THE EXTENSION OF THE MULTIPLE COMPARISONS TEST TO LATTICE DESIGNS

by

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I. INTRODUCTION

In many investigations, an experimenter is interested in comparing the effects of a number of experimental treatments, such as yields of several new agriculture varieties, results of several different production procedures, or effects of various raw materials on a final product. In this type of investigation, an F test of the mean square for treatments is often used to test the hypothesis that all treatment means are homogeneous. Given that this hypothesis can be rejected, the experimenter would like to make decisions about the significances of individual differences among treatment means considered a pair at a time.

The Multiple Comparisons Test was developed by Duncan (1951) to provide a basis for decisions of the type mentioned above. This test was constructed for experimental designs in which treatment means are independent and have constant variances. Thus, application of this technique has been limited to the more elementary designs such as randomized blocks and Latin squares.

At present, a large number of investigations are being conducted with incomplete block designs. As the treatment means are not independent in these designs, the Multiple Comparisons Test, as originally developed, can not be applied. The object of this thesis and a companion thesis by

Sanders (1953) is to determine what modifications of the Multiple Comparisons Test are necessary to apply this procedure to comparisons of treatment means in various incomplete block designs. This thesis is concerned with the development of test procedures for simple, triple, and balanced lattices, whereas, Sander's thesis develops test procedures for rectangular lattices.

In showing that a significant difference exists between two independent means $(\ldots, \bar{x}_i, \ldots, \bar{x}_j, \ldots)$ in a set of n treatment means by the Multiple Comparisons Test, a series [1] of F tests is required for the variances of all treatment combinations enveloping[2] $\bar{x}_i$ and $\bar{x}_j$. These tests are made to ensure that the two means do not belong to any group or subgroup of homogeneous treatment means. These F tests employ ratios of the type, $F = s_p^2 / s_{\bar{x}}^2$ where:

s_p^2 is the mean square for the p means involved and

$s_{\bar{x}}^2$ is the mean square of a treatment mean based on the whole experiment.

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(1) In the actual application of the Multiple Comparisons test, procedures have been developed which simplify and eliminate the need for testing all enveloping treatment combinations individually.

(2) The terminology 'enveloping $\bar{x}_i$ and $\bar{x}_j$', signifies means which includes both $\bar{x}_i$ and $\bar{x}_j$. For example, if the series (a,b,c,d) represents four means, then (a, b, c); (a, c); (a, c, d); and (a, b, c, d) are all enveloping combinations of treatments a and c.

The mean square, s_p^2, is calculated from the formula, $\sum_{i=1}^{p} (\bar{x}_i - \bar{x})^2/(p-1)$, where $\bar{x}$ is the mean of the p treatments in question. F ratios using s_p^2 in the numerator are valid only when the treatment means are independent. When treatment means are correlated, as they are in lattice designs, these ratios must be modified in some manner before they can be used in the Multiple Comparisons Test.

The end result of this thesis is to show that the Multiple Comparisons Test may be applied directly to correlated means in simple, triple, and balanced lattice designs, provided the mean square for a treatment mean, $s_{\bar{x}}^2$, is replaced by one half of the average variance, $\bar{V}_d$, of a difference between two treatment means. This result is demonstrated by showing that ratios of the form $s_p^2/\frac{1}{2}\bar{V}_d$ can be treated as F ratios. In doing this, two forms of approximations are involved.

The first approximation results from the assumption that covariances between adjusted treatment means are constant. The second approximation is that of using estimated variances instead of the true variances in calculating the weighting factors required to obtain adjusted treatment means. These weighting factors are also needed in the recovery of interblock information.

The first approximation is not necessary in a completely balanced design, as the covariances between treatment means are equal. Even in unbalanced designs, however, where both approximations have to be made, the effect of these approximations should not be very serious. In any event, there are good precedents[3] for accepting these approximations. The second approximation is always made in incomplete block designs employing recovery of interblock information.

To summarize, the same rules developed in applying the Multiple Comparisons Test to a set of independent means may be used in testing differences between adjusted means in lattice designs with one modification. This modification is the replacement of the variance of a treatment mean, $s_{\bar{x}}^2$ by half the average variance of a difference between two means, $\frac{1}{2}\bar{V}_d$. This result corresponds to that obtained by Sanders (1953) in his investigation of the Multiple Comparisons Test applied to rectangular lattice arrangements.

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(3) i. The first assumption is implicitly accepted whenever an average variance of a treatment difference is computed as suggested by Cochran and Cox (1950), chapter 10 or by Kempthorne (1952), p. 460.

ii. The use of estimated variances in computing weighting factors is discussed in Kempthorne (1952), section 22.2.1.

II. LATTICE DESIGNS

2.1 Introduction

An experimenter is often faced with the necessity of comparing a large number of treatments within the confines of a limited amount of homogeneous testing material. For example, he may wish to compare a large number of varieties on a limited area of homogeneous land or the number of treatments may be limited by the production capacity of a machine, by the supply of homogeneous raw materials, or by the length of time available in a working period. In such cases, the experimenter may often use experimental arrangements known as incomplete block designs.

Incomplete block designs are a series of arrangements in which the number of treatments in a block or group is less than the total number of treatments being investigated. For instance, it might be necessary to test the quality of forty-nine types of experimental enamels on an endurance machine which cannot test forty-nine specimens within one uniform testing area. If at least seven specimens can be placed within the largest uniform area of exposure, then one possible arrangement would be to test the enamels in incomplete blocks or groups of seven panels each. Such an arrangement would constitute an incomplete block design. With proper allocation of treatments in each of the incomplete blocks, a design of this type will give better precision for comparisons of the treatments, especially among those treatments occurring together in the same block, than will corresponding complete block arrangements. The precision gained by reducing the block size through the use of incomplete block designs is improved greatly when a method known as recovery of inter-block information is used.

Lattices are a large group of incomplete block designs which are developed from methods previously derived for factorial experiments. By establishing a correspondence between a lattice design and a factorial experiment, methods for reducing the block size and subsequent analysis of the lattice become much clearer. This correspondence also aids in developing the variances and covariances of treatment estimates. These variances and covariances are needed in the problem of modifying the Multiple Comparisons Test to assess the significance of differences among treatment means. Because of the correspondence between lattices and factorial designs, treatments in a lattice are often termed pseudo-factorial treatment combinations.

2.2 Pseudo-Factorial Combinations for a Four-Treatment Design

To illustrate the principles involved in arranging treatments in incomplete blocks a hypothetical experiment of four treatments is considered. Suppose that the four treatments under study are four temperatures at which glycerin is being nitrated and only two samples can be nitrated in an eight-hour shift. An acceptable arrangement would be to nitrate two samples on each of two shifts. In this study, each shift may be considered as an experimental block. A correspondence may be established between the four treatments under investigation and a set of pseudo-factorial treatment combinations in a 2^2 factorial experiment; that is, an experiment with two factors each at two levels.

The four treatments may be related to factorial treatment combinations as shown in Table 2.1.

Table 2.1

Equivalence Between Treatments and Pseudo-Factorial Combinations

T_1:	T_{00}	T_3:	T_{01}
T_2:	T_{10}	T_4:	T_{11}

In factorial notation, T_{ij} indicates the combination of factor a at its i^{th} level with factor b at its j^{th} level. Thus, T_{01} is the application of factor a at its lower level with factor b at its higher level.

Letting y_{ij} represent the result of applying treatment T_{ij}, the main effects of factors a and b are denoted by

$$A = \tfrac{1}{2}\left[y_{11} - y_{01} + y_{10} - y_{00}\right]$$

and

$$B = \frac{1}{2}\left[y_{11} - y_{10} + y_{01} - y_{00}\right] .$$

The interaction of factors a and b is denoted by

$$AB = \frac{1}{2}\left[y_{11} - y_{10} + y_{00} - y_{01}\right] .$$

A summary of these relationships together with the relationship for the mean, μ, is given in Table 2.2.

Table 2.2

<u>The Mean, Main Effects, and Interaction in Terms of Treatment Combinations</u>

	Results for Treatment Combinations			
Effect	y_{00}	y_{10}	y_{01}	y_{11}
4μ	+	+	+	+
2A	-	+	-	+
2B	-	-	+	+
2AB	+	-	-	+

2.3 <u>Arrangement of Treatments in Incomplete Blocks</u>

Through the techniques of confounding effects and interactions, the original treatments may be arranged in blocks of two treatments each. This may be accomplished by confounding the main effects, A and B, and the interaction, AB, in three replicates, as shown in Table 2.3.

Table 2.3

A 2^2 Pseudo-Factorial Experiment with A, B, and AB Partially Confounded

	Replicate I			Replicate II			Replicate III	
(1)	T_{00}	T_{01}	(3)	T_{00}	T_{10}	(5)	T_{00}	T_{11}
(2)	T_{10}	T_{11}	(4)	T_{01}	T_{11}	(6)	T_{10}	T_{01}
	A Effect Confounded			B Effect Confounded			AB Interaction Confounded	

( ) Block or Shift Numbers.

