EFFECTS OF SHEAR DEFORMATIONS ON THE VIBRATIONAL FREQUENCIES OF WIDE-FLANGED STRUCTURES

by

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IV. SYMBOL LIST

A	cross-sectional areas, square inches
A_s	shear-carrying area, square inches
a_N, b_N, c_N	coefficients of series for deflections
A, B	arbitrary constants in differential equation for $Y(y)$
b	flange half-width, inches
D	$\frac{Et^3}{12(1-\mu^2)}$, flexural stiffness of a plate, pound-inches
E	Young's modulus of elasticity, pounds per square inch
$\bar{E}$	total energy in a nondissipating system, inch-pounds
G	shear modulus of structural material, pounds per square inch
h	beam depth, inches
I	moment of inertia, $(\text{inches})^4$
k_B^2	frequency parameter, $\frac{A_T m\omega^2 l^4}{EI_T}$
k_{RI}^2	rotary inertia parameter, $\frac{r^2}{l^2}$
k_s^2	transverse shear parameter, $\frac{EI_T}{A_s G l^2}$
K^2	shear lag parameter, $\frac{E}{G}\frac{b^2}{l^2}$
l	length of beam, inches
m	mass density of beam material, pounds per cubic inch
t	thickness of flange material, inches
T	kinetic energy, inch-pounds
$u(x,y)$	shear lag deformation in flanges, inches
V	potential energy, inch-pounds
$w(x)$	beam bending deflection, inches

$\bar{w}(x,y)$ relative flange bending deflection, inches

x distance from end of beam, inches

y distance from web center line, inches

$Y(y)$ arbitrary shear lag variation in the y direction

z distance from flange midplane, inches

γ shear angle of rotation

γ_{xy} shear strain in the flanges

δ calculus of variations operator

∇^4 Laplacian operator, $\frac{\partial^4}{\partial x^4} + \frac{2\,\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$

ϵ_x, ϵ_y strains in the flanges

μ Poisson's ratio of the beam material

ω circular frequency of vibration, radians/second

σ_x, σ_y stresses in the flanges, pounds/square inch

τ_{xy} shear stress in the flange, pounds/square inch

Φ energy expression defined by equation (13)

$\psi(x)$ transverse shear rotation of the beam

$\big|_x$ evaluated at the limits of x

Subscripts

1 referring to the web

2 referring to the flange

N referring to the Nth natural mode

H referring to the homogeneous solution

P referring to the particular solution

T referring to the total beam

max referring to the maximum value of a quantity

V. INTRODUCTION

In most engineering design, where an answer is needed immediately to a particular problem, little time is available for developing a theory to fit the situation. Instead, the practicing engineer is led either to use the most elementary solution to the problem at hand or to estimate the effects of a more rigorous solution. Such has been the case with beam vibration problems.

There have been a number of recent papers on the vibration of uniform beams and box beams. (See refs. 1 - 6.) In these reports and articles, two principal theories are given consideration. One of these theories is the so-called elementary theory, while the other is now termed the Timoshenko theory.

In the case of elementary theory (refs. 1, 3, and 5) the cross sections of the beam remain perpendicular to the neutral axis and deflect an amount w. The deflection of the neutral axis is allowed to vary along the length of the beam, thus at each cross section the neutral axis has a slope of amount $\frac{dw}{dx}$. This type of analysis yields a fourth-order differential equation in w, without a second-order term. In the case of Timoshenko theory (refs. 1, 2, 3, and 6), however, the beam deflection is increased by allowing the beam to shear along its length; that is, the cross sections of the beam are allowed to rotate with respect to the neutral axis. Then the slope of each cross section becomes $\frac{dw}{dx} - \gamma$, where γ is the transverse shear strain. Also the sections of the beam not only have a transverse inertia but they also have an inertia of rotation. The beam equation thus adds rotational inertia terms and "transverse shear" terms. The differential equation is still a fourth-order equation in w, but additional terms and second-order terms appear. These terms only slightly complicate the solution to the problem, since the equation does not change order.

