GRAPHS AND TABLES FOR THE ANALYSIS AND DESIGN OF CURVED CONCRETE BEAMS

by

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II. ACKNOWLEDGMENTS

The author is indebted to Professor W. W. Payne, both for his suggestion of a topic worthy of investigation and his leadership, guidance and assistance during the investigation and during the writing of this thesis. Professor Payne spent many hours advising the author, helping him and giving assistance toward the understanding of the complicated behavior of concrete.

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III. INTRODUCTION

Torsion is almost invariably a secondary effect in reinforced concrete building and the subject has not received the same attention as bending, transverse shear, or diagonal tension.

This thesis is concerned with concrete members in which the torsional stress has a major effect on the section. That is, the magnitude of the torsional stress is such that it cannot be neglected; such is the case of curved beam in a horizontal plane.

Research work can be generally divided into two main classes:

1. That of a purely academic nature.
2. The investigation of a practical problem with the object of producing data to solve problems.

Each of these has its part in the acquisition of knowledge, and often work done from a purely academic standpoint is of great assistance to a person endeavoring to produce data to solve practical problems. In the past an engineer designing a member to resist twisting moment would not have made it of concrete, but today there are many cases where the designer finds himself faced with the problem of designing a concrete section to resist torsion. The standard text gives little information in this case.

Concrete members are directly subjected to torsional stresses in the following cases:

1. Curved beams in a horizontal plane.
2. Beams supporting a cantilever balcony.
3. Screw piles.

In addition to these cases there are other cases where it appears at first that torsion does not occur, but closer examination shows that torsion does exist. For example:

1. Beams and slabs with alternate panel loadings.
2. Exterior floor beams.
3. Framed structures.

Generally speaking, we may find members subjected to torsion in every modern structural form.

This thesis deals only with circularly curved horizontal beams. The curvature of the beam adds new dimensions of analysis to the ordinary analysis of beams because of the torsional moment that is induced by the vertical load. Therefore it is necessary to design such members for both bending and torsion.

In 1914 the first theory of bow girder was published by Gibson[7*] for a circular beam with symmetrical loading.

A few years later Robert B. Moorman[8] presented a semigraphical method of analysis used for any fixed end curved beam. The method is expedient in the case of symmetrically loaded beams of a constant cross-section.

The most outstanding paper that the author has reviewed is "The Beam Curved in a Plane", by Professor Pippard[5] who presented a theoretical analysis for curved beams. The Pippard approach for deriving the equations, as well as many symbols and signs, will be used in this thesis.

* Numbers refer to listing in Bibliography.

The purpose of this thesis is to reduce the amount of labor involved in the computations accompanying the analysis of horizontally curved beams, and for frames containing curved horizontal beams. This is accomplished by offering a comprehensive derivation of the equations for cantilever and fixed end beams, with concentrated and uniformly distributed loads. A general expression for the bending moment, torsion, and shear at the fixed end and midpoint of the beam has been completely derived in this paper.

Besides the complete derivation of the expression for the ordinate of the influence lines, the investigation includes a table representing coefficients for the bending moment, twisting moment, and shear force for a series of beams. The beams are subtended by the following angles: 30, 45, 60, 75, 90, 105, 120, 135, 150, 165, and 180 degrees.

In beams subjected to pure bending, the value of the bending stress depends upon EI; while in beams subjected to pure torsion, the value of the torsional stress depends upon the value of GJ.

In the case of beams subjected to combined bending and torsional moments the bending and torsional stress is a function of both EI and GJ. The ratio of EI/GJ in this discussion is designated by K which is called the stiffness ratio. The value of K is a function of both the geometry of the section and the properties of the material.

Graphs have been drawn to show the relation between the angles subtended by the arcs of the beams and the coefficient for bending moment, torsion, and shear force. If the equations derived in this discussion are solved for a particular value of the variables, the result will not be accurate unless the calculation is carried to five places past the decimal. It is the opinion of the author that plotting the results on graphs is much easier and safer. The graphs also show clearly the effect of the stiffness ratio on the bending moment, torsion and shear for beams subtended by the various angles.

Finally, examples are presented to illustrate the application of this thesis to the analysis and design of concrete curved members.

It is also felt that part of the material presented in this thesis is useful to designers who work with other materials as well as concrete.

Past papers on this subject have lacked information about concrete. Steel was the only material with which they dealt. Some have shown examples of beams subjected to pure torsion, which seldom occurs in practice.

With the information presented in this thesis, the engineer might be less reluctant to undertake problems dealing with the design of curved horizontal concrete beams or concrete beams subjected to bending moments, torsion, and shear.

IV. REVIEW OF LITERATURE

In 1932 Osterbloom[11] in a paper published by the American Concrete Institute, stated, "One may assume the simple case of complete fixity at the supports and, also, a uniform loading, and, thereby, cover the major part of the problem actually occurring in practice." He developed an equation for the bending moment at the center of a uniformily loaded circular beam, and expressions for the bending and torsion at the supports. He developed a table for the moment factors for the bending and torsion for circular beams of several lengths. The value of the stiffness ratio (K) used for rectangular sections was as follows:*

$$K = 0.65\left(1 + \frac{d^2}{b^2}\right)$$

where $K = \frac{EI}{GJ}$

d = depth of the section

b = width of the section

The above formula is based on the assumption that the ratio of the modulus of elasticity (E) to the modulus of rigidity (G) is equal to 2.35, and Sant Venant's approximate formula,

$$J = \frac{A^4}{40\ Ip}$$

where Ip = polar moment of inertia

A = the area of the section

J = the stiffness rigidity factor.

* Due to the difficulty in typing the symbols used in the equations of this thesis, symbols with a dot above ($\dot{s}$) indicate that the symbol is a subscript.

In 1937, a paper was published by Robert Moorman[2] for solving curved beams loaded with concentrated loads, using a semi-graphical method. The method is applicable to beams of any curvature, circular, parabolic, elliptical, etc.; it gives accurate results only when the beam is symmetrically loaded.

Veluntinei's paper in 1950 was a contribution to the general approach to the analysis of curved beams on several supports. He derived expressions for stiffness factors and carry-over factors for beams subjected to torsion. Veluntinei's major contribution was to extend the principle of moment distribution to members resisting combined bending and torsion.

The method of analysis presented by Pippard and Baker[13] in 1942 is probably the most general method for analyzing circular beams fixed at the ends, loaded with uniformly distributed loads and concentrated loads.

Concerning the design of concrete beams subjected to torsion, David Andersen[1] in 1938 published his first paper "Design of Reinforced Concrete in Torsion" in the American Society of Civil Engineering Journal. Andersen ran many tests on beams subjecting square and circular sections to torsion. His research work yielded a formula for designing spiral reinforcements to resist torsional stress for square or circular sections. The formula stated that:

$$\frac{As}{p} = (\dot{V}e - S\dot{c})^2(3Ve^2 + 2\dot{V}eS\dot{c} + S\dot{c}^2)\,\frac{.P\dot{e}^3}{32.V\dot{e}^3.P\dot{a}^2.S\dot{a}}$$

where As = the sectional area of spirals

p = pitch of the spiral

$V\dot{e}$ = shearing stress produced by the torque T

$S\dot{c}$ = permissible unit shearing stress for plain concrete

$P\dot{e}$ = least dimension of the section

$P\dot{s}$ = radial distance from the center of the section
to the center of spirals

$S\dot{s}$ = allowable stress in steel.

In 1941, W. T. Marshall and N. R. Tembe[9] published the results of tests in the Civil Engineering laboratories in Guild College. The tests were performed on the following specimens:

1. Plain concrete circular sections
2. Plain concrete rectangular sections
3. Reinforced concrete rectangular sections
4. Plain and reinforced "T" sections
5. Plain and reinforced "L" sections.

The reinforcement used in the rectangular sections to resist torsional stress were three types:

a. Longitudinal bars alone.
b. Longitudinal bars with vertical stirrups.
c. Longitudinal bars and 45° spirals.

Cubes and tension specimens were made to obtain the compression, tension, modulus of elasticity and modulus of rigidity for the concrete used. After analyzing the results they conclude the following:

1. The value of maximum torsional stress depends on the ratio of the sides of the section.

2. The maximum torsional rigidity of the reinforced section is the same as that of a plain section of the same dimensions.

3. The maximum torsional stress tends to increase as the ratio of the sides increase.

4. In the elastic range, the relation between torque and the angle of twist is the same for plain and reinforced sections of the same size, as shown in Figure 1 and Figure 2.

Turner and Davies[15] published a paper, "Plain and Reinforced Concrete in Torsion". They investigated the behavior of "T" and "L" sections of concrete beams under torsional stress. They assumed that the "T" and "L" sections consist of several rectangles; the torque taken by the "T" or "L" section is given by the formula:

$$T = \frac{4A}{PD}(R_1 + R_2 + \ldots R_n)$$

where A = total area of the section

P = perimeter of the section

D = diameter of the inscribed circle

R_1, R_2 and . . . R_n = torque taken by individual rectangles alone.

Tests show that the above equation is good only for a "T" section.

Turner and Davies suggested another empirical formula for designing a rectangular section subjected to torsion and reinforced by longitudinal bars and stirrups. The formula states that for equal percentages of longitudinal and lateral steel

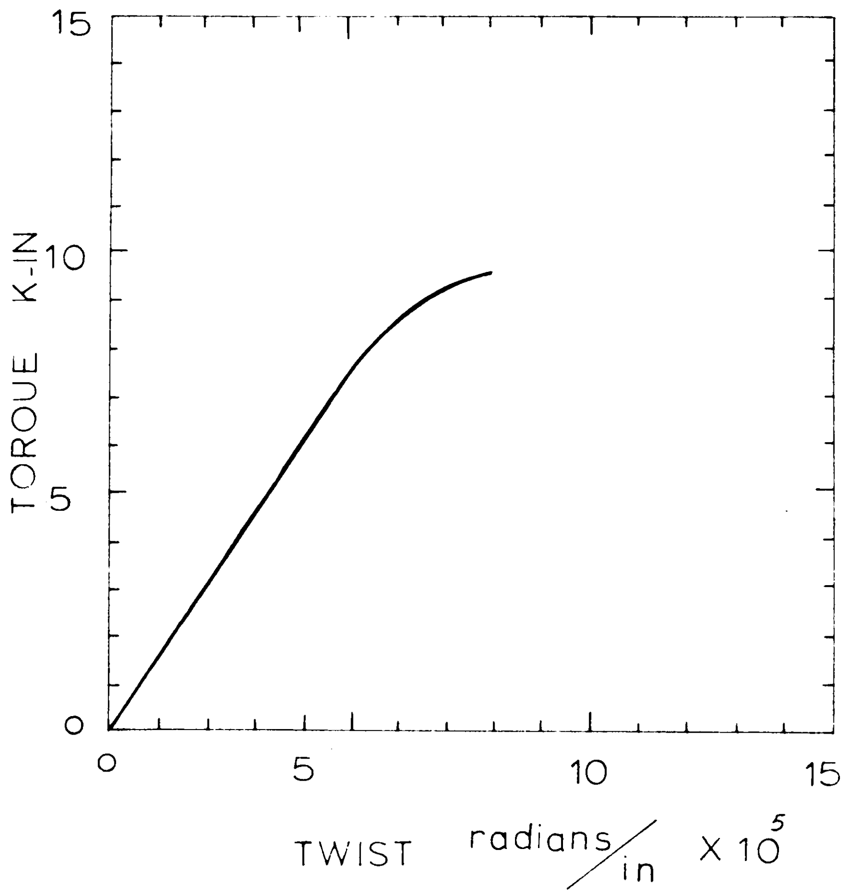


Fig. 1 - Typical Torque Twistgraph for Plain Concrete Members

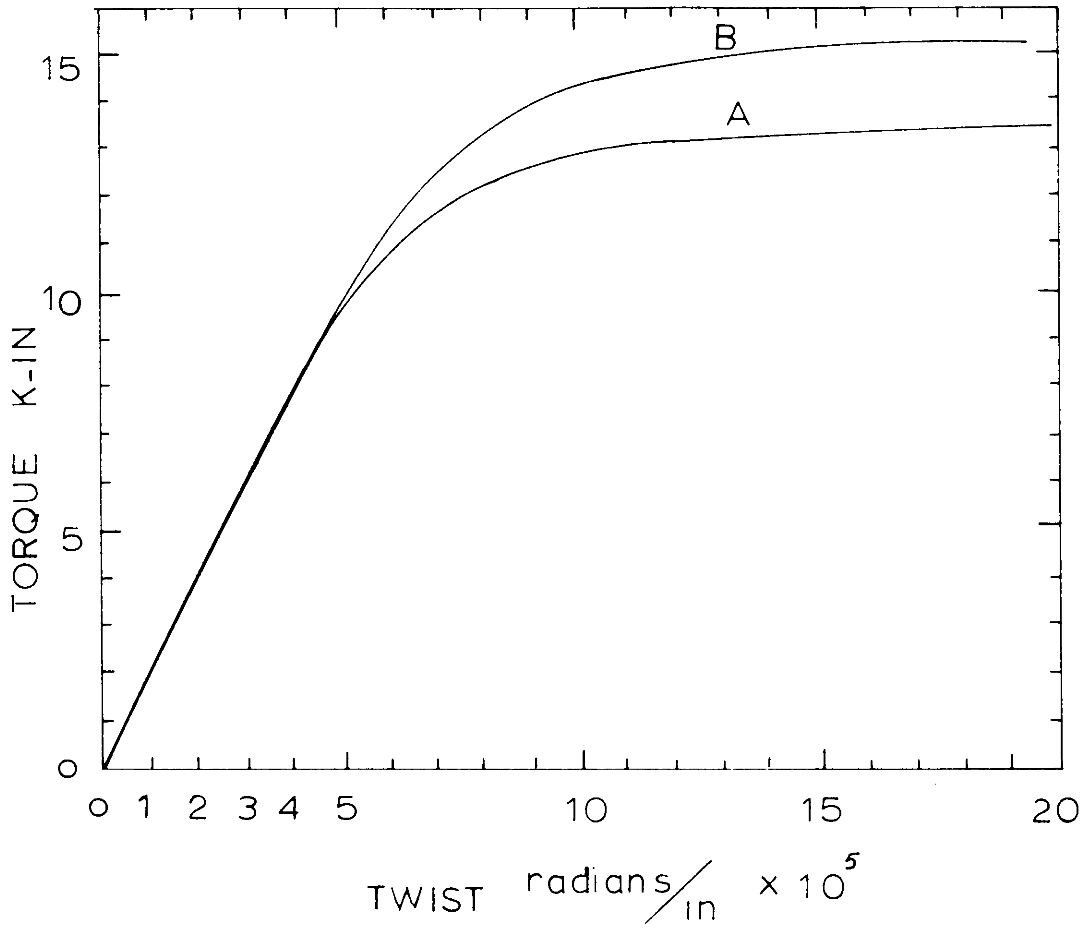


Fig. 2 - Typical Torque-Twist Graph for Reinforced Concrete Member

A - 4" x 6" specimen reinforced by spiral reinforcement

B - 4" x 6" specimen reinforced by longitudinal bars at the corners and vertical stirrups.

$$T\dot{r} = T\dot{c}\,(1 + 0.25\,p)$$

where $T\dot{r}$ = torque taken by reinforced section

$T\dot{c}$ = torque taken by concrete alone

p = total percentage of steel

The above formula is good when the percentage of steel is not greater than 1.5. In cases where the ratio of steel to the concrete is greater than 1.5 per cent, W. T. Marshall[9] suggests another empirical formula which agrees well with the experiment. This formula states:

$$T\dot{r} = T\dot{c}\,(1.33 + 0.1\,p)$$

In 1952 Henry Cowan[4] derived a mathematical proof for helical binders to resist torsional stress by assuming the concrete as an elastic and isotropic material.

In 1960 Henry Cowan[5] published a paper in the Proceedings of the American Concrete Institute on concrete beams subjected to pure bending, pure torsion and combined bending and torsion. The sections of the beams used were circular, rectangular and "T" shaped. He derived formulas for plain concrete sections subjected to torsion. The formula he derived for rectangular sections is as follows:

$$S\dot{t} = \frac{5T}{d\,b^2}$$

where $S\dot{t}$ = the diagonal tensile stress

T = the torsional moment on the section

b = width of the section

d = depth of the section

The above formula is the same as Timoshenko's[14] formula for torsional stress in rectangular sections if d/b = 1; it is correct for square sections and somewhat conservative for rectangular sections of a type normally used in concrete construction.

H. Cowan[5] ran several experiments on plain and reinforced concrete beams subjected to combined bending and torsion. He found that the bending moment in reinforced concrete sections does not reduce the capacity of the section to resist torsion as shown in Fig.3; while torsional moment capacity of plain sections under the effects of bending moment is greatly reduced as shown in Fig. 4.

Some of the specifications used for design in this thesis are recommended by the new Australian Code from Cowan [5], and others are from the results of experimental research conducted on concrete in the past.

The coefficient for torsional stiffness factors used in this thesis for the various sections of beams are those given by Timoshenko and Lessel[14] in "Applied Theory of Elasticity".

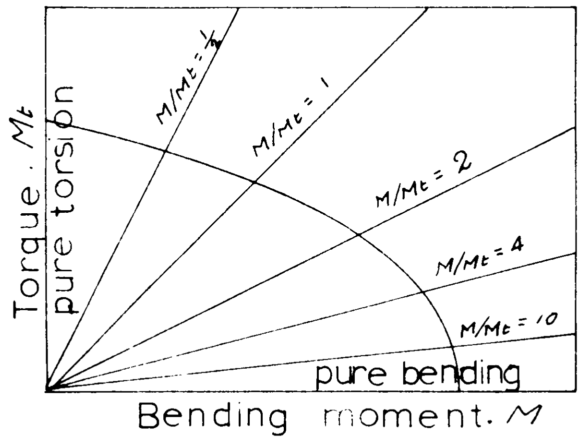


Fig. 3 - Relation Between Ultimate Bending Moment and Ultimate Twisting Moment for Plain Concrete

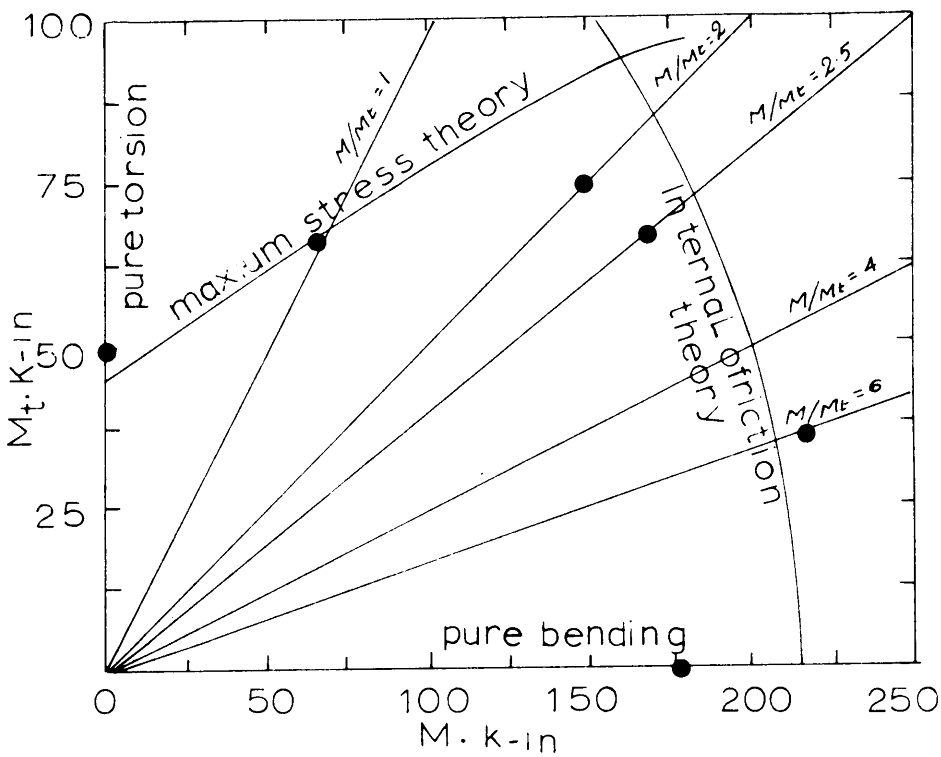


Fig. 4 - Relation Between Bending Moment and Twisting Moment for Reinforced Concrete at the Elastic Limit

The Symbol "O" Represents Experimental Result

V. NOTATION

A_s Area of tension reinforcement.

A_s' Area of compression reinforcement.

A_{sv} Area of vertical steel.

A_{sh} Area of longitudinal steel.

b Width of the beam.

b' Width of reinforcement cage.

C_1 Coefficient for bending moment in cantilever beam loaded with uniformly distributed load.

C_2 Coefficient for torsional moment for a cantilever beam loaded with uniformly distributed load.

C_3 Coefficient for bending moment for cantilever beam loaded with concentrated load.

C_4 Coefficient for torsional moment for a cantilever beam loaded with concentrated load.

C_5 Coefficient for mid point bending moment for fixed end curved beam loaded with uniformly distributed load.

C_6 Coefficient for end point bending moment for fixed end curved beam loaded with uniformly distributed load.

C_7 Coefficient for torsional moment at the end for a curved beam loaded with uniformly distributed load.

C_8 Coefficient for maximum torsional moment for a fixed end beam loaded with uniformly distributed load.

C_9 Coefficient for the support at the ends for fixed end beam loaded with uniformly distributed load.

d Depth of the beam.

d' Depth of reinforcement cage.

E Modulus of elasticity in direct stress.

EI Flexural stiffness.

F_o Vertical reaction at the supports.

f_c Stress in concrete.

f_c' Ultimate strength of concrete.

G Modulus of rigidity.

GJ Torsional stiffness.

I Principle moment of inertia of the cross-sectional area with respect to the axis of bending.