A full set of replicates with a different effect confounded in each replicate is called a basic repetition. For example, a complete design might consist of several repetitions of replicates I, II, and III in Table 2.3.

The concept of a basic repetition affords a basis for classifying the lattices into simple, triple, and so forth, up to balanced designs. In the simple lattice, only two effects are confounded and a basic repetition consists of two replicates. If replicates I and II from Table 2.3 are repeated one or more times, then the design so obtained would be classed as a simple lattice. When the basic repetition contains three replicates, i.e., confounding three different effects, the design is classed as a triple lattice. Similarly, in larger experiments with $k > 2$ (k being the number of units per block), it is possible to design four, five, or up to $(k + 1)$-tuple lattices by confounding four, five, or up to $(k + 1)$ effects in each repetition. When all possible $(k + 1)$ effects are confounded the design is called a balanced lattice and in these designs every pair of treatments occurs together once in the same incomplete block.

If all three replicates in Table 2.3 are used as a basic repetition, the design may be classed as a triple lattice. Also, in this particular case, the design is balanced, since the basic repetition contains all possible $(k + 1)$ confoundings (since $k = 2$).

2.4 The Model for Lattice Designs

In these designs, the yield of a particular treatment is assumed to be made up of the added effects of the general mean, replicate, treatment, block, and experimental unit which is associated with the particular yield. This assumption is given by

$$y_{fg(ij)} = \mu + \rho_f + \beta_{fg} + \pi_{(ij)} + \epsilon_{fg(ij)} \qquad (2.1)$$

where:

(i) $y_{fg(ij)}$ is the yield of the $(ij)^{th}$ treatment in block g of replicate f,

(ii) μ is the expected mean for the experiment,

(iii) ρ_f is the effect for replicate f,

(iv) β_{fg} is the effect for the g^{th} block in replicate f,

(v) $\pi_{(ij)}$ is the effect for the $(ij)^{th}$ treatment, and

(vi) $\epsilon_{fg(ij)}$ is the error of the $(ij)^{th}$ treatment in block g of replicate f.

The quantities μ, ρ, and π are fixed unknown parameters, whereas, both β and ϵ are assumed to be independent and normally distributed variates with means of zero and variances σ_b^2 and σ_e^2, respectively.

2.5 The Variances of Estimates of Pseudo-Factorial Effects

The treatment estimates in a lattice may be expressed as linear functions of independent pseudo-factorial effects. Once the variances of these effects are known, the variances of the treatment estimates may be determined with little additional difficulty from the linear relationships. For this reason, the variances of the estimated factorial effects are obtained in this section.

In a lattice design, each pseudo-factorial effect has two types of estimates. One estimate may be obtained from replicates in which the effect is not confounded and the other type of estimate from replicates in which the estimate is confounded. The A effect estimated from replicates II or III is of the first type.

Let $\hat{A}_{II}$ denote the estimate of the A effect from replicate II, that is,

(2.2) $$A_{II} = \tfrac{1}{2}[-y_{00} + y_{10} - y_{01} + y_{11}].$$

Now, keeping in mind the arrangement of the treatments in replicate II (see Table 2.3) and the model (2.1), $\hat{A}_{II}$ is equal to the true A effect, $A = \frac{1}{2}[\pi_{00} + \pi_{10} - \pi_{01} + \pi_{11}]$, plus an error of

$$\tfrac{1}{2}[-\beta_3-\epsilon_{3(00)}+\beta_3+\epsilon_{3(10)}-\beta_4-\epsilon_{4(01)}+\beta_4+\epsilon_{4(11)}]$$
$$= \tfrac{1}{2}[-\epsilon_{3(00)}+\epsilon_{3(10)}-\epsilon_{4(01)}+\epsilon_{4(11)}].$$

The variance, $V(\hat{A}_{II})$, for $\hat{A}_{II}$ is therefore,

(2.3) $$V(\hat{A}_{II}) = \frac{1}{4}[\sigma_e^2 + \sigma_e^2 + \sigma_e^2 + \sigma_e^2] = \sigma_e^2.$$

Similarly, the variance, $V(\hat{A}_{III})$, of the estimate of A from replicate III is also equal to σ_e^2.

The second type of estimate for the A effect may be obtained from replicate I and in this replicate, the A effect is estimated from interblock comparisons. From equations (2.1) and (2.2) and a knowledge of the treatment arrangement in replicate I (found in Table 2.3), an estimate, $\hat{A}_I$, of the A effect from replicate I is equal to the true A effect plus an error of

(2.4) $$\tfrac{1}{2}[-\beta_1-\epsilon_{1(00)}-\beta_1-\epsilon_{1(01)}+\beta_2+\epsilon_{2(10)}+\beta_2+\epsilon_{2(11)}]$$
$$= \tfrac{1}{2}[-2\beta_1+2\beta_2-\epsilon_{1(00)}-\epsilon_{1(01)}+\epsilon_{2(10)}+\epsilon_{2(11)}].$$

Thus,

(2.5) $$V(\hat{A}_I) = \frac{1}{4}[4\sigma_b^2+4\sigma_b^2+4\sigma_e^2] = 2\sigma_b^2 + \sigma_e^2.$$

Now, if weights W and W' are defined to be

(2.6)
$$W = \frac{1}{\sigma_e^2} \quad \text{and}$$
$$W' = \frac{1}{\sigma_e^2 + 2\sigma_b^2} ,$$

then the best estimate of the A effect combining information from all three replicates is the weighted mean

(2.7) $$\hat{A} = \frac{W'\,\hat{A}_I + W\,\hat{A}_{II} + W\,\hat{A}_{III}}{(W' + 2\,W)} .$$

The variance of $\hat{A}$ is

$$V(\hat{A}) = \frac{(W')^2 V(\hat{A}_I) + (W)^2 V(\hat{A}_{II}) + (W)^2 V(\hat{A}_{III})}{(W' + 2\,W)^2} = \frac{1}{(W'+2W)} .$$

If $V(\hat{A})$ is calculated from r repetitions, then

(2.8) $$V(\hat{A}) = \frac{1}{r(W' + 2W)} .$$

In a similar manner using $\hat{B}_I$, $\hat{B}_{II}$, and $\hat{B}_{III}$ to represent estimates of the B effect from replicates I, II, and III and $\hat{AB}_I$, $\hat{AB}_{II}$, and $\hat{AB}_{III}$ estimates of the interaction from these replicates, the variances of the various effects are

$$V(\hat{B}_I) = \sigma_e^2 \qquad V(\hat{AB}_I) = \sigma_e^2$$
$$V(\hat{B}_{II}) = \sigma_e^2 + 2\,\sigma_b^2 \qquad V(\hat{AB}_{II}) = \sigma_e^2$$
$$V(\hat{B}_{III}) = \sigma_e^2 \qquad V(\hat{AB}_{III}) = \sigma_e^2 + 2\,\sigma_b^2 .$$

Weighted means $\hat{B}$ and $\hat{AB}$, similar to $\hat{A}$, are

$$\hat{B} = \frac{W\,\hat{B}_I + W'\,\hat{B}_{II} + W\,\hat{B}_{III}}{(W' + 2W)} \quad \text{and}$$

$$\hat{AB} = \frac{W\,\hat{AB}_{I} + W\,\hat{AB}_{II} + W'\,\hat{AB}_{III}}{(W' + 2W)} .$$

Variances of these estimates obtained from r replicates are also given by

$$\frac{1}{r(W' + 2W)} .$$

The variance, $V(\hat{\mu})$, of an estimate, $\hat{\mu}$, of the general mean may be found much the same way as the variances of the factorial effects. The estimate of μ in any replicate is

$$\hat{\mu}_f = \frac{1}{4}[y_{00} + y_{01} + y_{10} + y_{11}], \quad f = I, II, III$$

and thus, $\hat{\mu}_I$ is equal to the true mean, μ, plus an error of

$$\text{(2.9)} \quad \begin{aligned} &\frac{1}{4}[\beta_1+\epsilon_1(00)+\beta_1+\epsilon_1(01)+\beta_2+\epsilon_2(10)+\beta_2+\epsilon_2(11)] \\ &= \frac{1}{4}[2\beta_1+2\beta_2+\epsilon_1(00)+\epsilon_1(01)+\epsilon_2(10)+\epsilon_2(11)] . \end{aligned}$$

Therefore,

$$V(\hat{\mu}_I) = \frac{1}{16}[4\sigma_b^2+4\sigma_b^2+4\sigma_e^2] = \frac{1}{4}[2\sigma_b^2+\sigma_e^2] .$$

Thence, the variance of $\hat{\mu}$ for an experiment with r repetitions is equal to $\frac{1}{4}[\frac{2\sigma_b^2 + \sigma_e^2}{3r}]$ or in terms of W',

$$V(\hat{\mu}) = \frac{1}{12\,r\,W'} .$$

The covariances of the estimates $\hat{A}$, $\hat{B}$, $\hat{AB}$, and $\hat{\mu}$ are readily found to be zero. For example, the covariance of $\hat{A}$ and $\hat{\mu}$, Cov $(\hat{A}_I\hat{\mu}_I)$, from replicate I may be found by multiplying the coefficients in the expressions, (2.5) and (2.9), of the error in $\hat{A}_I$ and $\hat{\mu}_I$. Thus,

$$\begin{aligned} \text{Cov}\,(\hat{A}_I\ \hat{\mu}_I) &= \frac{1}{2} \times \frac{1}{4}[-4\sigma_b^2 + 4\sigma_b^2 - \sigma_e^2 + \sigma_e^2 - \sigma_e^2 + \sigma_e^2] \\ &= 0 . \end{aligned}$$

If the same type of expressions are obtained for the covariances of the remaining pairs of factorial effects, they will also have a value of zero. From individual results like these, it follows that the covariances of pairs of factorial effects estimated from all the replicates in an experiment are also zero.