In both of these theories, the flange of the beam has the same motion as the top of the web and acts as a concentrated mass only. It appears, however, that in a practical case the flange has motion relative to the web and also can have motion within its plane. These effects can be incorporated into the equations with a corresponding increase in the order of the equations and some difficulty in solution. The in-plane motion of the flange is primarily a shear lag effect. That is, the flange deflection in its plane in the longitudinal direction is the same as the web where the flange and web join but farther out from the web the deflection tapers off or lags. This is due to the fact that the web puts a shear loading at the center of the flange and at the free edges of the flange the shear is zero. Therefore, this unbalance in shear load causes a varying deflection across the flange. In some specific instances, the effects of all these shear distortions can change the frequency of any particular mode (even as low as the first mode) by an appreciable amount, and a ready reference including shear distortions would be welcome. This thesis was written in order to provide the practicing engineer with a ready reference from which he may determine the frequency of vibration of his particular beam with a minimum of time and effort on his part and still include shear effects. For the simply-supported case, the tables may be used directly to determine the frequency of the first three modes. This computation is illustrated in an example case. For other boundary conditions, deflections appropriate to those boundary conditions must be substituted into the energy expressions and a variational procedure performed before the frequencies can be obtained. This process is accelerated by having the energy expression in terms of displacements as is given herein. Also the effect of relative bending of the flange may be investigated further by attempting various solutions of the equations including this effect. The author would like to take this opportunity to thank his faculty advisor, Professor F. J. Maher, for his help in the preparation of this thesis. Not only did Professor Maher suggest the problem, but he also aided immeasurably in suggesting the manner of presentation and in clarifying several inadequacies of the original versions.

VI. STATEMENT OF THE PROBLEM

The investigation of shear effects is a diffi ult problem to solve in an exact manner without using Fourier series where, in the final solution, a series of terms must be summed or, even worse, a series result that includes the frequency. This necessitates an iteration procedure, which is time consuming. This thesis derives the equations of motion for a wide-flanged structure including the effects of "shear-lag" deflection of the flanges, transverse shear deformation of the web, and the longitudinal inertia of the flanges. Also the equation incorporating the bending effect of the flanges is derived in an appendix. In most cases the solution of these equations would often necessitate approximate means of solutions. However, the example shown herein for simply-supported boundary conditions is solved exactly and a closed-form frequency equation determined. This reduces the problem to a simple trial and error solution no more difficult than that for a simple beam. This one feature allows a much better evaluation of the effects than had been previously available.

VII. ASSUMPTIONS

Since the principal effect being evaluated herein is the effect of shear deformations, the deflections included must be at least all those pertinent to shear. In figure 1 all the dimensions of the structure are shown together with the displacements being considered.

In this investigation the thickness of the flanges is considered small in comparison to the depth of the structure, and thus the stress distribution across the flange thickness may be considered uniform. This assumption is not a very serious restriction since in many cases the depth of the beam is appreciable and the flange need not be excessively thick in order to increase the moment of inertia to the value required in any particular design. Also Hooke's law is assumed to be valid as is generally the case in engineering structures. The entire beam deflects an amount w but, due to the transverse shear freedom allowed, the total rotation of the beam is now considered to be the sum of two rotations, or

$$\psi = \frac{dw}{dx} - \gamma \tag{1}$$

where $\frac{dw}{dx}$ is the slope of a cross section due to bending and γ is the shear angle of rotation. This shear angle is an additional freedom beyond the elementary theory allowing the shearing of the beam along its length.

Also in the flanges of this structure, a deflection u is allowed in addition to the usual beam bending deflection. This additional deflection is the shear lag deformation of the flanges. Since the deflections of the flanges are considered small in relationship to the

