Analysis of Composite Laminates with Matrix Cracks

by

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Thesis submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of
Master of Science
in
Engineering Mechanics

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August 1987
Blacksburg, Virginia
Analysis of the effects of matrix cracking on composite laminates is a well-known problem which has attracted considerable attention for the past decade. An approximate analytical solution is introduced in this thesis to study this type of problem.

The subjects of primary concern are the degradation of effective laminate properties, such as axial stiffness, Poisson’s ratio, shear modulus, and coefficient of thermal expansion, as a function of crack density and the axial stress redistribution due to the existence of matrix cracks. Both transverse cracks (2-D problem) and cross (transverse and longitudinal) cracks (3-D problem) are studied. Results for graphite/epoxy cross-ply laminates are presented and compared to those of other approaches. Some other materials, for instance, glass/epoxy, are also studied. The results and comparisons will appear where appropriate. In general, the agreement between the results of the present analysis and those of other approaches, in particular, the finite element method, is good for the lower crack density. The present study shows that Poisson’s ratio may be a good indicator of the degree of damage for a cracked laminate.
Acknowledgements

This study was supported by the CIT Institute of Materials Science and Engineering through Grant MAT-86-018. The author would like to express his deep gratitude to Dr. Aboudi and Dr. Herakovich for their guidance and advice. The author is also indebted to Dr. Pindera for some valuable suggestions. A special thanks goes to [Name] for her cooperation during this study.

The author would also like to dedicate this thesis to his wife [Name]. Without her loving care and understanding, the author would not have been able to finish his thesis in eleven months.
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1.0 Introduction and Literature Review

1.1 Introduction

It has been evident for a number of years that laminated fibrous composites are a promising alternative for conventional structural materials. The potential of composites to be the candidate for advanced structures is their high specific stiffness and strength and low coefficient of thermal expansion. Laminated composites provide engineers the capability to tailor the orientation, thickness, and stacking sequence of fiber reinforced laminae for a specific application. This, of course, has great advantage over the use of conventional homogeneous materials such as aluminum and steel.

One of the most significant differences between fibrous composite laminates and traditional engineering materials is that the damage development and failure of the former are much more complicated. For the latter, failure usually results from the propagation of a single flaw. However, unlike the conventional materials, several modes of damage, such as matrix cracking, fiber breakage, fiber-matrix debonding, and delamination, have been observed and may account for the eventual failure of composite laminates. Because of this complexity in damage modes, no efficient failure criterion, corresponding to fracture mechanics for isotropic materials, has been developed for com-
posites. On the other hand, it may be possible to predict the residual material properties and strength of composite laminates if the damage modes are completely understood.

Consider a cross-ply laminate subjected to uniaxial quasi-static or cyclic loading. The early stage of damage development is dominated by matrix cracking in the off-axis plies. These matrix cracks typically develop in the fiber direction and extend across the entire laminate from edge to edge. The number of cracks increases monotonically with increase of load or number of cycles until a saturation crack density is reached. In this state of damage development, the cracks are uniformly distributed throughout the laminate. Such damage causes the redistribution of stresses and reduction of effective properties. Accordingly, subsequent damage may be initiated. Since the transverse cracks extend in a specific direction only, it is possible to employ a two dimensional analysis to study this type of problem. After the transverse cracks fully develop, the next possible damage is longitudinal cracks in the 0° plies along the fiber direction. The longitudinal cracks together with the transverse cracks in the off-axis plies form a system of cross cracks. Again, this system of cracks will result in stress redistribution and property degradation and so on. The study of this type of problem involves a three dimensional analysis.

It is a standard design procedure to ensure that the components of a structure can endure the service loading assuming that some specific defects exist within the structure. Usually these assumed defects are the smallest cracks or voids which can be inspected by some nondestructive testing techniques. Since the effective material properties of a composite laminate can be measured during service without further degrading the material, it is believed that they may serve as indicators of the degree of damage in cracked laminates.

In the present research, an approximate analytical procedure based on Aboudi's model [1] for cracked solids is applied to study the effects of matrix cracks on composite laminates. As mentioned previously, this study is concerned with predicting the degradation of effective properties of a cracked laminate as an indicator of its degree of damage, namely, crack density. The properties of interest are axial stiffness, Poisson's ratio, shear modulus, and coefficient of thermal expansion.

Introduction and Literature Review
(CTE). Since the stresses are responsible for the initiation of subsequent damage, it is also very important to understand the redistribution of stresses caused by the matrix cracks. However, in order to keep this analysis within a reasonable bound, only the distribution of axial stress is presented in this study. The damage modes considered include transverse cracks in the off-axis plies and cross (transverse and longitudinal) cracks. Three types of symmetric cross-ply laminates are investigated. The material under consideration is graphite/epoxy (for properties, see Appendix A), but not exclusively. The results and comparisons for other materials are presented where appropriate.

1.2 Literature Review

The analysis of transverse cracking in composite laminates has been a well-known problem for the past decade. Both analytical studies and experimental investigations have been made on this subject. A thorough review of all the previous works would be exhausting; therefore, only those contributions which are directly related to the present study will be given in the following review. A more extensive literature review can be found in Adams et al. [2] and Highsmith [3].

At the early stage of damage studies, several investigations were made to observe the development of transverse cracking in the off-axis plies of a composite laminate. Most of the studies were concerned with determining the threshold stress or strain for cracking and the spacing between the adjacent cracks. For instance, Garret and Baily [4] studied the relationship between the matrix cracking in the inner 90° plies and the constraint in the outer 0° plies of a cross-ply composite laminate. They reported that the threshold strain for transverse cracking depended on the thickness of the 90° layers. An important achievement at this stage was the identification of the characteristic damage state (CDS) which was done by Reifsnider and his co-workers [5]. At this specific damage state, the transverse cracks reached a saturation crack density which implied a uniform distribution...
of cracks throughout the laminate. Reifsnider [6] also proposed a one dimensional shear lag model which can be used to predict the saturation crack density.

After the initial qualitative investigations of damage development, the effects of transverse cracks on the damaged laminate became the topic of primary concern. Typical results on this subject were given by Highsmith and Reifsnider [7]. They examined experimentally the mechanical response of several types of glass/epoxy laminates under monotonic and cyclic loading. It was noted that the transverse crack density increased with the increase of load or number of cycles until a saturation crack density was reached. Fairly large reduction in axial stiffness was observed for a [0/90], laminate. Significant degradation in Poisson's ratio in some other laminates were also reported. On the contrary, only very little decrease in shear modulus was detected. The absence of shear modulus degradation was attributed to the local nature of the strains measured by the strain gages [7]. In addition to experimental observations, the authors also used a shear lag analysis to predict the reduction of axial stiffness as a function of the transverse crack density.

Laws et. al. [8] developed a procedure based on the self-consistent scheme to obtain an estimated stiffness of cracked plies. By using this value in classical lamination theory, they calculated the effective stiffness of the damaged laminate.

Adams and Herakovich [9] investigated the degradation in coefficient of thermal expansion (CTE) for some cross-ply and quasi-isotropic graphite/epoxy laminates using a finite element analysis. The damage modes considered in this study were transverse cracking and delamination. It was shown that transverse cracks could have a significant influence on CTE, but delaminations, located symmetrically about the midplane of the laminate, had no effect on thermal expansion.

A variational method was adopted by Vasil'ev et. al. [10] and Hashin [11] to study the elastic property degradation and stress distribution in a cracked cross-ply laminate. It was assumed that the axial stress in each ply was constant through the ply thickness, including the region near the
crack tip where there was apparently a singularity in axial stress. Hashin’s prediction gave a lower bound of the degraded axial stiffness and shear modulus.

Talreja [12] used a different approach to formulate this type of problem. He characterized the damage mode by a vector field which was based on a continuum damage theory and derived relations between the elastic moduli and the magnitude of the field vectors. He fitted his coefficients with a set of measured data and found good agreement between experimental results and those predicted by his theory. In addition, he also noticed that there could be a large degradation in Poisson’s ratio while the change in stiffness was not as significant.

Sun and Jen [13] studied the effects of transverse cracks for [0\textsubscript{m}/90\textsubscript{n}], and [90\textsubscript{n}/0\textsubscript{m}], laminates using a finite element analysis. They observed that the inside or outside placement of the 90° layers had very little effect on the axial stress distribution; but if a soft and tough adhesive film was placed along the interface between 0° and 90° layers, the stress distribution was changed and, consequently, the strength of the laminate improved.

Altus et al. [14] used a finite difference method to analyze a hybrid composite laminate with transverse cracks in the outer 90° layers. They noted compressive axial stress at the outer surface near the crack when the laminate was subjected to a tensile loading. They attributed this compression to the bending effect due to the deformation in geometry and considered it as a factor to halt the process of transverse cracking.

A thorough study of the effects of transverse cracks on graphite/epoxy cross-ply laminates was recently done by Strauss and Herakovich [15] using a finite element analysis. The effective properties of concern in this study were axial stiffness, Poisson’s ratio, shear modulus, and CTE. Also, the stress distributions under different types of loading were presented in their analysis. In the subsequent chapters, the results of present study will be compared to theirs in a great detail.
Unlike the transverse crack problem, only very few investigations have studied the effects of cross cracks; i.e., transverse and longitudinal cracks, on the cross-ply laminates. A recent study was presented by Highsmith and Reifsnider [16]. This analysis was based on a structural theory proposed by Pagano [17] which was developed assuming that the inplane stresses of each ply would vary linearly through the thickness of that layer. Equilibrium considerations determined the order of variation of the other stress components. Reissner's variational principle yielded a set of differential equations. An approximate solution in terms of Chebyshev polynomials, in conjunction with imposed boundary conditions, provided a system of overdetermined algebraic equations which was solved by least squares. Details of this method can be found in Highsmith [3].

Aboudi [1] proposed an approximate analytical solution to study the problem of cracked solids. He simulated the displacement field with a second order series expansion. By considering the equilibrium and continuity conditions, and imposing the appropriate loading and boundary conditions, he calculated the effective elastic moduli of a solid with doubly periodical cracks. The present analysis is an extension of his approach. Furthermore, this method has been generalized to a three dimensional analysis for the cross crack problem in the present study.

In the following chapters, both transverse cracks (2-D problem) and cross cracks (3-D problem) are studied. The basic assumptions and general formulation for laminates with transverse cracks are given in the next chapter. The method of solution is also outlined in the same chapter. In Chapter 3, the problem of a laminate with transverse cracks subjected to uniaxial tension is formulated. The governing equations together with the relevant boundary and continuity conditions are presented. The axial stiffness, Poisson's ratio, and axial stress distribution of the damaged laminate are predicted. Comparison between the present results and others ([7] [11] [14] [15] [18]) is given. The corresponding problem under pure shear is analyzed in Chapter 4. The subject of concern is the degradation of shear modulus. The results of the present analysis are compared to those of the finite element predictions [15]. In Chapter 5, the thermal response of damaged cross-ply laminates is studied. The coefficient of thermal expansion is predicted by imposing a negative thermal loading. Comparison is also made between the present results and those of [15]. The
analysis of the cross crack problem is presented in Chapter 6. The variations of axial stiffness and Poisson's ratio are studied. The distributions of axial displacement and axial stress are also predicted. The present results are compared to those provided by Highsmith [3]. Conclusions from the previous chapters are summarized in Chapter 7.
2.0 The Transverse Crack Problem

2.1 Basic Assumptions

Consider a symmetric cross-ply laminate with transverse cracks as shown in Fig. 1. A Cartesian coordinate system is introduced such that the $x_1$-axis is normal to the planes of the transverse cracks. In order to construct a reasonable model for the analysis of this problem, the following assumptions are made:

1. Each ply of the laminate is homogeneous, orthotropic or transversely isotropic.
2. Adjacent layers are perfectly bonded together.
3. The dimensions of the laminate are infinite in the $x_1$ and $x_2$ directions.
4. The dimensions and properties of the laminate are symmetric with respect to its midplane; i.e., $x_3 = 0$ plane.
5. Cracks occur only in the $90^\circ$ layers, and develop completely through the thickness ($x_3$ direction) and the width ($x_2$ direction) of those layers.
6. Transverse cracks occur periodically along the $x_1$ direction; i.e., the spacing between any two adjacent cracks is uniform.
Figure 1. Symmetric Cross-Ply Laminate with Transverse Cracks
According to the assumptions listed above, the cracked laminate can be treated as a repeating cell (Fig. 2) which allows for a two dimensional generalized plane strain analysis. Since the planes \(x_3 = 0\) and \(x_1 = 0\) are planes of symmetry, it is possible to characterize the entire damage state by one-quarter of the repeating cell. However, the traction-free boundary conditions at the top and bottom surfaces are preferred in the subsequent formulation. Therefore, it is decided to take one half of the repeating cell as the modeled region. Because different layers may have different thickness and properties, it is more convenient to use local coordinates for each layer. In the present analysis, three sets of local coordinates \((x_1, x_2, \bar{x}_3)\) are introduced to the modeled region as shown in Fig. 3 where \(\alpha = 1, 2, 3\) denotes the layer number counted from the bottom of the laminate.

### 2.2 General Formulation

Because of the arrangement of the modeled region, it isn't possible to specify the traction boundary conditions in the \(x_1\) direction at \(x_1 = 0\) and \(x_1 = \frac{\ell_1}{2}\), except at the crack surface. Therefore, a displacement formulation is used to formulate this problem.

For a generalized plane strain analysis, let \(u_{ij}^{(\alpha)}\) and \(\sigma_{ij}^{(\alpha)}\) be the displacements and stress components in the \(\alpha\) layer with \(i, j = 1 \text{ and } 3\). Assume the expression of \(u_{ij}^{(\alpha)}\) in the form of series expansion in \(\bar{x}_3^{(\alpha)}\). By using the strain-displacement relations and the constitutive equations, \(\sigma_{ij}^{(\alpha)}\) can be expressed in terms of the expanded form of \(u_{ij}^{(\alpha)}\). In each layer, the equilibrium conditions must be satisfied. The equilibrium equations for a two dimensional analysis are partial differential equations with two coordinate variables. Integrating the equilibrium equations through the thickness of each layer by parts, we can eliminate the \(\bar{x}_3^{(\alpha)}\) dependence. The resulting boundary stress terms can also be eliminated by using the traction continuity conditions at the interfaces and the traction-free boundary conditions at the outer surfaces. As a result, the equilibrium equations reduce to a set of ordinary differential equations which have dependence on \(x_1\) only. These equilibrium equations,
Figure 2. Repeating Cell for a Laminate with Transverse Cracks
Figure 3. Modeled Region with Local Coordinates (2-D Analysis)
together with the displacement continuity conditions at the interfaces, provide sufficient equations for the unknown field variables from the assumed expansion of \( u^{(n)} \). Finally, by imposing the appropriate loading and boundary conditions and expressing them in the expanded form, a complete set of governing equations for the problem is obtained.

### 2.3 Method of Solution

It is quite common to use series expansions to solve elasticity problems. Fourier series may be the most popular method. However, any orthogonal set of functions can be used as the solution of differential equations. Since \( x_\alpha \) in the local coordinates is within the limits of \( \pm \frac{d_2}{2} \), it is preferable to choose a simple series expansion such as Legendre polynomials for the solution to this problem.

The first three Legendre polynomials are:

\[
P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1)
\]

Higher order polynomials can be determined by using the Rodrigues' formula:

\[
P_k(x) = \frac{1}{2^k k!} \frac{d^k}{dx^k} [(x^2 - 1)^k]
\]

In order to limit this study to a reasonable extent, Legendre polynomials up to the second order only are chosen. In fact, in the subsequent chapters it can be seen that the second order Legendre expansion already gives quite good results.

For the laminate in this study, the displacement field of the \( a \) layer can be assumed as a second order Legendre expansion in \( x_\alpha^{(n)} \) with the dimensional form shown below:
where the field variables, \( \Lambda_k^{(a)} = (\varphi_k^{(a)}, \psi_k^{(a)}) \) \((k = 0, 1, 2)\), are functions of \( x_1 \) only, and the repeated superscript or subscript \( a \) does not imply summation. Substituting (2.3) into the formulation described in the previous section, we will get a set of linear ordinary differential equations in terms of the unknown field variables \( \varphi_k^{(a)}, \psi_k^{(a)} \). It can be seen that the governing equations within the modeled region are second order differential equations while the boundary condition expressions are first order only. The derivatives of the field variables can be approximated by finite difference expressions. The interval of interest, \([0, \frac{\ell_1}{2}]\), is divided uniformly into \( N \) subintervals with the starting and ending points labeled 0 and \( N \), respectively. The length of each subinterval is then given by:

\[
\Delta x_1 = \frac{(\ell_1/2)}{N} \tag{2.4}
\]

The finite difference approximations for the first and second order derivatives of the field variables (for instance, \( \varphi_k^{(a)} \)) at any point \( n \) within the modeled region can be written as:

\[
\frac{\partial \varphi_k^{(a)}}{\partial x_1}|_n \approx \frac{\varphi_k^{(a)}|_{n+1} - \varphi_k^{(a)}|_{n-1}}{2\Delta x_1},
\]

\[
\frac{\partial^2 \varphi_k^{(a)}}{\partial x_1^2}|_n \approx \frac{\varphi_k^{(a)}|_{n+1} - 2\varphi_k^{(a)}|_n + \varphi_k^{(a)}|_{n-1}}{(\Delta x_1)^2} \tag{2.5}
\]

where \( n = 1, 2, ..., N-1 \). On the boundary, the first order derivatives at \( n = 0 \) and \( N \) can be approximated by the forward and backward difference expressions, respectively, as shown below:

\[
\frac{\partial \varphi_k^{(a)}}{\partial x_1}|_0 \approx \frac{\varphi_k^{(a)}|_{1} - \varphi_k^{(a)}|_{0}}{\Delta x_1},
\]

\[
\frac{\partial \varphi_k^{(a)}}{\partial x_1}|_N \approx \frac{\varphi_k^{(a)}|_{N} - \varphi_k^{(a)}|_{N-1}}{\Delta x_1} \tag{2.6}
\]
By substituting the difference expressions given by (2.4)-(2.6) for the derivatives of the field variables in the governing equations mentioned in the formulation, a banded matrix for the resulting linear algebraic equations is obtained as shown in Fig. 4. Consequently, the unknown field variables at points along $0 \leq x_i \leq \frac{L_i}{2}$ can be determined on the computer by any numerical banded matrix solver.

In the subsequent chapters, the damaged cross-ply laminates are studied under three types of loading. They are uniaxial tension, pure shear, and thermal loading. By the assumptions in the previous section and the nature of loading conditions, the first and the third cases are generalized plane strain analyses. Although the pure shear case is not a state of generalized plane strain, a similar methodology can be adopted to solve this problem. In order to keep the integrity of each individual problem, the detailed formulations are given in each chapter. Furthermore, an extension of the above mentioned formulation to the three dimensional analysis is presented in a later chapter.
Figure 4. Banded Structure of the Resultant Matrix
3.0 Transverse Cracks under Uniaxial Tension

3.1 Formulation

By the assumptions in the last chapter, a generalized plane strain analysis can be used for a damaged laminate under uniaxial tension. Let \( u_\ell^{(g)} \) and \( \sigma_\ell^{(g)} \) be the displacements and stress components in the \( \alpha \) layer with \( i, j = 1 \) and 3. By consideration of symmetry and the traction-free conditions at the outer surfaces and the crack surface, the following boundary conditions are imposed on the modeled region (see Fig. 5):

\[
\begin{align*}
\sigma_{31}^{(1)} &= \sigma_{33}^{(1)} = 0, & 0 \leq x_1 \leq \frac{\ell_1}{2}, & x_3^{(1)} = -\frac{d_1}{2} \\
\sigma_{31}^{(3)} &= \sigma_{33}^{(3)} = 0, & 0 \leq x_1 \leq \frac{\ell_1}{2}, & x_3^{(3)} = \frac{d_3}{2}
\end{align*}
\]  
(3.1)

and

\[
\begin{align*}
u_1^{(g)} = 0, & \quad \sigma_{13}^{(g)} = 0, & x_1 = 0, & \quad |x_3^{(g)}| \leq \frac{d_\alpha}{2}
\end{align*}
\]  
(3.3)
Figure 5. Boundary Conditions under Uniaxial Tension
\[ u_1^{(1)} = U, \quad \sigma_{13}^{(1)} = 0, \quad x_1 = \frac{\ell_1}{2}, \quad |\bar{x}_3^{(1)}| \leq \frac{d_1}{2} \]  \hfill (3.4)

\[ \sigma_{11}^{(2)} = \sigma_{13}^{(2)} = 0, \quad x_1 = \frac{\ell_1}{2}, \quad |\bar{x}_3^{(2)}| \leq \frac{d_2}{2} \]  \hfill (3.5)

\[ u_1^{(3)} = U, \quad \sigma_{13}^{(3)} = 0, \quad x_1 = \frac{\ell_1}{2}, \quad |\bar{x}_3^{(3)}| \leq \frac{d_3}{2} \]  \hfill (3.6)

where \( U \) is the uniform displacement applied to one end of the modeled region. For simplicity, \( U \) is set to be 1. Also, for a composite laminate, we have to take the traction and displacement continuity conditions at the interfaces into account. These conditions can be expressed as:

\[ \sigma_{3j}^{(1)}|_{\bar{x}_3^{(1)}} = \frac{d_1}{2} = \sigma_{3j}^{(2)}|_{\bar{x}_3^{(2)}} = -\frac{d_2}{2}, \quad \sigma_{3j}^{(2)}|_{\bar{x}_3^{(2)}} = \frac{d_2}{2} = \sigma_{3j}^{(3)}|_{\bar{x}_3^{(3)}} = -\frac{d_3}{2} \]  \hfill (3.7)

\[ u_1^{(1)}|_{\bar{x}_3^{(1)}} = \frac{d_1}{2} = u_1^{(2)}|_{\bar{x}_3^{(2)}} = -\frac{d_2}{2}, \quad u_1^{(2)}|_{\bar{x}_3^{(2)}} = \frac{d_2}{2} = u_1^{(3)}|_{\bar{x}_3^{(3)}} = -\frac{d_3}{2} \]  \hfill (3.8)

The displacement field in the \( \alpha \) layer is assumed in the form of the second order Legendre expansion in \( \bar{x}_3^{(\alpha)} \) as follows:

\[ u_1^{(\alpha)} = \phi_0^{(\alpha)} + \phi_1^{(\alpha)} \bar{x}_3^{(\alpha)} + \frac{1}{2} \phi_2^{(\alpha)} [3(\bar{x}_3^{(\alpha)})^2 - \frac{d_3^2}{4}], \]

\[ u_3^{(\alpha)} = \psi_0^{(\alpha)} + \psi_1^{(\alpha)} \bar{x}_3^{(\alpha)} + \frac{1}{2} \psi_2^{(\alpha)} [3(\bar{x}_3^{(\alpha)})^2 - \frac{d_3^2}{4}] \]  \hfill (3.9)

where the field variables, \( \phi_k^{(\alpha)}, \psi_k^{(\alpha)} \) (\( k = 0, 1, 2 \)), are functions of \( x_1 \) only.

