Kinematics of Beam Flexure Four-Bar Linkages With Applications in a Compound Bow

by

Matthew Earle Palmer

Thesis submitted to the Faculty of the Virginia Polytechnic Institute and State University in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

in

Mechanical Engineering

APPROVED:

C. F. Reinholtz, Chairman

R. G. Mitchiner

P. H. Tidwell

December 1996
Blacksburg, VA

Keywords: Kinematics, Four-Bar Linkages, Synthesis, Archery, Compound Bows, Bows, Beams
Kinematics of Beam Flexure Four-Bar Linkages With Applications in a Compound Bow

by

Matthew Earle Palmer

Dr. Charles F. Reinholtz, Chairman

Abstract

This thesis is a study in the application of kinematics coupled with elastic body mechanics. Most studies in kinematics assume all mechanism links to be inelastic. Furthermore, the methods of kinematic synthesis have generally been developed to meet requirements of displacement, velocity and acceleration. The work presented in this thesis differs in two important aspects. First, one grounded link of a four-bar linkage is replaced by a cantilevered beam in flexure to produce a force generating mechanism. Second, the synthesis method presented here allows the generation of these mechanisms in closed form for prescribed force generation.

A compound archery bow that incorporates four-bar linkages has been developed as an example. This design relies on the non-linear mechanical advantage of the four-bar linkage and the bow mechanics to provide a resistance curve that is more compatible with the human strength curve. In addition, by modifying the bow kinematics, more potential energy can be stored, and thus potentially more kinetic energy can be transferred to the arrow than with previous bows.
Dedication

This work would never have been completed without the love and dedication of my mother and father. They never let me give up and were always supportive.

To: Barbara E. and Frederick E. Palmer
Acknowledgments

I have to thank Dr. Charles Reinholtz for his encouragement and for the patience in listening to some of my crazy ideas. I couldn’t have chosen anyone more suited to be my advisor. My thanks also go out to Dr. Paul Tidwell and Dr. Reginald Mitchiner for their input to my thesis as my committee members.

I also have to thank Dr. Robert Comparin and the Virginia Tech Formula S.A.E. Team. While my involvement with that team might have caused my delay in getting this thesis finished, the experiences have taught me as much about engineering as all my classes.

In addition, this work could not have proceeded as well without the help of Randy Soper. Finally, this thesis would not have been written without the encouragement of William Zabaronick, a true friend and slave driver.
Table of Contents

Title Page i
Abstract ii
Dedication iii
Acknowledgments iv
Table of Contents v
List of Figures vii
Nomenclature viii

Chapter 1 Introduction 1

Chapter 2 Literature Review
  2.1 Human Strength 3
  2.2 Bows 6
  2.3 Kinematics 9
  2.4 Rules 11

Chapter 3 Theory
  3.1 Introduction 13
  3.2 Bows 16
  3.3 Planar Four-Bar Linkages 24
  3.4 Linkage Synthesis 24
  3.5 Spring Force and Modeling 27
<table>
<thead>
<tr>
<th>Chapter 4</th>
<th>Bow Design</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1 Introduction</td>
<td>33</td>
</tr>
<tr>
<td>4.2 Conceptual Bow Design</td>
<td>33</td>
</tr>
<tr>
<td>4.3 Strength Curve and Transformation to Resistance Curve</td>
<td>36</td>
</tr>
<tr>
<td>4.4 Linkage Synthesis</td>
<td>40</td>
</tr>
<tr>
<td>4.5 Validity Analysis</td>
<td>44</td>
</tr>
<tr>
<td>4.6 Strength Considerations</td>
<td>46</td>
</tr>
<tr>
<td>4.7 Deflection Considerations</td>
<td>48</td>
</tr>
<tr>
<td>4.8 Adaptability</td>
<td>49</td>
</tr>
<tr>
<td>4.9 Comparison of Energy Storage</td>
<td>50</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chapter 5</th>
<th>Conclusions</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1 Conclusions</td>
<td>52</td>
</tr>
<tr>
<td>5.2 Future Work</td>
<td>53</td>
</tr>
</tbody>
</table>

References | 56 |

Appendix A - Bow Mechanism Force Transformation Program | 58 |
Appendix B - Modified Mathcad Four-Bar Linkage Synthesis Module | 61 |
Appendix C - Rules | 76 |

Vita | 80 |
List of Figures

Figure 1: Strength and Resistance Curves For Extension of Calf 4
Figure 2: First Compound Bow, 1966 5
Figure 3: Longbow, Recurve Bow, Compound Bow 7
Figure 4: Kinematic Model of a Planar Four-Bar Linkage 14
Figure 5: Beam Flexure Four-Bar Linkage 15
Figure 6: Longbow Force Curve versus Example Human Strength 17
Figure 7: Recurve Bow Limbs, Rest and Draw Positions 18
Figure 8: Recurve Bow Draw Weight versus Human Strength 19
Figure 9: Various Wrapping Cams Used in Compound Bows 20
Figure 10: Another Compound Bow Design 20
Figure 11: Cam and Eccentric Round Wheel Bows versus Human 22
Figure 12: Proposed Resistance Curve For New Bow versus Human 23
Figure 13: Kinematic Model of Torsional Sprung Four-Bar 25
Figure 14: Beam Motion Model 28
Figure 15: Bow Design, Rest and Full Draw 34
Figure 16: Close up of New Bow Design at Rest Position 35
Figure 17: Strength Curve Data 39
Figure 18: Solution Linkage 41
Figure 19: Resistance Curve of the Solution 42
Figure 20: Draw Weight of the New Bow 43
Figure 21: Crank Slider Substitute 44
Figure 22: Comparison of Force Curves 45
Figure 23: New Bow Force Curve versus Cam Bow of Same Weight 51
3.6 Link Defect and Transmission Angle
3.7 Dynamics

Chapter 4 Bow Design
4.1 Introduction
4.2 Conceptual Bow Design
4.3 Strength Curve and Transformation to Resistance Curve
4.4 Linkage Synthesis
4.5 Validity Analysis
4.6 Strength Considerations
4.7 Deflection Considerations
4.8 Adaptability
4.9 Comparison of Energy Storage

Chapter 5 Conclusions
5.1 Conclusions
5.2 Future Work

References

Appendix A - Bow Mechanism Force Transformation Program
Appendix B - Modified Mathcad Four-Bar Linkage Synthesis Module
Appendix C - Rules

Vita
Nomenclature

b
Depth of beam

d
Beam thickness; \( d = \frac{h}{2} \)

E
Modulus of Elasticity

\( F_{in} \)
Force into the Four-Bar Mechanism, provided by user

\( F_{out} \)
Force providing resistance in Four-Bar Mechanism

h
Height of beam

I
Moment of Inertia

\( k_{T} \)
Torsional Rigidty

L
Length of Beam

\( L_1 \)
Length of ground link of Four-Bar Mechanism

\( L_2 \)
Length of input link of Four-Bar Mechanism

\( L_3 \)
Length of coupler link of Four-Bar Mechanism

\( L_4 \)
Length of output link of Four-Bar Mechanism, also length of beam

M
Moment

P
Point load

R
Non-dimensional resistance curve

\( T_{in} \)
Torque input, provided by user

\( T_{out} \)
Torque providing resistance in Four-Bar Mechanism

\( Y_s \)
Yield Strength

\( \beta \)
Input arm angle for Four-Bar Mechanism

\( \Delta \)
Deflection of the free end of the beam

\( \varepsilon \)
Pre-load angle for torsional spring

\( \Theta \)
Angle defined by the displacement and length of beam

\( \theta \)

\( \theta_2 \)
Angle of link 2 relative to ground link

\( \theta_3 \)
Angle of link 3 relative to ground link

\( \theta_4 \)
Angle of link 4 relative to ground link

\( \theta_{in} \)
Angle between user defined input angle, \( \beta \), and the generated angle \( \theta_2 \)

\( \theta_w \)
Angle between user defined output angle, \( \Phi \), and the generated angle \( \theta_4 \)

\( \Phi \)
Output angle (start defined by user but displacement generated)
Chapter 1  Introduction

The intent of this thesis is to show how engineering synthesis and analysis has been used to improve one of the human race’s oldest tools, the archery bow. The primary objective of this work was to develop a bow that makes better use of a human archer’s ability to do work which is stored as potential energy. Modern compound bows were studied to understand what advances had already been made, how energy was stored and released and as to how the bow performance was rated. Literature research also showed that the people likely to use the bow have to follow rules or laws that would limit what could be changed.

This work attempts to improve upon the current design of compound bow by using a four-bar linkage to produce a non-linear mechanical advantage needed. Methods were developed for synthesizing these mechanisms for force generation. These four-bar linkages differ from conventional four-bar linkages in that a cantilevered beam is substituted for one of the ground links in the mechanism. Methods were developed for
the easy synthesis of these mechanisms from prescribed curves defining the force output of the mechanism.
Chapter 2 Literature Search

2.1 Kinesiology and Human Strength

*Kinesiology* is the study of the mechanics of muscles and bones and the interactions to produce motion and force (*Logan* and *McKinney*, 1982). One of the major points of kinesiology is the discovery that biological strength is non-linear and varies through the range of motion of a limb. During the 1960’s, the Nautilus exercise equipment popularized the development of exercise machines that produced a *resistance curve* specifically tailored to closely match the *strength curve* of humans for specific exercises (*Soper*, 1995). A *strength curve* is a relationship of human strength in a limb for a particular motion versus the motion of the limb, see Figure 1. A *resistance curve* is the non-linear force curve required to move a mechanism through its range of motion. By tailoring exercise machines to reproduce the human strength curves for specific exercises, Nautilus provided a way for people to do more work during an exercise than by performing the same exercise using free weights (*Soper* 1995).
Another person who also used this discovery but may not have known he was using it was H. Wilbur Allen. In January of 1966, Allen invented the compound bow (Figure 2) (McKinney, McKinney, 1990). This bow used principles of kinematics to create a bow that better matched human strength. This bow stored more energy and was easier to shoot than previous bows.

More recently, Galloway, Gayle, Posey, Crowe and Jones (1995) presented a paper investigating a redesign of the compound bow. Their work include the measurement of the human strength curve for an archery bow draw both statically and dynamically, as well as many new ideas concerning the kinematics of the bow and energy storage.
Figure 2: 1st Compound Bow

McKinney and McKinney (1996)
2.2 Bows

In 1545, Roger Ascham wrote the first known English work on Archery, *Toxophilus*, *The Schole of Shooting*. This was a comprehensive study of the use of the bow for hunting and as a weapon. Ascham describes exercises to improve shooting skill and overall health. This however is not the first description of bows. The origin of the bow and arrow will forever be in doubt due to the lack of evidence because, being made of natural materials, those original bows decomposed long ago. However, some historians place the origin of the bow some 25,000 years ago with the Aurignacian people (Schuyler, 1970) from evidence such as cave drawings.