J Torsional stiffness factor (it is the polar moment of inertia for special case of a circular section).

K Stiffness ratio = $\frac{EI}{GJ}$

M Bending moment. The subscript indicates its location.

N Number of stirrups.

O Center of curvature of the beam.

R Radius of curvature of the center line of the beam.

S Horizontal tie spacing.

T Twisting moment (torque). The subscript indicates its location.

U_b Strain energy due to bending.

U_t Strain energy due to torsion.

U_e Total strain energy.

v Unit shear stress in concrete.

W Concentrated load.

w Intensity of loading of a distributed load.

Angular distance from the support to the location of the concentrated load.

t Angular distance of a point of zero torsion from the center line of the beam.

b Angular distance of point of zero bending from the center line of the beam.

Angular distance of any section of the member measured counterclockwise from the support.

Angle subtended by the arc of the beam.

Constant depending on the ratio b/d.

VI. ASSUMPTIONS

The following assumptions are made:

1. Hooke's law is valid.
2. Castigliano's theorems are valid.
3. The angle of twist per unit length of the beam varies as T/GJ.
4. The angle of bending per unit length of the beam varies as M/EI.
5. The total strain energy due to shearing force is neglected.
6. The deformation due to vertical shear forces are neglected.
7. There are no temperature stresses.
8. There is no yielding of the support.

VII. SIGN CONVENTION

The sign convention adopted for obtaining expressions for bending moment, twisting moment, and shear in this thesis will be as follows:

If the beam is viewed from the front, a positive moment at the section is one that tends to make the portion of the beam between the section under consideration and the support convex upward.

A positive torque at the section under consideration is one that tends to turn the viewed section clockwise.

A positive shearing force at the section is one that tends to raise the viewed section.

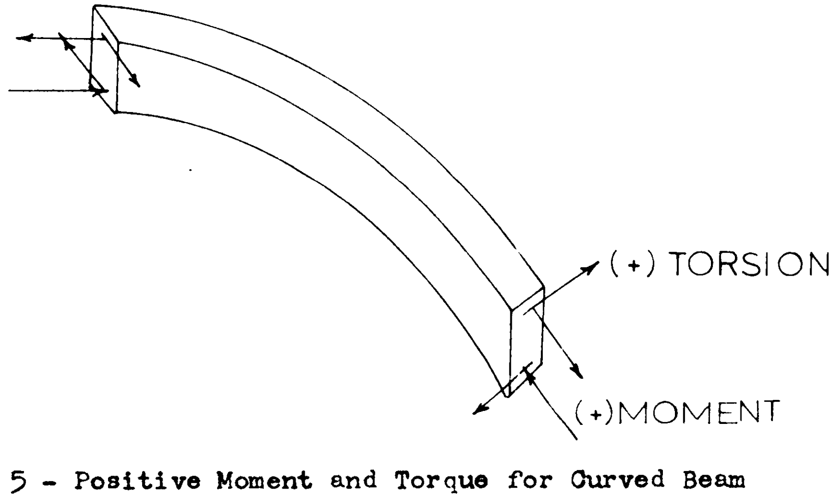


Fig. 5 - Positive Moment and Torque for Curved Beam

VIII. ANALYSIS AND DERIVATION OF EQUATIONS

For the purpose of analysis, two kinds of beams will be considered:

1. Determinant beams which are represented by cantilever beams.
2. Indeterminant beams.

The beam may be indeterminant to the first degree such as a curved beam fixed at the ends and loaded with a uniformly distributed load, or to a higher degree of indeterminacy, depending upon the type of loading and ends conditions.

A. Cantilever Beam Loaded with Uniformly Distributed Load.

Consider the case shown in Fig. 6 and assume the following:

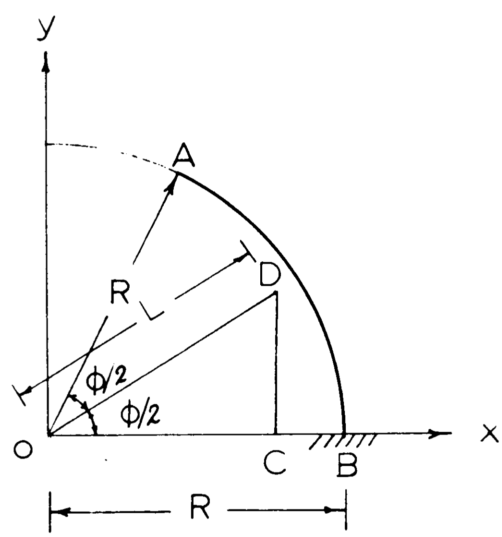


Fig. 6 - Cantilever Beam Loaded with Uniformly Distributed Load

ϕ = the angle subtended by the arc of the beam.

w = the intensity of load on the beam.

M_o = the bending moment at the fixed end.

T_o = the torsional moment at the fixed end.

L = the distance from the center of gravity of the arc to the center of curvature.

From statics

$$L = \frac{R \sin \frac{\phi}{2}}{\frac{\phi}{2}} = \frac{2 R \sin \frac{\phi}{2}}{\phi}$$

Total load on the beam = $w R \phi$

$$M_o = w R \phi \times DC$$

but $DC = L \sin \frac{\phi}{2} = \frac{2R}{\phi} \sin^2 \frac{\phi}{2}$

$$\therefore M_o = w R \phi \times \frac{2R}{\phi} \sin^2 \frac{\phi}{2}$$

or $M_o = w R^2 (1 - \cos \phi)$

Assuming that $(1 - \cos \phi) = C_1$

then $M_o = w R^2 C_1$ . . . (1)

$$T_o = w R \phi \times CB$$

but $CB = OB - OC = R - L \cos \frac{\phi}{2} = R - \frac{2R \sin \frac{\phi}{2}}{\phi} \cos \frac{\phi}{2}$

$$CB = R(1 - \sin \phi)$$

$\therefore T_o = w R \phi \times R(1 - \frac{1}{\phi} \sin \phi = w R^2(\phi - \sin \phi).$

Assuming that $(\phi - \sin \phi) = C_2$

$\therefore T_o = w R^2 C_2$ . . . (2)

Fig. 7 shows the relation between C_1, C_2 and $\emptyset$, the angle subtended by the arc of the beam.

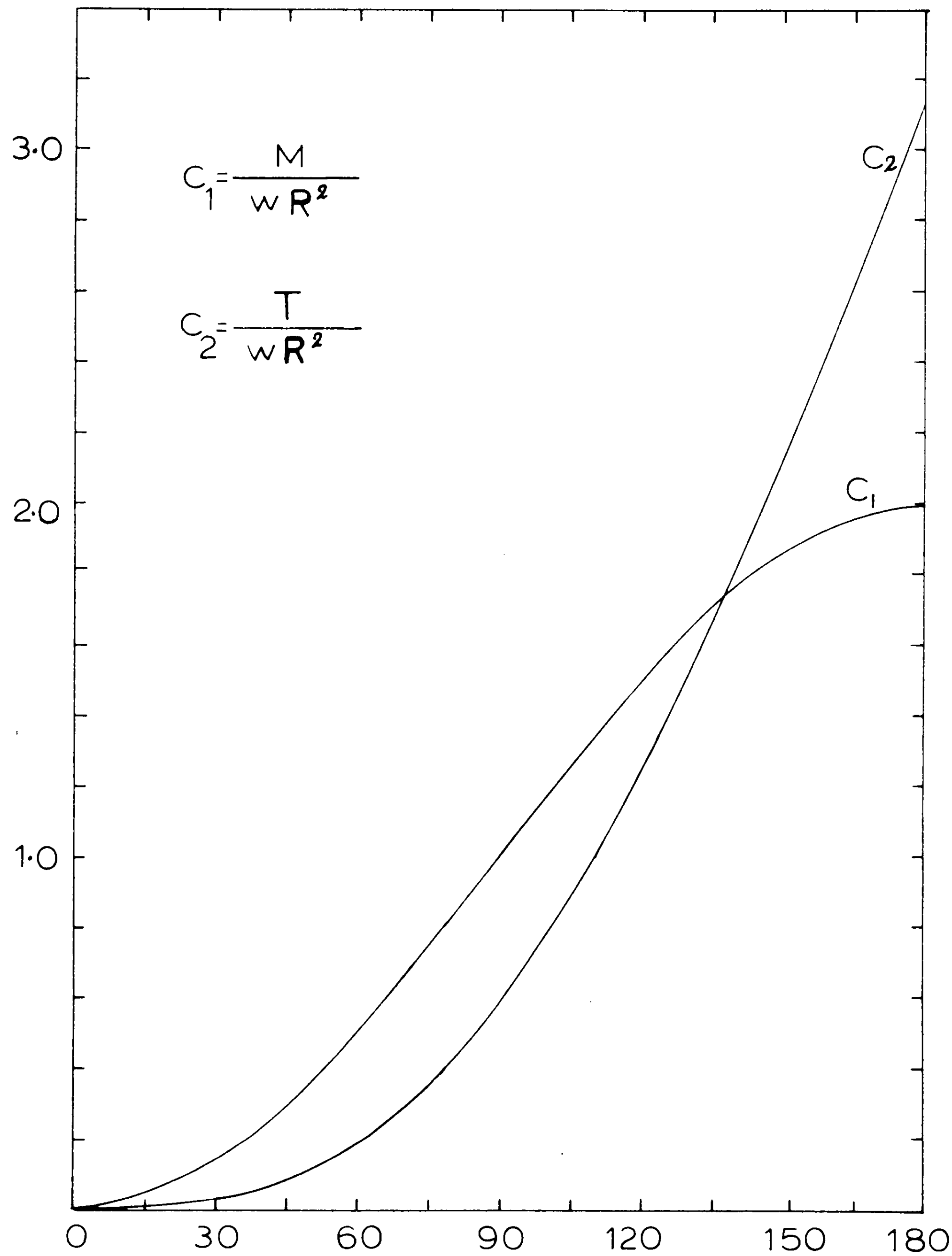


Fig. 7 - Coefficient for Bending Moment (M) and Torsional Moment (T) vs $\emptyset$ for a Cantilever Beam Loaded with Uniformly Distributed Load.

B. Cantilever Beam with Concentrated Load.

Consider the cantilever beam in Fig. 8 with a concentrated load W at point A.

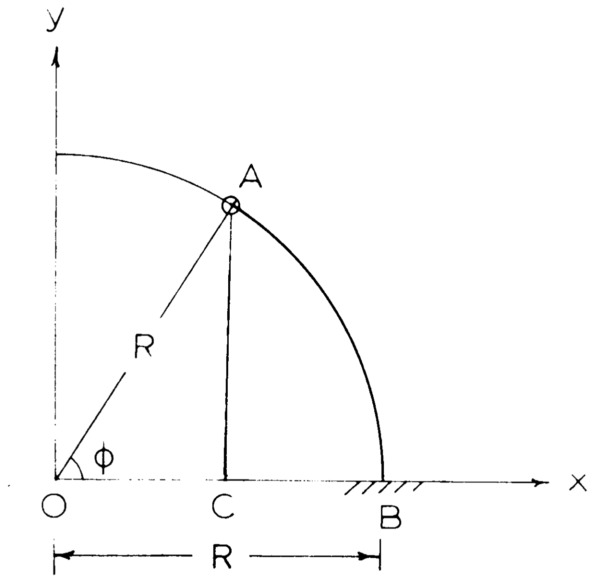


Fig. 8 - Cantilever Beam Loaded with Concentrated Load

The expression for bending moment Mo, and torque To at the end of the beam is as follows:

$M\dot{o} = W \times AC$

but $AC = R \sin \phi$

$\therefore\ M\dot{o} = WR \sin \phi$

Assuming that $\sin \phi = C_3$

$\therefore\ M\dot{o} = W R C_3$ . . . (3)

$T\dot{o} = W \times CB$

but $CB = OB - OC = R - R \cos \phi$

$\therefore\ T\dot{o} = W R (1 - \cos \phi)$

Assuming that $(1 - \cos \phi) = C_4$

$T\dot{o} = W R C_4$ . . . (4)

Fig. 9 shows the relation between C_3, C_4 and ϕ, the angle subtended by the arc of the beam.

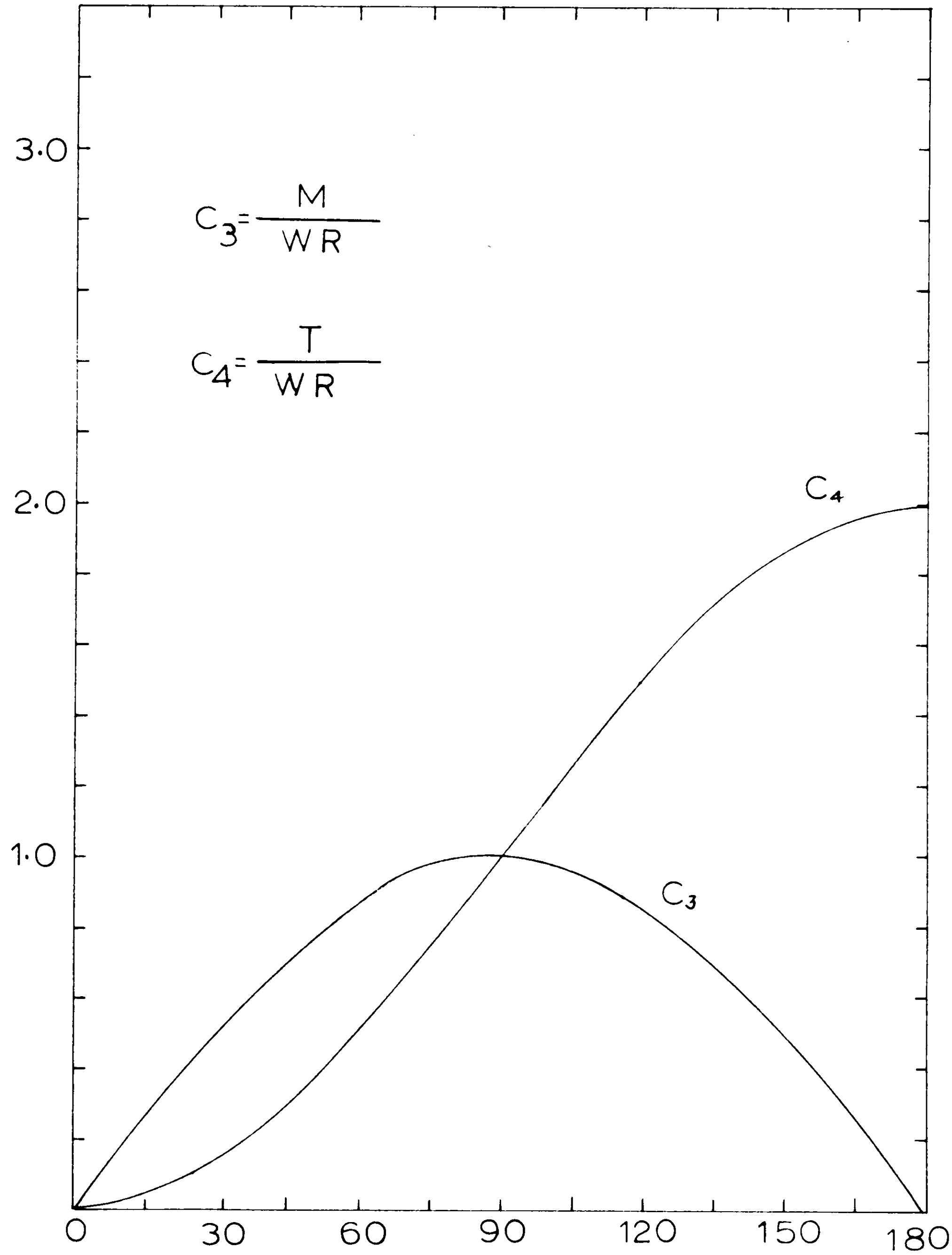


Fig. 9 - Coefficient for Bending Moment (M) and Torsional Moment (T) vs ϕ for a Cantilever Beam Loaded with Concentrated Load.

C. Fixed End Beam with Uniformly Distributed Load over the Entire Beam.

Let the beam in Fig. 10 be loaded with a uniformly distributed load and have equal fixity of the supports. Due to the symmetrical conditions

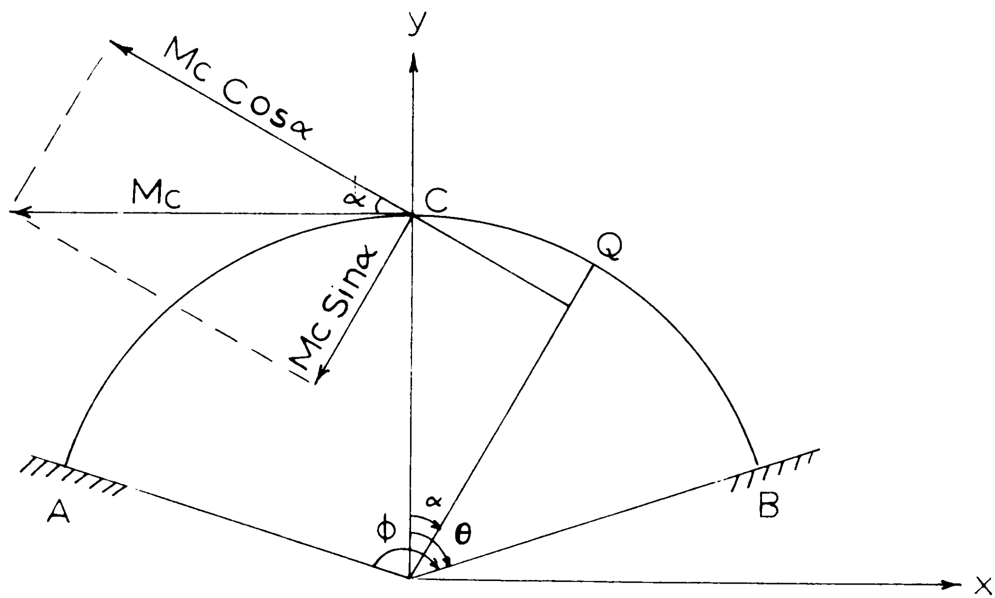


Fig. 10 - Horizontal Curved Beam Fixed at the Ends and Loaded with Uniformly Distributed Load.

of the beam there will be no torsion and no shear at the midpoint.

Therefore consider the portion CB as a cantilever beam, with bending moment M_c acting at the end.

Assume the following:

M_Q = the bending moment at point Q where Q is any point along the portion CB

M_b = the bending moment at Q due the uniformly distributed load of the portion CQ

T_Q = the torsional moment at Q

T_t = the torsional moment at Q due to the uniformly distributed load along the portion OQ

$$\therefore M_Q = M_o \cos\alpha + M_b$$

but $M_b = 2W R^2 \sin^2 \frac{\alpha}{2}$ (See equation 1)

$$\therefore M_Q = M_o \cos\alpha + 2W R^2 \sin^2 \frac{\alpha}{2} \quad \ldots (5)$$

$$T_Q = M_o \sin\alpha + T_t$$

but $T_t = 2wR^2 \left(\frac{\alpha}{2} - \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)$ (See equation 2)

$$\therefore T_Q = M_o \sin\alpha + 2\,wR^2 \left(\frac{\alpha}{2} - \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right) \quad \ldots (6)$$

It is now necessary to obtain M_o.

Assume U_e = the total strain energy in the beam.

$$\therefore U_e = \frac{1}{2EI}\int_0^B M_Q^2\, ds + \frac{1}{2GJ}\int_0^B T_Q^2\, ds \quad \ldots (7)$$

The rotation due to the effect of Mo at the end of the beam is equated to zero.