Since the laminate is symmetric with respect to its midplane, \( d_1 \) is equal to \( d_3 \). In addition, we have the following symmetry conditions in displacements:

\[ u_1^{(2)}(x_1, \bar{x}_3^{(2)}) = u_1^{(2)}(x_1, -\bar{x}_3^{(2)}), \]
\[ u_3^{(2)}(x_1, x_3^{(2)}) = -u_3^{(2)}(x_1, -x_3^{(2)}) \] (3.10)

and for any \( x_0^{(1)} = x_0^{(3)} \),

\[ u_1^{(1)}(x_1, x_3^{(1)}) = u_1^{(3)}(x_1, -x_3^{(3)}), \]

\[ u_3^{(1)}(x_1, x_3^{(1)}) = -u_3^{(3)}(x_1, -x_3^{(3)}) \] (3.11)

By (3.9)-(3.10), the expressions for the displacements in each layer can be written as:

\[ u_1^{(1)} = \varphi_0^{(1)} + \varphi_1^{(1)}x_3^{(1)} + \frac{1}{2}\psi_2^{(1)}[3(x_3^{(1)})^2 - \frac{d_1^2}{4}], \]

\[ u_3^{(1)} = \psi_0^{(1)} + \psi_1^{(1)}x_3^{(1)} + \frac{1}{2}\psi_2^{(1)}[3(x_3^{(1)})^2 - \frac{d_1^2}{4}] \] (3.12)

\[ u_1^{(2)} = \varphi_0^{(2)} + \frac{1}{2}\varphi_2^{(2)}[3(x_3^{(2)})^2 - \frac{d_2^2}{4}], \]

\[ u_3^{(2)} = \psi_1^{(2)}x_3^{(2)} \] (3.13)

\[ u_1^{(3)} = \varphi_0^{(3)} + \varphi_1^{(3)}x_3^{(3)} + \frac{1}{2}\varphi_2^{(3)}[3(x_3^{(3)})^2 - \frac{d_3^2}{4}], \]

\[ u_3^{(3)} = \psi_0^{(3)} + \psi_1^{(3)}x_3^{(3)} + \frac{1}{2}\psi_2^{(3)}[3(x_3^{(3)})^2 - \frac{d_3^2}{4}] \] (3.14)

and from (3.11), there are some relations between the field variables of \( \alpha = 1 \) layer and \( \alpha = 3 \) layer:

\[ \varphi_0^{(3)} = \varphi_0^{(1)}, \quad \varphi_1^{(3)} = -\varphi_1^{(1)}, \quad \varphi_2^{(3)} = \varphi_2^{(1)}, \]

\[ \psi_0^{(3)} = -\psi_0^{(1)}, \quad \psi_1^{(3)} = \psi_1^{(1)}, \quad \psi_2^{(3)} = -\psi_2^{(1)} \] (3.15)

As a result, the displacements and stresses of \( \alpha = 3 \) layer can be expressed in terms of the field variables of \( \alpha = 1 \) layer. From (3.12) to (3.15), there are nine unknown field variables:
The components of the linear strain tensor are given by:

\[
\varepsilon_{ij}^{(a)} = \frac{1}{2} (\partial u_i^{(a)} + \partial u_j^{(a)})
\]

(3.17)

where \( \partial_i = \partial / \partial x_i \), and \( \partial_3 = \partial / \partial \xi^3 \).

By (3.12)-(3.13) and (3.17), the strain components can be expressed as follows:

\[
\varepsilon_{11}^{(1)} = \partial_1 \phi_0^{(1)} + \partial_1 \phi_1^{(1)} \bar{x}_3^{(1)} + \frac{1}{2} \partial_1 \phi_2^{(1)} [3(\bar{x}_3^{(1)})^2 - \frac{d_1^2}{4}],
\]

\[
\varepsilon_{33}^{(1)} = \psi_1^{(1)} + 3 \psi_2^{(1)} \bar{x}_3^{(1)},
\]

\[
\varepsilon_{13}^{(1)} = \frac{1}{2} (\phi_1^{(1)} + \partial_1 \psi_0^{(1)}) + (3 \phi_2^{(1)} + \partial_1 \psi_1^{(1)}) \bar{x}_3^{(1)} + \frac{1}{2} \partial_1 \psi_2^{(1)} [3(\bar{x}_3^{(1)})^2 - \frac{d_1^2}{4}]
\]

(3.18)

\[
\varepsilon_{11}^{(2)} = \partial_1 \phi_0^{(2)} + \frac{1}{2} \partial_1 \phi_2^{(2)} [3(\bar{x}_3^{(2)})^2 - \frac{d_2^2}{4}],
\]

\[
\varepsilon_{33}^{(2)} = \psi_1^{(2)},
\]

\[
\varepsilon_{13}^{(2)} = \frac{1}{2} (3 \phi_2^{(2)} + \partial_1 \psi_1^{(2)}) \bar{x}_3^{(2)}
\]

(3.19)

The constitutive equation for each layer is given as:

\[
\sigma^{(a)} \sim C^{(a)} \varepsilon^{(a)}
\]

(3.20)

where

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\( \sigma^{(a)} = [\sigma_{11}^{(a)}, \sigma_{22}^{(a)}, \sigma_{33}^{(a)}, \sigma_{12}^{(a)}, \sigma_{13}^{(a)}] \), \( \epsilon^{(a)} = [\epsilon_{11}^{(a)}, \epsilon_{22}^{(a)}, \epsilon_{33}^{(a)}, 2\epsilon_{12}^{(a)}, 2\epsilon_{13}^{(a)}] \), 

and

\[
\tilde{C}^{(a)} = \begin{bmatrix}
C_{11}^{(a)} & C_{12}^{(a)} & C_{13}^{(a)} & 0 & 0 & 0 \\
C_{12}^{(a)} & C_{22}^{(a)} & C_{23}^{(a)} & 0 & 0 & 0 \\
C_{13}^{(a)} & C_{23}^{(a)} & C_{33}^{(a)} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44}^{(a)} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55}^{(a)} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}^{(a)} 
\end{bmatrix}
\]

For a transversely isotropic lamina with the axis of anisotropy in \( x_1 \) direction, the components of \( \tilde{C} \) can be related to the engineering properties as shown below:

\( C_{11} = E_{11} + 4Kv_{12}^2, \)

\( C_{22} = C_{33} = K + G_{23}, \)

\( C_{12} = C_{13} = 2Kv_{12}, \)

\( C_{23} = K - G_{23}, \)

\( C_{44} = G_{23}, \)

\( C_{55} = C_{66} = G_{12} \) \hspace{1cm} (3.21)

where

\[
K = \frac{E_{11}}{4\left[\frac{(1 - v_{23})}{2} E_{11} - v_{12}^2 \right]}
\]

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is the plane strain bulk modulus.

By (3.20), the inplane stresses in each layer can be expressed as:

\[ \sigma_{11}^{(a)} = C_{11}^{(a)}\varepsilon_{11}^{(a)} + C_{12}^{(a)}\varepsilon_{22}^{(a)} + C_{13}^{(a)}\varepsilon_{33}^{(a)}, \]

\[ \sigma_{33}^{(a)} = C_{13}^{(a)}\varepsilon_{11}^{(a)} + C_{23}^{(a)}\varepsilon_{22}^{(a)} + C_{33}^{(a)}\varepsilon_{33}^{(a)}, \]

\[ \sigma_{13}^{(a)} = C_{66}^{(a)}(2\varepsilon_{13}^{(a)}) \] (3.22)

and the out-of-plane normal stress is:

\[ \sigma_{22}^{(a)} = C_{12}^{(a)}\varepsilon_{11}^{(a)} + C_{22}^{(a)}\varepsilon_{22}^{(a)} + C_{23}^{(a)}\varepsilon_{33}^{(a)} \] (3.23)

The stress resultant in the \( x_2 \) direction is defined as:

\[ N_{22} = \int_{-\frac{d_1}{2}}^{\frac{d_1}{2}} \sigma_{22}^{(1)}d\bar{x}_3^{(1)} + \int_{-\frac{d_2}{2}}^{\frac{d_2}{2}} \sigma_{22}^{(2)}d\bar{x}_3^{(2)} + \int_{-\frac{d_3}{2}}^{\frac{d_3}{2}} \sigma_{22}^{(3)}d\bar{x}_3^{(3)} \] (3.24)

For a generalized plane strain analysis, \( \varepsilon_{22}^{(a)} \) is constant everywhere within the modeled region. Let \( \varepsilon_{22} \) denote this constant out-of-plane strain. Since \( N_{22} \) is zero for uniaxial tension, \( \varepsilon_{22} \) can be found by substituting (3.18)-(3.19) and (3.23) into (3.24) and by setting \( N_{22} \) equal to zero. As a result, the out-of-plane strain can be written as:

\[ \bar{\varepsilon}_{22} = -\frac{2d_1(C_{12}^{(1)}\partial_1\varphi_0^{(1)} + C_{23}^{(1)}\psi_1^{(1)}) + d_2(C_{12}^{(2)}\partial_1\varphi_0^{(2)} + C_{23}^{(2)}\psi_1^{(2)})}{2d_1C_{12}^{(1)} + d_2C_{22}^{(2)}} \] (3.25)

By substituting (3.18)-(3.19) and (3.25) into (3.22), the explicit expressions for the inplane stresses in each layer can be written as:

\[ \sigma_{11}^{(1)} = (A_{11}^{(1)}\partial_1\varphi_0^{(1)} + A_{13}^{(1)}\psi_1^{(1)} + B_{11}^{(1)}\partial_1\varphi_0^{(2)} + B_{13}^{(1)}\psi_1^{(2)}) \]

\[ + (C_{11}^{(1)}\partial_1\varphi_1^{(1)} + 3C_{13}^{(1)}\psi_2^{(1)})\bar{x}_3^{(1)} + \frac{1}{2}C_{11}^{(1)}\partial_1\varphi_2^{(1)}[3(\bar{x}_3^{(1)})^2 - \frac{d_1^2}{4}] \]
\[ \sigma_{33}^{(1)} = (A_{33}^{(1)} \partial_1 \psi_0^{(1)} + A_{33}^{(1)} \psi_1^{(1)} + B_{31}^{(2)} \partial_1 \psi_0^{(2)} + B_{33}^{(2)} \psi_1^{(2)}) + (C_{13}^{(1)} \partial_1 \psi_1^{(1)} + 3C_{33}^{(1)} \psi_2^{(1)}) \bar{x}_3^{(1)} + \frac{1}{2} C_{13}^{(1)} \partial_1 \psi_2^{(1)} [3(\bar{x}_3^{(1)})^2 - \frac{d_1^2}{4}], \]

\[ \sigma_{13}^{(1)} = C_{66}^{(1)} (\psi_1^{(1)} + \partial_1 \psi_0^{(1)}) + C_{66}^{(1)} (3\psi_2^{(1)} + \partial_1 \psi_1^{(1)}) \bar{x}_3^{(1)} + \frac{1}{2} C_{66}^{(1)} \partial_1 \psi_2^{(1)} [3(\bar{x}_3^{(1)})^2 - \frac{d_1^2}{4}] \] (3.26)

\[ \sigma_{11}^{(2)} = (B_{11}^{(1)} \partial_1 \psi_0^{(1)} + B_{13}^{(1)} \psi_1^{(1)} + A_{11}^{(2)} \partial_1 \psi_0^{(2)} + A_{13}^{(2)} \psi_1^{(2)}) + \frac{1}{2} C_{11}^{(2)} \partial_1 \psi_2^{(2)} [3(\bar{x}_3^{(2)})^2 - \frac{d_2^2}{4}], \]

\[ \sigma_{33}^{(2)} = (B_{31}^{(1)} \partial_1 \psi_0^{(1)} + B_{33}^{(1)} \psi_1^{(1)} + A_{31}^{(2)} \partial_1 \psi_0^{(2)} + A_{33}^{(2)} \psi_1^{(2)}) + \frac{1}{2} C_{31}^{(2)} \partial_1 \psi_2^{(2)} [3(\bar{x}_3^{(2)})^2 - \frac{d_2^2}{4}], \]

\[ \sigma_{13}^{(2)} = C_{66}^{(2)} (3\psi_2^{(2)} + \partial_1 \psi_1^{(2)}) \bar{x}_3^{(2)} \] (3.27)

where

\[ A_{11}^{(1)} = C_{11}^{(1)} - \frac{2d_1 C_{12}^{(1)} C_{12}^{(1)}}{2d_1 C_{22}^{(1)} + d_2 C_{22}^{(2)}}, \quad A_{13}^{(1)} = C_{13}^{(1)} - \frac{2d_1 C_{12}^{(1)} C_{33}^{(1)}}{2d_1 C_{22}^{(1)} + d_2 C_{22}^{(2)}}, \]

\[ A_{33}^{(1)} = C_{33}^{(1)} - \frac{2d_1 C_{23}^{(1)} C_{23}^{(1)}}{2d_1 C_{22}^{(1)} + d_2 C_{22}^{(2)}}, \]

\[ B_{11}^{(1)} = - \frac{2d_1 C_{12}^{(1)} C_{22}^{(2)}}{2d_1 C_{22}^{(1)} + d_2 C_{22}^{(2)}}, \quad B_{13}^{(1)} = - \frac{2d_1 C_{23}^{(1)} C_{12}^{(2)}}{2d_1 C_{22}^{(1)} + d_2 C_{22}^{(2)}}, \]

\[ B_{31}^{(1)} = - \frac{2d_1 C_{12}^{(1)} C_{23}^{(2)}}{2d_1 C_{22}^{(1)} + d_2 C_{22}^{(2)}}, \quad B_{33}^{(1)} = - \frac{2d_1 C_{23}^{(1)} C_{23}^{(2)}}{2d_1 C_{22}^{(1)} + d_2 C_{22}^{(2)}}, \] (3.28)

\[ A_{11}^{(2)} = C_{11}^{(2)} - \frac{d_2 C_{12}^{(2)} C_{12}^{(2)}}{2d_1 C_{22}^{(1)} + d_2 C_{22}^{(2)}}, \quad A_{13}^{(2)} = C_{13}^{(2)} - \frac{d_2 C_{12}^{(2)} C_{33}^{(2)}}{2d_1 C_{22}^{(1)} + d_2 C_{22}^{(2)}}, \]

\[ A_{33}^{(2)} = C_{33}^{(2)} - \frac{d_2 C_{23}^{(2)} C_{23}^{(2)}}{2d_1 C_{22}^{(1)} + d_2 C_{22}^{(2)}}, \]

\[ B_{11}^{(2)} = - \frac{d_2 C_{12}^{(2)} C_{12}^{(2)}}{2d_1 C_{22}^{(1)} + d_2 C_{22}^{(2)}}, \quad B_{13}^{(2)} = - \frac{d_2 C_{23}^{(2)} C_{12}^{(2)}}{2d_1 C_{22}^{(1)} + d_2 C_{22}^{(2)}}. \]
The reduced equilibrium equations for each layer are written as:

$$\delta_1 \sigma_{ij}^{(a)} + \delta_3 \sigma_{3j}^{(a)} = 0 \quad (3.30)$$

Introduce integration as:

$$\frac{1}{d_a} \int_{d_a}^{d_a} \delta_1 \sigma_{ij}^{(a)} + \delta_3 \sigma_{3j}^{(a)} \, (x_3^{(a)})^n \, dx_3^{(a)} = 0 \quad (3.31)$$

Integrating by parts, we obtain:

$$\delta_i S_{ij}^{(n)} = -n S_{ij}^{(n-1)} + \frac{1}{d_a} \left( \frac{dx_3^{(a)}}{2} \right)^n \sigma_{ij}^{(a)} \left|_{x_3^{(a)}}^{-1} \right. - \frac{d_a}{2} - (-1)^n \sigma_{ij}^{(a)} \left|_{x_3^{(a)}}^{-1} \right. = - \frac{d_a}{2} \quad (3.32)$$

where

$$S_{ij}^{(n)} = \frac{1}{d_a} \int_{d_a}^{d_a} \sigma_{ij}^{(a)} (x_3^{(a)})^n \, dx_3^{(a)}, \quad n = 0, 1, 2 \quad (3.33)$$

The expressions of $S_{ij}^{(n)}$ can be expanded explicitly by substituting (3.26)-(3.27) for $\sigma_{ij}^{(a)}$ in (3.33):

$$(0) S_{11}^{(1)} = A_{11}^{(1)} \partial_1 \phi_0^{(1)} + A_{13}^{(1)} \psi_1^{(1)} + B_{11}^{(2)} \partial_1 \phi_0^{(2)} + B_{13}^{(2)} \psi_1^{(2)};$$

$$(0) S_{33}^{(1)} = A_{13}^{(1)} \partial_1 \phi_0^{(1)} + A_{33}^{(1)} \psi_1^{(1)} + B_{31}^{(2)} \partial_1 \phi_0^{(2)} + B_{33}^{(2)} \psi_1^{(2)};$$

$$(0) S_{13}^{(1)} = C_{66}^{(1)} (\phi_1^{(1)} + \partial_1 \psi_0^{(1)});$$

$$(0) S_{11}^{(2)} = B_{11}^{(1)} \partial_1 \phi_0^{(1)} + B_{13}^{(1)} \psi_1^{(1)} + A_{11}^{(2)} \partial_1 \phi_0^{(2)} + A_{13}^{(2)} \psi_1^{(2)};$$

$$(0) S_{33}^{(2)} = B_{31}^{(1)} \partial_1 \phi_0^{(1)} + B_{33}^{(1)} \psi_1^{(1)} + A_{31}^{(2)} \partial_1 \phi_0^{(2)} + A_{33}^{(2)} \psi_1^{(2)};$$

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\[
S^{(1)}_{11} = \frac{d_1^2}{12} (C^{(1)}_{11} \partial_1 \phi_1^{(1)} + 3C^{(1)}_{13} \psi_2^{(1)}),
\]
\[
S^{(1)}_{33} = \frac{d_1^2}{12} (C^{(1)}_{13} \partial_3 \phi_1^{(1)} + 3C^{(1)}_{33} \psi_2^{(1)}),
\]
\[
S^{(1)}_{13} = \frac{d_1^2}{12} C^{(1)}_{66} (3 \phi_2^{(1)} + \partial_1 \psi_1^{(1)}),
\]
\[
S^{(2)}_{11} = \frac{d_2^2}{12} (C^{(2)}_{11} \partial_1 \phi_2^{(2)} + 3C^{(2)}_{13} \psi_2^{(2)}),
\]
\[
S^{(2)}_{33} = \frac{d_2^2}{12} (C^{(2)}_{13} \partial_3 \phi_2^{(2)} + 3C^{(2)}_{33} \psi_2^{(2)}),
\]
\[
S^{(2)}_{13} = \frac{d_2^2}{12} C^{(2)}_{66} (3 \phi_2^{(2)} + \partial_1 \psi_2^{(2)}),
\]

and all other \(S^{(n)}_{ij}\) are zero.

Using the traction continuity conditions in (3.7) and the traction free conditions in (3.1)-(3.2), and noticing the symmetry conditions in the \(\alpha = 2\) layer and the relations between \(\alpha = 1\) and \(\alpha = 3\) layers, we can eliminate the interfacial and boundary stress terms in (3.32). Therefore, the following equations can be derived from (3.32) by linear operations with different values of \(n\):

\[
2d_1 \partial_1 S^{(1)}_{11} + d_2 \partial_1 S^{(2)}_{11} = 0,
\]

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\[ d_2^2 \delta_1 S_{11}^{(0)} - 4 \delta_1 S_{11}^{(2)} + 8 S_{13}^{(1)} = 0, \]

\[ 4d_1 \delta_1 S_{11}^{(1)} - 4 S_{13}^{(1)} + d_2 \delta_1 S_{11}^{(2)} = 0, \]

\[ 8d_1 S_{11}^{(1)} - 16 S_{13}^{(1)} + d_1 d_2 \delta_1 S_{11}^{(2)} = 0, \]

\[ d_1 \delta_1 S_{13}^{(0)} - \delta_1 S_{13}^{(2)} + S_{33}^{(2)} = 0, \]

\[ 2 \delta_1 S_{13}^{(1)} - 2S_{33}^{(1)} - \delta_1 S_{13}^{(2)} + S_{33}^{(2)} = 0, \]

\[ 4d_1 \delta_1 S_{13}^{(1)} - 8 S_{33}^{(1)} - d_1 \delta_1 S_{13}^{(2)} + d_1 S_{33}^{(2)} = 0 \] (3.35)

Substituting (3.34) for \[S_{ij}^{(n)}\] in (3.35), we obtain seven equations in terms of \[\phi_1^{(n)}\] and \[\psi_1^{(n)}\] as shown below:

\[(2d_1 A_{11}^{(1)} + d_2 B_{11}^{(1)}) \delta_1^2 \phi_0^{(1)} + (2d_1 A_{13}^{(1)} + d_2 B_{13}^{(1)}) \delta_1 \psi_1^{(1)} + \]

\[(2d_1 B_{11}^{(2)} + d_2 A_{11}^{(2)}) \delta_1^2 \phi_0^{(2)} + (2d_1 B_{13}^{(2)} + d_2 A_{13}^{(2)}) \delta_1 \psi_1^{(2)} = 0, \]

\[20B_{11}^{(1)} \delta_1^2 \phi_0^{(1)} + 20B_{13}^{(1)} \delta_1 \psi_1^{(1)} + 20A_{11}^{(2)} \delta_1^2 \phi_0^{(2)} + 60C_{56}^{(2)} \psi_2^{(2)} - \]

\[d_2^2 C_{11}^{(2)} \delta_1^2 \psi_2^{(2)} + 20(A_{13}^{(2)} + C_{56}^{(2)}) \delta_1 \psi_1^{(2)} = 0, \]

\[3d_2 B_{11}^{(1)} \delta_1^2 \phi_0^{(1)} + d_1^2 C_{11}^{(1)} \delta_1^2 \phi_1^{(1)} - 12C_{56}^{(1)} \phi_1^{(1)} - 12C_{60}^{(1)} \delta_1 \psi_0^{(1)} + 3d_2 B_{13}^{(1)} \delta_1 \psi_1^{(1)} + \]

\[3d_2^2 C_{11}^{(1)} \delta_1 \psi_2^{(1)} + 3d_2 A_{11}^{(2)} \delta_1 \phi_0^{(2)} + 3d_2 A_{13}^{(2)} \delta_1 \psi_1^{(2)} = 0, \]
(10d_1 A_{11}^{(1)} + 15d_2 B_{11}^{(1)}) \phi_0^{(1)} + d_1^2 C_{11}^{(1)} \phi_0^{(1)} + 60d_1 C_{66}^{(1)} \phi_2^{(1)} +

(10d_1 A_{13}^{(1)} - 20d_1 C_{66}^{(1)} + 15d_2 B_{13}^{(1)}) \phi_1^{(1)} + (10d_1 B_{11}^{(2)} + 15d_2 A_{11}^{(2)}) \phi_0^{(2)} +

(10d_1 B_{13}^{(2)} + 15d_2 A_{13}^{(2)}) \phi_1^{(2)} = 0,

12B_{31}^{(1)} \phi_0^{(1)} + 12d_1 C_{66}^{(1)} \phi_1^{(1)} + 12d_1 C_{66}^{(1)} \phi_0^{(1)} + 12B_{33}^{(1)} \phi_1^{(1)} +

12A_{13}^{(2)} \phi_0^{(2)} - 3d_2^2 C_{66}^{(2)} \phi_0^{(2)} + 12A_{33}^{(2)} \phi_1^{(2)} - d_2^2 C_{66}^{(2)} \phi_1^{(2)} = 0.