2.6 Treatment Variances and Covariances in a Balanced Lattice

At this point, the main advantage in the pseudo-factorial approach to lattice designs becomes evident. Since the treatment estimates, t_{ij}, may be expressed as a linear function of the pseudo-factorial effects and since estimates of the effects are independent and their variances are known, it is fairly simple to obtain variances and covariances of the treatment estimates from these expressions.

Treatment estimates may be expressed in terms of pseudo-factorial effects by recalling the correspondence between treatments and factorial combinations (Table 2.1) and inverting the relations in Table 2.2. The results of this inversion are shown in Table 2.4.

Table 2.4

Treatment Estimates Expressed in Terms of Estimates of Pseudo-Factorial Effects and Interaction

Treatments	Factorial Effects and Interaction			
	μ	A	B	AB
t_1	+	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$
t_2	+	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
t_3	+	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$
t_4	+	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$

The variance, K_1, of any treatment estimate is easily obtained from Table 2.4, since

$$K_1 = V(\mu) + \frac{1}{4} V(A) + \frac{1}{4} V(B) + \frac{1}{4} V(AB) .$$

Then, by inserting the variances of the estimates, $\frac{1}{r(W'+2W)}$ for the variance of an effect and $\frac{1}{12\ r\ W'}$ for $V(\mu)$, K_1 becomes

$$K_1 = \frac{1}{12\ r\ W'} + \frac{3}{4r(W'+2W)} .$$

In a like manner, the covariance, K_2, of t_1 and t_2 is

$$K_2 = V(\mu) - \frac{1}{4} V(A) + \frac{1}{4} V(B) - \frac{1}{4} V(AB)$$

thence,

$$K_2 = \frac{1}{12\ r\ W'} - \frac{1}{4r(W'+2W)} .$$

Since each effect and interaction has the same expected variance in a balanced design, the covariance of any two treatments is also given by K_2.

In conclusion, variances and covariances for the four treatments in this example are as presented in Table 2.4.

Table 2.4

The Variance - Covariance Matrix of Treatment Estimates in a Balanced Lattice of Four Treatments

	t_1	t_2	t_3	t_4
t_1	K_1	K_2	K_2	K_2
t_2		K_1	K_2	K_2
t_3			K_1	K_2
t_4				K_1

III. DEVELOPMENT OF VARIANCES AND COVARIANCES OF TREATMENT ESTIMATES IN MORE COMPLEX LATTICE DESIGNS

3.1 Introduction

In the previous section, a simple design of four treatments was chosen to illustrate the general method for deriving the variances and covariances of treatment estimates. The method consisted of expressing the treatments as linear functions of independent pseudo-factorial effects and obtaining variances and covariances of the treatment estimates from these linear functions.

In this section, variances and covariances of treatment estimates from a design more complex than the previous example will be investigated, that is, one for which the corresponding factorial experiment would contain factors at more than two levels. A lattice of nine treatments is treated in this discussion. This lattice is the simplest arrangement in which the general techniques for developing variances and covariances of treatment estimates from factorial relationships can be exhibited.

3.2 Factorial Correspondence For A Lattice of Nine Treatments

As in section II, a correspondence may be established between the treatments and a hypothetical factorial experiment. The nine treatments correspond to a factorial experiment conducted with two factors, each at three levels and this correspondence is shown in Table 3.1.

Table 3.1

Pseudo-Factorial Combinations for a Lattice of Nine Treatments

$T_1 : T_{00}$	$T_2 : T_{01}$	$T_3 : T_{02}$
$T_4 : T_{10}$	$T_5 : T_{11}$	$T_6 : T_{12}$
$T_7 : T_{20}$	$T_8 : T_{21}$	$T_9 : T_{22}$

As before, the coordinates in the subscripts of T_{ij} signify the combination of the i^{th} level of factor a with the j^{th} level of factor b.

The main effect of factor a (the A effect) may be represented by comparisons among the means of the three rows in Table 3.1, that is, comparisons among those points for which $i = 0,: 1,: 2$. Thus, the A effect is based on comparisons of the three means:

$$A_0 = \frac{1}{3}\,[y_{00} + y_{01} + y_{02}]$$

$$A_1 = \frac{1}{3}\,[y_{10} + y_{11} + y_{12}]$$

$$A_2 = \frac{1}{3}\,[y_{20} + y_{21} + y_{22}]\ ,$$

when y_{ij} is the yield of treatment T_{ij}. In a similar manner, the main effect of factor b (the B effect) may be obtained by comparisons among the points for which $j = 0,\ :1,\ :2$ or by comparisons among the means of the three columns in Table 3.1. These means are given by

$$B_0 = \frac{1}{3}\,[y_{00} + y_{10} + y_{20}]$$

$$B_1 = \frac{1}{3}\,[y_{01} + y_{11} + y_{21}]$$

$$B_2 = \frac{1}{3}\,[y_{02} + y_{12} + y_{22}]\ .$$

The interaction between the two factors a and b will be comprised of comparisons within two groups of means. The first group consists of elements of the form AB_p, which defines means of points for which[1] $i + j = p \bmod 3$.

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(1) p mod 3 is an abbreviation for p modulo 3 and defines any number, which, when divided by 3, leaves a remainder of p. For example, $5 = 2 \bmod 3$, since 5 divided by 3 leaves a remainder of 2.

These are

$$AB_0 = \frac{1}{3}\,[y_{00} + y_{12} + y_{21}]$$

$$AB_1 = \frac{1}{3}\,[y_{10} + y_{01} + y_{22}]$$

$$AB_2 = \frac{1}{3}\,[y_{02} + y_{20} + y_{11}] \ .$$

A second group of means for the AB interaction is defined by AB_p^2, denoting means of the points for which $i+2j=p$ mod 3. These are

$$AB_0^2 = \frac{1}{3}\,[y_{00} + y_{11} + y_{22}]$$

$$AB_1^2 = \frac{1}{3}\,[y_{10} + y_{02} + y_{21}]$$

$$AB_2^2 = \frac{1}{3}\,[y_{20} + y_{01} + y_{12}] \ .$$

The means given above are linear functions of treatment combinations chosen so that the variation among these means is a measure of the pseudo-factorial effects. A summary of the foregoing linear relations together with the relation for μ is presented in Table 3.2.

The pseudo-factorial-effect-means as defined by Kempthorne (1952) differ slightly from the definitions adopted here. Kempthorne defines effect-means as deviations from μ. If the effect-means as defined by Kempthorne are indicated by (A_p), (B_p) etc., then the relation between our notation and Kempthorne's is

Table 3.2

Pseudo-Factorial Effects Expressed in Treatment Yields

(Lattice, Nine Treatments)

Pseudo-Factorial Effects	Treatment Yields								
	y_{00}	y_{01}	y_{02}	y_{10}	y_{11}	y_{12}	y_{20}	y_{21}	y_{22}
$3\ A_0$	+	+	+						
$3\ A_1$				+	+	+			
$3\ A_2$							+	+	+
$3\ B_0$	+			+			+		
$3\ B_1$		+			+			+	
$3\ B_2$			+			+			+
$3\ AB_0$	+					+		+	
$3\ AB_1$		+		+					+
$3\ AB_2$			+		+		+		
$3\ AB_0^2$	+				+				+
$3\ AB_1^2$			+	+				+	
$3\ AB_2^2$		+				+	+		
$9\ \mu$	+	+	+	+	+	+	+	+	+

$$
(3.1)\quad
\begin{aligned}
(A_p) &= A_p - \mu, \\
(B_p) &= B_p - \mu, \\
(AB_p) &= AB_p - \mu, \text{ and} \\
(AB_p^2) &= AB_p^2 - \mu.
\end{aligned}
$$

From the relationships in Table 3.2 and the expression given in (3.1), the estimate of a treatment in terms of the effects and interactions is

$$(3.2)\quad t_{ij} = \hat{A}_i + \hat{B}_i + \widehat{AB}_{i+j} + \widehat{AB}^2_{i+2j} - 3\hat{\mu},$$

where all subscripts are reduced modulo 3. As an example, the estimate of treatment (1,2) is given by

$$t_{12} = \hat{A}_1 + \hat{B}_2 + \widehat{AB}_0 + \widehat{AB}^2_2 - 3\hat{\mu}.$$

Table 3.3 is a summary of the treatment estimates expressed in terms of pseudo-factorial-effect-means and is obtained from (3.2).