Applying Castigliano's theorem

$$\frac{\partial U_e}{\partial M_o} = 0$$

Taking the partial derivative of equation 7 with respect to Mo

$$0 = \frac{1}{E}\int_0^B M_Q \frac{M_Q}{M_o}\, ds + \frac{1}{GJ}\int_0^B T_Q \frac{T_Q}{M_o}\, ds \quad \ldots (8)$$

but $ds = r\, d\alpha$

$$\frac{M_Q}{M_o} = \cos\alpha$$

$$\frac{\delta T_{\dot{Q}}}{\delta M_{\dot{o}}} = \sin\alpha$$

Substituting the above values in equation 8

$$0 = \frac{1}{EI}\int_{0}^{B}(M\dot{o}\cos\alpha + 2wR^2\sin^2\frac{\alpha}{2})\cos\alpha\, d\alpha$$

$$+ \frac{1}{GJ}\int_{0}^{B} M\dot{o}\sin\alpha + 2wR^2\left(\frac{\alpha}{2} - \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)\ \sin\alpha\, d\alpha$$

or $$\frac{1}{EI}\int_{0}^{B} M\dot{o}\cos^2\alpha\, d\alpha - \frac{2wR^2}{EI}\int_{0}^{B}\sin^2\frac{\alpha}{2}\cos\alpha\, d\alpha + \frac{M\dot{o}}{GJ}\int_{0}^{B}\sin^2\alpha\, d\alpha$$

$$- \frac{2wR^2}{EI}\int_{0}^{B}\frac{\alpha}{2}\sin\alpha\, d\alpha + \frac{2wR^2}{GJ}\int_{C}^{\theta}\sin\frac{\alpha}{2}\ \cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\cdot d\alpha = 0$$

or $$-\frac{M\dot{o}}{EI}\int_{0}^{\theta}\cos^2\alpha\, d\alpha + \frac{M\dot{o}}{GJ}\int_{0}^{\theta}\sin^2\alpha\ d\alpha = \frac{2wR^2}{EI}\int_{0}^{\theta}\sin^2\frac{\alpha}{2}\ \cos\alpha\ d\alpha$$

$$+ \frac{wR^2}{GJ}\int_{0}^{\theta}\sin\alpha\, d\alpha - \frac{2wR^2}{GJ}\int_{0}^{\theta}\sin\frac{\alpha}{2}\ \cos\frac{\alpha}{2}\ \sin\alpha\ .\ d\alpha$$

Integrating and collecting terms

$$\frac{M\dot{o}}{EI}\left\{\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta\right\} + \frac{M\dot{o}}{GJ}\left\{\frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta\right\}$$

$$= \frac{wR^2}{GJ}\left\{\sin\theta - \theta\cos\theta\right\} + \frac{wR^2}{EI}\left\{\sin\theta - \frac{1}{2}\sin\theta\cos\theta - \frac{\theta}{2}\right\}$$

$$- \frac{wR^2}{GJ}\left\{\frac{\theta}{2} - \frac{1}{2}\cos\theta\sin\theta\right\}$$

Multiplying the equation by EI and assume that $\frac{EI}{GJ} = K$ and collecting similar terms

$$M\dot{c}\left\{\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta + K\frac{\theta}{2} - \frac{K}{2}\sin\theta\cos\theta\right\}$$

$$= wR\left\{K\sin\theta - \frac{\theta}{2} - K\theta\cos\theta + \frac{K}{2}\cos\theta\sin\theta - \frac{K}{2}\theta\right.$$

$$\left.+ \sin\theta - \frac{1}{2}\sin\theta\cos\theta\right\}$$

or $$M\dot{c}\left\{(1-K)\sin\theta\cos\theta + \theta(1+K)\right\} = wR^2\left\{(K+1)\sin\theta + \frac{1}{2}(K-1)\right.$$

$$\left.\sin\theta\cos\theta - \frac{\theta}{2}(K+1) - K\theta\cos\theta\right\}$$

$$\therefore -M\dot{c} = \frac{wR^2\left\{2(K+1)\,\mathrm{Sin}\,\theta + (K-1)\sin\theta\cos\theta - \theta(K+1) - 2K\theta\cos\theta\right\}}{\left\{(1-K)\sin\theta\cos\theta + \theta(1+K)\right\}}$$

$$M\dot{o} = wR^2\left\{\frac{(K-1)\theta - (K-1)\sin\theta\cos\theta}{(K-1)\theta - (K-1)\sin\theta\cos\theta} - \frac{(K+1)\sin\theta - 2K\theta\cos\theta}{(K+1)\theta - (K-1)\sin\theta\cos\theta}\right\}$$

Assuming that

$$U = \frac{2(K+1)\sin\theta - 2K\theta\cos\theta}{(K+1)\theta - (K-1)\sin\theta\cos\theta} = \frac{4(K+1)\sin\theta - 4K\theta\cos\theta}{2\theta(K+1) - (K-1)\sin 2\theta}$$

but $\theta = \frac{\phi}{2}$

$$\therefore \quad U = \frac{4(K+1)\sin\frac{\phi}{2} - 2\phi K\cos\frac{\phi}{2}}{\phi(K+1) - (K-1)\sin\phi}$$

$$\therefore M\dot{c} = wR^2(1-U)$$

Assuming that $(1 - U) = C_5$, then

$$M\dot{c} = wR^2C_5$$

Fig. 11 shows the relation between C_5 and ϕ, the angle subtended by the arc of the beam.

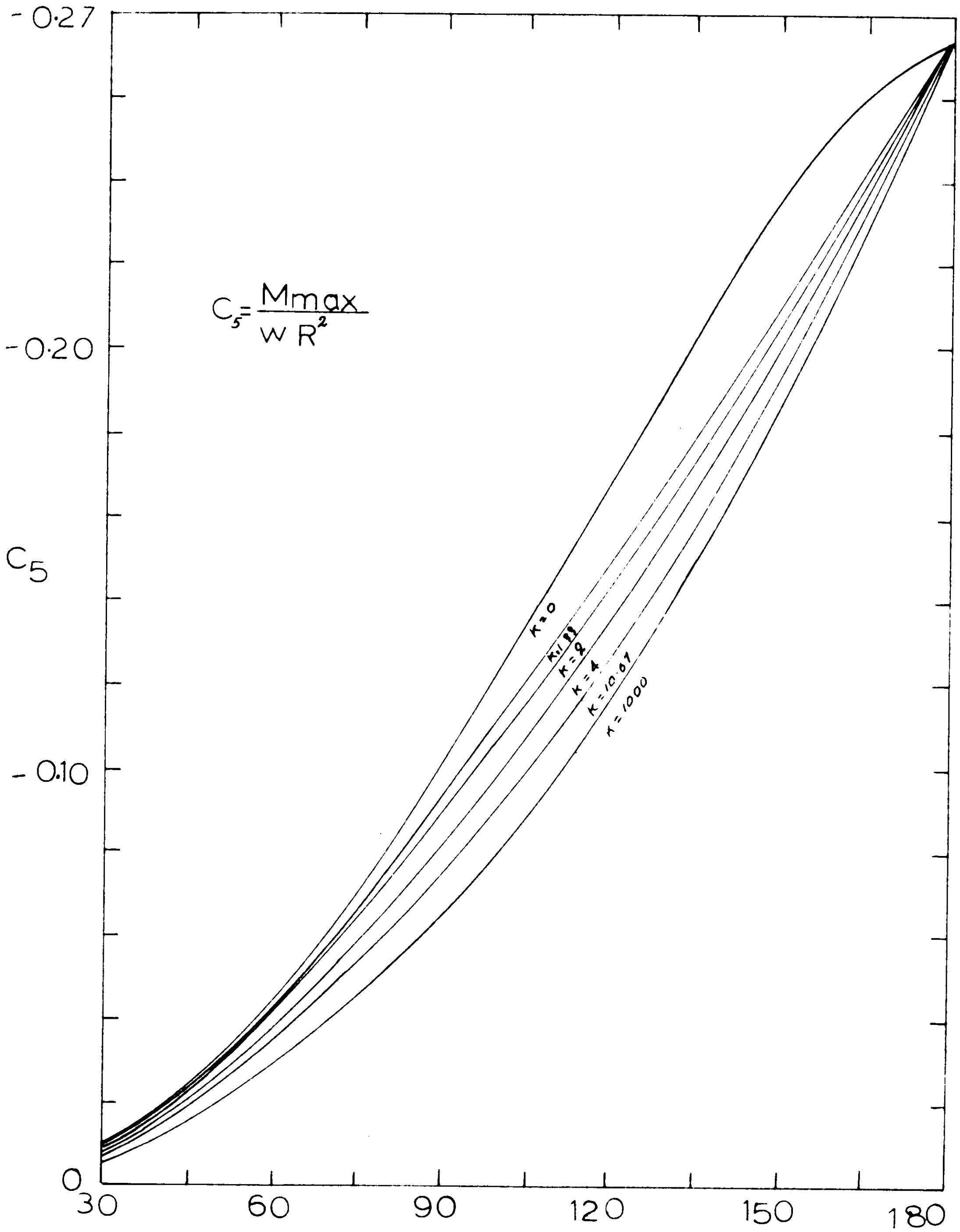


Fig. 11 - Mid Point Bending Moment (Mċ) vs Ø for Fixed End Beam with Uniformly Distributed Load.

Substituting the value of Mc in equation 5

$$M_Q = wR^2 (1 - U) \cos\alpha + 2\, wR^2 \sin^2 \frac{\alpha}{2}$$

$$= wR^2 (\cos\alpha - U \cos\alpha + \sin^2 \frac{\alpha}{2})$$

$$= wR^2 (1 - U \cos\alpha) \qquad \ldots (10)$$

In the same way with equation 6

$$T_Q = Mc \sin\alpha + 2\, wR^2 (\frac{\alpha}{2} - \sin\frac{\alpha}{2} \cos\frac{\alpha}{2})$$

$$\therefore\ T_Q = wR^2 (1 - U) \sin\alpha + 2wR^2 (\frac{\alpha}{2} - \sin\frac{\alpha}{2} \cos\frac{\alpha}{2})$$

$$= wR^2 (\sin\alpha - U \sin\alpha + \alpha - 2 \sin\frac{\alpha}{2} \cos\frac{\alpha}{2})$$

$$= wR^2 (\sin\alpha - U \sin\alpha + \quad - \sin\alpha)$$

$$= wR^2 (\alpha - U \sin\alpha) \qquad \ldots (11)$$

The moment at the supports can be obtained by substituting $\frac{\phi}{2}$ instead of

$$\therefore\ M_o = wR^2 (1 - U \cos\frac{\phi}{2})$$

Assuming that $(1 - U \cos\frac{\phi}{2}) = C_6$

$$M_o = wR^2\, C_6 \qquad \ldots (12)$$

Fig. 12 shows the relation between C_6 and ϕ, the angle subtended by the arc of the beam.

The torsion at the supports can be obtained by substituting $\frac{\phi}{2}$ instead of α in equation 11

$$T_o = wR^2 (\frac{\phi}{2} - U \sin\frac{\phi}{2})$$

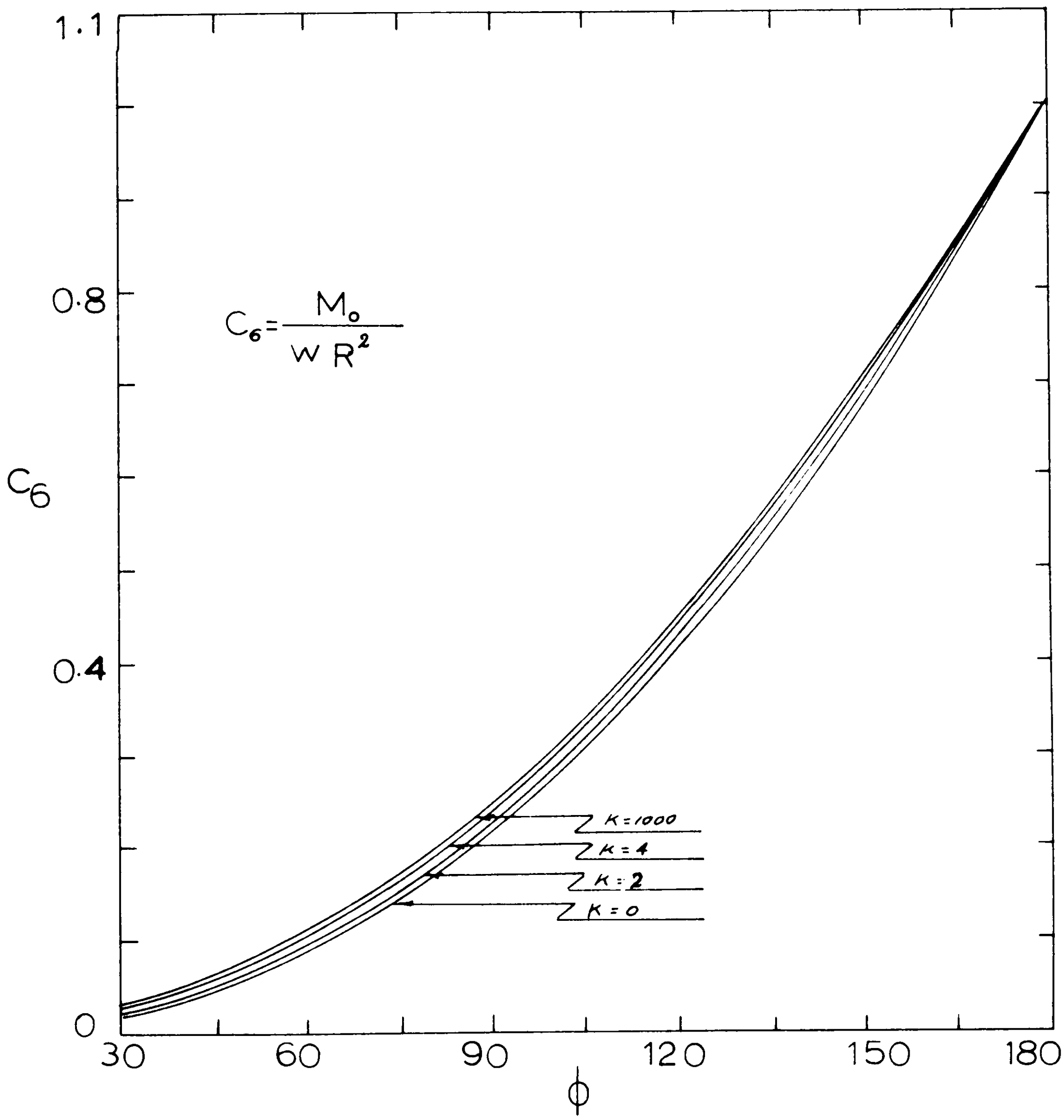


Fig. 12 - Coefficient for End Point Bending Moment (M_o) vs ϕ for Fixed End Beam with a Uniformly Distributed Load.

Assuming that $(\frac{\phi}{2} - U \sin \frac{\phi}{2}) = C_7$

$$T_o = wR^2 C_7 \quad \ldots (13)$$

Fig. 13 shows the relation between the torque factor C_7 and ϕ, the angle subtended by the arc of the beam.

It is quite clear that from the simple form of equation 12 and equation 13 that the moments may be quickly calculated as soon as U is known.

The equations so far derived may now be used to calculate the magnitude as well as the points of location of the maximum and zero moment.

For the maximum bending, differentiate equation 10 and equate to zero

$$M_Q = wR^2 (1 - U \cos \alpha)$$

$$\underline{MQ} = wR^2 (U \sin \alpha) = 0$$

$\therefore \quad \alpha = 0$ or

Substituting for = 0 in equation 10

$$M_{max} = wR^2 (1 - U) \quad \ldots (14)$$

Maximum bending moment is therefore at the mid point of the beam. There is also a maximum moment at each of the supports, but this is not found by differentiation because the curve of the bending moment is discontinuous at these points. The maximum moment at the supports is

$$M_o = wR^2 (1 - U \cos \frac{\phi}{2})$$

The maximum torsional moment is found in the following manner: Differentiate equation 5 and equate to zero.

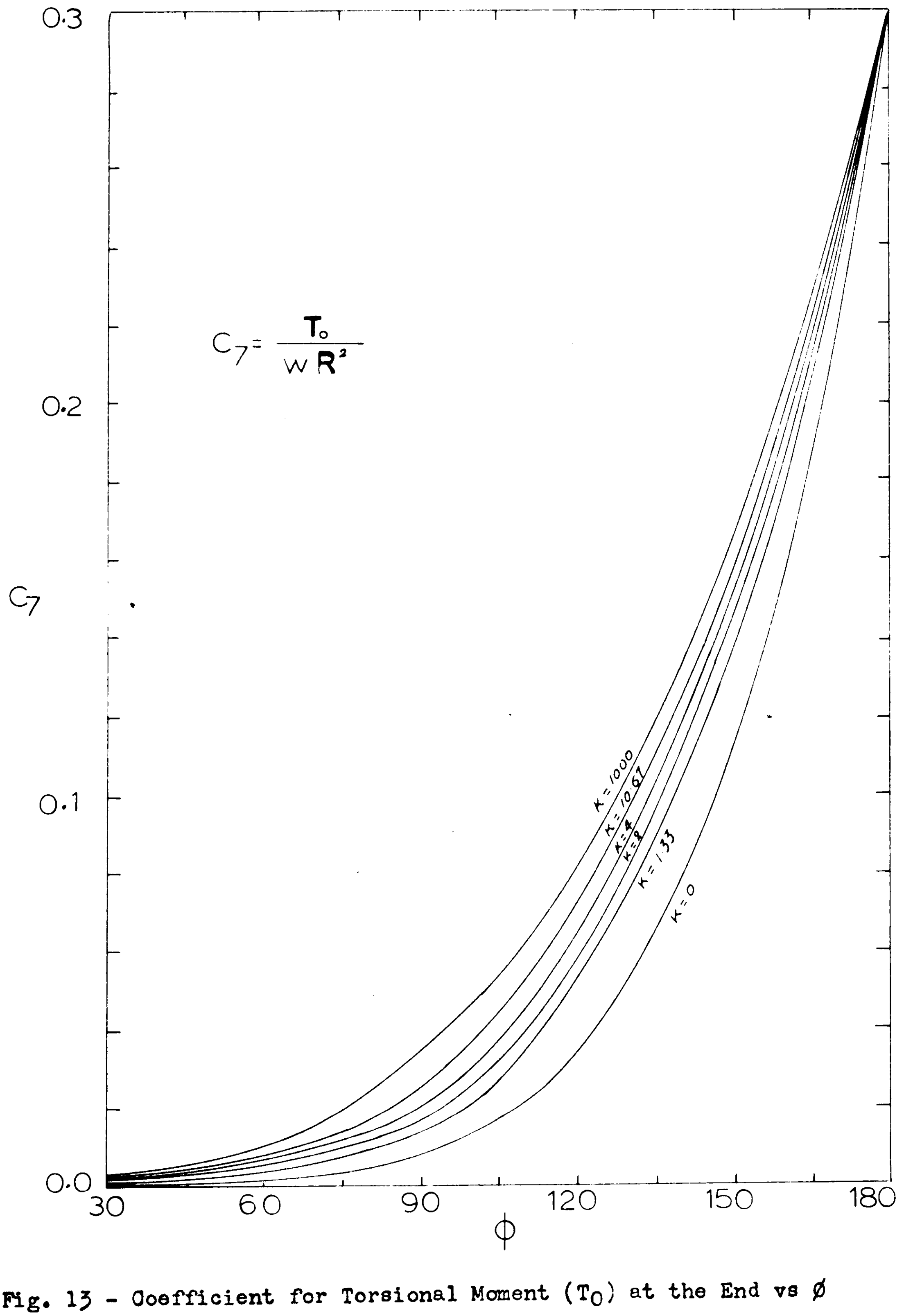


Fig. 13 - Coefficient for Torsional Moment (T_O) at the End vs ϕ for Fixed End Beam with a Uniformly Distributed Load.

$$T_Q = wR^2 (\alpha - U \sin\alpha)$$

$$\underline{T_Q} = wR^2 (1 - U \cos\alpha) = 0$$

$$\therefore U \cos\alpha = 1$$

$$\therefore \cos\alpha = \frac{1}{U}$$

$$\therefore \alpha = \cos^{-1} \frac{1}{U} \qquad \ldots (14)$$

Substituting $= \cos^{-1} \frac{1}{U}$ in equation 5

$$T_{max} = wR^2(\cos^{-1} \frac{1}{U} - U^2 - 1) \qquad \ldots (15)$$

For torsion as well as the moment another maximum occurs at the supports

$$T_{max} = T_o = wR^2(\frac{\phi}{2} - U \sin \frac{\phi}{2})$$

Assuming that $(\cos^{-1} \frac{1}{U} - \sqrt{U^2 - 1}) = C_8$

and substituting this value in equation 15

$$T_{max} = wR^2 C_8$$

Fig. 14 shows the relation between C_8 and ϕ, the angle subtended by the arc of the beam.

To calculate the point of zero moment, equate equation 5 to zero

$$wR^2 (1 - U \cos\alpha) = 0$$

$$\therefore \cos\alpha = \frac{1}{U}$$

$$\therefore \alpha_b = \cos^{-1} \frac{1}{U} \qquad \ldots (16)$$

It is shown from equation 15 and equation 16 that the point of zero moment and maximum torsion occur at the same point.

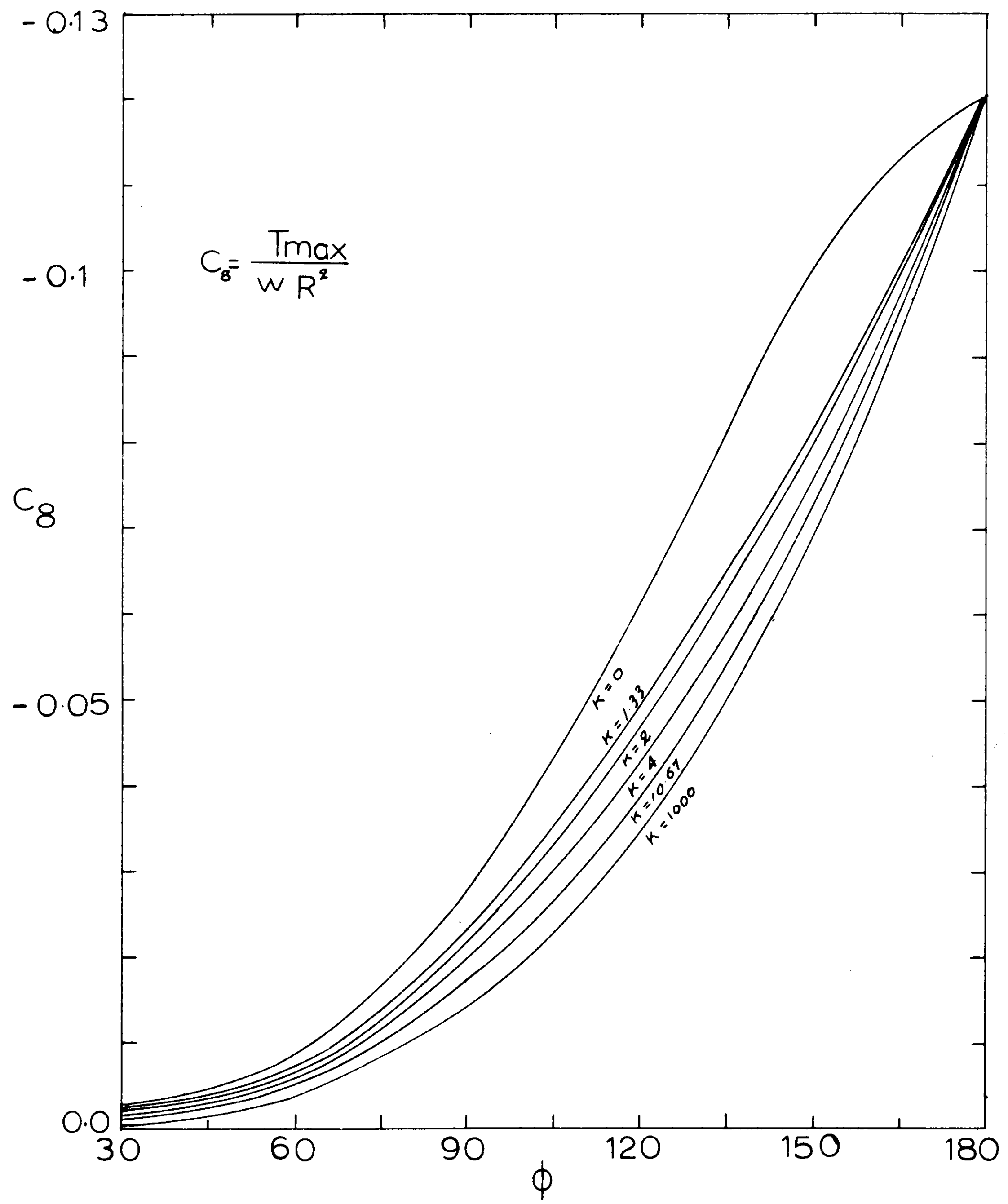


Fig. 14 - Coefficient for the Maximum Torsional Moment vs ϕ for Fixed End Beam with a Uniformly Distributed Load.