12(B_{31}^{(1)} - 2A_{13}^{(1)}) \phi_0^{(1)} + 6d_1^2 C_{66}^{(1)} \phi_1^{(1)} + 2d_1^2 C_{66}^{(1)} \phi_1^{(1)} +

12(B_{33}^{(1)} - 2A_{33}^{(1)}) \phi_1^{(1)} + 12(A_{13}^{(2)} - 2B_{31}^{(2)}) \phi_0^{(2)} - 3d_2^2 C_{66}^{(2)} \phi_2^{(2)} -

d_2^2 C_{66}^{(2)} \phi_2^{(2)} + 12(A_{33}^{(2)} - 2B_{33}^{(2)}) \phi_1^{(2)} = 0.

60B_{31}^{(1)} \phi_0^{(1)} + 20d_1 (C_{66}^{(1)} - 2C_{13}^{(1)}) \phi_1^{(1)} + 20d_1 C_{66}^{(1)} \phi_1^{(1)} + 60B_{33}^{(1)} \phi_1^{(1)} +

2d_1^3 C_{66}^{(1)} \phi_2^{(1)} - 120d_1 C_{33}^{(1)} \phi_2^{(1)} + 60A_{13}^{(2)} \phi_0^{(2)} -

15d_2^2 C_{66}^{(2)} \phi_2^{(2)} - 5d_2^2 C_{66}^{(2)} \phi_2^{(2)} + 60A_{33}^{(2)} \phi_1^{(2)} = 0 \quad (3.36)

In addition, by substituting (3.12)-(3.13) into (3.8), the displacement continuity conditions will give us two more equations as follows:

4\phi_0^{(1)} + 2d_1 \phi_1^{(1)} + d_1^2 \phi_2^{(1)} - 2d_1 \phi_0^{(2)} - d_2^2 \phi_2^{(2)} = 0,

4\psi_1^{(1)} + 2d_1 \psi_1^{(1)} + d_1^2 \psi_2^{(1)} + 2d_2 \psi_2^{(2)} = 0 \quad (3.37)
(3.36), together with (3.37), gives a set of nine linear homogeneous equations for nine unknown field variables listed in (3.16). These equations must be supplemented with the boundary conditions given in (3.3)-(3.5). By substituting (3.12)-(3.13) and (3.26)-(3.27) into (3.3)-(3.5), the exact boundary conditions can be expressed as:

\[ \varphi_0^{(1)} = \varphi_1^{(1)} = \varphi_2^{(1)} = \varphi_0^{(2)} = \varphi_2^{(2)} = 0, \]
\[ \partial_1 \psi_0^{(1)} = \partial_1 \psi_1^{(1)} = \partial_1 \psi_2^{(1)} = \partial_1 \psi_1^{(2)} = 0, \quad \text{at } x_1 = 0 \] (3.38)

and

\[ \varphi_0^{(1)} = 1, \quad \varphi_1^{(1)} = \varphi_2^{(1)} = 0, \]
\[ \partial_1 \psi_0^{(1)} = \partial_1 \psi_1^{(1)} = \partial_1 \psi_2^{(1)} = \partial_1 \psi_2^{(2)} = 0, \]
\[ B_{11}^{(1)} \partial_1 \varphi_0^{(1)} + B_{11}^{(1)} \psi_1^{(1)} + A_{11}^{(2)} \partial_1 \varphi_0^{(2)} + A_{11}^{(2)} \psi_1^{(2)} = 0, \]
\[ 3\varphi_2^{(2)} + \partial_1 \psi_1^{(2)} = 0, \quad \text{at } x_1 = \frac{t_1}{2} \] (3.39)

Approximating the derivatives by difference expressions, we can obtain a banded matrix for a system of linear algebraic equations and solve for the unknown field variables at points along \( 0 \leq x_1 \leq \frac{t_1}{2} \).

### 3.2 Results and Comparisons

For uniaxial tensile loading, the effective properties of primary concern are the axial stiffness and Poisson's ratio. The distribution of axial stress is also predicted in this study. In order to compare the results with those of other approaches, three types of symmetric cross-ply laminates:
[0/90]_n, [0/90]_n, [0/90]_n are investigated. The materials studied are graphite/epoxy (T300/5208) and glass/epoxy. The material properties and ply thickness can be found in Appendix A.

### 3.2.1 Axial Stiffness

From (3.34), the average axial stress in each layer can be determined. Therefore, the average axial stress of the laminate is given by:

\[
\bar{\sigma}_{11} = \frac{\sigma_{11}^{(0)} + \sigma_{11}^{(6)}}{2d_1 + d_2}
\]

Since the displacement at one end of the modeled region is set to be 1, the average axial strain becomes:

\[
\bar{\varepsilon}_{11} = \frac{1}{(\varepsilon_{1/2})}
\]

The effective axial stiffness of the damaged laminate is then defined as:

\[
E_{11}^* = \frac{\bar{\sigma}_{11}}{\bar{\varepsilon}_{11}}
\]

The degradation of the effective axial stiffness with respect to the transverse crack density (TCD) for three types of graphite/epoxy laminates is shown in Fig. 6. The results of the finite element method [15] are also given for comparison. In order to observe the detail of the comparison, the y-axis is scaled from 0.7 to 1. Accordingly, the agreement between results may look worse than it should be. The horizontal segments in this figure indicate the results predicted by the ply discount method in which the axial (x_1 direction) stiffness of the 90° layers has been reduced to zero. All of the results have been normalized by the undamaged effective axial stiffness (E_{11}^o) of the corre-
Figure 6. Degradation of Axial Stiffness for Graphite/Epoxy Laminates with Transverse Cracks
Table 1. Degradation of Axial Stiffness for Laminates with Transverse Cracks

<table>
<thead>
<tr>
<th>Condition</th>
<th>$[0/90]_s$</th>
<th>$[0_2/90_2]_s$</th>
<th>$[0/90_3]_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Undamaged</td>
<td>102 GPa</td>
<td>71.8 GPa</td>
<td>41.3 GPa</td>
</tr>
<tr>
<td>Crack Density = 3.2 mm$^{-1}$</td>
<td>100 GPa</td>
<td>67.5 GPa</td>
<td>34.7 GPa</td>
</tr>
<tr>
<td>Degradation</td>
<td>2 %</td>
<td>6 %</td>
<td>16 %</td>
</tr>
<tr>
<td>Ply Discount</td>
<td>99.6 GPa</td>
<td>66.4 GPa</td>
<td>33.2 GPa</td>
</tr>
<tr>
<td>Degradation</td>
<td>2 %</td>
<td>8 %</td>
<td>20 %</td>
</tr>
</tbody>
</table>

The present results show that $E_{11}$ degrades when the TCD increases and becomes saturated as the TCD approaches 1.6 mm$^{-1}$. Since the ratio of orthotropicity for graphite/epoxy is very high ($E_{11}/E_{22} = 12.3$), only a small portion of the loading is carried by the 90° layers. This is the reason the reduction in the effective stiffness is not significant for those graphite/epoxy laminates. Follow the same reasoning, we may say that the residual stiffness of damaged cross-ply laminates is a strong function of the percentage of the 0° layers. This phenomenon can be clearly observed from the results shown in this figure.

The comparison in Fig. 6 shows that the results of the present analysis and the FEM agree well with each other (the biggest difference at TCD = 2.6 mm$^{-1}$ for $[0/90]_s$ is less than 3%). Also, it can be seen that the agreement is even better when the TCD is lower or the percentage of the 0° plies is higher. Both results approach the value predicted by the ply discount method at very high TCD (the biggest difference appears at TCD = 3.2 mm$^{-1}$ for $[0/90_3]_s$ is about 3.5%). As a general
tendency, the degradation of the effective axial stiffness predicted by the present analysis is smaller and reaches an asymptote earlier than that by the FEM.

Fig. 7 shows the comparison between the present analysis, FEM [15], experimental data [7], and a variational method [11] for a glass/epoxy [0/90]_4 laminate. The result of the ply discount method is also given. Note that the TCD in this figure ranges from 0 to 0.8 mm\(^{-1}\) only, and all of the results are normalized with respect to the effective stiffness of the undamaged laminate (E\(_{11}^o\) = 20.3 GPa or 2.95 Msi). Generally speaking, the results fall into two categories: the present analysis is close to the FEM while the variational prediction agrees better with the experimental data. The variational approach provides a lower bound on the effective moduli. Since Highsmith et. al. [7] noticed that the transverse cracking was not the exclusive damage mode present in the test specimens, the experimental data may exhibit degradation in effective stiffness caused by other damage in addition to the transverse cracking. This may explain why the variational prediction is closer to the experimental observation.

Comparing Fig. 7 to Fig. 6, one may notice that the reduction in E\(_{11}\) of glass/epoxy is significantly larger than that of graphite/epoxy for the same laminate [0/90]_4. The reason is the ratio of orthotropy of glass/epoxy (E\(_{11}\)/E\(_{22}\) = 3.2) is much lower than that of graphite/epoxy (12.3). Thus, the 90° layers of a glass/epoxy laminate should carry a higher percentage of the loading and cause more reduction in effective stiffness when transverse cracks develop. The higher degradation in stiffness is one reason why glass/epoxy was chosen for the experimental study [7].

Additional comparison for a glass/epoxy [0/90]_4 laminate is presented in Fig. 8. Note that the properties and ply thickness for this laminate are taken from [1] and some of them are slightly different from those listed in Appendix A (E\(_{11}\) = 40GPa, E\(_{22}\) = 11GPa, G\(_{12}\) = 5GPa, ply thickness = 0.125 mm, the others are the same). The curve shown in this figure is the degradation of the effective stiffness predicted by the present analysis while the symbols represent the experimental data given in [18]. The undamaged stiffness used for normalization is 25.7 GPa (3.73 Msi). The figure
Figure 7. Degradation of Axial Stiffness for a Glass/Epoxy [0/90]s Laminate with Transverse Cracks

Transverse Cracks under Uniaxial Tension
Figure 8. Degradation of Axial Stiffness for a Glass/Epoxy [0/90]s Laminate with Transverse Cracks

Transverse Cracks under Uniaxial Tension
shows excellent agreement between theory and experiment if one notices that the y-axis has been
enlarged to a scale from 0.7 to 1.

3.2.2 Poisson’s Ratio

Since we adopt a generalized plane strain analysis, the out-of-plane strain can be found by (3.25).
Consequently, the effective Poisson’s ratio can be determined as:

\[ v_{12}^{*} = -\frac{\bar{\varepsilon}_{22}}{\bar{\varepsilon}_{11}} \]  (3.43)

where \( \bar{\varepsilon}_{11} \) is defined in (3.41).

Due to the degradation in axial stiffness mentioned in the last section, the axial stress \( (\sigma_{11}) \) in each
layer is reduced as transverse cracks develop. Thus, the negative transverse strain \( (\varepsilon_{22}) \) given by the
Poisson’s effect becomes smaller and the effective Poisson’s ratio decreases when the TCD in-
creases. The results presented in Fig. 9 for graphite/epoxy coincide with this reasoning and show
a dramatic change in the Poisson’s ratio as a function of the TCD. The tendency of degradation
in \( v_{12}^{*} \) is more or less similar to that in \( E_{11} \), but the extent of reduction in the Poisson’s ratio is much
greater than that in the axial stiffness. As a general observation, \( v_{12} \) drops rapidly before the TCD
reaches 0.8 mm\(^{-1}\) and approaches an asymptote after the TCD equals 2.4 mm\(^{-1}\). The undamaged
Poisson’s ratio \( (v_{12}^{*}) \) and the result of the ply discount method for each laminate are given in Table
2. By comparing Table 2 to Table 1, it can be seen that the degradation in the Poisson’s ratio is
several times that of the axial stiffness. Thus, \( v_{12}^{*} \) is obviously a better indicator of damage than
\( E_{11} \).

The results of the FEM [15] and the Poisson’s ratios predicted by the ply discount method are also
shown in Fig. 9 for comparison. The agreement between results for \( v_{12}^{*} \) is not as good as is found
Figure 9. Degradation of Poisson's Ratio for Graphite/Epoxy Laminates with Transverse Cracks
Table 2. Degradation of Poisson’s Ratio for Laminates with Transverse Cracks

<table>
<thead>
<tr>
<th>Condition</th>
<th>[0°/90°]ₜ</th>
<th>[0°/90°]ₜ</th>
<th>[0°/90°]ₜ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Undamaged</td>
<td>0.0627</td>
<td>0.0361</td>
<td>0.0253</td>
</tr>
<tr>
<td>Crack Density = 3.2 mm⁻¹</td>
<td>0.0508</td>
<td>0.0217</td>
<td>0.00988</td>
</tr>
<tr>
<td>Degradation</td>
<td>19 %</td>
<td>40 %</td>
<td>61 %</td>
</tr>
<tr>
<td>Ply Discount</td>
<td>0.0472</td>
<td>0.0181</td>
<td>0.00635</td>
</tr>
<tr>
<td>Degradation</td>
<td>25 %</td>
<td>50 %</td>
<td>75 %</td>
</tr>
</tbody>
</table>

for $E_{11}$. Again, it can be seen that the present analysis predicts less degradation and earlier saturation than does the FEM.

### 3.2.3 Axial Stress

The distribution of the axial stress in each layer can be determined from the expressions of stresses in (3.26)-(3.27). The results at different locations for a graphite/epoxy [0₂/90₂]ₜ laminate are given and compared to those predicted by the FEM [15] in Fig. 10-14. Since the coordinate system used for the present analysis is different from that for the FEM [15], a set of coordinate $(x'_1, x'_3)$ is introduced. All of the results are obtained assuming a 0.1% axial strain loading and are normalized by the uncracked axial stress in the 90° layers ($σ_{11} = 10.7$ MPa or 1.55 Ksi). The crack spacing $(δ_{1})$ studied is 4.064 mm (0.16 in).
Figure 10. Axial Stress Distribution along the Midplane of Cracked [0/90]s
Figure 11. Axial Stress Distribution along the 0/90 Interface of Cracked [0/90]s (in 90° layer)
Figure 12. Axial Stress Distribution along the 0°/90° Interface of Cracked [0°/90°2]s (in 0° layer)

Transverse Cracks under Uniaxial Tension
Figure 13. Through-the-thickness Distribution of Axial Stress along the Crack Surface
Figure 14. Through-the-thickness Distribution of Axial Stress along the Midplane between Cracks
Fig. 10 shows the stress distribution along the midplane of the laminate. Since $x'_1 = 0$ represents the crack surface, both results predict the axial stress at that point is zero. At the region away from the crack surface, the stress increases smoothly and reaches a saturation value at $x'_1 = \frac{\ell_i}{2}$. Generally speaking, the agreement between the present analysis and the FEM is very good except near the crack surface. It is very interesting that the present analysis predicts compressive stress in the vicinity of the crack surface while the FEM does not. The explanation of this compressive stress is the bending associated with the deformation of the cracked laminate. In fact, the FEM does exhibit this compressive stress for smaller crack spacing [15]. This local compression will be examined in detail later.

The results shown in Fig. 11 are the distribution of axial stress in the 90° layer along the interface between 0° and 90° layers. In this figure, it is evident that the stresses predicted by the two approaches are very different near the crack surface. The present analysis predicts zero stress at the crack surface while the FEM shows a singular behavior in the axial stress. Since the crack surface is a traction-free surface, it is obvious that the former gives a better result at this location. The results are in better agreement away from the crack surface and finally approach the same far field stress at the midpoint between cracks ($x'_1 = \frac{\ell_i}{2}$). The stress distribution in the 0° layer along the same interface is presented in Fig. 12. This figure shows quite good agreement between the results for most of the region except very near the crack. Both approaches predict the stress will behave singularly at the crack tip ($x'_1 = 0$).

Figures 13 and 14 show the through-the-thickness distribution of axial stress at $x'_1 = 0$ and $x'_1 = \frac{\ell_i}{2}$, respectively. In Fig. 13, the stress is zero through the 90° layer because of the traction-free boundary conditions at the crack surface. However, the FEM predicts singular stress near the crack tip (in the 90° layer) as mentioned before. In the 0° layer, the present analysis predicts a parabolic stress distribution with some singular behavior at the crack tip. Similarly, FEM's result shows this singular behavior at the same location, but with a higher order of singularity. In the remaining region of the 0° layer, the FEM predicts a quite uniform distribution in axial stress. Fig.
14 shows excellent agreement between the present analysis and the FEM. Both results predict a uniform stress distribution in the 90° and 0° layers.

In order to better understand the compressive stress mentioned before, Fig. 15 is presented. This figure shows the axial stress distribution in the 90° layer for different values of $x'$. It can be seen that the compressive stress decreases as away from the midplane and diminishes before $x'$ reaches $d_2/4$. Since the stress near the crack surface changes from compression to tension as $x'$ increases, it is reasonable to say that this phenomenon is an effect of bending which is resulted from the opening of the crack. A schematic diagram is shown in Fig. 16 to illustrate this deformation.

The same type of stress distribution has also been observed by a finite difference approach at the outer surface of a cracked [90/Al/90] hybrid composite laminate [14]. The transverse cracks are located in the 90° plies (graphite/epoxy) which are the outer layers of the laminate. The material properties and ply thickness are listed in Table 3 [19]. The results given by the present analysis and the FEM for a crack spacing $\ell_1 = 12$ mm (0.47 in) are shown in Fig. 17 and compared to the data taken from [14]. In spite of some difference between results, the agreement is fairly good. The magnitudes of the compressive stress predicted by those approaches are very close to each other. Furthermore, in a recent research on cracked cross-ply laminates, Sun and Jen [13] noted that the stress distributions in $[0_m/90_n]$, laminates were similar to those of $[90_n/0_m]$ laminates. This observation, together with the agreement shown in Fig. 17, should be considered as another evidence for the existence of the compressive stress near the crack surface.
Figure 15. Axial Stress Distribution at Different Locations
Table 3. Material Properties and Ply Thickness of [90/Al/90]

<table>
<thead>
<tr>
<th>Property</th>
<th>Graphite/Epoxy</th>
<th>Aluminum</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{11}$</td>
<td>150. GPa</td>
<td>73.0 GPa</td>
</tr>
<tr>
<td>$E_{22}$</td>
<td>10.0 GPa</td>
<td>73.0 GPa</td>
</tr>
<tr>
<td>$v_{12}$</td>
<td>0.27</td>
<td>0.33</td>
</tr>
<tr>
<td>$v_{23}$</td>
<td>0.39</td>
<td>0.33</td>
</tr>
<tr>
<td>$G_{12}$</td>
<td>6.00 GPa</td>
<td>27.4 GPa</td>
</tr>
<tr>
<td>Ply Thickness</td>
<td>1 mm</td>
<td>4 mm</td>
</tr>
</tbody>
</table>
Figure 16. Schematic Diagram of Crack Opening
Figure 17. Axial Stress Distribution at the Outer Surface of Cracked [90°/Al/90] Transverse Cracks under Uniaxial Tension
4.0 Transverse Cracks under Pure Shear

4.1 Formulation

Suppose that the damaged laminate is subjected to a plane shear loading; i.e., shear in $x_1 - x_2$ plane.

Considerations of symmetry and the traction-free conditions at the outer surfaces and the crack surface indicate that the following boundary conditions should be imposed on the modeled region (see Fig. 18):

\[ \sigma_{32}^{(1)} = 0, \quad 0 \leq x_1 \leq \frac{\ell_1}{2}, \quad \bar{x}_{3}^{(1)} = -\frac{d_1}{2} \]  

(4.1)

\[ \sigma_{32}^{(3)} = 0, \quad 0 \leq x_1 \leq \frac{\ell_1}{2}, \quad \bar{x}_{3}^{(3)} = \frac{d_3}{2} \]  

(4.2)

\[ u_2^{(a)} = 0, \quad x_1 = 0, \quad |\bar{x}_{3}^{(a)}| \leq \frac{d_a}{2} \]  

(4.3)

\[ u_2^{(l)} = V, \quad x_1 = \frac{\ell_1}{2}, \quad |\bar{x}_{3}^{(l)}| \leq \frac{d_1}{2} \]  

(4.4)
Figure 18. Boundary Conditions under Pure Shear
\[ \sigma_{12}^{(2)} = 0, \quad x_1 = \frac{\epsilon}{2}, \quad |x_3^{(2)}| \leq \frac{d_2}{2} \] (4.5)

\[ u_2^{(3)} = V, \quad x_1 = \frac{\epsilon}{2}, \quad |x_3^{(3)}| \leq \frac{d_3}{2} \] (4.6)

where \( V \) is the uniform displacement applied to the laminate. For simplicity, \( V \) is set to be 1. Also, for a composite laminate, the traction and displacement continuity conditions at the interfaces have to be taken into account. These conditions can be expressed as:

\[ \sigma_{32}^{(1)|x_3^2} = \frac{d_1}{2} = \sigma_{32}^{(2)|x_3^2} = -\frac{d_2}{2}, \quad \sigma_{32}^{(3)|x_3^2} = \frac{d_2}{2} = \sigma_{32}^{(3)|x_3^3} = -\frac{d_3}{2} \] (4.7)

\[ u_2^{(1)|x_3^2} = \frac{d_1}{2} = u_2^{(2)|x_3^2} = -\frac{d_2}{2}, \quad u_2^{(2)|x_3^2} = \frac{d_2}{2} = u_2^{(3)|x_3^3} = -\frac{d_3}{2} \] (4.8)

It should be noted that this problem has no dependence on \( x_2 \) because the laminate is infinite long in \( x_2 \) direction.