Around 5000 years ago, the Egyptians established the bow as the primary weapon of war (McKinney, McKinney, 1990). This bow was probably a simple bow similar to the English longbow or the bow used by the American Indians. It was 3200 years before the bow underwent any significant changes. Around 1800 B.C. the Assyrians started using the first composite recurve bows (Haywood, Lewis, 1989). Through the use of leather, horn, wood and a recurve shape, the Assyrians created a bow that better utilized the human strength curve. Over the next 3700 years, those bows evolved into to the modern longbow and recurve bow (Figure 3).
Figure 3: Longbow, Recurve Bow, Compound Bow

Haywood and Lewis (1989)
Some terms that are used in archery need to be defined. These definitions are from Haywood and Lewis (1989) and McKinney and McKinney (1990).

Anchor Point - a fixed position against the body to which the draw hand is brought, string in hand, usually determined by aiming method, varies from archer to archer.

Belly - The side of the bow facing the archer at full draw; face.

Back - The side of the bow away from archer at full draw.

Compound Bow - A bow using a cable system attached to eccentric pulleys (or cams) mounted at the limb tips, producing peak resistance at mid-draw, then dropping off to a hold weight, that is less than the peak weight, at full draw length.

Draw - The act of pulling arrow back and storing energy in the bow.

Draw Length - Length (usually measured in inches) measured from the string along the arrow to the point where the arrow crosses the back of the bow.

Full Draw Length - Draw length of archer when at anchor point.

Let-off - the difference in force, or percentage of peak weight, between the peak weight and the hold weight of a compound bow (Wise, 1992).

Limb - The energy-storing parts of a bow, above and below the riser section.

Riser - The center, handle portion of the bow exclusive of the limbs; also: part of the bow that does not store energy and includes the handle and
arrow shelf and serves as the main support structure

Tackle - That equipment required to store energy, take aim and propel an arrow

Weight, Draw Weight - The number of pounds of force required to
draw the bowstring a given distance

In 1966, as mentioned earlier, the compound bow was invented. Designed to make bow hunting easier on the archer (Haywood, Lewis, 1989), this bow approximated the human strength curve much better than the recurve bows. Today there are dozens of companies that make and market compound bows.

2.3 Kinematics

Kinematics is the study and design of mechanisms from a point of view that focuses on the motion requirements. A mechanism is any device that uses linkages, gears, gear trains or cams to produce a specific motion (Mabie and Reinholtz, 1987).

Tidwell (1995) created a treatise on wrapping-cam mechanisms. Wrapping-cam mechanism are mechanism that use a flexible link such as a cable, chain or a belt, in contact with a cam for the purpose of producing a non-linear variation in the motion between the output and input of the system. These mechanisms are used by Nautilus in exercise machines and in compound bows to produce a non-linear resistance curve. Tidwell’s work formalized the definition of wrapping-cam mechanisms and developed
theories to synthesize mechanisms for force generation. The non-linear mechanical advantage of wrapping cam mechanisms are used to change the force curve of the compound bow to a closer match to the strength curve of a human through the draw motion.

The works presented by Soper (1995) and Soper, Scardina, Tidwell, Reinholtz, and Lo Presti (1995) are extremely useful in the context of bow design. These works present a method of directly synthesizing planar four-bar linkages for force generation. These works provided the foundation for this thesis. Soper’s work allows the free selection of several parameters and produces, in closed form, one infinity of solution linkages that satisfy a given force curve requirement.

Garguilo’s work (1995) is an excellent example of what can be done with planar four-bar linkages. In his work, Garguilo uses theories presented by Soper, et al. (1995), to design a mechanism that demonstrates the effectiveness of non-linear mechanical advantage in an exercise device. By using a planar four-bar linkage to match the strength curve, Garguilo was able to increase the amount of work done in an exercise of the wrist. Some other examples of force generating linkage mechanisms are given in Goodman(1965), Nathan (1985) and Okada(1986).
By comparison, the bow designed by Galloway, et al. (1995), uses a linear coil spring, a
system of pulleys, and a cam/linkage system to produce a resistance curve that is a good
match for the human strength curve.

2.4 Archery Rules

The rules of archery are defined by several organizations. The target archery rules are
defined by the National Field Archery Association in the United States of America. The
constitution of the NFAA has a section which defines the rules for equipment in field
(that is outdoor) archery target competition. These rules exist to insure that the bow used
is indeed a bow and not a crossbow or a gun. These rules also insure that there is no
unfair advantage provided by the tackle used by any archer. A portion of the 1992 rules
for one governing body is included in Appendix C.

In addition, hunting archery is governed by state laws in the United States. These laws
not only define the hunting seasons, but also define legal archery tackle. These laws
separate what makes a bow different from a crossbow or a gun. In addition, some states
require a certain draw weight or energy storage to hunt specific game (McKinney and

The purpose of these rules is to disallow any unfair advantage that the archer might have
over his opponent (in target archery) or over the game animal he is hunting. These rules
disallow “cocking mechanisms” which allow the hunter to rest indefinitely after storing the energy in the bow, as well as defining that the hunter must provide all the energy used by the bow to propel the arrow. The draw weight requirement in the hunting laws are to increase the probability that a successful shot kills rather than maims.

These rules do permit the use of compound bows in hunting, fishing, and target archery. The only competition where compound bows are not allowed is the Olympics. This means that an improved compound bow has a large market potential. Further evidence for the marketability of compound bows can be found by noting that companies such as Martin, Onieda, Darton, Xi, High Country, and PSE all make compound bows of the cam and eccentric wheel type.

After close examination of the rules, it became clear that as long as the bow did not do anything except store energy provided by the archer in one motion, and require that the archer be the only thing preventing the release of the arrow while aiming is performed, then the improved bow would qualify for all archery where compound bows are currently allowed. Thus compound bows (already an improvement over all non-mechanism bows) could be improved and yet still manage to fit into the rules the governing bodies define for those bows.
Chapter 3 Theory

3.1 Introduction

The four-bar linkage is used in applications ranging from automobile suspension to aircraft landing gear. These mechanisms are useful because of their ability to provide path generation, function generation and force generation through the inherent non-linearity of the motion of the mechanism.

The four-bar linkage has the ability to provide a non-linear mechanical advantage that is extremely adaptable. Soper (1995) presents a method of synthesizing four-bar linkages which use various methods to provide the required resistance on the output link of the four-bar linkages including and specializing in weighted grounded-link four-bar linkages, weighted coupler link four-bar linkages, and some spring-loaded mechanisms including torsional-sprung grounded-link four-bar linkages.
By using wrapping-cams, the resistance curve, or draw curve, of a compound bow can be matched more closely to the human strength curve. However, cams are more expensive to manufacture compared to four-bar linkages (Mabie and Reinholtz, 1987). The concept proposed here is to design a compound bow that uses four-bar linkages instead of cams, and, in doing so, to extend the work presented by Soper (1995).
Figure 5: Beam Flexure Four-Bar Linkage

The planar four-bar-linkage investigated here for use in the compound bow is termed the Beam Flexure Four-Bar Linkage, and is a planar four-bar linkage that uses a cantilevered beam in flexure in place of one of the ground links. The free end of the cantilevered beam can be approximated by circular arc motion over a small range of deflection, thus approximating a rigid link grounded through a revolute joint. The beam also serves as an energy storage device. By proper design of this linkage, energy can be stored and released in a controlled manner, providing the desired bow resistance curve.
3.2 Bows

Galloway, et al. (1995), measured the human strength curves of several subjects, both isokinetically and dynamically, and found that the human strength curve was different from the resistance curves of compound bows, and radically different from resistance curves of recurve and long bows.

Compound bows, as defined earlier, are bows that, through mechanical means, have non-linear force curves. A long bow (Figure 3) has an almost linear force curve as shown in Figure 6. As the archer draws the arrow, the required force climbs almost linearly with draw length. Draw length is measured along the arrow from the string to the back of the bow. The same figure shows a graph of the human strength curve versus the draw weight of a longbow.

The intersection of this curve and the longbow draw curve is the point that the human archer would find uncomfortable to draw the bow beyond. It is also likely that after many shots, the archer would not be able to draw the bow beyond this point. Usually, the archer must hold the arrow at full draw for a few moments to aim. Unfortunately, muscles tend to be unsteady when pushed to the strength limit and this unsteadiness adversely affect the archer's aim. This forces the archer to choose a bow that has a force curve that reaches a draw weight a few pounds lighter than his maximum strength. This is illustrated by the longbow curve in Fig. 6. That curve represents a longbow chosen by
a person with a 28 inch draw length and, as can be seen at that draw length, the bow is about 3 lbs. lighter than the person’s static strength.

![Graph showing force in pounds versus drawlength in inches]

**Figure 6: Longbow Force Curve versus Example Human Strength**
All data from existing bows taken from Wise, 1992. The data for the human is representative of a data sample from Galloway et al., 1995.

Since these graphs are measured in force on the abscissa and distance on the ordinate, the area under the curve represents stored energy. Figure 6 illustrates that a longbow does not take full advantage of the amount of energy that a human can produce in the archery draw motion, especially when the time to aim requires an archer to choose a bow with a lower draw weight.
The first bows to attempt to use more of this energy were recurve bows (Fig. 3). These bows are designed such that the string contacts the bow limbs at points that are curved convex toward the shooter. This reduces the leverage that the bow string has at the beginning of the draw against the bow limbs, because the string contacts the bow limbs at a point closer to the handle, increasing the stiffness of the system. (Fig. 7). This enables the archer to do more work at the beginning of the draw and store more energy in the bow for a given draw weight as shown in Fig. 8.

Figure 7: Recurve Bow Limbs, Rest and Draw Positions
Haywood and Lewis (1989)

The advantage of storing more energy in a bow of a given draw weight is obvious. The initial velocity of an arrow of a given mass shot from any bow is a function of only the stored energy of the bow and the efficiency of the bow in converting the stored energy into motion. By increasing the stored energy, the kinetic energy, and thus the velocity, of the arrow is increased. The velocity of the arrow is important to archers for two different reasons. The higher the velocity of the arrow, the flatter its trajectory after release, allowing the archer to make longer shots with more precision and accuracy. In
addition, for a given arrow mass, a higher velocity correlates directly to a higher energy of impact. This causes more damage to the game, making more killing shots possible.

![Graph showing Force in Pounds vs Draw Length in Inches]

**Figure 8: Recurve Bow Draw Weight versus Human Strength**

The recurve bow was a great improvement over the longbow, and became the standard bow for archery. The recurve bow is still the kind used by archers in the Olympics (Wise, 1992).

