = 5 \). For all of the wave geometries, the current investigation predicts a fiber direction compression failure, \(-\sigma_s\). The previous investigation predicts a fiber direction compression failure for the wave geometries with \( \delta = 1/2 \) and 1, but an interlaminar shear failure, \( \tau_{ns} \), for the wave with \( \delta = 2 \). For the first two wave geometries, \( \delta = 1/2 \) and 1, the current investigation predicts lower failure pressure ratios than those predicted by the previous investigation, implying that the current investigation predicts fiber direction stresses due to layer waviness greater in magnitude than those predicted by the previous investigation. The fact that the
previous investigation predicts a lower failure pressure ratio and an interlaminar shear failure mode for the wave with $\bar{\delta}=2$, implies that the previous investigation predicts interlaminar shear stresses due to layer waviness which are greater in magnitude than those predicted by the current investigation. It is interesting to note that, referring back to Table 7.6 and Figure 7.5, interlaminar shear failure almost occurs for $\bar{\delta}=2$, i.e., failure pressure ratios of 0.648 and 0.646 for the interlaminar shear and fiber direction compression failure modes, respectively. It was also stated previously for the current investigation that interlaminar shear failure is expected to occur for wave amplitudes slightly greater than two.

<table>
<thead>
<tr>
<th>Wave Geometry</th>
<th>Failure Pressure Ratios ($P_{cr}/P_o$)</th>
<th>Current Work</th>
<th>Previous Work</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{\delta}=1/2, \bar{\lambda}=10$</td>
<td>0.897 ($-\sigma_s$)</td>
<td>0.948 ($-\sigma_s$)</td>
<td></td>
</tr>
<tr>
<td>$\bar{\delta}=1, \bar{\lambda}=10$</td>
<td>0.770 ($-\sigma_s$)</td>
<td>0.844 ($-\sigma_s$)</td>
<td></td>
</tr>
<tr>
<td>$\bar{\delta}=2, \bar{\lambda}=10$</td>
<td>0.584 ($-\sigma_s$)</td>
<td>0.571 ($\tau_{ns}$)</td>
<td></td>
</tr>
</tbody>
</table>

* failure pressure ratios for a wave located at the inner radius of a cylinder with $R/H=5$.

From the results presented in this section, it can be concluded that the current and previous investigations predict similar behavior due to layer waviness. The current investigation predicts a greater influence by the fiber direction compressive stresses, whereas the previous investigation predicts a greater influence by the interlaminar shear stresses. From the more accurate representation of the true cylindrical geometry by the current investigation, it is felt that the stresses due to layer waviness are more accurately

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predicted by the current investigation, but that results presented in the previous investigation, including additional effects not considered in the current investigation, i.e., thermal effects and additional wave geometry effects, should not be abandoned.

This concludes the discussion of cylinder response due to layer waviness. The following chapter summarizes the important conclusions made throughout the discussion. In addition, aspects of layer waviness which have not been considered in the current and in previous investigations are outlined as recommendations for future work.
CHAPTER 9

CONCLUSIONS AND RECOMMENDATIONS

This study has investigated the influence of layer waviness in thick cross-ply composite cylinders subjected to hydrostatic pressure. The cylinders considered contain 104 total layers with a layup of $[90/(90/0/90)_{17}]_S$, where a $0^\circ$ layer is taken to be in the axial direction. Layer waviness in only the circumferential direction has been considered and was present in a single isolated group of 14 layers in an otherwise perfect cylinder. The influences of wave amplitude, wave location, and cylinder geometry on the stresses within and near the wavy layers have been evaluated. Failure has been studied in detail. In addition, the buckling response of the perfect cylinder has been evaluated and compared with the response of the cylinder with included layer waviness.

9.1 CONCLUSIONS

A summary of conclusions is presented in this section regarding modeling of the cylinder and layer waviness, and regarding the response predicted by these models. The summary of the conclusions is arranged in a manner similar to that used in the previous chapters, namely, the influence of layer waviness on the stress distributions within the cylinder, the failure response of the cylinder due to layer waviness, and the correlation with previous work regarding layer waviness.
9.1.1 Conclusions Regarding the Modeling of Layer Waviness

General

1. The results presented in Chapter 5 indicate that the finite element models used in this investigation provide and accurate representation of the stress response due to layer waviness.

2. The models offer flexibility in the representation of layer waviness and lend themselves to further investigations to include aspects of layer waviness not considered in this investigation.

Specific

3. Reduced integration elements predict response due to layer waviness very similar to that predicted by the much more computationally intensive full integration elements. This is true, at least, for the cylinder and wave geometries considered in this investigation, i.e., cylinder geometries of R/H=5, 10, and 20, wave amplitudes of two layer thicknesses or less, and half wave lengths of ten layer thicknesses.

4. Modeling the layer waviness using a circumferential segment at least three half wave lengths in circumferential length predicts response very similar to that predicted by segments with greater circumferential length. The shorter segment lengths are adequate for modeling the cylinder and wave geometries considered in this investigation.

5. Half-thickness and full-thickness models predict virtually identical response for the cylinder and wave geometries considered in this investigation.

6. A finite element model incorporating the three modeling simplifications, i.e., reduced integration elements, a circumferential segment, and a half-thickness cylinder wall, predict stresses which exhibit good convergence at locations within
the model where failure is predicted to occur. Poor convergence is exhibited at a small number of locations throughout the model, but the stresses at these locations are overshadowed by the much larger stresses elsewhere in the model.

9.1.2 Conclusions Regarding the Stress State due to Layer Waviness

1. Wave-induced perturbations in the interlaminar normal, interlaminar shear, and fiber direction stresses are localized radially to within 10 layers of the wavy region. Perturbations in the interlaminar normal and interlaminar shear stresses are localized circumferentially to within two half wave lengths of the end of the wave. The extent of the perturbation in the fiber direction stresses in the circumferential direction is unknown, but is suspected to be at least 10 half wave lengths from the end of the wave.

2. The maximum values of interlaminar normal, interlaminar shear, and fiber direction stress always occur within the wavy region, i.e., within the 14 layer wavy sublamine and in the range $0.0 < \theta/\theta^* < 1.0$.

3. The magnitude of the interlaminar normal, interlaminar shear, and fiber direction stresses increases with increasing wave amplitude, and with increasing cylinder radius.

4. The wave's radial location within the cylinder wall does not make a significant impact on the character of the interlaminar normal, interlaminar shear, and fiber direction stresses.

5. The distribution of interlaminar normal, interlaminar shear, and fiber direction stress is independent of wave amplitude for a given wave location and cylinder geometry. Likewise, for a given wave location and wave amplitude, the stress distributions are independent of cylinder radius.
6. Two geometric characteristics, the midinterface of the wavy sublamine and the inflection point of the wave, play a significant role in the locations of the maximum interlaminar normal, interlaminar shear, and fiber direction stresses.

7. The two adjacent circumferential layers at the midradius of the wavy sublamine are in many ways analogous to a bent beam, surrounded by material, and loaded in compression.

8. Layer waviness can be interpreted as a stress concentration with an associated stress concentration factor.

9.3 Conclusions Regarding Failure

1. The largest wave amplitudes are seen to be most detrimental to the pressure capacity of the cylinder. Reductions to 52% of the cylinder’s original pressure capacity are seen for the worst case of layer waviness considered in this investigation, i.e., a wave with $\delta = 2$ and $\lambda = 10$ located at the outer radius of a cylinder with $R/H = 20$.

2. Fiber direction stresses dictate failure for all cylinder geometries and for most of the wave geometries considered in this investigation. However, for wave amplitudes greater than $\delta \geq 1.75$, interlaminar shear failure is seen to overshadow fiber direction compression failure for a wave at the outer radius of a cylinder with $R/H = 20$. The same occurs for the cylinder with $R/H = 10$ and a wave amplitude greater than $\delta \geq 1.85$. Similar overshadowing of fiber direction compression failure by interlaminar shear failure is suspected for other cylinder geometries for wave amplitudes $\delta > 2$.

3. Interlaminar normal tension failure was not predicted to occur for any of the cylinder and wave geometries considered in this investigation. It is suspected,
however, that interlaminar normal tension failure will occur in cylinders with wave amplitudes $\delta > 2$. This type of failure is anticipated to occur first in cylinders with smaller radii and with waves located at the outer radius of these cylinders.

4. Interlaminar normal compression failure was not a factor for any of the cylinder and wave geometries considered in this investigation. This type of failure may or may not occur for larger wave amplitudes, and if it does, it will likely occur in cylinders with larger radii.

5. Failure of the perfect cylinder due to buckling is significant in only one of the perfect cylinder geometries considered in this investigation, i.e., $R/H=20$, $L/R=1$. When compared with failure due to layer waviness, it was seen that for wave amplitudes less than $\delta = 1.2$, failure of the perfect cylinder due to buckling overshadowed material failure of the cylinder with wavy layers. For wave amplitudes greater than $\delta = 1.2$, failure due to layer waviness was seen to dominate. For all other cylinder geometries, material failure due to layer waviness was seen to overshadow failure of the perfect cylinder due to buckling. It is, however, anticipated that layer waviness may decrease the buckling strength of the cylinder, and therefore should be considered as a potential failure mechanism.

