Computer Aided Analyses of Symmetrically loaded Thin Cylindrical Shell using Transfer Matrix Method

by

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COMPUTER AIDED ANALYSES OF SYMMETRICALLY LOADED THIN CYLINDRICAL SHELL USING TRANSFER MATRIX METHOD

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(ABSTRACT)

Thin shells as a structural form have made important contribution to the development of several branches of engineering, e.g., chemical, power, structural engineering, etc. The aim of this research is to analyze symmetrically loaded thin cylindrical shell structures using transfer matrix method. A shell is a body bounded by a double curved surface, the thickness of the shell generally being smaller than its radius. A shell is classified thin if the ratio of the thickness to radius is less than twenty.

The analyses of the shell is done using the transfer matrix method. The theory of transfer matrix method is discussed and the analogy between shell and beam on elastic foundation is proved. This analogy between shell and beam is used as a foundation to analyze the shell. Several different loading conditions, e.g., uniform and variable pressure load, ring load etc. are considered. The displacement, shear, angle, moment and various stresses like maximum normal stress, hoop stress, longitudinal stress etc. are found. Shell structures with ring stiffeners are also studied. Several test examples are analyzed and the results are compared to published data.

To my mother

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Nomenclature

A_s Equivalent shear area (length^2)

a Radius of cylinder (length)

D Flexural rigidity = $\dfrac{Et^3}{12(1-\nu^2)}$ (force · length)

D_x Flexural rigidity (force · length)

D_v Shear Stiffness (force / length)

d_i Inside diameter (length)

d_o Outside diameter (length)

E Modulus of elasticity (force / length ^{2})

$[F]$ Field transfer matrix

F_i Transverse loading intensity (force / length)

G Shear Modulus of elasticity (force / length ^{2})

I Moment of inertia about centroidal axis (length ^{4})

K_{ss} Shear stress correction factor

k Elastic foundation stiffness (force / length ^{2})

k^* Rotary foundation modulus (force · length / length)

L Length

M Bending moment (force · length)

M_i Moment intensity (force · length / length)

M_T Thermal Moment (force · length)

m Mass

$[P]$ Point transfer matrix

P Axial force (force)

p Internal pressure (force / length^2)

R Radius of the cylinder (length)

R_0 Outside radius (length)

R_i inside radius (length)

r_y Radius of gyration of cross sectional area about y axis

T Temperature (degrees)

t Thickness of shell (length)

$[U]$ Overall transfer matrix

V	Shear force (force)
w	Lateral Displacement (length)
$\{Z\}$	State vector
α	Coefficient of thermal expansion (length / length · degrees)
θ	Angle or slope of the deflection curve (radian)
ν	Poisson's ratio
κ	Change in curvature
τ	Shear stress component
σ	Direct stress component
σ_L	Longitudinal Stress due to end plates
σ_H	Hoop stress
$\sigma_{B,L}$	Longitudinal stress due to bending
$\sigma_{B,C}$	Circumferential stress due to bending
γ	Shearing strain component
ε	Strain component
ρ	Mass per unit area (force · time^2 / length^3)
ω	Natural frequency (radian/sec)
ω_n	Natural frequency (radian/sec)

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Chapter 1

Introduction

1.1 Definition

The aim of this research is to analyze *thin shells* using the transfer matrix method. A shell is a body bounded by a double curved surface and the distance between the surfaces is the thickness of the shell; the thickness generally being small compared to the radius. For the purpose of analysis, shells are classified as either thick or thin shells. If R is the radius of curvature of the middle surface and δ the thickness of the shell, then the shell is considered *thin* if

$$\text{maximum}\ (\delta / R) \leq 1/20$$

For a broad range of applications like aircraft, ships, rollers, pressure vessels etc., the shell thickness lies in the range of

$$1/1000 < (\delta / R) < 1/50$$

1.2 Application

Thin shells as a structural form have made an important contribution to the development of several different branches of engineering. These include power, chemical and structural engineering, vehicle body structures, boat construction and other miscellaneous applications. The development of steam power depended upon the construction of thin shell structures (boilers) to a certain extent. Pressure vessels and associated pipework are key components in thermal and nuclear power plant and in all branches of chemical and petroleum industries. The use of thin-shelled structures for vehicles has similarly revolutionized the construction of aircraft and motor vehicles. The analysis of thin-shell structures; i.e., deformation and various stresses induced under different loading conditions is, therefore, very important.

1.3 Methods of Analysis

Over the years, several methods have been used to analyze the shell structures. These are the so-called classical continuum method, the finite element method, and transfer matrix method. In the classical continuum method, the shell is considered to be a continuous surface. The differential equation for this continuum structure have been developed and are solved by standard methods. The arbitrary constants involved in the solution are then determined so that the solution satisfies the boundary conditions of the problem. In the finite element analysis the shell surface is assumed to be built up from a series of small elements connected only at the nodes of the elements and suitably oriented so that they approximate to the overall geometric shape of the shell. The elastic behavior of the shell is found by deriving a relationship between forces and displacement at the nodes. Using this force-displacement relationship to satisfy the equilibrium and compatibility conditions at each node, an overall stiffness matrix is found in terms of unknown nodal displacements.

Inversion of this overall stiffness matrix then determines the nodal displacement which, in turn, may then be used to determine the stress resultants in the shell.

A third method, the transfer matrix method, is based upon the fact that a complex structure may be divided into simple sections with specific structural properties. These properties are expressed in matrix form. A state vector, consisting of variables like deflection (w), angle (θ), moment (M), shear (V), etc. describes the "state" of deflection and of internal loads in the structure in response to various loading conditions. The matrix that relates the state vector on one side of the section to the other is called a transfer matrix . There are two types of transfer matrices; a field transfer matrix and a point transfer matrix. The field transfer matrix transfers the state vector along the length of a section, while a point transfer matrix transfers the state vector across a point parameter such as a point load, point moment, etc. A complex structure is simplified by modeling it as a series of field and point transfer matrices. To find the response of the structure to various loading and support conditions, the field and point transfer matrices are successively multiplied. The resulting transfer matrix contains all the information about the variables of interest, which can be solved for when the boundary conditions are applied.

There are several advantages and disadvantages for each method. The classical continuum method is too cumbersome for practical application since solutions cannot be found for some practical configurations. Though the finite element method provides accurate results, it is very time consuming and requires massive computing power and extensive memory since the size of the stiffness matrix depends on the number of degrees of freedom.

The transfer matrix method is simple and very useful for simple systems having a chain topology i.e., a member that could be modeled as having a number of elements linked together end to end in the form of a chain. But it is less useful for branched or coupled system.

In the case of the analysis of a thin-cylindrical shell, we consider asymmetric shell structures whose geometry is not very complicated and where there are generally no rigid intermediate conditions. Hence, it is more efficient to use the transfer matrix method. The basic idea is to model this cylindrical shell as a continuum beam on elastic foundation. The internal or external pressure can be considered as uniform loading. The research also includes the analysis of thin-cylindrical shell with stiffener rings.

Chapter 2

Literature Survey

2.1 Thin Shells

The theory of shell structures is an extensive subject. It has been a part of the branch of structural mechanics branch for over a hundred years and there exists extensive literature in this field.

Loads applied on the shell are carried by a combination of bending and stretching actions which vary over the surface. One of the major difficulties in the theory of shell structures is to find a relatively simple way of describing the interaction between the two effects. According to Rayleigh [1] the deformation of a thin hemispherical bowl would be inextensible. Hence, he developed a simple method of analysis which took into account only the strain energy of bending in the shell. In 1888, Love [2] noticed the inaccuracies in Rayleigh's theory of shell and corrected them. He argued that for thin shells stretching dominates over bending. He assumed small strains and small thickness to radius ratios. At this time Love had not grasped the strong contrast between the behavior of open and closed shells. Moreover, Love did not discuss how to write the relations between forces, moments and the displacements of the middle surface.

Lamb [3] solved Love's general equations in the simple case of a cylindrical shell and demonstrated the possibility of a relatively narrow boundary layer in which there was a rapid spatial transition between bending and stretching effects, with the width of the boundary layer being determined by the interaction between these effects. Novozhilov [4] presented an analysis of homogenous orthotropic cylindrical shells. Later, Schnaell and Bush [5] and Thielmann et al.[6] presented several theoretical analysis limited to orthotropic shells. Cheng and Ho [7], Jones and Morgan [8] and Jones [9] presented general linear theoretical solutions to anisotropic cylinders. All the theories presented above assumed shells as infinitely rigid in transverse direction by neglecting transverse strains. These theories underestimate deflection and stresses and overestimate natural frequencies and buckling loads.

Simultaneously, some work on the development of thin shell theory was also being done in Russia. The deficiencies in the theory of shells given by Love and others were studies in the Russian school and Galerkin gave formulas for shells from the general theory of elasticity. It provided the ground work for the mathematically rigorous theory of thin shells. A.I. Lur'e [10] [11] first deduced the equation for the theory of thin shells, but he was not able to give the necessary criterion to simplify his formulae.

In 1973 Flügge [12] published a book where he presented a clear understanding of the mechanics of shells, from the formulation of the differential equations to the discussion of the results of analysis. He studied direct and bending stresses in a cylindrical shell and a general shell of revolution. He also covered buckling of shells and briefly introduced forces and deformation in circular rings. In 1977, Gould [13] presented the geometric description of the surface and, considering equilibrium, he introduced the membrane theory of shells.

Using energy principles, he presented the flexural theory of plates and bending theory of shells.

2.2 Transfer Matrix Method

The transfer matrix method was first used by Myklestad [14] in 1944. Though at that time it was a tabular method it marked the birth of what came to be known as the transfer matrix method. This method was used for vibration analysis of airplane wings and fuselage, bridges, critical speed of shafts, etc. This method was successful in finding the higher modes of vibration and it was not necessary to assume the shape of the modeshapes. Myklestad used his tabular method to model an airplane wing. He divided the wing into sections, each considered to be massless and the mass being lumped at the end of the sections. After applying the appropriate boundary condition he then applied a fourth-order beam equation and iterated until he found the natural frequency. The procedure he used was as follows: apply the boundary condition at the left end of the section and then using the beam equation move to the right. If at the end the right-end boundary condition was satisfied, then the frequency is a natural frequency; if not, the residual was recorded and another frequency was tried. Then the residuals were plotted against the respective frequencies and the natural frequency were the ones where residual were zero.

Then in 1945, M. A. Prohl [15] used a similar tabular method to find the critical speed of flexible rotors. He also used a fourth-order beam equation and since this equation applies to flexible rotors the procedure was very similar to the one used by Myklestad.

In 1950, W. T. Thompson [16] improved Myklestad tabular method and coined the term transfer matrix. Myklestad has assumed lumped mass at discrete locations along the beam. The stiffness coefficient for each section of the beam between station was determined from the actual stiffness curve of the beam by using the area moment principle. The geometry and equilibrium equation were then transformed to a tabular computational scheme which was very tedious. Thompson introduced the transfer matrix method which avoided this tedious algebraic substitution. Also, Thompson kept the same variables of interest on the left side as the right. He then applied boundary conditions to the complete matrix, which resulted into much simpler means of computing the natural frequency.

In 1956, K. Marguerre [17] worked on vibration and stability problems of beams analyzed by transfer matrix method. In transfer matrix method it was found that to solve an eigenvalue problem it was necessary that the second-order sub determinant must become zero. It was also found that systematic difficulties arose when there was intermediate support conditions or pin joints. Even though these supports or hinges are elastic and, therefore, can be taken into account by a transfer matrix, the coefficient of the equation for the eigenvalues appeared to be a difference of two almost identical large numbers. Marguerre introduced an associated matrix, called a delta matrix, which overcame both these difficulties. In 1956

Ehrich [18] tried applying the transfer matrix method to find the vibration modes of non uniform discs.

In 1960, Pestel and Leckie [19] published a paper which provided a comprehensive understanding of the transfer matrix. In this paper they defined the state vector, field and point transfer matrices, etc. The paper presented a procedure to obtain the overall or global transfer matrix by deriving the state vector on the right side in terms of state vector on the left side. Transfer matrices for various systems like branches, point parameters and elastic supports that were very stiff (nearly rigid) were also introduced. Numerical difficulties arose while dealing with elastic supports that were very stiff and the paper tried to investigate solutions for this difficulty.

In 1963 Pestel and Leckie [20] published the first book dealing with transfer matrix methods. It dealt with transfer matrices for very simple to complex systems with intermediate rigid conditions and releases. The book also introduced transfer matrices dealing with free and forced damped vibrations. It explained methods to calculate natural frequencies and normal modes of a system and then using these normal modes to obtain the response of the system to time varying forces and displacement. A means of deriving a transfer matrix from a stiffness matrix was also covered. A large catalogue of transfer matrices used for beams was also given. Several examples were provided to illustrate the usefulness of the transfer matrix method.

Pilkey [21] [22] published two manuals in 1969 which basically dealt with in depth study of response of structural members of arbitrary geometry under different loading conditions. The structural members considered included strings, bars, arches, membranes, spring mass systems, beams, thin walled beams, plates, thin shells, grillages and simple frames. Loading conditions like point force, point moment, distributed load, torsion etc. were treated. This manual is of special interest in the work here as it provides fundamental procedure to analyze thin-shelled structures. It illustrates how a beam on an elastic foundation can be used as an equivalent model for thin-shell cylinders. It also provides with some special transfer matrices e.g. transfer matrix for stiffener rings and shell with layers made of different material. The reader is warned that there were typographical errors in the transfer matrix given for the stiffener ring in reference [22].

The manual contains all essential geometry and loading conditions to take care of simple and complicated systems and to find their solution in the form of displacement, forces, critical loads, frequencies and stresses. It presents the transfer matrix method as a simple and efficient way to obtain analytical solutions for the response of structural members. A computer program called LSD (Linear Solution Development) supplemented the manual. It was written in FORTRAN and though being very versatile it required the user to be well versed with computer programming as several subroutines for input of beam geometry and loading conditions were supposed to be written by the user. Also some subroutines for some special transfer matrices like in-span indeterminate and time dependent loading had to be written by the user. This was the first comprehensive computer program written to automate the analysis of structural members using transfer matrix method. It dealt with several transfer matrices given in the manual and was able to handle beams, rotors, plates, bars etc.

In 1975 Pilkey and Pilkey [23] collected the existing computer programs dealing with shock and vibration and published a review of these in a book. The book had about fifteen programs dealing with analysis of beams. The methods used in these programs were: transfer matrix method, finite element method and finite difference method. SPIN was one of the notable programs which was capable of handling static and dynamic response, calculate natural frequencies and also various stresses. It allowed rotary and linear springs as point parameters but could not handle in-span rigid supports.

Realizing that the LSD program was difficult to operate Pilkey developed another more efficient and powerful program called BEAMRESPONSE in 1974. This program did not require the user to write subroutines and was much more user friendly. The only input required by the user was the geometry of the structure to be analyzed. BEAMRESPONSE had the capacity to handle in-span indeterminate, beams with shear deformation, axial loads etc. It also analyzed structures on elastic or higher order foundation.

In 1978, Pilkey and Chang [24] published another book dealing with transfer matrices. This book had extensive collection of transfer matrices for beams, shafts, disks, plates, shells, etc. Also relevant to the work here is to note that the typographical errors for stiffener ring in thin cylindrical shells, which were present in his earlier work, had been corrected.

In 1994, Pilkey [25] published yet another book which provides formulas for stresses, displacement, buckling loads and natural frequencies. It provides information for static, stability, and dynamic analysis of beams, plates and shells with very general mechanical or thermal loading.

Chapter 3

Thin shell: Theory

3.1 Introduction

3.1.1 Coordinate system

The following Figure 1 defines the coordinate system used for the various derivations. The x- coordinate lies along the axial length of the shell, the y- coordinate is along the circumferential direction and z- coordinate is normal to the shell surface. The z- coordinate is positive when it is directed away from the center of the shell's positive outward normal. The θ coordinate is defined as the circumferential angle of the shell element. The x-, z- and θ coordinate system is right handed and the center of the coordinate axis is taken as the left end of the semi infinite shell structure.

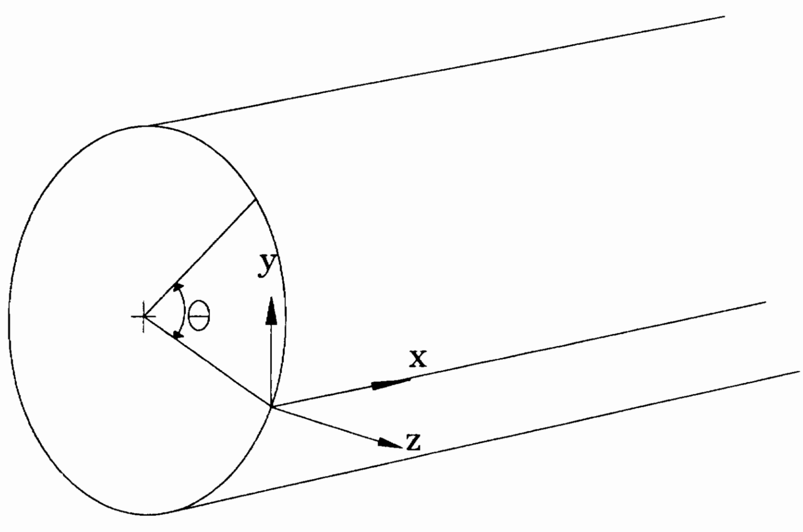

Figure 1. Coordinate system of shell structure

3.1.2 Assumptions

The assumptions made for the analysis are as follows:

1. Material of shell is homogeneous and isotropic.

2. Material obeys Hooke's law.

3. The displacements are small in comparison with the thickness of the shell.

3.2 Shells under symmetrical loading

3.2.1 Equilibrium Equation

Practical applications of shells under symmetrical loading include cylindrical tanks loaded with either internal or external pressure, cylindrical rollers used for film making, etc.

Consider a semi-infinite shell structure with internal pressure as a loading condition. Here, a semi-infinite shell is considered because it would simplify the analysis. Later on this analysis can be extended to shell structure of finite length. The main objective here is to find the displacement and stresses arising along the shell structure. It is assumed that the displacements are small and that the shell is initially stress free (i.e. there are no initial stresses like those induced by temperature difference etc.). Thus all the stresses arising are due to external loading only. These other stresses (like those arising from temperature difference, etc.) can be accounted for by superposing the results found by this analysis, e.g., if the cylinder has end plates and is loaded by interior pressure, then the following analysis can be used to find the stresses due to the interior pressure and then the longitudinal stresses arising due to the end plates can be superimposed, or added, to the results.

Referring to Figure 2 let ε_X and ε_θ be the longitudinal and circumferential strains respectively. They are positive when the shell element elongates and are negative when it contracts. κ_X and κ_θ are the longitudinal and circumferential changes in curvature of the shell. $\kappa_{\theta X}$ is the change in angular direction, which is zero in our case because of the symmetry. N_x and N_θ are the forces in x- and θ direction. They are measured per unit length.

The different variables of displacement are the radial component, *w*, the longitudinal component, *u*, and the circumferential component, *v*. The circumferential component *v* is zero since there is no loading which may rotate the shell. The radial component, *w*, is the main displacement component and the sign convention is positive when the displacement is away from the center of the shell axis.

The forces acting in the z- direction are the pressure loading $(padxd\theta)$, shear force $(\frac{dQ_x}{dx}adxd\theta)$ and the component $N_\theta dxd\theta$ of the force per length of cylinder N_θ. The pressure p and shear force are acting in the upward direction while the component of N_θ is acting downwards.

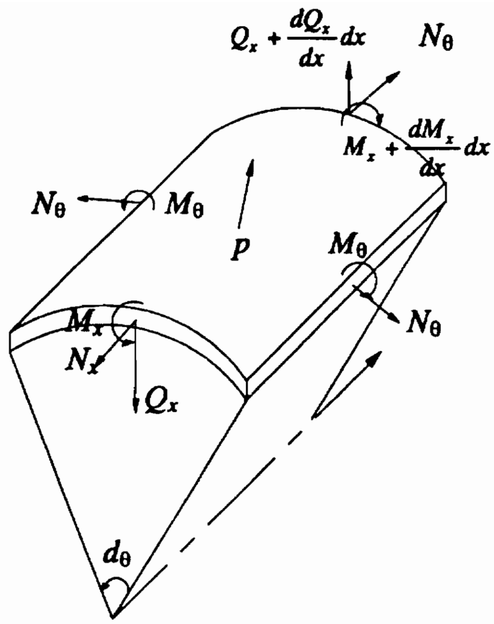

Figure 2. Various loads in a shell element with internal pressure

Therefore the equilibrium equation becomes

$$\frac{dQ_x}{dx} adxd\theta + padxd\theta = N_\theta dxd\theta \tag{3.1}$$

Therefore,

$$\frac{dQ_x}{dx} - \frac{N_\theta}{a} = -p \tag{3.2}$$

Similarly, taking the moments about an axis tangential to the circumferential direction gives

$$\frac{dM_x}{dx} - Q_x = 0 \tag{3.3}$$

substituting the value of Q_x from Eq. (3.3) to Eq. (3.2) results in

$$\frac{d^2 M_x}{dx^2} - \frac{N_\theta}{a} = -p \tag{3.4}$$

3.2.2 Analogy between Shell and Beam on elastic foundation

Consider that, because of the forces applied, the circumference of the shell is displaced outward by w and, hence, the radius of the shell changes from a to $a+w$. Thus, the circumferential strain, defined as the change in the circumference ratioed to original circumference is given by

$$\varepsilon_\theta = \frac{2\pi\,(a+w) - 2\pi a}{2\pi a} = \frac{w}{a} \tag{3.5}$$

and the change in the longitudinal curvature for small displacements is given by

$$\kappa_x = -\frac{d^2 w}{dx^2} \tag{3.6}$$

The change in the circumferential curvature is negligible in comparison to the change in the longitudinal curvature and hence is neglected.

According to Hooke's Law we have

$$\varepsilon_\theta = \frac{N_\theta}{Et} + \frac{\nu N_x}{Et} \tag{3.7}$$

but $N_x = 0$ in this case, therefore,

$$\varepsilon_\theta = \frac{N_\theta}{Et} \tag{3.8}$$

Now taking the moment about the center line of the element in discussion, we get

$$\kappa_x = M_x / D \tag{3.9}$$

Therefore,

$$M_x = \kappa_x D \tag{3.10}$$

where $$D = \frac{Et^3}{12(1-\nu^2)}$$

Combining Eq. (3.10) and Eq. (3.6) gives

$$M_x = -D\frac{d^2w}{dx^2} \tag{3.11}$$

and combining Eq. (3.5) and Eq. (3.8) gives

$$N_\theta = \frac{Et}{a}w \tag{3.12}$$

Substitute the Eq. (3.11) and Eq. (3.12) into Eq. (3.4) to obtain

$$D\frac{d^4w}{dx^4} + \left(\frac{Et}{a^2}\right)w = p \tag{3.13}$$

and since here we are concerned only with edge loads, $p = 0$

Therefore,

$$\frac{d^4w}{dx^4} + \left(\frac{Et}{Da^2}\right)w = 0 \tag{3.14}$$

The above equation is analogous to the governing equation for a "**Beam on Elastic Foundation**" as given by Pilkey [24].