The recurve bow still did not match the human strength curve and did not take full advantage of the archers ability to do work on the bow and thus was not storing as much energy as could be stored. The next advancement in bow technology came in 1966 with
the invention of the compound bow which was shown in Figure 2 (McKinney and McKinney, 1990). The compound bow come in many sizes, draw weights, and all rely on non-linear mechanical advantage provided by either eccentric wheels, wrapping cams (cammed bows), or a combination of both.

Figure 9: Various Wrapping Cams Used in Compound Bows, Wise (1992)

Figure 10: Another Compound Bow Design, Schuyler (1970)

In addition to the wrapping cam mechanisms, some compound bows have pulleys incorporated in the design to reduce the movement of the limb tips. These wrapping-cam
mechanisms and the pulleys are used to change the force curve of the bow so that a peak force is reached somewhere in the middle of the draw, and then to reduce that force to a hold weight that is less than the peak force and allows the archer to aim with less fatigue.

Most compound bows do a good job of approximating the human strength curve. Due to the kinematics of the bow, however, all of the bows start with a zero initial force at the beginning of the draw, and since the force is near zero, very little work is done in the initial stages of the draw and release. This is because of the kinematics of the string. Because the string is perpendicular to the draw force at the beginning of the draw, it cannot provide any force to oppose that draw force.

Two typical compound bow force curves are plotted in Figure 11. This figure graphs the force curves of typical eccentric wheel and cammed bows along with a representative human strength curve.

For the new bow design, a force curve was developed to improve upon the cammed, compound bow's force curve. The new curve increases the amount of energy that can be stored in the bow and was developed using four points, the initial force at the beginning of the draw, the peak force and location, the let-off and location, and a final point to define the end of the draw. The initial draw force was chosen to be 50% of the peak force to prevent an archer from feeling like he or she was pulling against a wall. The let-
off in current bow designs varies from 70% of the peak force (McKinney and McKinney, 1990) to 15% of the peak weight (Galloway, et al., 1995).

![Force vs Drawlength Graph](image)

Figure 11: Cam and Eccentric Round Wheel Bows versus Human Strength

A let-off was chosen to be 60% of the peak force for this design at a draw length of 28 in (711.2 mm), which is average for archers (wise 1992). Then a point was defined to increase the force after the let-off to inform the archer that full draw has been achieved. This is to establish consistency of the energy in individual shots made by the archer which affects precision (Wise, 1992). The peak force location, the beginning force and location, and the location of the final point are somewhat arbitrary, but the location of the beginning of the draw can affect accuracy of the shots. A quadratic was used to define
the curve from the beginning of the draw to the peak. A 5th order curve was then used to define the rest of the curve using all the points and the slopes at the peak and the valley. The two curves were matched at the peak in position and slope. Figure 12 shows the desired force curve for the new compound bow design. The increase in stored energy was accomplished by modifying the kinematics of the compound bow using beam-flexure four-bar linkages. The dip in the force at the end of the draw is desirable to facilitate easy aiming.

![Diagram](image)

**Figure.** 12 Proposed Resistance Curve For New Bow vs. Human
3.3 Planar Four-Bar Linkages

The design of the new compound bow relies on the beam flexure four-bar linkage. The stiffness of the cantilevered beam provides the resistance to make the four-bar linkage a force generating linkage (see Figure 5).

This class of mechanism is not normally easy to synthesize in a closed form. However, a torsional-sprung ground-link four-bar linkage can be directly synthesized using the methods presented by Soper (1995) and is kinematically similar to the beam flexure four-bar linkage. To synthesize the beam flexure four-bar linkage, first a torsional-sprung ground-link four-bar linkage is synthesized for the prescribed resistance curve and for a given $k_T$ (torsional rigidity of the spring). The $k_T$ value is then used to design a beam that replaces the fourth link in the mechanism, creating the beam flexure four-bar linkage.

3.4 Linkage Synthesis

Synthesizing a force generating torsional-sprung ground-link planar four-bar linkage is directly form the work presented by Soper (1995). The kinematic model for such a linkage is shown in Figure 13. Given a force curve like the one presented in section 3.2 (Figure 12), the first step in the synthesis is to define the resistance curve. The input
moment curve is found using trigonometry, the length of the input arm and the defined force curve.

Figure 13: Kinematic Model of the Torsional Sprung Four-Bar Linkage

The torque in, $T_{in}$, is nondimensionalized with respect to the torsional stiffness of the spring $k_T$, defining the resistance curve, $R$.

Eq. 3.4.1 \[ R = \frac{T_{in}}{k_T} \]
If the unstretched angle of the spring is defined by \( \varepsilon \), then the torque on the output link is

\[
T_{\text{out}} = k_T (\theta_f - \varepsilon)
\]

Eq. 3.4.2

Using virtual work, the relationship between \( \theta_2 \) and \( \theta_4 \) can be found.

\[
T_{\text{in}} \cdot \delta \theta_2 - T_{\text{out}} \cdot \delta \theta_4 = 0
\]

Eq. 3.4.3

\[
T_{\text{in}} \delta \theta_2 = T_{\text{out}} \delta \theta_4
\]

By assuming that the mechanism is moved slowly and smoothly, dynamic effects can be disregarded and the mechanism can be assumed static at each point in its motion. In this way, virtual displacements can be said to be equivalent to actual displacements. Then equation 3.4.3 can be integrated over the range of motion

\[
\int T_{\text{in}} d\theta_2 = \int T_{\text{out}} d\theta_4
\]

Eq. 3.4.4

\[
\int R d\theta_2 = \int (\theta_f - \varepsilon) d\theta_4
\]

Eq. 3.4.5

Then, by defining \( A_R \) as the integral of \( R d\theta_2 \) and simplifying

\[
\theta_4^2 - 2 \varepsilon \theta_4 - (2A_R + \theta_{d_2}^2 - 2 \varepsilon \theta_{d_4}) = 0
\]

Eq. 3.4.6

26
An explicit equation for $\theta_4$ as a function of the area under the resistance curve can be
found analytically using the quadratic formula. In this way, the force generation
synthesis problem has been transformed into a function generation synthesis. This
function generation is a common kinematic synthesis problem that can be solved using
any one of several standard methods. Solutions to this problem are presented by Soper

This method allows the designer to see a family of potential linkages that match the
function to be generated at four design points. Among these the designer may choose the
linkage that best fits other criteria, such as strain, size limits, etc.

3.5 Spring Force and Modeling

Once the torsional-sprung mechanism is synthesized, a substitution of a cantilevered
beam can be made for the fourth link and torsional spring, provided certain assumptions
about the cantilevered beam are satisfied. If the ratio of length of the beam to height and
width of the beam is on the order of approximately 10:1 or greater, a classic beam theory
model can be applied.
For a simple, uniform, rectangular, cantilevered beam, the classic equation for the displacement of the free end with a concentrated load, \( P \), located at the free end of the beam is:

Eq. 3.5.1 \[ \Delta = \frac{PL^3}{3EI} \] (Cook and Young, 1985)

\( \Delta \) is assumed to be in the direction of the component of the load \( P \) perpendicular to the undisplaced length of the beam. This equation only gives the displacement of the beam perpendicular to the length \( L \). If one assumes small displacements along the length of the beam, the motion of the beam can be approximated by a rigid beam rotating about a revolute joint at the position of the cantilevered support.

![Figure 14: Beam Motion Model](image.png)
This allows the motion of the end of the beam to be calculated using only the perpendicular component of the force applied to the end of the beam, and gives the deflection of the beam as an angle of rotation:

\[
\Theta = \sin^{-1} \left( \frac{\Delta}{L} \right)
\]

Eq. 3.5.2

Since \( \Delta \) is known:

\[
\Theta = \sin^{-1} \left( \frac{PL^2}{3EI} \right)
\]

Eq. 3.5.3

This gives the angular displacement of the end of a beam for a given perpendicular load. This is not directly useful in the synthesis method presented earlier, since this has the end load, \( P \), of the beam included in the equation. This load can be eliminated as follows.

The equation for the stiffness of a torsional spring is:

\[
k_T = \frac{M}{\Theta}
\]

Eq. 3.5.4

This can be used with the displacement of the beam to model the cantilevered beam, the fourth link in a four-bar linkage, as a non-bending member with a torsional spring attached. Combining Eq. 3.5.4 and 3.5.3 gives
Eq. 3.5.5 \[ k_T = \frac{M}{\sin^{-1}(PL^2/3EI)} \]

By setting

Eq. 3.5.6 \[ M = PL \]

the equation becomes

Eq. 3.5.7 \[ k_T = \frac{PL}{\sin^{-1}(PL^2/3EI)} \]

Next, the small angle sine substitution is used to eliminate the transcendental term. The sine of an angle can be approximated by the measure of the angle in radians if the angle is small enough. This approximation is good for angles of -20 degrees to +20 degrees. The error in such an approximation is 2.06% at 20 degrees and 0.5% at 10 degrees.

Using this approximation, Equation 3.5.7 becomes

Eq. 3.5.8 \[ k_T = \frac{PL}{(PL^2/3EI)} \]

which simplifies to:
Eq. 3.5.9 \[ k_T = \frac{3EI}{L} \]

This result gives the "torsional" stiffness of the beam in terms of its length, modulus of elasticity, \( E \), and moment of Inertia \( I \). Once a value for \( k_T \) is chosen to use in the synthesis method and a four-bar linkage is synthesized, \( L \), the length of the grounded link will be known. Knowing \( k_T \) and \( L \), the beam can be designed using Equation 3.5.10.

Eq. 3.5.10 \[ I = \frac{k_T L}{3E} \]

3.6 Link Defect and Transmission Angle

Whenever planar four-bar linkages are synthesized, the linkage motion must be examined across the entire range of motion. This is because there is a possibility that either the mechanism passes through a point where the no torque transmission can occur, a stationary point, or that the mechanism cannot be assembled between one or more positions of the motion. These problems can be avoided by analyzing the entire range of motion of the mechanism. In this case, a Mathcad model was used to graphically analyze the motion.
3.7 Dynamics

In the work by Soper (1995), it was assumed that dynamics would not affect the resistance curve of the mechanism. This is true if the four-bar linkage being synthesized is used in a way that allows only slow, smooth motion.

In the application of a compound bow, this will only be true during the draw of the arrow. However, when the arrow is released, velocities and accelerations will be high. The inertia of the mechanism may play a large part in the conversion of the potential energy into kinetic energy of the arrow. By keeping the inertia of the moving part of this mechanism low, more potential energy can be converted to kinetic energy of the arrow.