9.1.4 Conclusions Regarding Previous Work with Layer Waviness

1. It is felt that the previous work by Telegadas and Hyer [10,11] underpredicts the fiber direction stresses due to layer waviness and slightly overpredicts the interlaminar shear stresses due to layer waviness.
9.2 RECOMMENDATIONS FOR FURTHER RESEARCH

Several interesting aspects of layer waviness are felt to be of importance, but were not considered in this investigation. The first two of these have to do with extensions of the current models. The last involves a much more in depth study of the buckling issue.

1. This investigation modeled layer waviness in only the circumferential direction of the cylinder. Previous work has been done to model layer waviness in only the axial direction of the cylinder [13,14]. It warrants further investigation to determine whether the combined effect of layer waviness in the circumferential and axial directions will further reduce the pressure capacity of the cylinder. The scheme used to model layer waviness in this investigation could easily be modified for use in such a study.

2. The issue of multiple isolated waves and their interaction should be investigated to determine their effect on the pressure capacity of the cylinder. Many of the tools used in the current modeling scheme already are capable of modeling this multiple wave problem.

3. The issue of failure due to buckling was treated very superficially in this investigation, and only in the context of a perfect cylinder. It is felt, however, that buckling of the imperfect cylinder, i.e., a cylinder with included layer waviness, may be a very important issue and one which certainly warrants further study.
REFERENCES


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19. ABAQUS Theory manual, ABAQUS Version 4-6, p.4.5.3-3.


22. Tsai, S. W., Composites Design, 3rd Ed., Think Composites, 3033 Locust Camp Road (Box 581), Dayton, Oh 45419, 1987, pp. 10-4, 10-9.


APPENDIX A

WAVYREGN AND CYLINDER PCL CODES

This appendix contains a listing of the PCL codes used in generating the finite element models for the wave and surrounding cylinder. These codes are known to be compatible with PATRAN versions 2.4 and 2.5.

Wave

!FUNCTION GEN_14_WVY_LYRS
!" 
! * This function generates a section of wavy layers
! * 14 layers thick by a user defined number of elements in the
! * theta direction with two elements per layer.
! * 
! * Input: th_bg_d is the angle section begins at specified
! *       in degrees.
! * th_st_d is theta star or the angle at the end of the
! * wavy section specified in degrees
! * inrad is the inner radius of the wavy section
! * meanrad is the mean radius of the wavy region
! * lyrthick is the nominal layer thickness
! * elthick is the thickness of each element
! * numtheta is the number of elements in the theta
! * direction(wavy region only)
! * deltabar is the dimensionless amplitude of the wave
! * (really delta/t using blue report notation)
! * lambdabar is the dimensionless half wave length of the wave
! * (really lambda/t using blue report notation)
! * numelrad is the total number of elements in the
! * radial direction in the wavy region
! * numellyr is the user specified number of elements
! * per layer in the wavy region
! * numlyrs is the total number of layers which make up
! * the wavy region (14 in this case)
! * 
! * Notas: The factors for computing the
! * displacements from the nominal thickness of each layer
! * are 1/4,1/2,1/2,3/4,1,1,1,1,3/4,1/2,1/2,1/4 resp.
! * This should be used before any grli,or pa are created.
! */
! INTEGER i,j,k,m,grid1, grid2, numtheta, numelrad, numellyr, numlyrs
! INTEGER msizee(12),msize2(30),kcount(5),ngfeg, numnam, nset(15), istat
! INTEGER nstabl(100,2,15), mpc(2)
! STRING ina[7], inb[7], ina1[3], ina2[3], inb1[3], inb2[3]
! REAL dwl,th_d, fac, th_d, disp, rad, th_bg_d, th_st_d, inrad, meanrad, elthick, ang_d
! REAL pl, deltabar, lambdabar, lyrthick, th_bg, th_st, lambd
! STRING pidno[3], hpno[4], applybcs[1]
! INTEGER pid, nhpat, wavetype, prvgrid, prvline, prvpat, prvhpatt
!
! /* setup parameters */
!
SET,TOL,0.001
SET,LINES,0
SET,LABELS,OFF
SET,MWEIGHT,1
CORD,1,CYL.,0/0/0/0/-1/0/1/1
!
! numlyrs=14
!
! pi=math_acos(-1.0)
!
! UI_WRITE("Enter the type of wave being modelled")
! wavetype=UI_READ_INTEGER("1) Half Wave   2) Full Wave")
!
! deltabar=UI_READ_REAL("Enter a value for delta/t (DeltaBAR")
!
! lamdabar=UI_READ_REAL("Enter a value for lambda/t (LamdaBAR")
!
! th_bg_d=UI_READ_REAL("Enter a theta location (in degrees) for the symmetry line of the WAVE")
!
! th_bg=th_bg_d*pi/180.0
!
! inrad=UI_READ_REAL("Enter the inner radius of the WAVY region in inches")
!
! lyrthick=UI_READ_REAL("Enter the nominal layer thickness")
!
! numlyryr=UI_READ_INTEGER("Enter the number of elements per layer")
!
! numtheta=UI_READ_INTEGER("Enter the number of elements in the theta direction in the WAVY region")
!
! lamda=lamdabar*lyrthick
!
! meanrad=inrad+numlyrs/2*lyrthick
!
! /* If a full wave is requested, a mirror half wave is created */
! FOR (m=1 to wavetype by 1)
! IF (m=1) THEN
! th_st=th_bg+lamda/meanrad
! ELSE IF (m=2) THEN
! th_st=th_bg-lamda/meanrad
! END IF
!
! numelrad=numlyrs*numlyryr
!
! th_st_d=th_st*pi/180.0/}

Appendix A
! elthick=lyrthick/numellyr
!
! del_th_d=(th_st_d - th_bg_d)/(numtheta*1.0)
!
! /* Get maxid counts from the database */
! DB_ENTITY_COUNTS(MSIZE,MSIZE,ICOUNT,NGFEG,NUMNAM, @
!           NSET,NSTABL,MPC,ISTAT)
! IF(ISTAT  /= 0) THEN
! WRITE('Error from DB_ENTITY_COUNTS,ISTAT)
! RETURN
! END IF
!
! /* Extracting the necessary counts from the database count arrays */
! prvgrid=msize(1)
! prvline=msize(2)
! prvpat=msize(3)
! prvhpnt=msize(4)
!
! /* Determining the layer displacement factors */
! FOR (i=1 to 15 by 1)
! FOR (k=1 to numellyr by 1)
!
! IF (i=1) THEN
!   fac=0.0+((k-1)*.25/numellyr)
! ELSE IF (i=2) THEN
!   fac=0.25+((k-1)*.25/numellyr)
! ELSE IF (i=3) THEN
!   fac=0.50
! ELSE IF (i=4) THEN
!   fac=0.5+((k-1)*.25/numellyr)
! ELSE IF (i=5) THEN
!   fac=0.75+((k-1)*.25/numellyr)
! ELSE IF (i=6 || i=7 || i=8 || i=9) THEN
!   fac=1.0
! ELSE IF (i=10) THEN
!   fac=1.0-((k-1)*.25/numellyr)
! ELSE IF (i=11) THEN
!   fac=0.75-((k-1)*.25/numellyr)
! ELSE IF (i=12) THEN
!   fac=0.5
! ELSE IF (i=13) THEN
!   fac=0.5-((k-1)*.25/numellyr)
! ELSE IF (i=14) THEN
!   fac=0.25-((k-1)*.25/numellyr)
! ELSE IF (i=15) THEN
!   fac=0.0
! END IF
!
! IF ((i-1)+k < 16) THEN
! FOR (j=1 to numtheta+1 by 1)
th_d=(j-1)*del_th_d
ang_d=180.0*th_d/(th_st_d-th_bg_d)
disp=deltabar*({ tac}'0.5*ryrthick)'(1+mth_cossd(ang_d))
rad=((i-1)*numelryn+(k-1))'elthick + inrad + disp
GR.#/F1,'rad'/th_d+th_bg_d'
END FOR
END IF
END FOR
END FOR

/* GENERATE LINES BETWEEN GRIDS */
FOR (i=1 to numelrad+1 by 1)
FOR (j=1 to numtheta by 1)
grid1=prvgrid+(i-1)*(numtheta+1) + j
grid2=grid1 + 1
LI,#.2G,,'grid1',,'grid2'
END FOR
END FOR

/* GENERATE PATCHES */
hpat=numtheta*numelrad
FOR (i=prvline+1 to prvline+numtheta*(numelrad-1)+1 by numtheta)
ina1=st_from_integer(i)
ina2=st_from_integer(i+numtheta-1)
lnb1=st_from_integer(i+numtheta)
lnb2=st_from_integer(i+2*numtheta-1)
ina=ina1//"T":/ina2
lnb=lnb1//"T":/lnb2
PA,,'numtheta',2L,,'ina',,'lnb'
END FOR

/* GENERATE HYPERPATCHES */
HPAT,prvhpat+1'T'prvhpat+nhpat',EXTRUDE,.//0.01////////,1,'prvpat+1'T'prvpat+nhpat'

/* MESH EACH HYPERPATCH */
FOR (pid=prvhpat+1 to prvhpat+numtheta*numelrad by 1)
pidno=st_from_integer(pid)
hpno="H'/pidno
MESH,'hpno',HE/20/1,NUMBER,1/1/1/1/1,'pid',1
END FOR

applybcs=UI_READ_STRING("Apply Displacement Boundary Conditions?(y/n)")
IF (wave_type==1 && applybcs=="y") THEN
/* Applying Displacement Boundary Conditions */
/* Fixing theta at left boundary - BC set ID 1 */
FOR (i=1 to numelrad*numtheta by numtheta)
DFEG,H,i,DISP.,0///,1,F3,1
END FOR

Appendix A
! END IF

! IF (applybc="y") THEN
! */ Fixing z along the backface - BC set ID 2 */
! DFEQ.H1T#.DISP.//0//2,F6,1
! END IF

! END FOR

!