Table 3.3

The Relationship of Treatment Estimates to Estimates of Pseudo-Factorial Effects for a Lattice of Nine Treatments

Treatments	Pseudo-Factorial Effects												
	$\hat{A}_0$	$\hat{A}_1$	$\hat{A}_2$	$\hat{B}_0$	$\hat{B}_1$	$\hat{B}_2$	$\widehat{AB}_0$	$\widehat{AB}_1$	$\widehat{AB}_2$	$\widehat{AB}^2_0$	$\widehat{AB}^2_1$	$\widehat{AB}^2_2$	$\hat{\mu}$
t_{00}	+			+			+			+			-3
t_{01}	+				+			+				+	-3
t_{02}	+					+			+		+		-3
t_{10}		+		+				+			+		-3
t_{11}		+			+				+	+			-3
t_{12}		+				+	+					+	-3
t_{20}			+	+					+			+	-3
t_{21}			+		+		+				+		-3
t_{22}			+			+		+		+			-3

3.3 Arrangement of a Lattice of Nine Treatments in Blocks of Three

In a 3^2 pseudo-factorial experiment four confoundings are possible and these confoundings, which are indicated by Table 3.2, yield the replicates shown in Table 3.4.

Table 2.4

Four Confoundings in a 3^2 Factorial Experiment[1]

	Replicate I			Replicate II	
(1)	T_{00} T_{01} T_{02}	A_0	(4)	T_{00} T_{10} T_{20}	B_0
(2)	T_{10} T_{11} T_{12}	A_1	(5)	T_{01} T_{11} T_{21}	B_1
(3)	T_{20} T_{21} T_{22}	A_2	(6)	T_{02} T_{12} T_{22}	B_2
A Effect Confounded			B Effect Confounded		
	Replicate III			Replicate IV	
(7)	T_{00} T_{12} T_{21}	AB_0	(10)	T_{00} T_{11} T_{22}	AB^2_0
(8)	T_{01} T_{10} T_{22}	AB_1	(11)	T_{02} T_{10} T_{21}	AB^2_1
(9)	T_{02} T_{11} T_{20}	AB_2	(12)	T_{01} T_{12} T_{20}	AB^2_2
AB Effect Confounded			AB^2 Effect Confounded		

(1) Numbers in parentheses denote block number.

3.4 Variances and Covariances for Treatment Estimates in a Balanced Lattice

In section II, the variances and covariances of treatment estimates were very simply derived from independent factorial effects. Unfortunately, the pseudo-factorial-effect-means in expression (3.2), are not independent. However, the effect-means can be defined in terms of independent effect-comparisons. Thus, if A-effect-comparisons, A' and A'', are defined as,

$$\text{(3.3)} \quad \begin{aligned} A' &= A_0 - A_1, \\ A'' &= A_0 + A_1 - 2A_2 \end{aligned}$$

then, the treatments may be expressed in terms of independent pseudo-effect-comparisons. This may be accomplished from the inversion of the definitions (3.3) together with the definition of the mean, $\mu = \frac{1}{3}[A_0+A_1+A_2]$. The A-effect-means may be written in terms of effect-comparisons as shown below:

$$\text{(3.4)} \quad \begin{aligned} A_0 &= \mu + \frac{1}{2} A' + \frac{1}{6} A'' \\ A_1 &= \mu - \frac{1}{2} A' + \frac{1}{6} A'' \\ A_2 &= \mu - \frac{1}{3} A'' \; . \end{aligned}$$

Similarly, the remaining effect-means may be expressed in terms of effect-comparisons from the following:

$$B' = B_0 - B_1$$
$$B'' = B_0 + B_1 - 2B_2$$

$$(3.5) \qquad \begin{aligned} AB' &= AB_0 - AB_1 \\ AB'' &= AB_0 + AB_1 - 2AB_2 \end{aligned}$$

$$AB^{2\prime} = AB_0^2 - AB_1^2$$
$$AB^{2\prime\prime} = AB_0^2 + AB_1^2 - 2AB_2^2$$

Table 3.5 summarizes the expressions for effect-means in terms of effect-comparisons.

The treatment estimates, t_{ij}, may now be obtained in terms of estimates of independent pseudo-factorial-effect-comparisons. If $\hat{A}'$, $\hat{B}'$, $\widehat{AB}'$, $\widehat{AB}^{2\prime}$, $\hat{A}''$, $\hat{B}''$, $\widehat{AB}''$, and $\widehat{AB}^{2\prime\prime}$ denote estimates of effect-comparisons A', B', AB', $AB^{2\prime}$, A'', B'', AB'', and $AB^{2\prime\prime}$, respectively, then the expression for t_{00},

$$\frac{1}{2}[\hat{A}' + \hat{B}' + \widehat{AB}' + \widehat{AB}^{2\prime}] + \frac{1}{6}[\hat{A}'' + \hat{B}'' + \widehat{AB}'' + \widehat{AB}^{2\prime\prime}] + \hat{\mu},$$

is obtained by substituting estimates of the effect-comparisons for estimates of the effect-means in expression 3.2. (The relations between effect-means and effect-comparisons are given in Table 3.5.) The remaining expressions for the treatment estimates in terms of estimates of effect-comparisons are found in the same manner and a summary of these expressions is in Table 3.6.

Table 3.5

Effect-Means and Effect-Comparisons for a 3^2 Pseudo-Factorial Experiment

	μ	A'	A''	B'	B''	AB'	AB''	AB^{2}'	AB^{2}''
A_0	+	$\frac{1}{2}$	$\frac{1}{6}$						
A_1	+	$-\frac{1}{2}$	$\frac{1}{6}$						
A_2	+		$-\frac{1}{3}$						
B_0	+			$\frac{1}{2}$	$\frac{1}{6}$				
B_1	+			$-\frac{1}{2}$	$\frac{1}{6}$				
B_2	+				$-\frac{1}{3}$				
AB_0	+					$\frac{1}{2}$	$\frac{1}{6}$		
AB_1	+					$-\frac{1}{2}$	$\frac{1}{6}$		
AB_2	+						$-\frac{1}{3}$		
AB^2_0	+							$\frac{1}{2}$	$\frac{1}{6}$
AB^2_1	+							$-\frac{1}{2}$	$\frac{1}{6}$
AB^2_2	+								$-\frac{1}{3}$

Table 3.6

Relationship of Treatment Estimates to Estimates of Pseudo-Factorial-Effect-Comparisons in a Lattice of Nine Treatments

Treatment Estimates	$\hat{\mu}$	Pseudo-Factorial-Effect-Comparison Estimates							
		$\hat{A}'$	$\hat{B}'$	$\widehat{AB}'$	$\widehat{AB^2}'$	$\hat{A}''$	$\hat{B}''$	$\widehat{AB}''$	$\widehat{AB^2}''$
t_{00}	+	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
t_{01}	+	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{2}{6}$
t_{02}	+	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{2}{6}$	$-\frac{2}{6}$	$\frac{1}{6}$
t_{10}	+	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
t_{11}	+	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{2}{6}$	$\frac{1}{6}$
t_{12}	+	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{6}$	$-\frac{2}{6}$	$\frac{1}{6}$	$-\frac{2}{6}$
t_{20}	+	0	$\frac{1}{2}$	0	0	$-\frac{2}{6}$	$\frac{1}{6}$	$-\frac{2}{6}$	$-\frac{2}{6}$
t_{21}	+	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
t_{22}	+	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{2}{6}$	$-\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

The variances for estimates of effect-comparisons are found in much the same way as were the variances for pseudo-factorial effects in the 2^2 factorial experiment. For instance, an estimate of the A' effect-comparison from replicate I, $\hat{A}_I'$, within which the A effect is confounded (Table 3.5), is equal to the true A' comparison plus an error of

$$\frac{1}{3}[3\beta_1+\epsilon_1(00)+\epsilon_1(01)+\epsilon_1(02)]-\frac{1}{3}\ [3\beta_2+\epsilon_2(10)+\epsilon_2(11)+\epsilon_2(12)]$$

and therefore,

$$V(\hat{A}_I') = \frac{2}{3}[\,3\sigma_b^2 + \sigma_e^2\,] = \frac{2}{3W'} ,$$

where

$$W' = \frac{1}{3\sigma_b^2 + \sigma_e^2} .$$

Defining $\hat{B}_f'$, $\hat{AB}_f'$, $\hat{AB}_f^{2\prime}$, $\hat{A}_f''$, $\hat{B}_f''$, $\hat{AB}_f''$, and $\hat{AB}_f^{2\prime\prime}$ as estimates from replicate f of the effect-comparisons B', AB', $AB^{2\prime}$, A'', B'', AB'', and $AB^{2\prime\prime}$, respectively, then the variances of these estimates are found in the same manner as the variance for $\hat{A}_I'$ and are presented in Table 3.7.