A dynamic study of the inertia of the system would be complex and is beyond the scope of this work. In Chapter 4, the kinematic design of a compound bow is demonstrated in detail. A discussion of possible dynamic studies is included in Section 5.2, Future Work.
Chapter 4 New Bow Design

4.1 Introduction

Archery bows are usually sized to the draw length of the archer (Mckinney and Mckinney, 1990 and Haywood and Lewis, 1989), synthesizing the entire mechanism is impractical, because that would require that each bow be custom designed and built for each individual archer. It was decided to start with a general design of the mechanics of the bow chosen from an informal study of the kinematics of a design chosen and synthesize the energy storage mechanism. This would allow a system to be created which would allow semi-custom bows to be created quickly and at a reasonable cost.

4.2 Conceptual Bow Design

The final design for the bow is shown in Figures 15, 16. This design is comprised of two arms, 18 inches long, constructed of triangular aluminum sections or composite
Figure 15: Bow Design, Rest and Full Draw
Figure 16: Close up of New Bow Design at Rest Position
links. The arms are attached to a backbone, the handle riser, by revolute (hinge) joints. Typically the handle riser in compound bows is constructed from cast magnesium or aluminum. Each arm would be attached to a cantilevered beam by a loop of steel cable. The cantilevered beams would likely be made of a composite material such as an epoxy-glass mixture. Composite beams are used with success in current designs and seem to have the right combination of energy storage and damping. As stated earlier, this mechanism, that is the arm of the bow, the cable, the cantilevered beam, and the backbone of the bow would make up the four-bar linkage responsible for the tailored non-linear mechanical advantage curve. The bow design completed by attaching the string to the free ends of the two arms.

4.3 Strength Curve and Transformation to Resistance Curve

In studying the work of Galloway, et al. (1995), three points became clear. First, the current design of compound bow was not using the full strength of human beings. Second, the data collected for that paper represented only a small sample size. Third, the data did not fully agree with current design philosophy. The let-off of the compound bow design in that paper did not agree with current designs (that design had a let-off weight of 15% of the peak weight, whereas current designs have a let-off weight of 50 - 70% of peak weight, McKinney and McKinney, 1990).
Although limited in scope, the study by Galloway, et al., did indicate that current designs could be improved by storing more energy in the beginning of the draw. This can be accomplished in the new design of bow by pre-loading the arms of the bow with mechanical stops, and attaching the string to the arms so that, with the bow at rest, the string is not taut, but slack. When the archer draws the arrow the string gets taut. This allows the draw weight to be greater than zero at the start of the draw. The mechanical stops are not shown in Figs. 15 and 16, but would either consist of a cable or strap that would limit the motion of the arm during the release of the arrow.

The force curve presented in Chapter 3 (Figure 12), was used to synthesize the new bow design. The draw weight stays near the peak weight for a larger portion of the draw than any existing bow. The let-off follows a typical compound bow let-off. This was done to retain energy at the end of the draw and because the strength curves taken statically by Galloway, et al. (1995) seem to indicate more strength at the end of the draw motion than the dynamic strength curves taken. Since the end of the draw is where the archer stops moving to aim and release the arrow, the static strength seemed more appropriate.

A Fortran program was developed (Appendix A) to manipulate specific parameters used in the design of bows. These parameters were draw weight (peak), draw length (including start point and anchor point relative to the back of the bow), and bow size (length of the measurement from tip of one arm, where the string attaches, to the other arm). In addition, the start angle of the string (gamma) and the start angle of the arms of
the bow (rho) were parameters chosen to be specified by the user of the program to manipulate the kinematics and the forces in the bow. Those two parameters would normally be something set in the final design of the bow and not modified from user to user. The kinematics of the bow was analyzed by the program as a crank-slider mechanism. This can be seen if one takes the bow and cut it in half along the axis of symmetry (see Figure 15). The program only analyzes the motion of the string and arms of the bow.

For a bow of 60.0 inch (1524.0 mm) length and a 28 inch (711.2 mm) draw (an average bow), the design chosen had arms 18.0 inches (457.2 mm) long that were oriented at an angle of 15.0 degrees from vertical. The string was chosen to start an angle of 10.0 degrees from vertical at the rest position of the bow. The draw length begins at 8.0 inches (203.2 mm). This gives the location of the arm pivots 1.868 inches (47.447 mm) ahead of the point from which the draw length is measured and 12.613 inches (320.370 mm) from the centerline of the bow in the vertical direction. The point from which the draw length is measured is usually the farthest most point from the archer. On most bows this is the back of the bow at the arrow rest --- this is the traditional draw length measure point. The draw weight was chosen to be 40.0 lbs. (178 N).

This design was chosen as a compromise between minimizing the motion of the arms, the length of the arm and maximizing the angle the string makes with arrow at full draw. The minimization of arm motion was to minimize the inertia of the bow. The string
angle is important because small an angle will pinch the fingers of the draw hand and affect the release and subsequent precision of the shot (Wise, 1992). The kinematics of the bow made this difficult, because of the interaction of those three design parameters. Minimizing the length of the arms decreases the angle the string makes with the arrow, and this increases the angle the arm have to rotate. While the focus of this work was not the dynamics of the mechanism, the inertia of the system was kept in mind and thus the arm length and the motion of those arms were minimized.

Using the program listed in Appendix A, the force curve was transformed to a moment curve for the arms of the bow (Figure 17).

![Graph showing the transformed resistance curve and fifth order polynomial fit.](image)

Figure 17: Strength Curve Data
4.4 Linkage Synthesis

A MathCad 5.0 model for torsional sprung four-bar linkage was adapted from a model for weighted ground-link model created by Soper, Scardina, Tidwell, and Reinholtz and presented by Soper (1995). This model is presented in Appendix B.

The torsional spring constant, $k_T$, was chosen to be 75 ft-lbs. (102 N-m) per degree and a pre-load angle of -2.0 degrees was selected. This value for $k_T$ allowed the most flexibility in the synthesis of the four-bar linkages. That value for $k_T$ and the transformed force curve were entered into the model. That model then converted the moment curve to a resistance curve by dividing by $k_T$. After several dozen iterations of the synthesis program changing design points $k_T$, and trying to match the output force curve to the desired force curve, it was determined that the linkage was not quite capable of reproducing the given force curve exactly, but, with the addition of mechanical stops, the design is quite acceptable. The best design that was achieved is presented here.

This linkage has the ground link ($L_1$) oriented 30 degrees from horizontal and scaled to 9.0 inches long ($L_1 = 9.0$ in. (228.6 mm) and oriented 30 from horizontal). Both of these parameters are free choices in the method presented by Soper (1995) and do not affect the output resistance curve. The resulting lengths and orientations of the other links are:
\[ L_2 = 3.140 \text{ in. (79.76 mm)} \]
\[ L_3 = 9.517 \text{ in. (241.73 mm)} \]
\[ L_4 = 5.467 \text{ in. (138.86 mm)} \]
\[ \theta_{in} = 316.813^\circ \text{deg} \]
\[ \theta_{w} = 100.278^\circ \text{deg} \]

The angle \( \theta_{in} \) is the angle between the bow arm and the input link \( L_2 \). The angle \( \theta_w \) is the angle of orientation of the cantilevered beam at rest (\( L_4 \)) which is pre-loaded -2.0 degrees.

Figure 18: Solution Linkage
For this length, the pre-load deflection of the beam is 0.19 inches (4.83 mm). The total deflection of the beam for the entire motion of the bow is approximately 0.69 inches (17.53 mm). The resistance curve produced by this mechanism and the resulting force curve produced by the bow are shown in Figure 19 and Figure 20, respectively.

![Graph showing resistance curve of solution](image)

**Figure 19: Resistance Curve of Solution**

As can be seen from these graphs, the mechanism does not provide the increase in draw weight at the end of the draw nor does it provide any way to keep the archer from over drawing the bow and causing the mechanism to toggle (i.e., the angle between the input link and the coupler link goes through 180 degrees, causing the moment to change signs
and thus direction). The incorporation of an adjustable mechanical stop for each arm solves both problems. The most likely position for these stops would be just below the arm of the bow as illustrated in Fig. 16. The design would also need an adjustable stop to hold the arms of the bow at the pre-loaded, rest position (not illustrated).

Figure 20: Draw Weight of the New Bow
4.5 Validity Analysis

To determine the validity of the substitution of the cantilevered beam for a pinned link and torsion spring, the synthesized linkage was analyzed using classical beam theory and kinematics. From Beer and Johnston (1981), classic beam theory states that the deflection of the end of a cantilevered beam under the influence of a point load is only in the direction perpendicular to the axis of the beam, and that there is no deflection in direction parallel to the unbent length of the beam. Since the mechanism is already

![Diagram of Crank Slider Substitution](image)

Figure 21: Crank Slider Substitution

synthesized, this theory can be used with the existing mechanism to analyze and compare the force curve predicted by the Mathcad model and the one produced using beam theory.
Substituting the classic beam model into the slider-crank mechanism model, the mechanism is reoriented such that the beam is parallel to the y axis and the origin of the reference plain is at the pivot between the input link and ground. The problem of analyzing the kinematics becomes one of analyzing the kinematics of a crank-slider mechanism with the force at the slider produced by the deflection of the beam.

Figure 22: Comparison of Force Curves

Mabie and Reinholtz (1987) provide an analysis method for these mechanisms. The analysis steps are as follows. First, calculate the force at the slider using Eq 3.5.1 and
position analysis. Second use trigonometry and the position analysis to transform the force required to deflect the beam the amount $\Delta$, from the position analysis, through the mechanism to a moment at the input link, and then on through the bow mechanics to get a force required to draw the bow. As can be seen in Figure 22, for this problem the synthesis method described here agrees almost identically to the analysis using classic beam theory. The average error is for this synthesis method over the range of motion of the bow (8 - 31 inches, 203.2 - 787.4 mm) is 0.3 % and the maximum error is 5.3 %.

4.6 Strength Considerations.

After generating this linkage and integrating it into the bow design, a preliminary strength analysis was performed. The two primary areas of interest were link 3 (the coupler link) and the beam. These were of interest because the ratio of motion between the arm and the beam was on the order of 7.5:1. Knowing that the ratio of forces would be similar to the inverse of the ratio of motion it was obvious that these two members would be highly loaded.

The Mathcad model provides a method of finding the internal force in link 3, the coupler link. The peak force was found to be 1168 lbs. (5195.5 N). The 1987 Aircraft Spruce and Specialty Company catalogue was consulted and it was found that aircraft quality cable (already used in existing compound bows) had sufficient strength and stiffness for use as
the coupler link. A loop of 7x19 1/8 inch (3.175 mm) stainless steel cable has a breaking strength of 4000 lbs. (17790 N); a single cable breaks at 2000 lbs. (8895 N)

The cantilevered beam is loaded by the coupler link such that the loading is nearly perpendicular to the free end of the beam (as a classical point load) for the entire range of motion. Thus the stress in the beam can be analyzed using classical methods like those presented in Beer and Johnston (1981) or Cook and Young (1985). These methods result in a bending equation relating the modulus of rigidity (E), yield strength (Yₖ), and the ratio of length to height.