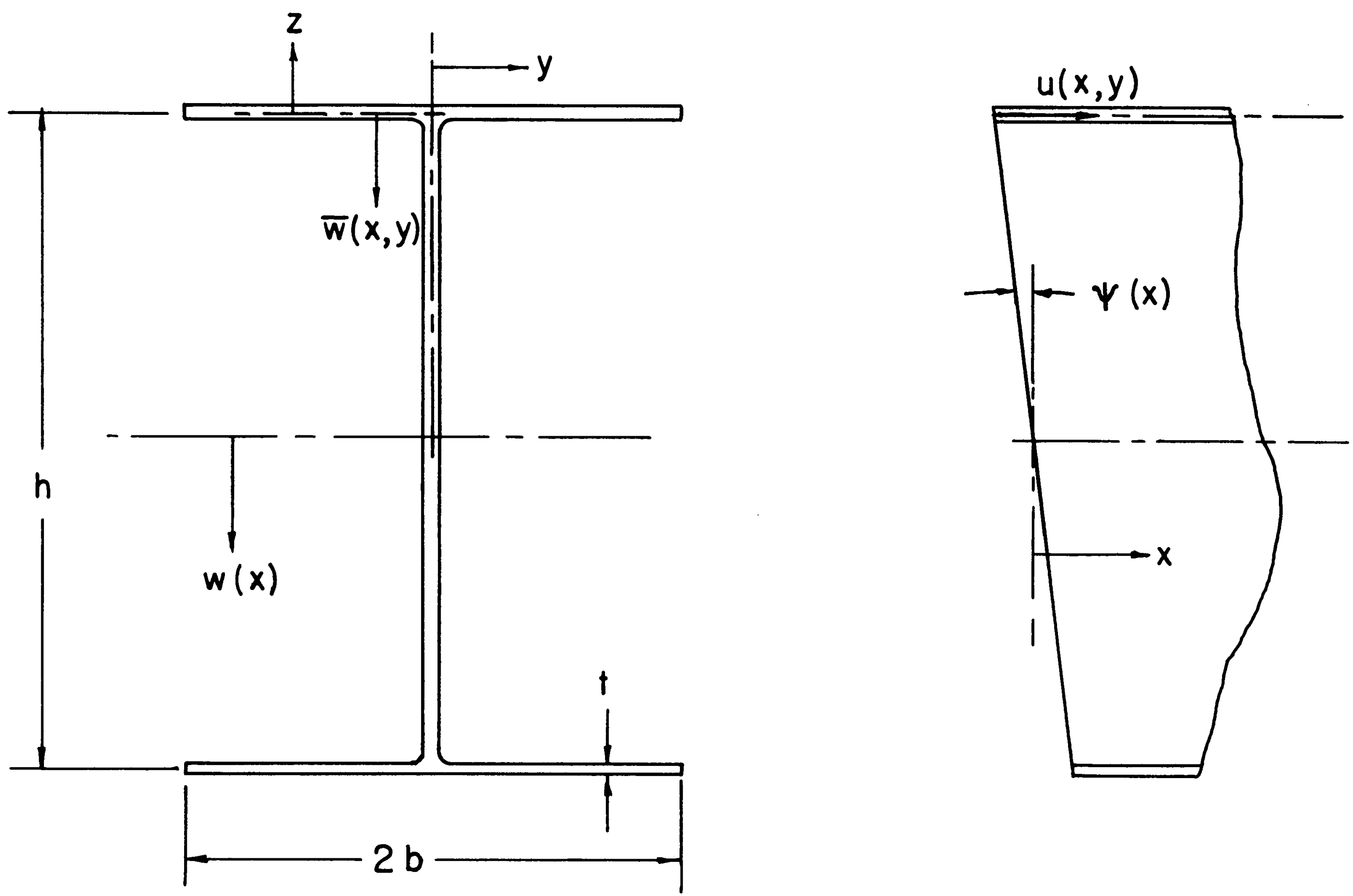

Figure 1.- Coordinate system and notations.

over-all dimensions of the beam, the displacement of the flanges inward toward the web are neglected. This deflection is the usual v displacement of plane elasticity or plate theory. Due to the shear lag deformation u, the originally plane cross sections of the structure do not remain plane but distort lengthwise along the beam. The warping of the cross sections that might be caused by relative bending of the flanges with respect to the web has been considered in derivation of the equations but not in the numerical example for two reasons. One reason is that the effect of shear deformation is the principal point of interest and can be studied much more effectively without the encumbrance of an additional infinity of modes and frequencies over the double infinity already included. Also, this effect is an essentially separate one and it can be investigated by itself to ascertain an idea of its importance.

VIII. ENERGY EXPRESSIONS

To derive the differential equations and the associated boundary conditions in general, an energy method will be employed for convenience. It is known that in a conservative system the total energy $\bar{E}$ is constant or

$$\delta\bar{E} = 0 \tag{2}$$

where δ denotes the variational operation (ref. 7).

Now if we assume that a natural vibration is taking place in a periodic fashion and, with no dissipative constraints included, then in terms of potential and kinetic energies we have

$$\delta\bar{E} = \delta(P.E. + K.E.) = 0 \tag{3}$$

When the beam reaches its extreme position all elements of the beam reach their maximum amplitudes simultaneously and the energy becomes all potential energy. Also, when the beam passes the equilibrium position during this vibration all elements have zero displacements or all the energy is kinetic. Now since the total energy of the system is known to be constant

$$P.E. + K.E. = \bar{E} = \text{constant} = P.E._{max} = K.E._{max} \tag{4}$$

All that needs to be found are the expressions for the maximum kinetic and potential energies.