Table 3.7

The Variances of Estimates of Effect-Comparisons

Effect-Comparison	In Replicates in which the Comparison is Confounded	In Replicates in which the Comparison is Unconfounded
$\hat{A}_f',\hat{B}_f',\hat{AB}_f',\hat{AB}_f^{2\prime}$	$\frac{2}{3W'}$	$\frac{2}{3W}$
$\hat{A}_f'',\hat{B}_f'',\hat{AB}_f'',\hat{AB}_f^{2\prime\prime}$	$\frac{2}{W'}$	$\frac{2}{W}$

Working along the same lines, the variance of $\hat{\mu}_f$, an estimate from replicate f of μ, is

$$V(\hat{\mu}_f) = \frac{1}{9W'} .$$

Combining the estimates of A' from all four replicates shown in Table 3.5, the weighted estimate for A' takes the form of

$$\hat{A}' = \frac{\frac{3W'}{2}\hat{A}'_I + \frac{3W}{2}\hat{A}'_{II} + \frac{3W}{2}\hat{A}'_{III} + \frac{3W}{2}\hat{A}'_{IV}}{\frac{3}{2}(W' + 3W)}$$

$$= \frac{W'\hat{A}'_I + W\hat{A}'_{II} + W\hat{A}'_{III} + W\hat{A}'_{IV}}{(W' + 3W)} ,$$

and, the variance of $\hat{A}'$ in r repetitions is

$$V(\hat{A}') = \frac{2}{3r(W' + 3W)} .$$

If $\hat{B}'$, $\widehat{AB}'$, $\widehat{AB^2}'$, $\hat{A}''$, $\hat{B}''$, $\widehat{AB}''$, and $\widehat{AB^2}''$ are defined in the same way as $\hat{A}'$, then their variances are as follows:

a. the $V(\hat{B}')$, $V(\widehat{AB}')$, and $V(\widehat{AB^2}')$ are equal to $\frac{2}{3r(W' + 3W)}$ and

b. the $V(\hat{A}'')$, $V(\hat{B}'')$, $V(\widehat{AB}'')$, and $V(\widehat{AB^2}'')$ are equal to $\frac{2}{r(W' + 3W)}$.

The variance of $\hat{\mu}$ for this design is

$$V(\hat{\mu}) = \frac{1}{36\, r\, W'} .$$

Covariances of pairs of the estimated effect-comparisons can be shown to be zero by a method similar to the one used to show zero covariances for the estimated effects in section 2.5. For example, the covariance of $\hat{A}'_I$ and $\hat{B}'_I$ is obtained by first expressing the effect-comparisons in terms of yields

$$A'_I = \tfrac{1}{2}[y_{00} + y_{01} + y_{02} - y_{10} - y_{11} - y_{12} + 0 + 0 + 0]$$

$$B'_I = \tfrac{1}{2}[y_{00} - y_{01} + 0 + y_{10} - y_{11} + 0 + y_{20} - y_{21} + 0].$$

Then, remembering the treatment arrangement given Table 3.4 and the model (2.1), $\hat{A}'_I$ is equal to the true A'_I,

$$A'_I = \tfrac{1}{2}[\pi_{00} + \pi_{01} + \pi_{02} - \pi_{10} - \pi_{11} - \pi_{12}],$$

plus an error of

$$\tfrac{1}{2}[\beta_1 + \epsilon_{1(00)} + \beta_1 + \epsilon_{1(01)} + \beta_1 + \epsilon_{1(02)} - \beta_2 - \epsilon_{2(10)}$$
$$- \beta_2 - \epsilon_{2(11)} - \beta_2 - \epsilon_{2(12)}]$$
$$= \tfrac{1}{2}[3\beta_1 + \epsilon_{1(00)} + \epsilon_{1(01)} + \epsilon_{1(02)} - 3\beta_2 - \epsilon_{2(10)}$$
$$- \epsilon_{2(11)} - \epsilon_{2(12)}].$$

A similar expression may be found for $\hat{B}'_I$, thus, $\hat{B}'_I$ is equal to the true B'_I plus an error of

$$\tfrac{1}{2}[\epsilon_{1(00)} - \epsilon_{1(01)} + \epsilon_{2(10)} - \epsilon_{2(11)} + \epsilon_{3(20)} - \epsilon_{3(21)}].$$

The covariance of $\hat{A}'_I$ and $\hat{B}'_I$ is therefore,

$$\text{Cov }(A'_I\ B'_I) = \frac{1}{4}[\ \sigma_e^2 - \sigma_e^2 - \sigma_e^2 + \sigma_e^2\ ] = 0.$$

From this and similar results the covariance of $\hat{A}'$ and $\hat{B}'$ from the complete design is $\text{Cov }(\hat{A}'\ \hat{B}') = 0$.

Working from Table 3.6, showing the linear expression of treatment estimates in terms of estimates of effect-comparisons, the variance of a treatment estimate, say t_{00}, for r replicates is

$$V(t_{00}) = \frac{1}{4}[V(\hat{A}') + V(\hat{B}') + V(\widehat{AB}') + V(\widehat{AB^2}')] + \frac{1}{36}[V(\hat{A}'')$$

$$+ V(\hat{B}'') + V(\widehat{AB}'') + V(\widehat{AB^2}'')] + V(\hat{\mu})$$

$$= \frac{1}{4}\left\{4\left[\frac{2}{3r(W'+3W)}\right]\right\} + \frac{1}{36}\left\{4\left[\frac{2}{r(W'+3W)}\right]\right\} + \frac{1}{36\ r\ W'}$$

$$= \frac{8}{9r(W'+3W)} + \frac{1}{36\ r\ W'} .$$

An examination of the remaining variances for treatment estimates reveals that each t_{ij} has the same variance which may be denoted by

$$V = \frac{8}{9r(W'+3W)} + \frac{1}{36\ r\ W'} .$$

The covariance of pairs of treatment estimates may also be found with the aid of Table 3.6. Thus, the covariance of t_{00} and t_{22} is

$$\text{Cov }(t_{00}\ t_{22}) = V(\hat{\mu}) - \frac{1}{4}\ V(\widehat{AB}') + \frac{1}{4}\ V(\widehat{AB^2}') - \frac{2}{36}[V(\hat{A}'')$$

$$+ V(\hat{B}'')] + \frac{1}{36}[V(\widehat{AB}'') + V(\widehat{AB^2}'')]$$

$$= \frac{1}{36\ r\ W'} - \frac{1}{9r(W'+3W)} .$$

In a similar way, the covariance of any pair of treatment estimates is found to be

$$C = \frac{1}{36\ r\ W'} + \frac{1}{9r(W'+3W)} ,$$

since all pseudo-factorial-effect-means are equally confounded.

In summary, the variance-covariance matrix for treatment estimates in this design is as presented in Table 3.8.

Table 3.8

Variance-Covariance Matrix for Treatment Estimates in a Balanced Lattice of Nine Treatments

	t_{00}	t_{01}	t_{02}	t_{10}	t_{11}	t_{12}	t_{20}	t_{21}	t_{22}
t_{00}	V	C	C	C	C	C	C	C	C
t_{01}		V	C	C	C	C	C	C	C
t_{02}			V	C	C	C	C	C	C
t_{10}				V	C	C	C	C	C
t_{11}					V	C	C	C	C
t_{12}						V	C	C	C
t_{20}							V	C	C
t_{21}								V	C
t_{22}									V

Note. $V = \frac{1}{9r}\left[\frac{1}{4W'} + \frac{8}{W'+3W}\right]$

$$C = \frac{1}{9r}\left[\frac{1}{4W'} - \frac{1}{W'+3W}\right]$$

3.5 Variances and Covariances for Treatments in a Simple Lattice

Continuing the previous example of nine treatments, a simple lattice would contain replicates I and II (see Table 3.4 and again in Table 3.9 below) in each basic repetition.

Table 3.9

Arrangement of Nine Treatments in a Simple Lattice

Replicate I					Replicate II				
(1)	T_{00}	T_{01}	T_{02}	A_0	(4)	T_{00}	T_{10}	T_{20}	B_0
(2)	T_{10}	T_{11}	T_{12}	A_1	(5)	T_{01}	T_{11}	T_{21}	B_1
(3)	T_{20}	T_{21}	T_{22}	A_2	(6)	T_{02}	T_{12}	T_{22}	B_2
A Effect Confounded					B Effect Confounded				

In this design only two pseudo-factorial-effects, A and B, are confounded. Using the variances of the effect-comparisons developed in the last subsection, the weighted mean for A' is

$$\hat{A}' = \frac{W' \hat{A}'_{I} + W \hat{A}'_{II}}{W' + W} ,$$

and thence,

$$V(\hat{A}') = \frac{2}{3(W'+W)r} \text{ for r repetitions.}$$

Since B' is also confounded, $\hat{B}'$ also has variance of $\frac{2}{3r(W'+W)}$. Similarly, the variances of $\hat{A}''$ and $\hat{B}''$ can be shown to be $\frac{2}{r(W'+W)}$.

