A COMPARISON OF DIGITAL BEACON RECEIVER FREQUENCY ESTIMATORS

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Thesis submitted to the Faculty of
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Master of Science
in
Electrical Engineering

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January 1993
Blacksburg, Virginia
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(ABSTRACT)

Two algorithms for estimating the frequency and power of the carriers of 20 GHz and 30 GHz satellite signals are compared. Both algorithms operate on a prefiltered sequence generated by lowpass filtering followed by signal decimation for the purpose of sampling rate reduction. The lowpass filtering is accomplished via the overlap-add method of FIR filtering using the FFT. Carrier frequency prediction and tracking is accomplished with a Kalman predictor, for which the frequency drift process is modeled via polynomial extrapolation. The Kalman predictor operates on frequency measurements provided by one of two frequency estimators.

One of the frequency estimation algorithms, a refinement of the DFT-automatic frequency control technique, uses the Chirp-Transform algorithm in its aim for the maximum likelihood estimate of frequency and power. The averaged periodogram is computed from the prefiltered sequence and is used to measure the frequency of the drifting frequency signal as well as its power. One of the disadvantages of this algorithm is the bias present in the estimation of power. The bias can be removed only with knowledge of the noise power. The algorithm has the advantage of being almost exclusively a convolution and therefore is accomplished with minimal computation via the FFT.

An alternative parametric approach to frequency estimation is also investigated. In this approach the weighted least-squares modified Yule-Walker method of autoregressive model estimation is used on the prefiltered sequence to
yield frequency estimates. Power estimation is accomplished next via modal
decomposition of the estimated correlation sequence. The advantage of this
approach is that for slowly varying frequency drift paths (24 hour cycle) the
frequency estimates exhibit MSE approximately 3 dB less than the Chirp-
Transform algorithm over a wide range of SNR. There are two disadvantages to
the parametric algorithm. First the parametric algorithm estimates power with
MSE approximately 2 dB greater than the nonparametric algorithm. Secondly the
algorithm is more complicated than the nonparametric Chirp-Transform algorithm
because it requires matrix inversions and the determination of the roots of a
polynomial.

For the digital beacon receiver problem investigated here both algorithms
perform similarly in two important respects. First both algorithms can lock onto a
carrier signal whose frequency is drifting at the rate of 5 Hertz per second in a noise
environment corresponding to a 15 dB/Hz SNR. Secondly both algorithms can
make unbiased frequency estimates of the carrier signal allowing the receiver to
track the carrier at 7 dB/Hz SNR. Both algorithms attain the Cramer-Rao bound
for estimation of constant frequency sinusoids. For a simulated satellite signal with
maximum frequency drift of 5 Hertz per second the Kalman frequency predictor is
able to reduce the problem to nearly that of the constant frequency case so that the
resulting performance corresponds to the Cramer-Rao bound for estimation of
constant frequency sinusoids.

Where computational considerations are critical the nonparametric
algorithm is preferred. In fact, unless the superior accuracy of the frequency
prediction afforded by the parametric algorithm is paramount, the nonparametric
algorithm is to be chosen.
ACKNOWLEDGEMENTS

I would like to thank those who contributed to this thesis. Without their support this work would not have been possible.

I would like to first thank the SATCOM group at Virginia Tech for asking me to consider new solutions to this problem as well as providing the initial funding for this thesis. Their willingness to look into new approaches to this problem provided me with the opportunity to learn a great deal about estimation and helped me in my own interest in the theory and practice of statistical signal processing.

Many thanks to Dr. Beex without whom the quality of this document would be much lacking. I relied greatly on his insight and knowledge to understand the various topics necessary to complete this work. His standard of excellence has been a great example to me. I am also thankful for his willingness to advise me on this thesis, I would not have learned nearly as much had it not been for Dr. Beex.

Thanks to Dr. Pratt who helped in explaining the nature of the ACTS signals and also for being willing to sit in judgment on this work. Thanks also to Dr. VanLandingham who introduced me to stochastic processes through two courses offered at Virginia Tech. His teaching methods and broad knowledge of the field made random processes a very interesting and enjoyable topic. This launched me into the study of estimation and for this I am grateful.

I would lastly like to thank this University and the Department of Electrical Engineering for the great atmosphere and friendly service which so many of the University employees have shown graduate students.

I would like to dedicate this thesis to my grandmother Sadie Karouz. Her dedication, love, commitment and faith have brought us, her grandchildren, so much, both intangible and otherwise. I am sure that the countless good fortunes I have stumbled over are the treasures Sadie buried in my path.
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1.0 INTRODUCTION

1.1 THE PROBLEM OF BEACON MEASUREMENT

The need to find, measure and track a radio frequency sinusoid in band limited noise has received much attention in the past 10 years. Indeed the desire to measure atmospheric attenuation and phase distortion of RF signals from a satellite has given rise to a number of analog receiver designs which can measure the amplitude of a sinusoid down to a SNR of 17 dB per 1 Hz bandwidth [1]. That is, if a sinusoid has power equal to 1 Watt (W) then measurements of this power can be made with noise power at the level of .0199 W/Hz in the bandwidth centered around the sinusoidal frequency. This follows from

\[ 10\log \frac{|A|^2}{2N_o} = 17 \text{ dB/Hz} \]

where \( N_o \) is the noise power per Hz and the signal power is \( P = \frac{|A|^2}{2} \). SNR is therefore defined in dB/Hz since \( N_o \) is a noise power density. This is common in the literature [1] of beacon receivers.

Measurements of signal power from the Olympus satellite have been made and correlated with rain statistics by researchers at Virginia Polytechnic Institute and State University. The effect of atmospheric scintillation on propagation path attenuation of RF signals is being studied as part of NASA's interest in this phenomenon. Measurement of signal power to within .1 dB at a 2 Hz sampling rate has been suggested as more than sufficient for such studies [1]. The necessity of obtaining attenuation statistics for diverse and various propagation paths will surely continue with the recent growth and interest in mobile radio and its need for terrestrial path propagation statistics. The need for receivers which make such measurements, commonly termed "beacon receivers," is therefore well established. Figure 1.1 illustrates the fading process.

With the ever increasing affordability, speed and processing power available from integrated circuits interest has developed in implementing digital signal processing techniques in the design of beacon receivers; we will call the latter digital beacon receivers (DBR). This has led to a number of beacon receiver designs based in part or wholly on discrete time signal processing algorithms [8,9].
It has been postulated that with discrete time signal processing algorithms a receiver can measure a sinusoid at a SNR of 10 dB/Hz [1]. With such apparent promise and affordability an investigation into the fundamental principles of digital beacon receivers and some promising discrete time algorithms is in order. The development, analysis and comparison of discrete time algorithms to find, measure and track a sinusoid in bandlimited noise will be considered.

![Power Fade Illustration](image)

Figure 1.1 Power Fade Illustration

The sinusoids to be used as benchmarks in our investigation will be modeled from those generated in the NASA Advanced Communication Technology Satellite (ACTS), which will be used as a signal source in the collection of atmospheric attenuation statistics at various locations in the United States. At each location a receiving terminal will receive the 20 and 30 GHz RF signals and in successive stages mix and filter (bandlimit) the RF signal to a center frequency of 10.7 MHz and a bandwidth of 200 kHz. Within this bandwidth resides the signal to be measured. Figure 1.2 is an illustration of a typical satellite receiving process. Rain and other atmospheric phenomena cause the satellite signal to be attenuated. Atmospheric noise is also present and is a contributor to degradation of the input SNR. The other significant contributor of noise is the satellite receiver itself. The amplification of signal causes a further decrease in SNR. Our interest will be confined to algorithms on which the DBR can be based. For a more complete appraisal and presentation of the ACTS receiving system the reader is referred to
the work of Sylvester [23].

Using a simulation of the ACTS intermediate frequency signal we will compare and contrast two discrete time signal processing algorithms with the intent of making suggestions as to their usefulness and optimality.

Figure 1.2 Illustration of Satellite Receiving System

1.2 THE ACTS SIGNALS

To gain a greater appreciation for the problem of estimating the ACTS 20 GHz and 30 GHz signals we will use the parameters of these signals as provided by Jet Propulsion Laboratories [1] to generate replicas of these signals for analysis. By simulating these signals we can estimate their spectra and gain insights into some of the problems associated with measuring the power of the beacon carrier.

The ACTS signals are assumed to be centered at an intermediate frequency
(IF) of $f_c=10.7$ MHz which is synonymous with mixing and thus frequency translating the actual 20 GHz and 30 GHz signals to 10.7 MHz. Although beacon frequency drift will be a major consideration in the design of a DBR we will be able to see most of the problems associated with estimating the power in the ACTS signal by limiting our present observations to a constant carrier frequency.

By simulating the ACTS signals we will be able to view their spectra and see the type of signal we will eventually be estimating. This is done by generating, via computer simulation, the ACTS signal and computing an estimate of its power spectral density (PSD) in terms of an average periodogram (via the FFT) of these signals [7]. The sampling rate is assumed to be

$$f_s = \frac{f_c}{32} = 334375 \text{ Hz}$$

since as we shall see virtually all of the signal energy resides in a bandwidth $f_s$ around the carrier. We will further assume that the noise bandwidth is also $w=334375$ Hz; that is the signal is bandpass filtered and thus limited to $w$ Hz about the carrier. By averaging a number of these periodogram realizations we can obtain a close representation of the ACTS spectrum.

We will first develop the signal description for the ACTS 20 GHz and 30 GHz signals. The 20 GHz signal is as follows

$$s(t) = A \exp[j \beta f_0 t + \psi(t) + j \beta_1 d(t) \sin(2 \pi f_{1t}) + j \beta_2 \sin(2 \pi f_{2t})] + \sqrt{N_0 W} N_c(t)$$

(1.1a)

where $N_c(t)$ is assumed to be a complex Gaussian bandlimited (to $WHertz$ about 10.7 MHz) random process of unit variance. The frequency modulation parameters are presented in Table 1.1. Phase noise is represented by $\psi(t)$ . In our approximation of this signal will omit the phase noise, this results in the following approximation.

$$s(t) = A \exp[j \beta f_0 t + j \beta_1 d(t) \sin(2 \pi f_{1t}) + j \beta_2 \sin(2 \pi f_{2t})] + \sqrt{N_0 W} N_c(t)$$

(1.1b)
At a sampling rate of $f_s$ the following sequence results for the 20 GHz signal

$$s(n) = A \exp\left\{ \frac{2\pi f_1 n}{f_s} + j\beta_1 \sin\left( \frac{2\pi f_1 n}{f_s} \right) + j\beta_2 \sin\left( \frac{2\pi f_2 n}{f_s} \right) \right\} + \sqrt{N_o f_s} \mathcal{N}_c(n)$$  \hspace{1cm} (1.2)$$

where $\mathcal{N}_c(n)$ is a complex Gaussian white random process of unit variance. $\mathcal{N}_c(n)$ is white noise since the sampling rate $f_s$ equals the noise bandwidth $W$ (see Section 3.4). Ignoring the noise term of equation (1.1) the 20 GHz signal, with $d(t) = 1$, can be expanded into a series representation. Taking the Fourier transform of the series representation yields

$$\mathcal{F}\{s(t)\} = \begin{cases} 
A^k(\beta - f_{-k}) \ast \left( \sum_k J_k(\beta_1) \delta(f - kf_1) \right) \ast \left( \sum_l J_l(\beta_2) \delta(f - lf_2) \right) & \text{if } |f - f_{-c}| \leq W \\
0 & \text{if } |f - f_{-c}| \geq W 
\end{cases} \hspace{1cm} (1.3a)$$

where $J_k(\beta)$ is the Bessel function of the first kind of order $k$ and argument $\beta$ and ‘$\ast$’ denotes the operation of convolution. Carrying out the convolution and taking the magnitude squared this becomes

$$S_s(f) = \begin{cases} 
|A|^2 \sum_k \sum_l [J_l(\beta_2)J_k(\beta_1)]^2 \delta(f - f_{b} - kf_1 - lf_2) + N_0 & \text{if } |f - f_{-c}| \leq W \\
0 & \text{if } |f - f_{-c}| \geq W 
\end{cases} \hspace{1cm} (1.3b)$$

The harmonics of the greatest magnitude are determined from the Bessel
function values according to the following

\[ 20 \log |J_k(\beta_1)J_l(\beta_2)| \]  \hspace{1cm} (1.3c)

Due to the bandlimited nature of the received signal there are no harmonics present more than \(W/2\) from the center frequency. Table 1.2 is a listing of some of the values of the terms described by equation (1.3c).

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Table 1.2 Harmonic Power in dB Relative to Total Signal Power for the ACTS 20 GHz Signal

The 30 GHz signal is represented by

\[ s(t) = \mathcal{A} \exp[j2\pi f_b t] + \sqrt{N_o W} \mathcal{N}_c(t) \]  \hspace{1cm} (1.4)

and after sampling becomes

\[ s(n) = \mathcal{A} \exp\left\{\frac{2\pi f_b n}{f_s}\right\} + \sqrt{N_o f_s} \mathcal{N}_c(n) \]  \hspace{1cm} (1.5)

where \(\mathcal{N}_c(n)\) is a complex Gaussian white random process of unit variance. \(\mathcal{N}_c(n)\) is white noise since the sampling rate \(f_s\) equals the noise bandwidth \(W\) (see Section 3.4). The power spectrum for the 30 GHz signal is that for a pure complex sinusoid

\[ S_s(f) = |\mathcal{A}|^2 \delta(f-f_b) \]  \hspace{1cm} (1.6)
We will now consider the averaged periodogram as an estimate of the power spectral density for the ACTS 20 GHz and 30 GHz signals. The averaged periodogram is defined as [7]

\[
\hat{S}_p(f) = \frac{1}{L} \sum_{l=0}^{L-1} \frac{1}{N} \left| \sum_{n=0}^{N-1} s_{lN+n} e^{-j2\pi fn} \right|^2
\]  

(1.7a)

where \( s_m \) is a sequence of length \( LN \) and \( f \) is the normalized frequency defined on the interval \([-0.5, 0.5]\). If \( s_m \) is a sinusoid in white noise, as is the case for the 30 GHz signal, then the expected value of the peak value of equation (1.7a) is

\[
E \left[ \max_f [\hat{S}_p(f)] \right] = PN + N\sigma_s^2
\]

where \( P \) is the power of the complex sinusoid. The Fourier transform implicit in the squared term of equation (1.7a) is computed for a finite number of frequency values via the FFT. Since the FFT represents a uniform sampling of the Fourier transform the underlying spectral peaks will not generally be sampled perfectly.

For this reason the maximum value of the periodogram of equation (1.7a) is not necessarily the actual peak value of the true PSD. This is a common trade off considering the computational ease with which the FFT is obtained and as we will see is one of the obstacles associated with an FFT based DBR. With an FFT length of \( N=4096 \) and a bandwidth of \( 334375 \) Hz the frequency resolution will be \( 81.6 \) Hz. Therefore the carrier frequency must equal one of these 4096 frequencies in order for the power of the carrier to be estimated without bias beyond the expected value of the noise variance [7]. To see this the discrete averaged periodogram is as follows

\[
\hat{S}_p(k) = \frac{1}{L} \sum_{l=0}^{L-1} \frac{1}{N} \left| \sum_{n=0}^{N-1} s_{lN+n} e^{-j\frac{2\pi kn}{N}} \right|^2 \quad k = 0, 1, \ldots, 4095
\]  

(1.7b)

Thus only at the discrete normalized frequencies \( k/N \) is the value of the periodogram known. To view the periodogram we will use the following decibel scale
\[ \hat{S}_p(k)_{dB} = 10 \log[\hat{S}_p(k)] \]

The average periodograms were generated without the presence of noise and with data lengths corresponding to approximately .3 seconds of data. Figures 1.3 and 1.4 therefore represent an average of 32 periodograms each of length 4096. In addition these spectra have been normalized relative to total signal energy which in this case is the data length \( N \). This is because the average power per sample is

\[ |A|^2 = 1 \]

Defining \( k \) as in equation (1.7b) Figures 1.3 and 1.4 were generated via

\[ \hat{S}_{N,p}(k) = \frac{1}{N} \sum_{l=0}^{L-1} \sum_{n=0}^{N-1} s_{IN+n} e^{-j \frac{2\pi kn}{N}} \quad (1.7c) \]

with the expected value of the peak value of equation (1.7c) given by

\[ E[\max_k(\hat{S}_{N,p}(k))] = P + \frac{N_0 f_s}{N} \]

where \( P \) is the power of the complex sinusoid and for the 30 GHz signal this is

\[ P = |A|^2 = 1 \]

The beacon frequency \( f_b \) was set at the constant value of \( f_c = 10.7 \) MHz. The resulting 30 GHz signal of equation (1.4) is then represented by the digital DC signal

\[ s_{IN+n} = 1^n \]

Figure 1.3 is a normalized averaged periodogram, equation (1.7c), of the ACTS 30 GHz signal and Figure 1.4 is a normalized averaged periodogram of the ACTS 20 GHz signal. The normalized averaged periodogram estimates the power at the carrier frequency for \( k = 0 \). The carrier power represented by the largest peak in Figures 1.3 and 1.4 is sampled at its maximum value and is represented in the periodogram as the tallest peak. Figure 1.3 also depicts the computational limits of
the computers floating point processor. The true PSD of the 30 GHz signal simulated has no power at frequencies other than the carrier, however a 'floor' is present at 250 dB due solely to the roundoff errors inherent in computation with finite precision.

Figure 1.3 ACTS 30 GHz Signal Spectrum

Figure 1.4, as was previously mentioned, represents an average of 32 periodograms each of length 4096. This averaging minimizes the variance of the spectrum due to the relatively slow rate of the 1kbit/sec BPSK random binary stream. Therefore the power spectral density of the BPSK process is closer to its expected value in Figure 1.4 than if less data had been used to depict the 20 GHz spectrum. From Figure 1.4 we can see that the spectrum of the 20 GHz signal matches the theoretical spectrum as described by equation (1.5b) and Tables 1.1 and 1.2. Each of the harmonic peaks of Table 1.2 is present at the frequency of equation (1.5b) and Table 1.1. There are small deviations in the frequency and magnitude values.
of equation (1.5b) and Figure 1.4. These are due to two primary reasons. First, as mentioned above, the spectral plot of Figure 1.4 does not sample at the exact frequency of the harmonic peaks but rather is off by as much as 40 Hz and thus the peaks are reduced by as much as 3 dB. This would not occur if all the harmonics of equation (1.5b) were equal to the periodogram values of equation (1.7). As an example Figure 1.4 shows a peak at 19 kHz with peak value of approximately -11 dB. This peak corresponds to $k,l=0,1$ and has an actual value in the true PSD of 10.17 dB as listed in Table 1.2. Second, the power in the harmonics related to the 64 kHz BPSK signal is spread due to the PSD of $d(t)$. The power spectral density of $d(t)$ is given by [10]

$$S_d(f) = T \text{sinc}^2(\frac{f}{T})$$

Unlike equation (1.5b), where $d(t)$ is set to 1, Figure 1.4 accurately represents the random nature of the BPSK tones.

Figure 1.4 ACTS 20 GHz Signal Spectrum
These results match closely the spectral plots of General Electric [2,3]. It should be noticed that less than half of the signal power (2.82 dB) is in the modulation tones since the carrier is 2.82 dB below the total signal power which in this case is $|A|^2=1$. We can use these results as a benchmark when we seek to construct the complex signal of equation (1.3) from samples of the actual IF signal.

For spectral plots of the signals in the presence of noise, Figures 1.5 and 1.6, the same length of data was used as for Figures 1.3 and 1.4. The noise level is approximately 5 dB to 10 dB above the "clear sky" conditions of 35 to 45 dB and thus is 30 dB below the signal power in a 1 Hz band about the carrier [1]. That is the noise power density is as follows

$$N_0=10^{-30/10}=0.0010\,W/Hz$$  \hspace{1cm} (1.8)

and the average signal power, as mentioned above, is 1 W.

![Figure 1.5a ACTS 30 GHz Signal Spectra First Realization SNR=30 dB/Hz](image-url)
Figure 1.5b  ACTS 30 GHz Signal Spectra Second Realization SNR=30 dB/Hz

Data lengths corresponding to approximately .3 seconds of signal were used for the averaged periodogram of the signal with additive noise (Figures 1.5 and 1.6). This corresponds to an average of $L=82$ periodograms taken from contiguous sequences of length $N=4096$ to form a spectral estimate. The spectrum is scaled by the bandwidth or sampling frequency $f_s$ and the 'noise floor' is apparent at -30 dB for this non-normalized frequency scale. Figures 1.5 and 1.6 were generated via the following normalized averaged periodogram

$$\hat{S}_{f_s,p}(k) = \frac{1}{L f_s} \sum_{l=0}^{L-1} \frac{1}{N} \left| \sum_{n=0}^{N-1} s_{lN+n} e^{-\frac{j 2 \pi kn}{N}} \right|^2$$

(1.7d)

The signal power is therefore represented by $10 \log |N/f_s| = -19.1$ dB, since $A=1$ in
equations (1.3) and (1.4). To see this simply substitute \( N^2 \) for the magnitude squared term of equation (1.7d) (since at \( k=0 \) the Periodogram attains its maximum for this DC signal). However due to the fact that the noise variance causes variance in the periodogram the sinusoidal power is estimated with variance. In addition the scaled periodogram is a biased estimator of sinusoidal power [11]. The bias in this estimator equals \( N_0 \) [11]. Thus the expected value of the peak of Figures 1.5a and 1.5b is

\[
E \left[ \max_k S_{f_x}(k) \right] = 10 \log \left[ N_0 + \frac{N}{f_s^2} \right] = -18.7 \text{ dB} \tag{1.9a}
\]

Figures 1.5a and 1.5b are for statistically independent realizations of the ACTS 30 GHz signal in noise and are examples of the type of variance present in the periodogram as a power estimator. The peak values of the spectrum appear well above the -30 dB noise floor. The actual spectral peaks of these realizations are also present without bias since the DFT based periodogram of equation (1.7d) samples the periodogram at the beacon frequency exactly, the beacon frequency being an exact multiple of the sampling frequency.