Assume a displacement field of the \( \alpha \) layer as follows:

\[ u_1^{(\alpha)} = u_3^{(\alpha)} = 0 \] (4.9)

and

\[ u_2^{(\alpha)} = \chi_0^{(\alpha)} + \chi_1^{(\alpha)} x_3^{(\alpha)} + \frac{1}{2} \chi_2^{(\alpha)} [3(\bar{x}_3^{(\alpha)})^2 - \frac{d_3^2}{4}] \] (4.10)

where the field variables, \( \chi_k^{(\alpha)} \) (\( k = 0, 1, 2 \)), depend on \( x_1 \) only.

Since the laminate is symmetric with respect to its midplane, \( d_1 \) is equal to \( d_3 \). Furthermore, we have the following symmetry conditions in displacements:

\[ u_2^{(2)}(x_1, \bar{x}_3^{(2)}) = u_2^{(2)}(x_1, -\bar{x}_3^{(2)}) \] (4.11)
and for any $\bar{x}_i^{(1)} = \bar{x}_i^{(3)}$,

$$u_2^{(1)}(x_1, \bar{x}_3^{(1)}) = u_2^{(3)}(x_1, -\bar{x}_3^{(3)}) \quad (4.12)$$

By (4.10)-(4.11), the displacements in the $x_2$ direction of each layer can be expressed as:

$$u_2^{(1)} = \chi_0^{(1)} + \chi_1^{(1)}\bar{x}_3^{(1)} + \frac{1}{2}\chi_2^{(1)}[3(\bar{x}_3^{(1)})^2 - \frac{d_1^2}{4}]$$

$$u_2^{(2)} = \chi_0^{(2)} + \frac{1}{2}\chi_2^{(2)}[3(\bar{x}_3^{(2)})^2 - \frac{d_2^2}{4}] \quad (4.13)$$

$$u_2^{(3)} = \chi_0^{(3)} + \chi_1^{(3)}\bar{x}_3^{(3)} + \frac{1}{2}\chi_2^{(3)}[3(\bar{x}_3^{(3)})^2 - \frac{d_3^2}{4}] \quad (4.14)$$

and from (4.12), there are some relations between the field variables of the $\alpha = 1$ layer and the $\alpha = 3$ layer:

$$\chi_0^{(3)} = \chi_0^{(1)}, \quad \chi_1^{(3)} = -\chi_1^{(1)}, \quad \chi_2^{(3)} = \chi_2^{(1)} \quad (4.15)$$

As a result, we can express the displacements and stresses of the $\alpha = 3$ layer in terms of the field variables of the $\alpha = 1$ layer. From (4.13) to (4.16), there are five unknowns:

$$\chi_0^{(1)}, \chi_1^{(1)}, \chi_2^{(1)}, \chi_0^{(2)}, \chi_2^{(2)} \quad (4.16)$$

which must be determined.

From the strain-displacement relation and (4.13)-(4.14), the strain components can be expressed as follows:

$$\varepsilon_{23}^{(1)} = \frac{1}{2}(\chi_1^{(1)} + 3\chi_2^{(1)}\bar{x}_3^{(1)})$$

$$\varepsilon_{12}^{(1)} = \frac{1}{2}(\partial_1\chi_0^{(1)} + \partial_1\chi_1^{(1)}\bar{x}_3^{(1)} + \frac{1}{2}\partial_1\chi_2^{(1)}[3(\bar{x}_3^{(1)})^2 - \frac{d_1^2}{4}])$$
The constitutive equation for orthotropic materials under pure shear is given as:

\[
\sigma_{23}^{(a)} = C_{44}^{(a)} (2\varepsilon_{23}^{(a)}),
\]

\[
\sigma_{12}^{(a)} = C_{55}^{(a)} (2\varepsilon_{12}^{(a)}).
\]  

From (4.18)-(4.19), the shear stresses in each layer can be expressed as:

\[
\sigma_{23}^{(1)} = C_{44}^{(1)} (\chi_{1}^{(1)} + 3\chi_{3}^{(1)}),
\]

\[
\sigma_{12}^{(1)} = C_{55}^{(1)} (\partial_{1}\chi_{0}^{(1)} + \partial_{1}\chi_{3}^{(1)} + \frac{1}{2} \partial_{1}\chi_{2}^{(1)} [3\chi_{3}^{(1)}]^2 - \frac{d_{2}^2}{4}],
\]

\[
\sigma_{23}^{(2)} = 3C_{44}^{(2)} \chi_{2}^{(2)} \chi_{3}^{(2)},
\]

\[
\sigma_{12}^{(2)} = C_{55}^{(2)} (\partial_{1}\chi_{0}^{(2)} + \frac{1}{2} \partial_{1}\chi_{2}^{(2)} [3\chi_{3}^{(2)}]^2 - \frac{d_{2}^2}{4}].
\]  

The nontrivial equilibrium equation in each layer for plane shear can be written as:

\[
\partial_{1}\sigma_{12}^{(a)} + \partial_{3}\sigma_{23}^{(a)} = 0
\]

where \(\partial_{i} = \partial/\partial x_{i}\), and \(\partial_{3} = \partial/\partial x_{3}^{(a)}\).

Integrating (4.21) over the thickness of each layer gives:

\[
\frac{1}{d_{a}} \int_{d_{a}/2}^{d_{a}/2} \left[ \partial_{1}\sigma_{12}^{(a)} + \partial_{3}\sigma_{23}^{(a)} \right] (x_{3}^{(a)})^{n} dx_{3}^{(a)} = 0
\]  

Transverse Cracks under Pure Shear
and integrating by parts gives:

\[
\frac{\partial}{\partial x_1} S_{12}^{(n)} - n S_{23}^{(n)} + \frac{1}{d_{13}} \left( \frac{d_{13}}{2} \right)^n \left[ \frac{\partial}{\partial x_1} \sigma_2^{(n)} \right]_{x_3} = \frac{d_{23}}{2} - (-1)^n \frac{\partial}{\partial x_1} \sigma_3^{(n)} \right]_{x_3} = - \frac{d_{23}}{2} ] = 0 \tag{4.23}
\]

where

\[
S_i^{(n)} = \frac{1}{d_{13}} \int_{a_i}^{b_i} \sigma_i^{(n)}(x_3) \left( x_3 \right)^n d x_3, \quad n = 0, 1, 2 \tag{4.24}
\]

The expressions of \( S_i^{(n)} \) can be expanded explicitly by substituting (4.20) for \( \sigma_i^{(n)} \) in (4.24):

\[
S_{12}^{(0)} = C_{44} x_1^{(1)},
\]

\[
S_{12}^{(1)} = C_{55} x_0^{(1)},
\]

\[
S_{12}^{(2)} = C_{55} x_0^{(2)},
\]

\[
S_{23}^{(1)} = \frac{d_1^2}{4} C_{44} x_2^{(1)},
\]

\[
S_{12}^{(1)} = \frac{d_1^2}{12} C_{55} x_1^{(1)},
\]

\[
S_{23}^{(2)} = \frac{d_1^2}{4} C_{44} x_2^{(2)},
\]

\[
S_{12}^{(2)} = \frac{d_1^2}{12} C_{55} \left( \partial_1 x_0^{(1)} + \frac{d_2^2}{10} \partial_1 x_1^{(2)} \right),
\]

\[
S_{12}^{(2)} = \frac{d_1^2}{12} C_{55} \left( \partial_1 x_0^{(2)} + \frac{d_2^2}{10} \partial_1 x_1^{(2)} \right) \tag{4.25}
\]

Transverse Cracks under Pure Shear
and all other $S_{ij}^{(n)}$ are zero.

By using the traction continuity conditions in (4.7) and the traction-free conditions in (4.1)-(4.2), and noticing the symmetry conditions in the $a = 2$ layer and the relations between the $a = 1$ and $a = 3$ layers (4.16), the interfacial and boundary stress terms in (4.23) can be eliminated. Therefore, the following equations can be derived from (4.23) by linear operations with different values of $n$:

$$2d_1 \partial_1 S_{12}^{(1)} + d_2 \partial_1 S_{12}^{(2)} = 0,$$

$$d_1 \partial_1 S_{12}^{(1)} - 2\partial_1 S_{12}^{(1)} + 2S_{23}^{(1)} = 0,$$

$$d_1 \partial_1 S_{12}^{(1)} - d_1 S_{23}^{(1)} - 2d_1 S_{12}^{(1)} + 4S_{23}^{(1)} = 0,$$

$$2d_2 \partial_1 S_{12}^{(2)} + d_1 \partial_1 S_{12}^{(2)} - 4d_2 S_{23}^{(2)} - 2d_1 S_{23}^{(2)} = 0 \tag{4.26}$$

Substituting (4.25) for $S_{ij}^{(n)}$ in (4.26), we obtain four equations in terms of $\chi_k^{(n)}$ as shown below:

$$2d_1 C_{55}^{(1)} \partial_1^2 \chi_0^{(1)} + d_2 C_{55}^{(2)} \partial_1^2 \chi_0^{(2)} = 0,$$

$$6d_1 C_{55}^{(1)} \partial_1 \chi_0^{(1)} - \frac{1}{4} C_{55}^{(1)} \partial_1^2 \chi_1^{(1)} + 12C_{44}^{(1)} \chi_1^{(1)} = 0,$$

$$10d_1 C_{55}^{(1)} \partial_1 \chi_0^{(1)} - 5d_1 C_{55}^{(1)} \partial_1^2 \chi_1^{(1)} + 60C_{44}^{(1)} \chi_1^{(1)} + \frac{1}{4} C_{55}^{(1)} \partial_1^2 \chi_2^{(1)} - 60d_1 C_{44}^{(1)} \chi_2^{(1)} = 0,$$

$$20d_1 C_{55}^{(1)} \partial_1 \chi_0^{(1)} + 2d_1 C_{55}^{(1)} \partial_1^2 \chi_2^{(1)} - 120d_1 C_{44}^{(1)} \chi_2^{(1)} +$$

$$10d_2 C_{55}^{(2)} \partial_1 \chi_0^{(2)} + \frac{1}{4} C_{55}^{(2)} \partial_1^2 \chi_2^{(2)} - 60d_2 C_{44}^{(2)} \chi_2^{(2)} = 0 \tag{4.27}$$

In addition, by substituting (4.13)-(4.14) into (4.8), the displacement continuity condition gives one more equation as follows:

$$4\chi_0^{(1)} + 2d_1 \chi_1^{(1)} + d_1^2 \chi_2^{(1)} - 4\chi_0^{(2)} - d_2^2 \chi_2^{(2)} = 0 \tag{4.28}$$
(4.27), together with (4.28), provides a set of five linear homogeneous equations for the five unknown field variables listed in (4.17). These equations should be supplemented with the boundary conditions given in (4.3)-(4.5). By substituting (4.13)-(4.14), (4.20) into (4.3)-(4.5), the exact boundary conditions can be expressed as:

\[ \chi_0^{(1)} = \chi_1^{(1)} = \chi_2^{(1)} = \chi_0^{(2)} = \chi_2^{(2)} = 0, \quad \text{at } x_1 = 0 \]  

(4.29)

and

\[ \chi_0^{(1)} = 1, \quad \chi_1^{(1)} = \chi_2^{(1)} = 0, \quad \partial_1 \chi_0^{(2)} = \partial_1 \chi_2^{(2)} = 0, \quad \text{at } x_1 = \frac{\ell_1}{2} \]  

(4.30)

Consequently, the unknown field variables at points in the interval \([0, \frac{\ell_1}{2}]\) can be determined by approximating the derivatives as difference expressions and by using the banded matrix structure as mentioned in the previous chapter.

### 4.2 Results and Comparisons

#### 4.2.1 Shear Modulus

The effective property of interest for pure shear is, of course, shear modulus. Since we apply unit displacement at one end of the modeled region, the average shear strain becomes:

\[ \bar{\gamma}_{12} = \frac{1}{(\ell_1/2)} \]  

(4.31)

The average shear stress of each layer is given by (4.25). Therefore, the average shear stress of the laminate can be defined as:
The effective shear modulus of the damaged laminate is then:

\[
G_{12}^* = \frac{\bar{\sigma}_{12}}{\bar{y}_{12}}
\]

(4.33)

The effective Poisson’s ratio and shear modulus are much larger than that of the undamaged laminate. The agreement between the present analysis and the FEM is excellent (the biggest difference at TCD = 2.6 mm−1 for [0/90₃], is about 5%). As expected, the degradation is higher for the laminates with a higher percentage of 90° layers.

4.2.2 Comparison among Effective Moduli

It is of interest to compare the differences in degradation of the effective moduli as a function of crack density for a damaged laminate. Fig. 20 shows a comparison of the degradation in axial stiffness, Poisson’s ratio, and shear modulus for a graphite/epoxy [0/90₃], laminate. It can be seen that the degradation in the effective Poisson’s ratio and shear modulus are much larger than that...
Figure 19. Degradation of Shear Modulus for Graphite/Epoxy Laminates with Transverse Cracks
Table 4. Degradation of Shear Modulus for Laminates with Transverse Cracks

<table>
<thead>
<tr>
<th>Condition</th>
<th>[0/90]_t</th>
<th>[0/90]_t</th>
<th>[0/90]_t</th>
</tr>
</thead>
<tbody>
<tr>
<td>Undamaged</td>
<td>5.65 GPa</td>
<td>5.65 GPa</td>
<td>5.65 GPa</td>
</tr>
<tr>
<td>Crack Density = 3.2 mm(^{-1}) Degradation</td>
<td>4.71 GPa</td>
<td>3.42 GPa</td>
<td>2.15 GPa</td>
</tr>
<tr>
<td></td>
<td>17 %</td>
<td>39 %</td>
<td>62 %</td>
</tr>
<tr>
<td>Ply Discount</td>
<td>4.24 GPa</td>
<td>2.82 GPa</td>
<td>1.41 GPa</td>
</tr>
<tr>
<td>Degradation</td>
<td>25 %</td>
<td>50 %</td>
<td>75 %</td>
</tr>
</tbody>
</table>

in the axial stiffness. Also, the response of $v_{12}$ and $G_{12}$ are similar to each other and have almost the same extent of degradation for the higher TCD. This tendency coincides with the results given by the ply discount method (see Table 2, 4). From the results shown in Fig. 20, we may conclude that the Poisson's ratio and shear modulus are better indicators of damage than axial stiffness. Since uniaxial tension is the simplest test and, from Fig. 20, the reduction in $v_{12}$ is larger for the lower TCD (which is of practical engineering interest), the Poisson's ratio may be the preferred property for a damage index of a cracked cross-ply laminate.
Figure 20. Comparison of Degradation in Effective Moduli for $[0/90]_s$ Transverse Cracks under Pure Shear
5.0 Transverse Cracks under Thermal Loading

5.1 Formulation

Suppose that the damaged laminate is subjected to a uniform thermal loading. By the assumptions in Chapter 2, it is reasonable to formulate this problem by a generalized plane strain analysis. Let \( u^{(o)} \) and \( \sigma^{(o)}_{ij} \) be the displacements and stress components in the \( \alpha \) layer with \( i, j = 1 \) and \( 3 \). By the considerations of symmetry and the traction-free conditions at the outer surfaces and the crack surface, the following boundary conditions should be imposed on the modeled region (see Fig. 21):

\[
\sigma_{31}^{(1)} = \sigma_{33}^{(1)} = 0, \quad 0 \leq x_1 \leq \frac{l_1}{2}, \quad x_3^{(1)} = -\frac{d_1}{2} \tag{5.1}
\]

\[
\sigma_{31}^{(3)} = \sigma_{33}^{(3)} = 0, \quad 0 \leq x_1 \leq \frac{l_1}{2}, \quad x_3^{(3)} = \frac{d_3}{2} \tag{5.2}
\]

and

\[
u_1^{(o)} = 0, \quad \sigma_{13}^{(o)} = 0, \quad x_1 = 0, \quad |x_3^{(o)}| \leq \frac{d_{\alpha}}{2} \tag{5.3}
\]
Figure 21. Boundary Conditions under Thermal Loading
\[ u_1^{(1)} = U, \quad \sigma_{13}^{(1)} = 0, \quad x_1 = \frac{d_1}{2}, \quad |\bar{x}_3^{(1)}| \leq \frac{d_1}{2} \]  

(5.4)

\[ \sigma_{11}^{(2)} = \sigma_{13}^{(2)} = 0, \quad x_1 = \frac{d_2}{2}, \quad |\bar{x}_3^{(2)}| \leq \frac{d_2}{2} \]  

(5.5)

\[ u_1^{(3)} = U, \quad \sigma_{13}^{(3)} = 0, \quad x_1 = \frac{d_3}{2}, \quad |\bar{x}_3^{(3)}| \leq \frac{d_3}{2} \]  

(5.6)

\[ N_{11} = \sum_{a=1}^{3} \int_{-d_a/2}^{d_a/2} \sigma_{11}^{(a)} \bar{x}_3^{(a)} = 0, \quad x_1 = \frac{d_1}{2} \]  

(5.7)

where \( U \) is presently an unknown constant and \( N_{11} \) is the stress resultant in \( x_1 \) direction. Also, for a composite laminate, the traction and displacement continuity conditions at the interfaces must be considered. These conditions can be expressed as:

\[ \sigma_{3j}^{(1)}|_{\bar{x}_3^{(1)}} = \frac{d_1}{2} = \sigma_{3j}^{(2)}|_{\bar{x}_3^{(2)}} = -\frac{d_2}{2}, \quad \sigma_{3j}^{(2)}|_{\bar{x}_3^{(3)}} = \frac{d_2}{2} = \sigma_{3j}^{(3)}|_{\bar{x}_3^{(3)}} = -\frac{d_3}{2} \]  

(5.8)

\[ u_1^{(1)}|_{\bar{x}_3^{(1)}} = \frac{d_1}{2} = u_1^{(2)}|_{\bar{x}_3^{(2)}} = -\frac{d_2}{2}, \quad u_1^{(2)}|_{\bar{x}_3^{(3)}} = \frac{d_2}{2} = u_1^{(3)}|_{\bar{x}_3^{(3)}} = -\frac{d_3}{2} \]  

(5.9)

Assume a displacement field of the \( \alpha \) layer in the form of a second order Legendre expansion in \( \bar{x}_3^{(\alpha)} \) as follows:

\[ u_1^{(\alpha)} = \varphi_0^{(\alpha)} + \varphi_1^{(\alpha)} \bar{x}_3^{(\alpha)} + \frac{1}{2} \varphi_2^{(\alpha)} [3(\bar{x}_3^{(\alpha)})^2 - \frac{d_\alpha^2}{4}], \]  

\[ u_3^{(\alpha)} = \psi_0^{(\alpha)} + \psi_1^{(\alpha)} \bar{x}_3^{(\alpha)} + \frac{1}{2} \psi_2^{(\alpha)} [3(\bar{x}_3^{(\alpha)})^2 - \frac{d_\alpha^2}{4}] \]  

(5.10)

where the field variables, \( \varphi_k^{(\alpha)}, \psi_k^{(\alpha)} (k = 0, 1, 2) \), have dependence on \( x_1 \) only.

Since the laminate is symmetric to its midplane, \( d_1 \) is equal to \( d_3 \). Furthermore, we have the following symmetry conditions in displacements:

Transverse Cracks under Thermal Loading
and for any $\bar{x}_i^{(t)} = \bar{x}_i^{(3)}$,

\[
\begin{align*}
\mathbf{u}_1^{(1)}(x_1, \bar{x}_3^{(1)}) &= \mathbf{u}_1^{(3)}(x_1, -\bar{x}_3^{(3)}), \\
\mathbf{u}_3^{(1)}(x_1, \bar{x}_3^{(1)}) &= -\mathbf{u}_3^{(1)}(x_1, -\bar{x}_3^{(3)})
\end{align*}
\]  

(5.12)

From (5.10)-(5.11), the displacements of each layer can be expressed as:

\[
\begin{align*}
\mathbf{u}_1^{(1)} &= \phi_0^{(1)} + \phi_1^{(1)}\bar{x}_3^{(1)} + \frac{1}{2}\phi_2^{(1)}[3(\bar{x}_3^{(1)})^2 - \frac{d_1^2}{4}], \\
\mathbf{u}_3^{(1)} &= \psi_0^{(1)} + \psi_1^{(1)}\bar{x}_3^{(1)} + \frac{1}{2}\psi_2^{(1)}[3(\bar{x}_3^{(1)})^2 - \frac{d_1^2}{4}] \tag{5.13}
\end{align*}
\]

\[
\begin{align*}
\mathbf{u}_1^{(2)} &= \phi_0^{(2)} + \frac{1}{2}\phi_2^{(2)}[3(\bar{x}_3^{(2)})^2 - \frac{d_2^2}{4}], \\
\mathbf{u}_3^{(2)} &= \psi_1^{(2)}\bar{x}_3^{(2)} 	ag{5.14}
\end{align*}
\]

\[
\begin{align*}
\mathbf{u}_1^{(3)} &= \phi_0^{(3)} + \phi_1^{(3)}\bar{x}_3^{(3)} + \frac{1}{2}\phi_2^{(3)}[3(\bar{x}_3^{(3)})^2 - \frac{d_3^2}{4}], \\
\mathbf{u}_3^{(3)} &= \psi_0^{(3)} + \psi_1^{(3)}\bar{x}_3^{(3)} + \frac{1}{2}\psi_2^{(3)}[3(\bar{x}_3^{(3)})^2 - \frac{d_3^2}{4}] \tag{5.15}
\end{align*}
\]

and from (5.12), there are some relations between the field variables of the $\alpha = 1$ layer and $\alpha = 3$ layer:

\[
\begin{align*}
\phi_0^{(3)} &= \phi_0^{(1)}, & \phi_1^{(3)} &= -\phi_1^{(1)}, & \phi_2^{(3)} &= \phi_2^{(1)}, \\
\psi_0^{(3)} &= -\psi_0^{(1)}, & \psi_1^{(3)} &= \psi_1^{(1)}, & \psi_2^{(3)} &= -\psi_2^{(1)} \tag{5.16}
\end{align*}
\]
As a result, we can express displacements and stresses of the \( \alpha = 3 \) layer in terms of the field variables of the \( \alpha = 1 \) layer. From (5.13) to (5.16), there are nine unknowns:

\[
\varphi_0^{(1)}, \varphi_1^{(1)}, \varphi_2^{(1)}, \psi_0^{(1)}, \psi_1^{(1)}, \psi_2^{(1)}, \varphi_0^{(2)}, \varphi_2^{(2)}, \psi_1^{(2)}
\]  

which must be determined.