Now, simplifying the coefficient of w in Eq. (3.14)using the value of D from Eq. (3.10) substituting the value of D in Eq. (3.14)

$$\frac{Et}{Da^2} = \frac{12(1-\nu^2)}{a^2t^2} = \frac{4}{\mu^4} \tag{3.15}$$

where $$\mu = \frac{(at)^{1/2}}{\left(3(1-\nu^2)\right)^{1/4}}$$

This value of μ is taken so as to make it easier to solve the governing equation.

Hence, the governing equation, Eq. (3.14), becomes

$$\frac{d^4w}{dx^4} + \left(\frac{4}{\mu^4}\right)w = 0 \tag{3.16}$$

Assume the solution of the form $w = Ae^{\alpha x}$

Substituting this value in the above Eq. (3.16) and integrating we get

$$(\alpha\mu)^4 = -4$$

which has the roots

$$\alpha\mu = \pm 1 \pm i$$

and the general solution for this may be written as

$$w = A_1 \cosh(x/\mu)\cos(x/\mu) + A_2 \cosh(x/\mu)\sin(x/\mu)$$

$$+ A_3 \sinh(x/\mu)\sin(x/\mu) + A_4 \sinh(x/\mu)\cos(x/\mu)$$

or

$$w = A_1 e^{-\frac{x}{\mu}} \cos(x/\mu) + A_2 e^{-\frac{x}{\mu}} \sin(x/\mu) + A_3 e^{\frac{x}{\mu}} \cos(x/\mu) + A_4 e^{\frac{x}{\mu}} \sin(x/\mu) \qquad (3.17)$$

where A_1, A_2, A_3 and A_4 are arbitrary constants and their values will be determined according to the boundary conditions.

3.3 Stresses in Shell

The various stresses arising in a shell when it is loaded with either internal or external pressure and the cylindrical shell is closed at both ends by end plates are given as follows:

1. Longitudinal Stress due to end plates (σ_L)
2. Hoop stress (σ_H)
3. Longitudinal stress due to bending ($\sigma_{B,L}$)
4. Circumferential stress due to bending ($\sigma_{B,C}$)
5. Shear Stress due to transverse load (pressure) (τ)

At each end of the cylindrical shell the pressure acts on a circular area formed by the inside radius of the shell. But due to the end plates this pressure force is resisted by an area formed by the thickness of the cylindrical shell. Thus, the longitudinal stress arising is

$$\sigma_L = \frac{p d_i^2 \frac{\pi}{4}}{\frac{\pi}{4}(d_0^2 - d_i^2)} = \frac{p d_i^2}{(d_i^2 - d_0^2)} \tag{3.18}$$

When the shell expands it produces circumferential stress. This stress is called the hoop stress and it is constant through the thickness of the thin shell. It is expressed as

$$\sigma_H = E\varepsilon_\theta = \frac{Ew}{R} \tag{3.19}$$

Ideally, bending is neglected while dealing with thin shell elements. However, since the shell structure in this case has end plates, the bending stresses are significant and cannot be neglected. The bending causes longitudinal and circumferential stress in the shell structure. The longitudinal stress due to bending is found by the elementary flexure formula

$$\sigma_{B,L} = \pm\frac{Mc}{I}$$

Here $c = t/2$, therefore

$$\sigma_{B,L} = \pm \frac{Mt}{2I} \tag{3.20}$$

and the circumferential stress due to bending is given by

$$\sigma_{B,C} = \pm \nu \sigma_{B,L}$$

Therefore,

$$\sigma_{B,C} = \pm \nu \frac{Mt}{2I} \tag{3.21}$$

The nominal transverse shear stress at the neutral axis of the shell is given by

$$\tau = \frac{V}{t} K_{ss} \tag{3.22}$$

According to Shigley and Mitchell [27], the value of K_{ss} = shear stress correction factor = 2

3.4 Maximum shear stress theory

Maximum shear stress theory is used to calculate the maximum stress in the inside, middle and outside surfaces of the shell structure. This theory is safe to use as it gives conservative results. Let σ_{MSST} represent the equivalent normal stress according to the maximum shear stress theory.

According to Shigley and Mitchell [28], maximum shear stress theory states that *"yielding begins whenever the maximum shear stress in any mechanical element becomes equal to the maximum shear stress in a tension-test specimen of the same material when that specimen begins to yield."*

If the principal stresses are arranged so that $\sigma_1 > \sigma_2 > \sigma_3$ then this theory predicts that failure will occur when

$$\tau_{max} = \frac{S_y}{2} \quad \text{where} \quad S_y = \sigma_1 - 0$$

Now while actually calculating these stresses the direction of each stress, i.e. whether the stresses are compressive or tensile, depends on the surface being considered and whether the pressure loading is internal or external. For example, consider the case of internal pressure acting on a cylindrical shell with end plates as shown in the Figure 3.

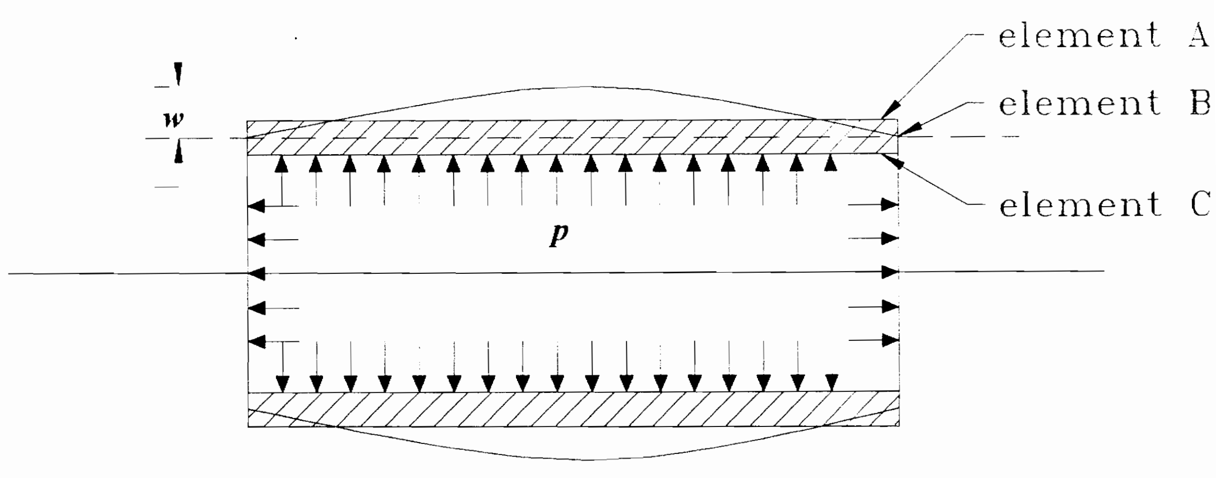

Figure 3. Cylindrical shell with end plates and internal pressure loading.

ELEMENT A: The element A is on the outside surface. The shell expands due to internal pressure and, hence, the outer surface is in tension. The stresses arising on this surface will be hoop, longitudinal, bending longitudinal and bending circumferential. The hoop stress will be tensile. As the end plates hold the ends together there will be some bending in the shell and the outer surface will be compressed by this action. The circumferential stress due to bending will be compressive. The internal pressure tends to move the end plates out and hence causes longitudinal stress which will be tensile. The longitudinal stress due to bending in this case will be compressive on the outside surface. Figure 4 shows the orientation of these stresses.

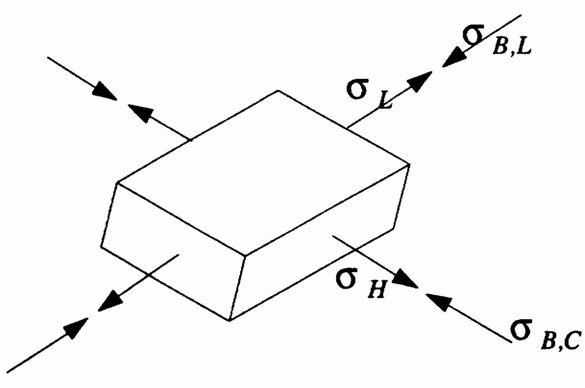

Figure 4. Stresses in the outer surface of shell loaded with internal pressure.

For this element the smallest principal stress $\sigma_3 = 0$ since the gauge pressure at the outside is zero. Also,

$$\sigma_1 = \text{larger of } (\sigma_L - \sigma_{B,L}) \quad \text{and} \quad (\sigma_H - \sigma_{B,C}).$$

The Mohr's circle for this condition is as shown in Figure 5.

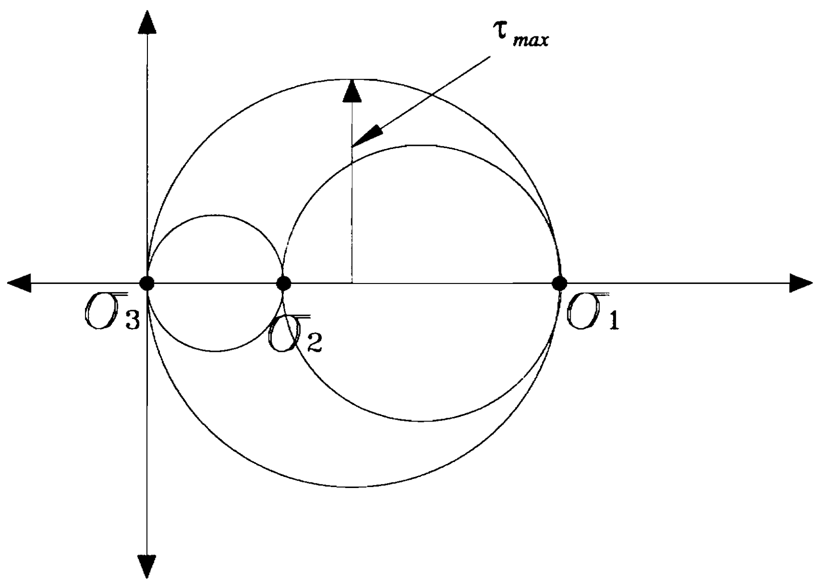

Figure 5. Mohr's circle for outside fiber

Thus $$\sigma_{MSST} = 2\tau_{MAX} = \sigma_1$$

there is no bending, so the stresses due to bending vanish. But there exists an internal pressure and maximum shear stress. According to standard thin shell theory there is no pressure on the neutral axis but in this case we are considering the pressure of $p/2$ at the neutral axis to be conservative. This makes the analysis three dimensional. For the purpose of understanding consider σ_L acting in the x- direction and σ_H acting in the y- direction. The internal pressure will act in the z- direction and the shear term will be τ_{xz}. Figure 6 further explains the above discussion.

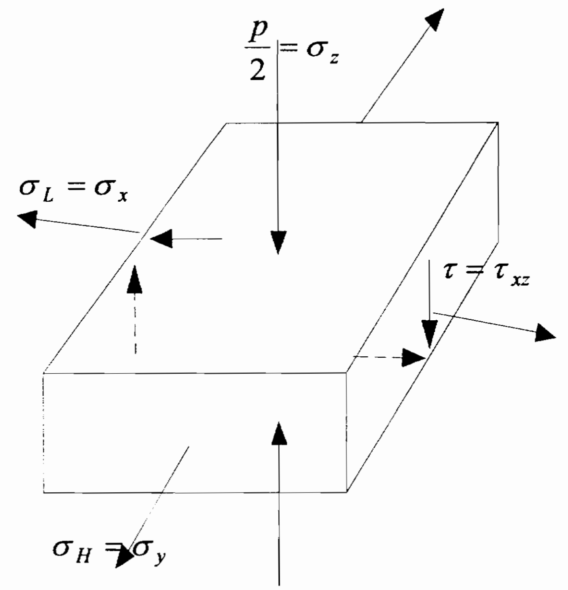

Figure 6. Stresses in the neutral surface of shell loaded with internal pressure.

Here hoop stress is one of the principal stresses as there is no shear associated with it. The other two principal stresses are found from the Mohr's circle for 3-dimensional stress state. Figure 7 shows the Mohr's circle for the above condition.

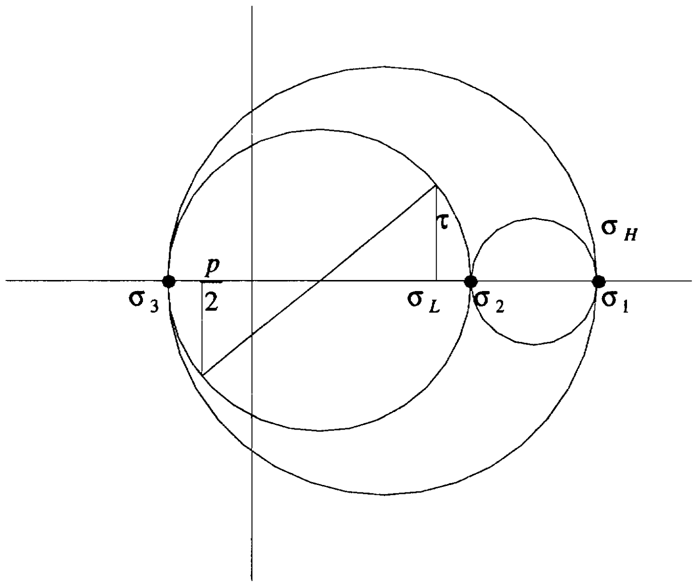

Figure 7. Mohr's circle for neutral surface

The following equation can be obtained from the Mohr's circle for the two remaining principal stresses.

$$\sigma_{2,3} = \frac{2\sigma_L - p}{4} \pm \sqrt{\left(\frac{2\sigma_L + p}{4}\right)^2 + \tau^2} \tag{3.23}$$

Here, $\sigma_1 = \sigma_H$

Thus,

$$\sigma_{MSST} = \sigma_1 - \sigma_3 \qquad \text{where, } \sigma_1 > \sigma_2 > \sigma_3$$

In the above discussion we have not taken into account the shear stress arising in the yz-plane due to the Poisson's effect. If we do consider this, then we can not use the Mohr's circle approach. Instead, one needs to use the following equation given by Shigley and Mitchell [29] to solve for the three principal stresses in 3-dimensional stress state. Here we solve for σ and the three roots of the following cubic equation are the 3 principal stresses.

$$\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma^2 + (\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\sigma$$
$$-(\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2) = 0 \tag{3.24}$$

Ideally it would be desirable to use the cubic equation and to consider the shear stress associated with the Poisson's ratio. But a computer code was being developed to solve this equation and it is easier to solve Eq. (3.23) against the cubic equation given in Eq. (3.24). So Eq. (3.23) has been used in the computer code called BEAM8. When tested for one particular case, as shown in Table 1 on the next page, the principal stresses obtained from Eq. (3.23) with τ_{yz} and Eq. (3.24) without τ_{yz} are nearly same. But the reader is warned to use caution while applying the Eq. (3.23) in the general case. Table 1 shows the comparison of Eq. (3.23) and Eq. (3.24) for one particular case.

σ_H =48.88 kpsi σ_L = -16.84 kpsi p = 0.75 kpsi τ_{xz} = 4.68 kpsi

$\tau_{yz} = 0.3\tau_{xz} = 0.3(4.68) = 1.40$ kpsi

Table 1. Comparison of principal stresses obtained by Eq. (3.23) with τ_{yz} and Eq. (3.24) without τ_{yz}.

	σ_1 kpsi	σ_2 kpsi	σ_3 kpsi	$\sigma_{MSST} = \sigma_1 - \sigma_3$ kpsi
Cubic Eq. (3.24)	48.88	1.88	-18.02	66.90
Mohr's circle Eq. (3.23)	48.88	1.19	-18.11	66.99

ELEMENT C: Element C is on the inside surface. The shell as a whole is expanding due to the internal pressure. The inside surface is thus under tension. The hoop and longitudinal stress will, hence, be tensile. Also there is some bending due to the end plates which causes tension in the inner surface. Thus, the bending stresses arising in the circumferential and longitudinal direction are tensile. A pressure force acts directly on the inside surface in the radial direction. Figure 8 below shows the orientation of these stresses.

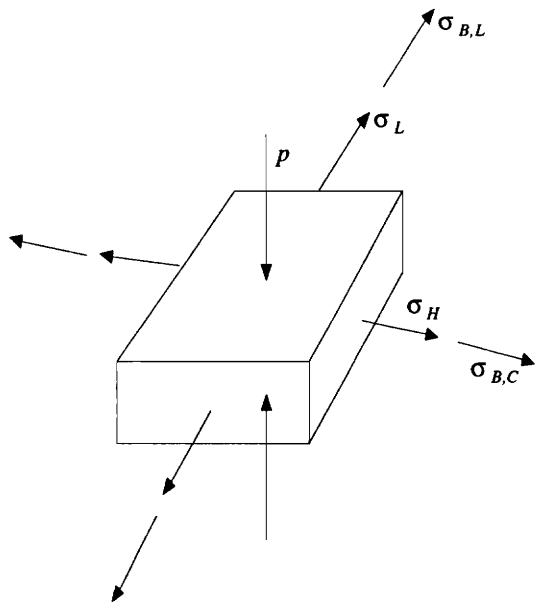

Figure 8. Stresses in the inside surface of shell loaded with internal pressure.

For this element the minimum principal stress $\sigma_3 = -p$

The maximum principal stress

$$\sigma_1 = \text{larger of } (\sigma_L + \sigma_{B,L}) \text{ and } (\sigma_H + \sigma_{B,C}).$$

The Mohr's circle for this condition is as shown in the Figure 9.

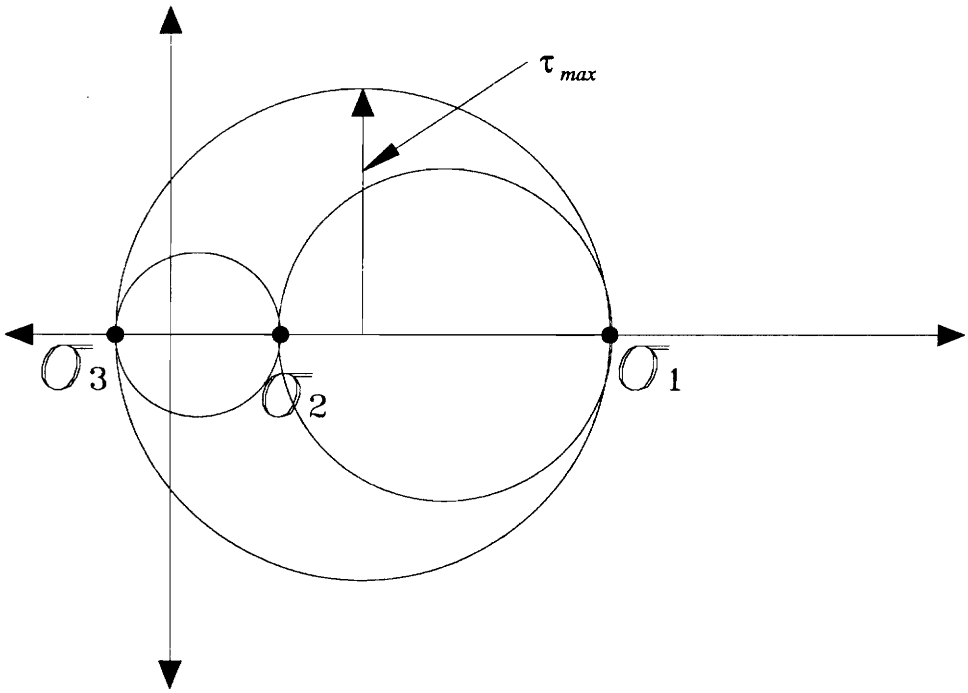

Figure 9. Mohr's circle for inside fiber

Thus $\sigma_{MSST} = 2\tau_{MAX} = \sigma_1 - \sigma_3$

Chapter 4

Transfer Matrix Method: Theory

4.1 Introduction

The fundamental idea behind the transfer matrix method is that a complex structure can be modeled as a combination of simple structures with a set of elastic and dynamic properties. Each of these simple sections can be presented in a matrix form and successive multiplication of matrices gives the final or global matrix which can be used to analyze the complete structure.

4.2 Coordinate System and Sign Convention

It is important to define a coordinate system for the transfer matrix method. The coordinate system followed here is the right-hand Cartesian coordinate system. We shall follow the sign convention followed by Pestel and Leckie [30]. If the structure is cut then the exposed face's outward normal points in a coordinate direction. If the outward normal points in the positive direction of x- axis, it is called a positive face. The outward normal from the other face, exposed by the cut, points in the negative x- axis direction. This is called a negative face. The displacements are positive if they coincide with the positive direction of the coordinate system. The signs of internal forces (shear and moment) are positive if their vectors are directed in the positive coordinate direction and are acting on the positive face or if the vectors are in the negative coordinate direction and are acting on the negative face. That is, the "force" pair sign equals the product of the face sign times the force vector sign. Figure 10 further explains the coordinate system.

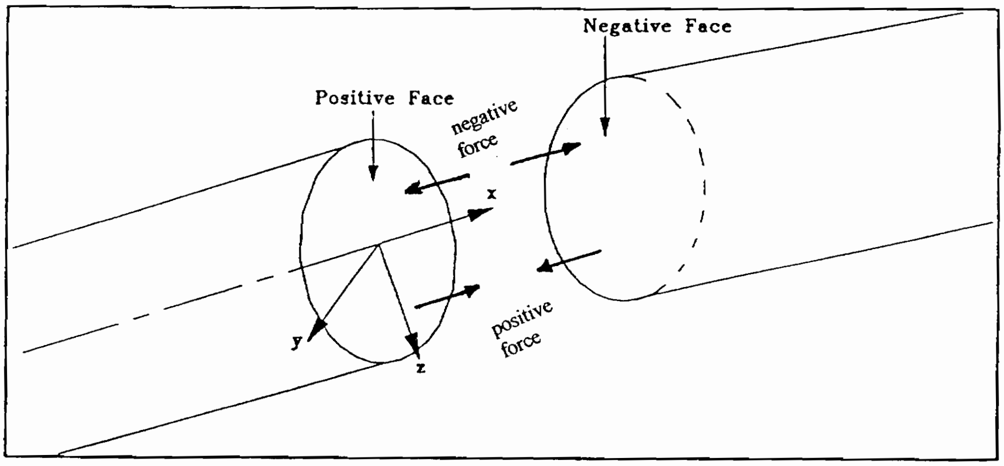

Figure 10. Coordinate system for transfer matrix method

4.3 General procedure and definitions

A complex structure can be divided into simple sections and matrices for these sections define their elastic and/or dynamic properties. A column vector, called a state vector, is made up of the variables that represent the response of the system. Depending upon the number of variables studied the state vector is made up of "deflection" and "internal forces". For example, in the study of transverse beam motion the state vector is made up of the transverse deflection (w), angle (θ), moment (M) and shear force (V). It may also include a variable for torsion and axial loading. A state vector is represented by $\{Z\}$. For a beam deflecting in one plane, the state vector is

$$\{Z\} = \begin{Bmatrix} w \\ \theta \\ M \\ V \\ 1 \end{Bmatrix}$$

The one (1) in the state vector acts as a marker to make provision for external forces in the associated transfer matrix.