For spectra of the ACTS 20 GHz signal Figures 1.6a and 1.6b represent realizations of averaged periodograms of this signal generated, as those of Figure 1.5, by equation (1.7d). The noise floor is again apparent at the level of -30 dB. The peak of the carrier spectral lobe is present above the 'floor' with expected value equal to

\[
E \left[ \max_k S_{f_x}(k) \right] = 10 \log \left[ N_0 + \frac{N \left| J_0(\beta_2)J_0(\beta_1) \right|^2}{f_s^2} \right] = -19.7 \text{ dB} \tag{1.9b}
\]

This result is obtained from equation (1.9a) and equation (1.5b). The decrease in this value compared with the 30 GHz averaged periodogram of equation (1.9a) is the magnitude squared term of the Bessel function product. This term accounts for the power present in the modulation tones. These realizations show variance in their estimation of the periodogram peak, each realization differing from the expected value in its estimation of the peak.
As in Figures 1.5a and 1.5b the variance is present not only in the estimation of the peak but also in the estimation of the noise level. The variance of the noise level appears to be approximately 1.5 dB. As previously stated these spectra exhibit no bias in their estimation of the frequency location of the carrier spectral peak. This is because the averaged periodogram in Figures 1.6a and 1.6b samples the carrier frequency exactly. Figure 1.6 also shows the noise level above the 64 kHz BPSK spectrum. The 19 KHz harmonics are barely noticeable above the noise level. We can thus recognize that if the frequency of the carrier is one of the 4096 frequencies sampled then the peak can be estimated without bias but with variance. This peak level however is a biased estimate of the sinusoidal power which underlies it as was shown in equation (1.9b).
If the carrier frequency is not at one of the 4096 discrete frequencies of the DFT then the spectral peak is not sampled exactly. In the case of the actual ACTS signals the carrier frequency is neither exactly known nor is it constant. Thus it will be instructive to observe the effect of an offset in the carrier frequency from the center frequency. Figures 1.7a and 1.7b were generated with the beacon frequency at 40 Hz from the center frequency

\[ f_b = f_c + 40 \text{ Hz} \]

In this way we will be able to see the effect of imperfect sampling of the spectral peak. The peak value of the periodogram of Figure 1.7a is at 0 Hz from the center frequency while the actual carrier frequency is at 40 Hz from the center frequency.
Figure 1.7a ACTS 30 GHz Signal Spectra First Realization with Carrier at 40 Hz from Center Frequency and SNR=30 dB/Hz

The frequency of the sinusoid would be estimated from this peak to be 0 Hz from the center frequency. In this case the carrier power can not be estimated accurately from the discrete averaged periodogram, since only the nearby frequencies 0 Hz and 81 Hz from center frequency are sampled. The estimate of the carrier power, with no knowledge of the actual carrier frequency, will therefore be biased by the Bartlett window [7] implicit in the periodogram estimator. Thus Figures 1.7a and 1.7b show that the discrete averaged periodogram is a biased estimator of frequency as well as power. This is apparent if we compare Figures 1.5a and 1.5b with Figures 1.7a and 1.7b. The latter figures show peaks approximately 3 dB below those of Figures 1.5a and 1.5b. There are algorithms from which more information can be culled from a periodogram regarding sinusoidal frequency [17] and power. Our effort is not to minimize the value of this
estimator but rather to bring out the limits of simple observation of its characteristics such as peak values and the discrete frequencies at which they occur.

![Graph](image)

**Figure 1.7b** ACTS 30 GHz Signal Spectra Second Realization with Carrier at 40 Hz from Center Frequency and SNR=30 dB/Hz

Similar spectra to those of Figures 1.7 are shown for the 20 GHz signal in Figures 1.8a and 1.8b. These plots exhibit similar bias as the periodograms of Figures 1.5a and 1.5b. The bias in peak value is present in these two realizations. The bias in frequency is again 40 Hz as in the case of Figures 1.7a and 1.7b. The peak values of these periodograms are markedly lower than those of Figures 1.6a and 1.6b. This is the result of having the actual carrier frequency between two evaluation points of the periodogram. The estimation of the carrier frequency is again biased since the peak occurs at the 0 Hz frequency where the discrete valued
periodogram achieves its maximum.

Figure 1.8a ACTS 20 GHz Signal Spectra First Realization with Carrier at 40 Hz from Center Frequency and SNR=30 dB/Hz

The difficulties associated with estimating the power of the carrier are now apparent. The peak of the averaged periodogram in all figures is at 0 Hz from the center frequency. Thus carrier frequency drift is a cause of bias in a periodogram based spectral estimate and it can be excessive with a low resolution (small $N$) periodogram. The resolution [7] $r_p$ is given by the following

$$r_p = \frac{f_s}{N}$$
This major shortcoming of the periodogram estimate of carrier power can be overcome in four ways. First, zero padding the signal would lead to a longer length FFT and thus more samples of the periodogram. Thus greater resolution of the averaged periodogram is obtained and the power at the carrier frequency is more closely estimated. The second approach is to sample the periodogram tightly only around the carrier. A number of algorithms are available to implement this approach such as the Goertzel algorithm [11], the Unequal resolution FFT [12], and the Chirp transform algorithm [11]. In this way the power in the beacon carrier can be more accurately estimated. Third, controlling the mixing frequency can ensure that the actual carrier frequency drift does not affect the translated and sampled carrier frequency beyond the magnitude of the innovations of the frequency drift predictor (see Section 3.1). Fourth, the sampling rate can be varied.
to ensure that the carrier frequency is a multiple of the sampling rate. A frequency predictor is required for the latter solution as well.

The other problem associated with this and any spectral estimate is that of excessive variance. Figures 1.5a through 1.8b show that the periodogram estimator has variance. These two statistically identical sequences produce spectra which differ by 1 dB (note the peak value of each) in their estimate of carrier signal power. This variance can be reduced by increasing the data length or by filtering (filtering is in effect an increase in data length). In either case there is the inherent limit of a 2 Hz bandwidth or similarly a data length corresponding to .5 seconds of observed signal (see Section 1.1). Data lengths greater than this will introduce a bias which may or may not be tolerable depending on the variance (noise level) present and the bandwidth of the actual power fade process. This does not however prohibit the use of larger data lengths for the estimation of the beacon carrier frequency. Since we can hypothesize that the carrier frequency drift process is of a bandwidth less than 2 Hz we can expect unbiased lower variance estimates of the carrier frequency with data lengths corresponding to an observation period in excess of .5 second.

The data length available is also dependent on the sampling rate and the maximum drift rate of the carrier frequency if a frequency predictor is not used. This is due to the fact that a drifting frequency beacon has power spread over a band of frequencies. This is not the case with a constant frequency complex sinusoid which has all of its power at a single frequency.

The loss of all phase information in the periodogram estimator is another disadvantage of using this estimator for the beacon measurement problem. In this regard the periodogram estimator is akin to the Harris algorithm [8]. Nevertheless phase information has been reported to be a parameter of interest [9]. The question arises as to whether accurate phase measurements, defined by S.M. Cherry [8], will be useful (indeed phase measurements have been reported as unnecessary [8]). Will the measurements be clouded with the effect of beacon phase noise and frequency drift? These considerations will not concern this investigation and phase measurements will not be considered in this research. Not surprisingly a great deal of the literature [8] addressing DBR designs is concerned only with power measurements.
Thus we see that the major problems associated with the beacon measurement problem are those of beacon frequency estimation, frequency control as well as low variance power measurement. The DBR design should address these problems.

1.3 THE PURPOSE OF THIS RESEARCH

We can now categorize the various qualities of the estimator we seek. First, unnecessary bias is to be avoided in our estimates of power. Our power estimates should also have as low a variance as possible. The main contributor of bias in the estimation of power is error in the estimation of the beacon frequency. For this reason the frequency estimator must exhibit minimum bias. Any bias in our estimate of the frequency will result in a corresponding bias in our power estimate. The same can be said for excessive variance in our frequency estimate. In short, any error in the estimate of the beacon carrier frequency will produce a corresponding bias in the beacon power measurement. While variance will be present in our power estimates bias is to be avoided.

One important component of the DBR is thus a good estimator for the beacon carrier frequency. We will therefore seek to find algorithms which can estimate the frequency of the carrier with zero bias and minimal variance. The three causes of beacon frequency estimation error are as follows. First the additive noise present in the bandpass signal. Second the frequency drift associated with the satellite movement relative to the receiver, the ACTS oscillator and the local oscillators of the receiver. Thirdly the presence of beacon phase noise. The problems associated with additive noise in the signal can be alleviated via filtering or other variance reduction methods. One such method is the computation of the correlation sequence for the signal. In this way the frequency information inherent in the signal is present but the variance of the correlation sequence estimate is greatly lower than that of the signal itself. The problems associated with frequency drift are many and are beyond the means of this writer to characterize completely. The nature of the frequency drift path becomes critical in determining the bounds on variance this causes upon the frequency estimator [18]. The approach to be taken is to maintain the carrier frequency at a near constant frequency via adaptive mixing. In this way the carrier is maintained in a narrow frequency band. In this
narrow band the frequency estimate will be of a lower variance than if the signal power is spread over the entire band in which the actual signal is drifting. The effect of phase noise on the beacon frequency estimate is more difficult to assess [8]. The approach we will take will be that of S.M. Cherry [9]. We will thus postulate an effective bandwidth of the beacon and seek frequency estimates with variance not exceeding this band.

As our approach to gaining insight into these problems and their solutions we will compare filtering with correlation sequence approximation as methods of variance reduction in frequency estimation. We will also be interested in, and explore, the importance of frequency drift estimation and seek to develop a model of this phenomenon. We will frame these inquiries by pursuing two different types of approaches. The first is non-parametric [7] and is based on the same Fourier methods as the periodogram. The considerations and problems of the periodogram estimator of Section 1.2 of this chapter will be similarly addressed in the non-parametric approach of Chapter 4. The parametric [7,13] approach will focus on the variance reduction of the correlation estimator and the principles of frequency estimation via autoregressive estimation from this correlation estimate. From these frequency estimates we will seek to decompose the correlation estimate into its modes. This approach termed modal decomposition [13] will be used to derive beacon power estimates from the estimate of the correlation sequence.

We will then have a basis for comparing the non-parametric and parametric approaches. We also would like to gain insight into some of the underlying reasons for the superior performance of either of the approaches. Along the way we will address the various issues pertaining to the beacon estimation problem.

1.4 LITERATURE REVIEW

As previously mentioned a number of beacon receivers have been designed that use digital signal processing techniques. In this section we would like to give a review of a number of these designs.

1.4.1 BAKER/BOSTIAN & SYLVESTER DIGITAL RECEIVERS

The Baker/Bostian algorithm was first proposed for the ACTS receiving
system and displays many of the features of analog/digital hybrid techniques. The receiver relies on direct digital synthesis of a sinusoid. This digital sinusoid is used to drive, via a D/A, a mixing stage which frequency translates the beacon carrier frequency to base band. The translated frequency signal is lowpass filtered and sampled before an FFT based frequency estimation algorithm is used to update the mixing frequency. Since any of the FFT outputs is in effect a filter and decimation stage the power measurement is easily taken from the FFT corresponding to the frequency measurement. A modification of this algorithm, the Sylvester algorithm, was actually chosen for the ACTS beacon receiver system [23]. The ACTS beacon receiver [23] employs a different digital filtering technique, namely multistage FIR filtering, than the Baker/Bostian receiver. In addition the Sylvester algorithm employs an analog prefilter after mixing and before sampling. Other than these modifications the algorithms are essentially similar. Beacon acquisition (the acquiring of an initial estimate of carrier frequency) is accomplished essentially through an FFT based algorithm similar to the averaged periodograms of Figures 1.3 through 1.8. The algorithm includes a novel approach to computing additional DFT values using analog mixing [23]. Figure 1.9 depicts the ACTS receiver due to Sylvester.

This algorithm does have shortcomings, the first being that the frequency update method cannot tolerate significant frequency drift. If the carrier frequency drifts at a rate greater than the closed loop bandwidth can tolerate the receiver will lose track. In addition the mixing frequency between updates is constant, thus even after translation the carrier frequency has the same drift rate as before. For this reason no improvement to the spectral spreading of the drifting frequency carrier is obtained and thus the observation bandwidth is further deleteriously increased. The detection algorithm is limited by the coarse bin width of the FFT.

![ACTS Receiver Block Diagram](image)

Figure 1.9 ACTS Receiver Block Diagram
1.4.2 SIGNAL PROCESSORS LIMITED

This algorithm first proposed direct digital mixing of the sampled beacon signal. Just as in the Baker/Bostian algorithm spectral estimation and thus frequency measurement is accomplished via the FFT. Power estimates are made from the maximum magnitude of the FFT based estimator. This receiver design employs a frequency tracking algorithm to determine the digital mixing frequency. The details of this frequency tracking algorithm were not available to the present writer. The major difference of this algorithm over the receiver of section 1.4.1 is the use of direct digital mixing. While direct digital mixing is computationally intensive the mixing signal is a relatively pure complex sinusoid. The only spectral spreading of this sinusoid is due to the clock (which is also an oscillator) which drives the direct digital synthesis. This mixing by a near pure sinusoid eliminates the spectral spreading prevalent with the analog mixers (since a local oscillator is eliminated from the feedback path. The effects of finite word length are not to be forgotten and these will introduce errors (an example of these is in Figure 1.3 above). These errors however seem to be broad band and do not tend to affect the narrowness of the spectral line.

1.4.3 CONCLUSIONS ON CURRENT DESIGNS

Other receivers such as the JPL Digital Receiver [8] are similar to the above Signal Processors Limited design. The JPL receiver design uses digital mixing for frequency control but uses an FFT for the first stage of filtering/decimation. All of the current digital receiver designs attempt to use the direct digital synthesis of the numerically controlled oscillator to maintain the carrier frequency at digital baseband. These receivers employ unspecified or crude frequency tracking algorithms to set the constant frequency of the mixing frequency signal. These receivers also rely heavily on the FFT. Indeed the detection algorithms of these receivers is based on the MLE (the periodogram) which is computationally efficient with the FFT. The improvement of the frequency tracking algorithm will be the subject of chapter 3 of this thesis and alternate digital detection algorithms from the FFT will be the subject of chapters 4 and 5.
2.0 SIGNAL GENERATION AND CONSTRUCTION

2.1 THE IMPORTANCE OF SIGNAL CONSTRUCTION

Generation of the 20 GHz and 30 GHz satellite signals is necessary to insure that the simulation of any of the DBR procedures under investigation accurately depicts its performance for the NASA ACTS satellite. It will be advantageous to construct the complex discrete time analytic signal corresponding to the IF bandlimited signal incorporating phase and magnitude into the signal to be processed [11]. In addition, the algorithms to be considered are general enough to be applied to the problem of analyzing complex sinusoids in complex white noise. In fact much of the available hardware is designed to process complex signals (such as the complex FFT). Construction of the complex analytic signal from the real IF bandlimited signal requires a Hilbert transformation, which like any realizable Hilbert transformer leads to spectral images which will need to be minimized. In-phase and quadrature sampling of a bandpass signal is a common approach to approximating a Hilbert transformation for narrowband signals [8,16] and, due to the ease of its implementation, will be considered here. An analysis of this Hilbert transformation approximation technique will be presented which will yield insights into its limitations and useful boundaries on the sampling frequency. The accuracy with which the resulting complex signal represents the complex envelope of the IF signal, and thus the beacon signal to be measured, will be considered. We will discover that the ratio of bandwidth $W$ to sampling frequency $f_s$ will need to be minimized. We will then proceed to analyze the nature of the sampled bandlimited noise. This will lead to a sampling algorithm which will reduce the degradation of the input SNR.

2.2 THE ACTS 30 GHz BEACON

We will now consider the analytic representation of the ACTS 30 GHz signals. This signal can be represented after mixing, and thus frequency translating, to a bandwidth $W$ centered at the frequency $f_c=10.7$ MHz as follows

$$r(t)=A(t)\cos[2\pi f_c t + \phi(t)] + N_W(t) \quad (2.1)$$

The noise $N_W(t)$ is assumed Gaussian and limited to the same bandwidth of $W$
centered at $f_c$. Neglecting the noise term, which will be dealt with later, this can be viewed as the real part of the following complex analytic signal

$$s(t) = A(t)e^{j2\pi f_ct} \tag{2.2a}$$

where

$$A(t) = A(t)e^{j\phi(t)} \tag{2.2b}$$

Since $A$ and $\phi$ and thus $A$ are bandlimited to 2 Hz [1] we can view them as approximately constant at sampling rates $f_s \gg 2$ Hz. Therefore, at a sampling frequency $f_s = 1/T_s$, the following discrete time signal results:

$$s(nT_s) \approx A e^{j2\pi f_b nT_s} \tag{2.3}$$

Defining the following change in parameter

$$\zeta = 2\pi f_b t$$

we can define the following

$$\sigma(\zeta) = s(t) = A e^{j\zeta} = A[\cos(\zeta) + j\sin(\zeta)] \tag{2.2c}$$

From the following trigonometric identity

$$\sin(\zeta) = \cos(\zeta + 2\pi N - \frac{\pi}{2})$$

we recognize that

$$Re\{\sigma(\zeta)\} = Im\{\sigma(\zeta + 2\pi N + \frac{\pi}{2})\} \quad N \in \text{Integers} \tag{2.4}$$

$$Im\{\sigma(\zeta)\} = Re\{\sigma(\zeta + 2\pi N - \frac{\pi}{2})\} \quad N \in \text{Integers} \tag{2.5}$$

Since the real part of equation (2.2) can be equated with the IF signal of equation (2.1) we can apply equation (2.5) to yield the following
\[ s(t) \simeq \text{Re}\{\sigma(\zeta)\} + j\text{Re}\{\sigma(\zeta+2\pi N-\frac{\pi}{2})\} \]  
\[ s(t) \simeq A[\cos(2\pi f_b t) + j\cos(2\pi f_b t+2\pi N-\frac{\pi}{2})] \]  
\[ s(t) \simeq A[\cos(2\pi f_b n T_s) + j\cos(2\pi f_b n T_s+N-\frac{1}{4})] \]

After sampling we have

\[ s(n) \simeq A[\cos(2\pi f_b n T_s) + j\cos(2\pi f_b n T_s+N-\frac{1}{4})] \]

Again it should be noted that the assumptions for this approximation are that \( A, \phi \) and \( f_b \) be bandlimited \( \ll f_s \) which of course is the case by hypothesis.

By the particular choice

\[ f_b T_s = N - \frac{1}{4} \]

equation (2.8) yields the following

\[ s(n) \simeq A[\cos(2\pi f_b n T_s) + j\cos(2\pi f_b n T_s(n+1))] \]

Equation (2.10) shows that the complex analytic signal can be constructed from samples of the IF signal. Define \( n \epsilon \text{even Integers} \), then the imaginary part of the signal can be viewed as odd samples and the real part as the even samples. Or

\[ s(n) = A[\cos(4\pi f_b n T_s) + j\cos(2\pi f_b n T_s(2n+1))] \]

This result is commonly known, namely that an approximate Hilbert transformation of a narrowband signal can be generated from samples taken of a time delayed version of the signal (which again amounts to inphase and quadrature sampling of the signal). If therefore \( T_s \) is chosen to be such a delay as described by equation (2.11) representing a 90 degree phase advance then the Hilbert transform of the bandlimited IF signal will be represented by the delayed samples. The sampling rate of the complex signal can then be reduced to the bandwidth rate via decimation, thus producing a wide band complex signal which represents the complex evelope of the IF signal at the frequency \( f_b \). We will now determine what occurs at frequencies other than \( f_b \).
This method of Hilbert transformation (complex signal construction) requires the sampling rate to be related to the beacon frequency as

\[ f_s = f_b / [N - \frac{1}{4}] \]  \hspace{1cm} (2.12)

The beacon frequency \( f_b \) is the unknown and time varying parameter to be estimated thus the sampling rate will never perfectly match equation (2.12). We must therefore consider the effect of sampling rate error on the analytic signal approximation. Consider \( f_{be} \) to be the estimate of the beacon frequency. Now define the frequency error as

\[ \Delta f_b = f_b - f_{be} \]  \hspace{1cm} (2.13)

we would then choose the sampling frequency

\[ f_s = f_{be} / [N - \frac{1}{4}] \]  \hspace{1cm} (2.14)

Substituting equations (2.13) and (2.14) into equation (2.11) yields

\[ s(n) = \mathcal{A} \left\{ \cos(\theta n) + j \cos \left( \theta n + 2\pi \left( f_{be} + \Delta f_b \right) N - \frac{1}{2} \right) \right\} \]  \hspace{1cm} (2.15)

where \( \theta = 4\pi f_b T_s \). Now define \( \psi \) as

\[ \psi = 2\pi \Delta f_b \frac{N - \frac{1}{2}}{f_{be}} = 2\pi \Delta f_b T_s \]  \hspace{1cm} (2.16)

so that we have

\[ 2\pi (f_{be} + \Delta f_b) [N - \frac{1}{4}] / f_{be} = 2\pi N - \pi / 2 + 2\pi \Delta f_b [N - \frac{1}{4}] / f_{be} = 2\pi N - \pi / 2 + \psi \]

which yields the following for equation (2.15)

\[ s(n) = \mathcal{A} \left\{ \cos(\theta n) + j \sin(\theta n + \psi) \right\} \]  \hspace{1cm} (2.17)

Had we chosen \([N + 1/4]\) in equation (2.14) we would have the following
\[ s(n) = A \left[ \cos(\theta n) - j \sin(\theta n + \psi) \right] \]  

\[ s(n) = \frac{A}{2} \left[ (1 + e^{j \psi}) e^{j \theta n} + (1 - e^{-j \psi}) e^{-j \theta n} \right] \]  

And equation (2.18) becomes

\[ s(n) = \frac{A}{2} \left[ (1 + e^{j \psi}) e^{-j \theta n} + (1 - e^{-j \psi}) e^{j \theta n} \right] \]

Thus the conjugate sequence present in the original signal is not totally eliminated but rather is attenuated. This conjugate sequence is represented by the following in equation (2.19)

\[ s^-(n) = \frac{A}{2} \left[ 1 - e^{j \psi} \right] e^{-j \theta n} \]  

This is symptomatic of any realizable Hilbert transformer. The magnitude of this residual image sequence equation (2.21) will increase with increasing \( \psi \) and therefore, from equation (2.16), with increasing deviation of the actual beacon frequency \( f_b \) from the expected beacon frequency \( f_{bc} \). Therefore \( \psi \) is to be minimized. Since \( \psi \) is proportional to \( N \) from equation (2.16), \( N=0,1 \) or 2 is preferable. If we are to have a constant sampling rate then we should set the sampling rate, as in equation (2.14), with \( f_{bc} = f_c \). This yields a minimal power loss to the negative frequency. The phase distortion is apparent as a multiplicative factor of \( [1 + e^{j \psi}] / 2 \) and is correctable along with the power by multiplying the measured signal by the inverse of this term. Choosing \( 1-1/4 \) sampling (\( N=1 \) in equation (2.14)) results in a relatively low \( \psi \) and provides a sampling frequency \( f_s \) which is realizable [13]. It should also be noted that the maximum \( \psi \) is related to the bandwidth \( W \) (see equation (2.16)) since the maximum frequency deviation \( \Delta f_b \) is bounded by half the bandwidth by hypothesis. This suggests that the bandwidth should be minimized. The analytic signal sampling rate is \( f_s / 2 \) since each sample is formed from two samples (an even and an odd) of the IF signal. This data rate can be reduced to the signal bandwidth/nyquist rate via decimation. Employing
equation (2.11) with a sampling rate $f_s$ as in equation (2.14) yields a complex
envelope representation of the bandlimited IF signal. Figure 2.1 depicts the power
loss of this Hilbert transformation for $N=1$. Figure 2.1 is a plot of the following
function

$$P_{loss} = \left| \frac{1 - e^{-j\psi}}{2} \right|^2$$

where $\psi$ is defined as in equation (2.16) and $\Delta f_b$ is the variable $P_{loss}$ is plotted
against.