! UI_WRITE("The subtended angle (in degrees) of the half wave is :",th_st_d-th_bg_d)
! END FUNCTION
Cylinder

!FUNCTION GENERATE_CYLINDER
!
! * This function generates a uniform cylinder whose parameters
! * are interactively defined by the user.
! *
! * This function has been written to allow for previous geometries
! * to be present in the model.(i.e. the wavy region can and should
! * exist before the surrounding perfect geometry)
! *
! * Input: thetabgn is the beginning angle for the cylinder in degrees.
! * thetacend is ending angle for the cylinder in degrees
! * inrad is the inner radius of the cylinder in inches
! * outrad is the outer radius of the cylinder in inches
! * hpthk is the hyperpatch thickness in inches
! * nhptcirc is the number of hyperpatches through the thick. of
! * the cylinder
! * nhptcyc is the number of hyperpatches in the cyl. in the
! * circumferential direction
! * prvgrid is the max grid ID from the previous geometry
! * gridno1 is the starting grid ID for the new geometry
! *
! (1+prvgrid)
! *
! * prvline is the max line ID from the previous geometry
! * lineno1 is the starting line ID for the new geometry
! * prvpat is the max patch ID form the previous geometry
! * patno1 is the starting patch ID for the new geometry
! * hpatno1 is the starting hpat ID for the new geometry
! *
! */
!

! INTEGER i,j,nhptthk,numpids,nhptcyc,gridno1,lineno1,hpat,lastpat
! INTEGER prvgrid,gridno1,prvline,lineno1,prvpat,patno1,prvpat,hpatno1
! INTEGER pat,a,patb,patc,patd,numpat,gridno2,gridno3,gridno4,gridno5,gridno6
! INTEGER nsize(12),nsize2(30),icount(5),ngfeg,numnam,nset(15),istat
! INTEGER nstabl(100,2,15),mpc(2)
! STRING ln[9],lnb[9],lna[4],lnb[4],lnb[4],lnb[4]
! REAL thetabgn,thetacend,nrad,outrad,hpthk,axlength,theta,gridrad,gridth
! REAL delthet,hpthbgn(100),hpthbnd(100)
! STRING elemtype[8],meshdata[19],hpatnos[300]
! STRING GR[2],LIN[5],PA[2],HP[2]
!
!
! /* Get maxid counts from the database */
! DB_ENTITY_COUNTS(MSIZE,MSIZE2,ICOUNT,NGFEG,NUMNAM, @
! NSET,NSTABL,MPC,ISTAT)
! IF (ISTAT /= 0) THEN
! WRITE("Error from DB_ENTITY_COUNTS",ISTAT)
! RETURN
! END IF

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/* Extracting the necessary counts from the database count arrays */
prvgrid=msize(1)
prvline=msize(2)
prvpat=msize(3)
prvhpat=msize(4)

/* setup parameters */

SET,TOL,0.001
SET,LINES,0
SET,LABELS,OFF
SET,MWEIGHT,1
CORD,1,CYL.,0/0/0/0/0/0/1/0/1/1

/* INTERACTIVE INPUT */

theta_beg=UI_READ_REAL("Enter the starting angle for the cylinder in degrees")
theta_end=UI_READ_REAL("Enter the ending angle for the cylinder in degrees")
inrad=UI_READ_REAL("Enter the inner radius for the cylinder in inches")
outrad=UI_READ_REAL("Enter the outer radius for the cylinder in inches")
axlength=UI_READ_REAL("Enter the axial length of the cylinder in inches")
hptthk=UI_READ_INTEGER("Enter the number of hyperpatches through the thickness of the cyl.")
hptthk=(outrad-inrad)/hptthk

/*

UI_WRITE("Choose a method of creating hyperpatches in the circumferential direction")
circconfig=UI_READ_INTEGER("1. Automatic (Generates 1 hpat/30 degrees) 2. User-defined")
*/

IF (circconfig == 1) THEN
  delttheta=theta_end-thetabgn
  IF (delttheta <= 30.0) THEN
    nhptcirc=1
  ELSE IF (delttheta > 30.0 && delttheta <= 60.0) THEN
    nhptcirc=2
  ELSE IF (delttheta > 60.0 && delttheta <= 90.0) THEN
    nhptcirc=3
  ELSE IF (delttheta > 90.0 && delttheta <= 120.0) THEN
    nhptcirc=4
  ELSE IF (delttheta > 120.0 && delttheta <= 150.0) THEN
    nhptcirc=5
  ELSE IF (delttheta > 150.0 && delttheta <= 180.0) THEN
    nhptcirc=6
  ELSE IF (delttheta > 180.0 && delttheta <= 210.0) THEN
    nhptcirc=7
  ELSE IF (delttheta > 210.0 && delttheta <= 240.0) THEN
    nhptcirc=8
  ELSE IF (delttheta > 240.0) THEN
    nhptcirc=9

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! nhptcirc=8
! ELSE IF (delthet > 240.0 && delthet <= 270.0) THEN
! nhptcirc=9
! ELSE IF (delthet > 270.0 && delthet <= 300.0) THEN
! nhptcirc=10
! ELSE IF (delthet > 300.0 && delthet <= 330.0) THEN
! nhptcirc=11
! ELSE IF (delthet > 330.0 && delthet <= 360.0) THEN
! nhptcirc=12
! END IF
!
! ELSE IF (circcfg == 2) THEN
! nhptcirc=UL_READ_INTEGER("Enter the number of hyperpatches in the circumferential direction")
! FOR (i=1 to nhptcirc by 1)
! hpthbggn(i)=UL_READ_REAL("Enter the starting angle for hyperpatch */str_from_integer(i))
! hpthendn(i)=UL_READ_REAL("Enter the ending angle for hyperpatch */str_from_integer(i))
! END FOR
! END IF
!
! /* GENERATE GRIDS */
!
! gridno1=prvgrid+1
!
! FOR (i=1 to nhpthik+1 by 1)
! FOR (j=1 to nhptcirc by 1)
! gridrad=inrad+(i-1)*hpthik
! IF (circcfg == 1) THEN
! gridth=thetabgn+(j-1)*30.0
! ELSE IF (circcfg == 2) THEN
! gridth=hpthbggn(j)
! END IF
! GR,#,F1,"gridrad '/' gridth"
! END FOR
! END FOR
!
! /* GENERATE LINES BY ARCING FROM GRIDS */
!
! lineno1=prvline+1
!
! FOR (i=1 to nhpthik+1 by 1)
! FOR (j=1 to nhptcirc by 1)
! gridno=(gridno1-1)+(i-1)*nhptcirc+j
! lineno=(lineno1-1)+(i-1)*nhptcirc+j
! IF (circcfg == 1) THEN
! IF (j < nhptcirc) THEN
! theta=30.0
! ELSE IF (j == nhptcirc) THEN
! theta=(thetase-n-thetabgn)-30.0*(nhptcirc-1)
! END IF
! END FOR
! ELSE IF (circcfg == 2) THEN
! theta=htthend(j)-htthbgn(j)
! END IF
Li, lino", ARC, 0/0/0/0/0/0/0/1/ theta, "//1," gridro"
! END FOR
! END FOR
!
! /* GENERATE PATCHES */
!
! ptno1=prtpat+1
! npt=nhptcir*nhptthk
! lastpat=patno1+npt-1
!
! j=0
! FOR (i=lino1 to lino1+nhptcir*(nhptthk-1) by nhptcir)
! i<sub>1</sub>=str_from_integer(i)
! i<sub>2</sub>=str_from_integer(i+nhptcir-1)
! ln1=str_from_integer(i+nhptcir)
! ln2=str_from_integer(i+2*nhptcir-1)
! in1=ln1/"T"//lna2
! in2=ln2/"T"//lnb2
! pata=pata1+nptcir*"j"
! patb=pata+(nhptcir-1)
! PA, pata T patb", 2L, lna1", lnb"
! j=j+1
! END FOR
!
!
! /* GENERATE HYPERPATCHES */
!
! hpatno1=prvhpatt+1
! nhpat=nhptcir*nhptthk
! lastpat=nhpat+nhptcir-1
!
! HPAT, "hpatno1" T lastpat", EXT, //axlength//0//1, "patno1" T lastpat"
! SET, LABH, ON
! PLOT
!
! /* MESH EACH HYPERPATCH */
!
! /!
! elemtype=UI_READ_STRING("Enter the element type. Example. HEX20/1")
! numpids=UI_READ_INTEGER("Enter the number of PID's associated with this model")
! UI_WRITE("The number option will be used to create the element mesh")
! nummesh=UI_READ_INTEGER("Enter the number of different mesh ratios to be used in this model")
!
! FOR (i=1 to nummesh by 1)
! meshdata=UI_READ_STRING("Enter one set of mesh data. Example: 3/13/1/1/R/A/T/1/O")
! IP (numpids == 1) THEN
! pidno=UI_READ_INTEGER("Enter a PID number for the model with the mesh data */meshdata)
! UI_WRITE("Enter the hyperpatch no.'s associated with the following:")
!
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! hpatnos=UL_READ_STRING("PID://str_from_integer(pidno)="/" Mesh data:"//meshdata)
   MESH,H"hpatnos",elemtype,NUMBER,meshdata,pidno,1
! ELSE
! FOR (j=1 to numpids by 1)
!  pidno=UL_READ_INTEGER("Enter a PID number for the model with the mesh data "/meshdata)
!  UI_WRITE("Enter the hyperpatch no.'s associated with the following:")
!  hpatnos=UL_READ_STRING("PID://str_from_integer(pidno)="/" Mesh data:"//meshdata)
   MESH,H"hpatnos",elemtype,NUMBER,meshdata,pidno,1
! END FOR
! END IF
! END FOR
!
!
! /* Applying Displacement Boundary Conditions */
!
! UI_WRITE("Enter the appropriate boundary conditions")
!
! IF (bcchoice == 1) THEN
! FOR (i=hpatno1 to hpatno1+nhptcirc*(nhptthk-1) by nhptcirc)
   DFEGH,T,DISP,/0///,1,F3,1
! END FOR
!
! ELSE IF (bcchoice == 2) THEN
! FOR (i=hpatno1+nhptcirc-1 to lasthpat by nhptcirc)
   DFEGH,T,DISP,/0///,1,F4,1
! END FOR
!
! ELSE IF (bcchoice == 3) THEN
   DFEGH,T"hpatno1"T"lasthpat",DISP,/0///,2,F6,1
!
! ELSE IF (bcchoice == 4) THEN
! FOR (i=hpatno1 to hpatno1+nhptcirc*(nhptthk-1) by nhptcirc)
   DFEGH,T,DISP,/0///,1,F3,1
! END FOR
! FOR (i=hpatno1+nhptcirc-1 to lasthpat by nhptcirc)
   DFEGH,T,DISP,/0///,1,F4,1
! END FOR
   DFEGH,T"hpatno1"T"lasthpat",DISP,/0///,2,F6,1
!
! ELSE IF (bcchoice == 5) THEN
! FOR (i=hpatno1 to hpatno1+nhptcirc*(nhptthk-1) by nhptcirc)
   DFEGH,T,DISP,/0///,1,F3,1
! END FOR
   DFEGH,T"hpatno1"T"lasthpat",DISP,/0///,2,F6,1
!
! ELSE IF (bcchoice == 6) THEN
! FOR (i=hpatno1+nhptcirc-1 to lasthpat by nhptcirc)
   DFEGH,T,DISP,/0///,1,F4,1
! END FOR