A field transfer matrix is a matrix used to express the state vector on the right side of a section that has a non zero length in terms of the state vector on the left side of the same vector. It is expressed matrixwise as $[F]$. A point transfer matrix, $[P]$, is used to model a zero-length element point parameters like external forces, moments, damping, etc.

A complex structure can be made of several sections. A transfer matrix is used to model each of these sections. This transfer matrix may be a field transfer matrix or a point transfer matrix. Consider for example a system of several spring-mass attached end to end as shown in Figure 11. (after Pestel and Leckie [31])

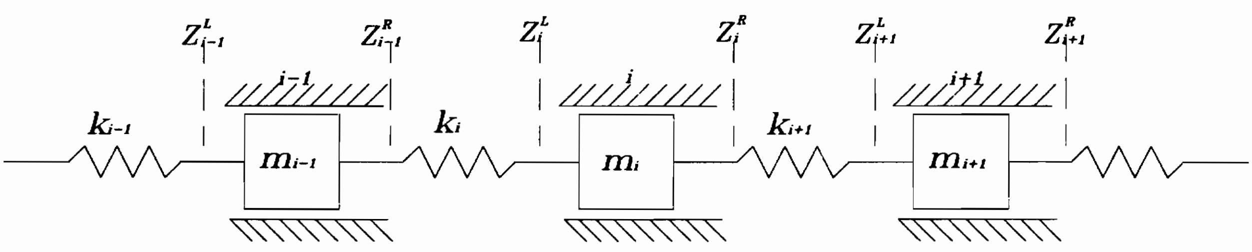

Figure 11. Spring - mass system

To calculate the response of the structure the matrices representing all the sections are assembled according to the left-to-right orientation of the structure. As a consequence, the matrices are assembled in an inverse order chain multiplication.

Let $[P]_i$ represent the point transfer matrix for the i^{th} mass (m_i) and $[F]_i$ represent the field transfer matrix for the i^{th} spring (k_i). Also the state vector just to the left side of a mass "m_i" is denoted as $\{Z\}_i^L$ and the state vector to the right of mass "m_i" is denoted as $\{Z\}_i^R$.

Thus, we can present the above system as follows

$$\{Z\}_{i-1}^{R} = [P]_{i-1}\{Z\}_{i-1}^{L}$$

$$\{Z\}_{i}^{L} = [F]_i\{Z\}_{i-1}^{R} = [F]_i[P]_{i-1}\{Z\}_{i-1}^{L}$$

$$\{Z\}_{i}^{R} = [P]_i\{Z\}_{i}^{L} = [P]_i[F]_i[P]_{i-1}\{Z\}_{i-1}^{L}$$

$$\{Z\}_{i+1}^{L} = [F]_{i+1}\{Z\}_{i}^{R} = [F]_{i+1}[P]_i[F]_i[P]_{i-1}\{Z\}_{i-1}^{L}$$

$$\{Z\}_{i+1}^{R} = [P]_{i+1}\{Z\}_{i+1}^{L} = [P]_{i+1}[F]_{i+1}[P]_i[F]_i[P]_{i-1}\{Z\}_{i-1}^{L}$$

etc.

Thus, a structure of "n" sections can be illustrated in the chain form as follows:

$$\{Z\}_n = [P]_n\,[F]_n\,[P]_{n-1}\,[F]_{n-1}\,.\,.\,.\,.\,.\,[P]_i\,[F]_i\,.\,.\,.\,.\,.\,.\,[P]_1\,[F]_1\,\{Z\}_0 = [U]\,\{Z\}_0 \qquad (4.1)$$

where $[U]$ is the global transfer matrix.

The multiplication of transfer matrices representing all the sections gives the final or global transfer matrix. Boundary conditions are then applied to this global transfer matrix to obtain the initial state vector $\{Z\}_0$. Figure 12 shows the various boundary conditions. After the initial state vector is found, i.e., the response of the structure on the left most end is found, this state vector is multiplied with the matrix for the section just to the right of it to get the response at the end of that element point. Response within a field element can be obtained by evaluation of the transfer matrix at the distance along the element where you desire the response. Successive multiplication to the matrices of the section just to the right allows us to find the response of the complete structure.

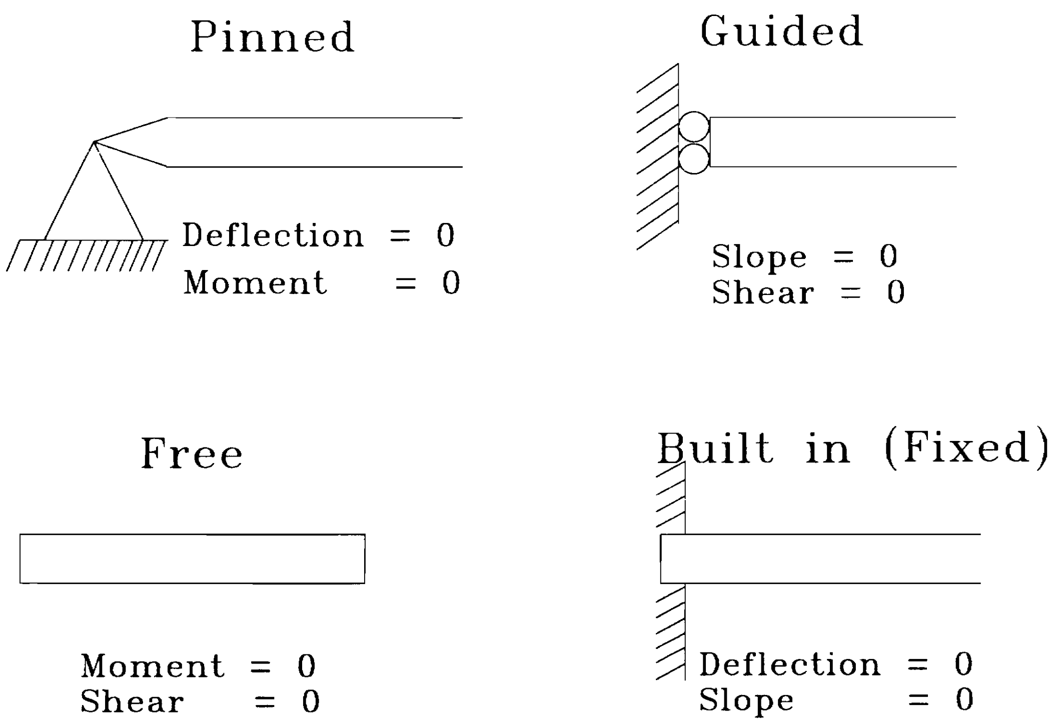

Figure 12. Various Boundary Conditions

Consider, for example, a spring-mass system analyzed by Pestel and Leckie [32]. Figure 13 shows the spring-mass system. State vector on the right of the mass m_l is denoted as $\{Z\}_l^R$ and that on the left is denoted as $\{Z\}_l^L$.

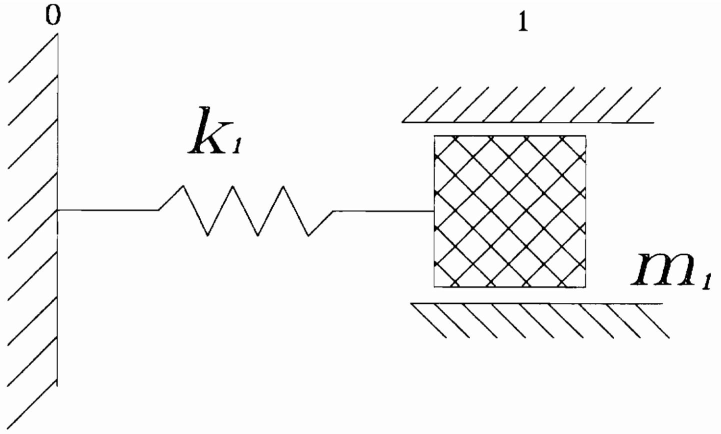

Figure 13. Spring-Mass System

Figure 14 shows the free-body diagram of the spring and Figure 15 shows the free-body diagram of the mass. The spring force on the left and right of spring k_1 is denoted by N_1^L and N_1^R respectively. "x" denotes the linear displacement.

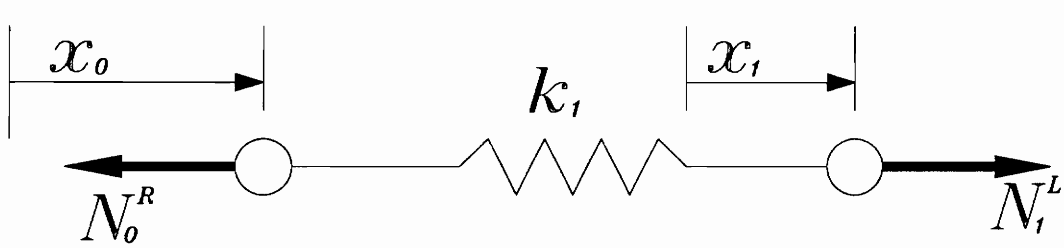

Figure 14. Free-body diagram of spring

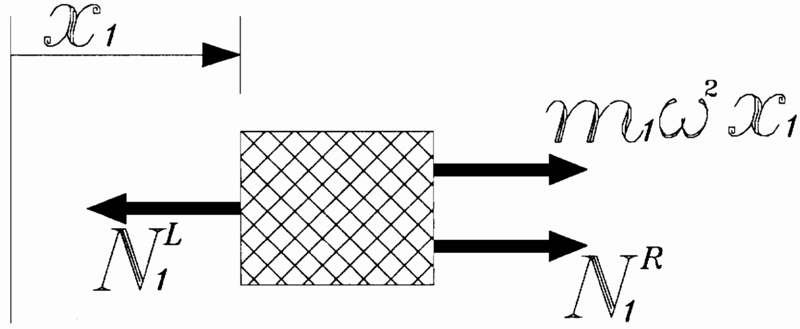

Figure 15. Free-body diagram of mass

Using equilibrium in the spring we get,

$$N_0^R = N_1^L = k_1(x_1 - x_0) \tag{4.2}$$

Therefore, the variables on the right side in the spring can be related to those on the left as

$$x_1 = x_0 + \frac{N_0^R}{k_1} \quad and \quad N_1^L = (0)x_0 + N_0^R \tag{4.3}$$

Rewriting the above Eq. (4.3) in the matrix form yields

$$\begin{Bmatrix} x \\ N \end{Bmatrix}_1^L = \begin{bmatrix} 1 & \frac{1}{k_1} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ N \end{Bmatrix}_0^R \tag{4.4}$$

which can be expressed in the symbolic form as

$$\{Z\}_1^L = [F]_l \ \{Z\}_0^R$$

Now for relating the state vectors on the two sides of the mass we have to consider the two internal forces and the inertia force. We assume the mass is rigid and of zero length. Hence, the deflection on both the sides of the mass is the same. Therefore,

$$x_1^R = x_1^L$$

Equilibrium yields

$$N_1^R = N_1^L - m_1\omega^2 x_1$$

which can be written in matrix form as

$$\begin{Bmatrix} x \\ N \end{Bmatrix}_1^R = \begin{bmatrix} 1 & 0 \\ -m_1\omega^2 & 1 \end{bmatrix} \begin{Bmatrix} x \\ N \end{Bmatrix}_1^L \tag{4.5}$$

Eq. (4.5) expressed in the symbolic form as

$$\{Z\}_1^R = [P]_1 \{Z\}_1^L$$

Substituting Eq. (4.4) in Eq. (4.5) with proper element subscripts, we get

$$\begin{Bmatrix} x \\ N \end{Bmatrix}_1^R = \begin{bmatrix} 1 & \dfrac{1}{k_1} \\ -m_1\omega^2 & 1-\dfrac{m_1\omega^2}{k_1} \end{bmatrix}_1 \begin{Bmatrix} x \\ N \end{Bmatrix}_0^R \tag{4.6}$$

Now applying the fixed boundary condition on the left and free on the right

$$x_0 = 0 \qquad\qquad N_1^R = 0 \tag{4.7}$$

Substituting Eq. (4.7) into Eq. (4.6) yields

$$x_1^R = \frac{N_0^R}{k_1} \tag{4.8}$$

$$0 = \left(1-\frac{m_1\omega^2}{k_1}\right)N_0^R \tag{4.9}$$

Since Eq. (4.8) has two unknowns, it can't be solved so one uses Eq. (4.9) from which one gets the natural frequency ω_n as

$$\omega_n = \sqrt{\frac{k_1}{m_1}} \tag{4.10}$$

Thus, this is the typical methodology for the eigenvalue extraction from a system using the transfer matrix method.

4.4 Equivalence of Shell deflection under symmetric load to a Beam on elastic foundation

It has already been shown in Chapter 2 that thin cylindrical shells can be modeled as beams on elastic foundation. More detailed analysis is given by Pilkey[22] in his "*Manual for the Response of Structural Members*".

The fundamental equation for a Timoshenko beam on elastic foundation as given by Pilkey [33] is as follows

$$
\begin{aligned}
&\frac{\partial w}{\partial x} = -\,\theta + \frac{V}{GA_s} \\
&\frac{\partial \theta}{\partial x} = \frac{M + M_T}{EI} \\
&\frac{\partial M}{\partial x} = V + (k^* - P)\,\theta - d\rho r_y^2 \frac{\partial \theta^2}{\partial t^2} - M_i(x,t) \\
&\frac{\partial V}{\partial x} = kw + \rho \frac{\partial^2 w}{\partial t^2} - F_i(x,t)
\end{aligned}
\tag{4.11}
$$

The fundamental equation of motion in first-order form for the radial, symmetric bending of a cylinder are: (after Pilkey [34])

$$\frac{\partial w}{\partial x} = -\theta + \frac{V}{D_V}$$

$$\frac{\partial \theta}{\partial x} = \frac{M + M_T}{D_x}$$

$$\frac{\partial M}{\partial x} = V - P\theta + \rho r_y^2 \frac{\partial \theta^2}{\partial t^2} - M_i(x,t) \tag{4.12}$$

$$\frac{\partial V}{\partial x} = \frac{Et}{R^2} w + \rho \frac{\partial^2 w}{\partial t^2} - F_i(x,t) + \frac{1}{R}(\nu P + \int_h E\alpha\Delta T dz)$$

Here the following loading variables change the units since all the loading conditions for a cylindrical shell are measured *per unit circumferential length of cylinder.*

V Shear force per unit circumferential length (force / length)

M Bending moment per unit circumferential length (force · length / length)

P Axial force per unit circumferential length (force / length)

By comparing the Eq. (4.11) to Eq. (4.12), one can draw the following comparison between the fundamental equation of beam on elastic foundation and cylinder equation. Table 2 shows the beam variables and its equivalent shell variables.

Table 2. Beam variables and their equivalent shell variables

Beam on elastic foundation equation	***Symmetric bending of cylinder equation***
w	w
θ	θ
M	M
V	V
P	P
M_T	M_T
F_i	$F_i + \frac{1}{R^2}(\nu P + \int_h E\alpha\Delta T dx)$
GA_s	D_V
EI	D_x
k	Et / R^2

4.5 Test of Equivalent Variables

To confirm the above equivalence, a test to compare the unit is performed. Consider for example a transfer matrix for an Euler-Bernoulli beam in bending with constant Young's modulus (E) and the area moment of inertia (I) as shown below. The first row and column represent the displacement terms. The second row and column represent the angle terms and similarly the third represent the shear terms and the fourth row and column represent moment terms.

$$\begin{bmatrix} 1 & L & \dfrac{L^2}{2EI} & \dfrac{L^3}{3!EI} \\ 0 & 1 & \dfrac{L}{EI} & \dfrac{L^2}{2EI} \\ 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.13}$$

To check the units consider the term $\left(\dfrac{L^2}{2EI}\right)$ in the beam form and the term $\left(\dfrac{L^2}{2D_x}\right)$ in the shell form. As seen above, this term lies at the intersection of the deflection row and moment column. Thus, if the units of this term are multiplied by units of moment (force·length) then we should get length, the unit of displacement. That is,

$$\frac{L^2}{2EI} \times M = w$$

Therefore,

$$\frac{\mathrm{in}^2}{\frac{\mathrm{lb}}{\mathrm{in}^2}\mathrm{in}^4} \times \mathrm{lb}\cdot\mathrm{in} = \mathrm{in}\ (\text{length})$$

Now check the units after making the substitution of D_x for EI.

$$D_x = Et^3 / 12\ (1\text{-}\nu^2\ )$$

Therefore the units of D_x will be $\dfrac{\frac{\mathrm{lb}}{\mathrm{in}^2} \times \mathrm{in}^3}{1} = \mathrm{lb}\cdot\mathrm{in}$

Now when making shell calculations the moments and shear are calculated per length of circumference. Hence the unit of moment M is $\dfrac{\mathrm{lb}\cdot\mathrm{in}}{\mathrm{in}} = \mathrm{lb}$

Substituting this in the term $\dfrac{L^2}{2EI} \times M \Rightarrow \dfrac{L^2}{2D_x} \times M$

one finds $\dfrac{\mathrm{in}^2}{\mathrm{lb}\cdot\mathrm{in}} \times \dfrac{\mathrm{lb}\cdot\mathrm{in}}{\mathrm{in}} = \mathrm{in}\ (\text{length})$

Hence $$\frac{L^2}{2D_x} \times M = w$$

Similar test may be done for the term $\frac{L^3}{3!EI}$ in beam form and $\frac{L^3}{3!D_x}$ for the equivalent shell. When the units of this term is multiplied by the units of shear we should get units for displacement.

For the beam:

$$\frac{\text{in}^3}{\frac{\text{lb}}{\text{in}^2}\text{in}^4} \times \text{lb} = \text{in (length)}$$

For the equivalent shell:

$$\frac{\text{in}^3}{\text{lb}\cdot\text{in}} \times \frac{\text{lb}}{\text{in}} = \text{in (length)}$$

Similar demonstrations can be given for other equivalences between the beam and shell transfer matrix elements. Thus, if the above substitution is made then all the transfer matrices used for beams on elastic foundation can be used for analyses of thin cylinders without concern for units problems except the consideration that all moments and shears are entered as per length of circumference.

4.6 Stiffener Rings

Many times stiffener rings are used to provide structural strength to the cylindrical structures. The transfer matrix for a stiffener ring is not present in beams. A transfer matrix for stiffener ring as given by Pilkey [35] is repeated in the following Eq. (4.14). Stiffener rings can be either on the inside or on the outside of the shell. Figure 17 shows the stiffener ring on the outside of the shell.

$$P_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \dfrac{Ec^3}{12R(\eta+\nu)} & 1 & 0 & 0 \\ \dfrac{Ec}{R(\eta+\nu)} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (4.14)$$

where, $$\eta = \frac{R_0^2 + R^2}{R_0^2 - R^2} \qquad (4.14a)$$

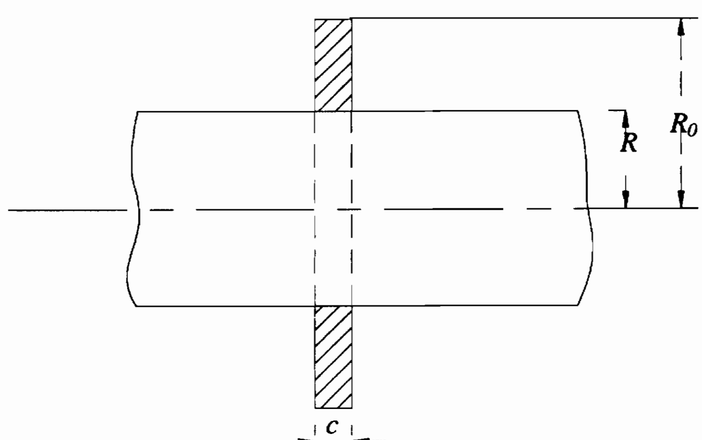

Figure 16. Stiffener ring on outside of cylindrical shell

Chapter 5

Test Examples

5.1 EXAMPLE 1.

Find the response of a thin-cylindrical shell structure with the following dimensions and material properties. The shell is subjected to a line circumferential load of 500 pounds directed inwards inwards at a distance of 10 in. from the left end.

Length	20 in	Inner diameter of shell	19.9 in
Outer diameter of shell	20.1 in	Poisson's ratio	0.3
Density	0.283 lb. / in^3	Modulus of elasticity	$30(10)^6$ psi

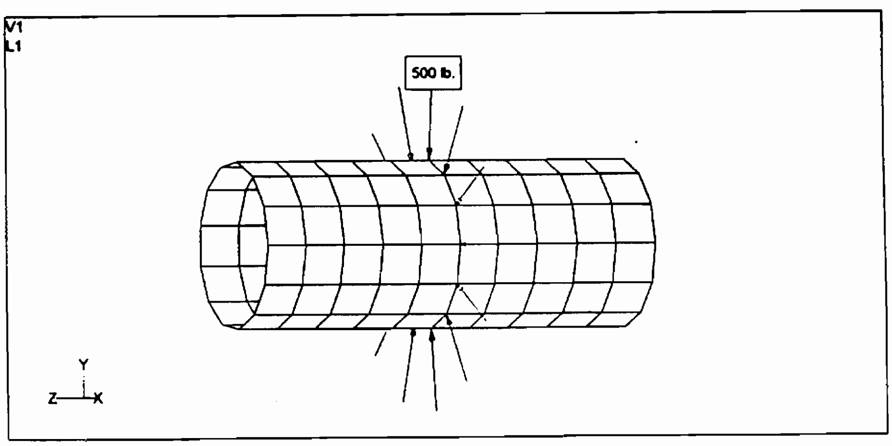

Figure 17 a. Shell subjected to Circumferential line load.

The above example was solved first using BEAM8 and the results were then compared to formulae given by Harvey [36]. Figure 17 shows the equivalent model in BEAM8 for the above example. Figure 18 shows the graphs of displacement, angle, moment and shear induced at different positions along the shell as a result of the circumferential line load.