![Figure 2.1 Power Loss Due to Sampling Rate Error](image)

The phase distortion of the Hilbert transformation is depicted in Figure 2.2. Figure
2.2 is a plot of the following function in degrees.

$$\arg \left[ \frac{1 + e^{j\psi}}{2} \right]$$
Figure 2.2 Phase Distortion Relative to Sampling Rate Error

To reduce the sampling rate to the bandwidth we decimate the complex analytic signal by a factor of $D$ to a rate of approximately 200 kHz. This results in the following representation of the beacon analytic signal

$$s(n) = \frac{A}{2} \left( [1+e^{j\psi}]e^{j\theta Dn} + [1-e^{j\psi}]e^{-j\theta Dn} \right) \quad (2.22)$$

with $f_s=14.267 \text{ MHz}$ and $D=36$, the bandwidth will be approximately 198 kHz. Calling $f_{sc}$ the final complex sampling rate we have the following

$$f_{sc} = \frac{f_s}{2D} \quad (2.23a)$$

The 30 GHz beacon can therefore be represented by sampling a sinusoid at $1-1/4$
cycles per sample, pairing successive samples to form complex samples, and
decimating this complex signal by 36 to reduce the sampling rate to the effective
bandwidth. In general the decimation rate should be chosen to be the largest
integer that satisfies

\[ D \leq \frac{f_s}{2W} \quad (2.23b) \]

The simulations presented will mimic this procedure exactly thus the
simulation of the 30 GHz time series is as follows

\[ s_D(n) = A(n) \cos \left( \frac{4\pi f_p n D}{f_s} \right) + jA \cos \left( \frac{2\pi f_b (2nD+1)}{f_s} \right) \quad (2.24) \]

2.3 THE ACTS 20 GHz BEACON

We are now ready to consider the 20 GHz signal which is represented by

\[ r(t) = A(t) \cos[2\pi f_1 t + \beta_1 d(t) \sin(2\pi f_1 t) + \beta_2 \sin(2\pi f_2 t) + \phi(t)] + N_W(t) \quad (2.25) \]

where \( A, f_b \) and \( \phi \) are the random parameters to be estimated. The modulation
tones (see Table 1.1) are apparent in the 20 GHz signal and are repeated here

\[ \beta_1 = \beta_2 = 0.79, \quad f_1 = 64 \text{ kHz}, \quad f_2 = \{19.27, \text{ or } 32 \text{ kHz (3 possible modes)}\} \]

and \( d(t) = \{1,-1\} \) at 1024 bit/sec.

The beacon frequency is bounded as follows

\[ f_c - \frac{W}{2} = f_c - 100 \text{ kHz} < f_b < f_c + 100 \text{ kHz} = f_c + \frac{W}{2} \]

The noise \( N_W(t) \) is assumed Gaussian and limited to the same bandwidth of \( W \)
centered at \( f_c \). In addition the maximum expected drift associated with the beacon
frequency \( f_b \) is 6 Hz/sec. This signal can be represented in the following way if we
assume \( d(t) \) to be constant and neglect the noise and phase terms

\[ r(t) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} J_k(\beta_1) J_l(\beta_2) A(t) \cos[2\pi (f_b - k f_1 - l f_2) t] \]
Employing the same method of signal construction as with the 30 GHz signal, see equation (2.11), we notice that the carrier $(k,l=0,0)$ will be represented in the same way as that developed for the 30 GHz beacon. We can also notice that the other tones $(k,l \neq 0,0)$ will cause an increase (or a decrease) in $\Delta f_b$ by the amount $k f_1 + l f_2$. We can therefore define a modulation deviation frequency as we did with the carrier tone

$$\Delta f_m = \Delta f_b + k f_1 + l f_2$$

For $(k,l \geq 2,2)$ we can see that this deviation becomes substantial. However since we have assumed that the signal is bandlimited to 200 kHz we will not be concerned with the Hilbert transformation of tones which are beyond this band. Also as was shown in Section 1.2 the power present in such tones $(k,l \geq 2,2)$ is very small, thus the conjugate image sequence is extremely small relative to the carrier. The signal construction is therefore the following for the 20 GHz signal

$$s(n) = A(n) \left[ \cos \left( \Sigma (2nD) \right) + j \cos \left( \Sigma (2nD+1) \right) \right]$$  \hspace{1cm} (2.26)

where

$$\Sigma (m) = \left[ \frac{2 \pi f_b m}{f_s} + \psi (m) \right]$$

$$\psi (m) = \left[ \frac{2 \pi f_b m}{f_s} + \beta_1 \right] \left[ \frac{m}{f_s} \right] \sin \left( \frac{2 \pi f_2 m}{f_s} \right) + \beta_2 \sin \left( \frac{2 \pi f_2 m}{f_s} \right)$$

Equation (2.26) is our approximation of the following actual analytic signal

$$s(n) = A(2n) \exp \left[ \frac{4 \pi f_b n D}{f_s} + j \beta_1 \frac{2 \pi n D}{f_s} \sin \left( \frac{4 \pi f_2 n D}{f_s} \right) + j \beta_2 \sin \left( \frac{4 \pi f_2 n D}{f_s} \right) \right]$$ \hspace{1cm} (2.27)

As just mentioned this FM signal can be represented as an infinite sum of
sinusoids each with its own deviation from the center frequency. Employing the same complex envelope construction of equation (2.25) each sinusoid will produce a conjugate image, equation (2.21), which will interfere/overlap with the spectra of other tones. For suitably chosen $f_s$ these conjugate tones are extremely small relative to the carrier, the worst case being that depicted in Table 1.1, and the spectral interference caused by them will not be stationary due to the drift of the beacon carrier frequency $f_b$. Thus we will gain nothing from calculating the countless $\psi$’s, (see Table 2.1), associated with each tone.

### 2.4 ANALYSIS OF BANDLIMITED NOISE

We are now ready to characterize the effect of the bandlimited noise ($N_W$ of equations (2.1) and (2.25)) on the signal construction. Consider the uniform (white) bandlimited spectrum of the IF noise and its inverse Fourier transform, the autocorrelation function $R_{NN}(t)$.

$$R_{NN}(t) = P(f) = \frac{N_o}{2} \left[ \text{Rect} \left( \frac{f-f_c}{W} \right) + \text{Rect} \left( \frac{f+f_c}{W} \right) \right]$$

(2.28)

where $\text{Rect}$ is the rectangular function defined as follows

$$\text{Rect}(\alpha) = \begin{cases} 
1 & \text{for } |\alpha| < \frac{1}{2} \\
0 & \text{for } |\alpha| > \frac{1}{2} 
\end{cases}$$

$W$ is the IF bandwidth, $f_c$ is the IF center frequency and $N_o$ is the noise power per Hz. The auto correlation is then evaluated to be

$$R_{NN}(t) = N_o W \text{sinc}(Wt) \cos(2\pi f_c t)$$

(2.29)

After sampling with $T_s = 1/f_s = \frac{N-1/4}{f_c}$ the result is

$$R_{NN}(m) = E \left[ n(k) n(k+m) \right] = N_o W \text{sinc}(WT_{sm}) \cos \left( \frac{m\pi}{2} \right)$$

(2.30a)

Now we define the complex noise corresponding to the analytic signal construction process. $N_c(n)$ is now considered the complex noise formed by pairing the adjacent
samples as described by equations (2.12) and (2.30b).

\[ N_c(k) = n(2k) + jn(2k+1) \quad (2.30b) \]

We can now readily evaluate the correlation, \( R_{NN}(m) \) for even and odd values of \( m \) in equation (2.30a). For even values of \( m \) \( R_{NN}(m) \) corresponds to the autocorrelation of \( Re\{N_c(n)\} \) or \( Im\{N_c(n)\} \). For odd values of \( m \) \( R_{NN}(m) \) corresponds to the crosscorrelation of \( Re\{N_c(n)\} \) and \( Im\{N_c(n)\} \). The latter crosscorrelation, \( m \) odd in equation (2.30a), is

\[ R_{Rr}(m) = E \left[ Re\{N_c(k)\} Im\{N_c(k+m)\} \right] = 0 \quad (2.31) \]

since \( \cos \left( \frac{m\pi}{2} \right) = 0 \) for \( m \) odd.

Thus, as is commonly known, \( Re\{N_c(k)\} \) and \( Im\{N_c(k)\} \) are orthogonal [14]. The autocorrelations, \( m \) even in equation (2.30) are as follows

\[ R_{RR}(m) = R_{II}(m) = E \left[ Re\{N_c(k)\} Im\{N_c(k)\} \right] = N_o W \text{sinc}(2 \pi T s m) \cos(m\pi) \quad (2.32) \]

This yields a correlation coefficient for the samples \( Re\{N_c(k)\} \) and \( Im\{N_c(k)\} \) as follows

\[ r_{RR}(m) = r_{II}(m) = \frac{R_{RR}(m)}{R_{RR}(0)} = \text{sinc}(2 \pi T s m) \cos(m\pi) \]

After decimation by \( D \) the autocorrelations of \( Re\{N_c(k)\} \) and \( Im\{N_c(k)\} \) become

\[ R_{RR}(m) = R_{II}(m) = N_o W \text{sinc}(2 \pi T D s m) \cos(Dm\pi) \quad (2.33a) \]

From which we can see that if \( D \) is chosen such that \( 2DT_s = 1/W \) or that the sampling rate of the complex noise is reduced to the bandwidth we have the following

\[ R_{RR}(m) = R_{II}(m) = N_o W \delta(m) \quad (2.33b) \]

This is exactly the situation we have already encountered in the construction
of the beacon envelope signal where we reduced the sampling rate via decimation to the bandwidth. Now we have an accurate way of modeling the noise. Equation (2.33b) tells us that the real and imaginary parts of the complex Gaussian noise are statistically independent and uncorrelated with variance $N_0 W$. We have assumed here that the noise is Gaussian, as in Section 1.2, and thus we have the following as our simulation model

$$
\text{Re}\{s(n)\} = A\left(\frac{2nD}{f_s}\right) \cos\left(\frac{4\pi f_b n D}{f_s} + \beta_1 d\left(\frac{2nD}{f_s}\right)\sin\left(\frac{4\pi f m_1 n D}{f_s}\right)\right) + \\
+ \beta_2 \sin\left(\frac{4\pi f m_2 n D}{f_s}\right) + \alpha\left(\frac{2nD}{f_s}\right) + N_0 W N_r(n)
$$

$$
\text{Im}\{s(n)\} = A\left(\frac{2nD+1}{f_s}\right) \cos\left(\frac{2\pi f_b (2nD+1)}{f_s} + \beta_1 d\left(\frac{2nD+1}{f_s}\right)\sin\left(\frac{2\pi f m_1 (2nD+1)}{f_s}\right)\right) + \\
+ \beta_2 \sin\left(\frac{2\pi f (nD+1)}{f_s}\right) + \alpha\left(\frac{nD}{f_s}\right) + N_0 W N_i(n)
$$

(2.34)

where $N_r(n)$ and $N_i(n)$ are white independent identically distributed Gaussian random variables with unit variance.

It is important to observe that under this sampling scheme and signal construction methodology the magnitude of the correlation coefficient of adjacent complex samples is less than 1. This would imply that some advantage could be gained by averaging these samples in an effort to decrease our input SNR. If the $D-1$ other sequences are discarded the signal is still uniquely represented. However the statistical value of the other $D-1$ sequences can be used to decrease the variance of the signal represented by equation (2.34) from $2N_0 W$. Since our goal is to estimate the values of $A$ and $f_b$, any decrease in noise variance will lead to a corresponding decrease in the variance of our estimates of $A$ and $f_b$. This as we will see can be accomplished rather simply and with virtually no computational effort. The importance of this reduction in variance can not be over-emphasized and will prove to be invaluable regardless of the algorithm chosen to measure the power of the beacon. It will be worth our while to analyze each of the $D$ statistically independent sequences and see how averaging (or adding) them can be accomplished and what benefits will result.

Still considering the 20 GHz signal we will attempt to tractably represent
the $D$ independent sequences. Let the sequence of complex samples at a rate of $f_s/2$ be viewed as $D$ time-interleaved sequences at a rate of $f_s/2D$. Thus the analytic sequence before decimation is seen as the partitioned or sectioned sequence described as follows

$$s_i^n = \mathcal{A} \exp \left[ j \frac{4\pi f_b(nD+i)}{f_s} + j\beta_1 d \left( \frac{2nD}{f_s} \right) \sin \left( \frac{4\pi f_{m1}(nD+i)}{f_s} \right) + j\beta_2 \sin \left( \frac{4\pi f_{m2}(nD+i)}{f_s} \right) \right]$$  \hspace{1cm} (2.35)$$

where $i \in [0,1, \ldots, D-1]$ indicates one of the individual sequences approximated by equation (2.34). It should be noted that we have now ignored the negligible effects due to our realizable Hilbert transformation and assume the perfect complex exponential description of equation (2.35). In addition we have replaced $n$ with $2n$ thus allowing $n$ to assume odd and even values. Assuming $\mathcal{A}$ is approximately constant over a period of $2D/f_s$ it requires no representation with respect to $i$. Also since $d$ operates at the relatively slow rate of 1024 bits/sec we will assume it to be constant between samples $i$. With the sampling period

$$T_s = \frac{1}{f_s} = \frac{1-25}{f_c} \text{ and } f_b = f_c + \Delta f$$  \hspace{1cm} (2.36)$$

The following results

$$s_i^n = \mathcal{A} \exp \left[ j \frac{3\pi f_b(nD+i)}{f_c} + j\beta_1 d_n \sin \left( \frac{3\pi f_{m1}(nD+i)}{f_c} \right) + j\beta_2 \sin \left( \frac{3\pi f_{m2}(nD+i)}{f_c} \right) \right]$$  \hspace{1cm} (2.37)$$

Defining the following

$$\Theta_{1,n}^i = \frac{3\pi f_{m1}(nD+i)}{f_c}$$

$$\Theta_{2,n}^i = \frac{3\pi f_{m2}(nD+i)}{f_c}$$

$$\Psi_n^i = \beta_1 d_n \sin \Theta_{1,n}^i + \beta_2 \sin \Theta_{2,n}^i$$  \hspace{1cm} (2.38)$$

this leads to: $0 < |\Psi_n^i| < \beta_1 + \beta_2$. Letting $f_b = f_c + \Delta f$ equation (2.37) results in
$$s_n^i = A \exp \left[ j3\pi nD + j3\pi i + j \frac{3\pi}{f_c} \Delta f (nD + i) + j\Psi_n^i \right]$$

(2.39)

Consider now the linear transformation or weighted average with $a_i$ to be determined

$$s_{a,n} = \sum_{i=0}^{D-1} a_i s_n^i = A \sum_{i=0}^{D-1} a_i \exp \left[ j3\pi nD + j3\pi i + j \frac{3\pi}{f_c} \Delta f (nD + i) + j\Psi_n^i \right]$$

(2.40)

Since $s_{a,n}$ represents the signal component we wish to maximize the modulus of $s_{a,n}$ which would require $a_i$ to be as follows

$$a_i = (-1)^i \exp \left[ -j \frac{3\pi \Delta f}{f_c} \frac{i}{\Psi_n} \right]$$

(2.41)

With the use of frequency tracking (see Chapter 3) a control loop can be employed to maintain the beacon frequency at the center of the band where $\Delta f = 0$. In this case we have

$$a_i = (-1)^i \exp \left[ -j\Psi_n^i \right]$$

(2.42a)

A close approximation and computationally burdenless estimate to equation (2.42a) is the following

$$a_i = (-1)^i$$

(2.42b)

For the 30 GHz signal we can recognize that $\Psi_n^1 = 0$. Thus for the 30 GHz beacon we have

$$s_{a,n} = (-1)^n D A \exp \left[ -j \frac{3\pi \Delta f nD}{f_c} \right] \sum_{i=0}^{D-1} a_i (-1)^i \exp \left[ -j \frac{3\pi i \Delta f}{f_c} \right]$$

(2.43)

Again a control loop which keeps $\Delta f \approx 0$ will be an optimal solution since $s_{a,n}$ is maximized under this condition. We then have the following

$$a_i = (-1)^i$$

(2.45)
Thus equation (2.43) is reduced to the following with \( D \) even and \( \phi_f \) defined as follows

\[
\phi_f = \sum_{i=0}^{D/2-1} \exp \left[ \frac{3\pi i \Delta f}{f_c} \right]
\]

\[
s_{a,n} = A \exp \left[ \frac{3\pi \Delta f n D}{f_c} \right] \phi_f
\]

(2.46)

We should note that with \( D \text{ odd} \) we would have the following result

\[
s_{a,n} = A \exp \left[ \frac{3\pi \Delta f n D}{f_c} + j3\pi n D \right] \phi_f
\]

The problem of estimating \( A \) could be considered one of envelope detection or demodulation if \( A \) were not so band limited (2 Hz). As we will see estimating \( A \) will be best accomplished by seeking a spectral estimate of \( s_{a,n} \). Equation (2.43) shows that the modulus of \( s_{n}^i \), our time series, can be increased by a factor of \( |\phi_f| \) simply by switching signs via \( a_i \) and adding \( D \) contiguous samples. This will yield a SNR increase since the magnitude of the correlation coefficients of these signals is less than 1. In addition the phase will also be skewed by the phase of \( \phi_f \). However, since \( \phi_f \) is dependent only on the frequency deviation of the beacon carrier from the center frequency and the maximum beacon frequency drift between estimates of \( A \) is relatively small the deviations in \( \phi_f \) are negligible. A more satisfying way of regarding this improvement is to view equation (2.43) as an FIR lowpass filter preceded by a digital multiply with the signal \((-1)^n\). The multiply changes the frequency of the signal so that it is near baseband allowing the adding of the signal samples to be constructive.

The simulations of DBR algorithms in this paper will not utilize this oversampling and averaging technique. This would require the generation of bandlimited noise. We will be content to test our algorithms on white noise as mentioned in equation (2.33b) and (2.34) since our main consideration is in comparing DBR algorithms.

2.5 EFFECT OF BEACON PHASE NOISE

The ACTS 20 GHz and 30 GHz signals exhibit phase noise [8] like any
oscillator. This phase noise results in a slight spreading of the power of the carrier. The resulting spectrum can be approximated by the following formula [8]

\[ S(f) = \frac{c}{(f - f_b)^3} \quad f \neq f_b \quad (2.47) \]

where \( c \) is an unknown constant and \( f \) is the frequency deviation from the 'actual' carrier frequency. Thus rather than being a delta impulse the ACTS signals are described as in equation (2.47). This spreading of power results in a degradation of SNR around the carrier. In this way beacon phase noise degrades the performance of a DBR [8]. Beacon phase noise also presents a problem to our simulation. We have ignored its effects however its presence will result in limits in the observation bandwidth [8] of the DBR algorithms. Most importantly for our study beacon phase noise is difficult to accurately simulate. To generate a signal with such a narrow bandwidth as equation (2.47) represents is a very difficult task. Even if a linear equation could be derived (and we know that one of sufficient degree could) the time available to generate the large amounts of data necessary to test our algorithms prohibits it. We must therefore limit ourselves in the generation of the ACTS signal. In addition we will find that the lower bounds on variance for our estimators are known for the sinusoid without phase noise. We will use the aforementioned generating equations which do not account for the phase noise present. This assumption will be kept in mind when we make decisions regarding observation bandwidths and any performance conclusions.

It has been reported [9] that the effective rms bandwidth of the Olympus beacons due to the presence of phase noise is less than 1 Hz. Although such accurate statistics could not be obtained regarding the ACTS beacons we will attempt to estimate the beacon frequency to within 1 Hz. This implies that much is to be gained by accurately measuring the beacon frequency. If we assume that the beacon carrier is confined to a 1 Hz bandwidth then theoretically the carrier power can be measured in this small bandwidth. The deleterious effects of beacon carrier frequency drift must therefore be mitigated if this goal is to be met.
3.0 A FREQUENCY TRACKING ALGORITHM

3.1 THE IMPORTANCE OF FREQUENCY TRACKING

The need to estimate and predict the frequency of the beacon carrier is well established (see Sections 1.2 and 2.4) and is commonly addressed in the literature of beacon receivers [8]. Knowledge of the instantaneous beacon frequency is important for two reasons. First it allows the DBR to control the center frequency of the observation bandwidth via digital or analog mixing. With this as a controllable parameter the beacon frequency can be maintained at a near constant frequency with the simple use of digital mixing. In this way the beacon carrier power is distributed in a much narrower band and thus the SNR in this band is increased. As a comparison consider a sinusoid which is drifting in frequency at a rate of 5 Hz/sec. Over a 2 second interval the power of this signal is spread over 10 Hz. Filtering this signal within this 10 Hz band will lead to a reduction of signal power. Thus no improvement of SNR can be gained beyond the 10 Hz band. It is this use of frequency prediction that we will rely on with the parametric and nonparametric DBR of the following chapters and an algorithm will be developed to accomplish this in this chapter. Secondly accurate power measurements of the beacon can be made only with a knowledge of the beacon carrier frequency. Thus if the variance of the frequency estimate can be lowered then more accurate (in this case unbiased) power measurements can be made.