To calculate the point of zero torsion, equate equation 5 to zero.

$$wR^2 \; (\alpha - U \sin \quad) = 0$$

$$\therefore U \sin \alpha = \alpha$$

$$\text{or} \quad \frac{\alpha t}{\sin \alpha t} = U \qquad \ldots (17)$$

The above equation must be solved by trial and error.

Fig. 15 shows the relation between the angular distance (α_b) for maximum torsion or zero moment and $\emptyset$, the angle subtended by the arc of the beam.

Fig. 16 shows the relation between the angular distance (α_t) for the zero torsion from the centerline of the beam and U.

The value of the reaction at the supports can be obtained by applying the equations of statics

$$2F_o = wR\emptyset$$

$$\therefore F_o = wR\frac{\emptyset}{2}$$

Substituting $\frac{\emptyset}{2}$ by C_9 then

$$F_o = wRC_9$$

Fig. 17 shows the relation between the coefficient at the support and $\emptyset$, the angle subtended by the arc of the beam.

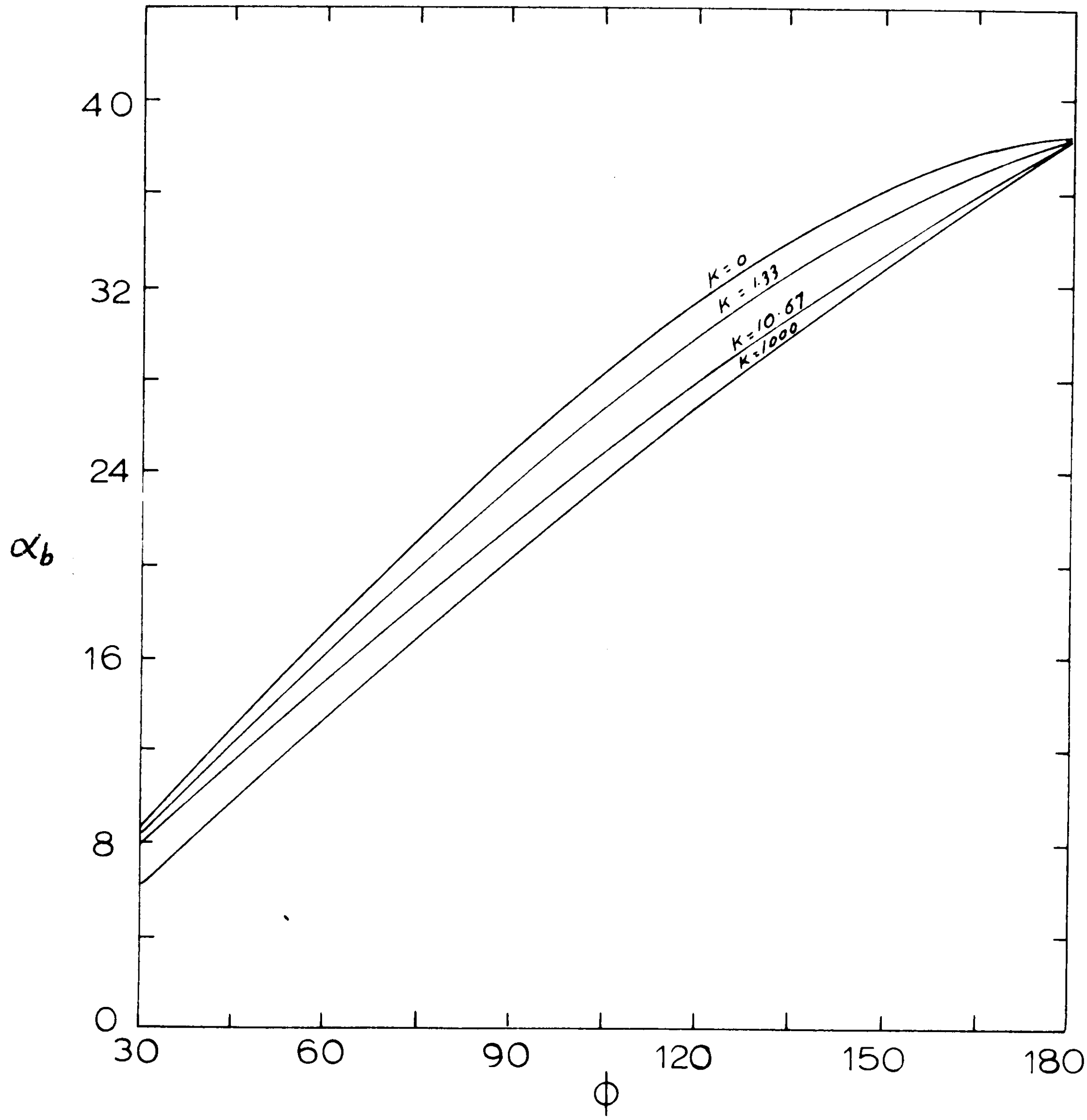


Fig. 15 - Angular Distance of Point of Zero Moment or Maximum Torsion from the Centerline of the Beam vs ϕ.

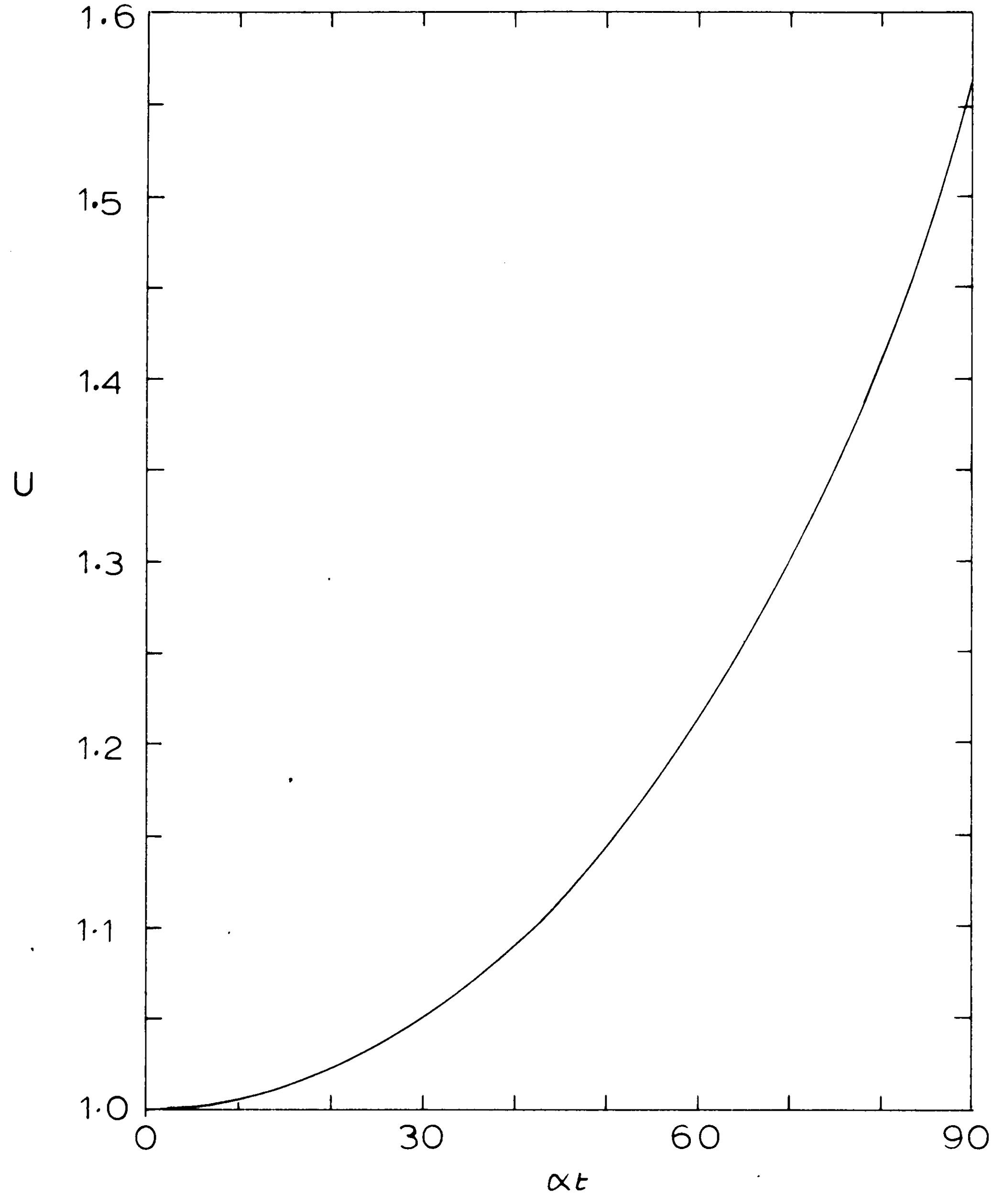


Fig. 16 - The Relation Between U and the Angular Distance of Zero Torsion from the Centerline of the Beam.

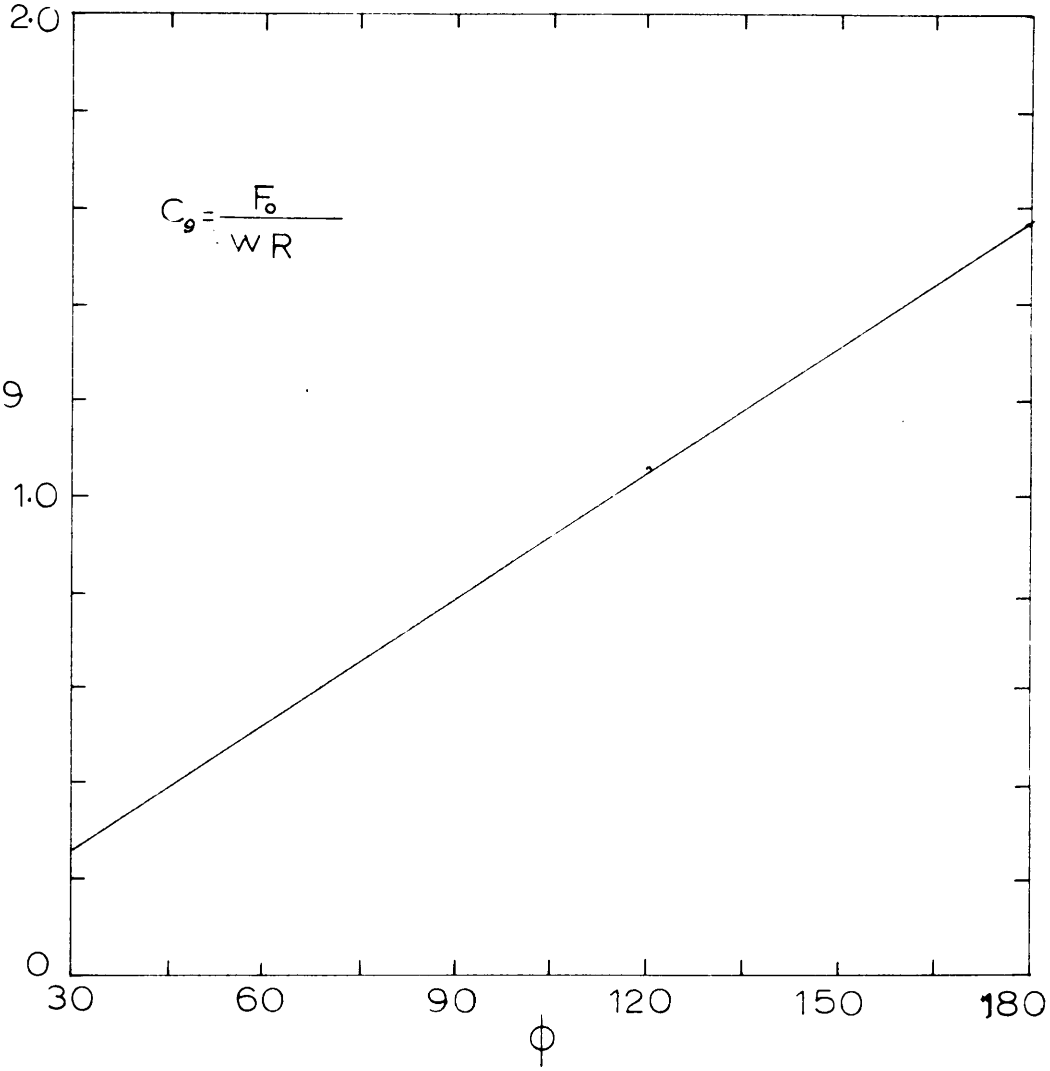


Fig. 17 - Coefficient of Shear Force at the End vs Ø for Fixed End Beam with a Uniformly Distributed Load.

D. Fixed End Beam Loaded with Concentrated Load.

Let the circular beam AB Fig. 18 subtend an angle ϕ at the center. W is a concentrated load normal to the plane of the beam.

is the angular distance from the support B to the point of application of the concentrated load.

Assume x is any point along the beam with angular distance θ from the support B. If the support at B is removed, its effect can be replaced by the bending moment M_o, the torque T_o and the shearing force F_o. These three unknowns will be taken as the unknown redundant elements.

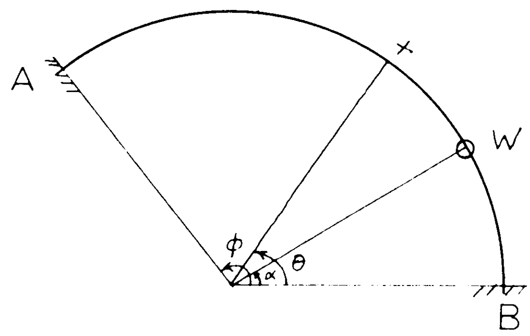


Fig. 18 - Horizontal Curve Beam Fixed at the Ends and Loaded with Concentrated Load.

Applying Castigilyano's theorem which states that, "The component of deflection of a point in direction of an action is equal to the first partial derivative of the total strain energy with respect to the action."

Assuming U_e is the total strain energy

$$\frac{\partial U_e}{\partial M_o} = \theta m$$

$$\frac{\partial U_e}{\partial T_o} = \theta t$$

$$\frac{\partial U_e}{\partial F_o} = \delta$$

But as the end B of the beam is fixed, then

$$\frac{\partial U_e}{\partial M_o} = 0$$

$$\frac{\partial U_e}{\partial T_o} = 0$$

$$\frac{\partial U_e}{\partial F_o} = 0$$

But the total strain energy of the beam is the sum of the components energies due to bending, torque and shear. The last term is very small compared with the other two, and can be neglected.

Denoting the strain energy due to bending by U_b and the strain energy due to the torsional moment by U_t

$$\frac{\partial U_b}{\partial M_o} + \frac{\partial U_t}{\partial M_o} = 0$$

$$\frac{\partial U_b}{\partial T_o} + \frac{\partial U_t}{\partial T_o} = 0$$

$$\frac{\partial U_b}{\partial T_o} + \frac{\partial U_t}{\partial T_o} = 0$$

If (ds) is a small element of the arc, EI is the flexural rigidity and GJ is the torsional rigidity of the beam.

$$U_b = \frac{1}{2EI}\int M_x^2 \, ds$$

$$U_t = \frac{1}{2GJ}\int T_x^2 \, ds$$

and the total energy $U_e = \frac{1}{2EI} \quad M_x^2 \, ds + \frac{1}{2GJ} \quad T_x^2 \, ds$

$$\frac{\partial U_e}{\partial M_o} = \frac{1}{EI}\int M_x \frac{M_x}{M_o} \, ds + \frac{1}{GJ}\int T_x \frac{T_x}{M_o} \, ds = 0 \qquad \ldots (18)$$

$$\frac{\partial U_e}{\partial T_o} = \frac{1}{EI}\int M_x \frac{M_x}{T_o} \, ds + \frac{1}{GJ}\int T_x \frac{T_x}{T_o} \, ds = 0 \qquad \ldots (19)$$

$$\frac{\partial U_e}{\partial F_o} = \frac{1}{EI}\int M_x \frac{M_x}{F_o} \, ds + \frac{1}{GJ}\int T_x \frac{T_x}{F_o} \, ds = 0 \qquad \ldots (20)$$

The induced bending moment M_x and torsional moment T_x at x due to the effect of M_o, T_o, F_o and the external load (W) will be illustrated by the following free body diagrams.

From Fig. 19 the effect of the end moment M_o on section x.

Moment = $M_o \cos \theta$

Torsion = $-M_o \sin \theta$

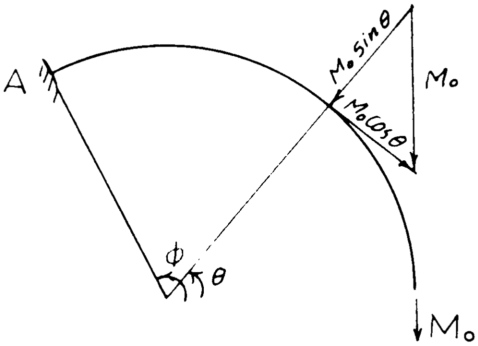


Fig. 19 - Effect of Bending Moment at the End on Any Section Along the Curved Beam.

The effect of T_o, Fig. 20, on section x is

Moment = $T_o \sin \theta$

Torsion = $T_o \cos \theta$

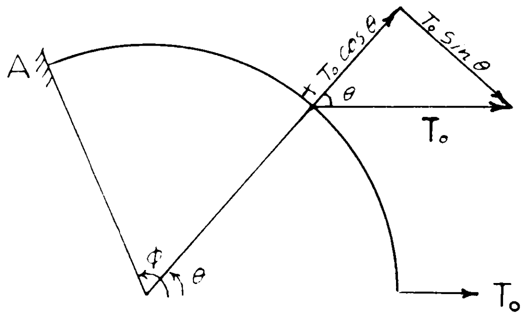


Fig. 20 - Effect of Torsional Moment at the End on Any Section Along the Curved Beam.

Fig. 21 shows the effect of the shear force F_o on section x.

Moment = $- F_o R \sin \theta$

Torque = $+ F_o R(1-\cos \theta)$

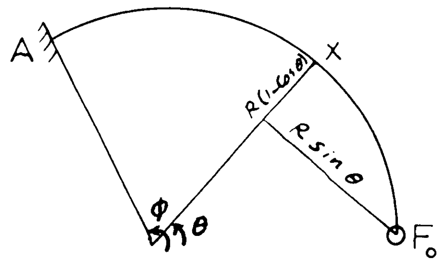


Fig. 21 - Effect of End Reaction on Any Section Along the Curved Beam.

Fig. 22 shows the effect of the concentrated load on section x.

Moment = $WR \sin(\theta - \alpha)$

Torsion = $- WR \quad 1 - \cos(\theta - \alpha)$

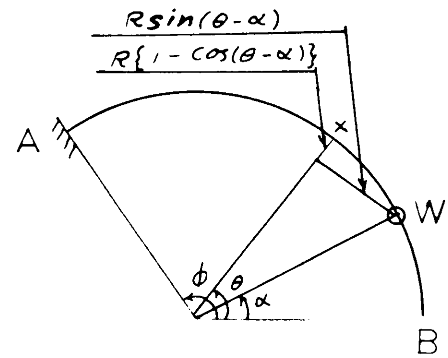


Fig. 22 - Effect of the Concentrated Load on Any Section Along the Curved Beam.