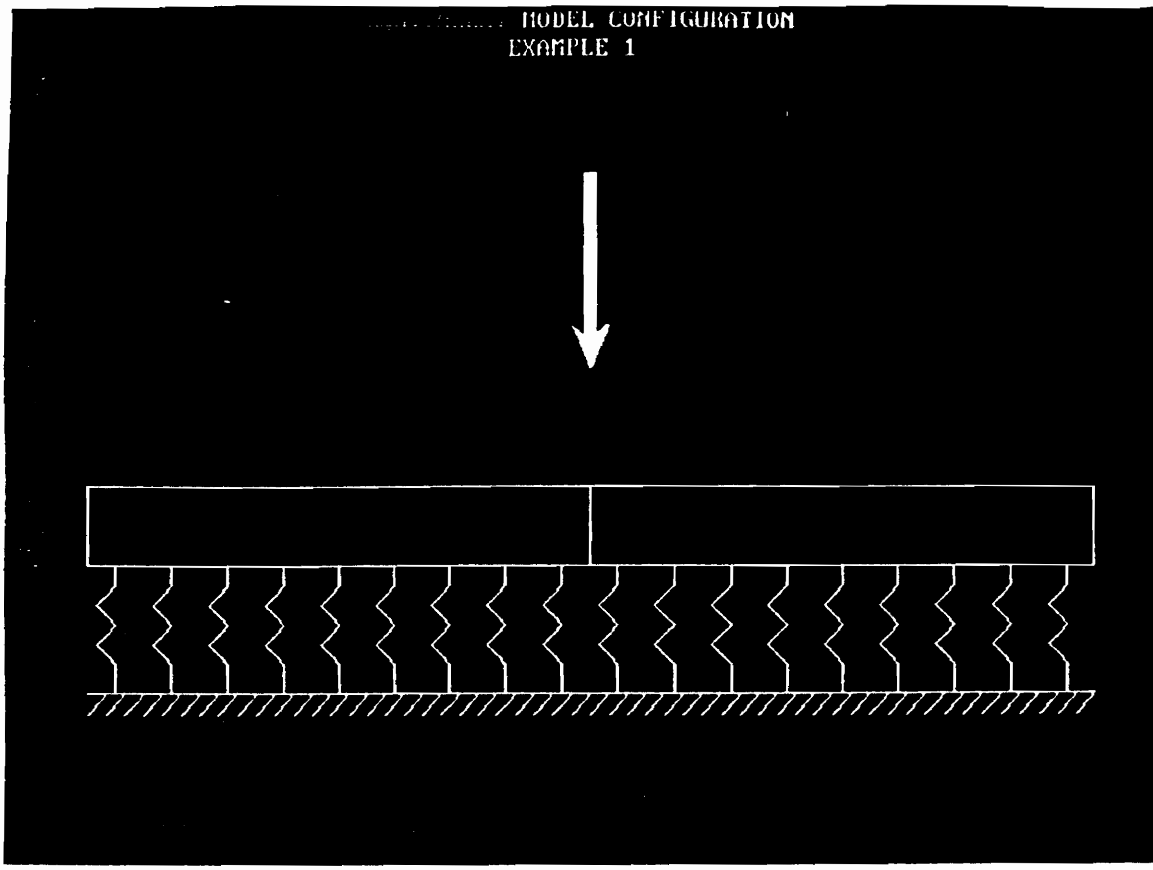

Figure 17 b. Equivalent model for a cylindrical shell with circumferential line load at the center.

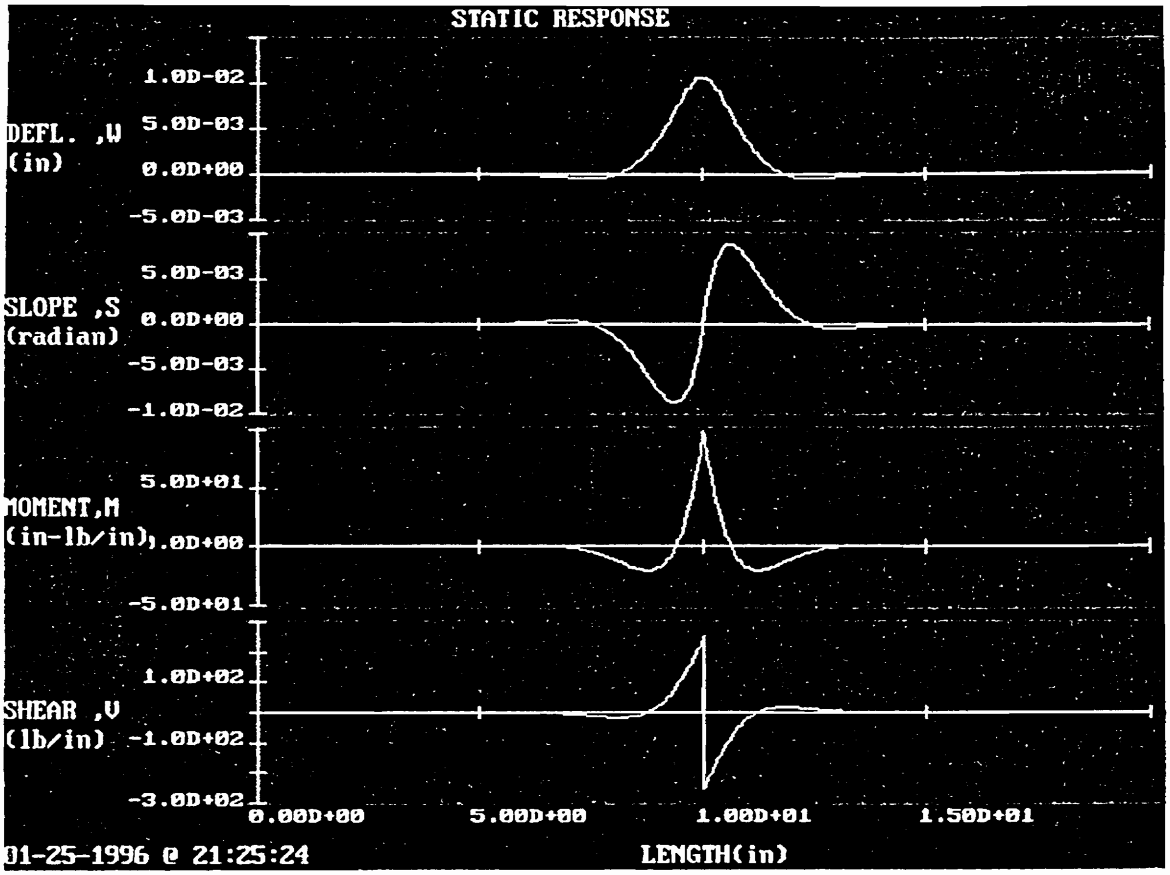

Figure 18. Graphs for displacement, angle, moment and shear for Example 1

The above example is equivalent to a beam on elastic foundation with point load at the center of its length. A study of beams on elastic foundation has been done by Harvey [37] and he has published the following results.

Consider a straight beam resting on a continuous supporting elastic foundation. It is subjected to a concentrated load P at the center of the beam. The beam deflects due to this load producing a continuos distributed reaction force q in the foundation proportional to the deflection, y of the beam at that point. Figure 19 (after Harvey [36]) below shows this.

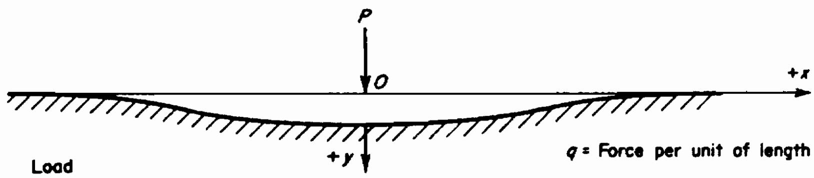

Figure 19. Point loading in a beam on an elastic foundation (after Harvey [36])

For an infinitely long beam with single concentrated load, Harvey has developed formulae to find the distance from the point of loading where the deflection, slope, moment and shear would go to zero. Figure 20 shows the position of zero deflection where

$$\beta = \sqrt[4]{\frac{k}{4EI}}$$

For the case here

$$\beta = \sqrt[4]{\frac{30000}{4 \times 3 \times 10^7 \times 9.1576 \times 10^{-5}}} = 1.285$$

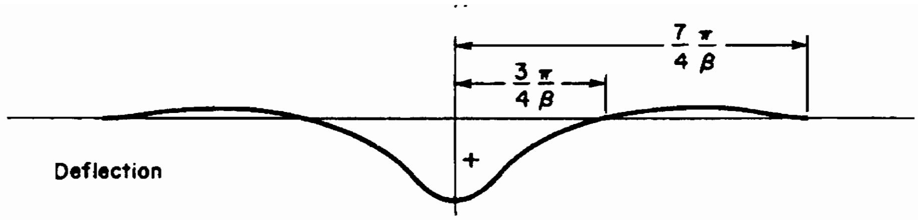

Figure 20. Deflection due to point load on beam on elastic foundation (after Harvey [36])

The position of zero deflection are found as

$$\frac{3\pi}{4\beta} = 1.832 \text{ in} \qquad \text{and} \qquad \frac{7\pi}{4\beta} = 4.276 \text{ in}$$

Therefore, when the point load is at the center (10 in) the zero displacement distance are as follows:

1. 10 - 1.832 = 8.167 (in.)
2. 10 + 1.832 = 11.832 (in.)
3. 10 - 4.276 = 5.724 (in.)
4. 10 + 4.276 = 14.276 (in.)

Each section of the above BEAM8 example is divided into 200 increments. A complete output of the displacement, slope, moment and shear is given in Appendix I. To match the results of BEAM8 exactly to the above results obtained by using Harvey's formulae it would be required to make more than 1000 increments of each section and may run into memory problems. Instead it can be seen in the following tables that the deflection curve changes sign between two points very near to the results found by Harvey. If one linearly interpolates the BEAM8 results to estimate the zero position, one finds excellent results. Table 3 shows comparison of results between BEAM8 and Harvey's formula for deflection.

Table 3. Comparison of results between BEAM8 and Harvey's formulae for the position of zero deflection

Harvey's Formula [36]	**BEAM8**		**Hand Calculations**
Length (inches)	Length (inches)	Deflection (inches)	Linear interpolated position (inches)
$10-\frac{7\pi}{4\beta}=5.724$	5.70 5.75	1.77e-06 -2.23e-06	5.722
$10-\frac{3\pi}{4\beta}=8.167$	8.15 8.20	-3.06e-05 6.36e-05	8.166
$10+\frac{3\pi}{4\beta}=11.833$	11.80 11.85	6.36e-05 -3.06e-05	11.834
$10+\frac{7\pi}{4\beta}=14.276$	14.25 14.30	-2.24e-06 1.77e-06	14.278

Calculations similar to that done for displacement are also done for slope, moment and shear. Figure 21 shows the position of zero slope due to point load on beam on elastic foundation and Table 4 shows comparison of results between BEAM8 and Harvey's formulae for zero slope position. Figure 22 shows the zero moment position due to point load on beam on elastic foundation and Table 5 shows comparison of results between BEAM8 and Harvey's formulae for zero moment position. Figure 23 shows the zero shear position due to point load on beam on elastic foundation and Table 6 shows comparison of results between BEAM8 and Harvey's formulae for zero shear position.

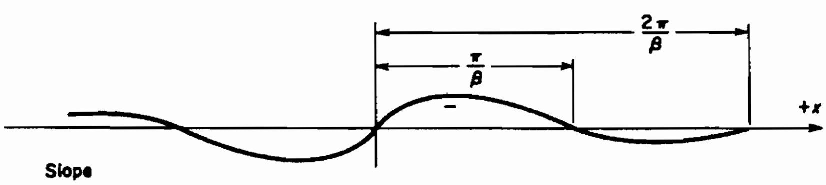

Figure 21. Slope due to point load on beam on elastic foundation (after Harvey [36])

Table 4. Comparison of results between BEAM8 and Harvey's formula for zero slope position

Harvey's Formula [36]	BEAM8		Hand Calculations
Length (inches)	Length (inches)	Slope (radians)	Linear interpolated position (inches)
$10-\frac{2\pi}{\beta}=5.113$	5.10 5.15	-7.74e-07 2.64e-06	5.111
$10-\frac{\pi}{\beta}=7.556$	7.55 7.60	9.03e-06 -7.12e-06	7.556
$10+\frac{\pi}{\beta}=12.444$	12.40 12.45	7.12e-05 -9.03e-06	12.444
$10+\frac{2\pi}{\beta}=14.887$	14.85 14.90	-2.64e-06 7.74e-07	14.888

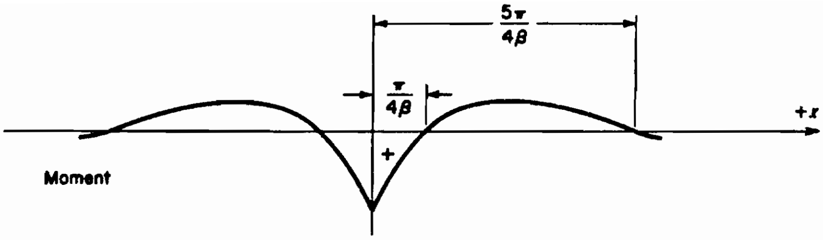

Figure 22. Moment due to point load on beam on elastic foundation (after Harvey [36])

Table 5. Comparison of results between BEAM8 and Harvey's formula for zero moment position

Harvey's Formula [36]	BEAM8		Hand Calculations
Length (inches)	Length (inches)	Moment (lb. / in.)	Linear interpolated position (inches)
$10-\frac{5\pi}{4\beta}=6.946$	6.90 6.95	1.47e-01 -1.77e-02	6.945
$10-\frac{\pi}{4\beta}=9.389$	9.35 9.40	-2.98e+00 9.00e-01	9.388
$10+\frac{\pi}{4\beta}=10.611$	10.60 10.65	9.00e-01 2.98e+00	10.612
$10+\frac{5\pi}{4\beta}=13.054$	13.05 13.10	-1.77e-02 1.47e-01	13.055

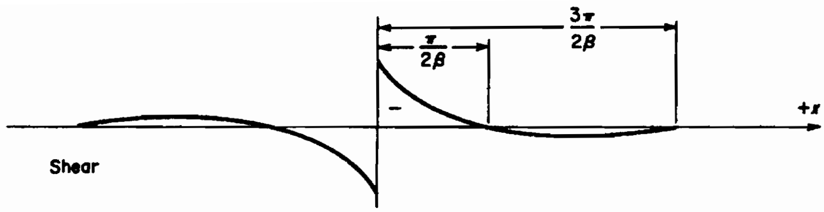

Figure 23. Shear due to point load on beam on elastic foundation (after Harvey [36])

Table 6. Comparison of results between BEAM8 and Harvey's formula for zero shear position

Harvey's Formula [36]	**BEAM8**		**Hand Calculations**
Length (inches)	Length (inches)	Shear (lb.)	Linear interpolated position (inches)
$10 - \frac{3\pi}{2\beta} = 6.335$	6.30 6.35	-4.73e-02 1.27e-02	6.339
$10 - \frac{\pi}{2\beta} = 8.778$	8.75 8.80	-1.80e+00 1.51e+00	8.777
$10 + \frac{\pi}{2\beta} = 11.222$	11.20 11.25	-1.51e+00 1.80e+00	11.223
$10 + \frac{3\pi}{2\beta} = 13.665$	13.65 13.70	4.73e-02 -9.37e-02	13.667

5.2 EXAMPLE 2

Analyze a thin cylindrical shell structure with internal pressure loading of 100 psi. The cylinder is fixed rigidly at both the ends. The physical and material properties are given below. It is required to find the deflection, moment shear and the various stresses arising in the shell on the inside, outside and the middle surface of the thin walled cylinder.

Length	20 in
Poisson's ratio	0.3
Density	0.283 lb. / in^3
Inner diameter of shell	71.75 in
Outer diameter of shell	72.25 in
Modulus of elasticity	$30(10)^6$ psi

The equivalent BEAM8 model for the above example is given in Figure 24. The result output using BEAM8 is given in Table 7. The graphs for displacement, slope, moment and shear is given in Figure 25.

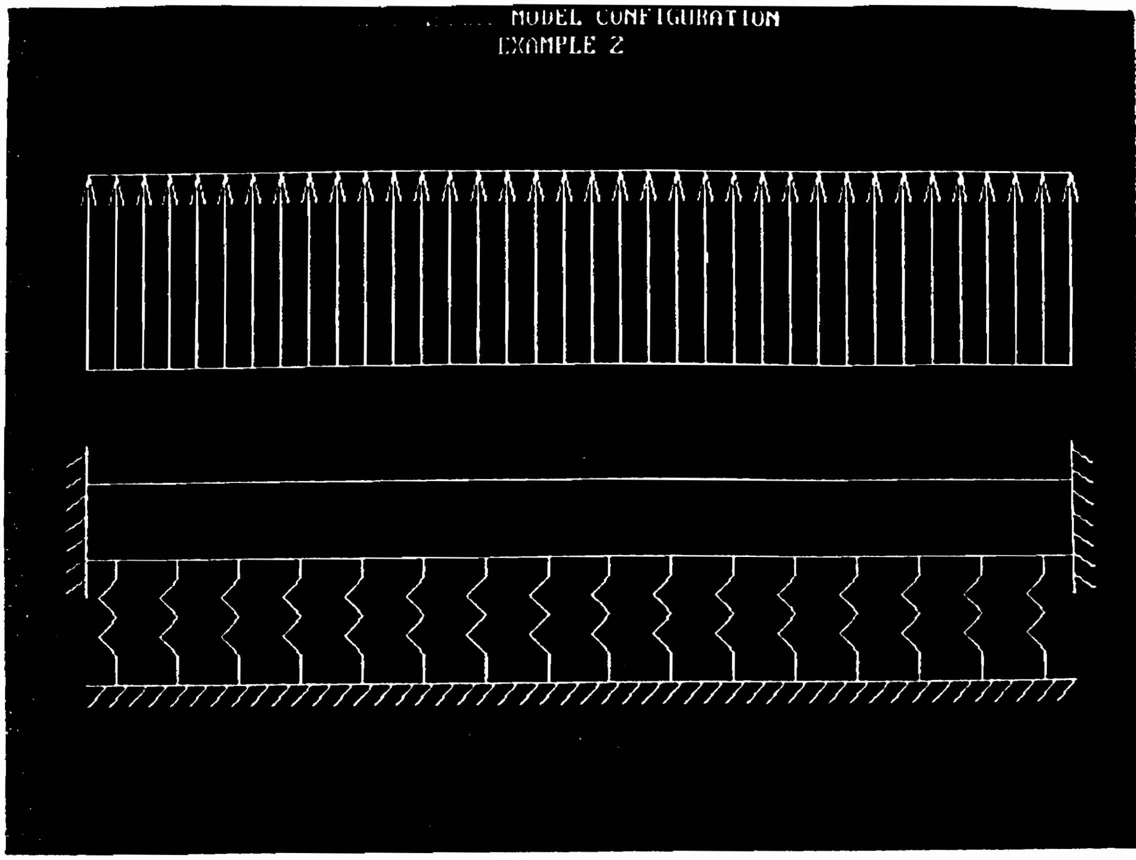

Figure 24. Equivalent model in BEAM8 for Example 2

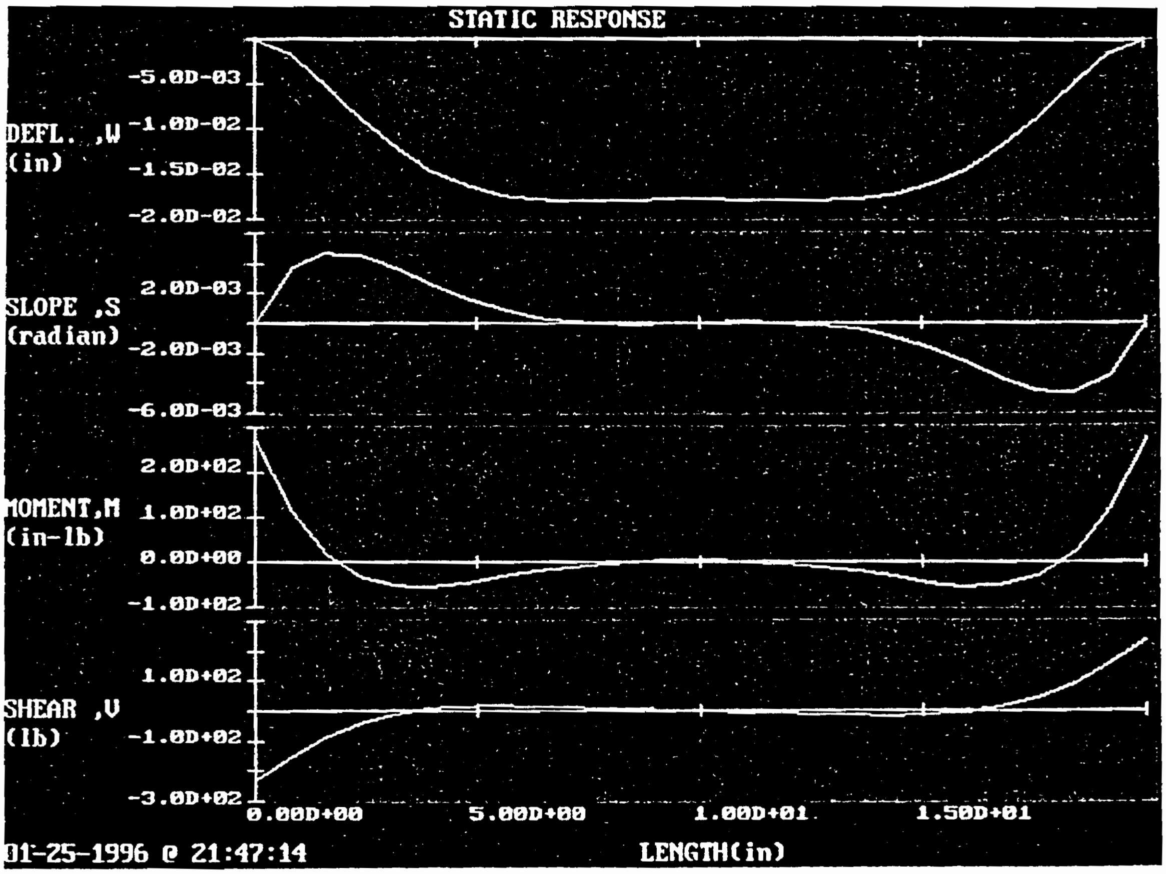

Figure 25. Graphs for Displacement, Slope, Moment and Shear for Example 2

Table 7. Displacement, Slope, Moment and Shear results for Example 2

```
                          EXAMPLE 2
                    *** STATIC RESPONSE ***
```

LENGTH (in)	DEFLECTION (in)	SLOPE (rad)	MOMENT (in-lb/in)	SHEAR (lb/in)
0.000D+00	0.000D+00	0.000D+00	2.722D+02	-2.334D+02
2.000D+00	-6.927D-03	4.744D-03	-1.181D+01	-6.483D+01
4.000D+00	-1.462D-02	2.632D-03	-5.548D+01	6.207D+00
6.000D+00	-1.765D-02	6.219D-04	-2.793D+01	1.557D+01
8.000D+00	-1.803D-02	-5.700D-05	-3.872D+00	7.834D+00
1.000D+01	-1.791D-02	5.551D-17	3.717D+00	-1.819D-12
1.200D+01	-1.803D-02	5.700D-05	-3.872D+00	-7.834D+00
1.400D+01	-1.765D-02	-6.219D-04	-2.793D+01	-1.557D+01
1.600D+01	-1.462D-02	-2.632D-03	-5.548D+01	-6.207D+00
1.800D+01	-6.927D-03	-4.744D-03	-1.181D+01	6.483D+01
2.000D+01	0.000D+00	0.000D+00	2.722D+02	2.334D+02

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The same example was analyzed using BEAMRESPONSE software developed by Pilkey [23]. The comparison of results for displacement, slope, moment and shear are given in Tables 8-12 respectively.