Our goal therefore is to minimize the variance of the beacon frequency estimate. This will be done by modeling the frequency drift path as an appropriate Gauss-Markov random variable. From this model a Kalman predictor will be used to extrapolate the beacon frequency for a future time period. This approach to prediction is common and has been successfully used for innumerable phenomena including economic demand forecasting [14]. In short the goal of this section is to decrease the variance of the frequency estimate from the variance of the measurements to the small variance innovations of the Kalman predictor.

With the Kalman estimator we will be able to use longer datalengths to make frequency estimates and thus reduce the variance of these estimates. With lower variance frequency estimates we will be able to make unbiased power estimates at lower SNR than without the Kalman estimator. This is due to the
fact that the carrier can be maintained via mixing in a small band over a longer period of time. The datalength used to estimate the frequency can therefore be increased to beyond that of the power estimate datalength. We will see that the limit on the frequency estimate datalength will depend on the time period over which the model of the drift process is valid.

Figure 3.1 illustrates a frequency drift path for the carrier of a typical satellite signal. The figure is meant only as an illustration and is not an actual realization of the ACTS signal.

![Frequency vs Time](image)

**Figure 3.1 Illustration of a Typical Frequency Drift Path**

### 3.2 A USEFUL MODEL OF FREQUENCY DRIFT

To permit development of a robust tracking algorithm two assumptions will be made:

A1. The beacon frequency is assumed continuous over time. That is all time derivatives of the instantaneous frequency are bounded.

This implies that the carrier frequency does not change discontinuously. It also implies that the beacon frequency $f(t)$ can be expanded into a Taylor series about the point $t_o$.

$$f(t) = \sum_{n=0}^{\infty} e_n(t_o) (t-t_o)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(t_o)}{n!} (t-t_o)^n \quad (3.1a)$$
Or more generally
\[ f^{(i)}(t) = \sum_{n=i}^{\infty} \frac{f^{(n)}(t_0)}{(n-i)!} (t-t_0)^{n-i} \] (3.1b)

where \( f^{(i)}(t) \) is the \( i^{th} \) derivative of the beacon frequency. Thus equation (3.1a) is the special case of equation (3.1b) with \( i=0 \).

A2. The first \( N \) derivatives are sufficient to characterize the beacon frequency over a given time period for a prespecified accuracy.

Or explicitly, that there exists \( N_0 \) such that given any \( N \geq N_0 \) then

\[ |f^{(i)}(t) - f_N^{(i)}(t)| \leq |\epsilon_{i,t}|, \quad \text{for any } |\epsilon_{i,t}| > 0 \]

where
\[ f_N^{(i)}(t) = \sum_{n=i}^{N} \frac{f^{(n)}(t_0)}{(n-i)!} (t-t_0)^{n-i} \] (3.2)

provided for some given \( \delta > 0, |t-t_0| < \delta \)

Therefore \( f_N^{(i)}(t) \) is an \((N-i)^{th}\) order polynomial approximation to \( f^{(i)}(t) \). Also the beacon frequency drift with respect to time is approximated by an \( N^{th} \) order polynomial. We can conveniently define an approximation error \( \epsilon_{i,t} \) such that

\[ f^{(i)}(t_0) = f_N^{(i)}(t_0) + \epsilon_{i,t_0} \] (3.3)

Thus from equation (3.2) if the first \( N \) Taylor series coefficients can be estimated then the beacon frequency \( f^{(i)}(t) \) can be extrapolated. Equation (3.3) provides an approximation to the \( N \) derivatives implicit in the coefficients of equation (3.2) provided the error terms can be approximated. The accuracy with which these parameters can be estimated will determine the accuracy with which \( f(t) \) can be estimated. The only assumption, as with any Kalman predictor, is that the model is valid. The assumptions we have made are general and applicable to most phenomena whose characteristics depend on forces such as temperature, as is the case with oscillators. The model does not include any periodic component relating to Doppler effects induced by the satellite's diurnal path. Tracking of these deterministic variations will be absorbed into the model of equation (3.2).
We will now develop an implementable recursive equation from the assumed model. Substituting (3.3) into (3.2) yields the following

\[
f_N^{(j)}(t) = \sum_{n=i}^{N} \frac{f_N^{(a)}(t_0)}{(n-i)!} t_{n-i} \quad (3.4)
\]

\[
= \sum_{n=i}^{N} \frac{f_N^{(a)}(t_0)}{(n-i)!} (t-t_0)^{n-i} + \sum_{n=i}^{N} \frac{\epsilon^{(a)}(t_0)}{(n-i)!} (t-t_0)^{n-i} \quad (3.5)
\]

Substituting \( t_0 = (k-1)T \) and \( t = kT \)

\[
f_N^{(j)}(k) = \sum_{n=i}^{N} \frac{f_N^{(a)}(k)}{(n-i)!} T^{n-i} + \epsilon_{i,k} \quad (3.7a)
\]

where we have defined

\[
\epsilon_{i,k} = \sum_{n=i}^{N} \frac{\epsilon^{(a)}(t_0)}{(n-i)!} T^{n-i} \quad (3.7b)
\]

From equation (3.7) a state transition matrix equation can be developed. First we define the frequency state and error vectors as follows

\[
 f_k = \begin{bmatrix} f_N^{(0)}(k) \\ f_N^{(1)}(k) \\ \vdots \\ f_N^{(N)}(k) \end{bmatrix} \quad e_k = \begin{bmatrix} \epsilon_{0,k} \\ \epsilon_{1,k} \\ \vdots \\ \epsilon_{N,k} \end{bmatrix} \quad (3.8)
\]
Then writing equation (3.7) for \( i = 0, \ldots, N \)

\[
f_{k+1} = A f_k + e_k \tag{3.9}
\]

The state matrix can be recognized as

\[
A = \begin{bmatrix}
1 & T & \frac{T^2}{2!} & \frac{T^3}{3!} & \cdots & \frac{T^N}{N!} \\
0 & 1 & T & \frac{T^2}{2!} & \cdots & \frac{T^{N-1}}{(N-1)!} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & T \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix} \tag{3.10}
\]

The actual instantaneous beacon frequency is then given by

\[
f(k) = c f_k \quad c = [1 \ 0 \ \ldots \ 0] \tag{3.11}
\]

Equations (3.9) and (3.11) describe the beacon carrier frequency drift as a Gauss-Markov state equation. The error vector \( e_k \) which can be viewed as driving the frequency drift process, is not necessarily white and is clearly unobservable. We will assume that the state model is accurate enough that \( e \) is zero mean. Let the covariance of this process be \( Q \) an \( N \times N \) positive definite matrix. Thus assuming \( e \) is white and the \((N-1)th\) derivative is fairly accurately modeled as linear over the sample period \( T \) then \( Q \) can be hypothesized to be

\[
Q = \begin{bmatrix}
0 & 0 \\
\vdots & \vdots \\
0 & 0 \ q
\end{bmatrix} \tag{3.12a}
\]
where $q$ is the variance of $e_{N,k}$ thus $e_k$ can be modeled as follows

$$
e_k = b \, e_{N,k} \quad \quad b = [0 \ldots 0 \, 1]^T \quad (3.12b)$$

It remains to determine the model order $N$. To do so we will hypothesize that the rate of frequency drift change is very small. This assumption coupled with the upper bound on the frequency drift rate can be succinctly stated as

$$f_N^{(2)}(k) \ll f_N^{(1)}(k) < 6 \quad (3.13)$$

Thus the carrier drift is assumed to be approximately linear over small intervals of time. We can now choose a model order of $N = 2$, which implies a rank of 3 for the transition matrix $A$.

3.3 THE KALMAN PREDICTOR AS A FREQUENCY TRACKER

We will now discuss the Kalman predictor as applicable to the above model. Let $\hat{f}(k)$ be the observation of the beacon carrier frequency at the $k^{th}$ instant. This observation has an associated measurement variance and can be modeled as

$$\hat{f}(k) = f(k) + w_k \quad (3.14a)$$

Substituting equation (3.11) into equation (3.14a) we have the following output equation

$$\ddot{f}(k) = c \, f_k + w_k \quad (3.14b)$$

where we will assume that $w$ is a zero mean Gaussian random variable with variance $R$. Thus $\hat{f}(k)$ is assumed to be an unbiased estimator. We seek to predict the beacon frequency based on a linear combination of past observations. From the state transition equation (3.9) $f$ is a Gauss-Markov random variable [5]. From the estimate the state vector $\hat{f}_k$ at the current instant the beacon frequency prediction $\hat{f}_{k+1/k}$ is found. Since $e$ is zero-mean we have the following

$$\hat{f}_{k+1/k} = A \hat{f}_k \quad (3.15)$$
It remains then to estimate the state vector \( \hat{\mathbf{x}}_k \) based on the latest observation \( \hat{\mathbf{f}}(k) \) and on the previous predicted estimate of the state vector \( \hat{\mathbf{x}}_{k-1} \) which incorporated all previous observations. Since for Gaussian distributed random variables the linear estimate that minimizes the mean square error (LMMSE) is both the maximum likelihood estimate (MLE) and the minimum mean square error estimate (MMSE), we seek the optimal linear combination of these two estimates [6]. The latter is expressed as

\[
\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{k-1} + \mathbf{k} [ \hat{\mathbf{f}}(k) - \mathbf{C} \hat{\mathbf{x}}_{k-1} ] \tag{3.16a}
\]

In equation (3.16a) we see the predictor/corrector structure of the Kalman predictor. The second term in equation (3.16a) is a correction to the state vector from what it would have been had the driving noise \( \mathbf{e} \) assumed its expected value, namely zero. The vector \( \mathbf{k} \), which is called the gain vector [5], will be defined shortly. Thus from equations (3.15) and (3.16) we can define an estimate of the driving noise vector \( \mathbf{e} \). First substituting (3.15) into (3.16) we have

\[
\hat{\mathbf{e}}_k = \mathbf{A} \hat{\mathbf{x}}_{k-1} + \mathbf{k} [ \hat{\mathbf{f}}(k) - \mathbf{C} \hat{\mathbf{x}}_{k-1} ] \tag{3.16b}
\]

From which we are able to define our estimate of the driving noise by comparison with equation (3.9)

\[
\hat{\mathbf{e}}_k = \mathbf{k} [ \hat{\mathbf{f}}(k) - \mathbf{C} \hat{\mathbf{x}}_{k-1} ] \tag{3.16c}
\]

Equation (3.16c) is thus a transformation of the measurement \( \hat{\mathbf{f}}(k) \) to an estimate of the driving noise vector \( \hat{\mathbf{e}}_k \). How much the observation affects the estimate of the driving noise is determined by the gain vector \( \mathbf{k} \). We can now also define an innovation \( \mathbf{v}_k \), associated with the random process \( \hat{\mathbf{f}}(k) \), which describes the unpredictable part of this random process

\[
\mathbf{v}_k = \hat{\mathbf{f}}(k) - \mathbf{C} \hat{\mathbf{x}}_{k-1} \tag{3.16d}
\]

It should be noted that this innovation is not associated strictly with \( \mathbf{f}(k) \), but rather with \( \hat{\mathbf{f}}(k) \) of equation (3.14b) and therefore includes the observation noise.

We still need to determine \( \mathbf{k} \), the correction vector or gain vector [5]. First
we define the state prediction error as follows [5]

\[ r_{k+1} = f_{k+1} - \hat{f}_{k+1/k} \]  

(3.17a)

Using (3.11) and (3.17a) the frequency prediction error is as follows

\[ r_{k+1} = f_{k+1} - \hat{f}_{k+1/k} = c r_{k+1} \]  

(3.17b)

With this definition of state error in equation (3.17a), we seek to minimize the mean square error (MSE) [5]

\[ \text{MSE} = \text{E} \left[ r_{k+1} r_{k+1}^T \right] \]  

(3.17c)

The minimization of the MSE [5] yields the now well-known Kalman gain and error covariance update equations [5]

\[ k_k = P_k c^T \left( c P_k c^T + R_k \right)^{-1} \]

\[ P_{k+1} = A \left[ I - k_k c \right] P_k A^T + Q_k \]  

(3.18)

\[ P_0 = qI \]

We therefore have the means of optimally (in the MMSE sense) estimating the present beacon frequency state from past measurements and the present measurement via equation (3.16b) and also of predicting the beacon frequency from this MLE of the present state via equation (3.15). We still need to determine the time varying parameters \( Q_k \) and \( R_k \). The accuracy of our estimates of \( Q_k \) and \( R_k \) will determine whether our Kalman predictor attains the MMSE. To estimate these parameters we will use the following unbiased minimum variance estimator

\[ \hat{R}_k = \frac{1}{M} \sum_{m=0}^{M} \left[ f(k-m) - \text{E}[f(k-m)] \right]^2 \]  

(3.19a)

An estimate of the bracketed quantity will be available to us as a difference frequency, see Section (4.4). We can thus reduce this estimator to the following
\[ \hat{R}_k = \frac{1}{M} \mathbf{df}^T \mathbf{df} \quad (3.19b) \]

\[ \mathbf{df} = [\hat{f}(k-m) - E[f(k-m)]] \quad m = 0, 1, \ldots, M \]

For our estimate of the driving noise variance we must estimate \( q \) in equation (3.12a). We use the fact that \( e \) is zero mean to yield the following

\[ q_k = \frac{1}{M-1} \sum_{m=0}^{M} [\hat{e}_{N,k-m}]^2 \quad (3.20) \]

The bracketed term is obtained by substituting equation (3.16c) into equation (3.12b). Figure 3.2 depicts this Kalman predictor.

![Figure 3.2 Diagram of Kalman Frequency Predictor](image)

The transient behavior of the frequency tracker is greatly improved by incorporating a limit on the estimate of the first derivative of the beacon frequency. Since we have apriori knowledge regarding the maximum frequency drift (6 Hz/sec).
we can use this in our frequency state prediction. If we limit the absolute value of \( \hat{f}_{k+1}^{(1)} \) to 6 then the performance of the tracker will be improved. Another improvement could be gained by operating the predictor at a faster rate during the first few seconds of operation. This would give a more accurate estimate of the state vector \( \hat{f}_{k+1} \) sooner.

As the computation associated with this predictor seems intensive, recall that the necessity of measuring the beacon carrier in a narrow band of observation implies that the carrier be in as narrow a band as possible over the observation period. With Kalman prediction of the frequency drift process the deleterious effects of the beacon carrier drift can be very nearly eliminated. We can now predict the frequency of the beacon carrier and use this prediction to translate the beacon carrier to digital baseband. We can also use the Kalman estimator of equation (3.16b) to decrease the variance of our beacon frequency estimates.

By using this predicted frequency to mix and thus frequency translate the carrier to digital baseband, we hypothesize that the carrier can be maintained in a 1 Hz band. We can use this predictor then to maintain frequency control for the DBR. The signal power will be maintained in a narrow band and thus the SNR will be greater in this band.

It should be noted that the effect of beacon phase noise has not been considered here. We assume that the spectral distribution of the carrier due to this phase noise is symmetric. This was implied in the assumption that the frequency measurements of equation (3.13b) were unbiased when made via a spectral estimator such as the periodogram. This type of frequency estimator is exactly the one to be considered in the nonparametric DBR which we will consider in Chapter 4 and is also the MLE for a single sinusoid in white noise [7].

### 3.4 DIGITAL MIXING WITH THE KALMAN FREQUENCY TRACKER

The Kalman predictor can therefore be used with a digital mixing scheme to maintain the carrier frequency at digital base band. We first note that the sampled carrier signal can be approximated on the interval \((kT, kT+T)\) by the following

\[
s_k = A \exp \left[ j2\pi \frac{f_k}{f_s} t_k \right]
\]  

(3.21)
Vector notation is used in equation (3.21) with the following definition of the exponential of a vector

\[ \exp[a_1, a_2, \ldots, a_N] = [\exp(a_1), \exp(a_2), \ldots, \exp(a_N)] \]

and the elementwise multiplication of 2 same sized vectors

\[ \mathbf{f}_k \mathbf{t}_k = [f_{k,1} \cdot t_{k,1}, f_{k,2} \cdot t_{k,2}, \ldots, f_{k,N} \cdot t_{k,N}] \]

and where \( s_k \) represents the sampled carrier signal over the aforementioned interval and \( f_k \) represents the sampled drifting carrier frequency signal over the same time interval. To translate this signal to digital base band we need to multiply each sample (element of the vector) by its normalized conjugate. The mixing signal is therefore given by the following

\[ m_k = \exp \left( -j2\pi \frac{f_k}{f_s} t_k \right) \quad (3.22) \]

we see that an estimate of \( f_k \) is used in equation (3.22). We have previously assumed that this drifting frequency process is linear over short time intervals. The frequency vector, \( \hat{f}_k \), can therefore be approximated as linear over the time period \( T \). The mixing signal of equation (3.22) is therefore a complex chirp signal. Thus the frequency segments will be linear with respect to time. This leads to the following sequence of mixing frequency values for a sampling rate of \( f_s \).

\[ \hat{f}_k = f_k^{(0)} + f_k^{(1)} \left[ 0, \frac{1}{f_s}, \ldots, \frac{N-1}{f_s} \right] \quad N = Tf_s \quad (3.23) \]

This can be understood as the equation of a line in the frequency-time plane. With the more similar notation of \( f_n = b + a t_n \) we note that \( t_n \) takes on discrete values with period equal to the sampling period. The actual mixing signal for the \( k^{th} \) sequence of length \( N \) is therefore the following

\[ m_k = \exp \left( -j2\pi \frac{\hat{f}_k}{f_s} t_k - \psi_k \right) \quad (3.24a) \]
where \( \psi_k \) is added to the phase of the mixing signal to maintain phase coherence between the \( k^{th} \) and \((k+1)^{th}\) mixing sequences and is therefore defined as follows

\[
\psi_{k+1} = \arg[m_{k,N+1}]
\]  

(3.24b)

where

\[
\arg[m_{k,N+1}] = \psi_k + 2\pi \frac{\hat{f}_{k,N+1}}{f_s} t_{k,N+1}
\]  

(3.24c)

This ensures the following

\[
\arg[m_{k-1,N}] = \arg[m_{k,0}]
\]

\( m_k \) is recognized as the normalized conjugate of the carrier signal within a multiplicative factor of

\[ e^{\psi_k} \]

We can conclude that with the Kalman predictor and digital mixing the carrier signal can be adaptively translated to digital base band. The only requirement of the algorithm is that we have a means of measuring the carrier frequency, see equations (3.14a) and (3.19b). The estimation of frequency from an observation of the signal is the topic of the next two chapters.
4.0 THE CHIRP TRANSFORM BEACON RECEIVER:
A Nonparametric Approach

4.1 INTRODUCTION TO THE NON-PARAMETRIC BEACON RECEIVER

It was shown in Section 1.2 that estimation of the beacon power required the evaluation of the periodogram at the beacon carrier frequency. Power estimates taken from the periodogram at frequencies away from the actual carrier frequency exhibited bias. In addition, determining the beacon carrier frequency as the frequency at which the periodogram attains its maximum value assumes that the periodogram can be computed for an infinite number of frequency values. This is the case even if the carrier frequency is known to be within a certain band. With a finite length DFT as the basis for the periodogram estimator the periodogram thus exhibits bias as a beacon frequency estimator as well. One of the options suggested was the zero-padding of the data prior to taking the FFT. In this way more values of the periodogram are evaluated and the beacon carrier frequency and power is estimated with less bias. This is suggested [7] as a method of obtaining frequency estimates of single sinusoids in white noise, a situation not too different from the DBR problem. This solution however results in the computation of an excessive number of DFT values, the overwhelming majority of which are not needed once the beacon carrier frequency is known to be within a certain band. Thus if the beacon frequency is known to within say 20 Hz then the DFT values outside this 20 Hz band need not be computed.

The computation of DFT values in narrow frequency bands has received some study and has been useful in many applications [4,12]. Indeed a number of algorithms have been proposed for this very purpose. The frequency warping approach [12] transforms the original signal to a new signal whose FFT equals the DFT values of the original signal on a warped frequency scale. In this way greater resolution can be obtained in a narrow band of interest. Another approach would be to compute the individual DFT values of interest directly via the Goertzel algorithm [11]. And lastly, the approach we will pursue, the Chirp Transform algorithm (CTA) [4] which computes in an efficient manner evenly spaced DFT values in a prespecified frequency band. It is precisely this high resolution spectral estimate which is required to attain the necessary observation bandwidth alluded to
4.2 THE CHIRP-TRANSFORM ALGORITHM FOR LARGE DATA LENGTHS

The chirp transform algorithm (CTA) was first proposed for narrow band spectral estimation in 1968 by Rabiner, Ryder and Schafer [4]. This algorithm employs an algebraic identity due to Bluestein to reduce the computation of evenly spaced Fourier transform values to a convolution. This convolution is efficiently computed via the Fast Fourier Transform (FFT). In the case of large data lengths sectioning is recommended. In this way the chirp transforms (CT) of fixed length contiguous sections can be combined to yield a CT of extremely long data lengths. The CTA provides the means to efficiently calculate densely packed z-Transform values on a small arc of the unit circle. Since it is necessary to evaluate the Fourier transform densely at and around the beacon carrier frequency in order to make accurate frequency and power estimates of the carrier in low SNR environments (see Section 2.2) the CTA is considered an alternative estimator for the DBR.

Following the suggestion of Rabiner [4] we will see that a further savings in computation can be realized. This more efficient algorithm for long data lengths will now be developed.