Appendix A
DFEG,H'hpato1'T'Iasthpat',DISP.,//0//,2,F6,1
!
ELSE IF (bcchoice == 7) THEN
! UI_WRITE("No Displacement Boundary Conditions were applied")
! END IF
!!
!! END FUNCTION
APPENDIX B

MATERIAL PROPERTY CALCULATION SOURCE LISTING

This appendix includes the source listing for ORSOLMTL, the FORTRAN code used to calculate modified material properties for the finite elements comprising the wavy region. These modified material properties are a result of the local fiber rotation and volume fraction which occurs in these elements.

```fortran
PROGRAM ORSOLMTL

  ! Areas which may need future modification are identified
  ! by ?????? above and below that area
  !
  ! Program to calculate the rotation angles needed for the *ORIENTATION
  ! card in the ABAQUS input file
  !
  ! The idea is to use a single material definition (stiffness
  ! matrix) and to use a single *ORIENTATION card for each of the elements
  !
  ! The perfect elements will all be referenced to a single cylindrical
  ! *ORIENTATION card while the wavy elements will each have its own
  ! *ORIENTATION card based on the rotation angle calculated in this program
  !
  ! This program will also output the necessary *ORIENTATION definitions
  ! to a file which can then be pasted into the ABAQUS input file
  !
  ! This program will be implemented after the initial model has been
  ! created in PATRAN (minus material definition) and after this
  ! initial model data has been converted to a preliminary ABAQUS
  ! input file via PATABA
  !
  ! This program will search through the preliminary ABAQUS input file
  ! from above, extracting the necessary data for calculation of the
  ! material angles and produce the *ORIENTATION cards
  ! corresponding to the elements in the wavy region
  !
  ! This program also creates the *SOLID SECTION cards for use with
  ! these *ORIENTATION definitions.
  !
  ! This program also calculates the volumes of each of the wavy
  ! elements for use in the determination of the modified material
  ! properties within the wavy elements
  !
  ! This program also uses the calculated element volumes and a
  ! modified volume fraction theory to calculate these new
  ! material properties. The modified material properties are
```

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used to calculate the stiffness(compliance) matrices for each
do of the wavy elements and these matrices are created for each of
des these elements using ABAQUS indexing scheme (different from
C Continuum indexing scheme).

This file is intended for use with ABAQUS 4.8

Created by tb 12/4/90

implicit real*8(a-h,o-z)
integer elnum,global,umlyrs,teilsyrs,relasyrs,wavloc,numnod
integer matid
real*8 lyrthick
character a1*1,a2*2,a3*3,a4*4,a5*6,a6*7,a7*10,a10*10,a17*17,
a21*21,a38*38,a40*40,rest*100
character*70 abqinp,orient,solid,matid,matprop
character ca1*3,ca2*3,yesno*1

parameter(maxnod=25000,maxel=3000)

dimension xn(maxnod),yn(maxnod),zn(maxnod)
dimension ncon(maxel,20)
dimension global(20,maxel),alpha(maxel),alphadeg(maxel)
dimension volwavy(maxel),voilper(maxel),volelta(maxel)
dimension volwavya(maxel),volwavyb(maxel)
dimension perconl(maxel)
dimension vfwavy(maxel),vfwmwy(maxel)
dimension E1(maxel),E2(maxel),E3(maxel),G12(maxel),G13(maxel)
dimension G23(maxel),V12(maxel),V13(maxel),V23(maxel)
dimension V21(maxel),V31(maxel),V32(maxel)
dimension circs(6,6,maxel),axs(6,6,maxel),circc(6,6,maxel)
dimension axc(6,6,maxel)
dimension circnpu(6,6),axnpu(6,6),axnrs(6,6),circnrs(6,6)
DIMENSION D(10,20),H(6,6),HINV(6,6),UNIT(10,10)

C -------------------------------------------------------------
write(6,'')
write(6,'')
write(6,'')
write(6,'')CAUTION:
write(6,'')Before running this program, you must include the
write(6,'')following lines in your ABAQUS .inp file
write(6,'')
write(6,'''WAVY ELEMENT CONNECTIVITY BEGINS HERE and
write(6,'''WAVY ELEMENT CONNECTIVITY ENDS HERE'
write(6,'')
write(6,'')on the line directly before the first wavy "ELEMENT
write(6,'') card and on the line immediately following the'

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write(6,*)'connectivity for the last wavy element.....'
write(6,*)'
write(6,*)'
write(6,*)'Hit <Return> to continue OR <Ctrl Q> to quit...
read(5,5)a1

format (a5)
write(6,*)'ABAQUS input (.inp) filename?'
read(5,7000)abqinp
write(6,*)'ABAQUS orientation definitions filename?'
read(5,7000)orient
write(6,*)'ABAQUS "SOLID SECTION card definitions filename?'
read(5,7000)solid
write(6,*)'ABAQUS material definitions filename?'
read(5,7000)material
write(6,*)'Material properties filename?'
read(5,7000)matprop

7000 format(a)

7001 format(4)
open(unit=7,file=abqinp,status='old')
open(unit=8,file=orient,status='new')
open(unit=9,file=solid,status='new')
open(unit=10,file=material,status='new')
open(unit=11,file=matprop,status='new')

* *
* defining the number of elements in the wavy region ONLY
* and the TOTAL number of nodes in the ENTIRE model
*
*
* Variables associated with the WAVY region:
*
* nu lyr is the number of layers in the wavy region
* t el sly r is the number of elements in the wavy region
* in the theta direction
* r el sly r is the number of elements per layer in the wavy
* region in the radial direction
* nu mel is the number of elements in the wavy region
*
*
* nu lyr=14
*
*
write(6,*)'Input the number of elements in the theta'
write(6,*)'direction for the WAVY REGION ONLY?'
read(5,(i2))telslyr
write(6,*)'Input the number of elements per layer'
write(6,*)'in the wavy region'
read(5,(i1))relyslyr

numel=nu lyr*telslyr*relyslyr
Determining the number of nodes in the ENTIRE model

write(6,'"Enter the number corresponding to the"
write(6,'"current ENTIRE model size:"
write(6,'"
write(6,'"1) 14layers with 1el/yr (14e1s high X 14e1s wide)"
write(6,'"2) 14layers with 2e1s/yr (28e1s high X 14e1s wide)"
write(6,'"3) 24layers with 2e1s/yr (48e1s high X 14e1s wide)"
write(6,'"4) 34layers with 2e1s/yr (68e1s high X 14e1s wide)"
write(6,'"5) 14layers with 2e1s/yr (28e1s high X 20e1s wide),"
write(6,'"- refined"
write(6,'"6) 102layers with 2e1s/yr (204e1s high X 14e1s wide)"
write(6,'"7) (full thickness centered wave model)"
write(6,'"8) 104layers with 2e1s/yr (208e1s high X 14e1s wide)"
write(6,'" (full thickness inner and outer wave models)"
write(6,'"9) 52layers with 2e1s/yr (104e1s high X 14e1s wide)"
write(6,'" (half thickness inner and outer wave models)"
read(5,"(i1)"nummodel
if (nummodel.eq. 1) then
  numnod=1515
else if (nummodel.eq. 2) then
  numnod=2957
else if (nummodel.eq. 3) then
  numnod=5017
else if (nummodel.eq. 4) then
  numnod=7077
else if (nummodel.eq. 5) then
  numnod=4163
else if (nummodel.eq. 6) then
  numnod=21085
else if (nummodel.eq. 7) then
  numnod=21497
else if (nummodel.eq. 8) then
  numnod=10785
endif