Position along the cylindrical shell (inches)	Displacement using BEAMRESPONSE (inches)	Displacement using BEAM8 (inches)
0.00	0.00e+00	0.00e+00
2.00	-6.93e-03	-6.93e-03
4.00	-1.46e-02	-1.46e-02
6.00	-1.77e-02	-1.77e-02
8.00	-1.80e-02	-1.80e-02
10.00	-1.79e-02	-1.79e-02
12.00	-1.80e-02	-1.80e-02
14.00	-1.77e-02	-1.76e-02
16.00	-1.46e-02	-1.46e-02
18.00	-6.93e-03	-6.93e-03
20.00	-1.14e-13	0.00e+00

Table 8. Comparison of Deflection variable between BEAM8 and BEAMRESPONSE

Table 9. Comparison of Slope variable between BEAM8 and BEAMRESPONSE

Position along the cylindrical shell (inches)	Slope using BEAMRESPONSE (radians)	Slope using BEAM8 (radians)
0.00	0.00e+00	0.00e+00
2.00	4.74e-03	4.74e-03
4.00	2.63e-03	2.63e-03
6.00	6.21e-04	6.22e-04
8.00	-5.72e-05	-5.70e-05
10.00	7.11e-15	5.55e-17
12.00	5.73e-05	5.70e-05
14.00	-6.21e-04	-6.22e-04
16.00	-2.63e-03	-2.63e-03
18.00	-4.74e-03	-4.74e-03
20.00	0.00e+00	0.00e+00

Table 10. Comparison of Moment variable between BEAM8 and BEAMRESPONSE

Position along the cylindrical shell (inches)	Moment using BEAMRESPONSE (lb. / in.)	Moment using BEAM8 (lb. / in.)
0.00	2.72e+02	2.72e+02
2.00	-1.18e+01	-1.18e+01
4.00	-5.55e+01	-5.55e+01
6.00	-2.79e+01	-2.79e+01
8.00	-3.86e+00	-3.87e+00
10.00	3.72e+00	3.72e+00
12.00	-3.86e+00	-3.87e+00
14.00	-2.79e+01	-2.79e+01
16.00	-5.55e+01	-5.55e+01
18.00	-1.18e+01	-1.18e+01
20.00	2.72e+02	2.72e+02

Table 11. Comparison of Shear variable between BEAM8 and BEAMRESPONSE

Position along the cylindrical shell (inches)	Shear using BEAMRESPONSE (lb.)	Shear using BEAM8 (lb.)
0.00	-2.33e+02	-2.33e+02
2.00	-6.48e+01	-6.48+01
4.00	6.22e+00	6.20e+00
6.00	1.56e+01	1.56e+01
8.00	7.83e+00	7.83e+00
10.00	-5.82e-11	-1.82e-12
12.00	-7.83e+00	-7.83e+00
14.00	-1.56e+01	-1.56e+01
16.00	-6.22e+00	-6.20e+00
18.00	6.48e+01	6.48e+01
20.00	2.33e+02	2.33e+02

EXAMPLE 2
*** STATIC RESPONSE ***

LENGTH (in)	DEFLECTION (in)	SLOPE (rad)	MOMENT (in-lb/in)	SHEAR (lb/in)
0.000D+00	0.000D+00	0.000D+00	2.722D+02	-2.334D+02
2.000D+00	-6.927D-03	4.744D-03	-1.181D+01	-6.483D+01
4.000D+00	-1.462D-02	2.632D-03	-5.548D+01	6.207D+00
6.000D+00	-1.765D-02	6.219D-04	-2.793D+01	1.557D+01
8.000D+00	-1.803D-02	-5.700D-05	-3.872D+00	7.834D+00
1.000D+01	-1.791D-02	5.551D-17	3.717D+00	-1.819D-12
1.200D+01	-1.803D-02	5.700D-05	-3.872D+00	-7.834D+00
1.400D+01	-1.765D-02	-6.219D-04	-2.793D+01	-1.557D+01
1.600D+01	-1.462D-02	-2.632D-03	-5.548D+01	-6.207D+00
1.800D+01	-6.927D-03	-4.744D-03	-1.181D+01	6.483D+01
2.000D+01	0.000D+00	0.000D+00	2.722D+02	2.334D+02

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Table 7. Displacement, Slope, Moment and Shear results for Example 2

The same example was analyzed using BEAMRESPONSE software developed by Pilkey [23]. The comparison of results for displacement, slope, moment and shear are given in Tables 8-12 respectively.

Position along the cylindrical shell (inches)	Displacement using BEAMRESPONSE (inches)	Displacement using BEAM8 (inches)
0.00	0.00e+00	0.00e+00
2.00	-6.93e-03	-6.93e-03
4.00	-1.46e-02	-1.46e-02
6.00	-1.77e-02	-1.77e-02
8.00	-1.80e-02	-1.80e-02
10.00	-1.79e-02	-1.79e-02
12.00	-1.80e-02	-1.80e-02
14.00	-1.77e-02	-1.76e-02
16.00	-1.46e-02	-1.46e-02
18.00	-6.93e-03	-6.93e-03
20.00	-1.14e-13	0.00e+00

Table 8. Comparison of Deflection variable between BEAM8 and BEAMRESPONSE

Position along the cylindrical shell (inches)	Slope using BEAMRESPONSE (radians)	Slope using BEAM8 (radians)
0.00	0.00e+00	0.00e+00
2.00	4.74e-03	4.74e-03
4.00	2.63e-03	2.63e-03
6.00	6.21e-04	6.22e-04
8.00	-5.72e-05	-5.70e-05
10.00	7.11e-15	5.55e-17
12.00	5.73e-05	5.70e-05
14.00	-6.21e-04	-6.22e-04
16.00	-2.63e-03	-2.63e-03
18.00	-4.74e-03	-4.74e-03
20.00	0.00e+00	0.00e+00

Table 9. Comparison of Slope variable between BEAM8 and BEAMRESPONSE

Position along the cylindrical shell (inches)	Moment using BEAMRESPONSE (lb. / in.)	Moment using BEAM8 (lb. / in.)
0.00	2.72e+02	2.72e+02
2.00	-1.18e+01	-1.18e+01
4.00	-5.55e+01	-5.55e+01
6.00	-2.79e+01	-2.79e+01
8.00	-3.86e+00	-3.87e+00
10.00	3.72e+00	3.72e+00
12.00	-3.86e+00	-3.87e+00
14.00	-2.79e+01	-2.79e+01
16.00	-5.55e+01	-5.55e+01
18.00	-1.18e+01	-1.18e+01
20.00	2.72e+02	2.72e+02

Table 10. Comparison of Moment variable between BEAM8 and BEAMRESPONSE

Position along the cylindrical shell (inches)	Shear using BEAMRESPONSE (lb.)	Shear using BEAM8 (lb.)
0.00	-2.33e+02	-2.33e+02
2.00	-6.48e+01	-6.48+01
4.00	6.22e+00	6.20e+00
6.00	1.56e+01	1.56e+01
8.00	7.83e+00	7.83e+00
10.00	-5.82e-11	-1.82e-12
12.00	-7.83e+00	-7.83e+00
14.00	-1.56e+01	-1.56e+01
16.00	-6.22e+00	-6.20e+00
18.00	6.48e+01	6.48e+01
20.00	2.33e+02	2.33e+02

Table 11. Comparison of Shear variable between BEAM8 and BEAMRESPONSE

The longitudinal and hoop stress are calculated using the formulae given below and discussed in Chapter 3.

Longitudinal stress: $$\sigma_L = \frac{pd_i^2}{(d_0^2 - d_i^2)}$$

Therefore, $$\sigma_L = \frac{100 \times (71.75)^2}{((72.25)^2 - (71.75)^2)} = 7150\,\text{psi}$$

The longitudinal stress calculated by BEAM8 is $\sigma_L = 7.150\,\text{kpsi} = 7150$ psi.

Hoop Stress: $$\sigma_H = \frac{Ew}{R}$$

The hoop stress depends on the displacement (w) at the location where the stress is being calculated. The hoop stress at discrete locations along the shell is calculated and compared to the results obtained by BEAM8 in Table 12. A complete output of BEAM8 for hoop stress is given in Appendix III. A complete output using BEAM8 of bending longitudinal, bending circumferential, normal stress calculated using maximum shear stress theory at inside, neutral and outside surface is given in Appendix III.

Table 12. Comparison of Hoop Stress between BEAM8 and theoretical results

Position along the cylindrical shell (inches)	Hoop stress using Theory (Chapter 3) (psi)	Hoop Stress using BEAM8 (psi)
0.00	0	0
2.00	-5783	-5813
4.00	-12250	-12270
6.00	-14750	-14810
8.00	-15083	-15130
10.00	-15000	-15030
12.00	-15083	-15130
14.00	-14750	-14810
16.00	-12250	-12270
18.00	-5783	-5813
20.00	0	0

5.3 EXAMPLE 3

Test the validity of results of the stiffener ring capability of BEAM8. The following dimensions and properties of cylindrical shell and stiffener ring is given. The cylinder is subjected to 100 psi of internal pressure.

CYLINDRICAL SHELL	
Length of cylindrical shell	20.00 in
Inner diameter of shell	71.75 in
Outer diameter of shell	72.25 in
Poisson's ratio	0.30
Density of shell material	0.283 lb./in^3
Modulus of Elasticity of shell	$30(10)^6$ psi
STIFFENER RING	
Inner diameter of stiffener ring	0.00 in
Outer diameter of stiffener ring	71.75 in
Width of stiffener ring	0.25 in
Poisson's ratio	0.30
Modulus of elasticity	$30(10)^6$ psi (case 2)
	$30(10)^7$ psi (case 3)
	$30(10)^8$ psi (case 4)
	$30(10)^{14}$ psi (case 5)

The idea behind this example is to first solve the example with fixed-fixed boundary condition. Then make the boundary condition free-free and place a stiffener ring of width equal to the thickness of the shell at each end of the cylindrical shell. First solve the example with the Modulus of elasticity of stiffener ring as $30(10)^6$ psi. Then increase the stiffness of the stiffener ring until it approaches the previous example with its fixed-fixed boundary conditions. As one increases the stiffness of the ring, the displacement, slope, moment and shear results should approach the results obtained by the fixed-fixed boundary condition.

Case 1. Fixed-Fixed boundary condition.

The results of the above case in the form of tables for displacement, slope, moment and shear are given in Appendix II.

Case 2. Free-Free boundary condition; Stiffener rings with stiffness of $30(10)^6$ psi

The results of the above case in the form of tables for displacement, slope, moment and shear are given in Appendix II.

Case 3. Free-Free boundary condition; Stiffener rings with stiffness of $30(10)^7$ psi

The results of the above case in the form of tables for displacement, slope, moment and shear are given in Appendix II.

Case 4. Free-Free boundary condition; Stiffener rings with stiffness of $30(10)^8$ psi

The results of the above case in the form of tables for displacement, slope, moment and shear are given in Appendix II.

Case 5. Free-Free boundary condition; Stiffener rings with stiffness of $30(10)^{14}$ psi

The results of the above case in the form of tables for displacement, slope, moment and shear are given in Appendix II.

Table 13 shows the comparison of displacement variable for various cases. Figure 26 shows the comparison of displacement variable for various cases. Table 14 shows the comparison of slope variable for various cases. Figure 27 shows the comparison of slope variable for various cases. Table 15 shows the comparison of moment variable for various cases. Figure 28 shows the comparison of moment variable for various cases. Table 16 shows the comparison of shear variable for various cases. Figure 29 shows the comparison of shear variable for various cases.

It can be seen from the various tables (Table 13-16) and graphs (Figures 26-29) that as we increase the stiffness of the stiffener ring, the results approach that of the fixed - fixed condition, thus verify the capability of the reinforcement ting to approach a known published solution.

Table 13. Comparison of Displacement variable for various cases of Example 3.

Position along the shell	Fixed-Fixed Boundary Condition psi	Stiffener Ring and Free-Free boundary condition Young's Modulus $30(10)^{14}$ psi	Young's Modulus $30(10)^{8}$ psi	Young's Modulus $30(10)^{7}$ psi	Young's Modulus $30(10)^{6}$ psi
	case 1	**case 5**	**case 4**	**case 3**	**case 2**
Length (in)	Displacement (in)				
0.00E+00	0.00E+00	1.12E-11	9.76E-06	6.89E-05	5.94E-04
1.00E+00	-2.36E-03	-2.36E-03	-3.53E-03	-5.93E-03	-6.55E-03
2.00E+00	-6.93E-03	-6.93E-03	-8.32E-03	-1.12E-02	-1.21E-02
3.00E+00	-1.13E-02	-1.13E-02	-1.25E-02	-1.49E-02	-1.57E-02
4.00E+00	-1.46E-02	-1.46E-02	-1.54E-02	-1.70E-02	-1.76E-02
5.00E+00	-1.66E-02	-1.66E-02	-1.71E-02	-1.80E-02	-1.83E-02
6.00E+00	-1.77E-02	-1.77E-02	-1.78E-02	-1.82E-02	-1.84E-02
7.00E+00	-1.80E-02	-1.80E-02	-1.80E-02	-1.81E-02	-1.81E-02
8.00E+00	-1.80E-02	-1.80E-02	-1.80E-02	-1.78E-02	-1.78E-02
9.00E+00	-1.80E-02	-1.80E-02	-1.79E-02	-1.77E-02	-1.76E-02
1.00E+01	-1.79E-02	-1.79E-02	-1.78E-02	-1.76E-02	-1.75E-02
1.10E+01	-1.80E-02	-1.80E-02	-1.79E-02	-1.77E-02	-1.76E-02
1.20E+01	-1.80E-02	-1.80E-02	-1.80E-02	-1.78E-02	-1.78E-02
1.30E+01	-1.80E-02	-1.80E-02	-1.80E-02	-1.81E-02	-1.81E-02
1.40E+01	-1.77E-02	-1.77E-02	-1.78E-02	-1.82E-02	-1.84E-02
1.50E+01	-1.66E-02	-1.66E-02	-1.71E-02	-1.80E-02	-1.83E-02
1.60E+01	-1.46E-02	-1.46E-02	-1.54E-02	-1.70E-02	-1.76E-02
1.70E+01	-1.13E-02	-1.13E-02	-1.25E-02	-1.49E-02	-1.57E-02
1.80E+01	-6.93E-03	-6.93E-03	-8.32E-03	-1.12E-02	-1.21E-02
1.90E+01	-2.36E-03	-2.36E-03	-3.53E-03	-5.93E-03	-6.55E-03
2.00E+01	-7.11E-15	1.12E-11	9.76E-06	6.89E-05	5.94E-04

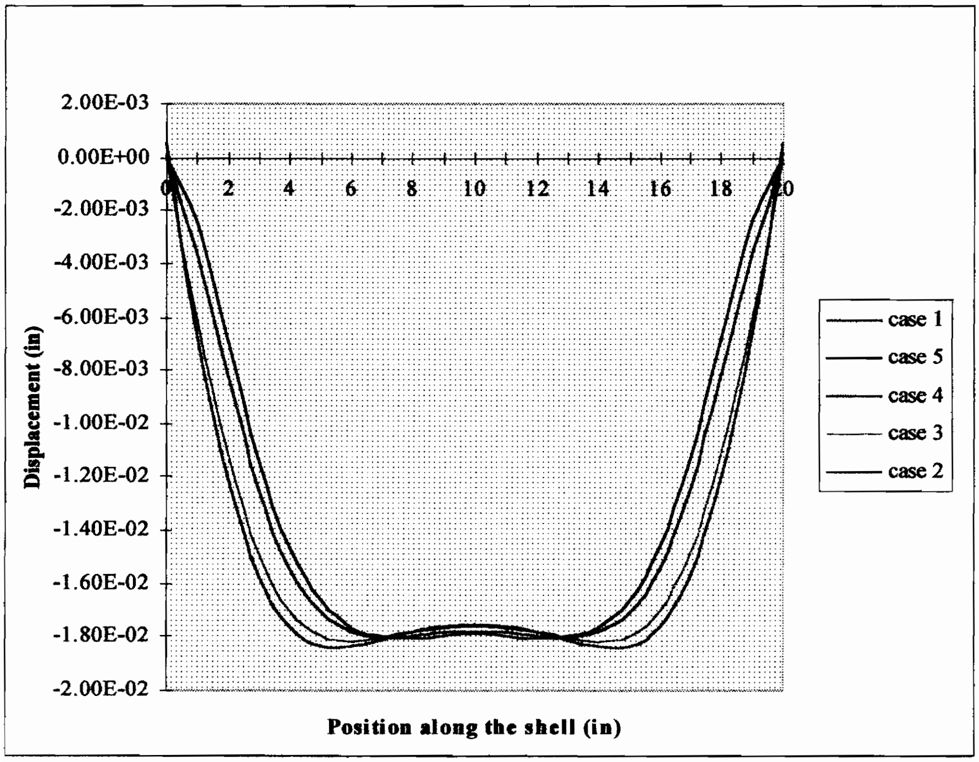

Figure 26. Graph showing comparison of Displacement variable for various cases of Example 3.

Table 14. Comparison of Slope variable for various cases of Example 3.

Position along the shell	Fixed-Fixed Boundary condition case 1	Stiffener Ring and Free-Free boundary condition Young's Modulus $30(10)^{14}$ case 5	Young's Modulus $30(10)^{8}$ case 4	Young's Modulus $30(10)^{7}$ case 3	Young's Modulus $30(10)^{6}$ case 2
	case 1	**case 5**	**case 4**	**case 3**	**case 2**
Length (in)	Slope (radians)				
0.00E+00	0.00E+00	2.50E-09	1.87E-03	5.74E-03	7.44E-03
1.00E+00	4.01E-03	4.01E-03	4.61E-03	5.87E-03	6.54E-03
2.00E+00	4.74E-03	4.74E-03	4.67E-03	4.52E-03	4.59E-03
3.00E+00	3.92E-03	3.92E-03	3.57E-03	2.86E-03	2.66E-03
4.00E+00	2.63E-03	2.63E-03	2.25E-03	1.47E-03	1.20E-03
5.00E+00	1.46E-03	1.46E-03	1.15E-03	5.26E-04	2.92E-04
6.00E+00	6.22E-04	6.22E-04	4.19E-04	9.78E-07	-1.65E-04
7.00E+00	1.41E-04	1.41E-04	2.60E-05	-2.12E-04	-3.13E-04
8.00E+00	-5.70E-05	-5.70E-05	-1.13E-04	-2.28E-04	-2.80E-04
9.00E+00	-7.23E-05	-7.23E-05	-9.31E-05	-1.36E-04	-1.57E-04
1.00E+01	5.55E-17	-8.33E-17	2.78E-17	-2.78E-17	-2.78E-17
1.10E+01	7.23E-05	7.23E-05	9.31E-05	1.36E-04	1.57E-04
1.20E+01	5.70E-05	5.70E-05	1.13E-04	2.28E-04	2.80E-04
1.30E+01	-1.41E-04	-1.41E-04	-2.60E-05	2.12E-04	3.13E-04
1.40E+01	-6.22E-04	-6.22E-04	-4.19E-04	-9.78E-07	1.65E-04
1.50E+01	-1.46E-03	-1.46E-03	-1.15E-03	-5.26E-04	-2.92E-04
1.60E+01	-2.63E-03	-2.63E-03	-2.25E-03	-1.47E-03	-1.20E-03
1.70E+01	-3.92E-03	-3.92E-03	-3.57E-03	-2.86E-03	-2.66E-03
1.80E+01	-4.74E-03	-4.74E-03	-4.67E-03	-4.52E-03	-4.59E-03
1.90E+01	-4.01E-03	-4.01E-03	-4.61E-03	-5.87E-03	-6.54E-03
2.00E+01	0.00E+00	-2.50E-09	-1.87E-03	-5.74E-03	-7.44E-03

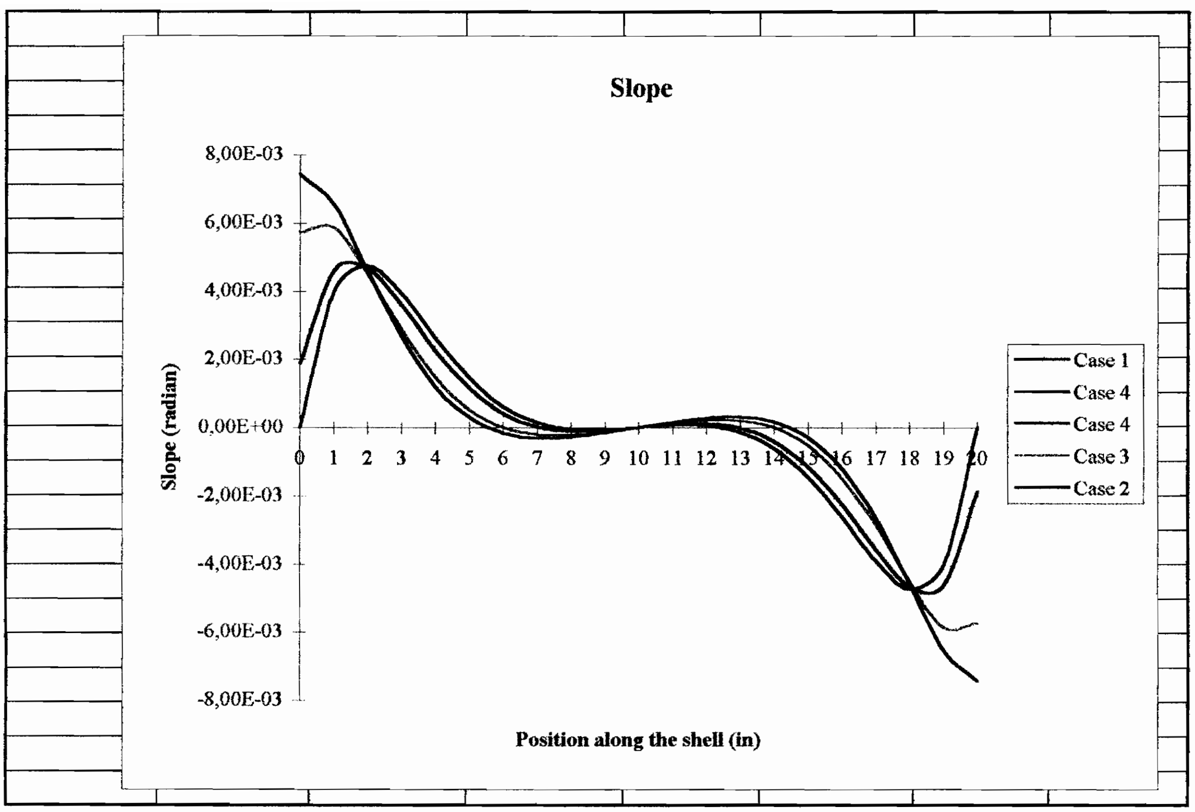

Figure 27. Graph showing comparison of Slope variable for various cases of Example 3.