The z-transform of the \( L \) length sequence \( s_l \) evaluated every \( \phi \) radians on the unit circle starting at \( \theta \) and ending at \( \theta + (M-1)\phi \) is described by the following

\[
S_k = \sum_{l=0}^{L-1} s_l e^{-j(\theta + k\phi)l} \quad k = \{0, 1, \ldots, M-1\} \quad (4.1)
\]

with \( p = e^{-j\theta}, W = e^{-j\phi}, l = rN + n \) where \( n = \{0, \ldots, N-1\}, r = \{0, \ldots, R-1\}, \) and \( L = RN \) we have the following

\[
S_k = \sum_{r=0}^{R-1} \sum_{n=0}^{N-1} s_{rN+n} p^{rN+n} W^{k(rN+n)} \quad k = \{0, 1, \ldots, M-1\} \quad (4.2)
\]
Letting $k$ be defined as above we have

$$S_k = \sum_{r=0}^{R-1} \sum_{n=0}^{N-1} p^{rN} W^{rkN} W_{rN+n}^{n} W^{kn} \quad (4.3)$$

Defining $s^r$ to be the $r^{th}$ contiguous subsection of length $N$ or $s_n^r = s_{rN+n}$ and using the following algebraic substitution,

$$nk = \frac{1}{2} [n^2 + k^2 - (k-n)^2]$$

the following results

$$S_k = \sum_{r=0}^{R-1} V_k^r \sum_{n=0}^{N-1} g_n^r h_{k-n} \quad (4.4a)$$

where

$$V_k^r = p^{rN} W^{rkN} W_{k^2}$$

$$g_n^r = s_n^r W_{n^2}^\frac{1}{2} \quad (4.4b)$$

$$h_n = W_{-\frac{n^2}{2}} \quad \text{for } n \in \{ -N+1, \ldots, M-1 \} \quad (4.4d)$$

Defining

$$y_k^r = \sum_{n=0}^{N-1} g_n^r h_{k-n} \quad (4.5a)$$

we have

$$S_k = \sum_{r=0}^{R-1} y_k^r V_k^r \quad (4.6a)$$

Thus $y^r$ is a vector associated with the $r^{th}$ subsection of data. We see that $y^r$ in
magnitude is equal to the magnitude of the Fourier Transform of the corresponding subsection of data. This is true since the magnitude of $y_k^r$ is identically equal to 1. To see this consider the case of $R=1$ thus

$$S_k = y_k \frac{k^2}{W^2}$$  \hspace{1cm} (4.6b)

and it follows that

$$|S_k|=|y_k|$$  \hspace{1cm} (4.6c)

Equation (4.6a) can be written as a vector inner product. First define

$$Y_k = \begin{bmatrix} y_k^0 \\ y_k^1 \\ \vdots \\ y_k^{R-1} \end{bmatrix}, \quad V_k = \begin{bmatrix} V_k^0 \\ V_k^1 \\ \vdots \\ V_k^{R-1} \end{bmatrix}, \quad Y = \begin{bmatrix} Y_0 & Y_1 & \cdots & Y_{M-1} \end{bmatrix}, \quad V = \begin{bmatrix} V_0 & V_1 & \cdots & V_{M-1} \end{bmatrix}, \quad S = \text{diagonal}[V^T Y]$$

then

$$S_k = \langle Y_k, V_k^* \rangle$$  \hspace{1cm} (4.6e)

or

$$S = \text{diagonal}[V^T Y]$$  \hspace{1cm} (4.6f)

The $r$ circular convolutions of equation (4.5a) are effectively computed via multiplication of the $p=N+M-1$ point FFT of $y_n^r$ and $h_n$. For computational efficiency it is necessary that $p=2^r \nu \in \text{Integers \ [4]}$. The improvement in computational efficiency of equations (4.5) and (4.6) is due to the fact that no post-
multiplies are computed for each individual CTA. This is accounted for in the definition of \( V_k^T \) which includes the factor

\[
\frac{k^2}{W^2}
\]

This modification results in a computational savings of \( RM \) complex multiplies over the approach of Rabiner, Ryder and Schafer [4] since there is no multiplication by the above factor for the \( R \) individual 'y' vectors of length \( M \).

As already mentioned this algorithm is well suited for the beacon estimation problem because with the CTA we are able to calculate the Fourier transform at and near the frequency of the beacon without the unnecessary computation of other transform values. In addition arbitrarily long data lengths can be used with a minimal storage requirement, see equation (4.6). There is also no restriction on the radian distance between transform values (\( \phi \) can be very small) other than finite word length considerations.

A simple comparison with the FFT shows the advantage of the CTA in this particular arena. With the FFT storage is required for \( P=RN \) signal values to yield a normalized frequency resolution of \( 2\pi/P \). However, with the use of the CTA storage on the order of \( 2N+RM \) is necessary for the same frequency resolution since data is processed in blocks of length \( N \). In the environment of low SNR the necessity of large data lengths becomes apparent and the advantage of the CTA is dramatic.

### 4.3. THE RECURSIVE CHIRP-TRANSFORM ALGORITHM

Truly remarkable savings result when we consider the CTA used as a mixed domain or recursive (frequency-time) spectral estimator as in the case with the DBR. First we define the 'y' vector associated with the \((m+r)^{th}\) data subsection

\[
y_k^{m+r} = \sum_{n=0}^{N-1} g_n^{m+r} h_{k-n} \tag{4.5b}
\]
In this way the Chirp Transform (CT) of the $RN$ length segment of data starting with the point $mN$ is as follows

$$S_k^m = \sum_{r=0}^{R-1} y_k^{m+r} V_k^r$$  \hspace{1cm} (4.7a)$$

Thus $S_k^m$ is the CT of the following section of an otherwise infinite length sequence

$$\{ s^r \} \text{ where } r=\{m,m+1, \ldots, m+R\} \text{ or similarly}$$

$$\{ s^r \}_m = \{ s_{mN}, s_{mN+1}, \ldots, s_{mN+RN-1} \}.$$  

Thus a spectral estimate is available for any section of data of length $RN$ starting at the point $mN$. We should note that the CT of this sequence will yield estimates of frequency, amplitude and phase which should be ascribed to the signal at the point $[m+\frac{R}{2}]N$. Thus the CTA is a noncausal estimator. Now consider the CT of the data length starting at point $mN+N$

$$\{ s^r \} \text{ where } r=\{m+1, \ldots, m+R+1\} \text{ or similarly}$$

$$\{ s^r \}_{m+1} = \{ s_{(m+1)N}, s_{(m+1)N+1}, \ldots, s_{(m+R+1)N-1} \}$$

This leads to the following CT

$$S_k^{m+1} = \sum_{r=0}^{R-1} y_k^{m+r+1} V_k^r$$  \hspace{1cm} (4.7b)$$

Note from comparison with equation (4.7a) to compute $S_k^{m+1}$ only the vector $y_k^{m+R}$ is required in addition to those previously calculated for $S_k^m$.

In this way a recursive CTA (RCTA) exhibits computational efficiency and low storage requirements. Figure 4.1 shows a block diagram of the RCTA for large data lengths with sectioning. First a section of data $s^r_n$ is acquired and premultiplied (see equation (4.4c)); this new sequence $g^r_n$ is then circularly convolved with the sequence $h_n$ (see equation (4.4d) and (4.5)) which is done efficiently via the FFT. The first $M$ points of the resulting circular convolution are
stored as the \( r^{th} \) \( y \) vector. The \( R \) most recent of these vectors are combined via equation (4.7) to produce the \( M \) transform values. This process continues with every new contiguous block of data; in the recursion the previous \( y_k^{r-R} \) vector is discarded when a new \( y_k^r \) vector comes in.

\[
\begin{align*}
S_h^r & \quad \text{premultiply} \quad g_h^r \quad \text{convolution} \quad y_k^r \\
S_k^r & \quad \text{vector dot} \quad V_k^r \quad k=\{0,1, \ldots, M-1\}
\end{align*}
\]

Figure 4.1 The Recursive Chirp Transform Algorithm

The computational savings available with the RCTA are now evident. The RCTA requires, as \( m \) approaches infinity, \((R+1)M+2N-1+2(N+M-1)\log_2(N+M-1)\) complex multiplies to compute the \( M \) DFT values from \( RN \) data points. To compute the FFT of the same length of data would require \( RN\log_2(RN) \) complex multiplies. Thus with \( N=4000, M=97, \) and \( R=50 \) the RCTA requires 3.2\% of the multiplies required by the FFT. Since only the \( M \) transform values around the beacon carrier are of interest, the RCTA with a frequency tracking algorithm is a good candidate for a DBR algorithm. Since the DFT, whether computed via the FFT or the RCTA, can be viewed as the output of a narrowband filter [7] we can consider a direct comparison with finite impulse response (FIR) narrowband filtering. For \( M=97 \) and \( N=4000 \) this corresponds to twentyfive narrowband filters operating at an output rate of 200 Hz assuming a data rate of 200 kHz. The length
of these filters would have to be approximately $RN$ to accomplish even a 1 Hz bandwidth. This would require $RN(M-1)$ complex multiplies to accomplish the same spectral resolution as the RCTA. The $-RN$ term comes from the assumption that the lowpass filter requires no multiplies but rather is a rectangular window filter with filter coefficients equal to unity. Therefore the RCTA requires .58% of the complex multiplies required by a direct FIR filtering approach. This computational savings is due to the RCTA being a convolution implemented via the FFT. Thus the RCTA is able to reap the benefits of both approaches without suffering the limits of either when used as a DBR algorithm. The CTA capitalizes on the computational efficiency of the FFT, yet does not require data to be processed more than once as would be the case with the FFT or FIR filter as a mixed domain/recursive spectral estimator. The CTA is therefore considered to be a good solution to the DBR problem.

4.4 A NON-PARAMETRIC BEACON RECEIVER BASED ON THE RECURSIVE CHIRP TRANSFORM ALGORITHM

Figure 4.2 is a diagram of a DBR based on the RCTA. Filtering is accomplished via the overlap and add method described in Oppenheim [14]. The FFT is used to minimize the computation associated with the implementation of these filters. The purpose of these filtering stages is twofold. The first four stages of filtering result in a decrease in the sampling rate when followed by decimation. The last filter decreases the bandwidth of the noise to a minimum around the carrier. The RCTA then operates on this signal in segments of length $N=120$. This corresponds to .155 seconds of signal. For the recursion, Figure 4.1, two of the resulting 'y' vectors are combined, $R=2$, resulting in DFT values from a length $RN=240$ (.31 seconds) of signal. After 10 seconds of operation the passband of H5 is reduced to 8 Hz. After 33 seconds the receiver enters steady state mode. Steady state mode differs from the aforementioned mode in only one important respect. Steady state frequency estimation is accomplished on an average of 3 periodograms ($3RN=720$) corresponding to .93 seconds of signal. The prediction period, ($T$ of Chapter 3), is therefore changed accordingly to .93 seconds.
Figures 4.3 through 4.7 show the frequency response of the decimation stage filters. The complex signal is first translated to baseband via digital multiplication. Lowpass filtering and sampling rate reduction is then accomplished on this mixed signal. The filtering is accomplished via the overlap and add method [12] and is computationally efficient with the use of the FFT. The filtered and decimated
sequence is then operated on by the RCTA. In this way the DFT values around baseband are computed and a spectral estimate in the form of a periodogram is computed. From this periodogram an estimate of the carrier frequency and beacon power is made. From the past frequency estimates a prediction of the carrier frequency is accomplished (see Chapter 3). The Kalman predictor is employed to provide frequency updates to a frequency mixer which translates the beacon carrier to baseband.

![First Stage Filter](image)

**Figure 4.3** Frequency Response of First Decimation Filter, \(f_s = 198 \text{ kHz}\)
Figure 4.4 Frequency Response of Second Decimation Filter, $f_s = 12.5 \text{ kHz}$

All of the decimation filters were derived in an ad-hoc manner. The approach was to use the FFT-IFFT method by first computing a suitably smooth function in the frequency domain. This smooth function is one for which the first derivative is continuous. This is done to eliminate the effect of discontinuities in the frequency domain representation of the decimation filters. It should be noted that discontinuities in the frequency domain are a major contributor to excessive length in the time domain response. The function chosen for the frequency domain magnitude response is that of a raised cosine. This function is uniformly sampled and the inverse FFT of it is computed. The resulting sequence is the finite length impulse response corresponding to the raised cosine. This sequence is then point by point multiplied with a suitable window function for the purpose of truncating the impulse response to the appropriate length. The transition band of decimation filter $H_1$ (Figure 4.3) is at approximately the normalized frequency .03. This easily
allows for sampling rate reduction by the factor of 16. Since the stop band attenuation exceeds 150 dB decimation incurred aliasing of noise is not problematic. Decimation filter H2 (Figure 4.4) has its transition band at approximately .15 which allows for a sampling rate reduction by the factor of 4. Thus the sampling rate after decimation by 4 following filtering via H2 is $f_s = 3.125$ kHz. Figure 4.5 depicts the frequency response of decimation filter H3 which shows the transition band at approximately the normalized frequency of .23. This allows for sampling rate reduction by the factor of 2.

![third stage filter](image)

**Figure 4.5 Frequency Response of Third Decimation Filter, $f_s = 3125$ Hz**

The frequency response of H4 is similar to that for H3. The order of H4 is recognizably lower than that of H3. H3 has a value of approximately -40 dB at the normalized frequency of .25. Therefore the spectrum of the aliased input signal at
0 Hz to H5 is attenuated by -120 dB (the sum of -80 dB and -40 dB) due to H3 and H4.

![Fourth Stage Filter](image)

**Figure 4.6 Frequency Response of Fourth Decimation Filter, \( f_s = 1562.5\) Hz**

Figure 4.7a depicts a passband of approximately 4 Hz while Figure 4.7b depicts a passband of over 20 Hz for filter H5. The passband is narrowed to that of Figure 4.7b once the carrier frequency is known to reside in the narrow 4 Hz passband. The filter depicted in Figure 4.7b is used during the initial moments of operation (12 seconds).
The passband of the Narrow Band Filter $H_5$ is depicted by the frequency response of Figure 4.7a. The stop band exhibits 80 dB rejection for frequencies greater than 5 Hz from DC. The receiver operates with the filter of Figure 4.7b for the first 12 seconds of operation. This is the time, determined by simulation, required for the receiver to reduce the frequency prediction error to less than 2 Hz for a SNR=15 dB/Hz. Many simulations were run to determine the minimum time to switch from the filter depicted in Figure 4.7b to that of Figure 4.7a; 12 seconds was found to be satisfactory.
Figure 4.7b Narrow Band Filter Frequency Response, Transient Mode, $f_s=781$ Hz

It remains to determine the parameters of the RCTA as a DBR. Used with a frequency tracking algorithm (whereby the beacon carrier frequency is maintained near digital baseband via analog or digital mixing) the RCTA can be used to make accurate estimates of the parameters of the beacon carrier. First we decide on a resolution bandwidth of $\nu=.15$ Hz. This will allow the spectral lobe of the carrier to be sampled sufficiently. Thus with $f_s = 200 \text{ kHz}$ and a total decimation rate of 256 the sampling frequency at the input of the RCTA will be 781 Hz. Thus $\phi$ is as follows

$$\phi = \frac{2\pi \times (.15)}{781}$$

(4.8)

With the beacon carrier frequency controlled to baseband and $M=129$ we choose $\theta$ to be as follows since we want baseband to be centered in our observation band
\[ \theta = \frac{(M+1)\phi}{2} = 65\phi \quad (4.9) \]

This results in a total observation bandwidth of 23 Hz. Thus the beacon frequency must be known to within 23 Hz before the RCTA can be used as a detector. With \( N=120 \) and \( R=2 \) defined as above the resulting output rate of the RCTA is approximately 3 samples per second.

To determine the estimate of the carrier amplitude and phase we simply choose the element of the vector \( S_k^m \) with the largest magnitude

\[ \hat{A}_m = \frac{1}{RN} S_k^m \quad (4.10a) \]

To determine the performance of the receiver we will be interested in the statistics of the following real valued commodity which equals twice the power of the ACTS signal

\[ |\hat{A}_m|^2 = \frac{1}{RN} \left[ \frac{1}{RN} |S_k^m|^2 \right] \quad (4.10b) \]

The bracketed term is recognized as the periodogram where

\[ |S_k^m| = \sup \left\{ |S_k^m| : k = \{0, 1, \ldots, M-1\} \right\} \quad (4.11) \]

To determine the carrier frequency we then use the following

\[ d\hat{f}_b = \frac{(\theta + \lambda\phi)f_s}{2\pi} \quad (4.12) \]

where \( f_s \) is the sampling rate into the CTA and is thus 781 Hz. This would result in a small bias associated with the \( v \) distance between CT samples. To eliminate this bias we could find the actual peak of the continuous transform

\[ S_{df}^m \]

as a function of the continuous frequency \( df \) where the periodogram is known at the following frequencies
\[ df_k = \frac{(\theta + k\phi)f_8}{2\pi} \]  \hspace{1cm} (4.13)

A number of algorithms have been proposed for this purpose [7,17]. The objective in them is simply to find the frequency at which the periodogram attains its maximum without computing additional periodogram values. This is the MLE for the case of a single sinusoid in white noise [7,17]. We will use a simple approach to this type of estimator. To do this we perform a least squares fit with a 2nd order polynomial. The computational burden is light and the result is an unbiased estimate of the beacon frequency. This results in

\[ \hat{df}_b = -\frac{a_1}{2a_2} \]  \hspace{1cm} (4.14)

with

\[
\begin{bmatrix}
a_2 \\
a_1 \\
a_0
\end{bmatrix}
\quad
F =
\begin{bmatrix}
df_{\lambda,2}^g & df_{\lambda,2} & 1 \\
\vdots & \vdots & \vdots \\
df_{\lambda+2}^g & df_{\lambda+2} & 1
\end{bmatrix}
\quad
s =
\begin{bmatrix}
S_{\lambda,2} \\
\vdots \\
S_{\lambda+2}
\end{bmatrix}
\]

and \( \mathbf{a} \) is the solution which minimizes the \( l_2 \) norm of \( \mathbf{e} \) in the following equation

\[ F\mathbf{a} = s + \mathbf{e} \]

or explicitly

\[ \mathbf{a} = (F^T F)^{-1} F^T s \]  \hspace{1cm} (4.15)

This frequency \( \hat{df}_b \) is actually an estimate of the difference between the beacon frequency measurement and the predicted frequency (hence the notation \( df \)) which was used to mix the signal to baseband. This frequency signal, \( \hat{df}_b \), will be a zero-mean random variable which will be used to control the frequency locked loop. This frequency can also be viewed as the sum of the measurement noise and the prediction error. Figure 4.8 is a diagram of the frequency control algorithm. This diagram shows why Figure 4.2 is a DBR with frequency tracking. A measurement of the actual beacon frequency can only be determined once the mixing frequency is added to the detected difference frequency. The mixing frequency and detected
difference frequency must correspond in time (or be synchronized). That is to measure the frequency of the carrier at some moment (say t-T) we need to know the frequency of the mixing signal at the same moment (t-T) and the estimate of the difference between these two frequencies at that same moment (t-T). If we add the mixing frequency corresponding to any other instant in time to the difference frequency corresponding to t-T seconds the measurement will be biased and we say that the frequency measurement is not synchronized.

![Frequency Control Diagram](image)

**Figure 4.8 Frequency Control Diagram**

Figure 4.9 depicts the synchronization of the DBR with Kalman prediction. The use of FIR filters becomes more critical when we consider the importance of constant group delay. Without constant group delay (linear phase) over the observation band we would not have the time invariant transition matrix which forms the basis for the Kalman predictor. The processing delay is depicted for both steady state and transient operation. The word synchronization is used here to denote the kalman algorithm predicting forward in time exactly the amount
necessary. If the Kalman predictor predicts for example 3 seconds ahead in time then the signal will be mixed with a mixing sinusoid of the wrong frequency (the mixing signal predicted and generated is for some future signal segment). This can cause serious instability if the synchronization is off by too much.

![Figure 4.9 Nonparametric Beacon Receiver Synchronization Diagram](image)

The DBR of Figure 4.2 is similar to the JPL beacon receiver [8]. Algorithmically these Fourier based DBRs rely on filtering techniques via the FFT to reduce the variance of the signal. The FFT is also used to frequency control the beacon carrier in both algorithms. The advantage of the RCTA is its extremely high resolution as a spectral estimator. This high resolution coupled with evaluation of long lengths of data equates to very accurate (unbiased and low variance) frequency estimates. These accurate frequency estimates coupled with
the Kalman frequency state estimator/predictor lead to the low observation bandwidth which is addressed in the literature [8,9].

4.5 RESULTS OF THE NONPARAMETRIC BEACON RECEIVER

The receiver of Figure 4.2 can be operated in two modes: 'transient' and 'steady state'. In the transient mode the receiver makes frequency and power estimates based on data corresponding to .31 seconds \(N=240\) of signal via the periodogram estimator. This is the mode of operation used during the initial seconds of operation. In the steady state mode frequency estimates are derived from an average of 3 periodograms corresponding to .93 seconds \(N=720\) of signal. Power estimates are still taken from periodograms corresponding to .31 seconds \(N=240\) of signal. The receiver was simulated to determine the lower bound on SNR for transient and steady state operation. The ACTS 30 GHz signal was simulated according to Chapter 3 with a frequency drift rate of 5 Hz/sec. and an initial frequency estimation error of 5 Hz. This frequency drift path is simulated as a 24 hour drift cycle. The results follow.