The components of the linear strain tensor are given by:

\[
\varepsilon_{ij}^{(a)} = \frac{1}{2}(\partial_i u_j^{(a)} + \partial_j u_i^{(a)})
\]  

where \( \partial_i = \partial/\partial x_i \) and \( \partial_5 = \partial/\partial \xi^a \).

By (5.13)-(5.14) and (5.18), the strain components can be expressed as:

\[
\varepsilon_{11}^{(1)} = \partial_1 \varphi_0^{(1)} + \partial_1 \varphi_1^{(1)} \bar{x}_3^{(1)} + \frac{1}{2} \partial_1 \varphi_2^{(1)} [3(\bar{x}_3^{(1)})^2 - \frac{d_1^2}{4}].
\]

\[
\varepsilon_{33}^{(1)} = \psi_1^{(1)} + 3 \psi_2^{(1)} \bar{x}_3^{(1)}.
\]

\[
\varepsilon_{13}^{(1)} = \frac{1}{2} ((\varphi_1^{(1)} + \partial_1 \psi_0^{(1)}) + (3 \varphi_2^{(1)} + \partial_1 \psi_1^{(1)}) \bar{x}_3^{(1)} + \frac{1}{2} \partial_1 \psi_2^{(1)} [3(\bar{x}_3^{(1)})^2 - \frac{d_1^2}{4}]).
\]

\[
\varepsilon_{11}^{(2)} = \partial_1 \varphi_0^{(2)} + \frac{1}{2} \partial_1 \varphi_2^{(2)} [3(\bar{x}_3^{(2)})^2 - \frac{d_2^2}{4}].
\]

\[
\varepsilon_{33}^{(2)} = \psi_1^{(2)}.
\]

\[
\varepsilon_{13}^{(2)} = \frac{1}{2} (3 \varphi_2^{(2)} + \partial_1 \psi_1^{(2)}) \bar{x}_3^{(2)}
\]

The thermalelastic constitutive equation for orthotropic materials is given as:

\[
\sigma^{(a)} = C^{(a)} \varepsilon^{(a)} - H^{(a)} \Delta T
\]
where

\[ \sigma^{(a)} = [\sigma_{11}^{(a)}, \sigma_{22}^{(a)}, \sigma_{33}^{(a)}, \sigma_{12}^{(a)}, \sigma_{13}^{(a)}, \sigma_{23}^{(a)}], \]

\[ \varepsilon^{(a)} = [\varepsilon_{11}^{(a)}, \varepsilon_{22}^{(a)}, \varepsilon_{33}^{(a)}, 2\varepsilon_{12}^{(a)}, 2\varepsilon_{13}^{(a)}, 2\varepsilon_{23}^{(a)}], \]

\[ H^{(a)} = [H_{11}^{(a)}, H_{22}^{(a)}, H_{33}^{(a)}, 0, 0, 0]. \]

and

\[ C^{(a)} = \begin{bmatrix} C_{11}^{(a)} & C_{12}^{(a)} & C_{13}^{(a)} & 0 & 0 & 0 \\ C_{12}^{(a)} & C_{22}^{(a)} & C_{23}^{(a)} & 0 & 0 & 0 \\ C_{13}^{(a)} & C_{23}^{(a)} & C_{33}^{(a)} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44}^{(a)} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55}^{(a)} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66}^{(a)} \end{bmatrix} \]

Furthermore, the components of \( H^{(a)} \) for the \( a \) layer can be expressed explicitly as follows:

\[ H_{11}^{(a)} = C_{11}^{(a)} \beta_{11}^{(a)} + C_{12}^{(a)} \beta_{22}^{(a)} + C_{13}^{(a)} \beta_{33}^{(a)}, \]

\[ H_{22}^{(a)} = C_{12}^{(a)} \beta_{11}^{(a)} + C_{22}^{(a)} \beta_{22}^{(a)} + C_{23}^{(a)} \beta_{33}^{(a)}, \]

\[ H_{33}^{(a)} = C_{13}^{(a)} \beta_{11}^{(a)} + C_{23}^{(a)} \beta_{22}^{(a)} + C_{33}^{(a)} \beta_{33}^{(a)} \]

(5.22)

where \( \beta_{11}^{(a)}, \beta_{22}^{(a)}, \beta_{33}^{(a)} \) are the coefficients of thermal expansion of the \( a \) layer in \( x_1, x_2, x_3 \) direction, respectively.

For simplicity, the \( \Delta T \) in (5.21) is set to be \(-1^\circ\). Thus, the inplane stresses can be expressed as:

\[ \sigma_{11}^{(a)} = C_{11}^{(a)} \varepsilon_{11}^{(a)} + C_{12}^{(a)} \varepsilon_{22}^{(a)} + C_{13}^{(a)} \varepsilon_{33}^{(a)} + H_{11}^{(a)}, \]
\begin{equation}
\sigma_{33}^{(a)} = C_{13}^{(a)} \varepsilon_{11}^{(a)} + C_{23}^{(a)} \varepsilon_{22}^{(a)} + C_{33}^{(a)} \varepsilon_{33}^{(a)} + H_{33}^{(a)},
\end{equation}

\begin{equation}
\sigma_{13}^{(a)} = C_{66}^{(a)} (2 \varepsilon_{13}^{(a)})
\end{equation}

and the out-of-plane normal stress is:

\begin{equation}
\sigma_{22}^{(a)} = C_{12}^{(a)} \varepsilon_{11}^{(a)} + C_{22}^{(a)} \varepsilon_{22}^{(a)} + C_{23}^{(a)} \varepsilon_{33}^{(a)} + H_{22}^{(a)}
\end{equation}

The stress resultant in the \( x_2 \) direction is defined as:

\begin{equation}
N_{22} = \int_{-d_{i/2}}^{d_{i/2}} \sigma_{22}^{(1)} d\bar{x}_3^{(1)} + \int_{-d_{2/2}}^{d_{2/2}} \sigma_{22}^{(2)} d\bar{x}_3^{(2)} + \int_{-d_{3/2}}^{d_{3/2}} \sigma_{22}^{(3)} d\bar{x}_3^{(3)}
\end{equation}

For a generalized plane strain analysis, \( \varepsilon_{22}^{(a)} \) is constant everywhere within the modeled region. Let \( \bar{\varepsilon}_{22} \) denote this out-of-plane strain. Since \( N_{22} \) is zero for uniform thermal loading, \( \bar{\varepsilon}_{22} \) can be found by substituting (5.19)-(5.20), (5.24) into (5.25) and by setting \( N_{22} \) equal to zero. As a result, the out-of-plane strain can be written as:

\begin{equation}
\bar{\varepsilon}_{22} = \frac{-2d_{1}(C_{12}^{(1)} \partial_{1} \varphi_{0}^{(1)} + C_{23}^{(1)} \psi_{1}^{(1)}) + d_{2}(C_{12}^{(2)} \partial_{1} \varphi_{0}^{(2)} + C_{23}^{(2)} \psi_{1}^{(2)}) + Q}{2d_{1} C_{22}^{(1)} + d_{2} C_{22}^{(2)}}
\end{equation}

where

\begin{equation}
Q = \frac{-2d_{1}H_{22}^{(1)} + d_{2}H_{22}^{(2)}}{2d_{1} C_{22}^{(1)} + d_{2} C_{22}^{(2)}}
\end{equation}

Substituting (5.19)-(5.20) and (5.26) into (5.23), we obtain the explicit expressions for the inplane stresses in each layer:

\begin{equation}
\sigma_{11}^{(1)} = (A_{11}^{(1)} \partial_{1} \varphi_{0}^{(1)} + A_{13}^{(1)} \psi_{1}^{(1)} + B_{11}^{(2)} \partial_{1} \varphi_{0}^{(2)} + B_{12}^{(2)} \psi_{1}^{(2)}) + (C_{11}^{(1)} \partial_{1} \varphi_{1}^{(1)} + 3C_{13}^{(1)} \psi_{2}^{(1)} \bar{x}_{3}^{(1)} +

\frac{1}{2} C_{11}^{(1)} \partial_{1} \varphi_{2}^{(1)} [3(\bar{x}_{3}^{(1)})^2 - \frac{d_{1}^2}{4}] + (C_{12}^{(1)} Q + H_{11}^{(1)}),
\end{equation}

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\[\sigma_{33}^{(1)} = (A_{13}^{(1)} \delta_1 \psi_0^{(1)} + A_{33}^{(1)} \psi_1^{(1)} + B_{31}^{(2)} \delta_1 \psi_0^{(2)} + B_{33}^{(2)} \psi_1^{(2)}) + (C_{13}^{(1)} \delta_1 \psi_1^{(1)} + 3C_{33}^{(1)} \psi_1^{(1)} \bar{x}_3^{(1)}) + \]
\[
\frac{1}{2} C_{13}^{(1)} \delta_1 \psi_2^{(1)} [3(\bar{x}_3^{(1)})^2 - \frac{d_1^2}{4}] + (C_{23}^{(1)} Q + H_{33}^{(1)}),
\]
\[\sigma_{13}^{(2)} = (B_{11}^{(1)} \delta_1 \psi_0^{(1)} + B_{13}^{(1)} \psi_1^{(1)} + A_{11}^{(2)} \delta_1 \psi_0^{(2)} + A_{13}^{(2)} \psi_1^{(2)}) + \frac{1}{2} C_{11}^{(2)} \delta_1 \psi_2^{(2)} [3(\bar{x}_3^{(2)})^2 - \frac{d_2^2}{4}] + (C_{12}^{(2)} Q + H_{13}^{(2)}),
\]
\[\sigma_{33}^{(2)} = (B_{31}^{(2)} \delta_1 \psi_0^{(1)} + B_{33}^{(2)} \psi_1^{(1)} + A_{31}^{(2)} \delta_1 \psi_0^{(2)} + A_{33}^{(2)} \psi_1^{(2)}) + \frac{1}{2} C_{31}^{(2)} \delta_1 \psi_2^{(2)} [3(\bar{x}_3^{(2)})^2 - \frac{d_2^2}{4}] + (C_{23}^{(2)} Q + H_{33}^{(2)}),
\]
\[\sigma_{13}^{(2)} = C_{13}^{(2)} [3 \psi_2^{(2)} + \delta_1 \psi_2^{(2)} \bar{x}_3^{(2)}]
\]

where

\[A_{11}^{(1)} = C_{11}^{(1)} - \frac{2d_1 C_{12}^{(1)} C_{12}^{(1)}}{2d_1 C_{12}^{(1)} + d_2 C_{22}^{(1)}}, \quad A_{13}^{(1)} = C_{13}^{(1)} - \frac{2d_1 C_{12}^{(1)} C_{23}^{(1)}}{2d_1 C_{12}^{(1)} + d_2 C_{22}^{(1)}},
\]
\[A_{33}^{(1)} = C_{33}^{(1)} - \frac{2d_1 C_{12}^{(1)} C_{33}^{(1)}}{2d_1 C_{12}^{(1)} + d_2 C_{22}^{(1)}},
\]
\[B_{11}^{(1)} = -\frac{2d_1 C_{12}^{(1)} C_{12}^{(1)}}{2d_1 C_{12}^{(1)} + d_2 C_{22}^{(1)}}, \quad B_{13}^{(1)} = -\frac{2d_1 C_{23}^{(1)} C_{12}^{(1)}}{2d_1 C_{12}^{(1)} + d_2 C_{22}^{(1)}},
\]
\[B_{31}^{(1)} = -\frac{2d_1 C_{12}^{(1)} C_{23}^{(1)}}{2d_1 C_{12}^{(1)} + d_2 C_{22}^{(1)}}, \quad B_{33}^{(1)} = -\frac{2d_1 C_{23}^{(1)} C_{23}^{(1)}}{2d_1 C_{12}^{(1)} + d_2 C_{22}^{(1)}},
\]

Transverse Cracks under Thermal Loading
The equilibrium equations for each layer are written as:

\[ \partial_1 \sigma_{ij}^{(a)} + \partial_3 \sigma_{ij}^{(a)} = 0 \]  

Integrating (5.32) over the thickness of each layer gives:

\[ \frac{1}{d_i} \int_{-\frac{d_{a/2}}{2}}^{\frac{d_{a/2}}{2}} \left[ \partial_1 \sigma_{ij}^{(a)} + \partial_3 \sigma_{ij}^{(a)} \right] (\bar{x}_3^{(a)}) d\bar{x}_3^{(a)} = 0 \]  

and integrating by parts, we obtain:

\[ \partial_i S_{ij}^{(a)} - n S_{ij}^{(a)} + \frac{1}{d_x} \left( \frac{d_{a/2}}{2} \right) \int_{-\frac{d_{a/2}}{2}}^{\frac{d_{a/2}}{2}} \sigma_{ij}^{(a)} (\bar{x}_3^{(a)}) \right] d\bar{x}_3^{(a)} = \frac{d_{a/2}}{2} - \left( \frac{n}{2} \right) \sigma_{ij}^{(a)} (\bar{x}_3^{(a)}) = - \frac{d_{a/2}}{2} \]  

where

\[ S_{ij}^{(a)} = \frac{1}{d_x} \int_{-\frac{d_{a/2}}{2}}^{\frac{d_{a/2}}{2}} \sigma_{ij}^{(a)} (\bar{x}_3^{(a)}) \right] d\bar{x}_3^{(a)} , \quad n = 0, 1, 2 \]  

The expressions of \( S_{ij}^{(a)} \) can be expanded explicitly by substituting (5.28)-(5.29) for \( \sigma_{ij}^{(a)} \) in (5.35):

\[ S_{11}^{(1)} = A_{11}^{(1)} \psi_0^{(1)} + A_{13}^{(1)} \psi_1^{(1)} + B_{11}^{(2)} \partial_1 \psi_0^{(2)} + B_{13}^{(2)} \psi_1^{(2)} + C_{12}^{(1)} Q + H_{11}^{(1)} \]

Transverse Cracks under Thermal Loading
\( S_{33}^{(1)} = A_{13}^{(1)} \partial_1 \phi_0^{(1)} + A_{33}^{(1)} \psi_1^{(1)} + B_{33}^{(2)} \partial_1 \phi_0^{(2)} + B_{33}^{(2)} \psi_1^{(2)} + C_{23}^{(1)} Q + H_{33}^{(1)}, \)

\( S_{13}^{(1)} = C_{66}^{(1)} (\phi_1^{(1)} + \partial_1 \psi_0^{(1)}), \)

\( S_{11}^{(2)} = B_{11}^{(1)} \partial_1 \phi_0^{(1)} + B_{13}^{(1)} \psi_1^{(1)} + A_{11}^{(2)} \partial_1 \phi_0^{(2)} + A_{13}^{(2)} \psi_1^{(2)} + C_{12}^{(2)} Q + H_{11}^{(2)}, \)

\( S_{33}^{(2)} = B_{31}^{(1)} \partial_1 \phi_0^{(1)} + B_{33}^{(2)} \psi_1^{(2)} + A_{31}^{(2)} \partial_1 \phi_0^{(2)} + A_{33}^{(2)} \psi_1^{(2)} + C_{23}^{(2)} Q + H_{33}^{(2)}, \)

\( S_{11}^{(1)} = \frac{d_1^2}{12} (C_{11}^{(1)} \partial_1 \phi_1^{(1)} + 3C_{13}^{(1)} \psi_2^{(1)}), \)

\( S_{33}^{(1)} = \frac{d_1^2}{12} (C_{13}^{(1)} \partial_1 \phi_1^{(1)} + 3C_{33}^{(1)} \psi_2^{(1)}), \)

\( S_{13}^{(1)} = \frac{d_1^2}{12} C_{66}^{(1)} (3\phi_2^{(1)} + \partial_1 \psi_1^{(1)}), \)

\( S_{13}^{(2)} = \frac{d_2^2}{12} C_{66}^{(2)} (3\phi_2^{(2)} + \partial_1 \psi_1^{(2)}), \)

\( S_{11}^{(2)} = \frac{d_1^2}{12} (A_{11}^{(1)} \partial_1 \phi_0^{(1)} + \frac{d_1^2}{10} C_{11}^{(1)} \partial_1 \phi_2^{(1)} + A_{13}^{(1)} \psi_1^{(1)} + B_{11}^{(2)} \partial_1 \phi_0^{(2)} + B_{13}^{(2)} \psi_1^{(2)} + C_{12}^{(1)} Q + H_{11}^{(1)}), \)

\( S_{33}^{(2)} = \frac{d_1^2}{12} (A_{13}^{(1)} \partial_1 \phi_0^{(1)} + \frac{d_1^2}{10} C_{13}^{(1)} \partial_1 \phi_2^{(1)} + A_{33}^{(1)} \psi_1^{(1)} + B_{31}^{(2)} \partial_1 \phi_0^{(2)} + B_{33}^{(2)} \psi_1^{(2)} + C_{23}^{(1)} Q + H_{33}^{(1)}), \)

\( S_{13}^{(2)} = \frac{d_1^2}{12} C_{66}^{(2)} (\phi_1^{(1)} + \partial_1 \psi_0^{(1)} + \frac{d_2^2}{10} \partial_1 \psi_2^{(1)}), \)

\( S_{11}^{(2)} = \frac{d_2^2}{12} (B_{11}^{(1)} \partial_1 \phi_0^{(1)} + B_{13}^{(1)} \psi_1^{(1)} + A_{11}^{(2)} \partial_1 \phi_0^{(2)} + \frac{d_2^2}{10} C_{11}^{(2)} \partial_1 \phi_2^{(2)} + A_{13}^{(2)} \psi_1^{(2)} + C_{12}^{(2)} Q + H_{11}^{(2)}), \)

\( S_{33}^{(2)} = \frac{d_2^2}{12} (B_{13}^{(1)} \partial_1 \phi_0^{(1)} + B_{33}^{(1)} \psi_1^{(1)} + A_{13}^{(2)} \partial_1 \phi_0^{(2)} + \frac{d_2^2}{10} C_{13}^{(2)} \partial_1 \phi_2^{(2)} + A_{33}^{(2)} \psi_1^{(2)} + C_{23}^{(2)} Q + H_{33}^{(2)}), \)

(5.36) Transverse Cracks under Thermal Loading
and all other \( S_{ij}^{(n)} \) are zero.

By using the traction continuity conditions in (5.8) and the traction-free conditions in (5.1)-(5.2), and noticing the symmetry conditions in the \( \alpha = 2 \) layer and the relations between the \( \alpha = 1 \) and \( \alpha = 3 \) layers (5.16), the interfacial and boundary stress terms in (5.34) can be eliminated. Therefore, the following equations can be derived from (5.34) by linear operations with different values of \( n \):

\[
2d_1 \partial_i S_{11}^{(0)} + d_2 \partial_i S_{11}^{(2)} = 0,
\]

\[
d_2^2 \partial_i S_{11}^{(0)} - 4 \partial_i S_{11}^{(2)} + 8S_{13}^{(2)} = 0,
\]

\[
4d_1 \partial_i S_{11}^{(1)} - 4S_{13}^{(1)} + d_2 \partial_i S_{11}^{(2)} = 0,
\]

\[
8 \partial_i S_{11}^{(1)} - 16S_{13}^{(1)} + d_1 d_2 \partial_i S_{11}^{(2)} = 0,
\]

\[
d_1 \partial_i S_{13}^{(0)} - \partial_i S_{13}^{(2)} - S_{33}^{(2)} = 0,
\]

\[
2 \partial_i S_{13}^{(1)} - 2S_{33}^{(1)} - \partial_i S_{13}^{(2)} + S_{33}^{(2)} = 0,
\]

\[
4 \partial_i S_{13}^{(1)} - 8S_{33}^{(1)} - d_1 \partial_i S_{13}^{(2)} + d_1 S_{33}^{(2)} = 0
\]

Substituting (5.36) for \( \delta_i \) in (5.37), we obtain seven equations in terms of \( \psi_i^{(0)} \) and \( \psi_i^{(0)} \) as shown below:

\[
(2d_1 A_{11}^{(1)} + d_2 B_{11}^{(1)}) \partial_i \phi_0^{(1)} + (2d_1 A_{13}^{(1)} + d_2 B_{13}^{(1)}) \partial_i \psi_1^{(1)} +
\]

\[
(2d_1 B_{11}^{(2)} + d_2 A_{11}^{(2)}) \partial_i \phi_0^{(2)} + (2d_1 B_{13}^{(2)} + d_2 A_{13}^{(2)}) \partial_i \psi_1^{(2)} = 0.
\]
\begin{align*}
20B^{(1)}_{11} \phi_{1}^{(1)} + 20B^{(1)}_{13} \psi_{1}^{(1)} + 20A^{(2)}_{11} \phi_{1}^{(2)} + 60C^{(2)}_{66} \phi_{2}^{(2)} - \\
d_{2}^{2}C^{(2)}_{11} \phi_{1}^{(2)} + 20(A^{(2)}_{13} + C^{(2)}_{66}) \phi_{1}^{(2)} = 0,
\end{align*}

\begin{align*}
3d_{2}B^{(1)}_{11} \phi_{0}^{(1)} + d_{1}^{2}C^{(1)}_{11} \phi_{1}^{(1)} - 12C^{(1)}_{66} \psi_{1}^{(1)} - 12C^{(1)}_{66} \psi_{0}^{(1)} + 3d_{2}B^{(1)}_{13} \phi_{1}^{(1)} + \\
3d_{2}C^{(1)}_{13} \phi_{1}^{(2)} + 3d_{2}A^{(2)}_{11} \phi_{0}^{(2)} + 3d_{2}A^{(2)}_{13} \phi_{1}^{(2)} = 0,
\end{align*}

\begin{align*}
(10d_{1}A^{(1)}_{11} + 15d_{2}B^{(1)}_{11}) \phi_{0}^{(1)} + d_{1}^{2}C^{(1)}_{11} \phi_{1}^{(1)} - 60d_{1}C^{(1)}_{66} \psi_{1}^{(1)} + \\
(10d_{1}A^{(1)}_{13} - 20d_{1}C^{(1)}_{66} + 15d_{2}B^{(1)}_{13}) \phi_{1}^{(1)} + (10d_{1}B^{(2)}_{11} + 15d_{2}A^{(2)}_{11}) \phi_{0}^{(2)} + \\
(10d_{1}B^{(2)}_{13} + 15d_{2}A^{(2)}_{13}) \phi_{1}^{(2)} = 0,
\end{align*}

\begin{align*}
12B^{(1)}_{31} \phi_{1}^{(1)} + 12d_{1}C^{(1)}_{66} \phi_{1}^{(1)} + 12d_{1}C^{(1)}_{66} \psi_{0}^{(1)} + 12B^{(1)}_{33} \psi_{1}^{(1)} + \\
12A^{(2)}_{13} \phi_{0}^{(2)} - 3d_{2}^{2}C^{(2)}_{66} \phi_{1}^{(2)} + 12A^{(2)}_{13} \psi_{1}^{(2)} - d_{2}^{2}C^{(2)}_{66} \psi_{1}^{(2)} = -12(C^{(2)}_{23} Q + H^{(2)}_{33}),
\end{align*}

\begin{align*}
12(B^{(1)}_{31} - 2A^{(1)}_{13}) \phi_{1}^{(1)} + 6d_{1}^{2}C^{(1)}_{66} \phi_{1}^{(1)} + 2d_{1}^{2}C^{(1)}_{66} \psi_{1}^{(1)} + \\
12(B^{(1)}_{33} - 2A^{(1)}_{33}) \psi_{1}^{(1)} + 12(A^{(2)}_{13} - 2B^{(2)}_{31}) \phi_{0}^{(2)} - 3d_{2}^{2}C^{(2)}_{66} \phi_{1}^{(2)} - \\
3d_{2}^{2}C^{(2)}_{66} \psi_{1}^{(2)} + 12(A^{(2)}_{33} - 2B^{(2)}_{33}) \psi_{1}^{(2)} = 24(C^{(1)}_{23} Q + H^{(1)}_{33}) - 12(C^{(2)}_{23} Q + H^{(2)}_{33}),
\end{align*}

\begin{align*}
60B^{(1)}_{31} \phi_{1}^{(1)} + 20d_{1}(C^{(1)}_{66} - 2C^{(1)}_{13}) \phi_{1}^{(1)} + 20d_{1}C^{(1)}_{66} \psi_{0}^{(1)} + 60B^{(1)}_{33} \psi_{1}^{(1)} + \\
2d_{1}^{2}C^{(1)}_{66} \psi_{2}^{(1)} - 120d_{1}C^{(1)}_{66} \psi_{1}^{(1)} + 60A^{(2)}_{13} \phi_{0}^{(2)} - \\
15d_{2}^{2}C^{(2)}_{66} \phi_{1}^{(2)} - 5d_{2}^{2}C^{(2)}_{66} \psi_{1}^{(2)} + 60A^{(2)}_{13} \psi_{1}^{(2)} = -60(C^{(2)}_{23} Q + H^{(2)}_{33})
\end{align*} \hfill (5.38)
In addition, by substituting (5.13)-(5.14) into (5.9), the displacement continuity conditions provide two more equations as follows:

\[ 4\psi_0^{(1)} + 2d_1\phi_1^{(1)} + d_1^2\phi_2^{(1)} - 4\psi_0^{(2)} - d_1^2\psi_2^{(2)} = 0, \]

\[ 4\psi_0^{(1)} + 2d_1\psi_1^{(1)} + d_1^2\psi_2^{(1)} + 2d_2\psi_2^{(2)} = 0 \]  \hspace{1cm} (5.39)