Table 15. Comparison of Moment variable for various cases of Example 3.

Position along the shell	Fixed-Fixed Boundary condition case 1	Stiffener Ring and Free-Free boundary condition Young's Modulus $30(10)^{14}$ case 5	Young's Modulus $30(10)^{8}$ case 4	Young's Modulus $30(10)^{7}$ case 3	Young's Modulus $30(10)^{6}$ case 2
	case 1	**case 5**	**case 4**	**case 3**	**case 2**
Length (in)	Moment (lb.in / in)				
0.00E+00	2.72E+02	2.72E+02	2.04E+02	6.24E+01	8.10E+00
1.00E+00	8.75E+01	8.75E+01	4.68E+01	-3.71E+01	-7.15E+01
2.00E+00	-1.18E+01	-1.18E+01	-3.09E+01	-7.05E+01	-8.83E+01
3.00E+00	-5.11E+01	-5.11E+01	-5.65E+01	-6.78E+01	-7.43E+01
4.00E+00	-5.55E+01	-5.55E+01	-5.38E+01	-5.05E+01	-5.06E+01
5.00E+00	-4.38E+01	-4.38E+01	-3.96E+01	-3.09E+01	-2.84E+01
6.00E+00	-2.79E+01	-2.79E+01	-2.37E+01	-1.50E+01	-1.20E+01
7.00E+00	-1.39E+01	-1.39E+01	-1.07E+01	-4.15E+00	-1.67E+00
8.00E+00	-3.87E+00	-3.87E+00	-1.90E+00	2.17E+00	3.85E+00
9.00E+00	1.87E+00	1.87E+00	2.97E+00	5.25E+00	6.30E+00
1.00E+01	3.72E+00	3.72E+00	4.51E+00	6.14E+00	6.97E+00
1.10E+01	1.87E+00	1.87E+00	2.97E+00	5.25E+00	6.30E+00
1.20E+01	-3.87E+00	-3.87E+00	-1.90E+00	2.17E+00	3.85E+00
1.30E+01	-1.39E+01	-1.39E+01	-1.07E+01	-4.15E+00	-1.67E+00
1.40E+01	-2.79E+01	-2.79E+01	-2.37E+01	-1.50E+01	-1.20E+01
1.50E+01	-4.38E+01	-4.38E+01	-3.96E+01	-3.09E+01	-2.84E+01
1.60E+01	-5.55E+01	-5.55E+01	-5.38E+01	-5.05E+01	-5.06E+01
1.70E+01	-5.11E+01	-5.11E+01	-5.65E+01	-6.78E+01	-7.43E+01
1.80E+01	-1.18E+01	-1.18E+01	-3.09E+01	-7.05E+01	-8.83E+01
1.90E+01	8.75E+01	8.75E+01	4.68E+01	-3.71E+01	-7.15E+01
2.00E+01	2.72E+02	2.72E+02	2.04E+02	6.24E+01	8.10E+00

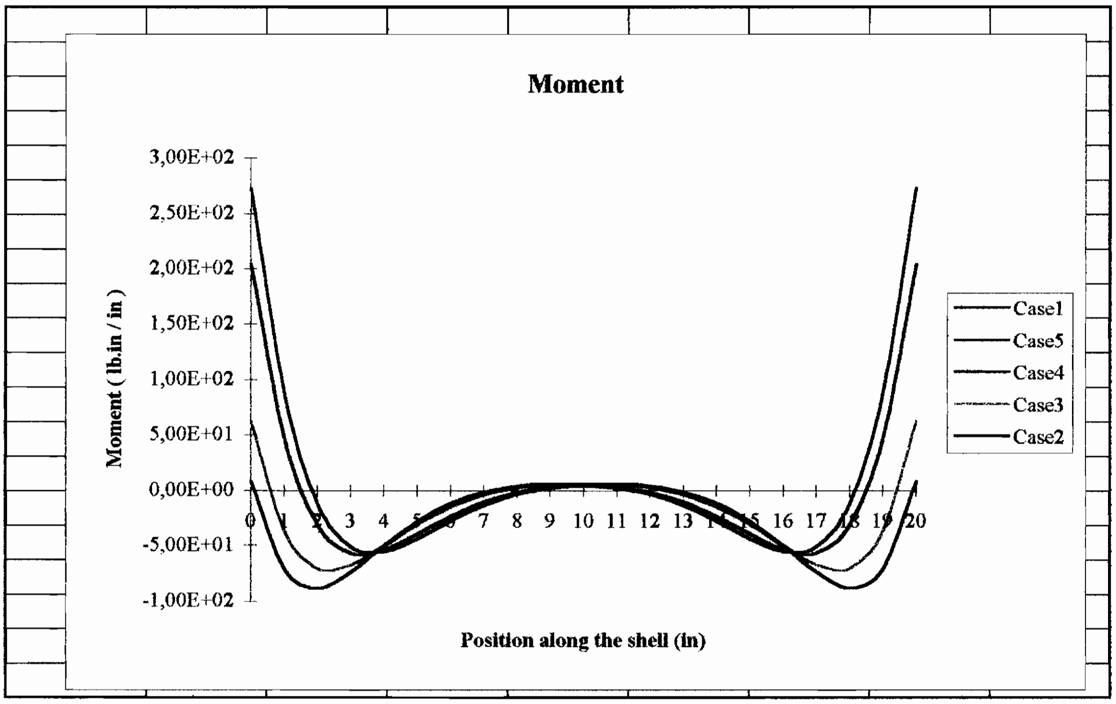

Figure 28. Graph showing comparison of Moment variable for various cases of Example 3.

Table 16. Comparison of Shear variable for various cases of Example 3.

Position along the shell	Fixed-Fixed Boundary condition	Stiffener Ring and Free-Free boundary condition			
		Young's Modulus $30(10)^{14}$ psi	Young's Modulus $30(10)^{8}$ psi	Young's Modulus $30(10)^{7}$ psi	Young's Modulus $30(10)^{6}$ psi
	case 1	**case 5**	**case 4**	**case 3**	**case 2**
Length (in)	Shear (lb./in)				
0.00E+00	-2.33E+02	-2.33E+02	-2.04E+02	-1.44E+02	-1.24E+02
1.00E+00	-1.38E+02	-1.38E+02	-1.13E+02	-6.09E+01	-4.19E+01
2.00E+00	-6.48E+01	-6.48E+01	-4.72E+01	-1.10E+01	3.05E+00
3.00E+00	-1.81E+01	-1.81E+01	-8.00E+00	1.28E+01	2.14E+01
4.00E+00	6.21E+00	6.21E+00	1.07E+01	1.99E+01	2.41E+01
5.00E+00	1.52E+01	1.52E+01	1.62E+01	1.83E+01	1.96E+01
6.00E+00	1.56E+01	1.56E+01	1.49E+01	1.34E+01	1.32E+01
7.00E+00	1.22E+01	1.22E+01	1.09E+01	8.38E+00	7.61E+00
8.00E+00	7.83E+00	7.83E+00	6.74E+00	4.49E+00	3.72E+00
9.00E+00	3.73E+00	3.73E+00	3.12E+00	1.86E+00	1.41E+00
1.00E+01	-1.82E-12	0.00E+00	-1.82E-12	0.00E+00	0.00E+00
1.10E+01	-3.73E+00	-3.73E+00	-3.12E+00	-1.86E+00	-1.41E+00
1.20E+01	-7.83E+00	-7.83E+00	-6.74E+00	-4.49E+00	-3.72E+00
1.30E+01	-1.22E+01	-1.22E+01	-1.09E+01	-8.38E+00	-7.61E+00
1.40E+01	-1.56E+01	-1.56E+01	-1.49E+01	-1.34E+01	-1.32E+01
1.50E+01	-1.52E+01	-1.52E+01	-1.62E+01	-1.83E+01	-1.96E+01
1.60E+01	-6.21E+00	-6.21E+00	-1.07E+01	-1.99E+01	-2.41E+01
1.70E+01	1.81E+01	1.81E+01	8.00E+00	-1.28E+01	-2.14E+01
1.80E+01	6.48E+01	6.48E+01	4.72E+01	1.10E+01	-3.05E+00
1.90E+01	1.38E+02	1.38E+02	1.13E+02	6.09E+01	4.19E+01
2.00E+01	2.33E+02	2.33E+02	2.04E+02	1.44E+02	1.24E+02

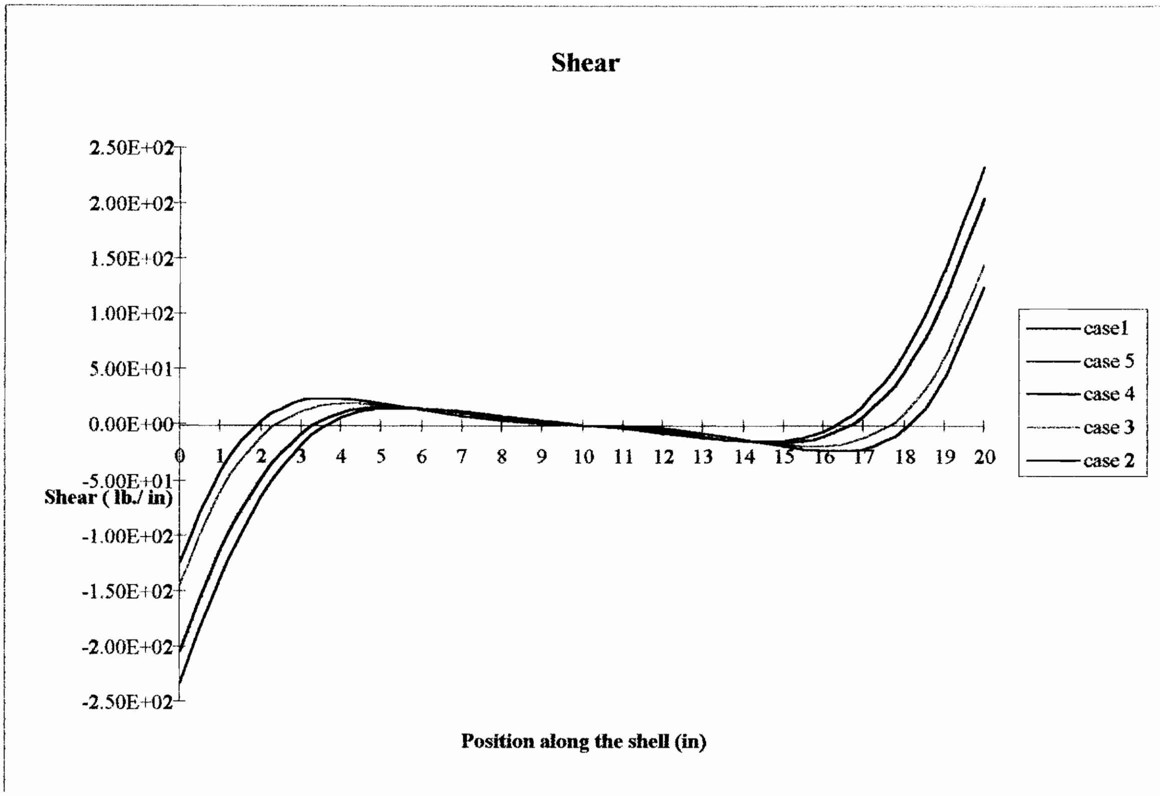

Figure 29. Graph showing comparison of Shear variable for various cases of Example 3.

Chapter 6

Conclusion and Recommendations

6.1 Conclusion

The purpose of this research was to analyze thin cylindrical shells using the transfer matrix method and generate a computer code to do the same. The literature review, verification and application of several concepts, computer codes and detailed analyses of stresses in the outer, neutral and inner surfaces are all vital to the completion of this research. A summary of major accomplishments of this research is as follows:

- Literature review including general shell theory, stresses in thin cylindrical shell, the transfer matrix method, computer codes using transfer matrix method and extensive catalogue of transfer matrices has been done.
- The concept of analyzing a thin cylindrical shell as a beam on an elastic foundation has been thoroughly explored and has been successfully applied to the research.
- In-depth study of various stresses arising in shell structures due to different loading conditions has been carried out. The stresses due to bending (bending longitudinal and bending circumferential), longitudinal, hoop and principal stress using maximum shear stress theory have been studied.
- A computer program (BEAM8) was developed to allow the user to easily enter and analyze the model. The results of the analyses are presented in a table as well as graphical mode for easy interpretation. A major improvement was made in the user

interface of BEAM8 to make it more logical and intuitive to use. However, it should be noted that at times the program has expected numerical problems. This happens when the stiffness of the elastic support is very large compared to the bending stiffness of the system. In this case the transfer matrix method breaks down. This is a well documented (Pestel and Leckie [39]) problem with transfer matrix and is not a bug within BEAM8.

- The computer program (BEAM8) was enhanced to handle stiffeners in the form of inside or outside circular stiffener rings.

6.2 Recommendations

The following recommendations are made to enhance the computer program and augment the analyses done in this research.

- Currently, loading in the form of internal or external pressure and circumferential line load is acceptable. But the program can be enhanced to include thermal loading and be capable of calculating thermal stresses. The thermal loading can be in the form of temperature difference in the outside and inside surface.
- Currently circular stiffener rings are allowed to provide in-span support to the cylindrical shell. Several other types of stiffeners should be included to account for the different practical applications. A very detailed study for various stiffener rings has been done by Olk [38] [39]. He has also given several transfer matrices for different types of shell stiffeners.

- Future work could include analyses of shells with different layers. Each of these layer could be made of a different material. Further discussion on this subject and transfer matrices capable of handling different layers has been given by Pilkey and Chang [24]
- It is also possible to analyze shells with variable thickness. Discussion on this subject and transfer matrix for analyzing shell with variable thickness is also given by Pilkey and Chang [24].

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APPENDIX I

Displacement, slope, moment and shear results using BEAM8 for Example 1.

LENGTH (in)	DEFLECTION (in)	SLOPE (Radian)	MOMENT (in-lb)	SHEAR (lb)
0.000D+00	1.075D-07	9.726D-08	0.000D+00	0.000D+00
5.000D-02	1.026D-07	9.728D-08	3.969D-06	1.575D-04
1.000D-01	9.773D-08	9.745D-08	1.563D-05	3.078D-04
1.500D-01	9.285D-08	9.790D-08	3.463D-05	4.507D-04
2.000D-01	8.793D-08	9.875D-08	6.058D-05	5.863D-04
2.500D-01	8.296D-08	1.001D-07	9.313D-05	7.145D-04
3.000D-01	7.791D-08	1.022D-07	1.319D-04	8.352D-04
3.500D-01	7.273D-08	1.050D-07	1.765D-04	9.482D-04
4.000D-01	6.740D-08	1.086D-07	2.266D-04	1.053D-03
4.500D-01	6.185D-08	1.133D-07	2.817D-04	1.150D-03
5.000D-01	5.605D-08	1.189D-07	3.415D-04	1.239D-03
5.500D-01	4.994D-08	1.257D-07	4.054D-04	1.318D-03
6.000D-01	4.346D-08	1.337D-07	4.731D-04	1.388D-03
6.500D-01	3.655D-08	1.430D-07	5.441D-04	1.448D-03
7.000D-01	2.914D-08	1.535D-07	6.178D-04	1.498D-03
7.500D-01	2.118D-08	1.655D-07	6.937D-04	1.536D-03
8.000D-01	1.258D-08	1.788D-07	7.712D-04	1.561D-03
8.500D-01	3.274D-09	1.935D-07	8.496D-04	1.573D-03
9.000D-01	-6.801D-09	2.097D-07	9.282D-04	1.570D-03
9.500D-01	-1.772D-08	2.273D-07	1.006D-03	1.552D-03
1.000D+00	-2.956D-08	2.463D-07	1.083D-03	1.517D-03
1.050D+00	-4.238D-08	2.667D-07	1.158D-03	1.463D-03
1.100D+00	-5.625D-08	2.885D-07	1.229D-03	1.389D-03
1.150D+00	-7.124D-08	3.114D-07	1.296D-03	1.294D-03
1.200D+00	-8.741D-08	3.356D-07	1.358D-03	1.175D-03
1.250D+00	-1.048D-07	3.608D-07	1.413D-03	1.031D-03
1.300D+00	-1.235D-07	3.870D-07	1.461D-03	8.597D-04
1.350D+00	-1.435D-07	4.139D-07	1.499D-03	6.596D-04
1.400D+00	-1.649D-07	4.415D-07	1.526D-03	4.284D-04
1.450D+00	-1.877D-07	4.694D-07	1.541D-03	1.641D-04
1.500D+00	-2.119D-07	4.975D-07	1.542D-03	-1.353D-04
1.550D+00	-2.374D-07	5.255D-07	1.527D-03	-4.721D-04
1.600D+00	-2.644D-07	5.530D-07	1.494D-03	-8.484D-04
1.650D+00	-2.927D-07	5.797D-07	1.441D-03	-1.266D-03
1.700D+00	-3.224D-07	6.053D-07	1.367D-03	-1.727D-03
1.750D+00	-3.532D-07	6.293D-07	1.268D-03	-2.234D-03
1.800D+00	-3.853D-07	6.513D-07	1.143D-03	-2.787D-03
1.850D+00	-4.183D-07	6.707D-07	9.884D-04	-3.390D-03
1.900D+00	-4.523D-07	6.871D-07	8.028D-04	-4.043D-03
1.950D+00	-4.870D-07	6.998D-07	5.832D-04	-4.747D-03
2.000D+00	-5.222D-07	7.081D-07	3.272D-04	-5.504D-03
2.050D+00	-5.577D-07	7.114D-07	3.196D-05	-6.314D-03
2.100D+00	-5.932D-07	7.090D-07	-3.051D-04	-7.177D-03
2.150D+00	-6.285D-07	7.000D-07	-6.866D-04	-8.093D-03
2.200D+00	-6.631D-07	6.837D-07	-1.115D-03	-9.062D-03
2.250D+00	-6.967D-07	6.592D-07	-1.594D-03	-1.008D-02
2.300D+00	-7.289D-07	6.254D-07	-2.124D-03	-1.115D-02
2.350D+00	-7.591D-07	5.815D-07	-2.710D-03	-1.227D-02
2.400D+00	-7.868D-07	5.264D-07	-3.352D-03	-1.343D-02
2.450D+00	-8.115D-07	4.591D-07	-4.053D-03	-1.463D-02
2.500D+00	-8.325D-07	3.785D-07	-4.815D-03	-1.586D-02
2.550D+00	-8.491D-07	2.835D-07	-5.640D-03	-1.712D-02

Appendix I continued....

Displacement, slope, moment and shear results using BEAM8 for Example 1.

2.600D+00	-8.606D-07	1.729D-07	-6.528D-03	-1.841D-02
2.650D+00	-8.661D-07	4.548D-08	-7.480D-03	-1.970D-02
2.700D+00	-8.649D-07	-9.982D-08	-8.498D-03	-2.100D-02
2.750D+00	-8.558D-07	-2.642D-07	-9.580D-03	-2.229D-02
2.800D+00	-8.381D-07	-4.489D-07	-1.073D-02	-2.356D-02
2.850D+00	-8.106D-07	-6.551D-07	-1.194D-02	-2.480D-02
2.900D+00	-7.722D-07	-8.838D-07	-1.321D-02	-2.599D-02
2.950D+00	-7.218D-07	-1.136D-06	-1.453D-02	-2.711D-02
3.000D+00	-6.582D-07	-1.413D-06	-1.592D-02	-2.815D-02
3.050D+00	-5.801D-07	-1.716D-06	-1.735D-02	-2.908D-02
3.100D+00	-4.862D-07	-2.045D-06	-1.882D-02	-2.988D-02
3.150D+00	-3.751D-07	-2.401D-06	-2.033D-02	-3.053D-02
3.200D+00	-2.456D-07	-2.785D-06	-2.187D-02	-3.100D-02
3.250D+00	-9.617D-08	-3.197D-06	-2.343D-02	-3.126D-02
3.300D+00	7.459D-08	-3.638D-06	-2.499D-02	-3.127D-02
3.350D+00	2.681D-07	-4.107D-06	-2.655D-02	-3.102D-02
3.400D+00	4.858D-07	-4.604D-06	-2.809D-02	-3.046D-02
3.450D+00	7.290D-07	-5.129D-06	-2.959D-02	-2.955D-02
3.500D+00	9.991D-07	-5.681D-06	-3.104D-02	-2.826D-02
3.550D+00	1.298D-06	-6.259D-06	-3.241D-02	-2.654D-02
3.600D+00	1.625D-06	-6.860D-06	-3.368D-02	-2.435D-02
3.650D+00	1.984D-06	-7.484D-06	-3.484D-02	-2.165D-02
3.700D+00	2.374D-06	-8.127D-06	-3.584D-02	-1.838D-02
3.750D+00	2.797D-06	-8.787D-06	-3.667D-02	-1.451D-02
3.800D+00	3.253D-06	-9.461D-06	-3.728D-02	-9.976D-03
3.850D+00	3.743D-06	-1.014D-05	-3.765D-02	-4.733D-03
3.900D+00	4.267D-06	-1.083D-05	-3.774D-02	1.271D-03
3.950D+00	4.826D-06	-1.151D-05	-3.751D-02	8.087D-03
4.000D+00	5.419D-06	-1.219D-05	-3.692D-02	1.577D-02
4.050D+00	6.045D-06	-1.286D-05	-3.592D-02	2.436D-02
4.100D+00	6.704D-06	-1.350D-05	-3.447D-02	3.392D-02
4.150D+00	7.394D-06	-1.411D-05	-3.251D-02	4.449D-02
4.200D+00	8.114D-06	-1.468D-05	-3.000D-02	5.612D-02
4.250D+00	8.861D-06	-1.520D-05	-2.688D-02	6.884D-02
4.300D+00	9.633D-06	-1.565D-05	-2.310D-02	8.271D-02
4.350D+00	1.043D-05	-1.603D-05	-1.859D-02	9.775D-02
4.400D+00	1.123D-05	-1.632D-05	-1.330D-02	1.140D-01
4.450D+00	1.206D-05	-1.651D-05	-7.169D-03	1.315D-01
4.500D+00	1.288D-05	-1.658D-05	-1.338D-04	1.502D-01
4.550D+00	1.371D-05	-1.651D-05	7.868D-03	1.701D-01
4.600D+00	1.453D-05	-1.629D-05	1.690D-02	1.913D-01
4.650D+00	1.534D-05	-1.589D-05	2.702D-02	2.137D-01
4.700D+00	1.612D-05	-1.530D-05	3.829D-02	2.373D-01
4.750D+00	1.686D-05	-1.449D-05	5.077D-02	2.620D-01
4.800D+00	1.756D-05	-1.344D-05	6.451D-02	2.879D-01
4.850D+00	1.820D-05	-1.213D-05	7.957D-02	3.147D-01
4.900D+00	1.877D-05	-1.054D-05	9.599D-02	3.424D-01
4.950D+00	1.925D-05	-8.629D-06	1.138D-01	3.710D-01
5.000D+00	1.963D-05	-6.384D-06	1.331D-01	4.001D-01
5.050D+00	1.988D-05	-3.775D-06	1.538D-01	4.298D-01
5.100D+00	2.000D-05	-7.748D-07	1.761D-01	4.597D-01
5.150D+00	1.995D-05	2.644D-06	1.998D-01	4.897D-01
5.200D+00	1.973D-05	6.508D-06	2.251D-01	5.195D-01
5.250D+00	1.930D-05	1.084D-05	2.518D-01	5.488D-01
5.300D+00	1.863D-05	1.568D-05	2.799D-01	5.773D-01

Appendix I continued....