This program is currently set up for a
model with the following number of wavy
elements and total nodes respectively
write(6,6)numel
write(6,6)numnod
write(6,'"Hit <Return> to continue or <Ctrl Q> to quit..."
read(5,5)a1
6  format(1x,i5)
c
--------------------------------------------------------------------------------
c initializing x,y, and z coords of the nodes
c
--------------------------------------------------------------------------------
do 10 i=1,maxnod
   xn(i)=0.d0
   yn(i)=0.d0
10  zn(i)=0.d0
--------------------------------------------------------------------------------
c an array global(20,numel) is used to store the global
c node numbers corresponding to the local
c node numbers 1:20 for each of the elements
c which make up the wavy region
c
c it is the local nodes # 9 and 11 which
c will be used to calculate the material orientation
c angles
c
c storing the conn. information in global(20,numel)
c
--------------------------------------------------------------------------------
c locating the data in the file
c
--------------------------------------------------------------------------------
 rewind 7
600  read(7,610)a40
610  format(a40)
   if(a40.eq.'** WAVY ELEMENT CONNECTIVITY BEGINS HERE') then
      read (7,610) a40
      write(6,'**beginning to read connectivity')
goto 650
   else
      goto 600
   endif
--------------------------------------------------------------------------------
c reading the data
c
--------------------------------------------------------------------------------
549  format (16i5)
650  read (7,649)elem,(global(j,elem),j=1,20)
   read (7,655) a38
   if (a38.eq.'** WAVY ELEMENT CONNECTIVITY ENDS HERE') then
      goto 660
   else
      goto 650
   endif
655  format (a38)
660  write(6,')'connectivity read in'
c
--------------------------------------------------------------------------------
c reading in x,y, and z coords. of all the nodes
c from the ABAQUS input file
c
--------------------------------------------------------------------------------
c locating the x,y,z data for the nodes of interest

Appendix B
c  ____________________________________________
rewind 7
490 read(7,500)a1,a4,arest
500 format(a1,a4,a95)
   if(a1.eq."**" and. a4.eq."NODE") then
      goto 550
   else
      goto 490
   endif

c  ____________________________________________

c reading the x,y,z data

c  ____________________________________________
550 write(6,"*"reading the x,y,z coords. of the nodes"
      do 560 i=1,numnod
         read(7,*)numx,numy,numz
         continue
      write(6,"*"x,y,z data for all nodes read in"

c  ____________________________________________
c calculating the material rotation angle for each of
  the wavy elements

c  ____________________________________________
   do 800 i=1,numel
      alpha(i)=atan((yn(global(9,i))-yn(global(11,i)))/
         + (xn(global(11,i))-xn(global(9,i))))
   continue
   alpha(i)=atan((yn(global(10,i))-yn(global(12,i)))/
      + (xn(global(12,i))-xn(global(10,i))))
   alpha(i)=alpha(i)*180.d0/acos(-1.d0)

800 continue


c c c


c Calculating the volumes of each of the wavy elements
  for use in modified volume fraction calculations

c The equation for the volume is equal to the equation
  for area (obtained from macsyma) times a unit thickness
  in the z direction (zthick)

c  ____________________________________________
c


c Note: The local midside node numbers 5,6,7,8 for a 2-D
  quad element correspond to local node numbers
  9,10,11,12 respectively in a 3-D brick element
  in ABAQUS. These substitutions have been made
  in the equation below for volwavy(wavy element
  volume) since the locations of the nodes are
  extracted from the ABAQUS input file which is
for 3-D elements. Had I calculated volume in
terms of 3-D elements rather than 2-D elements
times a unit thickness, this substitution would
not have been necessary.

zthick=0.01d0

do 1200 i=1,numel
   volwavya(i)=(xn(global(2,i))*a+y(n(global(9,i))
   a +9*y(n(global(10,i))
   a +2*y(n(global(12,i))+y(n(global(3,i)))-2*y(n(global(1,i))
   a +xn(global(12,i))*4*y(n(global(9,i))+3*y(n(global(10,i))
   a +4*y(n(global(11,i))-2*y(n(global(2,i))-2*y(n(global(3,i))
   a -yn(global(4,i))-9*y(n(global(1,i)))+xn(global(10,i))
   a *(4*y(n(global(9,i))-4*y(n(global(11,i))-6*y(n(global(12,i))
   a +9*y(n(global(2,i))+y(n(global(3,i))+2*y(n(global(4,i))
   a +2*y(n(global(1,i)))+xn(global(11,i))*8*y(n(global(9,i))
   a -2*y(n(global(10,i))+3*y(n(global(12,i))+2*y(n(global(2,i))
   a -yn(global(4,i)))+4*y(n(global(10,i))-4*y(n(global(12,i))
   a -8*y(n(global(2,i))+8*y(n(global(1,i)))xn(global(9,i))
   a +xn(global(11,i))*4*y(n(global(10,i))-4*y(n(global(12,i))
   a +xn(global(3,i)))*yn(global(10,i)))+2*y(n(global(12,i))
   a -yn(global(2,i)))+xn(global(4,i)))*(2*y(n(global(10,i))
   a +yn(global(12,i))+yn(global(1,i)))/12.0
volwavyb(i)=
   a +xn(global(12,i))*4*y(n(global(9,i)))+6*y(n(global(10,i))
   a +4*y(n(global(11,i))-2*y(n(global(2,i))-2*y(n(global(3,i))
   a -9*y(n(global(4,i)))-yn(global(1,i)))+xn(global(10,i))
   a *(4*y(n(global(9,i))-4*y(n(global(11,i))-6*y(n(global(12,i))
   a +yn(global(2,i)))+yn(global(3,i))+2*y(n(global(4,i))
   a +2*y(n(global(1,i)))+9*y(n(global(3,i))+2*y(n(global(4,i))
   a +2*y(n(global(1,i)))+4*y(n(global(10,i))-4*y(n(global(12,i))
   a *yn(global(9,i))+xn(global(11,i)))-4*y(n(global(10,i))
   a -4*y(n(global(12,i))-8*y(n(global(3,i))+8*y(n(global(4,i))
   a +xn(global(2,i)))*yn(global(10,i))+2*y(n(global(12,i))
   a -yn(global(3,i)))+xn(global(4,i)))*yn(global(10,i))
   a -8*y(n(global(11,i))+9*y(n(global(12,i))+2*y(n(global(3,i))
   a -yn(global(1,i))+xn(global(1,i)))*yn(global(10,i))
   a +yn(global(12,i))+yn(global(4,i)))+xn(global(3,i))
   a *(9-y(n(global(10,i))+8*y(n(global(11,i))+2*y(n(global(12,i))
   a +yn(global(2,i))-2*y(n(global(4,i)))))/12.0
volwavy(i)=zthick*(volwavya(i)+volwavyb(i))

1200 continue

Calculating the volume of the perfect elements corresponding to
the wavy elements prior to imperfection

write(6,"\n\nInput the number corresponding to the\n\nPresent case being studied:"

Appendix B
write(6,'*') 1 Wave located at the CENTER of 6.751 cylinder
write(6,'*') 1 Inner Radius of Wavy region = 4.089 in.
write(6,'*') 2 Outer Radius of Wavy region = 4.173 in.
write(6,'*') 2 Wave located at the INNER rad of 5:1 cylinder
write(6,'*') 2 Inner Radius of Wavy Region = 2.808 in.
write(6,'*') 2 Outer Radius of Wavy Region = 2.892 in.
write(6,'*') 3 Wave located at the OUTER rad of 5:1 cylinder
write(6,'*') 3 Inner Radius of Wavy Region = 3.348 in.
write(6,'*') 3 Outer Radius of Wavy Region = 3.432 in.
write(6,'*') 4 Wave located at the INNER rad of 10:1 cylinder
write(6,'*') 4 Inner Radius of Wavy Region = 5.928 in.
write(6,'*') 4 Outer Radius of Wavy Region = 6.012 in.
write(6,'*') 5 Wave located at the OUTER rad of 10:1 cylinder
write(6,'*') 5 Inner Radius of Wavy Region = 6.468 in.
write(6,'*') 5 Outer Radius of Wavy Region = 6.552 in.
write(6,'*') 6 Wave located at the INNER rad of 20:1 cylinder
write(6,'*') 6 Inner Radius of Wavy Region = 12.168 in.
write(6,'*') 6 Outer Radius of Wavy Region = 12.252 in.
write(6,'*') 7 Wave located at the OUTER rad of 20:1 cylinder
write(6,'*') 7 Inner Radius of Wavy Region = 12.708 in.
write(6,'*') 7 Outer Radius of Wavy Region = 12.792 in.

read(5, ('11')) wavloc
if (wavloc .eq. 1) then
  radin=4.089d0
else if (wavloc .eq. 2) then
  radin=2.808d0
else if (wavloc .eq. 3) then
  radin=3.348d0
else if (wavloc .eq. 4) then
  radin=5.928d0
else if (wavloc .eq. 5) then
  radin=6.468d0
else if (wavloc .eq. 6) then
  radin=12.168d0
else if (wavloc .eq. 7) then
  radin=12.708d0
endif

rellyr is known
tellyr is known
The volume of the perfect elements could be calculated in
two ways: 1) same way as the wavy elements, but reading
in the node locations from a perfect model
2) a percentage (based on theta dimension of

Appendix B
each element; of a ring calculated using the inner and outer radii of each element

At first at least, the second method will be used.