Eq 4.6.1   \[ Yₖ \geq \sigma = (3Ed/L^2)\Delta \]

From Equation 3.5.9, \( k_r = 3EI/L \). Equation 4.7.1 and the equation for the moment of inertia of a rectangular beam \( I = 1/I2 bh^3 \), can be used derived the following result:

Eq. 4.6.2   \[ d \leq (L^2 Yₖ) / (3 E \Delta Fₖ) \]

where \( d = h/2 \). If \( Fₖ \) (the factor of safety) is chosen to be 2.0, a rectangular cross section is chosen and glass epoxy is used for the beam \( (Yₖ = 150 \times 10^3 \text{ psi for tension and compression and E} = 7.8 \times 10^6 \text{ psi, Jones, 1975}) \) \( (Yₖ = 1.034 \times 10^9 \text{ Pa and E} = 5.38 \times 10^{10} \text{ Pa}) \) then:
\[ d < 0.139 \text{ in. (3.53 mm)} \text{ and thus } h = 2d < 0.278 \text{ in. (7.06 mm)} \]

A reasonable choice for \( h \), the height of the beam, would be approximately 3/16 in. (4.76 mm). A width of 0.383 in. (9.73 mm) give the beam the proper stiffness and a one use factor of safety of approximately 3. This study of strength of materials was not intended to determine final design parameters, it was more to determine if this design would be feasible from material point of view. The final design should include a full fatigue life analysis, but that was beyond the scope of this work.

The actual design of the beam would be longer than the working length by approximately 1.75 in. (44.45 mm). This would allow approximately 1.5 in. (38.1 mm) for clamping the fixed end and 0.25 in. (6.35 mm) for a molded hook to attach the loop of cable (the coupler link). It was also realized that the arm would have to be carefully designed to withstand the loading by the coupler link (the loop of cable). This should not be difficult, but it is a design detail that is not included in this work.

4.7 Deflection Considerations

Another concern was the non-linearity of the beam in bending. Classic beam theory is valid for deflections of the beam less than 10% of the length of the beam. This is because the small angle sine substitution is used in the derivation of the equations for
deflection of the beam. Timoshenko (1976), presents this derivation in detail. The ratio of the deflection of the beam and the length is 12.21%. This is obviously in the non-linear range of the beam. Using the derivations in Timoshenko, the beam will deflect approximately 2% less than predicted by the classic beam theory for the given shear loading at the end of the draw. This is favorable to the design as the synthesized mechanism produced more mechanical advantage and thus less force than specified near the end of the draw (see Figure 21), but the beam will be stiffer, offsetting the mechanical advantage.

4.8 Adaptability

One of the goals for this bow design was easy adaptability. One of the goals was to create a design that would be easily adaptable from archer to archer. This design is adjustable in draw weight by replacing the beams with ones of different stiffness. By changing the stiffness of the beams by 40%, the entire force curve is similarly changed by 40% without changing the shape of the curve. However, the draw length would only be marginally adjustable for each bow (the arm stops could be adjusted, but this would also affect the amount of let-off). To adjust the draw length by any large amount, the arms and the mechanism would have to be increased in size. It is believed that this is an an acceptable method, since this is what companies who manufacture bows do currently. Each bow is designed for a specific range of draw length, for example 26-28 in. (660.4 - 711.2 mm), and a certain peak weight.
4.9 Comparison of Energy Storage

Using piecewise integration, the data obtained from the Mathcad model and the data presented earlier for an existing compound bow (Figure 11) are used to find the amount of energy that can be stored in the new bow, and an existing bow. Comparing the new bow to a cam compound bow with the same draw length and peak weight, the energy stored is:

New Bow \hspace{1em} (42 lb (186.8 N) peak force) = 59.1 \text{ ft-lbs (80.1 N-m)}

Cam Bow \hspace{1em} (42 lb (186.8 N) peak force) = 51.0 \text{ ft-lbs (69.1 N-m)}

This gives a percentage gain in energy of 15.9% ! For a given arrow mass, and assuming that the new bow is as efficient as the existing compound bows, then the gain in arrow velocity is 7.65%. If the example compound bow shoots an arrow of a given mass 250 ft/s (76.2 m/s), the new bow would shoot the same arrow approximately 270 ft/s (82.3 m/s).
Figure 23: New Bow vs. Cam Bow

Figure 23: New Bow Force Curve vs. Cam Bow of Same Peak Weight
Chapter 5 Conclusions

5.1 Conclusions

It is clear from this work that four-bar linkages that use a cantilevered beam as the fourth
link (output link) are viable mechanisms for use in force generation. The methods
presented here are straightforward and make synthesizing the Beam-Flexure Four-Bar
Linkages simple.

The compound bow design holds much promise. After many iterations, it was found
that, for this application, the beam flexure four-bar linkage did not exactly produce the
desired resistance curve, however, the incorporation of mechanical stops enables a bow
design that is capable of matching the desired force curve well. As for attaining the goal
of storing more energy in the bow, this design “hits the bulls eye.” The results show it
does store more energy than the typical cammed compound bow (15.9% more) and a
large part of that is the use of the pre-loading of the bow while maintaining a slack string.
5.2 Future Work

This work should inspire others to investigate objects and tools that many consider fully developed. Concerning the bow design there is much work left to be done. The first is a more thorough and in-depth study of the kinesiology of the human body for the archery draw. A complete study will help to better define the non-dimensional resistance curve which is critical to the final design of the new bow.

From a mechanics of materials viewpoint, this bow is far from finished. The design of the bow arms need to be completed and optimized for low inertia and fatigue life. Optimizing the bow for low inertia will allow a high efficiency of energy transfer from potential to velocity of the arrow.

Once the design is finalized from an inertia and materials point of view, a dynamic study must be undertaken to see how efficient the new design is at converting the potential energy into kinetic energy of the bow. One method of doing this was presented by Suh and Radcliffe (1978). Suh and Radcliffe present a method for determining the time response of a mechanism to a given driving force or torque by equating the inertia of the system to that of an imaginary flywheel with a time dependent inertia, $I^*$, that has the same motion as one of the pinned links of the mechanism being analyzed (same $\alpha$, $\omega$, $\theta$). The method involves equating the imaginary flywheel’s kinetic energy and the kinetic
energy of the mechanism. From this equation $I^*$ can be calculated for the mechanism. In
the same manner, by equating the power inputs of the two systems, an equivalent time
dependent torque can be calculated, $T^*$. These values can then be used to calculate the
time response of the equivalent flywheel using numerical integration. While this method
would be relatively easy for this bow, it would not be so easy for a compound bow that
uses cables, cams, and pulleys. Thus, a comparison of this design's dynamics to that of
another bow could be quite complex and difficult. Another more constructive approach
would be to utilize some of the powerful dynamic modeling software available to
compare this bow to current designs.

Also, the design of the cantilevered beam needs to be investigated to determine if a
laminate or a metal spring would be more beneficial to the efficiency and strength of the
bow. Other ideas that might be useful include: using multi-leaf leaf springs in place of
the cantilevered beam to further tailor the force curve, shaping the beam to get non-
linear bending stiffness by tapering the beam, placing extra supports for the beam to
change the stiffness of the beam after a predetermined amount of displacement, etc. All
of these could be investigated to further increase the energy stored in the bow.

Finally, this work allows easy synthesis of yet another class of mechanism for use in
force generation. While not as adaptable as those weight-loaded mechanisms presented
by Soper (1995), these mechanisms are still simpler and cheaper to build than the
wrapping cams used in existing compound bows and presented in the work by Tidwell (1995).
References

*Aircraft Spruce and Specialty Company Catalogue*, 1987, Fullerton California.


Appendix A  Transformation Program

*234567
   real  gams,ts,la,ya,rhos,lr,ld,xp,yp,ls,faa,fap,m,pi,ln,rhon,
       gamm,A,B,C,Z,t,thetas,thetaa,ips,mom,strength,wrt
   integer i

   open (4,file='bow1.dat')
   open (9, file='bowout.dat')

* Establish output set

   write (9,*),'strength curve used is bow1.dat'

* Establish bow parameters

* *
*  gams = initial angle string half makes with arrow
*  rhos = initial angular deflection of arm from vertical
*   positive is toward shooter
*  ya   = half of the length of the bow (distance from end of
*        string to end of string)
*  la   = arm length
*  lr   = rest length (distance from back to string at rest)
*  ld   = traditional draw length
*  wt   = peak weight of draw

   gams = 10.0
   rhos = 15.0
   ya   = 30.0
   la   = 18.0
   lr   = 8.0
   ld   = 28.0
   wt   = 40.0

* Establish data set

   write (9,*),'units are: length-inches, force-lbs, angle-degrees'

   write (9,*), 'gams =',gams
   write (9,*), 'rhos =',rhos
   write (9,*), 'ya =',ya
   write (9,*), 'la =',la
   write (9,*), 'lr =',lr
write (9,*) 'ld  =', ld  
write (9,*) 'wt  =', wt  

pi = 3.14159265

* convert angles from degrees to radians

gams = gams * pi / 180.0
rhos = rhos * pi / 180.0

* Establish the location of the arm pivots xp, yp and lps = lp**2

xp = lr - (ya * sin(gams)) - (la * sin(rhos))
yp = ya - (la * cos(rhos))
lps = xp**2 + yp**2

write (9,*) 'xp  =', xp, ' yp  =', yp

* Find string length

ls = ya / cos(gams)

write (9,*) 'string length =', ls*2.0
write (9,50)

* Start loop

do 40 i=8,34,2

* Perform position analysis on bow mechanics by using mabie/reinholtz
* substitution. Z is intermediary term

ln = real(i)
Z = la**2 - ln**2 - lps - ls**2 + 2.0*ln*xp
A = Z + 2.0*ln*ls - 2.0*ls*xp
B = 4.0*ls*yp
C = Z - 2.0*ln*ls + 2.0*ls*xp
t = (0.0 - B + sqrt(B**2 - 4.0*A*C)) / (2.0*A)
gamm = atan(t) * 2.0 - pi/2.0
rhon = acos((ls*cos(gamm) - yp) / la)

* Read strength curve data
read (4,*) strength

* Calculate  ts = string tension
*      faa = axial force in arm
*      fap = perpendicular force at arm end
*      mom = moment around arm bearing

    ts = strength / (2.0 * sin(gamn)) * wt

    faa = ts * cos(rhon + gamn)
    fap = ts * sin(rhon + gamn)
    mom = l1a / 12.0 * fap

* Convert angles to degrees

    gamn = gamn * 180.0/π
    rhon = rhon * 180.0/π

* output data

    write (9,60) ln,gamn,rhon+90.0,ts,faa,fap,mom

40  continue

    stop

50  format(6x,'n',4x,'gamn',4x,'rhon',6x,'ts',5x,'faa',5x,'fap',5x,
        'mom')