4.5.1 TRANSIENT PERFORMANCE OF THE NONPARAMETRIC BEACON RECEIVER

To survey the nonparametric DBR in the transient mode a SNR=26 dB/Hz is used. Figure 4.10 is the first periodogram estimate computed from the DFT values of the RCTA. This periodogram is used to determine the frequency of the beacon carrier and the power estimate. The frequency measurement is determined as the frequency of the peak of the periodogram plus the mixing frequency as shown in Figure 4.8. For the periodogram of Figure 4.10 the initial frequency prediction error of 5 Hz is the frequency of the periodogram peak. Figure 4.11 shows the actual frequency prediction error. Figure 4.12 shows the estimate of the frequency measurement variance, equation (3.19). Figure 4.13 shows the estimate of the variance of the frequency drift driving noise, equation (3.20). Figure 4.14 depicts the power measurements for the first few moments of operation of the receiver.
The periodogram of Figure 4.10 obtains its maximum value at approximately 7.5 Hz. This is a value for $\hat{f}_m$ of Figure 4.8. The peak value of the periodogram of Figure 4.10 must be scaled for the estimation of power as in equation (4.11). By providing an estimate of frequency and power the periodogram is the basis for the nonparametric DBR.
Figure 4.11  Frequency Prediction Error, SNR=20 dB/Hz

Figure 4.11 shows the transient behavior of the frequency tracker with the RCTA as a detector. The algorithm requires approximately 10 seconds to predict the carrier frequency to within 1 Hz at a SNR= 20 dB/Hz. In these initial moments the Kalman tracker is not only compensating for frequency estimation errors but also for the frequency drift of 5 Hz/sec.
Figure 4.12 Estimate of Measurement Variance SNR=20 dB/Hz

Figure 4.12 shows the behavior of the measurement variance estimator as defined by equation (3.19). The estimator starts at a preset value until a history of measurements becomes available. At approximately 5 seconds this history is used to estimate the actual variance.
Figure 4.13 Estimate of Frequency Drift Driving Noise Variance, SNR=20 dB/Hz

Figure 4.13 depicts the estimate of the frequency drift driving noise variance as defined by equation (3.20). Initially the state estimate is assumed to be off and this accounts for the initial variance estimate. As the frequency state estimator begins to get close to the actual frequency state the error estimate approaches its steady state value. It can be seen in this Figure and Figure 4.12 that after approximately 10 seconds the state estimate is accurate.
Figure 4.14 Power Estimates in Decibels, SNR=20 dB/Hz

Figure 4.14 displays the estimate of the following quantity

\[ 20\log |A| \quad A = 1 \]

Variance is clearly present in this estimator but the small bias that is present is not readily evident. This bias of the periodogram estimator is given by the signal variance divided by the datalength [11]

\[ 10\log \frac{\text{No.}\text{81}}{N} \]

This is a disadvantage of the periodogram as an estimator of sinusoidal power. The periodogram is more suited to estimating power spectral density and thus the noise density is added to the power estimate.
With the nonparametric DBR in the transient mode the receiver can lock onto the simulated ACTS 30 GHz signal at a SNR=15 dB/Hz. At SNR lower than this the DBR can not lock on. The term 'lock on' is used to describe a prediction error of zero mean and variance less than 1 Hz$^2$ over a 5 second period. This is the performance with a 5 Hz/sec frequency drift rate. As mentioned before the frequency estimates are derived from periodograms corresponding to .31 seconds ($\tau=240$) of signal. Figures 4.15 and 4.16 show the frequency prediction error and detected difference frequency (d$\hat{f}$m of Figure 4.8) respectively for the DBR at a SNR = 15 dB/Hz.

![Graph showing frequency prediction error over time](image)

**Figure 4.15 Frequency Prediction Error with SNR= 15 dB/Hz**

The detected difference frequency of Figure 4.16 show that this measurement frequency is much less correlated with past measurements than the final prediction error of Figure 4.15. This demonstrates the Kalman frequency
predictors use of past observations to reduce the MSE.

![Graph of Detected Difference Frequency]

Figure 4.16 Detected Difference Frequency with SNR= 15 dB/Hz

4.5.2 STEADY STATE PERFORMANCE OF THE NONPARAMETRIC BEACON RECEIVER

To illustrate how low an SNR this algorithm can handle in the steady state mode the receiver described above was operated for an extended period of time and the noise level was increased by 5 dB every few seconds. The receiver was initialized at a SNR of 20 dB and then reduced to 15 dB/Hz next to 10 dB/Hz and finally to 5 dB/Hz. The results follow. Depicted graphs are the steady state operation of the receiver with the estimation of the frequency being from the averaged periodogram taken from data corresponding to .93 seconds (N=720) of signal. Power measurements are derived in the same manner as in the transient mode (from the periodogram corresponding to .31 seconds (N=240) of signal).
Thus the frequency estimate is from a mean of 3 periodograms each corresponding to .31 seconds \((N=240)\) of data.

Figure 4.17 Detected Difference Frequency with SNR=15 dB/Hz Through 5 dB/Hz

Figure 4.17 depicts the detected difference frequency (\(\hat{f}_m\) of Figure 4.8) of the DBR in the steady state mode. The SNR are marked on the Figure to show the instants where the SNR increases. The detected difference frequency of this Figure exhibits increasing variance as SNR decreases. The sampling interval of this process is .93 seconds.
Figure 4.18 Estimate Frequency Measurement Noise Variance at SNR=15 dB/Hz Through 5 dB/Hz

Figure 4.18 depicts the estimate of the measurement noise variance with decreasing SNR. The measurement variance is greater than 1 Hz$^2$ at a SNR of 5 dB/Hz.
Figure 4.19  Estimated Frequency Drift Driving Noise, SNR=15 dB/Hz - 5 dB/Hz

Figure 4.19 depicts the estimate of the frequency drift driving noise process. Since the driving noise is an innovation on the 2\textsuperscript{nd} derivative of the frequency the dimensions are as shown on the Figure.
Figure 4.20 Estimated Frequency Drift Driving Noise Variance at SNR=15 dB/Hz Through 5 dB/Hz

Figure 4.20 depicts the estimate of the driving noise variance decreasing as the SNR decreases. Since the measurements exhibit greater variance with decreasing SNR the Kalman gain, \( k \) of equation (3.16c), weighs the current state estimate more heavily and the variance of the state estimator is reduced correspondingly. The steady state operation of the nonparametric DBR shows that the recursive chirp transform can operate as a detector for the Kalman frequency predictor and digital mixer. The frequency estimates are from average periodograms corresponding to .93 seconds \( (N=790) \) of signal data. The following graph, Figure 4.21, depicts the behavior of the nonparametric DBR below 5dB/Hz. The detected difference frequency is shown for SNR at and below 5 dB/Hz.
Figure 4.21 Detected Difference Frequency at SNR= 15 dB/Hz Through 0 dB/Hz

4.6 CLOSING REMARKS

The RCTA-DBR was able to 'lock onto' (defined as a frequency prediction error less than 1 Hz after 10 seconds) a 5 Hz/sec drifting frequency ACTS 30 GHz signal at a SNR of 15 dB/Hz and was able to maintain lock (defined as a frequency prediction error less than 2 Hz in steady state operation) at SNR greater than 5 dB/Hz. The RCTA is an effective detection algorithm for the DBR and is able to provide frequency measurements as well as variance estimates of these frequency measurements for the Kalman predictor.
5.0 THE YULE WALKER AUTOREGRESSIVE BEACON RECEIVER: A Parametric Approach

5.1 BACKGROUND TO PARAMETRIC MODELING

The usefulness of linear models to characterize wide sense stationary (WSS) signals is well established [7]; synonymous it is well known that any spectra can be represented by a rational polynomial of sufficient degree [7]. In addition a great body of research has been undertaken to determine algorithms which can compute the coefficients of such a polynomial, termed the autoregressive (AR) coefficients, from a given data set. This approach to spectral estimation is commonly categorized as parametric spectral estimation [7], yet its connection with DFT analysis is well established [7,13]. From the autoregressive coefficients primary modes of the process can be determined [13] and the power in each mode can be found as well. In light of these considerations we will apply the tools of parametric spectral estimation to the DBR problem.

Inherent in every parametric approach is an estimate of the correlation sequence derived from the process to be parameterized. The variance associated with this covariance sequence estimate is the primary source of bias and variance in our estimation of power in the modes of the process. Consider the following estimator of a portion of the correlation sequence.

\[
\hat{r}_m = \frac{1}{N-|N-m|} \sum_{i=0}^{N-1} s_i \cdot s^*_{i+m} \quad m = \{1, \ldots, 2N-1\} \tag{5.1a}
\]

where \(s_i\) is a WSS random variable of length \(N\). The limits chosen for \(m\) in equation (5.1) are not arbitrary but correspond to the correlation of \(s_i\) with \(s_{i+N}\). This estimator is unbiased but not univariant with respect to \(m\) [7,13]. That is, the estimator's variance will not be the same for different lags. This trade off between bias and variance is inherent in correlation estimation with finite length observation of signals if the estimator to be efficient. This estimator is based on the assumption that the random variable is ergodic and thus the time series \(s_i\) can be used to estimate a statistical parameter \(r_m\). This estimator is based upon the following definition of the correlation
\[ r_m = E[s_i s_{i+m}^* \quad m=\{1, \ldots, 2N-1\}] \quad (5.1b) \]

We can now define a covariance of this correlation estimator as follows

\[ E = E[(\hat{r} - r) (\hat{r} - r)^H] \quad (5.2) \]

where \( E \) is a \( 2N-1 \) by \( 2N-1 \) Hermitian matrix.

### 5.2 Modal Decomposition as a Means of Power Estimation

Consider a random variable which is approximately WSS over some time period and is composed of complex sinusoids and complex zero-mean Gaussian white noise (uncorrelated with respect to itself and the sinusoids) as in the case of the ACTS signals. We will also consider the case of correlated noise as the necessity arises. This is equivalent to considering the ACTS signal as ergodic with respect to the autocorrelation over the same period. This assumption, while not necessarily a valid one, will be a working hypothesis for the formulation of an autocorrelation estimate of the ACTS signal. In this case the efficient unbiased estimator of correlation, equation (5.1), becomes with \( m \) as defined above,

\[ \hat{r}_m = \frac{1}{N-|N-m|} \sum_{i=0}^{N-1} (s_i + w_i) (s_{i+m}^* + w_{i+m}^*) \quad (5.3) \]

where \( s_i \) is a sinusoidal signal described for example by equation (2.10). If the noise is uncorrelated and zero-mean we have the following

\[ E\{\hat{r}_m\} = r_m = \frac{1}{N-|N-m|} \sum_{i=0}^{N-1} s_i s_{i+m}^* \quad (5.4a) \]

In this case the expectation of the correlation sequence for \( m \neq 0 \) can be exactly represented as follows [7].

\[ r_m = \sum_{i=1}^{T} P_i z_i^m \quad (5.5a) \]

where \( T \) is the number of sinusoids present in the signal and is equal to two for the
ACTS 30 GHz (recall from Chapter 2 that the complex sinusoid representing the 30GHz signal resulted from an imperfect Hilbert transform) and is $\infty$ for the 20 GHz case. If the noise is correlated we have the following

$$E\{r_m\} = r_m = r_n m + \frac{1}{N-|N-m|} \sum_{i=0}^{N-1} s_i \cdot s^*_i + m$$  \hspace{1cm} (5.4b)$$

where $r_n m$ is the correlation of the additive noise process $w$. The expectation of this correlation sequence can be represented as follows [7].

$$r_m = r_n m + \sum_{i=1}^{T} P_i z_i^m$$  \hspace{1cm} (5.5b)$$

where $z_i = e^{j2\pi f_i}$ and $f_i$ is the normalized frequency of the $i^{th}$ sinusoid. The power present in the process at the frequency $f_i$ is designated $P_i$. Equation (5.5a) represents $r$ as an element of a $T$ dimensional mode space. This amounts to a Fourier series expansion of $r_m$ [13] since $|z| = 1$. We can truncate the summation of equation (5.5b) to yield the following approximation.

$$r_m \approx r_n m + \sum_{i=1}^{L} P_i z_i^m$$  \hspace{1cm} (5.5c)$$

This is a projection of $r$ onto the $L+1$ dimensional mode space spanned by the following modes

$$\{ z_1, z_2, \ldots, z_L, r_n \}$$

where we have designated $r_n$ to be a mode. It should be noted that all the modes $z_i$ are orthogonal to one another. From (5.5b) it is evident that if $f_i$ and thus $z_i$ are known and an estimate of $r_m$ is available then the values of $P_i$ can be estimated [13] in the weighted least-squares sense. First approximate $r_n m$ as follows
\[ r_n m \approx \sigma^2 [ h_m \star h_{-m} ] = \sigma^2 \tilde{n}_m \]

where \( h \) is a finite length impulse response which approximates the impulse response of the process. In this way the noise is viewed as a moving average process. We can now represent equation (5.5b) in matrix form as follows

\[
\begin{bmatrix}
1 & \ldots & 1 & \tilde{n}_1 \\
z_1 & \ldots & z_L & \tilde{n}_2 \\
z_1^2 & \ldots & z_L^2 & \tilde{n}_3 \\
\vdots & \vdots & \vdots & \vdots \\
z_1^M & \ldots & z_L^M & \tilde{n}_M
\end{bmatrix}
\begin{bmatrix}
P_1 z_1 \\
P_2 z_2 \\
P_L z_L \\
\sigma^2
\end{bmatrix}
= 
\begin{bmatrix}
r_1 \\
r_2 \\
r_M
\end{bmatrix}
\]

Then with the following definitions

\[
V = 
\begin{bmatrix}
1 & \ldots & 1 & \tilde{n}_1 \\
z_1 & \ldots & z_L & \tilde{n}_2 \\
z_1^2 & \ldots & z_L^2 & \tilde{n}_3 \\
\vdots & \vdots & \vdots & \vdots \\
z_1^M & \ldots & z_L^M & \tilde{n}_M
\end{bmatrix}
\quad P = 
\begin{bmatrix}
P_1 z_1 \\
P_2 z_2 \\
P_L z_L \\
\sigma^2
\end{bmatrix}
\quad r = 
\begin{bmatrix}
r_1 \\
r_2 \\
r_M
\end{bmatrix}
\]

we have the generalized or weighted least squares solution [7,13] for \( P \)

\[
P = (V^H E^{-1} V)^{-1} V^H E^{-1} r
\]

(5.6a)

\( P \) is a complex valued vector whose components are equal to the power estimates in absolute value. This follows from the fact that the modes are located on the
unit circle. Therefore

\[ |A_{i_j}^2| = |P| = |P | z_i | \]  

(5.6b)

From equation (5.6a) it is apparent that only the modes \( z_i \) and the covariance of the correlation estimate \( E \) are necessary to estimate \( P \) from \( r \). We will first compute an approximation of \( E \). We will assume that the noise \( w_n \) is approximately white, thus from equations (5.2) and (5.3) we can define the following

\[ e_k = \hat{\tau}_k - \tau_k = \frac{1}{N - |N - k|} \sum_{i=0}^{N-1} (s_i + w_i) (s_{i+k}^* + w_{i+k}) - (s_i s_{i+k}^*) \]  

(5.7a)

which reduces to

\[ e_k = \frac{1}{N - |N - k|} \sum_{i=0}^{N-1} (s_i w_{i+k}^* + w_i s_{i+k}^* + w_i w_{i+k}^*) \]  

\( \{k = 1, \ldots, 2N-1\} \)

which yields the following for the error covariance matrix

\[ E = E\{e e^H\} \]  

(5.8a)

or explicitly

\[ E_{j,k} = G_{j,k} E \left( \sum_{l=0}^{N-1} (s_i w_{l+j}^* + w_i s_{l+j}^* + w_i w_{l+j}^*) \sum_{i=0}^{N-1} (s_i w_{i+k}^* + w_i s_{i+k}^* + w_i w_{i+k}^*) \right) \]  

(5.8b)

\[ = G_{j,k} \sum_{l=0}^{N-1} \sum_{i=0}^{N-1} E \left( s_i w_{l+j}^* + w_i s_{l+j}^* + w_i w_{l+j}^* \right) (s_i w_{i+k}^* + w_i s_{i+k}^* + w_i w_{i+k}^*) \]  

(5.8c)

which becomes

\[ = G_{j,k} \left\{ \sum_{l=0}^{N-1} \sum_{i=0}^{N-1} \left( s_i w_{l+j}^* s_i w_{i+k}^* + s_i w_{l+j}^* w_i s_{i+k}^* + s_i w_{l+j}^* w_i w_{i+k}^* + s_i w_{l+j}^* s_i s_{i+k}^* + s_i w_{l+j}^* s_i w_{i+k}^* + s_i w_{l+j}^* w_i s_{i+k}^* + s_i w_{l+j}^* w_i w_{i+k}^* + s_i w_{l+j}^* s_i s_{i+k}^* + s_i w_{l+j}^* s_i w_{i+k}^* + s_i w_{l+j}^* w_i s_{i+k}^* + s_i w_{l+j}^* w_i w_{i+k}^* \right) \right\} \]
\[ + w_i w_{i+j}^* w_{i+k}^* s_i s_{i+j} s_{i+k} + w_l w_{l+j}^* w_{l+k}^* w_{l+i} w_{l+i+k} \]  \hspace{1cm} (5.8d) 

where 

\[ G_{j,k} = \frac{1}{(N/[N-k]) (N/[N-j])} \]

Since the signal \( s \) is zero-mean and uncorrelated with the noise \( w_n \) the expected value of the 3rd, 6th, 7th and 8th terms are zero. The 1st, 2nd, 4th and 5th terms will have an expectation dependent on the signal autocorrelation. This could be estimated and iterated for optimal weighting, we however, will ignore these contributions to \( E \) in our approximation. This leaves the following as our rough approximation

\[ E_{j,k} = G_{j,k} E \left\{ \sum_{l=0}^{N-1} \sum_{i=0}^{N-1} w_l w_{l+j}^* w_{l+k}^* w_{l+i} \right\} \]  \hspace{1cm} (5.8e) 

If \( w_n \) is zero-mean and uncorrelated at any non-zero delay the following results

\[ E_{j,k} = \begin{cases} \frac{N \Delta}{(N/[N-k]) (N/[N-j])} & \text{for } j = k \\ 0 & \text{for } j \neq k \end{cases} \]  \hspace{1cm} (5.8f) 

where \( \Delta = E \{ w_j w_j^* w_j^* w_j \} \)

This approximation is therefore diagonal. If \( w_n \) is not white then the approximation is equal to equation (5.8e) and a useful identity for 4th order moments [6] yields

\[ E_{j,k} = G_{j,k} \sum_{l=0}^{N-1} \sum_{i=0}^{N-1} E [w_l w_{l+j}^*] E [w_{l+k}^* w_{l+i+k}] + E [w_{l+j}^* w_{l+i+k}] E [w_{l+k} w_{l+i+k}] \]

\[ + E [w_{l+i+j}^* w_{l+i+k}] E [w_{l+i}^* w_{l+i+k}] \]  \hspace{1cm} (5.8g)
and reduces to

\[ E_{j,k} = G_{j,k} \sigma^2 \sum_{i=0}^{N-1} \sum_{k=0}^{N-1} \tilde{n}_j \tilde{n}_{k} + \tilde{n}_{i+j-i} \tilde{n}_{k+i} + \tilde{n}_{i+j-k} \tilde{n}_{i} \]  

(5.8h)

The diagonal estimate of equation (5.8f), while crude, will be used here. This approximation is shown for a dimension of 3.

\[
E = \begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

5.3 FREQUENCY ESTIMATION VIA AUTOREGRESSIVE MODELING

It remains to estimate the modes \( z_i \) of the ACTS signal autocorrelation. The autoregressive coefficients \( \{ a_1, a_2, \ldots, a_p \} \) are defined as the sequence which satisfies the following signal auto-regression

\[
s_m = \sum_{i=1}^{p} a_i s_{m-i} + u_m \]  

(5.9)

where \( u_m \) is an uncorrelated minimum variance sequence such that (5.9) is satisfied. There is no mention of additive observation noise as is present in equation (5.3). Rather equation (5.9) is a compact linear description of the generation of the actual signal. A number of algorithms are available to solve for the set of autoregressive coefficients \( \{ a_1, a_2, \ldots, a_p \} \) but before we proceed to a solution we should note that equation (5.9) defines an autoregressive filter with poles at the solutions of the following polynomial equation

\[
\sum_{i=1}^{p} a_i z^i = 1 \]  

(5.10)

Let \( \{ z_1, z_2, \ldots, z_p \} \) be the solution set of equation (5.10) then let

\[
\text{arg}(z_i) = 2\pi f_i \]  

(5.11)
and if $s_m$ is known to consist of complex sinusoids then a reasonable approximation of the sinusoidal frequencies of the process $s_m$ is the set

$$\{ f_1, f_2, \ldots, f_p \}$$

And the modes can thus be estimated as follows

$$z_i = e^{i 2\pi f_i^t} \quad i \in \{1, p\}$$

It remains therefore only to estimate the autoregressive parameters $\{a_1, a_2, \ldots, a_p\}$ to determine the modes of the ACTS signal. It is suggested [7] that the modified covariance method be used for this application. It is also suggested [7] that an improvement to the modified Yule-Walker equation can be obtained based on a generalized-least squares solution. All of the methods rely on the fact that the autoregressive parameters fulfill the same regression equation with respect to the correlation as it does with respect to the signal itself. Namely

$$r_m = \sum_{i=1}^{p} a_i r_{m-i} \quad m \geq 1 \quad (5.12)$$

The modified Yule-Walker equation is therefore given as the expectation of the following

$$\begin{bmatrix}
\hat{r}_q & \hat{r}_{q-1} & \ldots & \hat{r}_{q-p+1} \\
\hat{r}_{q+1} & \hat{r}_q & \ldots & \hat{r}_{q-p+2} \\
\hat{r}_{q+2} & \hat{r}_{q+1} & \ldots & \hat{r}_{q-p+3} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{r}_{M-1} & \hat{r}_{M-2} & \ldots & \hat{r}_{M-p}
\end{bmatrix}
= \begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_p
\end{bmatrix} \begin{bmatrix}
\hat{r}_{q+1} \\
\hat{r}_{q+2} \\
\vdots \\
\hat{r}_M
\end{bmatrix} \quad (5.13a)$$

where $\hat{r}_i$ is an unbiased estimator of the autocorrelation. This can be written in compact notation as follows
\[ Ra = r \]  \hfill (5.19b)

The generalized or weighted least squares (WLS) solution to equation (5.13) is the following

\[ \hat{a} = (\hat{R}^H E^{-1} \hat{R})^{-1} \hat{R}^H E^{-1} \hat{r} \]  \hfill (5.14)

We should note that the subtle difference of the solution of equation (5.14) and the actual modified Yule-Walker equation is in the expectation inherent in the actual modified Yule-Walker equation. The variance of the estimates of the autocorrelation used in equation (5.13) will be the major source of variance in the parameter estimation of equation (5.14). The variance of the estimate is in turn a result of the additive noise present in the signal from which the correlation estimate is acquired (see equation (5.1)).