Determining the height(radial thickness) of each element

write(6,"Enter the nominal layer thickness'"
read(5,"lyrthick"

elheight=lyrthick/relsty

c
elwidth is the percentage of the element width to a full 360 degree circle

c
write(6,"Enter the subtended angle of the half wave'"
write(6,"\((\theta)\text{ star - }\theta\text{ begin})'"
read(5,"waveangl"

elwidth=(waveangl/relsty)/360.0d0

c
elthick is the thickness of the element in the z direction

c
write(6,"Enter the element thick. in the axial(Z) direction'"
read(5,"elthick"

c
pl=acos(-1.0d0)
k=0

do 1300 i=1,numlys*relsty
   rout=radi+*elheight
   rin=radi+(-(1)\text{)*elheight}
   do 1250 j=1,relsty
      volperf[k*relsty+j]=elthick*elwidth*pi*(rout**2-rin**2)
   do 1250
   continue
   k=k+1
1300 continue

c
c
------------------------------------------------------------------------------------------------------

CALCULATION OF MATERIAL PROPERTIES

------------------------------------------------------------------------------------------------------

Modified Volume Fraction Theory:

Calculating the modified material properties for each of the wavy elements

------------------------------------------------------------------------------------------------------

It is assumed that the addition or reduction in volume of each element is a result of increased or decreased resin only (i.e. no change in the fiber volume).

Therefore, the resulting fiber and matrix volume fractions are calculated as follows:

\[ v(f,w) = v(f,p)*v(T,p)/v(T,w) \]
where,

\[ v_{[f,w]} = \text{wavy element fiber volume fraction} \]
\[ v_{[f,p]} = \text{perf. element fiber volume fraction} \]
\[ V_{[T,p]} = \text{total volume of perfect element} \]
\[ V_{[T,w]} = \text{total volume of wavy element} \]
\[ v_{[m,w]} = \text{wavy element matrix volume fraction} \]

In order to calculate \( v_{[f,w]} \) and \( v_{[m,w]} \) we need to calculate the total volume of the perfect element and the total volume of the wavy element.

Note: The total volume of the perfect element varies as a function of radius!

In the interest of making the program as flexible as possible, I have included the option of defining the material properties in two ways:

1) Rotation of fibers only which rotates the fibers according to the shape of the wavy elements

2) Rotation and Volume Fraction change which accounts for the rotation of the fibers and the volume fraction changes in the wavy elements

write(6, *) 'Input the type of material definition'
write(6, *) 'you will be using:'
write(6, *) '1) Simple Rotation'
write(6, *) '2) Rotation and Volume Fraction'
read(5, (i1) 'matdef'

The volume fraction of the perfect laminate is assumed to be .65 and .35 for fiber and matrix volume fraction respectively

\[ \text{vfperf}=0.65d0 \]
\[ \text{vmperf}=0.35d0 \]

\[ \text{etay is Tsai's transverse stress partitioning parameter} \]
\[ \text{etas is Tsai's shear stress partitioning parameter} \]
\[ \text{etay}=516d0 \]
\[ \text{etas}=316d0 \]

write(6, *) 'Tsai developed the following values'
write(6, *) 'for the Transverse and Shear Stress'
write(6, *) 'Partitioning Parameters to be used'
write(6, *) 'in the modified volume fraction calculations:'
write(6, ('a7,15.3') 1) 'Ny=', etay
write(6, ('a7,15.3') 2) 'Nz=', etas
write(6,'*Would you like to change these? (y/n)'
read(5,'(a1)') yesno
if (yesno.eq. 'Y') then
  write(6,'*Enter a new value for Ny'
  read(5,'(F5.3)') etay
  write(6,'*Enter a new value for Ns'
  read(5,'(F5.3)') etas
end if

do 1500 i=1,numel
  if (matdef .eq. 1) then
    vfwavy(i)=vperf
    vmwavy(i)=vmperf
  end if
  E1(i)=19.1d6
  E2(i)=1.36d6
  E3(i)=1.34d6
  G12(i)=.8d6
  G13(i)=.8d6
  G23(i)=.5d6
  V12(i)=.279d0
  V13(i)=.279d0
  V23(i)=.34d0
  else if (matdef .eq. 2) then
    vfwavy(i)=vperf*volperf(i)*volwavy(i)
    vmwavy(i)=1.3d0-vfwavy(i)
  end if

  E1(i)=29.1d6*vfwavy(i)+.5d6*vmwavy(i)
  E2(i)=(1.d0+etay*vmwavy(i)/vfwavy(i))/(1.d0/2.61d6+
  a etay*(vmwavy(i)/vfwavy(i))/5d6)
  E3(i)=(1.d0+etay*vmwavy(i)/vfwavy(i))/(1.d0/2.514d6+
  a etay*(vmwavy(i)/vfwavy(i))/5d6)
  G12(i)=(1.d0+etas*(vmwavy(i)/vfwavy(i)))/
  a (1.d0/1.823d6+etas*(vmwavy(i)/vfwavy(i)))/
  a .186d6)
  G13(i)=G12(i)
  G23(i)=(.5d6/.8d6)*G12(i)
  V12(i)=.35d0*vfwavy(i)+.241d0*vmwavy(i)
  V13(i)=V12(i)
  V23(i)=(.34d0/.279d0)*V12(i)
  V21(i)=V12(i)*E2(i)/E1(i)
  V31(i)=V13(i)*E3(i)/E1(i)
  V32(i)=V23(i)*E3(i)/E2(i)

1500 continue

Assembling the Compliance Matrices and inverting to get the Stiffness Matrices
c circs(x,x) corresponds to the circ compliance matrix
c circc(x,x) corresponds to the circ stiffness matrix
c axis(x,x) corresponds to the axial compliance matrix
c axcs(x,x) corresponds to the axial stiffness matrix
c
Note: these matrices have been arranged according
to the ABAQUS indexing scheme (11,22,33,12,13,23)
c
-------------------------------------------------------------
do 1550 i=1,numel
do 1550 j=1,6
do 1550 k=1,6
circs(j,k,l)=0.d0
axs(j,k,l)=0.d0
1550 continue
do 1600 i=1,numel
circs(1,1,j)=1.d0/E3(i)
circs(1,2,j)=V13(i)/E1(i)
circs(1,3,j)=V23(i)/E2(i)
circs(2,1,j)=circs(1,2,i)
circs(2,2,j)=1.d0/E1(i)
circs(2,3,j)=V21(i)/E2(i)
circs(3,1,j)=circs(1,3,i)
circs(3,2,j)=circs(2,3,i)
circs(3,3,j)=1.d0/E2(i)
circs(4,4,j)=1.d0/G13(i)
circs(5,5,j)=1.d0/G23(i)
circs(6,6,j)=1.d0/G12(i)
c
axs(1,1,j)=1.d0/E3(i)
axs(1,2,j)=V23(i)/E2(i)
axs(1,3,j)=V13(i)/E1(i)
axs(2,1,j)=axs(1,2,i)
axs(2,2,j)=1.d0/E2(i)
axs(2,3,j)=V12(i)/E1(i)
axs(3,1,j)=axs(1,3,i)
axs(3,2,j)=axs(2,3,i)
axs(3,3,j)=1.d0/E1(i)
axs(4,4,j)=1.d0/G23(i)
axs(5,5,j)=1.d0/G13(i)
axs(6,6,j)=1.d0/G12(i)
c
do 1585 j=1,6
do 1585 k=1,6
circnput(j,k)=circs(j,k,l)
axnput(j,k)=axs(j,k,l)
1585 continue
c
call invert (circnput,cincwrs,6)
call invert (axnput,axnwrs,6)
do 1590 j=1,6
do 1590 k=1,6
  circv(j,k,i)=circvrs(j,k)
  axv(j,k,i)=axvrs(j,k)
1590 continue

1600 continue

c

c

c

c OUTPUT SECTION

*c

writing the *ORIENTATION, *SOLID SECTION, and
*ELASTIC definitions to files orient, solid,
and material whose names are user specified

write(10,'*The following stress partitioning parameters'
write(10,'*were used:')
write(10,'(a5,f5.3)')Ny=,elay
write(10,'(a5,f5.3)')Ns=,elas

i=0

do 1000 k=1,numlyrs
  do 1000 j=1,rellyr*tslyr
    j=(k-1)*rellyr*tslyr+1
1000 continue

*c

Use these for defining the wavy materials with
PID's 3 through numel(currently 224)+2 in wvy region
  if (i.le.7) write (ca1,'(i1)')i+2
  if (i.ge.8 .and. i.le.97) write (ca1,'(i2)')i+2
  if (i.ge.98) write (ca1,'(i3)')i+2