60  format(7f8.3)

    end
Appendix B

Torsional-Sprung-Grounded-Link Four-Bar Synthesis

Created on MathSoft Mathcad version 5

Problem Setup
Strength Curve Data:

<table>
<thead>
<tr>
<th>x</th>
<th>deg</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>104.895</td>
<td>36.571</td>
<td></td>
</tr>
<tr>
<td>106.214</td>
<td>44.239</td>
<td></td>
</tr>
<tr>
<td>107.558</td>
<td>50.645</td>
<td></td>
</tr>
<tr>
<td>108.924</td>
<td>55.844</td>
<td></td>
</tr>
<tr>
<td>110.310</td>
<td>59.883</td>
<td></td>
</tr>
<tr>
<td>111.715</td>
<td>62.791</td>
<td></td>
</tr>
<tr>
<td>113.138</td>
<td>64.597</td>
<td></td>
</tr>
<tr>
<td>114.579</td>
<td>65.319</td>
<td></td>
</tr>
<tr>
<td>116.039</td>
<td>64.992</td>
<td></td>
</tr>
<tr>
<td>117.518</td>
<td>64.157</td>
<td></td>
</tr>
<tr>
<td>119.017</td>
<td>63.257</td>
<td></td>
</tr>
<tr>
<td>120.536</td>
<td>62.105</td>
<td></td>
</tr>
<tr>
<td>122.078</td>
<td>60.497</td>
<td></td>
</tr>
<tr>
<td>123.643</td>
<td>58.244</td>
<td></td>
</tr>
<tr>
<td>125.233</td>
<td>55.223</td>
<td></td>
</tr>
<tr>
<td>126.850</td>
<td>51.421</td>
<td></td>
</tr>
<tr>
<td>128.496</td>
<td>48.957</td>
<td></td>
</tr>
<tr>
<td>130.173</td>
<td>42.148</td>
<td></td>
</tr>
<tr>
<td>131.885</td>
<td>37.479</td>
<td></td>
</tr>
<tr>
<td>133.634</td>
<td>33.660</td>
<td></td>
</tr>
<tr>
<td>135.424</td>
<td>31.623</td>
<td></td>
</tr>
<tr>
<td>137.259</td>
<td>32.000</td>
<td></td>
</tr>
<tr>
<td>139.144</td>
<td>37.611</td>
<td></td>
</tr>
<tr>
<td>141.085</td>
<td>48.446</td>
<td></td>
</tr>
</tbody>
</table>

\[
\text{degree} := 5
\]

The degree of the polynomial fit to the input strength curve data

\[
kt = \frac{75}{\text{deg}}
\]

The strength curve data here is nondimensionalized by the spring constant \(kt\) and is defined as the moment around the input arm pivot.

\[
\text{bowoutfi.dat}
\]

Discrete Data Points. A polynomial of degree "degree" will be fit to the data in order to develop a functional relationship

These are the extents of the input data points. They are used as integration limits below.

Setting up matrix operations

\[
\theta_{\min} = \min(x) \quad \text{\(\theta_{\min} = 104.895\ \text{deg}\)}
\]

\[
\theta_{\max} = \max(x) \quad \text{\(\theta_{\max} = 141.085\ \text{deg}\)}
\]

\[
N := \text{length}(x) \quad N = 24
\]

\[
m := 0..N - 1
\]
Curve Fitting

(Matrix Least Squares method, using the pseudo-inverse)

\[ k = 1 \text{ degree} \quad X_{m,0} = 1 \]

\[ X^{<k>} := \begin{pmatrix} x^k \end{pmatrix} \] Fills the \( k \)th column with the vectorized values of \( x^k \).

\[ b := (X^T \cdot X)^{-1} \cdot (X^T \cdot y) \] LEAST SQUARE POLYNOMIAL COEFFICIENTS

\[
\begin{bmatrix}
-91.931 \\
218.016 \\
207.024 \\
98.381 \\
-23.388 \\
2.224
\end{bmatrix}
\]

\[ S(\beta) = b_0 + \sum_{k} b_k \beta^k \] The nondimensional strength curve function

Figure A-1, Strength Curve Data
Input start angle and precision point locations:

\[
\begin{array}{c}
\theta_{\text{min}} \\
108.\text{deg} \\
117.\text{deg} \\
128.\text{deg} \\
140.\text{deg} \\
\theta_{\text{max}}
\end{array}
\quad \begin{array}{c}
141.085 \\
140 \\
128 \\
117 \\
108 \\
\beta_0 \text{ is the handle start angle. } \\
\beta_4 \text{ are four } "\text{Precision Points}" \text{ in the handle angle.}
\end{array}
\]

\[
\kappa = 1.5
\]

\[
\begin{array}{c}
S(\beta) \\
0.012 \\
0.015 \\
0.011 \\
0.01 \\
0.011
\end{array}
\quad A_R = \int_{\beta_0}^{\beta_4} S(\beta) \, d\beta
\]

\[
\begin{array}{c}
0 \\
3.279 \\
16.23 \\
31.247 \\
41.565 \\
42.693
\end{array}
\]

\[
A_R \text{ is the area under the strength curve for the values of } \beta \text{ given above. This is a nondimensional measure of the input work.}
\]

Knowns: \[ l_{\text{in}} := 18.0 \quad \text{The length of the handle.} \]

Free parameter choices: \[ l_w := 1.0 \quad \text{The length of the dummy arm and the start angle for the torsional spring and the unsprung angle.} \]

\[
\phi_0 := 0.0\text{.deg} \\
\varepsilon := -2.0\text{.deg}
\]

Synthesis Section

\[
\phi_q = \sqrt[\varepsilon^2 + (\phi_0)^2 - 2 \varepsilon \phi_0 + 2 \cdot A_R + \varepsilon}
\]

By using a dummy arm for the torsional spring, the model is able to synthesize the linkage from any convenient angle measures and calculate the rotational angle between the dummy arm and the fourth link.
Transformation to a four position body guidance problem:

\( n := 2 \ldots 4 \) 'n' is the number of the precision points. \[ \Phi = \begin{bmatrix} 0 \\ 0.785 \\ 2.754 \\ 4.309 \\ 5.185 \\ 5.275 \end{bmatrix} \] \( ^\circ \text{deg} \)

\[ \delta_n = e^{i(180 \text{ deg} - \Phi_n)} - e^{i(180 \text{ deg} - \Phi_1)} \]

\( \delta \) is a vector (complex valued).

\[ \begin{bmatrix} 0.001 + 0.034j \\ 0.003 + 0.061j \\ 0.004 + 0.077j \end{bmatrix} \]

\[ \alpha_n := \beta_n - \beta_1 + \Phi_1 - \Phi_n \]

\( \begin{bmatrix} \alpha_n \end{bmatrix} \) \( ^\circ \text{deg} \)

\[ \begin{bmatrix} 7.031 \\ 16.476 \\ 27.6 \end{bmatrix} \]

\( \alpha \) is the relative rotation of an Imaginary body between precision points.

**QUASI-LOOP-CLOSURE EQUATION**

\[ \Lambda_2 := \begin{bmatrix} i \alpha_3 - 1 & \delta_3 \\ i \alpha_4 - 1 & \delta_4 \end{bmatrix} \]

\( \Lambda_2 = 0.007 + 0.004j \)

\( \Delta \)'s are complex valued vector coefficients in an equation resembling loop closure in \( \psi \).

\[ \Lambda_3 := \begin{bmatrix} i \alpha_2 - 1 & \delta_2 \\ i \alpha_4 - 1 & \delta_4 \end{bmatrix} \]

\( \Lambda_3 = -0.007 - 0.003j \)

\[ \Lambda_4 := \begin{bmatrix} i \alpha_2 - 1 & \delta_2 \\ i \alpha_3 - 1 & \delta_3 \end{bmatrix} \]

\( \Lambda_4 = 0.002 + 9.817 \times 10^{-4}j \)

\[ \Lambda_1 = -\Lambda_2 - \Lambda_3 - \Lambda_4 \]

\( \Lambda_1 = -0.002 - 0.001j \)
Choose a range for iteration:

\[ \psi_2 = -2.5 \text{ deg., } -2.4 \text{ deg., } 3.5 \text{ deg} \]

Iterating on \( \psi_2 \) yields a solution set of all four bar linkages which achieve the precision point values given above.

Possible values of \( \psi_2 \) are 0 to 360 degrees. Each value will yield two solutions for the problem, a forward- and cross-closure solution. The second solutions may be viewed by changing the sign of the quadratic equation below.

Solving the quasi-loop closure problem:

\[
\Omega_1 := \arg(\Delta_1), \quad \Omega_2(\psi_2) = \psi_2 + \arg(\Delta_2)
\]

\[
r_1 := \left(\text{Re}(\Delta_1)^2 + \text{Im}(\Delta_1)^2\right)^{\frac{5}{2}}
\]

\[
r_2 := \left(\text{Re}(\Delta_2)^2 + \text{Im}(\Delta_2)^2\right)^{\frac{5}{2}}
\]

\[
r_3 := \left(\text{Re}(\Delta_3)^2 + \text{Im}(\Delta_3)^2\right)^{\frac{5}{2}}
\]

\[
r_4 := -\left(\text{Re}(\Delta_4)^2 + \text{Im}(\Delta_4)^2\right)^{\frac{5}{2}}
\]

\[
A(\psi_2) = r_3^2 - r_1^2 - r_2^2 - r_4^2 + 2 \cdot r_1 \cdot r_2 \cdot \cos(\Omega_2(\psi_2) - \Omega_1) + 2 \cdot r_1 \cdot r_4 - 2 \cdot r_2 \cdot r_4 \cdot \cos(\Omega_2(\psi_2) - 1
\]

\[
B(\psi_2) := 4 \cdot r_2 \cdot r_4 \cdot \sin(\Omega_2(\psi_2) - \Omega_1)
\]

\[
C(\psi_2) := r_3^2 - r_1^2 - r_2^2 - r_4^2 + 2 \cdot r_1 \cdot r_2 \cdot \cos(\Omega_2(\psi_2) - \Omega_1) - 2 \cdot r_1 \cdot r_4 + 2 \cdot r_2 \cdot r_4 \cdot \cos(\Omega_2(\psi_2) - 1
\]

\[
t(\psi_2) := \frac{-B(\psi_2) + \sqrt{B(\psi_2)^2 - 4 \cdot A(\psi_2) \cdot C(\psi_2)}}{2 \cdot A(\psi_2)}
\]

**SOLUTION SWITCHING:** Changing the sign on the quadratic equation here will result in the second set of solutions.
\[\Omega_4(\psi_2) = 2 \tan(\psi_2) + \Omega_1\]
\[\psi_4(\psi_2) = \Omega_4(\psi_2) - \arg(\Lambda_2)\]

\[T(\psi_2) = \frac{e^{i \cdot \psi_4(\psi_2)} - 1}{e^{i \cdot \psi_2} - 1}\]

\[Z(\psi_2) := \frac{\delta_4 - T(\psi_2) \cdot \delta_2}{e^{i \cdot \psi_2} - T(\psi_2) \cdot (e^{i \cdot \psi_2} - 1)}\]

\[M(\psi_2) := \frac{\delta_2 - (e^{i \cdot \psi_2} - 1) \cdot Z(\psi_2)}{e^{i \cdot \psi_2} - 1}\]

**SOLUTION DYAD** Vectors $Z$ and $M$ are the solution links to the body guidance problem. In body-guidance space, link $Z$ is the "rotating" link and link $M$ is the "fixed" link.