∴ The total bending moment Mx at section x is the sum of the component moments.

$$\therefore M_x = M_o \cos\theta + T_o \sin\theta - F_o R \sin\theta + WR \sin(\theta - \alpha)$$

$$\frac{\partial M_x}{\partial M_o} = \cos\theta$$

$$\frac{\partial M_x}{\partial T_o} = \sin\theta$$

$$\frac{\partial M_x}{\partial F_o} = - R \sin\theta$$

The torsional moment at x is the sum of the component torsions:

$$T_x = -M_o \sin\theta + T_o \cos\theta + F_o R(1 - \cos\theta) - WR\left\{1 - \cos(\theta - \alpha)\right\}$$

$$\therefore \frac{\partial T_x}{\partial M_o} = - \sin\theta$$

$$\frac{\partial T_x}{\partial T_o} = \cos\theta$$

$$\frac{\partial T_x}{\partial F_o} = R(1 - \cos\theta)$$

By substituting the above values in equations 14, 15 and 16 the following results:

$$\frac{1}{EI}\int\left\{M\dot{o} \cos \theta + T\dot{o} \sin \theta - F\dot{o}\, R \sin \theta + WR \sin (\theta - \ )\right\} \cos \theta \, d \theta$$

$$-\frac{1}{GJ}\int\left[- M\dot{o} \sin \theta + T\dot{o} \cos \theta + F\dot{o} \,.\, R(1 - \cos \theta) - WR\left\{1 - \cos (\theta - \alpha)\right\}\right]$$

$$\sin \theta \, d \theta = 0 \qquad \ldots (21)$$

$$\frac{1}{EI}\int\left\{M\dot{o} \cos \theta + T\dot{o} \sin \theta - F\dot{o}\, R \sin \theta + WR\left\{\sin (\theta - \alpha)\right\}\right\}\sin \theta \, d \theta$$

$$+ \frac{1}{GJ}\int\left[\left\{- M\dot{o} \sin \theta + T\dot{o} \cos \theta + F\dot{o}\, R\,(1 - \cos \theta) - WR\left\{1 - \cos(\theta - \alpha)\right\}\right]\right.$$

$$\cos \theta \, d \theta = 0 \qquad \ldots (22)$$

$$- \frac{1}{EI}\int\left\{M\dot{o} \cos \theta + T\dot{o} \sin \theta - F\dot{o}\, R \sin \theta + WR \sin (\theta - \alpha)\right\}\sin \theta \, d \theta$$

$$+ \frac{1}{GJ}\int\left[- M\dot{o} \sin \theta + T\dot{o} \cos \theta + F\dot{o}\, R(1-\cos \theta) - WR\left\{1 - \cos(\theta - \alpha)\right\}\right)$$

$$(1 - \cos \theta) \, d \theta = 0 \qquad \ldots (23)$$

Where the integration extends from $\theta = 0$ to $\theta = \emptyset$ for the term in Mo, To and Fo and from $\theta =$ to $\theta = \emptyset$ for W

For solution of equation 21

$$\frac{1}{EI}\int\overbrace{\left\{M\dot{o} \cos \theta + T\dot{o} \sin \theta - F\dot{o}R \sin \theta + WR \sin (\theta - \alpha)\right\} \cos\theta d \theta}^{A}$$

$$- \frac{1}{GJ}\int\left[\overbrace{-M\dot{o} \sin \theta + T\dot{o} \cos \theta + F\dot{o}R\,(1 - \cos \theta) - WR\left\{1 - \cos (\theta - \alpha)\right\}}^{B}\right]$$

$$\sin \theta \, d \theta$$

The solution of the first integral (A) is:

$$\frac{1}{EI}\int\left\{M\dot{o} \cos \theta + T\dot{o} \sin \theta - F\dot{o}R \sin \theta + WR \sin (\theta - \alpha)\right\} \cos \theta \, d \theta$$

$$= \frac{1}{EI}\left[\frac{M\dot{o}}{4} \left(\sin 2\theta + \frac{\theta}{2}\right) - T\dot{o} \cos 2\theta + \frac{F\dot{o}R}{4} \cos 2\theta\right]_{o}^{\emptyset}$$

$$+ \frac{1}{EI} WR\left\{\cos\alpha\left(-\frac{\cos 2\theta}{4}\right) - \sin\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right)\right\}_{\alpha}^{\phi}$$

$$= \frac{1}{4EI} \quad M\dot{o}(\sin 2\phi + 2\phi - T\dot{o}(\cos 2\phi - 1) + F\dot{o}R(\cos 2\phi - 1)$$

$$- \frac{WR}{4EI}\left[\cos\alpha \cos 2\phi - \cos\alpha + \sin\alpha(2\phi + \sin 2\phi)\right]$$

$$\therefore A = \frac{1}{4EI}\Big[M\dot{o}(\sin 2\theta + 2\phi) - T\dot{o}(\cos 2\phi - 1) + F\dot{o}R(\cos 2\phi - 1)$$

$$- WR \quad 2\phi \sin\alpha + \cos(2\phi-\alpha) + 2\phi\sin\alpha - \cos\alpha - 2\alpha\sin\alpha\}\Big]$$

The solution of the second term of the integral (B) is:

$$-\frac{1}{GJ}\int\Big[-M\dot{o}\sin\theta + T\dot{o}\cos\theta + F\dot{o}R(1-\cos\theta) - WR\{1-\cos(\theta-\alpha)\}\Big]$$

$$\sin\theta\, d\theta = -\frac{1}{GJ}\int_0^\phi\{-M\dot{o}\sin^2\theta + T\dot{o}\cos\theta\sin\theta + F\dot{o}R(\sin\theta - \cos\theta$$

$$\sin\theta)\; d\theta - \frac{1}{GJ}\int_\alpha^\phi - WR\{1 - (\cos\theta\cos\alpha + \sin\theta\sin\alpha\}\sin\theta\, d\theta$$

$$= -\frac{1}{GJ}\int_0^\phi\left\{\frac{M\dot{o}}{2}(\cos 2\theta - 1) + \frac{T\dot{o}}{2}\sin 2\theta + F\dot{o}R\left(\sin\theta - \frac{\sin 2\theta}{2}\right)\right\}d\theta$$

$$-\frac{1}{GJ}\int_\alpha^\phi -WR\left\{\sin\theta - \cos\alpha\frac{\sin 2\theta}{2} - \sin\alpha\left(\frac{1-\cos 2\alpha}{2}\right)\right\}d\theta$$

$$= -\frac{1}{GJ}\left\{\frac{M\dot{o}}{4}(\sin 2\theta - 2\theta) - \frac{T\dot{o}}{4}\cos 2\theta + \frac{F\dot{o}R}{4}(-4\cos\theta + \cos 2\theta)\right\}_0^\phi$$

$$-\frac{WR}{GJ}\left\{-\cos\theta + \frac{\cos\alpha\cos 2\theta}{4} - \frac{\sin\alpha}{4}(2\theta - \sin 2\theta)\right\}_\alpha^\phi$$

Substituting and collecting terms

$$B = \frac{1}{4GJ}\Big[\{M\dot{o}(2\phi - \sin 2\phi) + T\dot{o}(\cos 2\phi - 1) + F\dot{o}R(4\cos\phi - \cos 2\phi - 3)\}$$

$$+\{WR(\cos(2\phi-\alpha) - 4\cos\phi\cos\phi + 3\cos\alpha - 2(\phi-\alpha)\sin\alpha\}\Big]$$

$$\therefore A + B = M\dot{o}\;\frac{1}{EI}(\sin 2\phi + 2\phi) + \frac{1}{GJ}(2\phi - \sin 2\phi)$$

$$+ \dot{T}\left[\left(\frac{1}{EI} - \frac{1}{GJ}\right)(\cos 2\phi - 1)\right] + \dot{F}R\left[\frac{1}{EI}(\cos 2\phi - 1) + \frac{1}{GJ}(4\cos\phi - \cos 2\phi - 3)\right] - WR\left[\frac{1}{EI}\left\{\cos(2\phi - \alpha) + 2(\phi - \alpha)\sin\alpha - \cos\alpha\right\} + \frac{1}{GJ}\left\{4\cos\phi - 3\cos\alpha + 2(\phi - \alpha)\sin\alpha - \cos 2\phi - \alpha\right\}\right] = 0$$

Multiplying both sides of the equation by EI and substituting K for $\frac{EI}{GJ}$ in every term

$$\dot{M}\left\{2\phi\,(K + 1) - (K - 1)\sin 2\phi\right\} - \dot{T}\left\{(K-1)(1 - \cos 2\phi\right\}$$
$$+ \dot{F}R\left\{4K\cos\phi - (K - 1)\cos 2\phi - 1 - 3K\right\}$$
$$- WR\left\{2(K + 1)\phi\sin\alpha - 2(K + 1)\alpha\sin\alpha - (K-1)\cos 2\phi\cos\alpha - (K-1)\sin 2\phi\sin\alpha - (1 + 3K)\cos\alpha + 4K\cos\phi\right\} = 0 \quad \ldots (24)$$

For solution of equation 22

$$\frac{1}{EI}\int \overbrace{\left\{\dot{M}\cos\theta + \dot{T}\sin\theta - \dot{F}R\sin\theta + WR\sin(\theta - \alpha)\right\}\sin\theta\, d\theta}^{A}$$
$$+ \frac{1}{GJ}\int \overbrace{\left[-\dot{M}\sin\theta + \dot{T}\cos\theta + \dot{F}R(1 - \cos\theta) - WR\left\{1 - \cos(\theta - \alpha)\right\}\right]}^{B}$$
$$\cos\theta\, d\theta = 0$$

For solution of A

$$\frac{1}{EI}\int_0^{\phi}\left\{\dot{M}\cos\theta\sin\theta + \dot{T}\sin^2\theta - \dot{F}R\sin^2\theta\right\}d\theta$$

$$+ \frac{1}{EI}\int_{\alpha}^{\phi} WR\,(\cos\alpha\sin^2\theta - \sin\alpha\cos\theta\sin\theta)\}\, d\theta$$

$$= \frac{1}{EI}\left[-\frac{\dot{M}\cos 2\theta}{4}\right]_0^{\phi} + \dot{T}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\phi} - \dot{F}R\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\phi}$$

$$+ \frac{1}{EI}\, WR \left[\cos\alpha \left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right) + \frac{\sin\alpha \cos 2\theta}{4} \right]_{\alpha}^{\phi}$$

$$= \frac{1}{4EI} \Big[Mo(1 - \cos 2\phi) + To(2\theta - \sin 2\phi) + FoR(\sin 2\phi - 2\phi)$$

$$+ WR \Big\{ \cos\alpha\, (2\phi - \sin 2\alpha) - \sin\alpha \cos 2\alpha - \cos\alpha (2\alpha - \sin 2\alpha)$$

$$- \sin\alpha \cos 2\alpha \Big\} \Big] = \frac{1}{4E}\ Mo \Big[(1 - \cos 2\phi) + To(2\phi - \sin 2\phi) + FoR(\sin 2\phi - 2\phi)$$

$$+ WR \Big\{ 2(\phi - \alpha) + 2 \cos\phi \sin(\alpha - \phi) \Big\} \Big]$$

For solution of B

$$\frac{1}{GJ} \int_{0}^{\phi} \Big[- Mo \sin\theta + To \cos\theta + FoR(1 - \cos\theta) \cos\theta \, d\theta$$

$$- \frac{1}{GJ} \int_{\alpha}^{\phi} WR \Big\{ 1 - \cos(\theta - \alpha) \Big\} \cos\theta \, d\theta$$

$$= \frac{1}{GJ} \left[\frac{Mo}{4} \cos 2\theta + \frac{To}{4}(2\theta + \sin 2\theta) + \frac{FoR}{4} \Big\{ 4 \sin\theta - 2\theta - \sin 2\theta \Big\} \right]_{0}^{\phi}$$

$$- \frac{1}{GJ}\ WR \left[\sin\theta - \cos\alpha \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right) + \sin\alpha \frac{\cos 2\theta}{4} \right]_{\alpha}^{\phi}$$

$$= \frac{1}{4GJ} \Big[Mo(\cos 2\phi - 1) + To\,(2\phi + \sin 2\phi) + FoR\,(4 \sin\phi - 2\phi - \sin 2\phi)$$

$$- WR \Big\{ 4(\sin\phi - \sin\alpha) - 2(\phi - \alpha) \cos\alpha - \sin(2\phi - \alpha) + \sin\alpha \Big\} \Big]$$

By adding the solution of A and the solution of B and collecting the common terms:

$$\mathring{Mo} \left(\frac{1}{EI} - \frac{1}{GJ}\right)(1 - \cos 2\phi) + \mathring{To} \Big\{ \frac{1}{EI}(2\phi - \sin 2\phi) + \frac{1}{GJ}(2\phi + \sin 2\phi) \Big\}$$

$$+ \mathring{Fo}R \Big\{ \frac{1}{EI}(\sin 2\phi - 2\phi) + \frac{1}{GJ}(4 \sin\phi - \sin 2\phi - 2\phi \Big\}$$

$$+ WR \Big[\frac{1}{EI} \Big\{ 2(\phi - \alpha) \cos\alpha + 2 \cos\phi \sin(\alpha - \phi) \quad - \frac{1}{GJ} \Big\{ 4 \sin\phi - 3 \sin\alpha$$

$$- 2(\phi - \alpha) \cos\alpha - \sin(2\phi - \alpha) \Big\} \Big] = 0$$

Multiplying both sides of the equation by EI, substitute $\frac{EI}{GJ}$ for K and collecting the common terms:

$$M\dot{o}\left\{(K-1)(\cos 2\phi - 1)\right\} + T\dot{o}\left\{2\phi\,(K+1) + (K-1)\sin 2\phi\right\}$$

$$+ F\dot{o}R\left\{4K\sin\phi - (K-1)\sin 2\phi - 2\phi\,(K+1)\right\}$$

$$+ WR\left\{2\phi\,(K+1)\cos\alpha - 2\alpha\,(1+K)\cos\alpha - (K-1)\cos 2\phi\sin\alpha\right.$$

$$\left. -4K\sin\phi + (1+3K)\sin\alpha + (K-1)\sin 2\phi\cos\alpha\right\} = 0 \quad \ldots (25)$$

For solution of equation 23

$$0 = \frac{1}{EI}\int\left\{M\dot{o}\cos\theta + T\dot{o}\sin\theta - F\dot{o}R\sin\theta + WR\sin(\theta-\alpha)\right\}\sin\theta\, d\theta$$

$$+ \frac{1}{GJ}\int\left[-M\dot{o}\sin\theta + T\dot{o}\cos\theta + F\dot{o}R(1-\cos\theta) - WR\left\{1-\cos(\theta-\alpha)\right\}\right]$$

$$(1-\cos\theta)\,d\theta$$

Adding equation 18 to this equation

$$0 = \frac{1}{GJ}\int - M\dot{o}\sin\theta + T\dot{o}\cos\theta + F\dot{o}R(1-\cos\theta)\,d\theta - WR\left\{1-\cos\right.$$

$$\left.(\theta-\alpha)\right\}\,d\phi$$

$$\therefore\ \frac{1}{GJ}\int_{o}^{\phi} - M\dot{o}\sin\theta + T\dot{o}\cos\theta + F\dot{o}R(1-\cos\theta)\,d\theta$$

$$-\frac{1}{GJ}\int_{\alpha}^{\phi} WR\left\{1-\cos(\theta-\alpha)\right\}d\theta = 0$$

Dividing the equation by $\frac{1}{GJ}$ and integrating

$$-\Big[M\dot{o}\cos\theta\Big]_{o}^{\phi} + \Big[T\dot{o}\sin\theta\Big]_{o}^{\phi} - F\dot{o}R\ \ \theta - \sin\Big]_{o}^{\phi} + WR\Big[\theta - \sin(\theta-\alpha)\Big]^{\phi}$$

$\therefore$ $M\acute{o}\ (1 - \cos \phi) - T\acute{o} \sin \phi - F\acute{o}R\ (\phi - \sin \phi)$

$$+ WR \left\{ \phi - \alpha - \sin (\phi - \alpha) \right\} = 0 \qquad \ldots (26)$$

Solving equations 24, 25, and 26 simultaneously for various values of K, ϕ and for each point to get the results shown in Tables 1, 2, 3 and 4.

E. Fixed End Beam Loaded with Uniformly Distributed Load over the Entire or Part of the Beam.

If the girder carries a uniformly distributed load the terms in equation 21, 22, 23 and 24, 25, 26 for Mo, To and Fo are unchanged except those involving W, which require modification.

Let the beam AB in Fig. 23 carry a uniformly distributed load of intensity w. Let dm_x and dt_x be the bending moment and torque at any point x due to an element of load wds at C, the angular distance of C from OB being

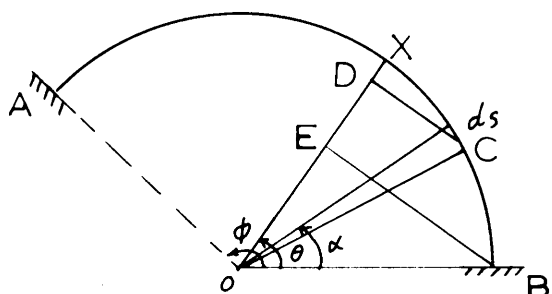


Fig. 23 - Horizontal Curved Beam Fixed at the Ends and Loaded with Uniformly Distributed Load Along Part of the Beam.

$$dm_x = CD \cdot wds = R^2 \sin (\theta - \alpha)\ d\alpha$$

$$dt_x = - XD \cdot wds = -R^2 w \left\{ 1 - \cos (\theta - \alpha) \right\} d\alpha$$

By integrating the distributed load along the beam, the bending moment and the torque at x due to the distributed load between B and x is then

Table 1. Fixed End Moment Coefficient for Concentrated Load for K = 1.33

K = 1.33

ϕ	α	$\frac{M_o}{WR}$	$\frac{T_o}{WR}$	$\frac{F_o}{W}$
	0	0	0	1
	15	0.1536	0.0020	0.8463
60	30	0.1398	0.0034	0.5000
	45	0.0526	0.0017	0.1534
	60	0	0	0
	0	0	0	1
	30	0.2064	0.0079	0.6518
75	45	0.1390	0.0071	0.3490
	60	0.0458	0.0028	0.0997
	75	0	0	0
	0	0	0	1
	30	0.2591	0.0136	0.7475
90	45	0.2231	0.0160	0.5000
	60	0.1321	0.0116	0.2524
	90	0	0	0
	0	0	0	1
	30	0.3020	0.0207	0.8115
105	60	0.2258	0.0263	0.3891
	90	0.0360	0.0054	0.0497
	105	0	0	0

Table 1. (cont'd)

K = 1.33

ϕ	α	$\frac{M_o}{WR}$	$\frac{T_o}{WR}$	$\frac{F_o}{W}$
120	0	0	0	1
	30	0.3386	0.0283	0.8581
	60	0.3151	0.0459	0.5000
	90	0.1151	0.0222	0.1441
	120	0	0	0
135	0	0	0	1
	30	0.3474	0.0495	0.8639
	60	0.3968	0.0697	0.5889
	90	0.2108	0.0494	0.2446
	120	0.0220	0.0121	0.0194
	135	0	0	0
150	0	0	0	1
	30	0.3897	0.0462	0.9124
	60	0.4698	0.0969	0.6605
	90	0.3112	0.0862	0.3394
	120	0.0953	0.0322	0.0875
	150	0	0	0

Table 1. (cont'd)

K = 1.33

ϕ	α	$\frac{M_o}{WR}$	$\frac{T_o}{WR}$	$\frac{F_o}{W}$
	0	0	0	1
	30	0.4093	0.0552	0.9307
	60	0.5336	0.1264	0.7184
165	90	0.4014	0.1309	0.4246
	120	0.1823	0.0712	0.1596
	150	0.0215	0.0101	0.0163
	165	0	0	0
	0	0	0	1
	30	0.4248	0.0667	0.9457
	60	0.5890	0.1668	0.7657
180	90	0.5000	0.1817	0.5000
	120	0.2515	0.1178	0.2362
	150	0.0748	0.0332	0.0510
	180	0	0	0

Table 2. Fixed End Moment Coefficients for Concentrated Load for K = 2

K = 2

ϕ	α	$\frac{M_o}{WR}$	$\frac{T_o}{WR}$	$\frac{F_o}{W}$
60	0	0	0	1
	15	0.1547	0.0024	0.6470
	30	0.1409	0.0041	0.5000
	45	0.0543	0.0022	0.1529
	60	0	0	0
75	0	0	0	1
	30	0.2047	0.0990	0.6526
	45	0.1431	0.0061	0.3465
	60	0.0456	0.0032	0.0989
	75	0	0	0
90	0	0	0	1
	30	0.2622	0.0151	0.7466
	45	0.2245	0.0162	0.5000
	60	0.1326	0.0064	0.2512
	90	0	0	0
105	0	0	0	1
	30	0.3043	0.0227	0.8131
	60	0.2265	0.0263	0.3912
	90	0.0355	0.0056	0.0453
	105	0	0	0

Table 2. (cont'd)

K = 2

ϕ	α	$\frac{M_o}{W R}$	$\frac{T_o}{W R}$	$\frac{F_o}{W}$
120	0	0	0	1
	30	0.3402	0.0302	0.8597
	60	0.3163	0.0493	0.5000
	90	0.1139	0.0227	0.1297
	120	0	0	0
135	0	0	0	1
	30	0.3514	0.0501	0.8674
	60	0.3981	0.0725	0.5844
	90	0.2109	0.0514	0.2389
	120	0.0236	0.0114	0.0197
	135	0	0	0
150	0	0	0	1
	30	0.3922	0.0501	0.9141
	60	0.4718	0.0984	0.6618
	90	0.3097	0.0944	0.3381
	120	0.0833	0.0141	0.0855
	150	0	0	0

Table 2. (cont'd)

K = 2

ϕ	α	$\frac{M_o}{WR}$	$\frac{T_o}{WR}$	$\frac{F_o}{W}$
165	0	0	0	1
	30	0.4118	0.0573	0.9323
	60	0.5363	0.1278	0.7201
	90	0.4018	0.1314	0.4239
	120	0.1793	0.0675	0.1571
	150	0.0207	0.0091	0.0156
	165	0	0	0
180	0	0	0	1
	30	0.4273	0.0674	0.9387
	60	0.5920	0.1676	0.7678
	90	0.5000	0.1816	0.5000
	120	0.2514	0.1168	0.2342
	150	0.0723	0.0325	0.0499
	180	0	0	0

Table 3. Fixed End Moment Coefficients for Concentrated Load for K = 4

K = 4

ϕ	α	$\frac{M_o}{WR}$	$\frac{T_o}{RW}$	$\frac{F_o}{W}$
60	0	0	0	1
	15	0.1582	0.0031	0.8493
	30	0.1448	0.0058	0.5000
	45	0.0529	0.0029	0.1512
	60	0	0	0
75	0	0	0	1
	30	0.2155	0.0137	0.6571
	45	0.1435	0.0116	0.3465
	60	0.0459	0.0048	0.0955
	75	0	0	0
90	0	0	0	1
	30	0.2698	0.0208	0.7532
	45	0.2305	0.0239	0.5000
	60	0.1332	0.0141	0.2463
	90	0	0	0
105	0	0	0	1
	30	0.3137	0.0291	0.8191
	60	0.2295	0.0308	0.3858
	90	0.336	0.0062	0.0487
	105	0	0	0