(5.38), together with (5.39), gives a set of nine linear homogeneous equations for the nine unknown field variables listed in (5.17). These equations should be supplemented with the boundary conditions given in (5.3)-(5.7). By substituting (5.13)-(5.14), (5.28)-(5.29) into (5.3)-(5.7), the exact boundary conditions can be expressed as:

\[ \psi_0^{(1)} = \psi_1^{(1)} = \psi_2^{(1)} = \psi_0^{(2)} = \psi_2^{(2)} = 0, \]

\[ \partial_x\psi_0^{(1)} = \partial_x\psi_1^{(1)} = \partial_x\psi_2^{(1)} = \partial_x\psi_2^{(2)} = 0, \quad \text{at} \quad x_1 = 0 \]  \hspace{1cm} (5.40)

and

\[ \phi_1^{(1)} = \phi_2^{(1)} = 0, \]

\[ \partial_x\phi_0^{(1)} = \partial_x\phi_1^{(1)} = \partial_x\phi_2^{(1)} = \partial_x\phi_2^{(2)} = 0, \]

\[ A_{11}\partial_x\phi_0^{(1)} + A_{13}\psi_1^{(1)} + B_{11}\partial_x\phi_0^{(2)} + B_{13}\psi_1^{(2)} = -(C_{12}Q + H_{11}^{(1)}), \]

\[ B_{11}\partial_x\phi_0^{(1)} + B_{13}\psi_1^{(1)} + A_{11}\partial_x\phi_0^{(2)} + A_{13}\psi_1^{(2)} = -(C_{12}Q + H_{11}^{(2)}), \]

\[ 3\psi_2^{(2)} + \partial_x\psi_2^{(2)} = 0, \quad \text{at} \quad x_1 = \frac{\xi_1}{2} \]  \hspace{1cm} (5.41)

It should be noted that the displacement \( U \) in (5.4) is an unknown constant in the thermal formulation and thus can not be used as a boundary condition in (5.41). This condition is replaced by the fact that the stress resultant \( N_{11} \) is zero.

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Approximating the derivatives of field variables in (5.38)-(5.41) by difference expressions, we can obtain a banded matrix as mentioned in the previous chapter. Therefore, the unknowns in (5.17) at points along $0 \leq x_i \leq \ell_i/2$ can be solved numerically on the computer.

### 5.2 Results and Comparisons

The axial elongation, $U$, of the modeled region due to the thermal loading can be determined by (5.13). The average axial thermal strain is then:

$$
\bar{\varepsilon}_{11} = \frac{U}{(\ell_i/2)}
$$

(5.42)

For a unit thermal loading $\Delta T = -1^\circ$, we can define the effective axial coefficient of thermal expansion as:

$$
\beta_{11}^* = \frac{\bar{\varepsilon}_{11}}{\Delta T} = -\bar{\varepsilon}_{11}
$$

(5.43)

The results for three types of graphite/epoxy laminates are presented in Fig. 22. The laminates studied are: $[0/90]_4$, $[0/90]_2$, $[0/90]_6$. The undamaged coefficient of thermal expansion ($\beta_{11}^*$) used for normalization for each laminate is given in Table 5. The figure shows there is a dramatic change in $\beta_{11}^*$ when the transverse cracks develop in the $90^\circ$ layers. It is not difficult to understand this significant reduction if one notices that the $\beta_{11}^*$ of graphite/epoxy is negative (see Appendix A). At the first glance, it may seem very strange that the degradation curve of $[0/90]_4$ intersects the others at approximately $TCD = 0.8 \text{ mm}^{-1}$. Furthermore, unlike the effective elastic moduli studied in the previous chapters, the $\beta_{11}^*$ of the laminate with a lower percentage of $90^\circ$ layers degrades more than those with a higher percentage. Referring to the unnormalized degradation curves in Fig. 23, this...
Figure 22. Degradation of CTE for Graphite/Epoxy Laminates with Transverse Cracks
phenomenon can be completely explained. Since the absolute value of axial (x₁ direction) CTE of 90° layers (24.4 με/°C) is much higher than that of 0° layers (-7.74 με/°C), the β₁₁ of a undamaged laminate is dominated by the 90° layers. This is why the undamaged β₁₁ of [0/90]ₜ is much larger than that of [0₂/90₂]ₜ, and so on. Once the transverse cracks develop, the laminate with the higher percentage of 90° layers is influenced more. Thus, as one can see in Fig. 23, the β₁₁ of [0/90]ₜ decreases much more than the other laminates at the initiation of damage development. This explains why the normalized degradation of [0/90]ₜ is the largest for the low TCD in Fig. 22. Because the undamaged β₁₁ of the [0/90]ₜ laminate is much larger than the others, its normalized degradation appears smaller when the effective property becomes saturated. However, the same phenomenon described above has also been observed in the FEM’s results [15].

Since the normalized degradation curves of different laminates are very close to each other for the lower TCD, comparisons between different approaches for three types of laminates are presented in Fig. 24-26, respectively. As a general tendency, the agreement between the present analysis and FEM is good for the lower TCD but not as good for TCD larger than 0.8 mm⁻¹. At the very high TCD, the present analysis predicts a much lower degradation than the ply discount method while
Figure 23. Degradation of CTE for Graphite/Epoxy Laminates with Transverse Cracks (unnormalized)
the FEM's results are closer to those of ply discount. On the other hand, the present analysis shows earlier saturation in degradation of $\beta_{11}$ which is more consistent with the experimental observation for saturation in crack density [2]. More work on this subject will be required before a thorough understanding is reached.
Figure 24. Comparison of CTE for Cracked [0\degree/90\degree]s
Figure 25. Comparison of CTE for Cracked $[0_2/90_2]$s
Figure 26. Comparison of CTE for Cracked [0/90]s

Transverse Cracks under Thermal Loading
6.0 3-D Analysis for The Cross Crack Problem

6.1 Introduction

The degradation of effective properties induced by transverse cracks in a symmetric cross-ply laminate is studied in previous chapters. During the actual load history of a composite laminate, damage modes other than transverse cracks may develop and may contribute to the eventual failure of the laminate. In the following study, we will restrict our attention to the cross crack problem.

Because of the difficulty and complexity of this problem, very few investigations have been performed to study the effects of cross cracks; i.e., transverse cracks in 90° layers and longitudinal cracks in 0° layers, on the cross-ply laminates. Previous results were presented by Highsmith and Reifsnider [16] using a critical element model. In this chapter, a 3-D analytical model, extended from its 2-D counterpart, is introduced to study the elastic properties of a symmetric cross-ply laminate with cross cracks under uniaxial tension (Fig. 27). The emphasis is placed on the variation of effective moduli as a function of transverse crack density (TCD = 1/ε₁) and longitudinal crack density (LCD = 1/ε₂). The effective properties studied here are axial stiffness and Poisson’s ratio.
Figure 27. Symmetric Cross-Ply Laminate with Cross Cracks
Also, the distributions of axial displacement and axial stress at a certain location are predicted and compared to the results of [3].

### 6.2 Assumptions

The basic assumptions for the cross crack problem are mostly the same as those made for the 2-D case in Chapter 2. However, additional statements must be made to define the longitudinal cracks in 0° layers. In order to have an overall idea of the assumptions for this problem, the previous assumptions are restated as follows:

1. Each ply of the laminate is homogeneous, orthotropic or transversely isotropic.
2. Adjacent layers are perfectly bonded together.
3. The dimensions of the laminate are infinite in the $x_1$ and $x_2$ directions.
4. The dimensions and properties of the laminate are symmetric with respect to its midplane; i.e., $x_3 = 0$ plane.
5. Transverse cracks occur only in the 90° layers, and longitudinal cracks occur only in the 0° layers. The cracks develop completely through the thickness ($x_3$ direction) and the width ($x_2$ direction) or length ($x_1$ direction) of those layers.
6. The cracks appear periodically along the $x_1$ and $x_2$ directions; i.e., the spacings between any two adjacent cracks are equal. The longitudinal crack spacing is not necessarily equal to the transverse crack spacing.

According to the assumptions listed above, the cracked laminate can be treated as a repeating cell (Fig. 28) which allows for a three dimensional analysis. Furthermore, because of the symmetric nature of this repeating cell, we can reduce the modeled region to one half of the cell only.
Figure 28. Repeating Cell for a Laminate with Cross Cracks
6.3 Formulation

Introduce three sets of local coordinates \((x_1, x_2, x_3^{(a)})\) to the modeled region as shown in Fig. 29 where \(a\) denotes the layer number counted from the bottom of the laminate. Let \(u_i^{(a)}\) and \(\sigma_{ij}^{(a)}\) be the displacements and stress components in the \(a\) layer with \(i,j = 1,2,3\). Suppose that the damaged laminate is subjected to a uniaxial tension at the infinity. By consideration of the symmetry and traction-free conditions at the outer surfaces and the crack surfaces, the following boundary conditions are imposed on the modeled region:

\[
\begin{align*}
\sigma_{31}^{(1)} = \sigma_{32}^{(1)} = \sigma_{33}^{(1)} &= 0, \quad 0 \leq x_1 \leq \frac{\ell_1}{2}, \quad |x_2| \leq \frac{\ell_2}{2}, \quad \vec{x}_3^{(1)} = -\frac{d_1}{2} \\
\sigma_{31}^{(3)} = \sigma_{32}^{(3)} = \sigma_{33}^{(3)} &= 0, \quad 0 \leq x_1 \leq \frac{\ell_1}{2}, \quad |x_2| \leq \frac{\ell_2}{2}, \quad \vec{x}_3^{(3)} = \frac{d_3}{2} \\
\sigma_{21}^{(1)} = \sigma_{22}^{(1)} = \sigma_{23}^{(1)} &= 0, \quad 0 \leq x_1 \leq \frac{\ell_1}{2}, \quad x_2 = \pm \frac{\ell_2}{2}, \quad |\vec{x}_3^{(1)}| \leq \frac{d_1}{2} \\
\sigma_{21}^{(2)} = \sigma_{22}^{(2)} &= 0, \quad 0 \leq x_1 \leq \frac{\ell_1}{2}, \quad x_2 = \pm \frac{\ell_2}{2}, \quad |\vec{x}_3^{(2)}| \leq \frac{d_2}{2} \\
\sigma_{22}^{(2)} x_2 = \frac{\ell_2}{2} = \sigma_{22}^{(2)} |x_2| = -\frac{\ell_2}{2}, \quad 0 \leq x_1 \leq \frac{\ell_1}{2}, \quad |\vec{x}_3^{(2)}| \leq \frac{d_2}{2} \\
\sigma_{21}^{(3)} = \sigma_{22}^{(3)} = \sigma_{23}^{(3)} &= 0, \quad 0 \leq x_1 \leq \frac{\ell_1}{2}, \quad x_2 = \pm \frac{\ell_2}{2}, \quad |\vec{x}_3^{(3)}| \leq \frac{d_3}{2} \\
\text{and} \\
u_i^{(a)} = 0, \quad \sigma_{12}^{(a)} = \sigma_{13}^{(a)} = 0, \quad x_1 = 0, \quad |x_2| \leq \frac{\ell_2}{2}, \quad |\vec{x}_3^{(a)}| \leq \frac{d_a}{2}
\end{align*}
\]
Figure 29. Modeled Region with Local Coordinates (3-D Analysis)
\[ u_1^{(1)} = U, \quad \sigma_{12}^{(1)} = \sigma_{13}^{(1)} = 0, \quad x_1 = \frac{\ell_1}{2}, \quad |x_2| \leq \frac{\ell_2}{2}, \quad |\bar{x}_3^{(1)}| \leq \frac{d_1}{2} \] (6.8)

\[ \sigma_{11}^{(2)} = \sigma_{12}^{(2)} = \sigma_{13}^{(2)} = 0, \quad x_1 = \frac{\ell_1}{2}, \quad |x_2| \leq \frac{\ell_2}{2}, \quad |\bar{x}_3^{(2)}| \leq \frac{d_2}{2} \] (6.9)

\[ u_1^{(3)} = U, \quad \sigma_{12}^{(3)} = \sigma_{13}^{(3)} = 0, \quad x_1 = \frac{\ell_1}{2}, \quad |x_2| \leq \frac{\ell_2}{2}, \quad |\bar{x}_3^{(3)}| \leq \frac{d_3}{2} \] (6.10)

where \( U \) is the uniform displacement applied to the laminate. For simplicity, \( U \) is set to be 1.

Also, for a composite laminate, the traction and displacement continuity conditions at the interfaces must be taken into account. These conditions can be expressed as:

\[ \sigma_{3j}^{(1)}|_{\bar{x}_3^{(1)}} = \frac{d_1}{2} = \sigma_{3j}^{(2)}|_{\bar{x}_3^{(2)}} = -\frac{d_2}{2}, \quad \sigma_{3j}^{(3)}|_{\bar{x}_3^{(3)}} = \frac{d_2}{2} = \sigma_{3j}^{(3)}|_{\bar{x}_3^{(3)}} = -\frac{d_3}{2} \] (6.11)

\[ u_1^{(1)}|_{\bar{x}_3^{(1)}} = \frac{d_1}{2} = u_1^{(2)}|_{\bar{x}_3^{(2)}} = -\frac{d_2}{2}, \quad u_1^{(3)}|_{\bar{x}_3^{(3)}} = \frac{d_2}{2} = u_1^{(3)}|_{\bar{x}_3^{(3)}} = -\frac{d_3}{2} \] (6.12)

By the assumptions, boundary conditions, and continuity conditions stated in the previous paragraphs, the problem is completely defined.

The displacement field of the \( \alpha \) layer is assumed in the form of the second order Legendre expansion in \( x_2 \) and \( \bar{x}_3^{(\alpha)} \) as follows:

\[ u^{(\alpha)} = \Lambda_0^{(\alpha)} + \Lambda_1^{(\alpha)}x_2 + \Lambda_2^{(\alpha)}\bar{x}_3^{(\alpha)} + \Lambda_3^{(\alpha)}x_2\bar{x}_3^{(\alpha)} + \] \[ + \frac{1}{2}\Lambda_4^{(\alpha)}[3(x_2)^2 - \frac{\ell_2^2}{4}] + \frac{1}{2}\Lambda_5^{(\alpha)}[3(\bar{x}_3^{(\alpha)})^2 - \frac{d_2^2}{4}] \] (6.13)

where the field variables, \( \Lambda_k^{(\alpha)} = (\varphi_k^{(\alpha)}, \chi_k^{(\alpha)}, \psi_k^{(\alpha)}) \) (\( k = 0 - 5 \)), are functions of \( x_1 \) only.

Since \( x_2 = 0 \) is a plane of symmetry for the modeled region, we have the following symmetry conditions in displacements:
Because the laminate is symmetric to its midplane \((x_3 = 0)\), \(d_1\) is equal to \(d_3\). In addition, there are more symmetry conditions for the \(\alpha = 2\) layer:

\[
\begin{align*}
\varphi_1(x_1, x_2, x_3) &= \varphi_2(x_1, x_2, -x_3), \\
\varphi_3(x_1, x_2, x_3) &= \varphi_4(x_1, x_2, -x_3), \\
\varphi_5(x_1, x_2, x_3) &= -\varphi_5(x_1, x_2, -x_3) \quad (6.15)
\end{align*}
\]

and for any \(\bar{x}_i = \bar{x}_0\), we have:

\[
\begin{align*}
\varphi_1(x_1, x_2, \bar{x}_0) &= \varphi_3(x_1, x_2, -\bar{x}_0), \\
\varphi_2(x_1, x_2, \bar{x}_0) &= \varphi_4(x_1, x_2, -\bar{x}_0), \\
\varphi_3(x_1, x_2, \bar{x}_0) &= \varphi_5(x_1, x_2, -\bar{x}_0) \quad (6.16)
\end{align*}
\]

Imposing (6.14)-(6.15) on (6.13), we obtain expressions for the displacements in each layer as shown below:

\[
\begin{align*}
\varphi_1 &= \varphi_0 + \varphi_2 x_3 + \frac{1}{2} \varphi_4 [3x_2^2 - \frac{d_2^2}{4}] + \frac{1}{2} \varphi_5 [3(\bar{x}_0)^2 - \frac{d_0^2}{4}], \\
\varphi_2 &= (x_1 + x_3 \bar{x}_3) x_2, \\
\varphi_3 &= \varphi_0 + \varphi_2 x_3 + \frac{1}{2} \varphi_4 [3x_2^2 - \frac{d_2^2}{4}] + \frac{1}{2} \varphi_5 [3(x_3)^2 - \frac{d_3^2}{4}] \quad (6.17)
\end{align*}
\]
\begin{align}
  u_1^{(2)} &= \varphi_0^{(2)} + \frac{1}{2} \varphi_4^{(2)}[3(x_2)^2 - \frac{\varepsilon_2^2}{4}] + \frac{1}{2} \varphi_5^{(2)}[3(x_3)^2 - \frac{d_3^2}{4}], \\
  u_2^{(2)} &= \chi_1^{(2)} x_2, \\
  u_3^{(2)} &= \psi_2^{(2)} x_3 
\end{align}

(6.18)

\begin{align}
  u_1^{(3)} &= \varphi_0^{(3)} + \varphi_2^{(3)} x_3 + \frac{1}{2} \varphi_4^{(3)}[3(x_2)^2 - \frac{\varepsilon_2^2}{4}] + \frac{1}{2} \varphi_5^{(3)}[3(x_3)^2 - \frac{d_3^2}{4}], \\
  u_2^{(3)} &= (\chi_1^{(3)} + \chi_3^{(3)} x_3) x_2, \\
  u_3^{(3)} &= \psi_0^{(3)} + \psi_2^{(3)} x_3 + \frac{1}{2} \psi_4^{(3)}[3(x_2)^2 - \frac{\varepsilon_2^2}{4}] + \frac{1}{2} \psi_5^{(3)}[3(x_3)^2 - \frac{d_3^2}{4}] 
\end{align}

(6.19)

Because of (6.16), there are some relations between the field variables of the \( \alpha = 1 \) layer and \( \alpha = 3 \) layers:

\begin{align}
  \varphi_0^{(3)} &= \varphi_0^{(1)}, & \varphi_2^{(3)} &= -\varphi_2^{(1)}, & \varphi_4^{(3)} &= \varphi_4^{(1)}, & \varphi_5^{(3)} &= \varphi_5^{(1)}, \\
  \chi_1^{(3)} &= \chi_1^{(1)}, & \chi_3^{(3)} &= -\chi_3^{(1)}, \\
  \psi_0^{(3)} &= -\psi_0^{(1)}, & \psi_2^{(3)} &= \psi_2^{(1)}, & \psi_4^{(3)} &= -\psi_4^{(1)}, & \psi_5^{(3)} &= -\psi_5^{(1)} 
\end{align}

(6.20)

As a result, the displacements and stresses of the \( \alpha = 3 \) layer can be expressed in terms of the field variables of the \( \alpha = 1 \) layer. From (6.17) to (6.20), there are fifteen unknowns:

\begin{align}
  \varphi_0^{(1)}, \varphi_2^{(1)}, \varphi_4^{(1)}, \varphi_5^{(1)}, \chi_1^{(1)}, \chi_3^{(1)}, \psi_0^{(1)}, \psi_2^{(1)}, \psi_4^{(1)}, \psi_5^{(1)}, \\
  \varphi_0^{(2)}, \varphi_2^{(2)}, \varphi_5^{(2)}, \chi_1^{(2)}, \psi_2^{(2)} 
\end{align}

(6.21)

which must be determined.

The components of the linear strain tensor are given by:

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\[ \varepsilon_{ij}^{(a)} = \frac{1}{2} (\partial_i u_j^{(a)} + \partial_j u_i^{(a)}) \] (6.22)

where \( \partial_1 = \partial / \partial x_1 \), \( \partial_2 = \partial / \partial x_2 \), and \( \partial_3 = \partial / \partial x_3 \).