### 5.4 A PARAMETRIC BEACON RECEIVER BASED ON THE MODIFIED YULE WALKER EQUATION

We are now ready to consider the weighted least-square modified Yule-Walker (WLSMYW) method as a parametric DBR algorithm. First we define, based on equation (5.1), the following estimate of the correlation sequence at the \( k \)th instant.

\[ \hat{r}_{m,k} = \frac{j}{L} \sum_{q=0}^{j} \sum_{i=0}^{N-1} s_{i+qN+kLN}^* s_{i+m+qN+kLN} \quad \forall \ m \in \{1, 2N-1\} \]  \hfill (5.14)

This estimator is based on the following

\[ r_{m,k} = E\left\{ \frac{j}{N-/N-m} \sum_{i=0}^{N-1} s_{i+kLN}^* s_{i+m+kLN} \right\} \]  \hfill (5.15)

which is just a reformulation of equation (5.4). The expectation is approximated as the mean since \( r \) is assumed distributed according to a Gaussian pdf. We can save computation by performing the unbiasing operation after the summation of correlations
\[ \hat{r}_{m,k} = \frac{1}{L(N_{-1}N-m)} \sum_{q=0}^{N-1} \sum_{i=0}^{N-1} s_{i+qN+kLN}s_{i+m+qN+kLN}^* \quad \forall \ m \in [1, 2N-1] \quad (5.16) \]

This represents an unbiased estimation of the autocorrelation based on \( LN \) data points. The \( k^{th} \) correlation estimate is to be associated with the signal at the point \((k+\frac{1}{2})LN\). This correlation estimate forms the basis for the parametric DBR. Figure 5.1 depicts an implementation of this estimator based on the computational efficiency of the FFT [15].

Figure 5.1 A Computationally Efficient Correlator
It should be noted that the larger \( L \) is in equation (5.16) the smaller the variance of this correlation estimator. The reduction in variance is at the expense of an increase in bias since our assumption of ergodicity is being further taxed. It is remembered that we seek to measure the power of the beacon with a period less than .5 seconds, thus power measurements should be made from correlations derived from datalengths corresponding to less than .5 seconds of data. This is necessary if the estimates are to be without bias and uncorrelated for power signals which have bandwidths of up to 2 Hz. For this reason the estimate of equation (5.16) will be used for power measurement with \( L=4 \) and \( N=60 \) corresponding to .31 seconds of data. The mode, representing the frequency, has a much more slowly varying nature and can therefore be measured from a correlation estimate which is derived from a longer data length, \( L=12 \) and \( N=60 \), corresponding to .93 seconds of signal data. Thus longer observation times are used for modal estimation. This will allow the frequency of the beacon to be known even when the power estimates exhibit large variance. The Kalman smoother can also be used to decrease the variance of the frequency estimates.

An improvement in the estimation of the power vector can be accomplished by keeping the dimension of the Vandermonde matrix small. In this way errors in the frequencies of the modes do not translate into inefficient use of the correlation estimate in equation (5.6). To see this note that the errors in the frequencies of the modes are multiplied by the row number of the Vandermonde matrix. Thus if \( M \) is chosen large in equation (5.6) then the modal frequency errors are multiplied by a large amount and the later rows of the Vandermonde matrix will not constructively contribute to the power estimate. To alleviate this phenomenon we divide the correlation estimate into a number of subsections and estimate the power from each subsection as in equation (5.6). The estimates are then averaged in a weighted sense to give a better estimate of the power. This can also be viewed as a partitioning of equation (5.6) into a number of equations to be solved separately.

Figure 5.2 shows a block diagram of the parametric DBR. Decimation filtering is identical to that for the receiver of Chapter 4. Just as in Chapter 4 the assumption is made here that the frequency of the carrier is known to within 20 Hz. After 12 seconds of operation the bandwidth of the narrow band filter H5 is reduced to that of Figure 4.7a.
Estimation of the signal modes is done from the AR parameters. The power present in each mode is used to determine the actual beacon carrier frequency. This is done by simply choosing the mode with the greatest power. A diagram of this process is shown in Figure 5.3. Correlations are derived from .31 seconds of signal ($L=4$, $N=60$) for the transient mode of operation.
In this way the frequency of the carrier is determined after the power at each mode is known. After a few seconds of operation the Kaiman predictor has a good estimate of the frequency. The receiver can then determine the signal mode as the mode which is a minimal distance from the expected location of the carrier mode. Figure 5.4 depicts this steady state detection algorithm. In the steady state mode of operation correlations are derived from .93 seconds of signal \((L=12, N=60)\) for frequency estimation and .31 seconds of signal \((L=4, N=60)\) for power estimation.
The bandwidth of the narrow band filter $H_5$ is reduced after 12 seconds of operation from its transient bandwidth, Figure 4.7b, to its steady state bandwidth, Figure 4.7a.

![Diagram: Steady State Parametric Estimation Algorithm](image)

Figure 5.4 Steady State Parametric Estimation Algorithm

Thus with the receiver in this steady state 'mode' the power estimate needs to be computed only for the beacon frequency mode. Thus the modal decomposition, equation (5.6a), becomes an overdetermined solution to a dimension 2 vector. In this mode the estimate of the carrier frequency is determined before the power in that mode is computed. It should be noted that the Kalman predictor can make an error in choosing the beacon mode. This will certainly occur at some lower SNR, exactly how low a SNR is to be determined by simulation.
5.5 RESULTS FOR A PARAMETRIC BEACON RECEIVER

The parametric receiver described in the previous section was simulated at various SNR with a frequency drift path of 5 Hz/sec which is identical to that used in Section 4.5 for the nonparametric DBR. The receiver was tested in its transient and steady state modes.

5.5.1 TRANSIENT PERFORMANCE FOR A PARAMETRIC BEACON RECEIVER

In the transient mode the receiver makes frequency estimates based on correlation estimates of equation (5.14) that is averaging is used reduce the variance of the correlation estimator. The correlation estimates are derived from signal observations corresponding to .31 seconds of data. The Figures that follow depict the behavior of the receiver at a SNR of 20 dB/Hz.

![Figure 5.5 Mode Plot SNR=20 dB/Hz](image)

Figure 5.5 is a mode plot of the prefiltered ACTS 30 GHz signal in the
transient mode. The beacon carrier mode is depicted as a '+' . The other two
types are visible as '.' and model the additive lowpass process. One of these
modes is seen at a magnitude just below 1 next to the signal mode, the other noise
mode is approximately at the origin.

![Graph of Detected Difference Frequency](image)

**Figure 5.6** Detected Difference Frequency, SNR=20 dB/Hz

Figure 5.6 depicts the detected difference frequency of the nonparametric
receiver. This graph shows larger variance in the first 12 seconds. This is due to
the wider bandwidth of H5 during the first few seconds. This Figure reveals the
importance of the bandwidth of H5 on the parametric estimation algorithm.
Figure 5.7 Estimated Frequency, SNR=20 dB/Hz

Figure 5.7 is the estimated frequency at a SNR of 20 dB/Hz. The frequency drift rate is noticeable as the slope of the curve. The wobbles in the curve correspond in time to the wide variations in the detected difference frequency of Figure 5.6. It can be noticed from Figure 5.7 that after approximately 10 seconds of operation the wobbles are significantly smaller than in the initial 10 seconds of operation.
The driving noise variance estimate shown above exhibits a sharp decline in the initial moments of operation as the frequency state estimate nears the actual frequency state.
Figure 5.9 Estimate of Frequency Measurement Variance at SNR=20 dB/Hz

Figure 5.9 depicts the estimate of the measurement noise variance for a SNR of 20 dB/Hz. The estimate starts at a preset value of 1 and reaches its steady state value for this SNR within 25 seconds.
Figure 5.10 Histogram of Power Estimates, SNR=20 dB/Hz

Figure 5.10 depicts a histogram of the power estimate at the SNR of 20 dB/Hz. Each sample (or occurrence) is derived from .31 seconds of data. This distribution is not noticeably nonGaussian.

The transient behavior of the parametric DBR was simulated at a SNR of 15 dB/Hz. The detected difference frequency is depicted in Figure 5.11. This difference frequency in Figure 5.11 shows that the variance of the frequency estimates for the parametric DBR is sensitive to the bandwidth of the final stage filter H5. At approximately 12 seconds the variance of the detected difference frequency appears to settle at a reduced level.
Figure 5.11 Detected Difference Frequency at SNR=15 dB/Hz

Figure 5.11 is the detected difference frequency for a realization at a SNR=15 dB/Hz. At approximately 15 seconds the detected difference frequency reaches its apparent steady state statistical behavior for the transient mode.

5.5.2 STEADY STATE PERFORMANCE FOR A PARAMETRIC BEACON RECEIVER

In the steady state mode the receiver makes frequency estimates based on averages of correlations which correspond to .155 seconds of signal. Thus 6 correlation sequences are averaged to result in a single correlation estimate which characterizes .93 seconds of signal. To depict the behavior of this steady state mode the receiver was initially operated at a SNR of 20 dB/Hz and the SNR was
decreased in 5 dB/Hz increments. The results are shown in Figure 5.14.

![Graph showing detected difference frequency over time with SNR levels from 15 dB/Hz to 0 dB/Hz.](image)

Figure 5.12 Detected Difference Frequency at SNR=15 dB/Hz Through 0 dB/Hz

Figure 5.12 shows the detected difference frequency at various SNR and the loss of lock is discernable as the portion where the detected difference frequency is visibly uncorrelated. The detected difference frequency after approximately 200 seconds is markedly uncorrelated and of greater variance.

5.6 CLOSING REMARKS

The WLSYW method of AR estimation performed well as a frequency detection algorithm. The algorithm presented above performed well in both the
transient and steady state modes for different numbers of signal modes modeling
the ACTS carrier. The writer, during countless simulations, comes to the
conclusion that 1 signal mode is sufficient for transient mode operation. A single
mode provides the algorithm with no fault decision as to the carrier mode. For
steady state operation at moderate SNR (25 dB/Hz Thru 10 dB/Hz) the algorithm
was best served with as many as 5 modes. At low SNR, below 10 dB/Hz, the
algorithm ran best modeling the data with few modes. In fact the writer feels 1
mode can give good results at all SNR over which the DBR is designed. The writer
recommends that the algorithm be implemented with the option of multiple mode
operation.
6.0 PERFORMANCE COMPARISON OF RESULTS FOR 30 GHz BEACON

6.1 BASIS FOR COMPARISON OF ALGORITHMS

The variance of the frequency prediction error is the measure of performance we will consider in our assessment of the non-parametric and parametric DBR. With the assumption that the frequency prediction was unbiased the MSE is the estimate of variance we will consider. To determine the performance of the algorithms at various SNR we will compute the MSE of the frequency prediction. We will also compare plots of the transient behavior of the frequency prediction error for both algorithms. For an assessment of the parametric and nonparametric power estimation procedures we will compare histograms of the corresponding estimates for various SNR. We will first compare the transient behavior and then the steady state behavior.

6.2 COMPARISON OF TRANSIENT PERFORMANCE FOR BEACON RECEIVERS

We will now compare the nonparametric and parametric beacon receivers in the transient mode of operation.

6.2.1 TRANSIENT PERFORMANCE OF THE NONPARAMETRIC BEACON RECEIVER

With the nonparametric DBR in the transient mode the receiver can lock onto the ACTS 30 GHz signal at a SNR=15 dB/Hz. At SNR lower than this the nonparametric DBR can not lock on. The term 'lock on' is used to describe a prediction error of zero mean and variance less than 1 over a 5 second period. This is the performance with a 5 Hz/sec frequency drift rate. As mentioned before the frequency estimates were derived from periodograms corresponding to .31 seconds of signal \( (N=240) \). Figure 6.1 shows the frequency prediction error for the nonparametric DBR at a SNR = 15 dB/Hz.
Figure 6.1 Nonparametric Beacon Receiver Transient Frequency Prediction Error

SNR=15 dB/Hz

6.2.2 TRANSIENT PERFORMANCE OF THE PARAMETRIC BEACON RECEIVER

The parametric DBR in the transient mode can lock onto the ACTS 30 GHz signal at a SNR=15 dB/Hz. As with the nonparametric DBR the simulation is of a 5 Hz/sec frequency drift rate. As mentioned before the frequency estimates were derived from the average of 2 correlations each corresponding to .155 seconds for a total observation period of .31 seconds \((N=24\theta)\) of signal. Figure 6.2 shows the frequency prediction error for the DBR at a SNR = 15 dB/Hz.
This prediction error is less than 1 Hz after approximately 10 seconds of operation. The behavior of the prediction error for the parametric DBR is poorer than that for the nonparametric DBR shown on Figure 6.1 for the first 15 seconds of operation. This reveals the fact that the parametric DBR frequency prediction error variance is greatly dependent on the bandwidth of the fifth stage bandwidth reduction filter. The bandwidth of the latter filter reduces the noise bandwidth to approximately 4 Hz at 12 seconds of operation, as explained in Section 5.4.

6.3 COMPARISON OF STEADY STATE PERFORMANCE FOR BEACON RECEIVERS

We will now compare the nonparametric and parametric beacon receivers in the steady state mode of operation.
6.3.1 STEADY STATE PERFORMANCE OF THE NONPARAMETRIC BEACON RECEIVER

To illustrate how low an SNR this algorithm can handle in the steady state mode the receiver described above was operated for an extended period of time (approximately 260 seconds) and the noise level was increased by 5 dB every few seconds. The receiver was initialized at a SNR of 20 dB which was then reduced to first 15 dB/Hz, next to 10 dB/Hz, then to 5 dB/Hz, and finally to 0 dB/Hz. The steady state operation of the nonparametric DBR failed to maintain lock at a SNR of 5 dB/Hz. The frequency estimates are from average periodograms corresponding to .93 seconds (N=240) of signal data. The frequency prediction error is shown for the SNR of 15 dB/Hz through 0 dB/Hz.

![Plot showing frequency prediction error over time for different SNRs from 15 dB/Hz to 0 dB/Hz.](image)

**Figure 6.3 Nonparametric Beacon Receiver Frequency Prediction Error**

**SNR= 15 dB/Hz Through 0 dB/Hz**
Figure 6.3 reveals that the frequency prediction error begins to increase beyond the bandwidth of H5 at a SNR of 5 dB/Hz. As will be shown in Section 6.5 the actual threshold SNR for this algorithm is 7 dB/Hz.

6.3.2 STEADY STATE PERFORMANCE OF THE PARAMETRIC BEACON RECEIVER

To illustrate how low an SNR the parametric DBR can handle in the steady state mode the receiver described above was operated for an extended period of time (approximately 300 seconds) and the noise level was increased by 5 dB every few seconds. The receiver was initialized at a SNR of 20 dB which was then reduced to first 15 dB/Hz, next to 10 dB/Hz, then to 5 dB/Hz, and finally to 0 dB/Hz. The steady state operation of the parametric DBR failed to maintain lock at a SNR of 5 dB/Hz. The frequency estimates are from averages of 6 correlations each corresponding to .155 seconds (N=120) of signal data. This results in an observation period of .93 seconds (N=720) (identical to the nonparametric DBR). The frequency prediction error is shown for the SNR of 15 dB/Hz through 0 dB/Hz.

Figure 6.4 depicts the frequency prediction error for the steady state test described above. This figure shows the prediction error drifting away from its zero mean state at 5 dB/Hz. The performance of the parametric and the nonparametric DBR was identical in terms of their threshold SNR. Both algorithms failed at 5 dB/Hz. Both algorithms could track the carrier at 7 dB/Hz although this result is not shown. In the case of nonparametric DBR the bandwidth of the final stage filter H5 could be increased to yield estimates of the frequency of the carrier. In the case of the parametric DBR increasing the bandwidth of the final stage filter H5 leads to a large increase in the variance of the frequency estimates. This is seen in Figure 6.2 where the variance of the prediction error for the first 12 seconds of operation (corresponding to the wider band filter of Figure (4.7b)) is markedly greater than that after 15 seconds of operation (corresponding to the narrow band filter of Figure (4.7a)).
6.4 POWER ESTIMATION RESULTS

In this section we will compare the empirical distributions of the power estimates for the nonparametric and parametric DBR algorithms for various SNR. The histograms that follow were from the receivers described in the previous section for a frequency drift path with a maximum drift rate of 5 Hz/sec.

6.4.1 POWER ESTIMATION PERFORMANCE FOR THE NONPARAMETRIC BEACON RECEIVER

Figures 6.5 through 6.8 represent histograms of power estimates for the nonparametric DBR. As has been stated in Section 1.2 this estimator is biased. The bias is repeated here
bias = \frac{N_0^{781}}{N}

The sampling rate into the RCTA periodogram estimator is present in the bias as the factor of 781 Hz. The periodogram data length is \( N=240 \). Thus the bias is seen as the noise power divided by the data length. Figure 6.5 is a histogram of power estimates at the SNR of 15 dB/Hz. The tails of the distribution are noticeably short with no support beyond 1 W from the true value. The distribution is approximately symmetric and appears approximately Gaussian at this and higher SNR. This histogram is of the estimator of the following quantity

\[
\left| A \right|^2 \bigg|_{A=1}
\]

Figure 6.5  Histogram of Power Estimates for Nonparametric Beacon Receiver
at SNR=15 dB/Hz
Figure 6.6 Histogram of Power Estimates For Nonparametric Beacon Receiver at SNR=10 dB/Hz

Figure 6.6 is a histogram of estimated power for a SNR of 10 dB/Hz. At this SNR however the distribution is clearly non-Gaussian. This low SNR causes an increase in the variance of the power estimates in comparison with that shown in Figure 6.5. The lack of symmetry of this distribution suggests that a linear estimator of the expected value will not be optimal. The true value is the same as in Figure 6.5 and is 1 W.

Figure 6.7 is a histogram of power estimates at the threshold level of 5 dB/Hz. The asymmetry is dramatic and the variance is extremely large. The leftmost tail of this distribution reaches the value of 8 W.
Figure 6.7 Histogram of Power Estimates for Nonparametric Beacon Receiver at SNR = 5 dB/Hz

The variance is large for the estimates represented by Figure 6.7 and the distribution is non-Gaussian. This is actually the break-down point for the nonparametric DBR. At this SNR the DBR loses lock on the carrier.

6.4.2 POWER ESTIMATION PERFORMANCE FOR THE PARAMETRIC BEACON RECEIVER

Figures 6.8 through 6.10 represent histograms of power estimates for the parametric DBR. As has been previously stated this estimator is unbiased. Figure 6.8 is a histogram of power estimates for the parametric DBR at a SNR of 15 dB/Hz.
Figure 6.8 Histogram of Power Estimates For Parametric Beacon Receiver at SNR=15 dB/Hz

Figure 6.8 is similar to the distribution of Figure 6.5 by comparison. The variance may be considered greater for the parametric case of Figure 6.8. The distribution appears to be approximately Gaussian with few occurrences beyond 2.5 W from the true value of 1 W.

Figure 6.9 depicts a histogram of power estimates for the parametric DBR at a SNR=10 dB/Hz. Similar to Figure 6.7 at this SNR however the distribution is clearly non-Gaussian. This low SNR causes an increase in variance which is dramatic in comparison with Figure 6.9. The nonsymmetry of this distribution suggests that a linear estimator of the expected value will not be optimal.
6.5 PERFORMANCE COMPARISON

Both the parametric and nonparametric DBR were able to lock onto a 5 Hz/sec drifting frequency ACTS 30 GHz signal at a SNR of 15 dB/Hz and were able to maintain lock at SNR greater than 5 dB/Hz for frequency measurements based on observation periods of .93 seconds ($N=720$). At and below this SNR the algorithms fail to maintain lock. The variance of the frequency prediction error is the measure of performance we will consider in our assessment of the non-parametric and parametric DBR. The two algorithms performed similarly in terms of power and frequency estimation.

Figure 6.10 shows the Cramer-Rao (CR) bound of frequency estimation for a
single sinusoid in white noise as a solid line. The MSE of the prediction frequency for the parametric and non-parametric DBR algorithms are also shown. These variances were computed as the MSE from the actual carrier frequency.

It is clear from Figure 6.10 that the parametric beacon receiver exhibits approximately 3 dB less variance than the nonparametric receiver over a wide range of SNR. At the SNR of 20 dB/Hz the parametric receiver outperforms the nonparametric receiver by approximately 8 dB. The reason for this is twofold. First the simulation at this SNR (SNR= 20 dB/Hz) was accomplished with 5 modes modeling the signal. At all other SNR on Figure 6.10 only one mode was used. The additional modes provide a closer modeling (in the weighted least squares sense) of the data and therefore more accurate frequency estimates. Secondly, the additional modes provide the steady state algorithm (Figure 5.4) with additional modes to choose from to determine the actual signal mode. The Kalman predictor therefore plays a larger role in the detection of the current frequency estimate for the parametric algorithm with multiple modes. This has the effect of introducing bias into the frequency estimate (since the Kalman is a MMSE estimator). While this estimator is biased it does not necessarily follow that the MSE depicted on Figure 6.10 will increase accordingly since for this simulation (the frequency drift path having an extremely small rate of acceleration) the bias introduced is negligible. The frequency prediction of the parametric receiver with one mode nevertheless outperforms the nonparametric receiver.

Differences can be attributed to two causes. First, the generic fact that the detected ACTS 30 GHz signal is not a pure sinusoid but rather a time-varying sinusoid. Secondly, the inefficient use of data that the two algorithms were based on. In the case of the nonparametric DBR the averaged periodogram was used for frequency estimation. This estimator is not the MLE and does not attain the CR bound. To express the variance of this estimator we first note the CR bound for frequency and the bound for the periodogram as a frequency estimator (since it is the MLE) from Kay [7]. This bound is, for a complex signal of amplitude $A=1$ and noise variance $2N_0 f_s$,

$$ crb(f, N) = \frac{6(2N_0 f_s)f_s^2}{N (N^2-1) \frac{3}{4\pi^2}} \quad (6.1) $$
The averaged periodogram is a mean of periodograms. To determine its variance \( \text{vap}(f) \) we assume statistical independence of periodograms and therefore have

\[
\text{vap}(f, N) = \frac{1}{L} \text{crb}(f, \frac{N}{L}) \quad (6.2a)
\]

where \( L \) is the number of periodograms averaged to estimate frequency. Substituting equation (6.1) into equation (6.2) with the assumption, \( \frac{N}{L} \gg 1 \), results in

\[
\text{vap}(f, N) \approx L^2 \text{crb}(f, N) \quad (6.2b)
\]

In the case of the nonparametric DBR \( L=3 \) and therefore, in decibels, the following results

\[
\text{vap}(f, N)_{dB} \approx 9.5 \text{ dB} + \text{crb}(f, N)_{dB} \quad (6.3a)
\]

where

\[
\text{crb}(f, N)_{dB} = 10\log(\text{crb}(f, N)) \quad (6.3b)
\]
Figure 6.10 Cramer-Rao Bound For Constant Frequency Sinusoid in White Noise with parametric and nonparametric MSE results, o=nonparametric, +=parametric

The AR approach suffers for the same reason. Correlation estimates were averaged to make an estimate of the AR parameters and thus the frequency. The parametric algorithm of Chapter 5 performed as well as the nonparametric algorithm for SNR > 5 dB/Hz.