Table 3. (cont'd)

K = 4

ϕ	α	$\frac{M_o}{WR}$	$\frac{T_o}{WR}$	$\frac{F_o}{W}$
120	0	0	0	1
	30	0.3498	0.0372	0.8671
	60	0.3265	0.0561	0.5000
	90	0.1093	0.0243	0.1356
	120	0	0	0
135	0	0	0	1
	30	0.3665	0.0521	0.8806
	60	0.4033	0.0836	0.5855
	90	0.2113	0.0694	0.2414
	120	0.0301	0.0117	0.208
	135	0	0	0
150	0	0	0	1
	30	0.4011	0.05105	0.9205
	60	0.4798	0.1083	0.6665
	90	0.3055	0.0894	0.3331
	120	0.0859	0.0212	0.0799
	150	0	0	0

Table 3. (cont'd)

K = 4

ϕ	α	$\frac{M_o}{WR}$	$\frac{T_o}{WR}$	$\frac{F_o}{W}$
165	0	0	0	1
	30	0.4203	0.0632	0.9379
	60	0.5456	0.1316	0.7267
	90	0.4041	0.1329	0.4207
	120	0.1688	0.0664	0.1505
	150	0.0176	0.0082	0.0135
	165	0	0	0
180	0	0	0	1
	30	0.4377	0.0699	0.9418
	60	0.6011	0.1704	0.7742
	90	0.5000	0.1616	0.5000
	120	0.2513	0.1132	0.2269
	150	0.0645	0.0302	0.0466
	180	0	0	0

Table 4. Fixed End Moment Coefficients for Concentrated Load for K = 10.67

K = 10.67

ϕ	α	$\frac{M_o}{WR}$	$\frac{T_o}{WR}$	$\frac{F_o}{W}$
60	0	0	0	1
	15	0.1632	0.0061	0.8530
	30	0.1510	0.0096	0.5000
	45	0.0562	0.0047	0.1488
	60	0	0	0
75	0	0	0	1
	30	0.2262	0.0194	0.6624
	45	0.1485	0.0162	0.3440
	60	0.0460	0.0059	0.0914
	75	0	0	0
90	0	0	0	1
	30	0.2805	0.0280	0.7599
	45	0.2379	0.0308	0.5000
	60	0.1341	0.0206	0.2401
	90	0	0	0
105	0	0	0	1
	30	0.3254	0.0373	0.8269
	60	0.2331	0.0418	0.3824
	90	0.0313	0.0071	0.0410
	105	0	0	0

Table 4. (cont'd)

K = 10.67

ϕ	α	$\frac{M_o}{WR}$	$\frac{T_o}{WR}$	$\frac{F_o}{W}$
120	0	0	0	1
	30	0.3610	0.0462	0.8728
	60	0.3265	0.0657	0.5000
	90	0.1035	0.0263	0.1270
	120	0	0	0
135	0	0	0	1
	30	0.3856	0.0547	0.8973
	60	0.4097	0.0935	0.5838
	90	0.2117	0.0892	0.2382
	120	0.0382	0.0112	0.0221
	135	0	0	0
150	0	0	0	1
	30	0.4125	0.0628	0.9286
	60	0.4897	0.1192	0.6728
	90	0.2990	0.0925	0.3271
	120	0.0764	0.0276	0.0719
	150	0	0	0

Table 4. (cont'd)

K = 10.67

ϕ	α	$\frac{M_o}{WR}$	$\frac{T_o}{WR}$	$\frac{F_o}{W}$
165	0	0	0	1
	30	0.4307	0.0710	0.9453
	60	0.5572	0.1367	0.7341
	90	0.4085	0.1350	0.4186
	120	0.1558	0.0618	0.1405
	150	0.0136	0.0064	0.0107
	165	0	0	0
180	0	0	0	1
	30	0.4456	0.0765	0.9578
	60	0.6157	0.1740	0.7830
	90	0.5000	0.1816	0.5000
	120	0.2512	0.1086	0.2176
	150	0.0543	0.0273	0.0421
	180	0	0	0

$$m_x^{\circ} = R^2 w \int_0^{\theta} \sin(\theta - \alpha)\, d\alpha = R^2 w\,(1 - \cos\theta)$$

$$t_x^{\circ} = -R^2 w \int_0^{\theta} \left\{ 1 - \cos(\theta - \alpha) \right\} d\alpha = -R^2 w\,(\theta - \sin\theta)$$

Note: If the beam is partially loaded with w then the lower limit of integration for α will be the angular distance from which the load started to θ.

Now instead of the terms for W in equations 21, 22 and 23 substitute respectively:

$$R^2 w \left[\frac{1}{EI} \int_0^{\phi} (1 - \cos\theta) \cos\theta\, d\theta + \frac{1}{GJ} \int_0^{\phi} (\theta - \sin\theta) \sin\theta\, d\theta \right]$$

$$R^2 w \left[\frac{1}{EI} \int_0^{\phi} (1 - \cos\theta) \sin\theta\, d\theta - \frac{1}{GJ} \int_0^{\phi} (\theta - \sin\theta) \cos\theta\, d\theta \right]$$

and

$$-R^2 w \left[\frac{1}{EI} \int_0^{\phi} (1 - \cos\theta) \sin\theta\, d\theta + \frac{1}{GJ} \int_0^{\phi} (\theta - \sin\theta)(1 - \cos\theta)\, d\theta \right]$$

Then the general equation for circular arc beam with uniformly distributed load along the entire beam will be

$$M_o \left[\frac{1}{EI}(2\phi + \sin 2\phi) + \frac{1}{GJ}(2\phi - \sin 2\phi) \right] + T_o \left[\left(\frac{1}{EI} - \frac{1}{GJ}\right)(1 - \cos 2\phi) \right]$$

$$+ F_o R \left[\frac{1}{EI}(\cos 2\phi - 1) + \frac{1}{GJ}(4\cos\phi - \cos 2\phi - 3) \right]$$

$$+ wR^2 \left[\frac{1}{EI}(4\sin\phi - 2\phi - \sin 2\phi) + \frac{1}{GJ}(4\sin\phi - 2\phi + \sin 2\phi - 4\phi\cos\phi) \right] = 0 \qquad \ldots (27)$$

$$M_o \left(\frac{1}{EI} - \frac{1}{GJ}\right)(1 - \cos 2\phi) + T_o \left[\frac{1}{EI}(2\phi - \sin 2\phi) + \frac{1}{GJ}(2\phi + \sin 2\phi)\right]$$

$$+ F_oR \left[\frac{1}{EI}(\sin 2\phi - 2\phi) + \frac{1}{GJ}(4 \sin \phi - \sin 2\phi - 2\phi)\right]$$

$$+ wR^2 \left[\frac{1}{EI}(3 + \cos 2\phi - 4 \cos \phi) - \frac{1}{GJ}(\cos 2\phi + 4 \cos \phi - 5 + 4\phi \sin \phi)\right]$$

$$= 0 \qquad \ldots (28)$$

$$M_o(1 - \cos \phi) - T_o \sin \phi - F_oR(\phi - \sin \phi) + wR^2\left(\frac{\phi^2}{2} + \cos \phi - 1\right) = 0 \qquad \ldots (29)$$

IX. THEORIES OF TORSION

If a prism is subjected to torque the motion of any particle in that prism consists of:

1. Rotation about the axis of twist in a plane perpendicular to the axis.
2. A movement parallel to the axis of twist.

Thus the plane sections will not generally remain plane and the problem of calculating the stress and strain in terms of the twisting moment is in the general case extremely difficult.

In the particular case of circular cross-sections, plane sections remain plane and the problem becomes simpler, giving the relation

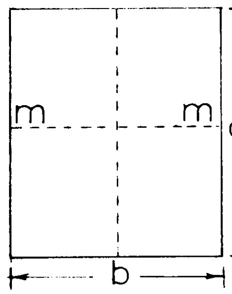
$$\frac{T}{J} = \frac{G\theta}{L}$$

With rectangular sections, however, plane sections do not remain plane and the problem is a difficult one. The classic solution of the general case is that of Sant-Venant[14] and the application of his theory to a rectangle of sides (b) and (d) in the case of an elastic material gives

$$T = S_1 \frac{\frac{bd^3}{3} - \frac{64b^4}{5}\left(\tan h\frac{\pi d}{2b} + \frac{1}{3^2}\tan h\frac{3\pi d}{2b} + \text{etc.}\right)}{b - \frac{8b}{2}\left(\text{Sec } h\frac{\pi d}{2b} + \frac{1}{3^2}\sec h\frac{3\pi d}{2b} + \text{etc.}\right)}$$

where S_1 = maximum shear stress

He also shows that due to the distortion of the cross-section, the maximum shear occurs at the midpoint of the longer side; and the torsional stress at points on the corners become zero.



The maximum stress will occur at the points m and m in the middle of the longer side.

The formula of Sant-Venant is complicated and others have presented simplifications. Sant-Venant[14] himself gives an empirical formula in which he states

$$T = \frac{b^2d^2}{1.8b + 3d} \quad S_1$$

Timoshenko[14] expresses the relationship as

$$S_1 = \frac{T}{k_1db^2}$$

in which k_1 is a constant depending on the magnitude ratio of d/b and the angle of twist can be obtained from the equation:

$$\theta = \frac{T\ .}{k_2Gb^2d}$$

where θ is the angle of twist, k_2 is a constant and is length of the beam.

Timoshenko gives values for k_1 and k_2 as shown.

$\frac{d}{b}$	1	1.5	1.75	2	2.5	3	4
k_1	.208	.231	.239	.246	.258	.267	.282
k_2	.141	.196	.214	.229	.249	.267	.281

Fig. 24 shows the relation between the ratio I/J and d/b by using Timoshenko's constants for rectangular sections.

The effect of torsion is to produce on an element a shear stress, tensile stress and a compressive stress. The failure of a material under torsion is governed by the ability of the material to resist shear, tension and compression. In a material which is weakest in shear the fracture is in a plane at right angles to the axis of twist. Such a fracture occurs in the case of mild steel. In a brittle material such as concrete failure occurs in tension, which in case of circular cross-sections, is along a helix inclined at 45° to the angle of twist.

W. T. Marshall[9] performed an experiment by twisting beams of rectangular cross-sections. He put a narrow strip of plaster along the center of the longer sides and no cracks appeared until the beams failed, at which time the cracks and failure occurred simultaneously. This proved that concrete does not act as an elastic material at failure. The tests showed that concrete is apparently much stronger in torsion when in a rectangular cross-section than that of a circular section.

From the above previous theoretical and experimental data, the author of this thesis makes the following important conclusions:

1. Since the relation between the torque and the angle of twist is almost the same for plain and reinforced concrete specimens of rectangular sections of the same

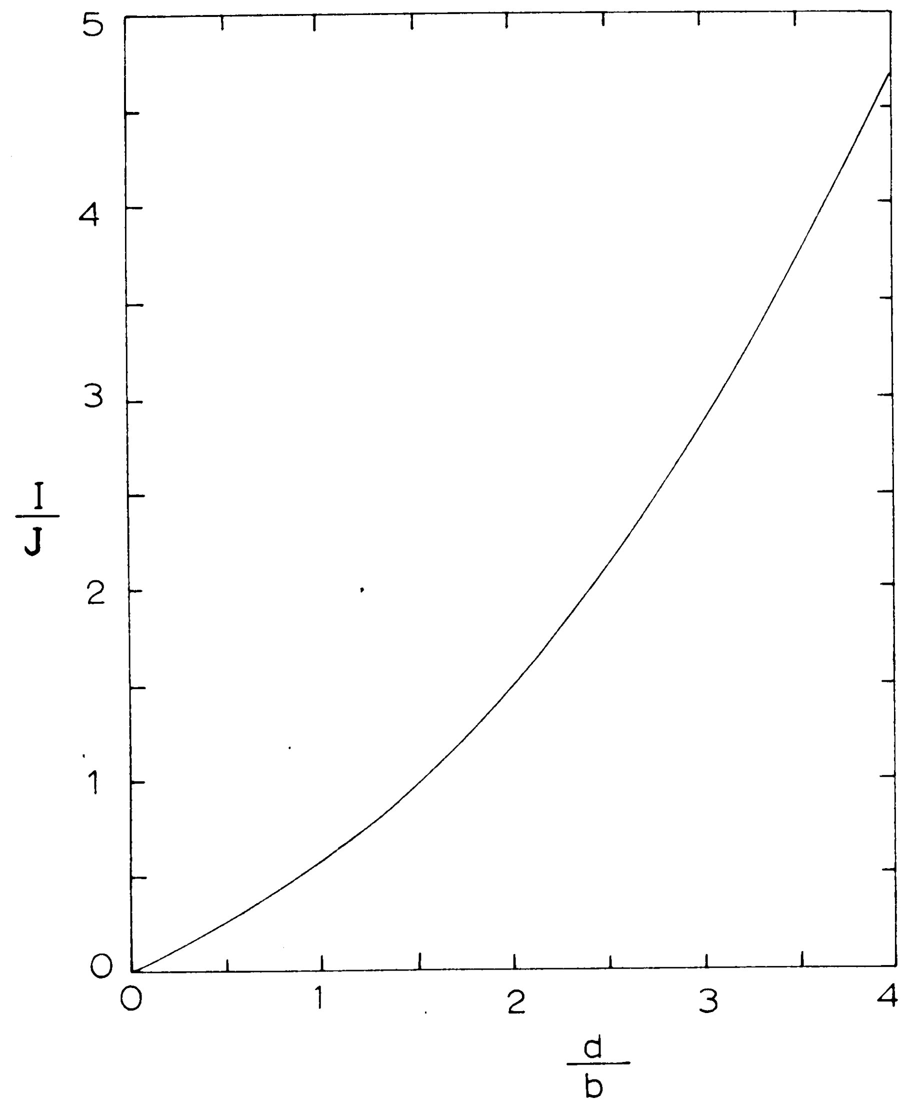


Fig. 24. I/J vs d/b for a Rectangular Section.

dimensions as shown in Fig. 1 and Fig. 2 respectively, then the modulus of rigidity G is the same in both cases, or the difference is insignificant; so if we use EI/GJ for plain concrete sections to stand for EI/GJ in reinforcement concrete sections of the same dimensions, the results will be correct or slightly on the safe side.

2. Fig. 12, 13 and 17 show that the variation in bending moment, shear and torsional moment is not related to variations in the value of K; variations in the values of K have little or no effect on the magnitude of these values. According to these figures, the magnitude of bending moment, shear and torsional moment is controlled by the radius of curvature, ϕ, and the type of loading.

From conclusion 2 above, after bending moment, shear and torsional moment are obtained, the rectangular section of concrete may be designed by either the conventional theory or the ultimate strength theory. For Illustrative Example 3 of this thesis, the author will use the conventional theory only.

Fig. 1 shows that the torque-twist graph indicates that the specimen behaves elastically to within two-thirds of the ultimate after which creep becomes very apparent, with failure occurring by developing 45° tension cracks.

The typical torque-twist curve, Fig. 2, shows three stages:

1. Straight line from zero up to about one-half the failure load.

2. Straight line which started just below failure.

3. A curved portion joining the two straight lines.

The first stage shows that the maximum torsional rigidity of plain concrete is the same as the reinforced section of the same dimensions. The third stage shows that the line is almost parallel with the angle of twist axis, showing that in this stage the beam possesses very little torsional rigidity. In this stage most of the load is taken by the reinforcement.

A. Torsional Shear Reinforcement.

If concrete is twisted, it fails along a spiral line at 45° to the axis of the material. The most effective reinforcement consists of a series of 45° spirals. Torsion differs from transverse shear in one important respect, and that is, diagonal tensile stresses exist on all four faces of a rectangular section subjected to torsion; whereas they only extend over two faces in the section subjected to direct shear. In rectangular sections subjected to torsion the maximum diagonal tensile stress occurs in the surface layer, and the stress at the core of the section is small. Direct shear sets up diagonal tensile stresses which act in the same direction on the two vertical faces of the beam, while diagonal tensile stresses due to torsion act in opposite directions on opposite faces. Hence, the diagonal tensile stresses due to torsion and due to direct shear partly cancel one another on one face and add to each other on the other face, which is the critical face in the design of the beams.

Assuming that

v_t = unit shear stress due to torsion

v_d = unit shear stress due to direct shear

then

$$v_{max} = v_t + v_d$$

The Australian Code, according to Cowan[5], limits the maximum permissible concrete stress in diagonal tension due to torsion to

$$v_t = 0.02\ f_c' + 20 \text{ psi} \leq 90 \text{ psi}$$

When this stress is exceeded, torsional reinforcement must be provided to take the excess diagonal tension, but irrespective of the amount of torsional reinforcement, the diagonal tension stress* must not exceed

$$v_{max} = 0.08\ f_c' + 80 \text{ psi}$$

The excess shear will be carried by the binder.

B. Design of Binder.

The problem of designing rectangular sections for torsion is complicated, partially because of the complexity of the theory of torsion for non-circular sections, and partially because the stress in shear reinforcement is not constant as in circular sections.

The Australian Code recommended formula was derived by H. Cowan[4] by equating the strain energy stored in the spiral and stored in the concrete in the diagonal compression area to the work done by the twisting moment in rotating the beam. This gives the following area for the spiral reinforcement for rectangular beams

---

* **The words shear stress and diagonal tension stress are used interchangeable - actually, the shear stress is used as a measure of the diagonal tension stress.**

$$a_s = \frac{2\, T_s\, s}{4 \lambda f_s\, d' b'}$$

where

a_s = cross-sectional area of one bar

b' = the width of the core

d' = the depth of the core

f_s = allowable stress in the steel

T_s = the torque carried by the reinforcement

λ = a function of d'/b'

In practice spiral reinforcement is seldom used in rectangular sections because of the following reasons:

1. Rectangular spirals are not easily made.
2. Two sets of spirals, right hand and left hand, are required in beams which are subjected to reversal of twisting moments.

It is most convenient that the shear reinforcement be provided in the form of vertical stirrups, to resist the vertical component of the diagonal tension, and longitudinal bars to resist the horizontal component of the diagonal tension.

If we assume a_{sv} is the area of the vertical stirrups and a_{sh} is the area of the horizontal bars

$$a_{sv} = \frac{a_s}{\sin 45^\circ} = \sqrt{2}\, a_s$$

$$a_{sh} = \frac{a_s}{\cos 45} = \sqrt{2}\, a_s$$

The total area $A_{sv} = 2\sqrt{2}\,a_s$

$$\text{but } a_s = \frac{2 \cdot T_s \cdot s}{4\lambda f_s\, b'd'}$$

$$\therefore A_{sv} = \frac{T_s \cdot s}{\lambda f_s\, b'd'}$$

where s is the spacing between vertical stirrups.

According to the Australian Code the value of λ does not vary greatly from 0.8 and it is sufficient for all practical purposes, therefore

$$A_{sv} = \frac{T_s \cdot s}{0.8\, f_s\, b'd'}$$

On a length of s the volume of steel in the longitudinal direction is $A_{sh} \cdot s$; the same amount of vertical steel is required to resist the vertical component of the diagonal tension. This steel contains one binder in a length of s and has a volume of $A_{sv}\,(b' + d')$. Therefore,

$$A_{sh} = \frac{A_{sv}(b' + d')}{s}$$

According to Cowan[5] the Australian Code requires at least four bars, one in each corner.

It is important to check the results by the empirical formulas derived by Marshall[9] or Turner[15] if the beam is subjected to torsion only.

C. Maximum Permissible Stress in Bending and Torsion

H. J. Cowan[5] performed two experiments, one on plain concrete and one on reinforced concrete, and the results are shown in Fig. 3 and

Fig. 4. The graphs show that in the reinforced concrete specimens, the bending moment does not reduce the capacity of the beam to resist torsion. In this respect the reinforced concrete beam behaves quite different from a steel beam or a plain concrete beam, when they are subjected to combined bending and torsional moments.

Fig. 4 shows two criteria of failure, one is the maximum stress theory of torsion, and the other is maximum internal friction theory between concrete particles. The latter is the maximum bending moment theory.

It is the opinion of the author of this thesis that if the ratio of the bending moment to the torsional moment is less than 2, then torsion is the most critical, and the design should be based on the maximum stress theory in torsion and checked by the internal friction theory; but if the ratio of the bending moment to the torsional moment is more than 2, then the bending moment is the most critical, and the section must first be designed by the internal friction theory and checked by the maximum stress theory of torsion; provided that the vertical shear stress is small in both cases.

X. ILLUSTRATIVE EXAMPLES

Illustrative Example 1

A beam carrying a uniformly distributed load of 600 plf (including the weight of the beam), the arc of the beam subtending an angle of 150°. The radius of the beam is 10 ft. and the stiffness ratio K = 3. Draw the bending and torsional moment diagrams for the beam.