From (6.17)-(6.18) and (6.22), the components of the strain tensor can be expressed as follows:

\[ \varepsilon_{11}^{(1)} = \partial_1 \phi_0^{(1)} + \partial_1 \phi_2^{(1)} x_3^{(1)} + \frac{1}{2} \partial_1 \phi_4^{(1)} [3(x_2)^2 - \frac{\ell_2^2}{4}] + \frac{1}{2} \partial_1 \psi_5^{(1)} [3(x_3^{(1)})^2 - \frac{d_1^2}{4}], \]

\[ \varepsilon_{22}^{(1)} = \chi_1^{(1)} + \chi_3^{(1)} x_3^{(1)}, \]

\[ \varepsilon_{33}^{(1)} = \psi_2^{(1)} + 3 \psi_3^{(1)} x_3^{(1)}, \]

\[ \varepsilon_{23}^{(1)} = \frac{1}{2} (\chi_3^{(1)} + \psi_4^{(1)}) x_2, \]

\[ \varepsilon_{12}^{(1)} = \frac{1}{2} (3 \phi_4^{(1)} + \partial_1 \chi_1^{(1)} + \partial_1 x_3^{(1)} x_3^{(1)}) x_2, \]

\[ \varepsilon_{13}^{(1)} = \frac{1}{2} (\phi_2^{(1)} + \partial_1 \psi_0^{(1)}) + \frac{1}{2} (3 \phi_5^{(1)} + \partial_1 \psi_2^{(1)}) x_3^{(1)} + \]

\[ \frac{1}{4} (\partial_1 \psi_4^{(1)} [3(x_2)^2 - \frac{\ell_2^2}{4}] + \partial_1 \psi_5^{(1)} [3(x_3^{(1)})^2 - \frac{d_1^2}{4}])] \] (6.23)

\[ \varepsilon_{11}^{(2)} = \partial_1 \phi_0^{(2)} + \frac{1}{2} \partial_1 \phi_2^{(2)} [3(x_2)^2 - \frac{\ell_2^2}{4}] + \frac{1}{2} \partial_1 \phi_4^{(2)} [3(x_3^{(2)})^2 - \frac{d_1^2}{4}], \]

\[ \varepsilon_{22}^{(2)} = \chi_1^{(2)}, \]

\[ \varepsilon_{33}^{(2)} = \psi_2^{(2)}, \]

\[ \varepsilon_{23}^{(2)} = 0, \]

\[ \varepsilon_{12}^{(2)} = \frac{1}{2} (3 \phi_4^{(2)} + \partial_1 \chi_1^{(2)}) x_2, \]
\[ \epsilon_{ij}^{(2)} = \frac{1}{2} (3\psi_3^{(2)} + \phi_1^{(2)}) \bar{\chi}_j^{(2)} \]  

(6.24)

The constitutive equation for each layer is given as:

\[ \sigma^{(a)} = \tilde{C}^{(a)} \tilde{\epsilon}^{(a)} \]  

(6.25)

where

\[ \sigma^{(a)} = [\sigma_{11}^{(a)}, \sigma_{22}^{(a)}, \sigma_{33}^{(a)}, \sigma_{12}^{(a)}, \sigma_{13}^{(a)}, \sigma_{23}^{(a)}], \quad \tilde{\epsilon}^{(a)} = [\epsilon_{11}^{(a)}, \epsilon_{22}^{(a)}, \epsilon_{33}^{(a)}, 2\epsilon_{23}^{(a)}, 2\epsilon_{12}^{(a)}, 2\epsilon_{13}^{(a)}] \]

and

\[ \tilde{C}^{(a)} = \begin{bmatrix}
C_{11}^{(a)} & C_{12}^{(a)} & C_{13}^{(a)} & 0 & 0 & 0 \\
C_{12}^{(a)} & C_{22}^{(a)} & C_{23}^{(a)} & 0 & 0 & 0 \\
C_{13}^{(a)} & C_{23}^{(a)} & C_{33}^{(a)} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44}^{(a)} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55}^{(a)} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}^{(a)}
\end{bmatrix} \]

From (6.23)-(6.25), the stresses in each layer can be expressed as:

\[ \sigma_{11}^{(1)} = (C_{11}^{(1)} \partial_1 \phi_0^{(1)} + C_{12}^{(1)} \chi_1^{(1)} + C_{13}^{(1)} \psi_2^{(1)}) + (C_{11}^{(1)} \partial_1 \phi_0^{(1)} + C_{12}^{(1)} \chi_3^{(1)} + 3C_{13}^{(1)} \psi_5^{(1)} \bar{\chi}_3^{(1)} + \]

\[ \frac{1}{2} C_{11}^{(1)} \partial_1 \phi_0^{(1)} [3(x_2)^2 - \frac{\ell_2^2}{4}] + \frac{1}{2} C_{11}^{(1)} \partial_1 \phi_0^{(1)} [3(\bar{x}_3^{(1)})^2 - \frac{d_1^2}{4}] \]

\[ \sigma_{22}^{(1)} = (C_{12}^{(1)} \partial_1 \phi_0^{(1)} + C_{22}^{(1)} \chi_1^{(1)} + C_{23}^{(1)} \psi_2^{(1)}) + (C_{12}^{(1)} \partial_1 \phi_0^{(1)} + C_{22}^{(1)} \chi_3^{(1)} + 3C_{23}^{(1)} \psi_5^{(1)} \bar{\chi}_3^{(1)} + \]

\[ \frac{1}{2} C_{12}^{(1)} \partial_1 \phi_0^{(1)} [3(x_2)^2 - \frac{\ell_2^2}{4}] + \frac{1}{2} C_{12}^{(1)} \partial_1 \phi_0^{(1)} [3(\bar{x}_3^{(1)})^2 - \frac{d_1^2}{4}] \]
\[ \sigma_{33}^{(1)} = (C_{13}\psi_0^{(1)} + C_{23}\chi_1^{(1)} + C_{33}\psi_2^{(1)}) + \frac{1}{2}C_{13}\partial_1\phi_1^{(1)}[3(\chi_2^{(1)})^2 - \frac{\varepsilon_2^2}{4}] + \frac{1}{2}C_{13}\partial_1\phi_1^{(1)}[3(\chi_3^{(1)})^2 - \frac{d_2^2}{4}], \]

\[ \sigma_{33}^{(1)} = C_{44}(\chi_3^{(1)} + 3\psi_4^{(1)})x_2, \]

\[ \sigma_{12}^{(1)} = C_{55}(3\psi_5^{(1)} + \partial_1\chi_1^{(1)} + \partial_1\chi_3^{(1)}\bar{x}_3^{(1)})x_2, \]

\[ \sigma_{13}^{(1)} = C_{66}(\psi_2^{(1)} + \partial_1\psi_0^{(1)}) + C_{66}(3\phi_5^{(1)} + \partial_1\psi_2^{(1)}\bar{x}_3^{(1)} + \frac{1}{2}C_{66}\partial_1\psi_3^{(1)}[3(\bar{x}_3^{(1)})^2 - \frac{d_2^2}{4}], \]

\[ \sigma_{11}^{(2)} = (C_{11}\psi_0^{(2)} + C_{12}\chi_1^{(2)} + C_{13}\psi_2^{(2)}) + \frac{1}{2}C_{11}\partial_1\phi_1^{(2)}[3(\chi_2^{(2)})^2 - \frac{\varepsilon_2^2}{4}] + \frac{1}{2}C_{11}\partial_1\phi_1^{(2)}[3(\chi_3^{(2)})^2 - \frac{d_2^2}{4}], \]

\[ \sigma_{22}^{(2)} = (C_{12}\psi_0^{(2)} + C_{22}\chi_1^{(2)} + C_{23}\psi_2^{(2)}) + \frac{1}{2}C_{12}\partial_1\phi_1^{(2)}[3(\chi_2^{(2)})^2 - \frac{\varepsilon_2^2}{4}] + \frac{1}{2}C_{12}\partial_1\phi_1^{(2)}[3(\chi_3^{(2)})^2 - \frac{d_2^2}{4}], \]

\[ \sigma_{33}^{(2)} = C_{33}(3\phi_4^{(2)} + \partial_1\chi_4^{(2)}\bar{x}_3^{(2)} + \frac{1}{2}C_{33}\partial_1\phi_1^{(2)}[3(\chi_3^{(2)})^2 - \frac{d_2^2}{4}], \]

\[ \sigma_{23}^{(2)} = 0, \]

\[ \sigma_{12}^{(2)} = C_{55}(3\psi_5^{(2)} + \partial_1\chi_1^{(2)})x_2, \]

\[ \sigma_{13}^{(2)} = C_{66}(3\psi_2^{(2)} + \partial_1\psi_2^{(2)}\bar{x}_3^{(2)} \] (6.27)
The equilibrium equations for each layer are written as:

$$\delta_i \sigma_{ij}^{(a)} = 0 \quad (6.28)$$

Introducing a double integration as:

$$\frac{1}{d_\alpha} \int_{-\ell_2}^{\ell_2} \int_{-d_{ij}/2}^{d_{ij}/2} \left[ \partial_1 \sigma_{ij}^{(a)} + \partial_2 \sigma_{ij}^{(a)} + \partial_3 \sigma_{ij}^{(a)} \right] x_2 \left( x_3^{(a)} \right)^n dx_2 dx_3^{(a)} = 0 \quad (6.29)$$

and integrating by parts, we obtain:

$$\partial_1 S_{ij}^{(a)} - mS_{ij}^{(a)} - nS_{ij}^{(a)} +$$

$$\frac{1}{d_\alpha \ell_2} \int_{-\ell_2}^{\ell_2} \int_{-d_{ij}/2}^{d_{ij}/2} \left[ \sigma_{ij}^{(a)} \right]_{x_2} = \frac{\ell_2}{2} - (-1)^m \sigma_{ij}^{(a)} \bigg|_{x_2} = - \frac{\ell_2}{2} x_3^{(a)} dx_2 = 0 \quad (6.30)$$

where

$$S_{ij}^{(a)} = \frac{1}{d_\alpha \ell_2} \int_{-\ell_2}^{\ell_2} \int_{-d_{ij}/2}^{d_{ij}/2} \sigma_{ij}^{(a)} x_2 \left( x_3^{(a)} \right)^n dx_2 dx_3^{(a)} \quad (6.31)$$

and

$$(m, n) = (0, 0), (0, 1), (1, 0), (1, 1), (0, 2), (2, 0)$$

The expressions of $S_{ij}^{(a)}$ can be expanded explicitly by substituting (6.26)-(6.27) for $\sigma_{ij}^{(a)}$ in (6.31):

$$S_{11}^{(1)} = C_{11}^{(1)} \partial_1 \varphi_0^{(1)} + C_{12}^{(1)} x_1^{(1)} + C_{13}^{(1)} \psi_2^{(1)}$$

$$S_{22}^{(1)} = C_{12}^{(1)} \partial_1 \varphi_0^{(1)} + C_{22}^{(1)} x_1^{(1)} + C_{23}^{(1)} \psi_2^{(1)}$$

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\begin{align*}
S_{13}^{(0)} &= C_{12}^{(1)} \varphi_2^{(1)} + C_{23}^{(1)} \chi_1^{(1)} + C_{33}^{(1)} \psi_2^{(1)}, \\
S_{13}^{(0)} &= C_{66}^{(1)} \varphi_0^{(1)} + \varphi_1 \psi_0^{(1)}, \\
S_{11}^{(0)} &= C_{11}^{(2)} \varphi_0^{(2)} + C_{12}^{(2)} \chi_1^{(2)} + C_{13}^{(2)} \psi_2^{(2)}, \\
S_{22}^{(0)} &= C_{12}^{(2)} \varphi_0^{(2)} + C_{22}^{(2)} \chi_1^{(2)} + C_{23}^{(2)} \psi_2^{(2)}, \\
S_{33}^{(0)} &= C_{13}^{(2)} \varphi_0^{(2)} + C_{23}^{(2)} \chi_1^{(2)} + C_{33}^{(2)} \psi_2^{(2)}, \\
S_{23}^{(1)} &= \frac{\varepsilon^2}{12} C_{44}^{(1)} \chi_3^{(1)} + 3 \psi_4^{(1)}, \\
S_{12}^{(1)} &= \frac{\varepsilon^2}{12} C_{55}^{(1)} (3 \varphi_4^{(1)} + \varphi_1 \chi_1^{(1)}), \\
S_{12}^{(2)} &= \frac{\varepsilon^2}{12} C_{55}^{(2)} (3 \varphi_4^{(2)} + \varphi_1 \chi_1^{(2)}), \\
S_{11}^{(1)} &= \frac{d^2}{12} (C_{11}^{(1)} \varphi_2^{(1)} + C_{12}^{(1)} \chi_3^{(1)} + 3 C_{13}^{(1)} \psi_3^{(1)}), \\
S_{22}^{(1)} &= \frac{d^2}{12} (C_{12}^{(1)} \varphi_2^{(1)} + C_{22}^{(1)} \chi_3^{(1)} + 3 C_{23}^{(1)} \psi_3^{(1)}), \\
S_{33}^{(1)} &= \frac{d^2}{12} (C_{13}^{(1)} \varphi_2^{(1)} + C_{23}^{(1)} \chi_3^{(1)} + 3 C_{33}^{(1)} \psi_3^{(1)}), \\
S_{13}^{(1)} &= \frac{d^2}{12} C_{66}^{(1)} (3 \varphi_3^{(1)} + \varphi_1 \psi_2^{(1)}), \\
S_{13}^{(2)} &= \frac{d^2}{12} C_{66}^{(2)} (3 \varphi_3^{(2)} + \varphi_1 \psi_2^{(2)}),
\end{align*}
\[ S_{12}^{(1)} = \frac{d_1^2 \epsilon_2}{12} C_{11}^{(1)} \partial_1 \chi_3^{(1)}, \]

\[ S_{11}^{(0)} = \frac{d_1^2}{12} (C_{11}^{(1)} \partial_1 \phi_0^{(1)} + C_{12}^{(1)} \chi_1^{(1)} + C_{13}^{(1)} \psi_2^{(1)}) + \frac{d_1^4}{120} C_{11}^{(1)} \partial_1 \phi_5^{(1)}, \]

\[ S_{22}^{(0)} = \frac{d_1^2}{12} (C_{12}^{(1)} \partial_1 \phi_0^{(1)} + C_{22}^{(1)} \chi_1^{(1)} + C_{23}^{(1)} \psi_2^{(1)}) + \frac{d_1^4}{120} C_{12}^{(1)} \partial_1 \phi_5^{(1)}, \]

\[ S_{33}^{(0)} = \frac{d_1^2}{12} (C_{13}^{(1)} \partial_1 \phi_0^{(1)} + C_{23}^{(1)} \chi_1^{(1)} + C_{33}^{(1)} \psi_2^{(1)}) + \frac{d_1^4}{120} C_{13}^{(1)} \partial_1 \phi_5^{(1)}, \]

\[ S_{13}^{(0)} = \frac{d_1^2}{12} C_{66}^{(1)} \psi_2^{(1)} + \partial_1 \psi_0^{(1)} \]

\[ + \frac{d_1^4}{120} C_{66}^{(1)} \partial_1 \psi_5^{(1)}, \]

\[ S_{11}^{(2)} = \frac{d_2^2}{12} (C_{11}^{(2)} \partial_1 \phi_0^{(2)} + C_{12}^{(2)} \chi_1^{(2)} + C_{13}^{(2)} \psi_2^{(2)}) + \frac{d_2^4}{120} C_{11}^{(2)} \partial_1 \phi_5^{(2)}, \]

\[ S_{22}^{(2)} = \frac{d_2^2}{12} (C_{12}^{(2)} \partial_1 \phi_0^{(2)} + C_{22}^{(2)} \chi_1^{(2)} + C_{23}^{(2)} \psi_2^{(2)}) + \frac{d_2^4}{120} C_{12}^{(2)} \partial_1 \phi_5^{(2)}, \]

\[ S_{33}^{(2)} = \frac{d_2^2}{12} (C_{13}^{(2)} \partial_1 \phi_0^{(2)} + C_{23}^{(2)} \chi_1^{(2)} + C_{33}^{(2)} \psi_2^{(2)}) + \frac{d_2^4}{120} C_{13}^{(2)} \partial_1 \phi_5^{(2)}, \]

\[ S_{11}^{(2)} = \frac{\epsilon_2^2}{12} (C_{11}^{(1)} \partial_1 \phi_0^{(1)} + C_{12}^{(1)} \chi_1^{(1)} + C_{13}^{(1)} \psi_2^{(1)}) + \frac{\epsilon_2^4}{120} C_{11}^{(1)} \partial_1 \phi_4^{(1)}, \]

\[ S_{22}^{(2)} = \frac{\epsilon_2^2}{12} (C_{12}^{(1)} \partial_1 \phi_0^{(1)} + C_{22}^{(1)} \chi_1^{(1)} + C_{23}^{(1)} \psi_2^{(1)}) + \frac{\epsilon_2^4}{120} C_{12}^{(1)} \partial_1 \phi_4^{(1)}, \]

\[ S_{33}^{(2)} = \frac{\epsilon_2^2}{12} (C_{13}^{(1)} \partial_1 \phi_0^{(1)} + C_{23}^{(1)} \chi_1^{(1)} + C_{33}^{(1)} \psi_2^{(1)}) + \frac{\epsilon_2^4}{120} C_{13}^{(1)} \partial_1 \phi_4^{(1)}, \]

\[ S_{13}^{(2)} = \frac{\epsilon_2^2}{12} C_{66}^{(1)} \psi_2^{(1)} + \partial_1 \psi_0^{(1)} \]

\[ + \frac{\epsilon_2^4}{120} C_{66}^{(1)} \partial_1 \psi_4^{(1)}, \]

\[ S_{11}^{(2)} = \frac{\epsilon_2^2}{12} (C_{11}^{(2)} \partial_1 \phi_0^{(2)} + C_{12}^{(2)} \chi_1^{(2)} + C_{13}^{(2)} \psi_2^{(2)}) + \frac{\epsilon_2^4}{120} C_{11}^{(2)} \partial_1 \phi_4^{(2)}, \]

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\[
\begin{align*}
\frac{(2,0)}{S_{22}^{(2)}} &= \frac{\nu_s^2}{12} (C_{12}^{(2)} \psi_0^{(2)} + C_{22}^{(2)} \psi_1^{(2)} + C_{33}^{(2)} \psi_2^{(2)} + \nu_s^2 C_{12}^{(2)} \psi_4^{(2)}), \\
\frac{(2,0)}{S_{33}^{(2)}} &= \frac{\nu_s^2}{12} (C_{12}^{(2)} \psi_0^{(2)} + C_{23}^{(2)} \psi_1^{(2)} + C_{33}^{(2)} \psi_2^{(2)} + \nu_s^2 C_{12}^{(2)} \psi_4^{(2)}),
\end{align*}
\]

and all other \(S_n^{(m)}\) are zero.

By using the traction continuity conditions in (6.11) and the boundary conditions in (6.1)-(6.6), and noticing the symmetry conditions in the \(\alpha = 2\) layer and the relations between the \(\alpha = 1\) and \(\alpha = 3\) layers (6.20), the interfacial and boundary stress terms in (6.30) can be eliminated. Therefore, the following equations can be derived from (6.30) by linear operations with various combinations of \((m, n)\):

\[
\begin{align*}
2d_1 \frac{(0,0)}{S_{11}^{(1)}} + d_2 \frac{(0,0)}{S_{11}^{(2)}} &= 0, \\
(0,1) \frac{(0,0)}{S_{13}^{(1)}} - 2(0,0) \frac{(0,1)}{S_{13}^{(2)}} - d_1 \frac{(0,0)}{S_{13}^{(2)}} + (0,0) \frac{(0,0)}{S_{33}^{(2)}} &= 0, \\
4d_1 \frac{(0,0)}{S_{11}^{(1)}} - 4(0,0) \frac{S_{13}^{(1)}}{S_{13}^{(2)}} + d_2 \frac{(0,0)}{S_{11}^{(2)}} &= 0, \\
8d_1 \frac{(0,0)}{S_{11}^{(1)}} - 16(0,0) \frac{S_{13}^{(1)}}{S_{13}^{(2)}} + d_1 d_2 \frac{(0,0)}{S_{11}^{(2)}} &= 0, \\
d_1 \frac{(0,0)}{S_{13}^{(1)}} - \frac{(0,0)}{S_{13}^{(2)}} + \frac{(0,0)}{S_{33}^{(2)}} &= 0, \\
d_2^2 \frac{(0,0)}{S_{11}^{(1)}} - 4(0,0) \frac{S_{13}^{(1)}}{S_{13}^{(2)}} + 8(0,0) \frac{S_{13}^{(1)}}{S_{13}^{(2)}} &= 0, \\
4d_1 \frac{(0,0)}{S_{13}^{(1)}} - 8(0,0) \frac{S_{33}^{(1)}}{S_{33}^{(2)}} - d_1 \frac{(0,0)}{S_{13}^{(2)}} + d_1 \frac{(0,0)}{S_{33}^{(2)}} &= 0, \\
2d_1 \frac{(1,0)}{S_{12}^{(1)}} - 2d_1 \frac{(0,0)}{S_{22}^{(1)}} + d_2 \frac{(1,0)}{S_{12}^{(2)}} - d_2 \frac{(0,0)}{S_{22}^{(2)}} &= 0,
\end{align*}
\]