*c

Use these for defining the wavy materials with
PID's 1 through numel(currently 224) in wvy region
  if (i.le.9) write (ca2,'(i1)')i
  if (i.ge.10 .and. i.le.99) write (ca2,'(i2)')i
  if (i.ge.100) write (ca2,'(i3)')i

if (i.le.9) write (ca2,'(i1)')i
  if (i.ge.10 .and. i.le.99) write (ca2,'(i2)')i
  if (i.ge.100) write (ca2,'(i3)')i

write (8,1100)'ORIENTATION, NAME=WAVY//ca2
write (8,1101)'0.,0.,1.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.'
write (8,1102) 3.,alphadeg(i)
write (9,1103)**

For simple Rotation -> No volume fraction changes

Appendix B
if (matdef .eq. 1) then
  
  if (k.eq.3 .or. k.eq.6 .or. k.eq.9 .or. k.eq.12) then
    write (9,1104)**SOLID SECTION, ELSET=PID'/ca1,
      ', MATERIAL=MID2, ORIENTATION=WAYVY'/ca2
  else
    write (9,1104)**SOLID SECTION, ELSET=PID'/ca1,
      ', MATERIAL=MID1, ORIENTATION=WAYVY'/ca2
  endif

  
  So that the mat def's. are written only once
  if (k .eq. 1 .and. j .eq. 1) then
    write(10,1105)**MATERIAL, NAME=MID1'
    write(10,1105)** CIRCUMFERENTIAL FIBERS'
    write(10,1105)**ELASTIC, TYPE=ORTHO'
    write(10,1106)axc(1,1,i),axc(1,2,i),axc(2,2,i),
      + axc(1,3,i),
      + axc(2,3,i),axc(3,3,i),axc(4,4,i),axc(5,5,i)
    write(10,1106)axc(6,6,i)
  end if

  
  else if (k .eq. 3 .and. j .eq. 1) then
    write(10,1105)**MATERIAL, NAME=MID2'
    write(10,1105)** AXIAL FIBERS'
    write(10,1105)**ELASTIC, TYPE=ORTHO'
    write(10,1106)axc(1,1,i),axc(1,2,i),axc(2,2,i),
      + axc(1,3,i),
      + axc(2,3,i),axc(3,3,i),axc(4,4,i),axc(5,5,i)
    write(10,1106)axc(6,6,i)
  end if

  
  For Volume Fraction change and Rotation

  else if (matdef .eq. 2) then
  
  write (9,1107)**SOLID SECTION, ELSET=PID'/ca1,
      ', MATERIAL= MID'/ca2,
      ', ORIENTATION=WAYVY'/ca2

  

  if (k.eq.3 .or. k.eq.6 .or. k.eq.9 .or. k.eq.12) then

  write(10,1105)**MATERIAL, NAME=MID'/ca2
  write(10,1105)** AXIAL FIBERS'
  write(10,1105)**ELASTIC, TYPE=ORTHO'
  write(10,1106)axc(1,1,i),axc(1,2,i),axc(2,2,i),
    + axc(1,3,i),
    + axc(2,3,i),axc(3,3,i),axc(4,4,i),axc(5,5,i)
  write(10,1106)axc(6,6,i)

Appendix B
else
  c
  write(10,1105)**MATERIAL, NAME=MID''/ca2
  write(10,1105)** CIRCUMFERENTIAL FIBERS'
  write(10,1105)**ELASTIC, TYPE=ORTHO'
  write(10,1106)circ(1,1,i),circ(1,2,i),circ(2,2,i),
  + circ(1,3,i),
  + circ(2,3,i),circ(3,3,i),circ(4,4,i),circ(5,5,i)
  write(10,1106)circ(6,6,i)
  c
  end if
  c
end if

1000 continue
1100 format(a26)
1101 format(a18)
1102 format(a3,f9.6)
1104 format(a28,a37)
1103 format(a2)
1105 format(a)
1106 format(8e10.4)
1107 format(a28,a17,a21)
  c
  ---------------------------------------------
  c  Testing the volume calculations
  c
  ---------------------------------------------
  write(11,1405)
  do 1400 i=1,numel
    voidelta(i)=volwavy(i)-volperf(i)
    perdelta(i)=(volwavy(i)-volperf(i))/volperf(i)*100
  write(11,1410)i,volwavy(i),volperf(i),voidelta(i),
  a perdelta(i),volwavy(i),vmwavy(i),E1(i),E2(i),
  a E3(i),G12(i),G13(i),G23(i),V12(i),
  a V13(i),V23(i)
  1400 continue
1405 format(1x,'volwavy','volperf','voidelta','% change',
  a 'fiber vol. frac.', 'matrix vol. frac.',
  a 'E1','E2','E3','G12','G13','G23','V12','V13','V23')
1410 format(i4,20(3x,d15.8))
  c
  stop
end
  c
  END OF MAIN PROGRAM
  c
  ---------------------------------------------
  c  Matrix Inversion Routine
  c
  ---------------------------------------------
  SUBROUTINE INVERT(H,HNVS,N)
  c

Appendix B
C FINDS THE INVERSE (HINVS) OF MATRIX H (N BY N) BY GAUSSIAN
C ELIMINATION WITH PIVOT ELEMENTS.
C
REAL*8 A,B,D,H,HINVS,QUOT,TEMP,UNIT
DIMENSION D(10,20),H(N,N),HINVS(N,N),UNIT(10,10)
C
DO 50000 I=1,N
DO 50000 J=1,N
50000 UNIT(I,J)=0.D0
   NT2=N*2
   NP1=N+1
   NM1=N-1
   DO 40000 I=1,N
40000 UNIT(I,I)=1.D0
   DO 41000 I=1,N
   DO 41000 J=1,N
   D(I,J)=H(I,J)
41000 D(I,N+J)=UNIT(I,J)
   DO 42000 J=1,NM1
   A=ABS(D(J,J))
   JP1=J+1
   DO 43000 I=1,JP1,N
   B=ABS(D(I,J))
   IF(A-B) 10000,43000,43000
10000 DO 44000 K=J,NT2
   TEMP=D(I,K)
   D(I,K)=D(J,K)
44000 D(I,K)=TEMP
43000 A=B
   DO 42000 I=JP1,N
   QUOT=D(I,J)/D(J,J)
   DO 42000 K=JP1,NT2
42000 D(I,K)=D(I,K)-QUOT*D(J,K)
   K=N
11000 I=K-1
12000 QUOT=D(I,K)/D(K,K)
   DO 45000 J=NP1,NT2
45000 D(I,J)=D(I,J)-QUOT*D(K,J)
   IF(I.EQ.1) GO TO 13000
   I=I-1
   GO TO 12000
13000 IF(K.EQ.2) GO TO 14000
   K=K-1
   GO TO 11000
14000 DO 46000 I=1,N
   DO 46000 J=1,N
46000 HINVS(I,J)=D(I,N+J)/D(I,J)
RETURN
END
APPENDIX C
SAMPLE ELEMENT VOLUME CALCULATIONS

This appendix contains sample finite element volume calculations for a cylinder with perfect geometry, i.e., no wave included. The volumes are calculated exactly and by the Jacobian method described in Chapter 4.

The exact volume is calculated using simple geometric equations for the volume of a circular ring. Since the finite element has a fixed circumferential length, its volume is a percentage of the volume of the entire ring. The exact volume of the finite element is given by:

\[ V = \frac{\theta_{\text{element}}}{360} \cdot \pi(r_o^2 - r_i^2) \cdot t_z \]  \hspace{1cm} (32)

where

- \( V \) = volume of the finite element
- \( \theta_{\text{element}} \) = circumferential length of the element in degrees
- \( r_o \) = outer radius of the element
- \( r_i \) = inner radius of the element
- \( t_z \) = axial thickness of the element

The Jacobian element volume calculations were performed with the FORTRAN program ORSOLMTL included in Appendix B. The details of the calculations can be seen there.

Sample element volumes are shown in Table C.1. Values are shown for both the exact and Jacobian methods. The percent error between the two methods is also shown. The maximum percent error between the two methods was seen to be approximately 0.03%.
### Table C.1

Comparison of Exact Finite Element Volumes vs. those Calculated by the Jacobian Method

<table>
<thead>
<tr>
<th>Element Geometry</th>
<th>Exact Volume (x10^-6 in^3)</th>
<th>Jacobian Calculated Volume (x10^-6 in^3)</th>
<th>% Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_0$(in.)</td>
<td>$r_i$(in.)</td>
<td>$\theta_{element}$(deg.)</td>
<td>$t_\phi$(in.)</td>
</tr>
<tr>
<td>4.092</td>
<td>4.089</td>
<td>0.1</td>
<td>0.01</td>
</tr>
<tr>
<td>4.095</td>
<td>4.092</td>
<td>0.1</td>
<td>0.01</td>
</tr>
<tr>
<td>4.098</td>
<td>4.095</td>
<td>0.1</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Appendix C
APPENDIX D
SAMPLE ABAQUS INPUT FILES

This appendix contains sample input files used by ABAQUS in the finite element analysis. The intent of this appendix is to give the reader an idea of all the necessary input definitions necessary for the ABAQUS analyses. Some sections have been abbreviated with '(a section has been removed from this point)'. The first input file is for a typical wavy layer model. The second input file is for a buckling analysis of the perfect cylinder.