Solution:

a. By using equations:

$$U = \frac{4(K + 1)\sin\frac{\phi}{2} - 2\phi K\cos\frac{\phi}{2}}{\phi\,(K + 1)-(K - 1)\sin\phi}$$

$$\phi = 150^\circ = 2.618 \text{ red}$$

$$\sin\frac{\phi}{2} = \sin 75 = 0.9659$$

$$\cos\frac{\phi}{2} = \cos 75 = 0.2588$$

$$\sin\phi = \sin 150 = 0.5$$

$$K = 3$$

Substituting these values in the above formula we get

$$U = 1.2024$$

Bending moment at the center

$$M_c = wR^2\,(1 - U)$$

$$= 600 \times 100\,(1 - 1.2024) = -12134 \text{ lb.ft.}$$

Bending moment at the supports

$$M_o = wR^2 \left(1 - U \cos \frac{\phi}{2}\right)$$

$$= 600 \times 100 \, (1 - 1.2024 \times .2588) = 41{,}329 \text{ lb.ft.}$$

Torsional moment at the ends

$$T_o = wR^2 \left(\frac{\phi}{2} - U \sin \frac{\phi}{2}\right)$$

$$= 600 \times 100 \, (1.3090 - 1.2024 \times .9659)$$

$$= 8856 \text{ lb. ft.}$$

Maximum torsional moment

$$T_{max} = wR^2 \left(\cos^{-1} \frac{1}{U} - \sqrt{1 - U^2}\right)$$

$$= 600 \times 100 \left(\cos^{-1} \frac{1}{1.2024} - \sqrt{.4457}\right)$$

$$(\cos^{-1} 0.83167 - .6676)$$

$$= 600 \times 100 \, (.58904 - .66760) = -4714 \text{ lb.ft.}$$

Point of zero moment or maximum torsion

$$\cos \; b = \frac{1}{U} = 0.83167$$

$$b = 33^\circ - 44'$$

Point of zero torsion

$$\frac{t}{\sin} = U = 1.2024$$

The solution by trial and error

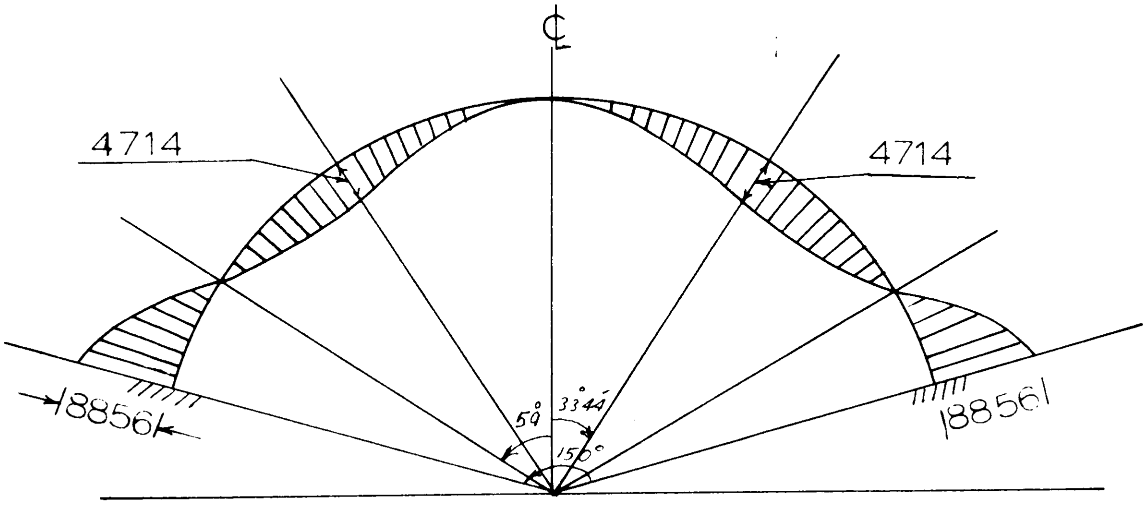
$$t = 59.5^\circ$$

b. Solution by using the graphs:

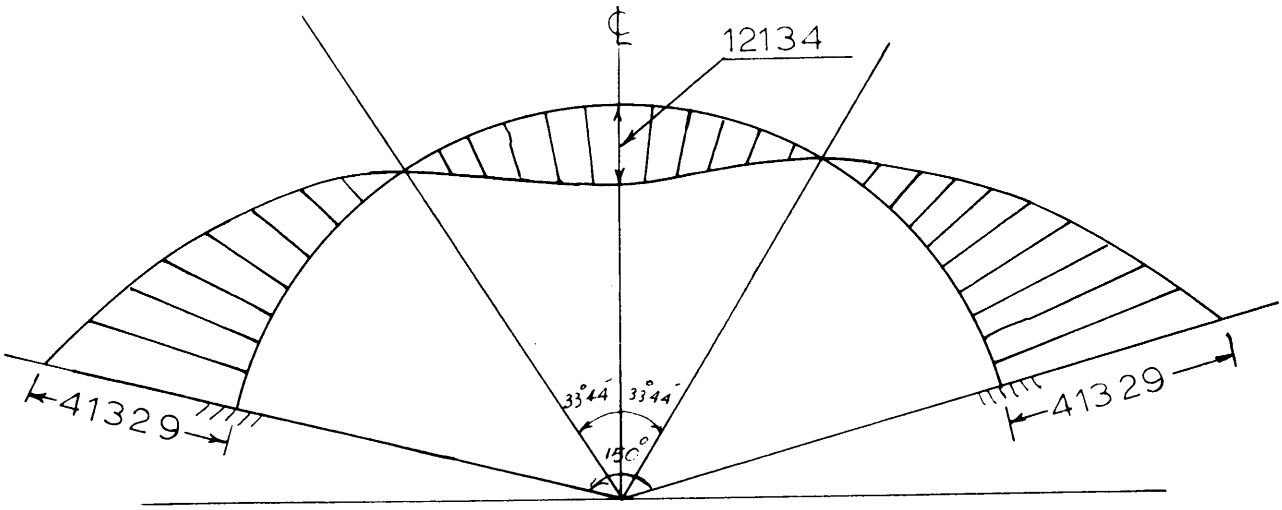
$\phi = 150$

$K = \;\; 3$

In this case interpolate between the curve K = 2 and K = 4.



Torsional Moment Diagram



Bending Moment Diagram

Moment at the center

$M_c = wR^2 \, C_5$

From the Figure, $C_5 = -0.202$

$\therefore M_c = 600 \times 100 \times 0.202 = -12120$ lb.ft.

Bending moment at the supports

$M_o = wR^2 \, C_6$

$C_6 = 0.69$

$M_o = 600 \times 100 \times (-.69) = 41400$ lb.ft.

Torsional moment at the ends

$T_o = wR^2 \, C_7$

$C_7 = .148$

$T_o = 600 \times 100 \times (.148) = 8880$ lb.ft.

Maximum torsional moment

$T_{max} = wR^2 \, C_8$

$C_8 = -0.079$

$T_{max} = 600 \times 100 \times 0.079 = -4740$ lb.ft.

Point of zero bending moment or maximum torsion

From the Figure for $\phi = 150$, $K = 3$, $b = 34^\circ$ from $\mathbb{C}_L$

Point of zero torsion

$t = 59^\circ$

Illustrative Example 2.

Draw the influence line for the bending moment, torsional moment and shear at the support for a fixed-end circular beam subtending an arc of 135°. The radius of curvature of the centerline is 12 ft. and the stiffness ratio $K = 7$.

Solution:

The tabulated values shown have been computed for a series of beams in which the radius of the circle is taken as unity. To obtain the value of the ordinate of the influence line for any function it is necessary to multiply the values given in the table by the given radius.

For the intermediate values of K, as the given value in this problem, it is possible to interpolate between the tabular values by observation, when only one dimension differs from those given in the table. When more than one differs, intermediate values can be obtained by plotting a curve or assuming a straight line variation.

From Table 3 the ordinate of the influence lines for K = 4 and $\phi = 135^{\circ}$ can be obtained. From Table 4 the ordinate of influence lines for K = 10.67 and $\phi = 135^{\circ}$ can be obtained. The interpolation will be carried out between these values as shown in the table below.

$\phi = 135$

α	K	$\frac{M_o}{R}$	$\frac{T_o}{R}$	F_o
30	4 7 10.67	0.3665 0.3751 0.3856	0.0521 0.0533 0.0547	0.8806 0.8881 0.8973
60	4 7 10.67	0.4033 0.4062 0.4097	0.0836 0.0881 0.0935	0.5855 0.5847 0.5838
90	4 7 10.67	0.2113 0.2115 0.2117	0.0694 0.0783 0.0892	0.2414 0.24 0.2382
120	4 7 10.67	0.0301 0.0337 0.0382	0.0117 0.0114 0.0112	0.0208 0.0213 0.0221

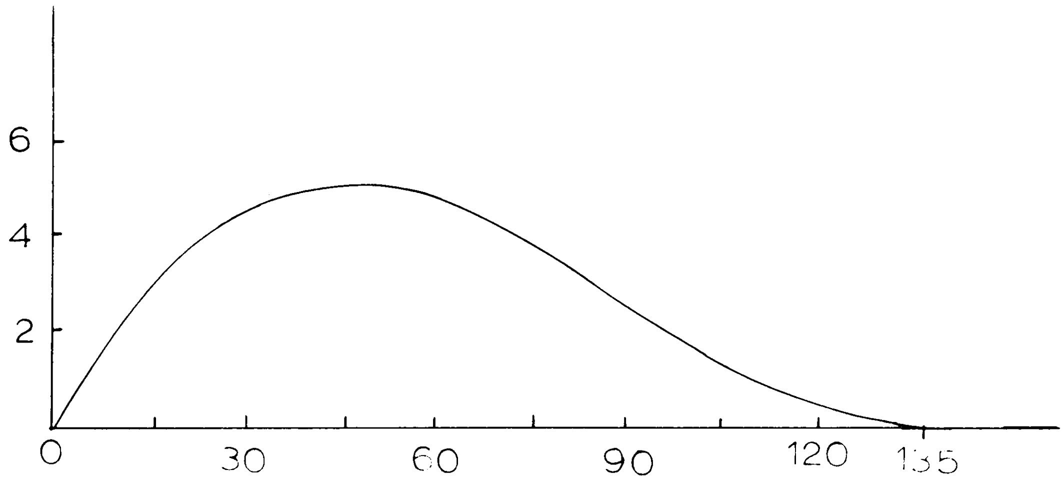
The values of the ordinate for the influence after interpolation is shown below in the table.

α	$\frac{M_o}{R}$	$\frac{T_o}{R}$	F_o
0	0.000	0.000	1.000
30	0.3751	0.0533	0.8881
60	0.4062	0.0881	0.5838
90	0.2115	0.0783	0.2400
120	0.0337	0.0114	0.0213
135	0.000	0.000	0.000

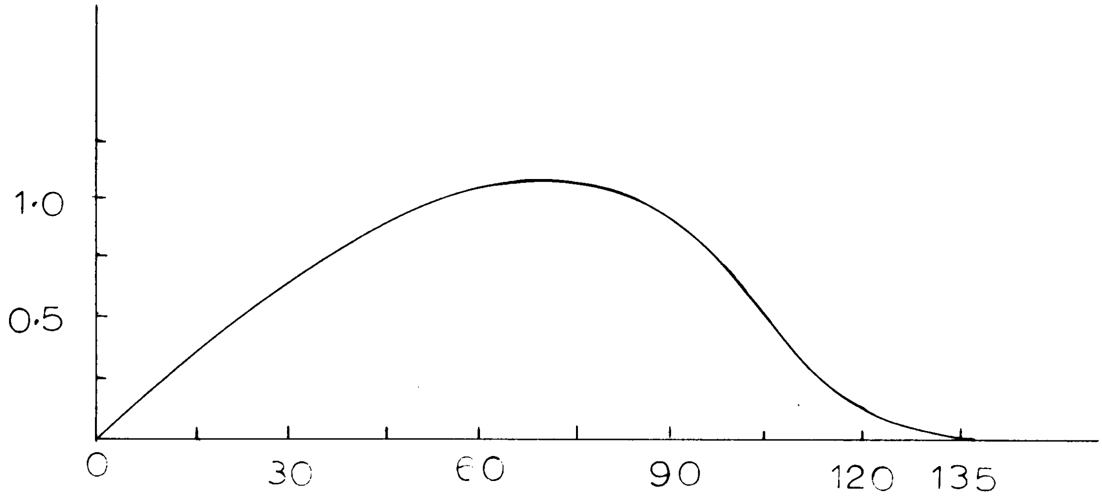
The above tabular value is true only when R is unity. To change it to the requirement of this problem, multiply the above values by R = 12 ft.

Ordinate for the influence lines when K = 7, ϕ = 135° and R = 12 ft.

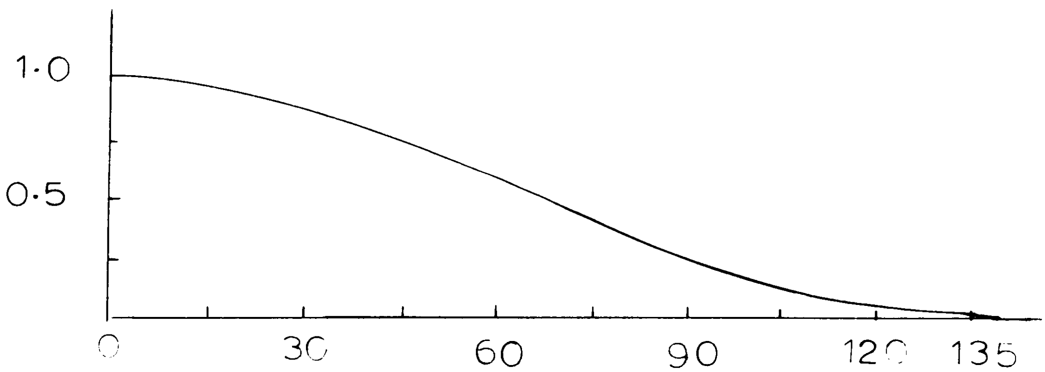
α	M_o	T_o	F_o
0	0.000	0.000	1.000
30	4.5012	0.6396	0.8881
60	4.8744	1.0572	0.5838
90	2.5380	0.9396	0.2400
120	0.4044	0.1368	0.0213
135	0.000	0.000	0.000



I.L. for the Bending Moment at the Support



I.L. for the Torsional Moment at the Support



I.L. for the Reaction Shear at the End

Illustrative Example 3.

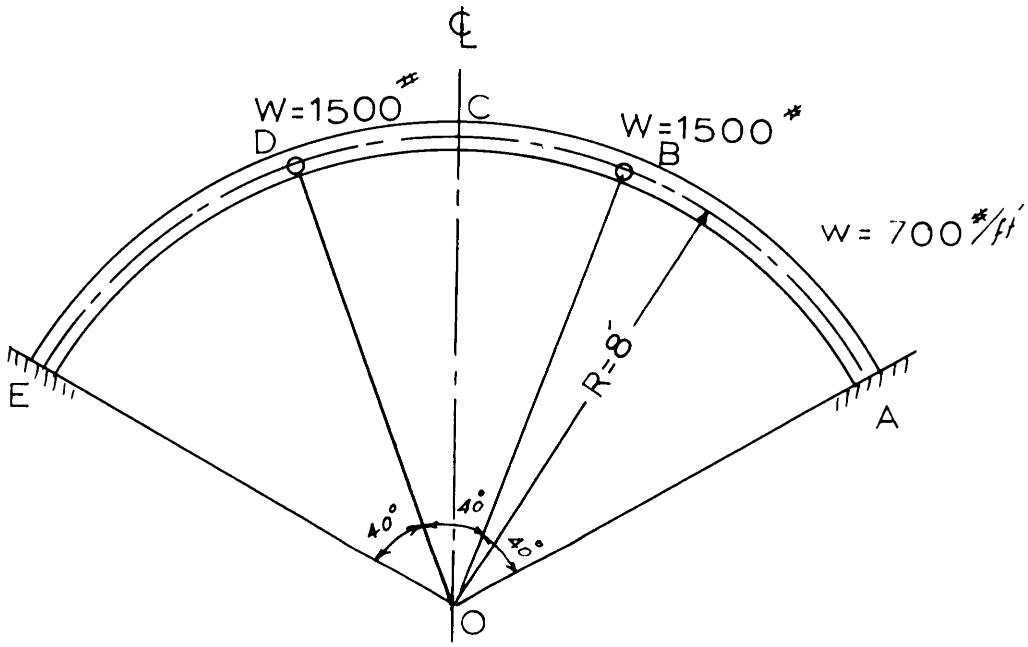
Design a concrete curved beam to carry a uniformly distributed load of 700 lb. per lin. ft. and two concentrated loads of 1500 lbs. as shown in the figure below. The radius of the beam is 8 ft., the angle subtended by the arc of the beam is 120°. Laboratory tests on the concrete gave 2.2 for ratio of modulus of elasticity to the modulus of rigidity.

$f_c' = 3000$ psi   1956 ACI Code

$f_s = 20{,}000$ psi

Solution:

Assume the weight of the beam is 200 lbs. per lin. ft. and the ratio of the depth to the width is 1.5. From Fig.24 , for d/b = 1.5,



$I/J = .96$

$K = \frac{E}{G}\frac{I}{J} = 2.2 \times .96 = 2.15$

$700 + 200 = 900$ lb. lin. ft. total uniform load

For the uniform load:

$M_o = wR^2 C_6$

C_6 from Fig. 12 = 0.43

$\therefore M_o = 900 \times 8^2 \times 0.43 = 24750$ ft. lb.

$T_o = wR^2 C_7$

C_7 from Fig. 13 = 0.058

$\therefore T_o = 900 \times 64 \times 0.058 = 3340$ ft. lb.

$F_o = wR \ C_9$

C_9 from Fig. 17 = 1.05

$\therefore F_o = 900 \times 8 \times 1.05 = 7550$ lb.

For the concentrated load:

From the symmetrical conditions $F_o = 1500$ lb.

Note: If the beam is not symmetrically loaded then F_o can be found from the tables as shown below.

Solve for each load separately; then interpolate between the rows and the columns as shown in the following tables.

M_{oA}, T_{oA}, and F_{oA} due to the load at point D

K = 2 $\quad \phi = 120$

α	$\frac{M_o}{wR}$	$\frac{T_o}{wR}$	$\frac{F_o}{w}$
60	0.3265	0.0561	0.500
80	0.1848	0.0338	0.2531
90	0.1139	0.0227	0.1297

K = 4 $\quad \phi = 120$

α	$\frac{M_o}{wR}$	$\frac{T_o}{wR}$	$\frac{F_o}{w}$
60	0.3265	0.0561	0.500
80	0.1817	0.0349	0.2571
90	0.1093	0.0243	0.1356

$\phi = 120 \quad = 80$

K	$\frac{M_o}{wR}$	$\frac{T_o}{wR}$	$\frac{F_o}{w}$
2	0.1848	0.0338	0.2531
2.15	0.1850	0.0339	0.2534
4	0.1817	0.0349	0.2571

M_{oA}, T_{oA} and F_{oA} due to load at point B

K = 2 ϕ = 120

α	$\frac{M_o}{WR}$	$\frac{T_o}{WR}$	$\frac{F_o}{W}$
30	0.3402	0.0302	0.8597
40	0.3322	0.0366	0.7398
60	0.3163	0.0493	0.500

K = 4 ϕ = 120

α	$\frac{M_o}{WR}$	$\frac{T_o}{WR}$	$\frac{F_o}{W}$
30	0.3498	0.0372	0.8671
40	0.3421	0.0435	0.7447
60	0.3265	0.0561	0.500

ϕ = 120 = 40

K	$\frac{M_o}{WR}$	$\frac{T_o}{WR}$	$\frac{F_o}{W}$
2	0.3322	0.0366	0.7398
2.15	0.3330	0.0371	0.7402
4	0.3421	0.0435	0.7447

Final coefficients for the ordinates of the influence lines due to effect of the load at B and D are as shown in the table below.

Angular Distance of the Load From the Support	$\frac{M_o}{WR}$	$\frac{T_o}{WR}$	$\frac{F_o}{W}$
40°	0.3330	0.0371	0.7402
80°	0.1850	0.0339	0.2534

$M_o = 1500 \times 8 \times (0.333 + 0.1850) = 6216$ ft. lb.

$T_o = 1500 \times 8 \times (0.0371 + 0.0339) = 852$ ft. lb.

$F_o = 1500 \times (0.7402 + 2534) = 1490.4$ lb.

The exact value of F_o = 1500 lb.; the difference between the two values exists because we assumed a straight line variation, and the error is about 1.0 per cent. Because of the symmetrical conditions the same bending moment, torsional moment and shear exist at the other end.

Total bending moment at A

$6216 + 24750 = 30{,}966$ ft. lb.

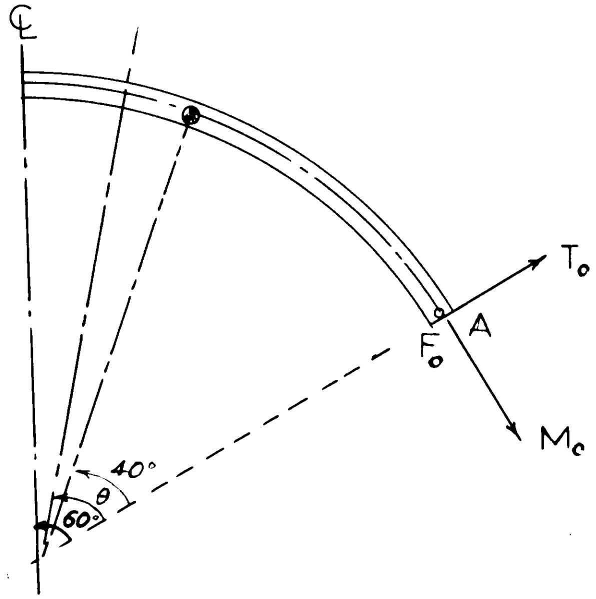
Total torsional moment at A

$852 + 3340 = 4192$ ft. lb.

Total shear force at B

$7550 + 1500 = 9050$ lb.

For simplicity analyze half of the beam



The general equation for bending moment and torsion at any section x beyond the concentrated load is as follows:

$$M_x = M_o \cos\theta + T_o \sin\theta - F_oR \sin\theta + R^2w(1 - \cos\theta)$$
$$+ WR \sin(\theta - 40)$$

$$T_x = - M_o \sin\theta + T_o \cos\theta + F_oR(1 - \cos\theta) - WR^2(\theta - \sin\theta)$$
$$- WR \;\; 1 - \cos(\theta - 40)$$

If the angular distance of the section under consideration is less than 40° from the support, the last term in both expressions will be zero.