The variances of the unconfounded pseudo-factorial-effect-comparisons $\widehat{AB}'$ and $\widehat{AB^2}'$ are $\frac{1}{3rW}$ and the variances of $\widehat{AB}''$ and $\widehat{AB}^{2}{''}$ are $\frac{1}{r\,W}$. The variance of $\hat{\mu}$ is found to be $\frac{1}{18\,r\,W'}$.

The variances and covariances for treatment estimates may now be found withthe aid of the linear relations in Table 3.6. As in the balanced design, the simple lattice has only one variance form and this variance is

$$V_2 = \frac{1}{9r}\left[\frac{1}{2W'} + \frac{4}{W'+W} + \frac{2}{W}\right].$$

Upon investigating the covariances of the treatment estimates, two covariance forms become evident. These covariance forms are:

i. For estimates of treatment pairs appearing together once in the same block, the covariance form is $C_{2,1} = \frac{1}{9r}\left[\frac{1}{2W'} + \frac{1}{W'+W} - \frac{1}{W}\right]$. This form is obtained from pairs of treatment estimates, $t_{ij}\ t_{mq}$, where either $i = m$ or $j = q$.

ii. For estimates of treatment pairs not appearing together in the same block, the covariance form is $C_{2,2} = \frac{1}{9r}\left[\frac{1}{2W'} - \frac{2}{W'+W} + \frac{1}{2W}\right]$ and developes from pairs of estimates $t_{ij}\ t_{mq}$, where either $i + j = m + q$ or $i + 2j = m + 2q$.

The variance - covariance matrix for the treatment estimates in this simple lattice is shown in Table 3.10.

Table 3.10

The Variance - Covariance Matrix for Treatment Estimates in a Simple Lattice of Nine Treatments

	t_{00}	t_{01}	t_{02}	t_{10}	t_{11}	t_{12}	t_{20}	t_{21}	t_{22}
t_{00}	V_2	$C_{2,1}$	$C_{2,1}$	$C_{2,1}$	$C_{2,2}$	$C_{2,2}$	$C_{2,1}$	$C_{2,2}$	$C_{2,2}$
t_{01}		V_2	$C_{2,1}$	$C_{2,2}$	$C_{2,1}$	$C_{2,2}$	$C_{2,2}$	$C_{2,1}$	$C_{2,2}$
t_{02}			V_2	$C_{2,2}$	$C_{2,2}$	$C_{2,1}$	$C_{2,2}$	$C_{2,2}$	$C_{2,1}$
t_{10}				V_2	$C_{2,1}$	$C_{2,1}$	$C_{2,1}$	$C_{2,2}$	$C_{2,2}$
t_{11}					V_2	$C_{2,1}$	$C_{2,2}$	$C_{2,1}$	$C_{2,2}$
t_{12}						V_2	$C_{2,1}$	$C_{2,2}$	$C_{2,1}$
t_{20}							V_2	$C_{2,1}$	$C_{2,1}$
t_{21}								V_2	$C_{2,1}$
t_{22}									V_2

Note. $V_2 = \frac{1}{9r}\left[\frac{1}{2W'} + \frac{4}{W'+W} + \frac{2}{W}\right]$, $C_{2,1} = \frac{1}{9r}\left[\frac{1}{2W'} + \frac{1}{W'+W} - \frac{1}{W}\right]$,

and $C_{2,2} = \frac{1}{9r}\left[\frac{1}{2W'} - \frac{2}{W'+W} + \frac{1}{2W}\right]$

3.6 Variances and Covariances for Treatment Estimates in a Triple Lattice

In triple lattice designs, three pseudo-factorial-effects are confounded and replicates I, II, and III (Table 3.4) are included in each repetition. In replicates of these types, the effect-means in A, B, and AB are confounded and since replicate IV is not included in this design, the effect-means in AB^2 are not confounded.

By proceeding as in the previous subsection, the variance of a treatment estimate is found to be

$$V_3 = \frac{1}{9r}\left[\frac{1}{3W'} + \frac{6}{W'+2W} + \frac{2}{3W}\right].$$

In the triple lattice, as in the simple lattice, two covariance forms exist:

i. For treatment pairs, t_{ij} t_{mq}, occurring once in the same block, that is, for pairs of treatments for which $i = m$, $j = q$, or $i + j = m + q$, the covariance is given by

$$C_{3,1} = \frac{1}{9r}\left[\frac{1}{3W'} - \frac{1}{3W}\right].$$

ii. For estimates of treatment pairs not appearing in the same block, that is, for treatment pairs t_{ij} t_{mq} where $i + 2j = m + 2q$, the covariance is

$$C_{3,2} = \frac{1}{9r}\left[\frac{1}{3W'} - \frac{3}{W'+2W} + \frac{2}{3W}\right].$$

The variances and covariances for treatment estimates in a triple lattice are presented in Table 3.11.

Table 3.11

The Variance - Covariance Matrix for Treatment Estimates in a Triple Lattice of Nine Treatments

	t_{00}	t_{01}	t_{02}	t_{10}	t_{11}	t_{12}	t_{20}	t_{21}	t_{22}
t_{00}	V_3	$C_{3,1}$	$C_{3,1}$	$C_{3,1}$	$C_{3,2}$	$C_{3,1}$	$C_{3,1}$	$C_{3,1}$	$C_{3,2}$
t_{01}		V_3	$C_{3,1}$	$C_{3,1}$	$C_{3,1}$	$C_{3,2}$	$C_{3,2}$	$C_{3,1}$	$C_{3,1}$
t_{02}			V_3	$C_{3,2}$	$C_{3,1}$	$C_{3,1}$	$C_{3,1}$	$C_{3,2}$	$C_{3,1}$
t_{10}				V_3	$C_{3,1}$	$C_{3,1}$	$C_{3,1}$	$C_{3,2}$	$C_{3,1}$
t_{11}					V_3	$C_{3,1}$	$C_{3,1}$	$C_{3,1}$	$C_{3,2}$
t_{12}						V_3	$C_{3,2}$	$C_{3,1}$	$C_{3,1}$
t_{20}							V_3	$C_{3,1}$	$C_{3,1}$
t_{21}								V_3	$C_{3,1}$
t_{22}									V_3

Note.

$$V_3 = \frac{1}{9r}\left[\frac{1}{3W'} + \frac{6}{W'+2W} + \frac{2}{3W}\right],$$

$$C_{3,1} = \frac{1}{9r}\left[\frac{1}{3W'} - \frac{1}{3W}\right], \text{ and}$$

$$C_{3,2} = \frac{1}{9r}\left[\frac{1}{3W'} - \frac{3}{W'+2W} + \frac{2}{3W}\right].$$

IV. AN F TEST FOR DIFFERENCES AMONG CORRELATED TREATMENT MEANS

It can readily be shown (see Sanders, 1953) that if correlated means have equal variances and constant covariances; that is, if the variance-covariance matrix, Σ_t, for the treatment estimates is in the form

$$\Sigma_t = \begin{bmatrix} V & C & \cdot & \cdot & \cdot & C & C \\ C & V & \cdot & \cdot & \cdot & C & C \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & & \cdot & \cdot & \cdot \\ C & C & \cdot & \cdot & \cdot & V & C \\ C & C & \cdot & \cdot & \cdot & C & V \end{bmatrix}, \text{ then the ratio}$$

$\sum_{i=1}^{p} (t_i - \bar{t})^2 \Big/ \frac{1}{2}\sigma_d^2$ has a chi-square distribution with (p-1) degrees of freedom, where:

$\sum_{i=1}^{p} (t_i - \bar{t})^2$ is the corrected sum of squares of the p means involved and

σ_d^2 is the expected variance of a treatment difference.

Thence, if an estimate V_d of σ_d^2 exists, such that, V_d is independent of $t_1, t_2, \ldots, t_p$ and $\frac{n_2 V_d}{\sigma_d^2}$ is distributed as chi-square with n_2 degrees of freedom, the ratio $\frac{1}{p-1} \sum_{i=1}^{p} (t_i - \bar{t})^2 \Big/ \frac{1}{2} V_d$ has an F distribution with (p-1) and n_2 degrees of freedom.

In effect, proper allowance is made for the correlation between treatment means by substituting $\frac{1}{2}V_d$ for the estimated variance, $s_{\bar{x}}^2$, of a treatment mean which is used in the F ratio $\frac{1}{p-1} \sum_{i=1}^{p} (\bar{x}_i - \bar{\bar{x}})^2 / s_{\bar{x}}^2$, when the means are independent. Thus, it follows that modified F ratios in the Multiple Comparisons Test provide a valid test for correlated means, if two assumptions are met:

assumption (i) $\frac{n_2 V_d}{\sigma_d^2}$ is distributed as chi-square with n_2 degrees of freedom and is independent of the treatment estimates and

assumption (ii) the treatment estimates have constant variances and constant covariances.