Solutions to the function generation (mechanical advantage) synthesis problem are obtained by re-inverting the mechanism.

\[Z'(\psi_2) := -Z(\psi_2) \cdot e^{i \cdot \Phi_1}\]
\[M'(\psi_2) := -M(\psi_2) \cdot e^{i \cdot \Phi_1}\]

Links 2 and 3 correspond to links $Z'$ and $M'$ respectively.

\[L_2(\psi_2) := \left(\text{Re}(Z(\psi_2))^2 + \text{Im}(Z(\psi_2))^2\right)^{\frac{1}{2}}\]
\[L_3(\psi_2) := \left(\text{Re}(M(\psi_2))^2 + \text{Im}(M(\psi_2))^2\right)^{\frac{1}{2}}\]

\[\theta_2(\psi_2) := \arg(Z'(\psi_2))\]
\[\theta_3(\psi_2) := \arg(M'(\psi_2))\]
BURMESTER POINT PAIRS

\[ X_1(\psi_2) = L_2(\psi_2) \cdot \cos(\theta_2(\psi_2)) \]
\[ X_2(\psi_2) = L_2(\psi_2) \cdot \cos(\theta_2(\psi_2)) + L_3(\psi_2) \cdot \cos(\theta_3(\psi_2)) \]

\[ Y_1(\psi_2) = L_2(\psi_2) \cdot \sin(\theta_2(\psi_2)) \]
\[ Y_2(\psi_2) = L_2(\psi_2) \cdot \sin(\theta_2(\psi_2)) + L_3(\psi_2) \cdot \sin(\theta_3(\psi_2)) \]

Figure A-2, Burmester Curves
CHOOSING THE SOLUTION

Choose a solution: \( \psi_2 := -0.5 \text{ deg} \)

Choose a rotation for the ground link: \( \chi := -30 \text{ deg} \)

Select the scale for the design: \( l_1 = 9 \)

\[
L_1 := 1 \quad l_2 = L_2(\psi_2) \cdot l_1 \quad l_3 := L_3(\psi_2) \cdot l_1
\]

\[
\theta_{in} := \beta_{in} - \theta_2(\psi_2) - \chi \quad \theta_4 := \arg \left( l_2 e^{i \theta_2(\psi_2)} + l_3 e^{i \theta_3(\psi_2)} - l_1 \right)
\]

\[
\theta_w := \theta_4 - \Phi_{w} + \chi
\]

\[
l_4 := \sqrt{\text{Re} \left( l_2 e^{i \theta_2(\psi_2)} + l_3 e^{i \theta_3(\psi_2)} - l_1 \right)^2 + \text{Im} \left( l_2 e^{i \theta_2(\psi_2)} + l_3 e^{i \theta_3(\psi_2)} \right)^2}
\]

THE SOLUTION DESIGN PARAMETERS

Fixed links and angles for the scaled solution

\( l_1 = 9 \quad l_{in} = 18 \quad \theta_{in} = 316.813 \text{ deg} \quad \chi = -30 \text{ deg} \)

\( l_2 = 3.138 \quad l_w = 12 \text{ ft}^{-1} \quad \theta_w = 100.278 \text{ deg} \)

\( l_3 = 9.517 \)

\( l_4 = 5.467 \)
Analysis Section

Now the solution fourbar linkage is plotted:

\[
\begin{align*}
\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} &= \begin{bmatrix} l_2 \cdot \cos(\theta_2(\psi_2) + \xi) \\ l_2 \cdot \sin(\theta_2(\psi_2) + \xi) \end{bmatrix}, \\
\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} &= \begin{bmatrix} x_2 + l_3 \cdot \cos(\theta_3(\psi_2) + \xi) \\ y_2 + l_3 \cdot \sin(\theta_3(\psi_2) + \xi) \end{bmatrix}, \\
\begin{bmatrix} x_4 \\ y_4 \end{bmatrix} &= \begin{bmatrix} l_1 \cdot \cos(\xi) \\ l_1 \cdot \sin(\xi) \end{bmatrix}, \\
\begin{bmatrix} x_{\text{in}} \\ y_{\text{in}} \end{bmatrix} &= \begin{bmatrix} l_{\text{in}} \cdot \cos(\beta_1) \\ l_{\text{in}} \cdot \sin(\beta_1) \end{bmatrix}, \\
\begin{bmatrix} x_{\text{out}} \\ y_{\text{out}} \end{bmatrix} &= \begin{bmatrix} l_1 \cdot \cos(\xi) + l_w \cdot \cos(\phi_1) \\ l_w \cdot \sin(\phi_1) + l_1 \cdot \sin(\xi) \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
m_1 &= \frac{y_2 - y_1}{x_2 - x_1}, \quad m_2 = \frac{y_3 - y_2}{x_3 - x_2}, \quad m_3 = \frac{y_4 - y_3}{x_4 - x_3}, \\
m_4 &= \frac{y_1 - y_4}{x_1 - x_4}, \quad m_{\text{in}} = \frac{y_{\text{in}} - y_1}{x_{\text{in}} - x_1}, \quad m_{\text{out}} = \frac{y_{\text{out}} - y_4}{x_{\text{out}} - x_4}, \\
\text{step1} &= \frac{x_2 - x_1}{10}, \quad \text{step2} = \frac{x_3 - x_2}{10}, \quad \text{step3} = \frac{x_4 - x_3}{10}, \\
\text{step4} &= \frac{x_1 - x_4}{10}, \quad \text{stepin} = \frac{x_{\text{in}} - x_1}{10}, \quad \text{stepout} = \frac{x_{\text{out}} - x_4}{10}
\end{align*}
\]

\[
\begin{align*}
\text{rangea} &= x_1 \cdot x_1 + \text{step1} \cdot x_2, \quad \text{rangec} = x_3 \cdot x_3 + \text{step3} \cdot x_4, \quad \text{rangee} = x_1 \cdot x_1 + \text{stepin} \cdot x_{\text{in}}, \\
\text{rangeb} &= x_2 \cdot x_2 + \text{step2} \cdot x_3, \quad \text{ranged} = x_4 \cdot x_4 + \text{step4} \cdot x_1, \quad \text{rangef} = x_4 \cdot x_4 + \text{stepout} \cdot x_{\text{out}}
\end{align*}
\]

\[
\begin{align*}
\text{drawlinka}(\text{rangea}) &= m_1 \cdot (\text{rangea} - x_1) + y_1, \quad \text{drawlinkc}(\text{rangec}) = m_3 \cdot (\text{rangec} - x_3) + y_3, \\
\text{drawlinkb}(\text{rangeb}) &= m_2 \cdot (\text{rangeb} - x_2) + y_2, \quad \text{drawlinkd}(\text{ranged}) = m_4 \cdot (\text{ranged} - x_4) + y_4, \\
\text{drawlinke}(\text{rangee}) &= m_{\text{in}} \cdot (\text{rangee} - x_1) + y_1, \quad \text{drawlinke}(\text{rangee}) = m_{\text{out}} \cdot (\text{rangef} - x_4) + y_4
\end{align*}
\]
Force analysis

Indicate Closure, $\xi = +1$ or $-1$ (the quadratic equation has two solutions, a "+" root and a "-" root)

\[ g = 1.4 \quad b_0 := \beta_0 \quad b_g := \beta_g \quad p_g := \Phi_g \quad Y_1 = y \quad Y_2 = x \]

Retaining precision point values

\[ k := 0, 1, . . . 40 \quad \beta_k := b_0 + k \quad \theta_{2_k} := \beta_k - \theta_{in} \quad \text{Iterating} \]
\[ E_3 = 2 \cdot 1 \cdot 2 \cdot 1 \cdot 3 \cdot \sin(\theta_2) - 2 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot \sin(\chi) \quad F_3 = 2 \cdot 1 \cdot 2 \cdot 1 \cdot 3 \cdot \cos(\theta_2) - 2 \cdot 1 \cdot 2 \cdot 1 \cdot \cos(\chi) \]

\[ G_3 = 1 \cdot 2^2 + 1 \cdot 3^2 + 1 \cdot 1^2 - 1 \cdot 4^2 + 2 \cdot 1 \cdot 2 \cdot 1 \cdot 4 \cdot \cos(\theta_2) \cdot \cos(\chi) - 2 \cdot 1 \cdot 2 \cdot 1 \cdot \sin(\chi) \cdot \sin(\theta_2) \]

\[ N_3 = -E_3 + \frac{5}{\sqrt{(E_3)^2 + (F_3)^2 - (G_3)^2}} \cdot \theta_3 = 2 \cdot \tan \left( \frac{N_3}{G_3 - F_3} \right) \]

![Graph](image_url)

**Figure A-4, Theta 3 vs. Beta**

\[ E_4 = -2 \cdot 1 \cdot 2 \cdot 1 \cdot 4 \cdot \sin(\theta_2) + 2 \cdot 1 \cdot 4 \cdot 1 \cdot \sin(\chi) \quad F_4 = -2 \cdot 1 \cdot 2 \cdot 1 \cdot 4 \cdot \cos(\theta_2) + 2 \cdot 1 \cdot 2 \cdot 1 \cdot \cos(\chi) \]

\[ G_4 = 1 \cdot 2^2 + 1 \cdot 4^2 + 1 \cdot 1^2 - 1 \cdot 3^2 - 2 \cdot 1 \cdot 2 \cdot 1 \cdot \cos(\chi) \cdot \cos(\theta_2) - 2 \cdot 1 \cdot 2 \cdot 1 \cdot \sin(\chi) \cdot \sin(\theta_2) \]

\[ N_4 = -E_4 + \frac{5}{\sqrt{(E_4)^2 + (F_4)^2 - (G_4)^2}} \quad \theta_4 = 2 \cdot \tan \left( \frac{N_4}{G_4 - F_4} \right) \]
\( \Phi_k = \theta_k^4 - \theta_w \)