Wavy Layer Model Input File

*HEADING
REDUCED INTEGRATION ELEMENTS FOR R/H=20 OUTER DBAR=2 WAVE
** NEUTRAL FILE GENERATED ON: 01-APR-92 16:56:01 PATABA VERSION: 3.1A
**
** NODE DEFINITIONS
**
*NODE, NSET=CID1
  1, 0.0000000000E+00, 0.1270800002E+02, 0.0000000000E+00
  2, 0.0000000000E+00, 0.1271024999E+02, 0.0000000000E+00
(a section has been removed from this point)
  2956, 0.167245731E+00, 0.127759037E+02, -0.9999999978E-02
  2957, 0.167265400E+00, 0.127774029E+02, -0.9999999978E-02
*TRANSFORM, TYPE=C, NSET=CID1
  0.000000E+00, 0.000000E+00, 0.000000E+00, 0.000000E+00, 0.000000E+00, -0.100000E+01
**
** NODE SETS FROM NAMED COMPONENTS
**
*NSET, NSET=FFACE
  1 2 3 4 5 6 7 8 21 22 23 24 25 33 34 35
(a section has been removed from this point)
  2915 2916 2917 2918 2919 2920 2921 2922 2923 2924 2925 2926
**
** ELEMENT DEFINITIONS
**
** WAVY ELEMENT CONNECTIVITY BEGINS HERE
*ELEMENT, TYPE=C3D20R, ELSET=PID1
  1 1 3 8 6 13 15 20 18 2 5 7 4 14 17 19
  16 9 10 12 11
*ELEMENT, TYPE=C3D20R, ELSET=PID2
  2 6 8 25 23 18 20 32 30 7 22 24 21 19 29 31
  28 11 12 27 26
(a section has been removed from this point)
*ELEMENT, TYPE=C3D20R, ELSET=PID224

228
224 1679 1740 1747 1686 1683 1744 1751 1690 1739 1745 1746 1684 1743 1749 1750
1688 1680 1741 1748 1687
** WAVY ELEMENT CONNECTIVITY ENDS HERE
*ELEMENT, TYPE=C3D20R , ELSET=PID225
225 95 97 1757 1755 102 104 1823 1821 96 1753 1756 1752 103 1819 1822
1818 98 99 1801 1800
(a section has been removed from this point)
344 2685 2687 2693 2691 2733 2735 2741 2739 2686 2689 2692 2688 2734 2737 2740
2736 2702 2703 2705 2704
*ELEMENT, TYPE=C3D20R , ELSET=PID226
345 283 344 2744 1871 287 348 2774 1919 343 2742 2743 1867 347 2772 2773
1915 284 345 2766 1903
(a section has been removed from this point)
392 2922 2579 2587 2926 2952 2645 2653 2956 2923 2584 2927 2924 2953 2650 2957
2954 2932 2604 2607 2933
**
** ELEMENT SETS FROM NAMED COMPONENTS
**
*SOLID SECTION, ELSET=PID1 , MATERIAL=MID1 , ORIENTATION=WAVY1
(a section has been removed from this point)
*SOLID SECTION, ELSET=PID224, MATERIAL=MID224, ORIENTATION=WAVY224
**
*SOLID SECTION, ELSET=PID225, MATERIAL=CIRC, ORIENTATION=CYL
**
*SOLID SECTION, ELSET=PID226, MATERIAL=AXIAL, ORIENTATION=CYL
**
** MATERIAL DEFINITIONS
*Material, NAME=MID1
** CIRCUMFERENTIAL FIBERS
*ELASTIC, TYPE=ORTHO
0.1129E+07 0.4706E+06 0.1326E+06 0.4268E+06 0.4735E+06 0.1139E+07 0.5131E+06
0.3207E+06
(a section has been removed from this point)
*MATERIAL, NAME=MID224
** CIRCUMFERENTIAL FIBERS
*ELASTIC, TYPE=ORTHO
0.1544E+07 0.5791E+06 0.1959E+06 0.5355E+06 0.5859E+06 0.1569E+07 0.8104E+06
0.5065E+06
0.8104E+06
**
** THE FOLLOWING TWO MATL DEFS. WERE MADE USING THE VOLUME FRACTION EQNS
** BUT HAD THE FIBER AND MATRIX VOLUME FRACTIONS FIXED AT .65 AND .35 RESP.
**
*MATERIAL, NAME=CIRC
** CIRCUMFERENTIAL FIBERS
*ELASTIC, TYPE=ORTHO
0.1530E+07 0.5754E+06 0.1941E+06 0.5318E+06 0.5821E+06 0.1554E+07 0.7996E+06
0.4998E+06
0.7996E+06
*MATERIAL, NAME=AXIAL
** AXIAL FIBERS
*ELASTIC, TYPE=ORTHO
0.1530E+07 0.5318E+06 0.1554E+07 0.5754E+06 0.5821E+06 0.1941E+06 0.4998E+06 0.7996E+06

Appendix D
0.7996E+06
**
*ORIENTATION, NAME=WAVY1
0.1.,0.,1.,0.,0.
3, 0.226834
(a section has been removed from this point)
*ORIENTATION, NAME=WAVY224
0.1.,0.,1.,0.,0.
3, 0.486615
**
*ORIENTATION, NAME=CYL, SYSTEM=C
0.0.,0.0.,0.0.,1.
3.0.
**
*STEP, LINEAR, AMP=RAMP
*STATIC
**
*DLOAD, OP=NEW
  217, P4, 1.00000000
(a section has been removed from this point)
  344, P4, 1.00000000
**
*CLOAD, OP=NEW
  2927, 3, 1.0710
**
*BOUNDARY, OP=NEW
  FFACE, 3., ____
    1., 2., 0.0
(a section has been removed from this point)
    2987, 3., 0.0
*EL FILE, POSITION=INTEGRATION POINTS
S
*EL PRINT, POSITION=INTEGRATION POINTS
S
*NODE FILE, GLOBAL=NO
U
*NODE PRINT, GLOBAL=NO
U
*END STEP
Buckling Model Input File

*HEADING
BUCKLE OF CLAMPED COMP CYL R/H=20 L/R=1 H=624 2ELTHK
** NEUTRAL FILE GENERATED ON: 20-MAR-92 11:22:58 PATABA VERSION: 3.1A
**
** NODE DEFINITIONS
**
*NODE, NSET=CID1
  1, 0.0000000000E+00, 0.1216800002E+02, 0.0000000000E+00
(a section has been removed from this point)
  2664, -0.111637127E+01, 0.127431965E+02, -0.124799995E+02
*TRANSFORM, TYPE=C, NSET=CID1
  0.00000E+00, 0.00000E+00, 0.00000E+00, 0.00000E+00, 0.00000E+00, -0.10000E+01
**
** NODE SETS FROM NAMED COMPONENTS
**
*NSET, NSET=FFACE
  1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
(a section has been removed from this point)
  2499 2500 2501 2502 2503 2504 2505 2506 2507 2508 2509 2510 2511 2512 2513
**
** ELEMENT DEFINITIONS
**
*ELEMENT, TYPE=C3D20R , ELSET=PID1
  1 1 1 11 9 42 44 52 50 2 7 10 6 43 48 51
  47 30 31 34 33
(a section has been removed from this point)
  432 2634 2636 210 208 2659 2661 251 249 2635 2639 209 2638 2660 2664 250
  2663 2644 2645 237 236
**
** ELEMENT PROPERTIES
**
*SOLID SECTION, ELSET=PID1, MATERIAL=BLUERPT, ORIENTATION=CYL
**
** MATERIAL DEFINITIONS
**
** SMEARED ENGINEERING PROPERTIES FOR 104 LAYER CYLINDER
** WITH A LAYUP OF [90(90/0/90)17]S
**
*MATERIAL, NAME=BLUERPT
*ELASTIC, TYPE=ENGINEERING CONSTANTS
  .14692E7,.13354E6,.7189E7,.038256,.07078,.0530,.66881E6,.56987E6
  .800E6
**
*ORIENTATION, NAME=CYL, SYSTEM=C
  0.,0.,0.,0.,0.,-1.
  3.0.
**
** MULTI-POINT CONSTRAINT
*EQUATION

Appendix D
**STEP, NLGEOM**

*STATIC, PTOL=0.1000E-02*

*DLOAD, OP=NEW*

2, P4, 100.000000

(a section has been removed from this point)

432, P4, 100.000000

**

*CLOAD, OP=NEW*

2514, 3, 51407.53

**

*BOUNDARY, OP=NEW*

1, 1, 0.0

1, 2, 0.0

(a section has been removed from this point)

2664, 1, 0.0

2664, 2, 0.0

2664, 3, 0.0

*END STEP*

**

*STEP*

*BUCKLE, DEAD*

*END STEP*

**

*STEP, NLGEOM*

*STATIC, PTOL=0.1000E-02*

*DLOAD*

2, P4, 50.00000000

(a section has been removed from this point)

432, P4, 50.00000000

**

*CLOAD*

2514, 3, 25703.77

**

*END STEP*

**

*STEP*

*BUCKLE, UVE*

1, 80

*NODE FILE, GLOBAL=NO U*

*NODE PRINT, GLOBAL=NO U*

*END STEP*
VITA

The author was born January 2, 1968 in Austin, Texas. After graduating from L. C. Anderson High School, he enrolled at The University of Texas at Austin in the fall of 1986. Upon completion of his Bachelor of Science degree in Mechanical Engineering in the spring of 1990, he enrolled for the fall semester of 1990 in the Master’s degree program at Virginia Polytechnic Institute and State University in the Engineering Science and Mechanics department. During his Master’s degree studies sponsored by the Office of Naval Research, he served an internship at David Taylor Research Center in Bethesda, Maryland.

Timothy L. Brown