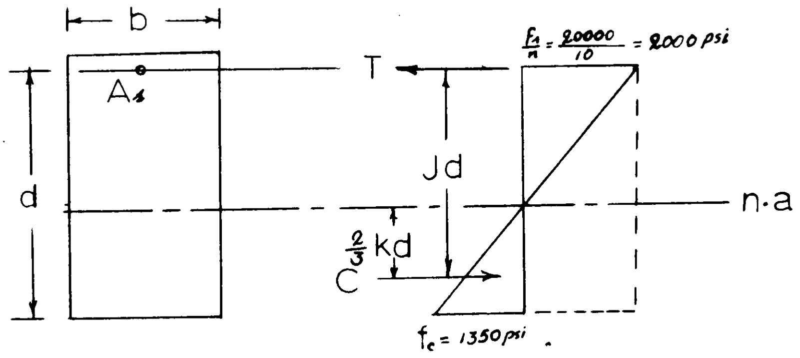
Angular Distance of the Section From the Support	Bending Moment	Torsional Moment
10°	19,527	- 0.200
20°	9,380	- 2,692
30°	0.430	- 3,513
40°	-6,636	- 2,948
50°	-9,686	- 1,488
60°	-10,683	0.000

$$\frac{M_{max}}{T_{max}} = \frac{30000}{4192} = 7.2 > 2$$

Therefore, the beam will fail in bending before failing in torsion.

Use balanced design method to obtain section to resist bending.

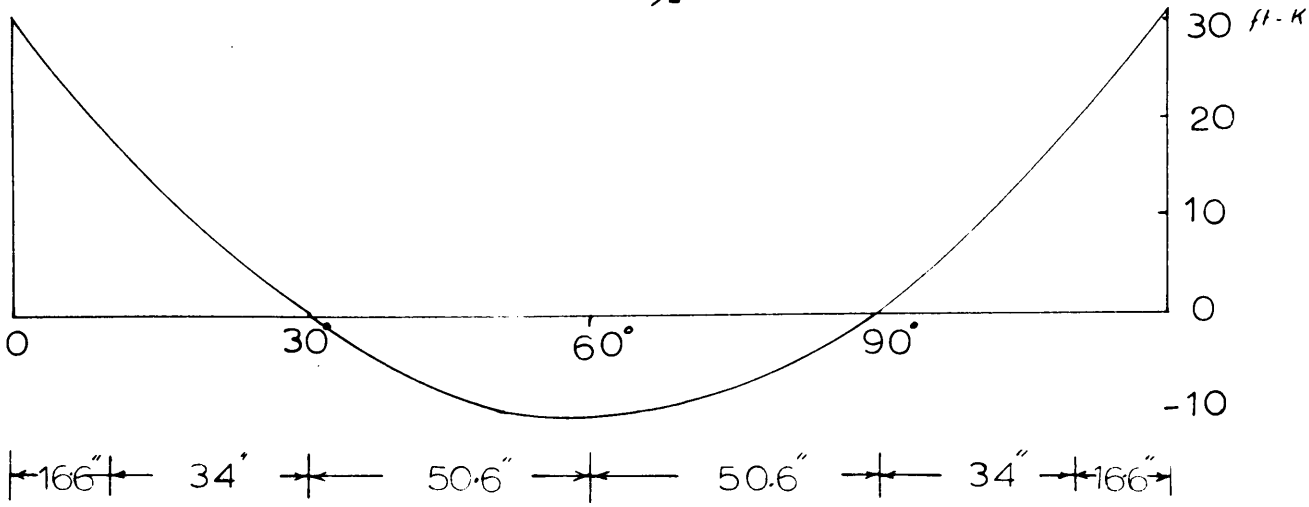
Design constants are: $f_c' = 1350$ psi $f_s = 20,000$ psi $n = 10$



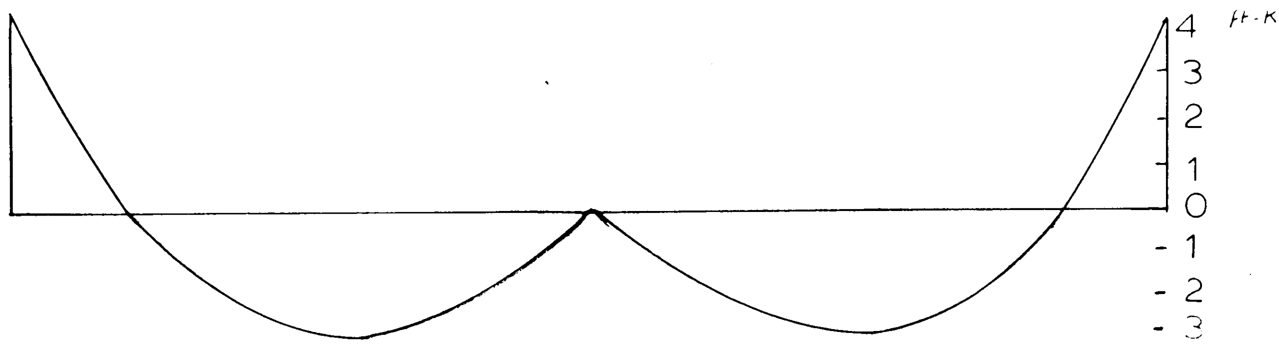
$\therefore kd = 0.403\ d$

$$jd = d - \frac{kd}{3} = 0.866\ d$$

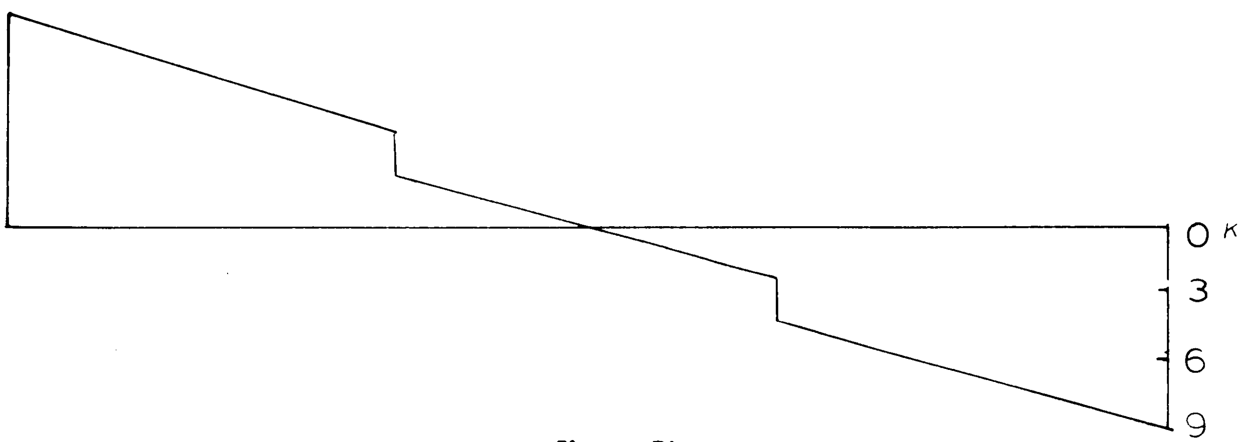
$$\therefore C = \frac{1350}{2} \times 0.403\ d \times b$$



Bending Moment Diagram



Torsional Moment Diagram



Shear Diagram

$$M = Cjd = \left(\frac{1350}{2} \times 0.403\ d \times b\right) \times 0.866\ d = 236\ bd^2$$

$$\therefore bd^2 = \frac{30192 \times 12}{236} = 1550\ \text{in.}^3$$

but $d = 1.5\ b$

$\therefore\ 2.25\ b^3 = 1550$

$b^3 = 690$

$b = 8.84$ use 9"

$\therefore\ d = 13.3$

$$A_s = \frac{M}{f_s jd} = \frac{30200 \times 12}{20000 \times .866 \times 13.3} = 1.59\ \text{in.}^2$$

Minimum steel required by the 1956 ACI Code

$0.005\ bd = 0.005 \times 9 \times 13.3 = .6 \quad 1.59$

Use 4 #6 bars $A_s = 1.76 > 1.59$ OK

Steel area required for maximum positive bending moment is

$$A_s = \frac{10,000 \times 12}{20,000 \times .866 \times 13.3} = 0.53\ \text{in.}^2$$

Use 2 #6 bars $A_s = 2 \times 0.44 = 0.88\ \text{in.}^2$

0.88 0.53 OK

The maximum torsional stress is

$$S_1 = \frac{T}{k_1 d\, b^2}$$

$T_{max} = 4192$ ft. lb.

$k_1 = .231$ for $d/b = 1.5$

$$\therefore S_1 = \frac{4192 \times 12}{.231 \times 13.3 \times 81} = 214\ \text{psi}$$

The maximum direct shear stress is

$$v = \frac{V}{bjd} = \frac{9050}{9 \times .866 \times 13.3} = 87 \text{ psi}$$

Maximum shear occurring in the beam is

$$v_{max} = 214 + 87 = 301 \text{ psi}$$

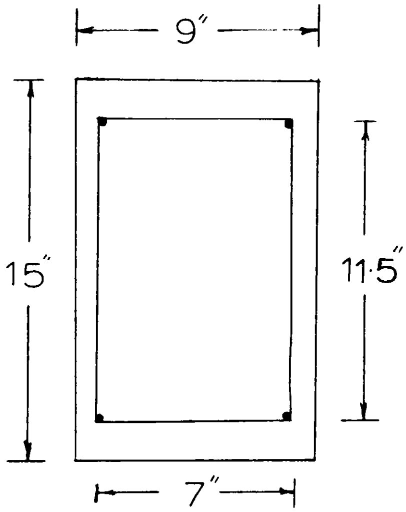
According to the recommendations of the Australian Code

$$v_{max} = 0.08\, f_c' + 80$$

$$= 0.08 \times 3000 + 80 = 320 > 301$$

Design of the binder

According to the recommendations of the Australian Code the torsion resisted by the concrete is



$$v_t = 0.02\, f_c' + 20 = 80 \text{ psi}$$

$$T_c = k_1 db^2 S = .231 \times 13.3 \times 81 \times 80$$

$$= 19{,}908 \text{ in.lb.}$$

Therefore, torsion resisted by the binder

$$T_b = 4192 \times 12 - 19{,}908$$

$$= 50{,}304 - 19{,}908 = 30{,}396 \text{ in.lb.}$$

Required steel area of the binder

$$A_{sv} = \frac{T_b s}{0.8\, f_s bd}$$

The Australian Code does not permit the stress of the steel resisting torsion to exceed 18,000 psi.

$$A_{sv} = \frac{30{,}396\ s}{0.8 \times 18000 \times 11.5 \times 7} = 0.02533\ s$$

Assume #4 bars to be used

$$\text{area} = 0.2 \text{ in.}^2$$

then the legs of two binders = 0.4 in.^{2}

$$\therefore s = \frac{0.4}{0.02533} = 15.6 \text{ in.} \qquad \text{Use 15 in.}$$

The area of the longitudinal stirrups

$$A_{sh} = A_{sv}\left(\frac{d' + b'}{s}\right)$$

$$A_{sh} = 0.02533\ (7 + 11.5) = 0.467 \text{ in.}^2$$

Use 4 #4 bars; one bar in each corner. As the maximum positive torsional moment is approximately equal to the maximum negative torsional moment, it is quite convenient to run 4 #4 bars, one in each corner, all over the beam.

Design of vertical stirrups to resist transverse shear.

Because the concrete was assumed to resist the allowable torsional shear stress, the transverse shear must be resisted by the stirrups alone.

$$\text{Minimum spacing} = s = \frac{A_v f_s}{Vb}$$

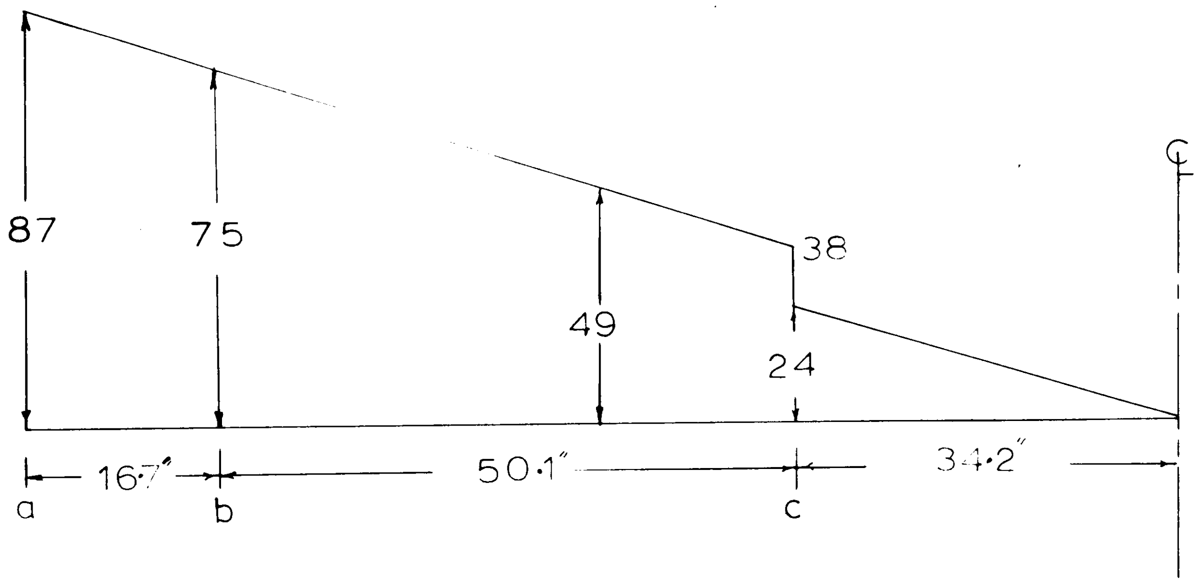
Assume the same size of stirrups as the binders, i.e. #4 bars.

$$A_v f_s = 0.2 \times 2 \times 18000 = 7200 \text{ lb.}$$

$$\therefore \frac{7700}{9 \times 87} = 9.2 \qquad \text{Use 9 in.}$$

Number of stirrups used for the part of the beam from the support up to point of zero torsion.

$$N = \frac{b(\text{area of unit shear diagram})}{A_v f_s}$$



Stirrups required in the section a - b.

$$\text{Area a - b} = \frac{(87 + 75)}{2} \times 16.7 = 1350$$

$$N = \frac{9 \times 1350}{7200} = 1.88 \quad \text{Use 2}$$

Total stirrups and binder for this section = 2 + 1 = 3.

The 2 stirrups and the 1 binder will each be placed 5 inches apart.

Note: the binder has four sides and the stirrup has three sides. For this problem the stirrups will also have four sides and be shaped like the binder.

Stirrups along the section b - c.

Minimum spacing is

$$s = \frac{A_v f_s}{v' b'} = \frac{7200}{75 \times 9} = 8.9 \text{ in. Use 8 in.}$$

Number of stirrups

$$N = \frac{b \text{ (area of unit shear diagram)}}{A_v f_s}$$

Area of unit shear diagram $= \left(\frac{75 + 38}{2}\right) \times 50.1 = 2830$

$\therefore N = \frac{9 \times 2830}{7200} = 3.5$ Use 4

The number of binders to resist the torsional stress is

$\frac{50.1}{15.6} = 3.2$ Use 4

Total number of stirrups and binders $= 4 + 4 = 8$

Binders and stirrups provided along the section b - c are 8 binders at 6 inches apart.

Torsional stress at section c.

$T = 1{,}488$ ft. lb.

$$S_1 = \frac{T}{k_1 d b^2} = \frac{1488 \times 12}{.231 \times 13.3 \times 81} = 71 \text{ psi}$$

Maximum shear stress at the section is

$71 + 24 = 95$ psi

Excess unit shear $= 95 - 80 = 15$ psi

Therefore

$$s = \frac{A_v f_s}{v'b} = \frac{7200}{15 \times 9} = 53 \text{ in.}$$

For convenience change the size of the stirrups. Use #2 bars, the area of the two legs = 0.1 in.^{2}.

$$\therefore s = \frac{0.1 \times 18000}{9 \times 15} = \frac{1800}{135} = 13.3"$$

Use #2 stirrups 10 inches apart.

Design of positive bending moment steel

$$A_s = \frac{M}{f_s jd}$$

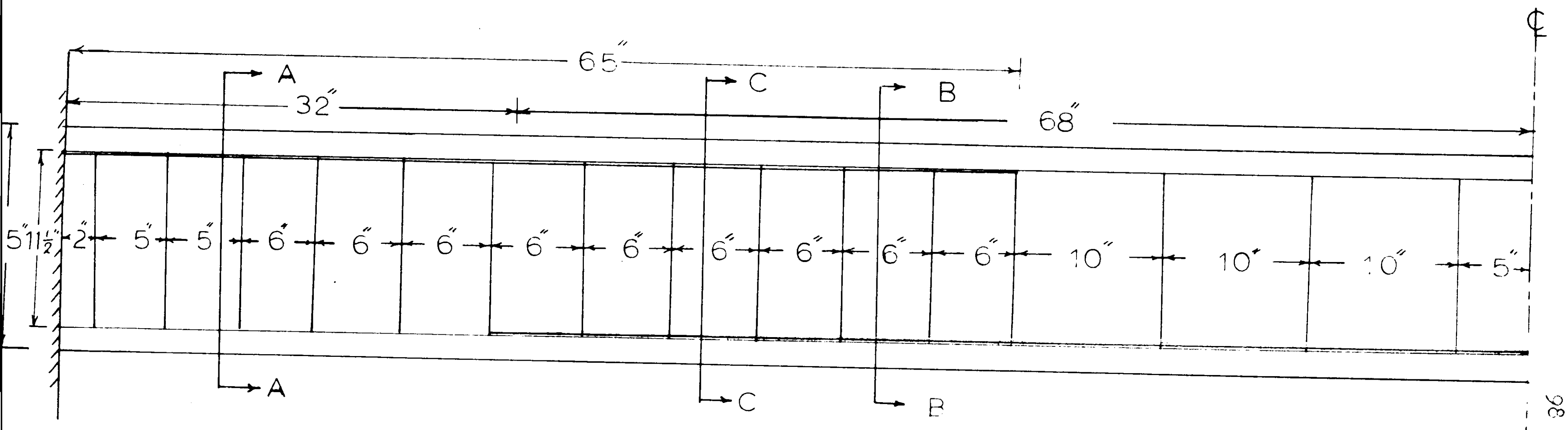
$$M = 10{,}683 \times 12 = 128{,}000 \text{ in. lb.}$$

$$A_s = \frac{128{,}000}{20000 \times .866 \times 13.3} = 0.557 \text{ in.}^2$$

Use 2 #5 bars. $A_s = 0.62 > 0.557$ OK

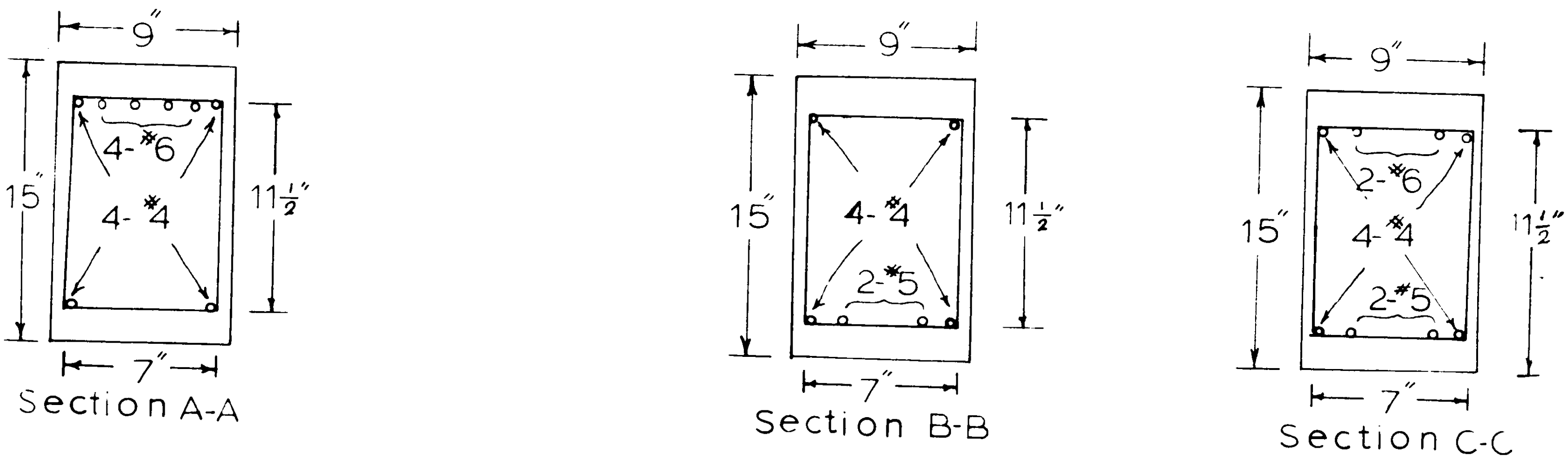
Detail of the Reinforcement

Elevation



The beam is symmetrical about the center line

distance along the center line of the beam



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ABSTRACT

This thesis presents an investigation and derivation of the expressions for bending moment, torsional moment, and shear at the ends and at any intermediate point along a circularly curved beam. The investigation includes both cantilever beams and fixed ends beams, loaded with a uniformly distributed load, concentrated loads or a combination of the two.

The solutions of the equations have been presented in a graphical form for the case of the uniformly distributed load, and a tabulated form representing the ordinate of the influence lines for the case of the concentrated loads. The graphs and tables cover a series of beams whose arcs are subtending central angles of 30, 45, 60, 75, 90, 120, 135, 150, 165 and 180 degrees; and whose stiffness ratios (K) are 1.33, 2, 4, and 10.67.

Special emphasis has been given to reinforced concrete curved beam design as based on the theories and experiments presented in the literature by Timoshenko and Cowan.

The investigation shows that many questions still remain to be answered in the design of reinforced concrete beams subjected to bending moment, torsional moment and shear; and there is a need for the ACI Code to give some criteria for such designs in the near future.