Figure 6.11 depicts the Cramer Rao bound for power estimation for a single pure sinusoid in white noise. This bound is an ad-hoc approximation derived from the magnitude bound [7]

$$\text{crb}(|A|) = \frac{N_0 f_s}{X}$$

(6.5)

Papoulis [21] provides the following approximation of the variance of a function of a random variable
\[ \text{var}(g) = \left| \theta'(E[x]) \right|^2 \text{var}(x) \]

from which we approximate the bound on \(|A|^2\) to be as follows

\[ \text{crb}(|A|^2) \approx 4 \frac{N_0\delta_s}{N} \]  \hspace{1cm} (6.6a)

Figure 6.11 AD-HOC Cramer-Rao Bound For Power estimation of a Constant Frequency Sinusoid in White Noise with parametric and nonparametric MSE results

\( \circ = \) nonparametric, \( + = \) parametric

From Figure 6.11 we can see that for decreasing SNR both algorithms exhibit MSE that increase relative to the ad-hoc CR bound previously mentioned in equation (6.5). This is due in part to the fact that the MSE is equal to the following

\[ \text{MSE} = \text{variance} + \text{bias}^2 \]
where we have previously defined the bias to be equal to the following

\[ \text{bias} = \frac{N_0 781}{N} \]

For the SNR of 7 dB/Hz we have \( N_0 = 10^{-7/10} \) and the bias is equal to

\[ \text{bias} = .69 \]

which is 69% of the value of \( |A|^2 \). It is apparent that both algorithms exhibit MSE below the ad-hoc CR bound. This is explained by the fact that the noise is not actually white. The narrow band filter (H5 of figures 4.2 and 5.2) greatly reduces the variance of the noise process. If we consider the filter as a moving average then the filter length is added to the data length to yield an effective data length for the estimation of the signal in white noise. Thus the following

\[ N_{ef} = N + L_H \quad (6.6b) \]

In the scheme we have proposed \( L_H = 905 \) and \( N = 240 \), this leads to the following approximation

\[ N_{ef} \approx 4.77N \quad (6.6c) \]

This increase in the effective data length is responsible for the lower MSE relative to the CR bound. To compute this we note that

\[ \left[ \frac{1}{4.77} \right] = -6.7 dB \quad (6.6d) \]

The MSE results of the nonparametric receiver are however approximately 4.5 dB less than the CR bound depicted in Figure 6.11. Thus for the effective data length used the receiver is not efficient (it should be if it were efficient -6.7 dB). This is not surprising since the narrowband filter (H5 of Figure 4.2) is not optimal (it is not the MLE) for estimating power. This is hypothesized as the cause of the approximate 2.2 dB difference.

One comment is in order regarding the pre-filtering accomplished in both
algorithms. The purpose of this filtering is to reduce the sampling frequency via subsequent decimation. This reduces the adverse effects of finite word length errors in the computation of frequency. This does not produce a significant degradation in the lower bound on variance beyond the effect of noise aliasing. To see this consider the input signal CR bound repeated here for convenience

\[
\text{crb}(f, N) = \frac{6(2N_0f_s)^2f_s^2}{N(N^2-1)4\pi^2}
\]  

(6.1)

Now consider the CR bound for the filtered and decimated signal

\[
\text{crb}(f, \frac{N}{D}) = \frac{6(2N_0f_s/D)^2(f_s/D)^2}{\frac{N}{D}\left[\frac{N^2}{D^2} - 1\right]}4\pi^2
\]  

(6.7a)

Equation (6.7a) is the same as equation (6.1) with \( \frac{N}{D} \) substituted for \( N \) and \( f_s/D \) substituted for \( f_s \). This becomes

\[
\text{crb}(f, \frac{N}{D}) = \frac{6(2N_0f_s)^2f_s^2}{N\left[\frac{N^2}{D^2} - 1\right]}4\pi^2
\]  

(6.7b)

The \( D^2 \) term in the denominator of equation (6.7b) in place of the 1 in equation (6.1) suggests a slight increase in the CR bound of frequency. In short the bandwidth of the noise process is not a determining factor in the estimation of frequency provided the observation period is much larger than the decimation rate. For the DBR simulated in the steady state mode, \( D=256 \) and \( N=184279 \) for an observation period of .93 seconds; this leads to the following CR bound for frequency estimation

\[
\text{crb}(f, \frac{N}{D}) = \frac{12N_0f_s^2}{4\pi^2}1.5979742 \times 10^{-16}
\]  

(6.8a)

The CR bound for frequency estimation for the same signal without the bandwidth reduction scheme becomes the following from equation (6.1)
\[ crb(f, N) = \frac{12N_0 f_s^3}{4\pi^2} \times 1.5979711 \times 10^{-16} \tag{6.8b} \]

By dividing equation (6.8a) by equation (6.8b) we can get an idea of the efficiency loss of the sampling rate reduction scheme employed. The quantity so defined is the relative efficiency of the sampling rate reduction scheme with respect to the maximum likelihood estimator

\[ \frac{crb(f, N)}{crb(f, N)} = 0.99998061 \tag{6.9a} \]

A value such as this is so close to 1 that we surmise that little is lost by decimation of the signal by a factor of 256. For the DBR simulated in the transient mode \( D=256 \) and \( N=61426 \) for an observation period of .31 seconds; this leads to the following relative efficiency

\[ \frac{crb(f, N)}{crb(f, N)} = 0.99982631 \tag{6.9b} \]

This number is again so close to 1 that the same conclusion can be reached regarding the effect of sampling rate reduction for an observation period of .31 seconds as an observation period of .93 seconds.
7.0 COMPARISON OF RESULTS FOR 20 GHz BEACON

7.1 RESULTS FOR THE 20 GHZ BEACON

The parametric and nonparametric receivers were also operated on the simulated ACTS 20 GHz beacon in both the transient and steady state modes in an identical fashion as the 30 GHz beacon simulations of the preceding chapter. The SNR described in this chapter are of the carrier to noise power rather than the total signal to noise power. In this way the power in the modulation tones is accounted for in our calculation of SNR. The noise power per Hertz is therefore (with A=1)

\[ N_o = 10^{(-SNR_{dB} - 2.82)/10} \]

This definition accounts for the 2.82 dB reduction in carrier power present in the 20GHz signal relative to the 30GHz signal. We will remember from chapter 1 that a little less than half of the total signal power is in the modulation tones.

7.2 COMPARISON OF TRANSIENT PERFORMANCE FOR BEACON RECEIVERS

We will now compare the nonparametric and parametric beacon receivers in the transient mode for the 20 GHz beacon.

7.2.1 TRANSIENT PERFORMANCE OF THE NONPARAMETRIC BEACON RECEIVER

With the nonparametric DBR in the transient mode the receiver can lock onto the ACTS 20 GHz signal at a SNR=15 dB/Hz. At SNR lower than this the nonparametric DBR can not lock on. The term 'lock on' is used to describe a prediction error of zero mean and variance less than 1 over a 5 second period. This is the performance with a 5 Hz/sec frequency drift rate. Figure 7.1 shows the
frequency prediction error for the nonparametric DBR at a SNR = 15 dB/Hz. A simple comparison of Figure 7.1 with Figure 6.1 shows that the parametric receiver does not perform as well with the 20 GHz beacon as it does with the 30 GHz beacon. The behavior of the frequency prediction error for the first few moments of operation in Figure 7.1 when compared with that of Figure 6.1 shows the degradation in performance caused by the modulation tones present in the 20 GHz signal. The frequency prediction error exhibits much larger variation in the initial seconds of operation with the 20 GHz beacon (note the spike at approximately 5 seconds). Figure 7.1 with its increased deviations from 0 Hz relative to Figure 6.1 typified the performance of the receiver in the transient mode with the 20 GHz beacon.

Figure 7.1 Nonparametric Beacon Receiver Transient Frequency Prediction Error with ACTS 20 GHz Beacon, SNR=15 dB/Hz
7.2.2 TRANSIENT PERFORMANCE OF THE PARAMETRIC BEACON RECEIVER

The parametric DBR in the transient mode can lock onto the ACTS 30 GHz signal at a SNR=15 dB/Hz. As with the nonparametric DBR the simulation is of a 5 Hz/sec frequency drift rate. Figure 7.2 shows the frequency prediction error for the DBR at a SNR = 15 dB/Hz. Unlike the 30 GHz simulation (Figure 6.2) the 20 GHz receiver simulation depicted in Figure 7.2 uses only one mode to model the beacon. After a number of simulations it was found that one mode gave the most consistent performance for the 20 GHz signal.

![Graph showing frequency prediction error over time.](image)

**Figure 7.2** Parametric Beacon Receiver Transient Frequency Prediction Error with ACTS 20 GHz Beacon, SNR=15 dB/Hz
This prediction error is less than 1 Hz after approximately 15 seconds of operation. The behavior of the prediction error for the parametric DBR of Figure 7.2 exhibits less variation than that for the nonparametric DBR shown on Figure 7.1. Also for the first 12 seconds of operation when the bandwidth of the narrowband filter $H_5$ is approximately 30 Hz the frequency prediction error is greater than after 12 seconds (when the bandwidth of $H_5$ has reduced to 4 Hz). This reveals the fact that the parametric DBR frequency prediction error variance is greatly dependent on the bandwidth of the fifth stage bandwidth reduction filter. The parametric receiver also suffers in performance due to the modulation tones of the 20 GHz beacon. This can be seen by comparing Figure 7.2 with Figure 6.2.

7.3 COMPARISON OF STEADY STATE PERFORMANCE FOR BEACON RECEIVERS

We will now compare the nonparametric and parametric beacon receivers in the steady state mode of operation with the 20 GHz beacon.

7.3.1 STEADY STATE PERFORMANCE OF THE NONPARAMETRIC BEACON RECEIVER

To illustrate how low an SNR this algorithm can handle in the steady state mode with the 20 GHz beacon the receiver was operated for an extended period of time (approximately 260 seconds) as in Chapter 6. The SNR was decreased at the instants marked in the Figure 7.3 in the same manner as the 30 GHz simulation of chapter 6. Figure 7.3 illustrates that the nonparametric receiver can track the 20 GHz beacon at a SNR of 7 dB/Hz as it could for the ACTS 30 GHz beacon.
Figure 7.3 Nonparametric Beacon Receiver Frequency Prediction Error with ACTS 20 GHz Beacon, SNR = 15 dB/Hz Through 0 dB/Hz

Figure 7.3 reveals that the frequency prediction error begins to increase beyond the bandwidth of H5 at a SNR of 5 dB/Hz. The prediction error is generally less than 1 Hz for an SNR of 7 dB/Hz.

7.3.2 STEADY STATE PERFORMANCE OF THE PARAMETRIC BEACON RECEIVER

To illustrate how low an SNR this algorithm can handle in the steady state mode with the 20 GHz beacon the receiver was operated for an extended period of time as in Chapter 6. Figure 7.4 illustrates that the parametric receiver can track the 20 GHz beacon at an SNR of 7 dB/Hz as it could for the ACTS 30 GHz beacon. The algorithm operated with one mode modeling the filtered ACTS 20
GHz beacon. The simulation of Figure 8.4 was not operated for a long enough duration at the SNR of 5 dB/Hz to determine its behavior at this SNR.

![Graph showing frequency prediction error](image)

**Figure 7.4 Parametric Beacon Receiver Frequency Prediction Error with ACTS 20 GHz Beacon, Steady State Behavior**

### 7.4 POWER ESTIMATION RESULTS

In this section we will compare the empirical distributions of the power estimates for the nonparametric and parametric DBR algorithms for various SNR with the ACTS 20 GHz signal. The histograms that follow were from the receivers described in the previous section for a frequency drift path with a maximum drift rate of 5 Hz/sec.

#### 7.4.1 POWER ESTIMATION PERFORMANCE FOR THE NONPARAMETRIC
BEACON RECEIVER

Figures 7.5 through 7.8 represent histograms of power estimates for the nonparametric DBR with the ACTS 20 GHz signal. Figure 7.5 is a histogram of power estimates at the SNR of 15 dB/Hz.

Figure 7.5 Histogram of Power Estimates for Nonparametric Beacon Receiver at SNR=15 dB/Hz
Figure 7.6 Histogram of Power Estimates For Nonparametric Beacon Receiver at SNR=10 dB/Hz

Figure 7.6 is a histogram of estimated power for a SNR of 10 dB/Hz. At this SNR however the distribution is clearly non-Gaussian. This low SNR causes an increase in the variance of the power estimates in comparison with that shown in Figure 7.5. The lack of symmetry of this distribution suggests that a linear estimator of the expected value will not be optimal. The true value is the same as in Figure 7.5 and is .522 W.

Figure 7.7 is a histogram of power estimates at an SNR of 7 dB/Hz. The asymmetry is dramatic and the variance is extremely large. The rightmost tail of this distribution reaches the value of 3.5 W.
Figure 7.7 Histogram of Power Estimates for Nonparametric Beacon Receiver at SNR = 5 dB/Hz

The variance is large for the estimates represented by Figure 7.7 and the distribution is non-Gaussian.

7.4.2 POWER ESTIMATION PERFORMANCE FOR THE PARAMETRIC BEACON RECEIVER

Figures 7.8 through 7.10 represent histograms of power estimates for the parametric DBR. As has been previously stated this estimator is unbiased. Figure 7.8 is a histogram of power estimates for the parametric DBR at a SNR of 15 dB/Hz.
Figure 7.8 Histogram of Power Estimates For Parametric Beacon Receiver at SNR=15 dB/Hz

Figure 7.8 is similar to the distribution of Figure 7.5 by comparison. The variance may be considered greater for the parametric case of Figure 7.8. The distribution appears to be approximately Gaussian with little support beyond .6 W from the true value of .522 W.

Figure 7.9 depicts a histogram of power estimates for the parametric DBR at a SNR of 10 dB/Hz. The distribution of Figure 7.8 is clearly non-Gaussian. The variance of Figure 7.8 is noticeably greater than Figure 7.9. The nonsymmetry of this distribution suggests that a linear estimator of the expected value will not be optimal.
7.5 CONCLUSIONS REGARDING RECEIVER PERFORMANCE WITH THE ACTS 20 GHz BEACON CARRIER

The parametric and nonparametric receiver designs are able to perform in the transient mode at the SNR of 15 dB/Hz and in the steady state mode at an SNR of 7 dB/Hz. The parametric receiver in the transient mode requires the use of one mode when operated with the 20 GHz beacon carrier. With more than one mode the parametric receiver performed erraticly and did not lock onto the carrier with probability 1.

Since the only difference between the 30 GHz beacon and the 20 GHz beacon is the modulation tones present in the 20 GHz beacon we can ascribe the degradation of the receiver performance in this chapter to the modulation tones.
8.0 CONCLUSIONS AND RECOMMENDATIONS

8.1 CONCLUSIONS

A number of observations and conclusions can be made from our investigation. Hilbert transformation of the ACTS 20 GHz and 30 GHz signals can be accomplished with inphase and quadrature sampling provided that two conditions are met. First the bandwidth of the signal is to be minimal. Secondly the sampling rate should be as large as possible.

The parametric and nonparametric DBR performed similarly. Both algorithms failed to track the 30 GHz carrier at 5 dB/Hz with an observation period of .93 seconds \((N=720)\). Both algorithms were able to lock onto a drifting frequency carrier at 15 dB/Hz with an observation period of .31 seconds \((N=240)\). The parametric DBR however obtained frequency estimates with variance approximately 3 dB lower than the nonparametric DBR at SNR greater than 10 dB/Hz.

The efficiency of the parametric and nonparametric algorithms is directly related to the use of averaging. For the nonparametric algorithm investigated steady state frequency estimation was based upon the averaged periodogram (an average of 3 periodograms) instead of the maximum likelihood estimator, the periodogram. This leads to a 9 dB increase in the variance of the frequency estimates. For the parametric algorithm even more averaging (an averaging of 6 correlations) is present and the performance of the receiver is correspondingly degraded. In spite of this handicap the parametric DBR outperformed the nonparametric DBR over a wide range of SNR in the arena of frequency prediction. In fact the parametric receiver is able to estimate frequency with a MSE approximately 3 dB less than the nonparametric approach. This result (a 3 dB improvement over the nonparametric receiver) was attained with a single mode model. With 5 modes modeling the data the parametric receiver is able to estimate frequency with a MSE approximately 7 dB less than the nonparametric algorithm.

The nonparametric receiver however outperformed the parametric receiver for the criterion of power estimation. The nonparametric receiver exhibited a MSE which was approximately 1.5 dB to 2 dB lower than that for the parametric
approach.

The deleterious effects of frequency drift can be effectively minimized with the use of Kalman prediction of the frequency drift path. The carrier signal can therefore be maintained in a very narrow band of 1 Hz with the use of Kalman prediction. The excellent results for the simulated ACTS signals were obtained for a 5 Hz/second frequency drift path. The instantaneous frequency is periodic with period 24 hours.

Sampling rate reduction is important for two reasons. First, it is important for the purpose of obtaining the narrow bandwidth filter necessary for the parametric estimation of frequency. In addition any finite word length effects on frequency estimation are reduced at lower sampling rates. Both of these considerations are necessary for the parametric DBR and less so for the nonparametric DBR. Sampling rate reduction does not significantly increase the lower bound of frequency estimation variance.

8.2 SUGGESTIONS FOR FUTURE WORK

The question remains as to what can be done to improve these algorithms. The first suggestion is that averaging of correlations and/or periodograms is to be avoided. The data must be used efficiently in the estimation of frequency and therefore longer correlation sequences must be used (parametric) or the periodogram (in lieu of the averaged periodogram) must be used (nonparametric).

An additional improvement can be gained in the parametric approach if the more accurate approximation of the autocorrelation covariance matrix is used. We chose a diagonal approximation which was computationally more efficient. The nonsparse matrix of equation (5.8h) would be more accurate and could improve the results of the parametric DBR. A further improvement could be gained if the nature in which the modified Yule-Walker equation was solved is considered. The equation is solved in the weighted least squares sense. Thus no account is made for the variance present in the correlation matrix estimate, \( \hat{R} \) of equation (5.14). Research has been undertaken [19,20] to develop algorithms which can find the AR parameters accounting for the errors present in the correlation matrix. In this approach [19,20] a perturbation matrix \( E \) is sought which when added to \( \hat{R} \) yields a
low rank approximation to R. Since R is a single sinusoid correlation matrix it has rank=1. In this way the AR parameters are more accurately estimated. These approaches rely on the singular value decomposition of the estimated correlation matrix and are computationally intensive. Another approach would be to perform Kalman prediction on the correlation estimates themselves. In this way the variance of the estimates is reduced prior to AR estimation. This would be problematic however. The determination of a transformation (transition matrix) for the correlation state would be difficult. The transition would be time varying and nonlinear. Other approaches are possible. One approach would be to compute the correlation estimate on data that is prefiltered by a low pass filter but not decimated. The correlation estimate can then be decimated to form the basis for a number of autoregressive coefficient estimates which can then be averaged or can produce a number of frequency estimates which can be averaged.

For the nonparametric approach the CTA used to compute a periodogram estimate of the unfiltered data is the MLE. Anything done in a DBR algorithm to approach this estimator is desirable. In this regard the last narrow band filter is unnecessary. The RCTA is statistically most efficiently when performed on unfiltered data (since the periodogram is the MLE for sinusoids in white noise). For this reason only sample rate reduction filtering is required and not necessarily as much sample rate reduction as is performed in the algorithms presented in this thesis. There are other nonparametric approaches to the detection problem which could be considered. One such approach is the pseudo Wigner distribution (PWD) which is the computation of a frequency-time spectrum (or distribution). If we consider the periodogram as an inner product of the transformed data then the PWD can be viewed as an outer product [22]. The advantages of the PWD is that each PWD provides not only frequency estimates but also estimates of the first derivative of the frequency. With the PWD the Kalman predictor could have two inputs rather than one.

A suggestion is in order regarding the simulation of the ACTS signal and the application of the algorithms to this signal. Since the necessary amount of data processed by these algorithms is excessive and effects of sampling rate reduction being known any future work can be greatly expedited by simulating the detection algorithms on low sampling rate signals. These low rate signals can be directly generated. In this way the decimation filtering common to the algorithms can be
avoided for the purpose of making performance comparisons.

The use of larger data lengths is also an option and should be considered. The algorithms presented estimated the frequency from data lengths corresponding to .93 seconds of data for frequency estimates and .31 seconds of data for power estimates. Any increase in these parameters will result in a corresponding improvement in the performance of the algorithms. It would be wise therefore to determine an upper bound on the time over which the frequency drift process can be assumed linear. This would result in a maximum allowable observation period for frequency estimation if linearly varying mixing frequency signals are used, see equation (3.22). In conclusion the data must be used efficiently in the estimation of frequency and as much data as possible must be used per frequency estimate.
REFERENCES


VITA

Born in Providence Rhode Island, June 13 1963, Paul John Gendron attended public high school in Seekonk Massachusetts. He earned the BSEE degree from the University of Massachusetts in May of 1985. He worked for the Portsmouth Naval Shipyard, Portsmouth New Hampshire, as a submarine design engineer for ships navigation, control and radio systems before beginning studies as a masters student. This current work is the result of his continuing interest in statistical signal processing and estimation theory. His other interests include history and economics.

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