Figure A-5, Theta 4 vs. Beta

Figure A-6, Function Generation
Calculate Resistance Curve, Internal Coupler Force and Pivot Bearing Forces

\[
\begin{align*}
\begin{bmatrix} \Gamma_{4_k} \\ \Gamma_{3_k} \end{bmatrix} &= \begin{bmatrix} 1_{4} \cdot \cos(\theta_{4_k}) - 1_{3} \cdot \cos(\theta_{3_k}) \\ 1_{4} \cdot \sin(\theta_{4_k}) - 1_{3} \cdot \sin(\theta_{3_k}) \end{bmatrix}^{-1} \begin{bmatrix} 1_{2} \cdot \cos(\theta_{2_k}) \\ 1_{2} \cdot \sin(\theta_{2_k}) \end{bmatrix} \\
R_k &= \begin{bmatrix} 1_{4_k} \cdot (\Phi_{k} - \varepsilon) \end{bmatrix} \\
F_{3_k} &= 12 \frac{R_k}{1_{2} \cdot \sin(\theta_{2_k} - \theta_{3_k})} \\
\sin(\theta_{2_0} - \theta_{3_0}) &= 0.457
\end{align*}
\]

\[
\text{Bout}_k = \sqrt{\left(F_{3_k} \cdot \cos(\theta_{3_k})\right)^2 + \left(F_{3_k} \cdot \sin(\theta_{3_k})\right)^2}
\]
Figure A-7, Resistance Curve of the Soln
Figure A-8, Internal Coupler Force
ARTICLE II. NFAA SHOOTING STYLES AND EQUIPMENT RULES

A. General:

1. A conventional bow of any type may be used provided it subscribes to the accepted principal and meaning of the word "bow" as used in archery competition, i.e., an instrument consisting of a handle (grip) riser and two flexible limbs, each ending in a tip with string nock. The bow is braced for use by a single bowstring attached directly between the two string nocks only. In operation it is held in one hand by the handle (grip) riser while the fingers of the other hand draw, hold back and release the string.

2. Compound bows may be used, provided:
   a) Basic design includes a handle riser (grip) and flexible limbs.
   b) Total arrow propelling energy is developed from a flexing of the materials employed in limb construction.
   c) Weight reduction factor is of no consequence.
   d) Bows which develop any portion of arrow propelling energy from sources "other than the limbs" shall not be allowed. This is not to be construed to mean that compound bows which employ other sources of arrow propelling energy, not specifically listed in this paragraph, will be allowed.
   e) The cables of the compound bow shall be considered as part of the string and all applicable string rules except color requirements shall apply.

3. This Paragraph Is Applicable Only To Competition On Unmarked Distance Tournaments: The use of a range finder is prohibited. At no time shall any device be allowed that would in any manner be an aid in establishing the distance of any shot. No archer may refer to any written memoranda that would aid in determining the distance to the target.

4. Any device that would allow the mass weight, or the draw weight of the bow to be relieved from either or both arms, at full draw, shall be declared illegal.

5. All overdraws shall be designed in such a fashion as to prevent the arrow from falling off the rest, endangering other competitors.
6. All Equipment rulings must be accompanied by an example of the item in question; to the assigned committee and for examination by the Board of Directors prior to voting.

B. Barebow:

1. Archers shooting Barebow style will use bow, arrows, strings and accessories free from any sights, marks or blemishes.
   a) String will be made of one or more strands. Strands will be of one consistent color of the archer's choice. The center serving on the string will be served with one layer of any material suitable to use, but material will be of one consistent size and one consistent color. Placement of a nock locator on the serving will be permitted.
   b) No written memoranda shall be allowed.

2. An adjustable arrow plate may be used to control the space between the arrow and the face of the sight window.

3. The use of stabilizers shall be permitted.

4. One consistent nocking point only is permitted.
   a) Nocking point shall be held by one or two nock locators, which shall be snap on type, shrink tubing, thread or dental floss, tied or served on the serving. Nocking point locators shall not extend more than one half inch (1/2") above or below the arrow nock when at full draw.

5. No mechanical device will be permitted other than one non-adjustable draw check and level mounted on the bow, neither of which may extend above the arrow. "Note: Mechanical type arrow rests and cushion plungers are legal."

6. Releases other than gloves, tabs, or fingers shall be deemed illegal.

7. All arrows shall be identical in length, weight, diameter and fletching, with allowance for wear and tear.

8. The ends or edges of laminated pieces appearing on the inside of the upper limb shall be considered a sighting mechanism.

9. No device of any type, including arrow rest, that may be used for sighting, may be used or attached to the archer's equipment.

10. The pylon (string clearance bar) will be allowed in this style if it is not located in the sight window.

11. Any part of the arrow rest extending more than 1/4 inch above the arrow is deemed illegal in the Barebow style.

12. An arrow plate extending more than 1/4 inch above the arrow is deemed illegal in the Barebow style.

C. Freestyle:

1. Any type of sight and its written memorandum may be used.

2. Any release aid may be used provided it is hand operated and supports the draw weight of the bow.

D. Freestyle Limited:

1. Any type of sight and its written memorandum may be used.

2. Release aids shall be limited to gloves, tabs and fingers.

E. Competitive Bowhunter:

1. This style of shooting is for those with heavy tackle equipment used during hunting activities. Junior Bowhunters shall not be recognized.

2. No device of any type (including arrow rest), that may be used for sighting, may be used or attached to the archer's equipment.

3. There shall be no device, mechanical or otherwise, in the sight window except the arrow rest and/or cushion plungers.
4. Any part of the arrow rest extending more than 1/4 inch above the arrow is deemed illegal in the Competitive Bowhunter style.

5. An arrow plate extending more than 1/4 inch above the arrow is deemed illegal in the Competitive Bowhunter style.

6. No clickers, drawchecks, or levels will be allowed. No laminations, marks, or blemishes may appear in the sight window or upper limb.

7. String shall be one color only. End serving and center serving may be of different colors than the string, but center serving must be of one color only. One consistent nocking point only is permitted. Nocking point locators shall not extend more than one half inch (1/2") above or below the arrow nock when at full draw. Any marks, ties or string attachment to the string (except brush buttons and silencers properly located) shall invalidate its use in this division.

8. One anchor point only is permitted.

9. An Archer shall touch the arrow when nocked with the index finger against the nock. Finger position may not be changed during competition. In cases of physical deformity or handicap, special dispensation shall be made.

10. Releases other than gloves, tabs, or fingers shall be deemed illegal.

11. Each time an archer shoots a round, all arrows shall be identical in length, weight, diameter and fletching with allowances for wear and tear.

12. The Field Captain, or his counterpart, shall be the final authority regarding equipment and style eligibility, and may reclassify at his discretion.

13. Brush buttons, string silencer, no less than 12 inches above or below the nocking point, and bow quiver installed on the opposite side of the sight window, with no part of the quiver or attachments visible in the sight window, are legal. One straight stabilizer, coupling device included if used, which cannot exceed 12 inches at any time, as measured from the back of the bow may be used in the Competitive Bowhunter style. No forked stabilizer or any counter balance will be legal.

14. The following broadhead standard will be followed whenever broadheads are authorized for tournaments:
   a) Male—7/8 inch cutting edge width (minimum).
   b) Female—3/4 inch cutting edge width (minimum).

15. There shall be no restrictions on the bow draw weight. Arrows must be equipped with a minimum of 125 grain points for men and a minimum of 100 grain points for women.

16. Any device for lengthening or shortening the draw length of an archer shall be prohibited.

17. An archer will not be permitted to change the draw weight of the bow during a round.

18. The pylon (string clearance bar) will be allowed in this shooting style if it is not located in the sight window.

19. No written memoranda shall be allowed.

20. All official NFAA rounds shall be considered official rounds for the Bowhunter style of shooting, and further all classification shall be based upon the Field and Hunter rounds.

F. Freestyle Bowhunter:
1. A sight with a maximum of 5 fixed reference points that must not be moved during a round. Pin sights are to be of straight stock from point of
anchor to sighting point, with only one sighting reference possible from each pin. Hooded pins or scopes cannot be used. The maximum sight extension measurement shall be 5", measured from the back of the bow at the center of attachment to the foremost part of the sight assembly, as measured on a horizontal plane. Lighted or illuminated sights (pins) are illegal.

2. Release aids will be permitted.
3. A kisser button or string peep sight will be permitted, but not both. Whichever is installed must be secured so as not to be movable between shots of different distances.
4. It will not be mandatory in this style of shooting to provide for other than one division for men and one division for women.
5. All rules of the Competitive Bowhunter shooting style, except those excluded by this section, shall also apply to the Freestyle Bowhunter shooting style.

G. Freestyle Limited Bowhunter:
1. A sight with a maximum of 5 fixed reference points that must not be moved during a round. Pin sights are to be of straight stock from point of anchor to sighting point, with only one sighting reference possible from each pin or reference point. Hooded pins or scopes cannot be used. The maximum sight extension measurement shall be 5", measured from the back of the bow at the center of attachment to the foremost part of the sight assembly, as measured on a horizontal plane. Lighted or illuminated sights (pins) are illegal.
2. Release aids will not be permitted.
3. A kisser button or string peep will be permitted, but not both. Whichever is installed must be secured so as not to be movable between shots of different distances.
4. It will not be mandatory in this style of shooting to provide for other than one division for men and one division for women.
5. All rules of the Competitive Bowhunter shooting style, except those excluded by this section, shall also apply to the Freestyle Limited Bowhunter shooting style.

H. Traditional:
1. Adults shooting traditional style will use recurve or long bows.
2. No sights, stabilizers or counter balance.
3. Arrow shafts with 125 grain points for men and 100 grain for women. The points must be commercially manufactured.
4. Arrows shall be identical in length, weight, color except for normal wear.
5. String will have single color middle serving.
6. One single nocking point only is permitted.
7. One or two nock locaters, which may be snap-on type shrink tubing, thread or dental floss tied or served on the serving.
8. Arrow rest no more than 1/4 inch above arrow.
9. One anchor point only is permitted.
10. An archer shall touch the arrow when nocked with the index finger against the nock. Finger position may not be changed during competition.
11. If it is not covered in this statement, it is deemed illegal.
Vita

Matthew E. Palmer was born on June 21, 1970 in Bethesda, Maryland. He graduated 3rd with honors from Giles High School (30 minute drive from Blacksburg) which included as much math and science as he could take.

He entered Virginia Polytechnic Institute and State University as a freshmen in 1988 and graduated with a Bachelor of Science in Aerospace Engineering five years later. After a year hiatus, He came back to Tech for a Master of Science in Mechanical Engineering.

He enjoys auto and motorcycle racing and modification, rock climbing, drawing, playing guitar, and on occasion, Engineering.