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Substituting \((6.32)\) for \(S_{ij}^p\) in \((6.33)\), we obtain ten equations in terms of the field variables \(\phi^p\), \(\chi^p\), \(\psi^p\) as shown below:

\[
2d_1 C_{11}(1)^1 \phi_0^{(1)} + 2d_1 C_{12}(1)^1 \phi_1^{(1)} + 2d_1 C_{13}(1)^1 \phi_2^{(1)} + \\
d_2 C_{11}(2)^1 \phi_0^{(2)} + d_2 C_{12}(2)^1 \phi_1^{(2)} + d_2 C_{13}(2)^1 \phi_2^{(2)} = 0,
\]

\[
24C_{11}(1)^1 \phi_0^{(1)} - 6d_2 C_{66}(1)^1 \phi_2^{(1)} + 24C_{12}(1)^1 \chi_1^{(1)} + 24C_{13}(1)^1 \psi_1^{(1)} - \\
12C_{13}(2)^1 \phi_0^{(2)} + 3d_2 C_{66}(2)^1 \phi_2^{(2)} + 12C_{23}(2)^1 \chi_1^{(2)} + 12C_{33}(2)^1 \psi_1^{(2)} - \\
d_2 C_{11}(2)^1 \phi_1^{(2)} + 12C_{12}(2)^1 \phi_2^{(2)} + 12C_{13}(2)^1 \psi_2^{(2)} + d_2 C_{11}(2)^1 \psi_2^{(2)} = 0,
\]

\[
d_1^2 C_{11}(1)^2 \phi_1^{(1)} - 12C_{12}(1)^2 \chi_1^{(1)} - 12C_{13}(1)^2 \psi_1^{(1)} + 3d_2 C_{11}(1)^2 \chi_1^{(1)} + \\
3d_2 C_{12}(1)^2 \chi_1^{(1)} + 3d_2 C_{13}(1)^2 \psi_1^{(1)} + 3d_2 C_{12}(1)^2 \psi_1^{(1)} = 0,
\]

\[
10d_2^2 C_{11}(1)^2 \phi_0^{(1)} + d_1^2 C_{11}(2)^2 \phi_0^{(1)} - 60d_2^2 C_{66}(1)^2 \phi_2^{(1)} + 10d_1^2 (C_{13}^{(1)} - 2C_{66}^{(1)}) \phi_1^{(1)} + \\
10d_1^2 C_{12}(1)^2 \chi_1^{(1)} + 15d_1 d_2 C_{11}(2)^2 \phi_0^{(2)} + 15d_1 d_2 C_{12}(2)^2 \phi_1^{(2)} + 15d_1 d_2 C_{13}(2)^2 \phi_0^{(2)} = 0,
\]

\[
12d_1 C_{66}(1)^2 \phi_2^{(1)} + 12d_1 C_{66}(1)^2 \psi_0^{(1)} - 3d_2^2 C_{66}(1)^2 \phi_2^{(2)} - d_2^2 C_{66}(1)^2 \psi_2^{(2)} = 0,
\]

\[
12C_{13}^{(2)} \phi_1^{(2)} + 12C_{23}^{(2)} \chi_1^{(2)} + 12C_{33}^{(2)} \psi_1^{(2)} = 0,
\]

\[
20C_{11}^{(2)} \phi_1^{(2)} + 60C_{66}^{(2)} \phi_2^{(2)} - d_2^2 C_{11}^{(2)} \phi_1^{(2)} + \\
20C_{12}^{(2)} \chi_1^{(2)} + 20(C_{13}^{(2)} + C_{66}^{(2)}) \phi_1^{(2)} = 0,
\]

\[
20d_1^2 (C_{66}^{(1)} - 2C_{13}^{(1)}) \phi_2^{(1)} - 40d_1^2 C_{23}^{(1)} \chi_1^{(1)} + 20d_1^2 C_{66}^{(1)} \phi_0^{(1)} + 2d_1^2 C_{66}^{(1)} \phi_0^{(1)} -
\]

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In addition, by substituting (6.17)-(6.18) into (6.12), the displacement continuity condition gives five more equations as follows:

\[ 4\phi_0^{(1)} + 2d_1\phi_2^{(1)} + d_1^2\psi_3^{(1)} - 4\phi_0^{(2)} - d_2^2\psi_5^{(2)} = 0, \]

\[ \psi_4^{(1)} - \psi_4^{(2)} = 0, \]

\[ 2\chi_1^{(1)} + d_1\chi_3^{(1)} - 2\chi_1^{(2)} = 0, \]

\[ 4\psi_0^{(1)} + 2d_1\psi_2^{(1)} + d_1^2\psi_3^{(1)} + 2d_2\psi_2^{(2)} = 0, \]

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(6.34), together with (6.35), provides a set of fifteen linear homogeneous equations for the fifteen unknown field variables listed in (6.21). These equations should be supplemented with the boundary conditions given in (6.7)-(6.9). By substituting (6.17)-(6.18) and (6.26)-(6.27) into (6.7)-(6.9), the exact boundary conditions can be expressed as:

\[
\begin{align*}
\phi_0^{(1)} = \varphi_2^{(1)} = \varphi_4^{(1)} = \varphi_5^{(1)} = \varphi_0^{(2)} = \varphi_4^{(2)} = \varphi_5^{(2)} &= 0, \\
\partial_1 x_1^{(1)} = \partial_1 x_3^{(1)} = \partial_1 x_1^{(2)} &= 0, \\
\partial_1 \psi_0^{(1)} = \partial_1 \psi_2^{(1)} = \psi_4^{(1)} = \partial_1 \psi_5^{(1)} = \partial_1 \psi_2^{(2)} &= 0, \quad \text{at } x_1 = 0 \\
\text{and} \\
\phi_0^{(1)} &= 1, \quad \varphi_2^{(1)} = \varphi_4^{(1)} = \varphi_5^{(1)} = 0, \\
\partial_1 x_1^{(1)} = \partial_1 x_3^{(1)} &= 0, \\
\partial_1 \psi_0^{(1)} = \partial_1 \psi_2^{(1)} = \psi_4^{(1)} = \partial_1 \psi_5^{(1)} &= 0, \\
\partial_1 \varphi_4^{(2)} = \partial_1 \varphi_5^{(2)} &= 0, \\
3\varphi_4^{(2)} + \partial_1 x_1^{(2)} &= 0, \\
3\varphi_5^{(2)} + \partial_1 \psi_2^{(2)} &= 0, \\
C_{11}^{(2)} \partial_1 \varphi_0^{(2)} + C_{12}^{(2)} x_1^{(2)} + C_{13}^{(2)} \psi_2^{(2)} &= 0, \quad \text{at } x_1 = \frac{\ell_1}{2} 
\end{align*}
\]

Approximating the derivatives of field variables by difference expressions, we obtain a banded matrix as mentioned in the previous chapter. Consequently, the unknowns in (6.21) at points in the interval \([0, \frac{\ell_1}{2}]\) can be solved numerically on the computer.
6.4 Results and Comparisons

In order to compare the results with those of [3], three types of symmetric cross-ply laminates, [0/90]s, [0/90]s, and [0/90]s, are studied in this analysis. The thickness of each ply of the laminate is 0.127 mm (0.005 in). The material under consideration is graphite/epoxy. The material properties are taken from [3] and they are slightly different from those shown in Appendix A (E11 = 128.2 GPa, E22 = 9.715 GPa, ν12 = 0.308, ν21 = 0.492, G12 = 5.388 GPa).

6.4.1 Axial Stiffness

The average axial stress for each layer can be determined from (6.32). Thus, the average axial stress of the laminate is given by:

\[
\bar{\sigma}_{11} = \frac{\sigma_{11}^{(0,0)} d_1 + \sigma_{11}^{(0,0)} d_2}{d_1 + d_2}
\]  

(6.38)

Since we apply unit displacement at one end of the modeled region, the average axial strain becomes:

\[
\bar{\varepsilon}_{11} = \frac{1}{(\varepsilon_{11}/2)}
\]  

(6.39)

The effective axial stiffness of the damaged laminate is then defined as:

\[
E_{11}^* = \frac{\bar{\sigma}_{11}}{\bar{\varepsilon}_{11}}
\]  

(6.40)
Figure 30. Degradation of Axial Stiffness for [0/90]s with Cross Cracks
Table 6. Degradation of Axial Stiffness for Cracked [0/90]s

<table>
<thead>
<tr>
<th>Damage Condition</th>
<th>Present</th>
<th>Ref.[3]</th>
<th>Lamination Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>Undamaged</td>
<td>50.2 GPa</td>
<td>---</td>
<td>49.5 GPa</td>
</tr>
<tr>
<td>Transverse Cracking</td>
<td>45.5 GPa</td>
<td>44.8 GPa</td>
<td>43.0 GPa</td>
</tr>
<tr>
<td>Cross Cracking</td>
<td>45.5 GPa</td>
<td>44.8 GPa</td>
<td>42.7 GPa</td>
</tr>
</tbody>
</table>

The degradation of axial stiffness of a damaged [0/90]s laminate with respect to the transverse crack density (TCD) and the longitudinal crack density (LCD) is shown in Fig. 30. The result in this figure has been normalized by the undamaged axial stiffness ($E_{11}^* = 69.3$ GPa or 10.1 Msi). Fig. 30 shows that the transverse cracks reduce the effective axial stiffness to some extent while the longitudinal cracks have no effect on the stiffness. Also, the results of the present analysis and [3] for a [0/90]s laminate are shown in Table 6. The second row (Transverse Cracking) of this table presents the effective stiffness of a laminate with transverse cracks only (TCD = 1.6 mm⁻¹ or 40 in⁻¹) while the third row (Cross Cracking) gives that of a laminate with cross cracks (TCD = 1.6 mm⁻¹ or 40 in⁻¹, LCD = 0.63 mm⁻¹ or 16 in⁻¹). The difference between the present analysis and [3] for the last case is only 1.4%.

6.4.2 Poisson’s Ratio

In order to obtain a definite value of the effective Poisson’s ratio for the damaged laminate, the lateral strain is defined as:

$$
\varepsilon_{22} = \frac{u_2|_{x_2} - \xi_2 - u_2|_{x_2}}{\xi_2} = -\frac{\xi_2}{\xi_2}
$$

(6.41)
Averaging $\dot{e}_{22}$ from $x_t = 0$ to $x_t = \frac{L_1}{2}$ gives the average lateral strain $\bar{e}_{22}$. Consequently, the effective Poisson’s ratio can be determined as:

$$\nu_{12} = -\frac{\bar{e}_{22}}{\bar{e}_{11}}$$

where $\bar{e}_{11}$ is defined in (6.39). The effective Poisson’s ratios at two different locations are examined in this study. The results for [0/90]_4 are shown in Fig. 31. The Poisson’s ratios at both locations are normalized with respect to the Poisson’s ratio of the uncracked laminate ($\nu_{12}^0 = 0.0434$).

In Fig. 31, both plots show that $\nu_{12}$ degrades monotonically when the transverse crack density increases. However, the response of $\nu_{12}$ with respect to the longitudinal crack density is surprisingly different. The Poisson’s ratio at the top surface increases to several times that of the uncracked Poisson’s ratio when the longitudinal crack density increases. This result seems reasonable since there is no constraint for the outer layers ($0^\circ$ layers) in the $x_1$ direction. This phenomenon can be explained by referring to the schematic diagram in Fig. 32. On the contrary, the Poisson’s ratio at the midplane decreases when the longitudinal cracks become denser. At the first glance, this behavior seems to be against the intuition. However, the ply discount method (set $E_{22}$ of $0^\circ$ layers to be zero) also shows the same tendency. By further investigation, we observe what is believed to be responsible for this phenomenon.

Due to the mismatch of the Poisson’s ratio and the perfect bonding between adjacent layers, there will be a biaxial stress state developed in each layer when the laminate is subjected to a uniaxial tension. For the cross-ply laminate in this study, there will be transverse tension (positive $\sigma_0^0$ and $\sigma_9^0$) induced in the $0^\circ$ layer while the transverse normal stress in the $90^\circ$ layer is compressive (negative $\sigma_9^0$). In this case, the transverse strain ($\varepsilon_0^0$) in each layer is given not only by its own Poisson’s effect ($-\nu^0 \frac{\sigma_0^0}{E_0}$), but also by the transverse tension or compression ($\frac{\sigma_9^0}{E_9}$) in that layer. Since the longitudinal cracks decrease the transverse tension ($\sigma_0^0$, $\sigma_9^0$) in the outer layers, the transverse compression ($\sigma_9^0$) in the $90^\circ$ layer will be reduced. As a result, the negative transverse strain ($\varepsilon_9^0$) in the $90^\circ$ layer decreases and the effective Poisson’s ratio in that layer be-
Figure 31. Variation of Poisson’s Ratio for [0/90]s with Cross Cracks
Figure 32. Schematic Diagram of Deformed Shape for a Laminate with Cross Cracks
comes smaller. According to the discussion above, the reduction of the Poisson's ratio with respect to the longitudinal cracks development is not really unreasonable. Moreover, since the same behavior has been observed as well in Highsmith's analysis (page 140 in [3]), this result is believed to be reliable.

In order to have a better understanding concerning the effects of cross cracks, the variations of the effective Poisson's ratio in the $x_3$ direction, for different crack densities, are given in Fig. 33. If only transverse cracks exist in the 90° layer (TCD = 3.2 mm$^{-1}$, LCD = 0 mm$^{-1}$), the problem reduces to a two dimensional generalized plane strain state as studied in the previous chapters. As a result, Poisson's ratio is constant through the thickness of the laminate. When longitudinal cracks develop (TCD = 3.2 mm$^{-1}$, LCD = 3.2 mm$^{-1}$), the constant Poisson's ratio in the 90° layer decreases, but the Poisson's ratio in the 0° layer increases linearly through the thickness. If the transverse cracks are removed from the 90° layer (TCD = 0 mm$^{-1}$, LCD = 3.2 mm$^{-1}$), the two linear segments shift simultaneously. This behavior reveals that the transverse cracks reduce the effective Poisson's ratio in both layers by the same amount. Comparing Fig. 33 to Fig. 31, one can find that they are consistent with each other.

The linearity of $v_{12}$ indicated in Fig. 33 can be explained as follows. Due to the periodicity of the repeating cell, the lateral surfaces ($x_2 = \pm \frac{\ell_2}{2}$) of the 90° layer in the modeled region are planes of symmetry between adjacent cells. Thus, these two surfaces will remain a plane after the loading is applied. This is the reason why the effective Poisson's ratio is constant through the thickness of the 90° layer. On the other hand, the distribution of the effective Poisson's ratio is linear through the thickness of the 0° layer because the expansion in (6.13) is up to the second order only and $u^{(2)}_{12}$ is symmetric to the $x_2 = 0$ plane. For higher order expansion, $v_{12}$ in 0° layer might be nonlinearly distributed.

The discussion above may help the experimentalists to decide how to measure the effective Poisson's ratio of a damaged laminate with cross cracks. Since $v_{12}$ at the top surface involves more local effects, it seems not a good idea to measure the effective Poisson's ratio of a damaged laminate
Figure 33. Variation of Poisson's Ratio through the Thickness of Cracked [0/90]s
by installing strain gauges at the top surface of the laminate—especially for the higher crack density. The more reliable method to measure the effective Poisson’s ratio would be to use an extensometer. The measured value would then correspond to the effective Poisson’s ratio in the 90° layer which is closer to the global behavior of the damaged laminate.

6.4.3 Axial Displacement

The axial displacement can be determined by (6.17)-(6.18) directly. The result at the top surface of a [0/90]₉ laminate is shown in Fig. 34a. The transverse crack is located in the 90° layer along the line \(\frac{x_1}{(\ell_1/2)} = 1\), while the longitudinal crack is located in the 0° layer along the line \(\frac{x_2}{(\ell_2/2)} = 1\). The lines \(\frac{x_1}{(\ell_1/2)} = 0\) and \(\frac{x_2}{(\ell_2/2)} = 0\) represent the midplanes between adjacent transverse cracks and longitudinal cracks, respectively. The displacement exhibits a linear distribution in the axial direction \(x_1\), which indicates a state of uniform axial strain and is consistent with the loading applied. The result of [3] for the same laminate is also given in Fig. 34b. The agreement between these two approaches is very good.

6.4.4 Axial Stress

The distribution of the axial stress in the 0° layer along the 0/90 interface of a cracked [0/90]₉ laminate, as predicted by the present analysis and [3], is presented in Fig. 35. The spacings between adjacent transverse cracks and longitudinal cracks are \(\ell_1 = 0.636\) mm (0.025 in) and \(\ell_2 = 1.588\) mm (0.0625 in), respectively. An applied axial strain of \(\bar{\varepsilon}_{11} = 1\%\) is assumed.

The present analysis shows that along the line at the midplane between two transverse cracks; i.e., \(x_1 = 0\), the axial stress is approximately 1192 MPa (173 Ksi). The stress increases with distance
Figure 34. Axial displacement along the top surface of cracked $[0/90]$s.
from the midplane and reaches a relatively high value of 1743 MPa (253 Ksi). On the other hand, the axial stresses predicted by [3] (Fig. 35b) range from 1205 MPa (175 Ksi) to 1488 MPa (216 Ksi). Thus, the percent increases of axial stress at the transverse crack tip are 46% by the present analysis and 23% by [3], respectively. It should be noted that this axial stress is responsible for the fiber breakage in the 0° layers. This is the reason why the fiber fracture occurs preferentially along the transverse crack tip when a cracked laminate is subjected to the uniaxial tension.

Further comparisons between the interfacial axial stresses predicted by these two approaches are given in Fig. 36 for a [0/90], laminate with \( \ell_1 = 0.482 \text{ mm (0.019 in)} \), \( \ell_2 = 1.588 \text{ mm (0.0625 in)} \), and in Fig. 37 for a [0/90], laminate with \( \ell_1 = 0.762 \text{ mm (0.03 in)} \), \( \ell_2 = 1.588 \text{ mm (0.0625 in)} \). In both cases the loading corresponds to an applied axial strain of 1%.

In general, the far field stresses given by both approaches agree quite well. However, near the transverse crack tip, the axial stresses predicted by [3] are less singular than those predicted by the present analysis. Since the through-the-thickness variation of inplane stresses in [3] is linear whereas that of the present analysis is quadratic, it is expected that the present results are more accurate. In addition, from fracture mechanics point of view, the axial stress should be singular near a crack tip. This trend of high gradient in stress is exhibited by the present results shown in Fig. 35-37.
Figure 35. Axial Stress Distribution along the 0/90 Interface of Cracked [0/90₂]₃s
Figure 36. Axial Stress Distribution along the 0/90 Interface of Cracked [0/90]s

3-D Analysis for The Cross Crack Problem
Figure 37. Axial Stress Distribution along the 0/90 Interface of Cracked [0/90]s
7.0 Conclusions

In this study, an approximate analytical solution has been given to analyze composite laminates with matrix cracks. Both transverse cracks (2-D problem) and cross cracks (3-D problem) are investigated. The analysis leads to a system of linear algebraic equations in the field variables which can be easily solved on the computer. This method is applied to damaged graphite/epoxy and glass/epoxy cross-ply laminates to study the effects of matrix cracks.

The subjects of primary concern are the degradation in effective material properties and the redistribution of axial stress. The effective properties investigated are axial stiffness, Poisson's ratio, shear modulus, and coefficient of thermal expansion. The results of the present study have been compared to those of finite element analysis, experimental investigation, and others in great detail. The results reveal that the effective Poisson's ratio of cracked laminates, for both the 2-D and 3-D cases, exhibits a remarkable behavior and can be used to determine the degree of damage of the laminate.

The advantages of the present approach over the others are its capability for easy programming and computer time saving. The conclusions of previous chapters are summarized in the following sections.
7.1 2-D Analysis of Laminates with Transverse Cracks

The basic assumptions and formulation are given. Three types of loading: uniaxial tension, pure shear, and thermal loading, are studied. The degradation in effective material properties and distribution of axial stress predicted by the present analysis are presented and compared to those by others.

1. The degradation in axial stiffness for graphite/epoxy laminates is not significant because of the high orthotropicity ratio \( \frac{E_{11}}{E_{22}} \). The agreement between the results of the present study and FEM is excellent.

2. Due to the low orthotropicity ratio and the high percentage of 90° layers, the reduction in axial stiffness of glass/epoxy \([0/90]_S\) can be more than 40%. The comparison of results of the present analysis, FEM, variational method, and experimental data is given. The variational method gives the lower bound for the degradation in axial stiffness. Since transverse cracks are not the exclusive damage in the test specimens, the variational prediction is closer to the experimental observation. A good agreement between the present prediction and experimental data in axial stiffness reduction for glass/epoxy \([0/90]_S\) is shown.

3. The Poisson's ratio decreases dramatically when transverse cracks develop. For a graphite/epoxy \([0/90]_S\) laminate, the reduction can be up to 60%. The present analysis predicts smaller degradation and earlier saturation in Poisson's ratio than the FEM does. The results agree quite well when the crack density is low.

4. The distribution of axial stress for a graphite/epoxy \([0_2/90_2]_S\) laminate under uniaxial tension is studied. The comparison between the present results and FEM's is given. The present analysis predicts that there is compressive axial stress near the crack surface while the FEM does not show this compression for the crack spacing considered. On the other hand, a cracked hybrid composite laminate with outer 90° layers is also investigated. It is found that the tendency of axial stress distribution is similar to that predicted by the present analysis for the
5. The effect of transverse cracks on shear modulus for graphite/epoxy cross-ply laminates is investigated. The reduction in shear modulus is significant. The extent and tendency of degradation in shear modulus are similar to that in Poisson's ratio. The agreement of the results given by the present analysis and FEM is excellent. Another comparison among the degradation in axial stiffness, Poisson's ratio, and shear modulus for a graphite/epoxy [0/90\textdegree], laminate is presented. It can be seen that the last two effective moduli are better indicators of damage than the axial stiffness. Because a tensile test is easier to perform, the Poisson's ratio is recommended as an index of damage for a cracked laminate.

6. The decrease in coefficient of thermal expansion (CTE) with respect to transverse crack density for graphite/epoxy laminates is studied. The reduction in CTE is very high. The percentage of degradation ranges from 83\% to 196\%. The agreement between the present results and FEM's is fairly good except for the higher crack density.

7.2 3-D Analysis of Laminates with Cross Cracks

Assumptions are made which lead to periodic arrays of transverse and longitudinal cracks. The formulation and method of solution are given in detail for a representative volume. The degradation in axial stiffness and Poisson's ratio are investigated. The axial displacement and stress for a laminate under uniaxial tension are also predicted. Comparison of results between the present analysis and [3] is given. The material under consideration is graphite/epoxy.

1. The degradation in axial stiffness is dominated by transverse cracks in the 90\degree layers. Longitudinal cracks have no effect on axial stiffness. The results of the present analysis agree well with those of [3].
2. The variation of Poisson's ratio in a laminate with cross cracks is remarkable. The Poisson's ratio degrades because of the existence of transverse cracks, but the effect of longitudinal cracks on Poisson's ratio is completely different at different locations. The Poisson's ratio at the top surface of the laminate increases dramatically, but that at the midplane decreases by a small amount when longitudinal cracks develop. This observation may give guidance to the experimentalists when measuring the Poisson's ratio of a cross-cracked laminate.

3. The axial displacement at the top surface of a cross-cracked laminate under uniaxial tension is distributed linearly. This displacement distribution implies a state of uniform axial strain which is consistent with the loading applied. The agreement between results of different approaches is good.

4. The distribution of axial stress in the 0° layer along the interface between 0° and 90° layers for laminates under uniaxial tension is studied. The results for three types of cross-ply laminates are presented and compared to those of [3]. The axial stresses predicted by the present analysis appear more singularly than those by [3] at the transverse crack tip. Since the through-the-thickness variation of inplane stresses in [3] is linear whereas that of the present analysis is quadratic, it is expected that the present results are more accurate.
References


Appendix A. Material Properties
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