V. APPLICATION OF THE MULTIPLE COMPARISONS TEST TO BALANCED, SIMPLE, and TRIPLE LATTICES

In the previous section a modification of the F ratios in the Multiple Comparisons Test was proposed which extends these procedures to comparisons of correlated means. When this modification is applied to correlated treatment means in a lattice design one or two approximations have to be accepted.

One approximation arises from the fact that the weights W and W', from which V_d is computed, must be estimated from the analysis of variance. This estimation offends assumption (i) above. However, the substitution of estimates for the true weights has been accepted in all analyses of incomplete block designs whenever the usual procedures for the recovery of interblock information are utilized and F and t tests have been used. The inaccuracies introduced by using estimates for W and W' (see Kempthorne, 1952, section 23.6 or Cochran and Cox, 1950, section 9.32) are not considered appreciable and in most lattice designs the degrees of freedom for estimating the weights are large enough for the resulting inaccuracy in estimation to be neglected.

A second approximation is necessary when the covariances for pairs of treatment estimates are not constant, as in simple (Table 3.10) or triple (Table 3.11) lattices. Since the variances and covariances of the treatment estimates are both constant in a balanced lattice (see Tables 2.4 and 3.8) and the variance-covariance matrix is in the required form, only the first approximation is necessary when the modified Multiple Comparisons Test procedures are used on treatments in this case. However, in the unbalanced designs, the variance-covariance matrices for the treatment estimates are not in the form required for the application of the modified F ratios and these matrices have to be altered to the required form with constant covariances.

One approach to this problem is to replace the covariances for the individual pairs of treatment estimates by an average covariance, which would convert the matrices into the required form

$$\begin{bmatrix} V & \overline{C} & \cdot & \cdot & \cdot & \overline{C} & \overline{C} \\ \overline{C} & V & \cdot & \cdot & \cdot & \overline{C} & \overline{C} \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & & \cdot & \cdot & \cdot \\ \overline{C} & \overline{C} & \cdot & \cdot & \cdot & V & \overline{C} \\ \overline{C} & \overline{C} & \cdot & \cdot & \cdot & \overline{C} & V \end{bmatrix}$$

and the modified Multiple Comparisons Test could then be applied. The use of an average covariance in this way is implicitly suggested by Kempthorne (1952, p. 460), Cochran and Cox (1950, chapter 10), and others when they use an average estimated variance, $\overline{V}_d$, of treatment differences, rather than estimated variances of individual treatment differences. Thus in applying the test to treatments in an unbalanced design, the proposed modification is to use $\sqrt{\frac{1}{2}\overline{V}_d}$ in place of the standard error, $s_{\overline{x}}$, of a treatment mean.

In getting the average covariance for an unbalanced design, the relative frequencies of occurrence of individual covariances are first obtained, and the different values of the covariances are weighted with these frequencies. The average covariance for the simple lattice in section 3.5 is found to be

$$\overline{C}_2 = \frac{1}{18r}\left[\frac{2}{2W'} - \frac{1}{W'+W} - \frac{1}{2W}\right],$$

whereas, the average covariance for the triple lattice in section 3.6 is

$$\overline{C}_3 = \frac{1}{36r}\left[\frac{4}{3W'} - \frac{3}{W'+2W} - \frac{1}{3W}\right].$$

The average variance, $\overline{\sigma}^2_d$, of a difference between two treatments, $t_i - t_j$, may then be found from the following relationship

$$(5.1) \quad \overline{\sigma}^2_d = 2\sigma^2_t - 2\overline{C} \quad .$$

For a balanced lattice of nine treatments, the average variance of treatment differences is

$$\sigma^2_d = 2\left\{\frac{1}{9r}\left[\frac{1}{4W'} + \frac{8}{W'+3W}\right] - \frac{1}{9r}\left[\frac{1}{4W'} - \frac{1}{W'+3W}\right]\right\}$$

$$= \frac{2}{r(W'+3W)} \quad .$$

For a simple lattice of nine treatments,

$$\sigma^2_d = 2\left\{\frac{1}{9r}\left[\frac{1}{2W'} + \frac{4}{W'+W} + \frac{2}{W}\right] - \frac{1}{18r}\left[\frac{2}{2W'} - \frac{1}{W'+W} - \frac{1}{2W}\right]\right\}$$

$$= \frac{1}{2r}\left[\frac{2}{W'+W} + \frac{1}{W}\right] \quad .$$

For a triple lattice of nine treatments,

$$\sigma^2_d = 2\left\{\frac{1}{9r}\left[\frac{1}{3W'} + \frac{6}{W'+2W} + \frac{2}{3W}\right] - \frac{1}{36r}\left[\frac{4}{3W'} - \frac{3}{W'+2W} - \frac{1}{3W}\right]\right\}$$

$$= \frac{1}{2r}\left[\frac{3}{W'+2W} + \frac{1}{3W}\right] \quad .$$

The general expressions for the average variances of treatment differences for lattices of k^2 treatments in r repetitions are:

(a) balanced lattice

$$(5.2) \quad \frac{2}{r(W'+k\,W)}$$

(b) simple lattice

(5.3) $\frac{2}{r(k+1)}\left[\frac{2}{W'+W}+\frac{k-1}{2W}\right]$

(c) triple lattice

(5.4) $\frac{2}{r(k+1)}\left[\frac{3}{W'+2W}+\frac{k-2}{3W}\right]$.

In the actual analysis, the weights W and W' are replaced by their estimated values w and w'. When this is done the estimate $\bar{V}_d$ is obtained in place of $\bar{\sigma}_d^2$ and the degrees of freedom, n_2, are assumed to be equal to the degrees of freedom for intrablock error.

The degree of approximation involved in using $\bar{V}_d$, depends on the heterogeneity of the various covariance values. When these values vary a great deal two other approaches to this problem are possible. Each procedure reduces the degree of approximation involved but involves more complex computations. The first solution is to calculate an average variance of a difference among the actual treatments involved in the numerator of each individual F ratio $\frac{1}{p-1}\sum_{i=1}^{p}\frac{(t_i-\bar{t})^2}{\frac{1}{2}\bar{V}_d}$. This would entail the computation of an average variance of a treatment difference for a large number of combinations and would involve a fair amount of work. A second solution, one that would be exact and would not involve the approximation (ii) at all, would be to invert the variance-covariance matrix for each combination of treatments involved. Then, exact F tests (begging assumption (i)) could be obtained from quadratic forms based on these matrices. The number of matrix inversions alone would make this method prohibitive.

If $\sqrt{\frac{1}{2}V_d}$ is used in place of the standard error of a mean in all the component F tests, the approximations involved would be no greater than those implicitly accepted in the usual uses of V_d and in most cases this procedure can be recommended.

VI. AN EXAMPLE OF THE MULTIPLE COMPARISONS TEST APPLIED TO A TRIPLE LATTICE

The actual application of the Multiple Comparisons test to a lattice design will be illustrated using data from an experiment presented by Cochran and Cox (1950, chapter 10). The original plan of the experiment was a balanced lattice designed to investigate the effects of nine feeding treatments on the growth rates of pigs. However, for the present purposes, the experiment has been abridged to form a triple lattice by omitting replicate I.

The results, weight gains (pounds per day) for a total of two pigs, the analysis of variance table, and the treatment means adjusted for differences among litters (blocks) are shown in Tables 6.1, 6.2, and 6.3, respectively. In Table 6.1, the blocks consist of sets of litter-mates. The treatment means in Table 6.3 have been ranked in ascending order for convenience in applying the multiple comparisons procedure.

Table 6.1

Gains in Weights (pounds per day) for a

Total of 2 Pigs

Replicate II

Blocks				Totals
4	(1) 1.19	(4) 1.20	(7) 1.15	3.54
5	(2) 2.26	(5) 1.07	(8) 1.45	4.78
6	(3) 2.12	(6) 2.03	(9) 1.63	5.78
				14.10

Replicate III

Blocks				Totals
7	(1) 1.81	(5) 1.16	(9) 1.11	4.08
8	(2) 1.76	(6) 2.16	(7) 1.80	5.72
9	(3) 1.71	(4) 1.57	(8) 1.13	4.41
				14.21

Replicate IV

Blocks				Totals
10	(1) 1.77	(6) 1.57	(8) 1.43	4.77
11	(2) 1.50	(4) 1.60	(9) 1.42	4.52
12	(3) 2.04	(5) 0.93	(7) 1.78	4.75
				14.04

The numbers in parentheses indicate treatments.

Table 6.2

Analysis of Variance for Total Growth Rate of 2 Pigs in a Triple Lattice Design

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	
Replicates	2	0.0016		
Treatments (unadjusted)	8	2.1395		
Blocks (adjusted)	6	1.0312	0.1719	B
Intrablock error	10	0.3936	0.03936	E
Total	26	3.5659		

Table 6.3

Adjusted Mean Gain in Weight (pounds per day) for a Total of 2 Pigs

t_5	t_8	t_9	t_7	t_4	t_1	t_2	t_6	t_3
0.955	1.320	1.335	1.646	1.660	1.733	1.741	1.814	1.913

Before the variance of a treatment difference can be computed, the two weighting factors, w and w', must be estimated. These weighting factors may be computed from formulas given in (6.1).