Displacement, slope, moment and shear results using BEAM8 for Example 1.

5.350D+00	1.772D-05	2.104D-05	3.095D-01	6.046D-01
5.400D+00	1.652D-05	2.695D-05	3.403D-01	6.303D-01
5.450D+00	1.501D-05	3.344D-05	3.725D-01	6.540D-01
5.500D+00	1.317D-05	4.052D-05	4.057D-01	6.751D-01
5.550D+00	1.095D-05	4.821D-05	4.399D-01	6.933D-01
5.600D+00	8.336D-06	5.654D-05	4.750D-01	7.078D-01
5.650D+00	5.287D-06	6.550D-05	5.106D-01	7.181D-01
5.700D+00	1.774D-06	7.512D-05	5.467D-01	7.234D-01
5.750D+00	-2.236D-06	8.540D-05	5.829D-01	7.231D-01
5.800D+00	-6.777D-06	9.634D-05	6.189D-01	7.165D-01
5.850D+00	-1.188D-05	1.079D-04	6.544D-01	7.025D-01
5.900D+00	-1.758D-05	1.202D-04	6.890D-01	6.805D-01
5.950D+00	-2.391D-05	1.330D-04	7.223D-01	6.495D-01
6.000D+00	-3.089D-05	1.464D-04	7.538D-01	6.085D-01
6.050D+00	-3.856D-05	1.604D-04	7.830D-01	5.565D-01
6.100D+00	-4.694D-05	1.749D-04	8.092D-01	4.924D-01
6.150D+00	-5.606D-05	1.899D-04	8.320D-01	4.153D-01
6.200D+00	-6.593D-05	2.052D-04	8.505D-01	3.239D-01
6.250D+00	-7.658D-05	2.208D-04	8.641D-01	2.171D-01
6.300D+00	-8.802D-05	2.366D-04	8.720D-01	9.374D-02
6.350D+00	-1.002D-04	2.525D-04	8.732D-01	-4.736D-02
6.400D+00	-1.133D-04	2.683D-04	8.669D-01	-2.074D-01
6.450D+00	-1.271D-04	2.840D-04	8.521D-01	-3.875D-01
6.500D+00	-1.417D-04	2.993D-04	8.278D-01	-5.890D-01
6.550D+00	-1.570D-04	3.141D-04	7.929D-01	-8.129D-01
6.600D+00	-1.731D-04	3.281D-04	7.461D-01	-1.060D+00
6.650D+00	-1.898D-04	3.411D-04	6.864D-01	-1.332D+00
6.700D+00	-2.071D-04	3.530D-04	6.125D-01	-1.630D+00
6.750D+00	-2.251D-04	3.633D-04	5.230D-01	-1.954D+00
6.800D+00	-2.434D-04	3.719D-04	4.166D-01	-2.305D+00
6.850D+00	-2.622D-04	3.784D-04	2.920D-01	-2.685D+00
6.900D+00	-2.812D-04	3.824D-04	1.477D-01	-3.092D+00
6.950D+00	-3.004D-04	3.836D-04	-1.774D-02	-3.528D+00
7.000D+00	-3.196D-04	3.816D-04	-2.057D-01	-3.993D+00
7.050D+00	-3.385D-04	3.760D-04	-4.176D-01	-4.487D+00
7.100D+00	-3.571D-04	3.663D-04	-6.548D-01	-5.009D+00
7.150D+00	-3.751D-04	3.520D-04	-9.189D-01	-5.558D+00
7.200D+00	-3.922D-04	3.327D-04	-1.211D+00	-6.134D+00
7.250D+00	-4.082D-04	3.077D-04	-1.533D+00	-6.734D+00
7.300D+00	-4.229D-04	2.767D-04	-1.885D+00	-7.358D+00
7.350D+00	-4.358D-04	2.389D-04	-2.269D+00	-8.002D+00
7.400D+00	-4.467D-04	1.939D-04	-2.685D+00	-8.664D+00
7.450D+00	-4.551D-04	1.410D-04	-3.135D+00	-9.341D+00
7.500D+00	-4.606D-04	7.958D-05	-3.620D+00	-1.003D+01
7.550D+00	-4.629D-04	9.036D-06	-4.138D+00	-1.072D+01
7.600D+00	-4.614D-04	-7.126D-05	-4.692D+00	-1.141D+01
7.650D+00	-4.556D-04	-1.620D-04	-5.280D+00	-1.210D+01
7.700D+00	-4.450D-04	-2.636D-04	-5.902D+00	-1.278D+01
7.750D+00	-4.290D-04	-3.770D-04	-6.557D+00	-1.343D+01
7.800D+00	-4.071D-04	-5.025D-04	-7.245D+00	-1.406D+01
7.850D+00	-3.785D-04	-6.409D-04	-7.963D+00	-1.465D+01
7.900D+00	-3.428D-04	-7.925D-04	-8.709D+00	-1.519D+01
7.950D+00	-2.991D-04	-9.580D-04	-9.481D+00	-1.568D+01
8.000D+00	-2.467D-04	-1.138D-03	-1.028D+01	-1.609D+01
8.050D+00	-1.850D-04	-1.332D-03	-1.109D+01	-1.641D+01

Appendix I continued....

Displacement, slope, moment and shear results using BEAM8 for Example 1.

8.100D+00	-1.133D-04	-1.542D-03	-1.192D+01	-1.664D+01
8.150D+00	-3.063D-05	-1.766D-03	-1.275D+01	-1.675D+01
8.200D+00	6.360D-05	-2.006D-03	-1.359D+01	-1.672D+01
8.250D+00	1.702D-04	-2.261D-03	-1.442D+01	-1.655D+01
8.300D+00	2.899D-04	-2.530D-03	-1.524D+01	-1.621D+01
8.350D+00	4.235D-04	-2.815D-03	-1.604D+01	-1.567D+01
8.400D+00	5.716D-04	-3.114D-03	-1.680D+01	-1.493D+01
8.450D+00	7.351D-04	-3.427D-03	-1.753D+01	-1.395D+01
8.500D+00	9.145D-04	-3.752D-03	-1.819D+01	-1.272D+01
8.550D+00	1.110D-03	-4.088D-03	-1.879D+01	-1.120D+01
8.600D+00	1.324D-03	-4.435D-03	-1.931D+01	-9.376D+00
8.650D+00	1.554D-03	-4.791D-03	-1.973D+01	-7.220D+00
8.700D+00	1.803D-03	-5.153D-03	-2.003D+01	-4.704D+00
8.750D+00	2.069D-03	-5.519D-03	-2.019D+01	-1.802D+00
8.800D+00	2.355D-03	-5.887D-03	-2.020D+01	1.513D+00
8.850D+00	2.658D-03	-6.253D-03	-2.003D+01	5.271D+00
8.900D+00	2.980D-03	-6.614D-03	-1.966D+01	9.497D+00
8.950D+00	3.319D-03	-6.967D-03	-1.907D+01	1.422D+01
9.000D+00	3.676D-03	-7.307D-03	-1.823D+01	1.946D+01
9.050D+00	4.050D-03	-7.629D-03	-1.712D+01	2.526D+01
9.100D+00	4.439D-03	-7.928D-03	-1.570D+01	3.162D+01
9.150D+00	4.842D-03	-8.199D-03	-1.395D+01	3.858D+01
9.200D+00	5.258D-03	-8.434D-03	-1.183D+01	4.615D+01
9.250D+00	5.685D-03	-8.627D-03	-9.320D+00	5.436D+01
9.300D+00	6.120D-03	-8.771D-03	-6.383D+00	6.321D+01
9.350D+00	6.561D-03	-8.857D-03	-2.988D+00	7.272D+01
9.400D+00	7.005D-03	-8.876D-03	9.002D-01	8.290D+01
9.450D+00	7.447D-03	-8.821D-03	5.313D+00	9.374D+01
9.500D+00	7.885D-03	-8.680D-03	1.028D+01	1.052D+02
9.550D+00	8.314D-03	-8.443D-03	1.585D+01	1.174D+02
9.600D+00	8.728D-03	-8.099D-03	2.203D+01	1.302D+02
9.650D+00	9.122D-03	-7.637D-03	2.887D+01	1.436D+02
9.700D+00	9.489D-03	-7.044D-03	3.640D+01	1.575D+02
9.750D+00	9.824D-03	-6.307D-03	4.464D+01	1.720D+02
9.800D+00	1.012D-02	-5.414D-03	5.361D+01	1.870D+02
9.850D+00	1.036D-02	-4.351D-03	6.334D+01	2.023D+02
9.900D+00	1.055D-02	-3.104D-03	7.385D+01	2.180D+02
9.950D+00	1.067D-02	-1.659D-03	8.515D+01	2.340D+02
1.000D+01	1.071D-02	2.927D-16	9.725D+01	2.500D+02
1.000D+01	1.071D-02	2.927D-16	9.725D+01	-2.500D+02
1.005D+01	1.067D-02	1.659D-03	8.515D+01	-2.340D+02
1.010D+01	1.055D-02	3.104D-03	7.385D+01	-2.180D+02
1.015D+01	1.036D-02	4.351D-03	6.334D+01	-2.023D+02
1.020D+01	1.012D-02	5.414D-03	5.361D+01	-1.870D+02
1.025D+01	9.824D-03	6.307D-03	4.464D+01	-1.720D+02
1.030D+01	9.489D-03	7.044D-03	3.640D+01	-1.575D+02
1.035D+01	9.122D-03	7.637D-03	2.887D+01	-1.436D+02
1.040D+01	8.728D-03	8.099D-03	2.203D+01	-1.302D+02
1.045D+01	8.314D-03	8.443D-03	1.585D+01	-1.174D+02
1.050D+01	7.885D-03	8.680D-03	1.028D+01	-1.052D+02
1.055D+01	7.447D-03	8.821D-03	5.313D+00	-9.374D+01
1.060D+01	7.005D-03	8.876D-03	9.002D-01	-8.290D+01
1.065D+01	6.561D-03	8.857D-03	-2.988D+00	-7.272D+01
1.070D+01	6.120D-03	8.771D-03	-6.383D+00	-6.321D+01
1.075D+01	5.685D-03	8.627D-03	-9.320D+00	-5.436D+01

Appendix I continued....

Displacement, slope, moment and shear results using BEAM8 for Example 1.

1.080D+01	5.258D-03	8.434D-03	-1.183D+01	-4.615D+01
1.085D+01	4.842D-03	8.199D-03	-1.395D+01	-3.858D+01
1.090D+01	4.439D-03	7.928D-03	-1.570D+01	-3.162D+01
1.095D+01	4.050D-03	7.629D-03	-1.712D+01	-2.526D+01
1.100D+01	3.676D-03	7.307D-03	-1.823D+01	-1.946D+01
1.105D+01	3.319D-03	6.967D-03	-1.907D+01	-1.422D+01
1.110D+01	2.980D-03	6.614D-03	-1.966D+01	-9.497D+00
1.115D+01	2.658D-03	6.253D-03	-2.003D+01	-5.271D+00
1.120D+01	2.355D-03	5.887D-03	-2.020D+01	-1.513D+00
1.125D+01	2.069D-03	5.519D-03	-2.019D+01	1.802D+00
1.130D+01	1.803D-03	5.153D-03	-2.003D+01	4.704D+00
1.135D+01	1.554D-03	4.791D-03	-1.973D+01	7.220D+00
1.140D+01	1.324D-03	4.435D-03	-1.931D+01	9.376D+00
1.145D+01	1.110D-03	4.088D-03	-1.879D+01	1.120D+01
1.150D+01	9.145D-04	3.752D-03	-1.819D+01	1.272D+01
1.155D+01	7.351D-04	3.427D-03	-1.753D+01	1.395D+01
1.160D+01	5.716D-04	3.114D-03	-1.680D+01	1.493D+01
1.165D+01	4.235D-04	2.815D-03	-1.604D+01	1.567D+01
1.170D+01	2.899D-04	2.530D-03	-1.524D+01	1.621D+01
1.175D+01	1.702D-04	2.261D-03	-1.442D+01	1.655D+01
1.180D+01	6.360D-05	2.006D-03	-1.359D+01	1.672D+01
1.185D+01	-3.063D-05	1.766D-03	-1.275D+01	1.675D+01
1.190D+01	-1.133D-04	1.542D-03	-1.192D+01	1.664D+01
1.195D+01	-1.850D-04	1.332D-03	-1.109D+01	1.641D+01
1.200D+01	-2.467D-04	1.138D-03	-1.028D+01	1.609D+01
1.205D+01	-2.991D-04	9.580D-04	-9.481D+00	1.568D+01
1.210D+01	-3.428D-04	7.925D-04	-8.709D+00	1.519D+01
1.215D+01	-3.785D-04	6.409D-04	-7.963D+00	1.465D+01
1.220D+01	-4.071D-04	5.025D-04	-7.245D+00	1.406D+01
1.225D+01	-4.290D-04	3.770D-04	-6.557D+00	1.343D+01
1.230D+01	-4.450D-04	2.636D-04	-5.902D+00	1.278D+01
1.235D+01	-4.556D-04	1.620D-04	-5.280D+00	1.210D+01
1.240D+01	-4.614D-04	7.126D-05	-4.692D+00	1.141D+01
1.245D+01	-4.629D-04	-9.036D-06	-4.138D+00	1.072D+01
1.250D+01	-4.606D-04	-7.958D-05	-3.620D+00	1.003D+01
1.255D+01	-4.551D-04	-1.410D-04	-3.135D+00	9.341D+00
1.260D+01	-4.467D-04	-1.939D-04	-2.685D+00	8.664D+00
1.265D+01	-4.358D-04	-2.389D-04	-2.269D+00	8.002D+00
1.270D+01	-4.229D-04	-2.767D-04	-1.885D+00	7.358D+00
1.275D+01	-4.082D-04	-3.077D-04	-1.533D+00	6.734D+00
1.280D+01	-3.922D-04	-3.327D-04	-1.211D+00	6.134D+00
1.285D+01	-3.751D-04	-3.520D-04	-9.189D-01	5.558D+00
1.290D+01	-3.571D-04	-3.663D-04	-6.548D-01	5.009D+00
1.295D+01	-3.385D-04	-3.760D-04	-4.176D-01	4.487D+00
1.300D+01	-3.196D-04	-3.816D-04	-2.057D-01	3.993D+00
1.305D+01	-3.004D-04	-3.836D-04	-1.774D-02	3.528D+00
1.310D+01	-2.812D-04	-3.824D-04	1.477D-01	3.092D+00
1.315D+01	-2.622D-04	-3.784D-04	2.920D-01	2.685D+00
1.320D+01	-2.434D-04	-3.719D-04	4.166D-01	2.305D+00
1.325D+01	-2.251D-04	-3.633D-04	5.230D-01	1.954D+00
1.330D+01	-2.071D-04	-3.530D-04	6.125D-01	1.630D+00
1.335D+01	-1.898D-04	-3.411D-04	6.864D-01	1.332D+00
1.340D+01	-1.731D-04	-3.281D-04	7.461D-01	1.060D+00
1.345D+01	-1.570D-04	-3.141D-04	7.929D-01	8.129D-01
1.350D+01	-1.417D-04	-2.993D-04	8.278D-01	5.890D-01

Appendix I continued....

Displacement, slope, moment and shear results using BEAM8 for Example 1.

1.355D+01	-1.271D-04	-2.840D-04	8.521D-01	3.875D-01
1.360D+01	-1.133D-04	-2.683D-04	8.669D-01	2.074D-01
1.365D+01	-1.002D-04	-2.525D-04	8.732D-01	4.736D-02
1.370D+01	-8.802D-05	-2.366D-04	8.720D-01	-9.374D-02
1.375D+01	-7.658D-05	-2.208D-04	8.641D-01	-2.171D-01
1.380D+01	-6.593D-05	-2.052D-04	8.505D-01	-3.239D-01
1.385D+01	-5.606D-05	-1.899D-04	8.320D-01	-4.153D-01
1.390D+01	-4.694D-05	-1.749D-04	8.092D-01	-4.924D-01
1.395D+01	-3.856D-05	-1.604D-04	7.830D-01	-5.565D-01
1.400D+01	-3.089D-05	-1.464D-04	7.538D-01	-6.085D-01
1.405D+01	-2.391D-05	-1.330D-04	7.223D-01	-6.495D-01
1.410D+01	-1.758D-05	-1.202D-04	6.890D-01	-6.805D-01
1.415D+01	-1.188D-05	-1.079D-04	6.544D-01	-7.025D-01
1.420D+01	-6.777D-06	-9.634D-05	6.189D-01	-7.165D-01
1.425D+01	-2.236D-06	-8.540D-05	5.829D-01	-7.231D-01
1.430D+01	1.774D-06	-7.512D-05	5.467D-01	-7.234D-01
1.435D+01	5.287D-06	-6.550D-05	5.106D-01	-7.181D-01
1.440D+01	8.336D-06	-5.654D-05	4.750D-01	-7.078D-01
1.445D+01	1.095D-05	-4.821D-05	4.399D-01	-6.933D-01
1.450D+01	1.317D-05	-4.052D-05	4.057D-01	-6.751D-01
1.455D+01	1.501D-05	-3.344D-05	3.725D-01	-6.540D-01
1.460D+01	1.652D-05	-2.695D-05	3.403D-01	-6.303D-01
1.465D+01	1.772D-05	-2.104D-05	3.095D-01	-6.046D-01
1.470D+01	1.863D-05	-1.568D-05	2.799D-01	-5.773D-01
1.475D+01	1.930D-05	-1.084D-05	2.518D-01	-5.488D-01
1.480D+01	1.973D-05	-6.508D-06	2.251D-01	-5.195D-01
1.485D+01	1.995D-05	-2.644D-06	1.998D-01	-4.897D-01
1.490D+01	2.000D-05	7.748D-07	1.761D-01	-4.597D-01
1.495D+01	1.988D-05	3.775D-06	1.538D-01	-4.298D-01
1.500D+01	1.963D-05	6.384D-06	1.331D-01	-4.001D-01
1.505D+01	1.925D-05	8.629D-06	1.138D-01	-3.710D-01
1.510D+01	1.877D-05	1.054D-05	9.599D-02	-3.424D-01
1.515D+01	1.820D-05	1.213D-05	7.957D-02	-3.147D-01
1.520D+01	1.756D-05	1.344D-05	6.451D-02	-2.879D-01
1.525D+01	1.686D-05	1.449D-05	5.077D-02	-2.620D-01
1.530D+01	1.612D-05	1.530D-05	3.829D-02	-2.373D-01
1.535D+01	1.534D-05	1.589D-05	2.702D-02	-2.137D-01
1.540D+01	1.453D-05	1.629D-05	1.690D-02	-1.913D-01
1.545D+01	1.371D-05	1.651D-05	7.868D-03	-1.701D-01
1.550D+01	1.288D-05	1.658D-05	-1.338D-04	-1.502D-01
1.555D+01	1.206D-05	1.651D-05	-7.169D-03	-1.315D-01
1.560D+01	1.123D-05	1.632D-05	-1.330D-02	-1.140D-01
1.565D+01	1.043D-05	1.603D-05	-1.859D-02	-9.775D-02
1.570D+01	9.633D-06	1.565D-05	-2.310D-02	-8.271D-02
1.575D+01	8.861D-06	1.520D-05	-2.688D-02	-6.884D-02
1.580D+01	8.114D-06	1.468D-05	-3.000D-02	-5.612D-02
1.585D+01	7.394D-06	1.411D-05	-3.251D-02	-4.449D-02
1.590D+01	6.704D-06	1.350D-05	-3.447D-02	-3.392D-02
1.595D+01	6.045D-06	1.286D-05	-3.592D-02	-2.436D-02
1.600D+01	5.419D-06	1.219D-05	-3.692D-02	-1.577D-02
1.605D+01	4.826D-06	1.151D-05	-3.751D-02	-8.087D-03
1.610D+01	4.267D-06	1.083D-05	-3.774D-02	-1.271D-03
1.615D+01	3.743D-06	1.014D-05	-3.765D-02	4.733D-03
1.620D+01	3.253D-06	9.461D-06	-3.728D-02	9.976D-03
1.625D+01	2.797D-06	8.787D-06	-3.667D-02	1.451D-02

Appendix I continued....

Displacement, slope, moment and shear results using BEAM8 for Example 1.

1.630D+01	2.374D-06	8.127D-06	-3.584D-02	1.838D-02
1.635D+01	1.984D-06	7.484D-06	-3.484D-02	2.165D-02
1.640D+01	1.625D-06	6.860D-06	-3.368D-02	2.435D-02
1.645D+01	1.298D-06	6.259D-06	-3.241D-02	2.654D-02
1.650D+01	9.991D-07	5.681D-06	-3.104D-02	2.826D-02
1.655D+01	7.290D-07	5.129D-06	-2.959D-02	2.955D-02
1.660D+01	4.858D-07	4.604D-06	-2.809D-02	3.046D-02
1.665D+01	2.681D-07	4.107D-06	-2.655D-02	3.102D-02
1.670D+01	7.459D-08	3.638D-06	-2.499D-02	3.127D-02
1.675D+01	-9.617D-08	3.197D-06	-2.343D-02	3.126D-02
1.680D+01	-2.456D-07	2.785D-06	-2.187D-02	3.100D-02
1.685D+01	-3.751D-07	2.401D-06	-2.033D-02	3.053D-02
1.690D+01	-4.862D-07	2.045D-06	-1.882D-02	2.988D-02
1.695D+01	-5.801D-07	1.716D-06	-1.735D-02	2.908D-02
1.700D+01	-6.582D-07	1.413D-06	-1.592D-02	2.815D-02
1.705D+01	-7.218D-07	1.136D-06	-1.453D-02	2.711D-02
1.710D+01	-7.722D-07	8.838D-07	-1.321D-02	2.599D-02
1.715D+01	-8.106D-07	6.551D-07	-1.194D-02	2.480D-02
1.720D+01	-8.381D-07	4.489D-07	-1.073D-02	2.356D-02
1.725D+01	-8.558D-07	2.642D-07	-9.580D-03	2.229D-02
1.730D+01	-8.649D-07	9.982D-08	-8.498D-03	2.100D-02
1.735D+01	-8.661D-07	-4.549D-08	-7.480D-03	1.970D-02
1.740D+01	-8.606D-07	-1.729D-07	-6.528D-03	1.841D-02
1.745D+01	-8.491D-07	-2.835D-07	-5.640D-03	1.712D-02
1.750D+01	-8.325D-07	-3.785D-07	-4.815D-03	1.586D-02
1.755D+01	-8.115D-07	-4.591D-07	-4.053D-03	1.463D-02
1.760D+01	-7.868D-07	-5.264D-07	-3.352D-03	1.343D-02
1.765D+01	-7.591D-07	-5.815D-07	-2.710D-03	1.227D-02
1.770D+01	-7.289D-07	-6.254D-07	-2.124D-03	1.115D-02
1.775D+01	-6.967D-07	-6.592D-07	-1.594D-03	1.008D-02
1.780D+01	-6.631D-07	-6.837D-07	-1.115D-03	9.062D-03
1.785D+01	-6.285D-07	-7.001D-07	-6.867D-04	8.093D-03
1.790D+01	-5.932D-07	-7.090D-07	-3.051D-04	7.177D-03
1.795D+01	-5.577D-07	-7.114D-07	3.192D-05	6.314D-03
1.800D+01	-5.222D-07	-7.081D-07	3.271D-04	5.504D-03
1.805D+01	-4.870D-07	-6.998D-07	5.832D-04	4.747D-03
1.810D+01	-4.523D-07	-6.871D-07	8.027D-04	4.043D-03
1.815D+01	-4.183D-07	-6.708D-07	9.883D-04	3.390D-03
1.820D+01	-3.852D-07	-6.513D-07	1.143D-03	2.787D-03
1.825D+01	-3.532D-07	-6.293D-07	1.268D-03	2.234D-03
1.830D+01	-3.223D-07	-6.053D-07	1.367D-03	1.727D-03
1.835D+01	-2.927D-07	-5.797D-07	1.441D-03	1.266D-03
1.840D+01	-2.644D-07	-5.530D-07	1.494D-03	8.485D-04
1.845D+01	-2.374D-07	-5.255D-07	1.527D-03	4.723D-04
1.850D+01	-2.118D-07	-4.975D-07	1.542D-03	1.355D-04
1.855D+01	-1.877D-07	-4.694D-07	1.541D-03	-1.640D-04
1.860D+01	-1.649D-07	-4.415D-07	1.526D-03	-4.282D-04
1.865D+01	-1.435D-07	-4.140D-07	1.499D-03	-6.594D-04
1.870D+01	-1.235D-07	-3.870D-07	1.461D-03	-8.595D-04
1.875D+01	-1.048D-07	-3.608D-07	1.413D-03	-1.031D-03
1.880D+01	-8.740D-08	-3.356D-07	1.358D-03	-1.175D-03
1.885D+01	-7.122D-08	-3.115D-07	1.296D-03	-1.293D-03
1.890D+01	-5.623D-08	-2.885D-07	1.229D-03	-1.389D-03
1.895D+01	-4.236D-08	-2.667D-07	1.158D-03	-1.463D-03
1.900D+01	-2.954D-08	-2.463D-07	1.083D-03	-1.516D-03

Appendix I continued....

Displacement, slope, moment and shear results using BEAM8 for Example 1.

1.905D+01	-1.770D-08	-2.273D-07	1.006D-03	-1.552D-03
1.910D+01	-6.780D-09	-2.097D-07	9.284D-04	-1.570D-03
1.915D+01	3.295D-09	-1.935D-07	8.497D-04	-1.572D-03
1.920D+01	1.260D-08	-1.788D-07	7.714D-04	-1.560D-03
1.925D+01	2.120D-08	-1.654D-07	6.939D-04	-1.535D-03
1.930D+01	2.917D-08	-1.535D-07	6.181D-04	-1.497D-03
1.935D+01	3.657D-08	-1.429D-07	5.444D-04	-1.448D-03
1.940D+01	4.348D-08	-1.337D-07	4.735D-04	-1.388D-03
1.945D+01	4.996D-08	-1.257D-07	4.058D-04	-1.318D-03
1.950D+01	5.607D-08	-1.189D-07	3.419D-04	-1.238D-03
1.955D+01	6.187D-08	-1.132D-07	2.821D-04	-1.150D-03
1.960D+01	6.741D-08	-1.086D-07	2.271D-04	-1.053D-03
1.965D+01	7.274D-08	-1.049D-07	1.770D-04	-9.474D-04
1.970D+01	7.791D-08	-1.021D-07	1.324D-04	-8.344D-04
1.975D+01	8.297D-08	-1.001D-07	9.371D-05	-7.137D-04
1.980D+01	8.793D-08	-9.866D-08	6.120D-05	-5.855D-04
1.985D+01	9.284D-08	-9.779D-08	3.528D-05	-4.500D-04
1.990D+01	9.772D-08	-9.733D-08	1.633D-05	-3.070D-04
1.995D+01	1.026D-07	-9.715D-08	4.696D-06	-1.568D-04
2.000D+01	1.074D-07	-9.712D-08	7.682D-07	6.976D-07

APPENDIX II

Displacement, slope, moment and shear results using BEAM8 for Example 3.

Case 1. Fixed-Fixed boundary condition

*** STATIC RESPONSE ***

LENGTH (in)	DEFLECTION (in)	SLOPE (Radian)	MOMENT (in-lb)	SHEAR (lb)
0.000D+00	0.000D+00	0.000D+00	2.722D+02	-2.334D+02
1.000D+00	-2.360D-03	4.005D-03	8.753D+01	-1.383D+02
2.000D+00	-6.927D-03	4.744D-03	-1.181D+01	-6.483D+01
3.000D+00	-1.133D-02	3.920D-03	-5.113D+01	-1.807D+01
4.000D+00	-1.462D-02	2.632D-03	-5.548D+01	6.207D+00
5.000D+00	-1.664D-02	1.458D-03	-4.380D+01	1.519D+01
6.000D+00	-1.765D-02	6.219D-04	-2.793D+01	1.557D+01
7.000D+00	-1.800D-02	1.414D-04	-1.389D+01	1.218D+01
8.000D+00	-1.803D-02	-5.700D-05	-3.872D+00	7.834D+00
9.000D+00	-1.795D-02	-7.232D-05	1.872D+00	3.729D+00
1.000D+01	-1.791D-02	-2.776D-17	3.717D+00	-2.728D-12
1.100D+01	-1.795D-02	7.232D-05	1.872D+00	-3.729D+00
1.200D+01	-1.803D-02	5.700D-05	-3.872D+00	-7.834D+00
1.300D+01	-1.800D-02	-1.414D-04	-1.389D+01	-1.218D+01
1.400D+01	-1.765D-02	-6.219D-04	-2.793D+01	-1.557D+01
1.500D+01	-1.664D-02	-1.458D-03	-4.380D+01	-1.519D+01
1.600D+01	-1.462D-02	-2.632D-03	-5.548D+01	-6.207D+00
1.700D+01	-1.133D-02	-3.920D-03	-5.113D+01	1.807D+01
1.800D+01	-6.927D-03	-4.744D-03	-1.181D+01	6.483D+01
1.900D+01	-2.360D-03	-4.005D-03	8.753D+01	1.383D+02
2.000D+01	-7.105D-15	0.000D+00	2.722D+02	2.334D+02

Appendix II continued .....

Displacement, slope, moment and shear results using BEAM8 for Example 3.

Case 2. Free-Free boundary condition; Stiffener ring with stiffness of 30(10)^6 psi

*** STATIC RESPONSE ***

LENGTH (in)	DEFLECTION (in)	SLOPE (Radian)	MOMENT (in-lb)	SHEAR (lb)
0.000D+00	5.943D-04	7.438D-03	0.000D+00	0.000D+00
0.000D+00	5.943D-04	7.438D-03	8.099D+00	-1.242D+02
1.000D+00	-6.548D-03	6.540D-03	-7.152D+01	-4.192D+01
2.000D+00	-1.214D-02	4.592D-03	-8.826D+01	3.047D+00
3.000D+00	-1.574D-02	2.663D-03	-7.429D+01	2.142D+01
4.000D+00	-1.763D-02	1.203D-03	-5.061D+01	2.414D+01
5.000D+00	-1.834D-02	2.920D-04	-2.838D+01	1.963D+01
6.000D+00	-1.837D-02	-1.653D-04	-1.195D+01	1.321D+01
7.000D+00	-1.811D-02	-3.129D-04	-1.667D+00	7.605D+00
8.000D+00	-1.780D-02	-2.800D-04	3.846D+00	3.719D+00
9.000D+00	-1.758D-02	-1.573D-04	6.304D+00	1.413D+00
1.000D+01	-1.750D-02	-1.110D-16	6.972D+00	9.095D-13
1.100D+01	-1.758D-02	1.573D-04	6.304D+00	-1.413D+00
1.200D+01	-1.780D-02	2.800D-04	3.846D+00	-3.719D+00
1.300D+01	-1.811D-02	3.129D-04	-1.667D+00	-7.605D+00
1.400D+01	-1.837D-02	1.653D-04	-1.195D+01	-1.321D+01
1.500D+01	-1.834D-02	-2.920D-04	-2.838D+01	-1.963D+01
1.600D+01	-1.763D-02	-1.203D-03	-5.061D+01	-2.414D+01
1.700D+01	-1.574D-02	-2.663D-03	-7.429D+01	-2.142D+01
1.800D+01	-1.214D-02	-4.592D-03	-8.826D+01	-3.047D+00
1.900D+01	-6.548D-03	-6.540D-03	-7.152D+01	4.192D+01
2.000D+01	5.943D-04	-7.438D-03	8.099D+00	1.242D+02
2.000D+01	5.943D-04	-7.438D-03	-6.210D-11	1.323D-09

Appendix II continued .....

Displacement, slope, moment and shear results using BEAM8 for Example 3.

Case 3. Free-Free boundary condition; Stiffener ring with stiffness of $30(10)^{7}$ psi

*** STATIC RESPONSE ***

LENGTH (in)	DEFLECTION (in)	SLOPE (Radian)	MOMENT (in-lb)	SHEAR (lb)
0.000D+00	6.887D-05	5.735D-03	0.000D+00	0.000D+00
0.000D+00	6.887D-05	5.735D-03	6.244D+01	-1.440D+02
1.000D+00	-5.925D-03	5.869D-03	-3.707D+01	-6.087D+01
2.000D+00	-1.118D-02	4.520D-03	-7.048D+01	-1.103D+01
3.000D+00	-1.487D-02	2.863D-03	-6.783D+01	1.278D+01
4.000D+00	-1.700D-02	1.471D-03	-5.047D+01	1.989D+01
5.000D+00	-1.796D-02	5.262D-04	-3.093D+01	1.827D+01
6.000D+00	-1.819D-02	9.783D-07	-1.498D+01	1.340D+01
7.000D+00	-1.807D-02	-2.121D-04	-4.153D+00	8.378D+00
8.000D+00	-1.784D-02	-2.277D-04	2.166D+00	4.485D+00
9.000D+00	-1.765D-02	-1.362D-04	5.247D+00	1.859D+00
1.000D+01	-1.758D-02	-8.327D-17	6.142D+00	9.095D-13
1.100D+01	-1.765D-02	1.362D-04	5.247D+00	-1.859D+00
1.200D+01	-1.784D-02	2.277D-04	2.166D+00	-4.485D+00
1.300D+01	-1.807D-02	2.121D-04	-4.153D+00	-8.378D+00
1.400D+01	-1.819D-02	-9.783D-07	-1.498D+01	-1.340D+01
1.500D+01	-1.796D-02	-5.262D-04	-3.093D+01	-1.827D+01
1.600D+01	-1.700D-02	-1.471D-03	-5.047D+01	-1.989D+01
1.700D+01	-1.487D-02	-2.863D-03	-6.783D+01	-1.278D+01
1.800D+01	-1.118D-02	-4.520D-03	-7.048D+01	1.103D+01
1.900D+01	-5.925D-03	-5.869D-03	-3.707D+01	6.087D+01
2.000D+01	6.887D-05	-5.735D-03	6.244D+01	1.440D+02
2.000D+01	6.887D-05	-5.735D-03	-5.154D-11	3.930D-09

Appendix II continued .....

Displacement, slope, moment and shear results using BEAM8 for Example 3.

Case 4. Free-Free boundary condition; Stiffener ring with stiffness of 30(10)^8 psi

*** STATIC RESPONSE ***

LENGTH (in)	DEFLECTION (in)	SLOPE (Radian)	MOMENT (in-lb)	SHEAR (lb)
0.000D+00	9.761D-06	1.870D-03	0.000D+00	0.000D+00
0.000D+00	9.761D-06	1.870D-03	2.036D+02	-2.041D+02
1.000D+00	-3.534D-03	4.610D-03	4.684D+01	-1.129D+02
2.000D+00	-8.322D-03	4.667D-03	-3.093D+01	-4.724D+01
3.000D+00	-1.249D-02	3.573D-03	-5.654D+01	-7.999D+00
4.000D+00	-1.540D-02	2.251D-03	-5.380D+01	1.067D+01
5.000D+00	-1.707D-02	1.153D-03	-3.957D+01	1.619D+01
6.000D+00	-1.783D-02	4.190D-04	-2.369D+01	1.485D+01
7.000D+00	-1.802D-02	2.604D-05	-1.070D+01	1.093D+01
8.000D+00	-1.796D-02	-1.126D-04	-1.901D+00	6.737D+00
9.000D+00	-1.785D-02	-9.311D-05	2.971D+00	3.116D+00
1.000D+01	-1.780D-02	0.000D+00	4.505D+00	-1.819D-12
1.100D+01	-1.785D-02	9.311D-05	2.971D+00	-3.116D+00
1.200D+01	-1.796D-02	1.126D-04	-1.901D+00	-6.737D+00
1.300D+01	-1.802D-02	-2.604D-05	-1.070D+01	-1.093D+01
1.400D+01	-1.783D-02	-4.190D-04	-2.369D+01	-1.485D+01
1.500D+01	-1.707D-02	-1.153D-03	-3.957D+01	-1.619D+01
1.600D+01	-1.540D-02	-2.251D-03	-5.380D+01	-1.067D+01
1.700D+01	-1.249D-02	-3.573D-03	-5.654D+01	7.999D+00
1.800D+01	-8.322D-03	-4.667D-03	-3.093D+01	4.724D+01
1.900D+01	-3.534D-03	-4.610D-03	4.684D+01	1.129D+02
2.000D+01	9.761D-06	-1.870D-03	2.036D+02	2.041D+02
2.000D+01	9.761D-06	-1.870D-03	6.149D-10	1.087D-07

Appendix II continued .....

Displacement, slope, moment and shear results using BEAM8 for Example 3.

Case 5. Free-Free boundary condition; Stiffener ring with stiffness of 30(10)^14 psi

*** STATIC RESPONSE ***

LENGTH (in)	DEFLECTION (in)	SLOPE (Radian)	MOMENT (in-lb)	SHEAR (lb)
0.000D+00	1.116D-11	2.500D-09	0.000D+00	0.000D+00
0.000D+00	1.116D-11	2.500D-09	2.722D+02	-2.334D+02
1.000D+00	-2.360D-03	4.005D-03	8.753D+01	-1.383D+02
2.000D+00	-6.927D-03	4.744D-03	-1.181D+01	-6.483D+01
3.000D+00	-1.133D-02	3.920D-03	-5.113D+01	-1.807D+01
4.000D+00	-1.462D-02	2.632D-03	-5.548D+01	6.207D+00
5.000D+00	-1.664D-02	1.458D-03	-4.380D+01	1.519D+01
6.000D+00	-1.765D-02	6.219D-04	-2.793D+01	1.557D+01
7.000D+00	-1.800D-02	1.414D-04	-1.389D+01	1.218D+01
8.000D+00	-1.803D-02	-5.700D-05	-3.872D+00	7.834D+00
9.000D+00	-1.795D-02	-7.232D-05	1.872D+00	3.729D+00
1.000D+01	-1.791D-02	5.551D-17	3.717D+00	-3.638D-12
1.100D+01	-1.795D-02	7.232D-05	1.872D+00	-3.729D+00
1.200D+01	-1.803D-02	5.700D-05	-3.872D+00	-7.834D+00
1.300D+01	-1.800D-02	-1.414D-04	-1.389D+01	-1.218D+01
1.400D+01	-1.765D-02	-6.219D-04	-2.793D+01	-1.557D+01
1.500D+01	-1.664D-02	-1.458D-03	-4.380D+01	-1.519D+01
1.600D+01	-1.462D-02	-2.632D-03	-5.548D+01	-6.207D+00
1.700D+01	-1.133D-02	-3.920D-03	-5.113D+01	1.807D+01
1.800D+01	-6.927D-03	-4.744D-03	-1.181D+01	6.483D+01
1.900D+01	-2.360D-03	-4.005D-03	8.753D+01	1.383D+02
2.000D+01	1.116D-11	-2.500D-09	2.722D+02	2.334D+02
2.000D+01	1.116D-11	-2.500D-09	5.026D-04	8.959D-02

APPENDIX III

Hoop stress results using BEAM8 for Example 2.

LENGTH (in)	HOOP STRESS (kpsi)	LONGITUDINAL STRESS (kpsi)
0.000D+00	0.000D+00	7.150D+00
2.000D+00	-5.813D+00	7.150D+00
4.000D+00	-1.227D+01	7.150D+00
6.000D+00	-1.481D+01	7.150D+00
8.000D+00	-1.513D+01	7.150D+00
1.000D+01	-1.503D+01	7.150D+00
1.200D+01	-1.513D+01	7.150D+00
1.400D+01	-1.481D+01	7.150D+00
1.600D+01	-1.227D+01	7.150D+00
1.800D+01	-5.813D+00	7.150D+00
2.000D+01	0.000D+00	7.150D+00

Press a key to continue...

Appendix III continued .....

Bending Longitudinal and bending circumferential stress using BEAM8 for Example 2.

LENGTH (in)	BENDING STRESS LONGITUDINAL (kpsi)	BENDING STRESS CIRCUMFERENTIAL (kpsi)
0.000D+00	7.839D+00	2.613D+01
2.000D+00	-3.401D-01	-1.134D+00
4.000D+00	-1.598D+00	-5.326D+00
6.000D+00	-8.045D-01	-2.682D+00
8.000D+00	-1.115D-01	-3.717D-01
1.000D+01	1.070D-01	3.568D-01
1.200D+01	-1.115D-01	-3.717D-01
1.400D+01	-8.045D-01	-2.682D+00
1.600D+01	-1.598D+00	-5.326D+00
1.800D+01	-3.401D-01	-1.134D+00
2.000D+01	7.839D+00	2.613D+01

Appendix III continued .....

Shear stress and Normal stress using maximum shear stress theory at inside and outside fiber for Example 2 using BEAM8.

NOTE : *MSST* = Maximum Shear Stress Theory

LENGTH (in)	INSIDE FIBER MAX. NORMAL STRESS USING *MSST* (kpsi)	OUTSIDE FIBER MAX. NORMAL STRESS USING *MSST* (kpsi)
0.000D+00	-6.672D+01	-7.839D+00
2.000D+00	-9.398D+01	8.284D+00
4.000D+00	-9.818D+01	1.248D+01
6.000D+00	-9.553D+01	9.832D+00
8.000D+00	-9.322D+01	7.522D+00
1.000D+01	-9.249D+01	6.793D+00
1.200D+01	-9.322D+01	7.522D+00
1.400D+01	-9.553D+01	9.832D+00
1.600D+01	-9.818D+01	1.248D+01
1.800D+01	-9.398D+01	8.284D+00
2.000D+01	-6.672D+01	-7.839D+00

Appendix III continued .....

Normal stress using maximum shear stress theory at neutral fiber for Example 2 using BEAM8.

***** STRESSES ON THE NEUTRAL FIBER *****

NOTE : *MSST* = Maximum Shear Stress Theory

LENGTH (in)	NOMINAL SHEAR (kpsi)	MAX. NORMAL STRESS USING *MSST* (kpsi)
0.000D+00	-1.867D+00	7.605D+00
2.000D+00	-5.186D-01	1.300D+01
4.000D+00	4.966D-02	1.942D+01
6.000D+00	1.246D-01	2.196D+01
8.000D+00	6.267D-02	2.228D+01
1.000D+01	-1.455D-14	2.218D+01
1.200D+01	-6.267D-02	2.228D+01
1.400D+01	-1.246D-01	2.196D+01
1.600D+01	-4.966D-02	1.942D+01
1.800D+01	5.186D-01	1.300D+01
2.000D+01	1.867D+00	7.605D+00

VITA

The author was born in Bhavnagar, India, on January 18, 1972. He attended Fatima Convent High School in Bhavnagar. He graduated First Class with Distinction in Bachelors in Mechanical Engineering from L. D. College of Engineering, India, in 1993. From August 1993 to August 1994, he was employed by an American multinational company, Ingersoll-Rand (I) Ltd. and worked as a Project Engineer. In August 1994 he came to Virginia Polytechnic Institute and State University to continue his education. He will receive his Masters of Science degree in Mechanical Engineering in February 1996.