(6.1) $w = \frac{1}{E}$ and $w' = \frac{3r-1}{3rB-E}$, where E and B are obtained from Table 6.2. Thus,

$$w = \frac{1}{0.03936} = 25.3807 \quad \text{and}$$

$$w' = \frac{3 - 1}{3(0.1719) - 0.03936} = 4.1990 \ .$$

From equation (5.4), the average variance of a treatment difference in a triple lattice is

$$\overline{V}_d = \frac{2}{k+1}\left[\frac{3}{2w+w'} + \frac{k-2}{3w}\right]$$

where k is the number of treatments included within a block. In this example, the average variance of a treatment difference is

$$\overline{V}_d = \frac{2}{3+1}\left[\frac{3}{2(25.3807) - 4.1990} + \frac{3 - 2}{3(25.3807)}\right]$$
$$= 0.03395 \ .$$

Thence, the effective standard error of a treatment mean for use in the Multiple Comparisons Test is $s=\sqrt{\frac{1}{2}\overline{V}_d}=0.1304$.

The Multiple Comparisons Test can now be applied to the adjusted treatment means using the rules given by Duncan (1951). Following his procedure the tests are conducted in two stages.

Stage 1

Step 1:

The first step is to establish the least significant ranges, $R_{p,n_2,\alpha}$. These ranges are computed by multiplying $s = 0.1304$ by the significant ranges, $R'_{p,n_2,\alpha}$, tabulated by Duncan (1951). The factors, $R'_{p,n_2,\alpha}$, depend on:

α, the desired significance level (a five-per cent level will be employed for this illustration),

n_2, the degrees of freedom connected with the intra-block error ($n_2 = 10$, see Table 6.2), and

p, the number of means in a combination for which R'_p is the range.

The significant ranges, $R'_{p,10,5}$, and corresponding least significant ranges, $R_{p,10,5} = s\ R'_{p,10,5}$, are listed in Table 6.4

Table 6.4

	Combination Size							
p	2	3	4	5	6	7	8	9
	Significant Ranges							
R'_p	3.151	3.444	3.691	3.908	4.101	4.28	4.44	4.58
	Least Significant Ranges							
R_p	0.4109	0.4491	0.4813	0.5096	0.5348	0.558	0.579	0.597

Step 2:

The range of the most extreme means, $t_5 - t_3$, is tested against R_9 (since $t_5 - t_3$ is the range of nine treatments). As $t_5 - t_3 = 0.955$ is greater than R_9 (=0.597), the difference is significant at the five-per-cent level and the procedure is continued by testing the range, $t_3 - t_8$ (=0.613), of the largest and the second smallest means against R_8 (=0.579). In this case, a significant difference exists and the testing is continued. The range between t_3 and each successively increasing mean is tested in turn, until, a range is found which is not greater than R_p. In the present set of tests, this difference is $t_3 - t_7$ (=0.267), which is not greater than R_6 (=0.5348). At this point, any decisions about treatments included between t_3 and t_7 are deferred to stage 2.

Similar procedures are employed to comparisons with each decreasing mean and with one exception, each difference is significant if it exceeds R_p; otherwise, the decision is deferred to stage 2. The exception to this rule is that no difference can be declared significant if the two means concerned are both contained in a subset of means with a non-significant range. Following this rule, the next step is testing the ranges between: the second largest and the smallest means, the second largest and the second smallest means, etc., until either a non-significant range is found or the range is contained within a larger subset (t_3 to t_7) which has a non-significant range. Each decreasing mean is tested in a similar way until the last possible comparison is made under the rules given above. A complete summary of stage 1 tests is presented below:

1. $t_3 - t_5 = 0.958 > R_9 = 0.597$, significant
   $t_3 - t_8 = 0.613 > R_8 = 0.579$, significant
   $t_3 - t_9 = 0.578 > R_7 = 0.558$, significant
   $t_3 - t_7 = 0.267 \not> R_6 = 0.5348$, deferred to stage 2

2. $t_6 - t_5 = 0.859 > R_8 = 0.579$, significant

   $t_6 - t_8 = 0.494 \ngtr R_7 = 0.558$, deferred to stage 2

3. $t_2 - t_5 = 0.786 > R_7 = 0.558$, significant

   $t_2 - t_8$, no test, contained within t_6 and t_8 (see 2 above)

4. $t_1 - t_5 = 0.778 > R_6 = 0.5348$, significant

   $t_1 - t_8$, no test, contained within t_6 and t_8 (see 2 above)

5. $t_4 - t_5 = 0.705 > R_5 = 0.5096$, significant

   $t_4 - t_8$, no test, contained within t_6 and t_8 (see 2 above)

6. $t_7 - t_5 = 0.691 > R_4 = 0.4813$, significant

   $t_7 - t_8$, no test, contained within t_6 and t_8 (see 2 above)

7. $t_9 - t_5 = 0.380 \ngtr R_3 = 0.4491$, deferred to stage 2.

At this point all stage one testing ceases, since the remaining treatment comparisons are contained in a larger subset having a non-significant range.

If a common bracket is placed beneath treatment means having a non-significant range in stage one testing, then the foregoing steps may be presented in a simple table.

Table 6.5

Significance Established between Means in Stage 1

t_5	t_8	t_9	t_7	t_4	t_1	t_2	t_6	t_3
0.955	1.320	1.335	1.646	1.660	1.733	1.741	1.814	1.913

Stage 2

In this stage, each of the subgroups left bracketed together at the end of stage one is examined further. A complete discussion of this portion of the Multiple Comparisons Test is presented by Duncan (1950, pp. 182-185). These procedures will be briefly illustrated with subgroup $t_8t_9t_7t_4t_1t_2t_6$ (see Table 6.5) which has exhibited a non-significant range in stage one.

The main purpose of the stage two tests is to see whether the range of the subgroup under consideration, in this case $t_6 - t_8$, is significant. For $t_6 - t_8$ to be significant, the sum of squares of each subset of means enveloping t_6 and t_8 must exceed a corresponding least significant sum of squares, $S_p = \frac{1}{2}R_p^2$. Table 6.6 shows the least significant sum of squares for the present example, which are calculated from the least significant ranges in Table 6.4. A least significant sum of squares for a subset of $p = 2$ means is not needed, since an identical test can be made more simply in terms of the range.

Table 6.6

Least Significant Sum of Squares (S_p)					
p	3	4	5	6	7
S_p	0.1003	0.1158	0.1298	0.1430	0.1557

Stage two testing starts by computing the sum of squares for the entire subgroup. For group $t_8t_9t_7t_4t_1t_2t_6$, the sum of squares equals $SS = 18.3145 - \frac{(11.249)^2}{7} = 0.2374$. This is greater than S_7 (=0.1557) and the test can proceed. For convenience, the sum of squares of the smallest subset, t_8t_6, is tested next. This is done by calculating the range $t_6 - t_8 = 0.494$, which exceeds R_2 (=0.4109). Thus, the sum of squares for t_8 and t_6 is significant and the test is continued.

Since $t_6 - t_8$ also exceeds R_4 (=0.4813), the sums of squares of all subsets enveloping t_8 and t_6 which contain four means or less may be concluded to be significant. The next step is to find the subset of five means enveloping t_8 and t_6 which has the smallest sum of squares. This subset is $t_8t_7t_4t_1t_6$, for which the sum of squares is $SS_{87416} = 0.1416$ exceeding S_5 (=0.1298). Thus, all subsets containing five means and enveloping t_8 and t_6 are significant. The last step in testing this subgroup is to examine subsets containing six means which envelope t_8 and t_6. The subset of six means having the smallest sum of squares is $t_8t_7t_4t_2t_1t_6$ and its sum of squares is SS_{874216} = 0.1510 exceeding S_6 (=0.1430). Therefore, all enveloping subsets containing six means have significant sum of squares. These tests ensure that all enveloping subsets of t_8 and t_6 have significant sums of squares and hence, $t_6 - t_8$ is significant. The bracket is now removed from under $t_8t_9t_7t_4t_2t_1t_6$ and new brackets are placed under $t_8t_9t_7t_4t_1t_2$ and $t_9t_7t_4t_2t_1t_6$. These latter groups are now eligible for further testing under stage one.

The same stage two procedure is applied to all subgroups left bracketed together at the end of stage one. After all subgroups have been completely tested, the two stages of testing may be summarized as in Table 6.7, which shows the final results for this example.

Table 6.7

Significance Established between Means at End of Stage 2

t_5	t_8	t_9	t_7	t_4	t_1	t_2	t_6	t_3
0.955	1.320	1.335	1.646	1.660	1.733	1.741	1.814	1.913

If desired a stage three procedure (Duncan, 1950, p. 186) could be applied to group $t_8t_9t_7t_4t_1t_2$, which has a significant sum of squares but does not have a significant range since the three means $t_8t_7t_2$ do not have a significant sum of squares. The stage three procedure will not be presented here, since the above discussion of stage one and stage two will serve the purpose of this section.

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