Due to bending of the web of the beam, the usual energy expressions for the web including transverse shear appear as

$$T_1 = \frac{mA_1}{2} \int_0^l (\dot{w})^2 \, dx$$

$$V_1 = \frac{EI_1}{2} \int_0^l \left(\frac{d\psi}{dx}\right)^2 dx + \frac{A_s G}{2} \int_0^l \left(\frac{dw}{dx} - \psi\right)^2 dx \qquad (5)$$

where dots denote differentiation with respect to time

m mass density of the beam material

A_1 area of the web

EI_1 bending stiffness of the web, and

$A_s G$ shear stiffness of the web

When rotational inertia is included, an energy term corresponding to the angular velocity of rotation of the web must be added. This inertial effect appears as

$$\bar{T}_1 = \frac{mI_1}{2} \int_0^l (\dot{\psi})^2 dx \qquad (6)$$

The above are the energy terms for the web portion of the beam.

For the flanges, both the longitudinal inertial energy and the strain energies are necessary. If the total deflection of a flange is $\bar{u}$, then the longitudinal inertia can be written

$$T_2 = 2\left(\frac{m}{2}\right) \int_{-\frac{t}{2}}^{+\frac{t}{2}} \int_{-b}^{+b} \int_0^l (\dot{\bar{u}})^2 \, dx \, dy \, dz \qquad (7)$$

where

t the thickness of the flange,

b the flange half-width,

$\bar{u}$ $u - \frac{h}{2}\psi$

and the transverse inertia can be written in the usual form as

$$\bar{T}_2 = \frac{mA_2}{2} \int_0^l (\dot{w})^2 \, dx \tag{8}$$

where A_2 = the total area of the flanges.

The flange strain energy using Hooke's law can be written in a general sense as an integral over one flange or

$$V_2 = \frac{1}{2} \int (\epsilon_x \sigma_x + \tau_{xy} \gamma_{xy}) dV = 2\left(\frac{1}{2}\right) \int_{-\frac{t}{2}}^{+\frac{t}{2}} \int_{-b}^{+b} \int_0^l \left[E\epsilon_x^2 + G\gamma_{xy}^2\right] dx\, dy\, dz \tag{9}$$

where

ϵ_x strain in the longitudinal direction in the flange

γ_{xy} the shear strain in the flange

These strains can be expressed in terms of the displacements by the relationships

$$\epsilon_x = \frac{\partial}{\partial x}\left(u - \frac{h}{2} \psi\right)$$

and

$$\gamma_{xy} = \frac{\partial u}{\partial y} \tag{10}$$

These expressions then totally define the energies for this problem, and the derivation of the differential equations and boundary conditions can be made from them. The changes in these flange energies due to inclusion of relative flange bending can be seen in appendix A.

IX. DERIVATION OF THE DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

In the previous section, it was shown that the maximum kinetic and potential energies in a nondissipative system were equal, or in other words the complimentary energy approach states

$$(P.E.)_{max} - (K.E.)_{max} = 0 \tag{11}$$

Using the energy expressions from the preceding section, equation (11) can be written in terms of displacements as

$$- \frac{mA_1}{2} \int_0^l \left[\dot{w}_{max}\right]^2 dx + \frac{EI_1}{2} \int_0^l \left(\frac{\partial\psi}{\partial x}\right)^2_{max} dx + \frac{A_sG}{2} \int_0^l \left(\frac{dw}{dx} - \psi\right)^2_{max} dx$$

$$- \frac{mI_1}{2} \int_0^l \left[\dot{\psi}_{max}\right]^2 dx - m \int_{-\frac{t}{2}}^{+\frac{t}{2}} \int_{-b}^{+b} \int_0^l \left[\dot{u} - \frac{h}{2}\dot{\psi}\right]^2_{max} dx\, dy\, dz$$

$$+ \int_{-\frac{t}{2}}^{+\frac{t}{2}} \int_{-b}^{+b} \int_0^l \left[E\left(\frac{\partial u}{\partial x} - \frac{h}{2}\frac{\partial\psi}{\partial x}\right)^2 + G\left(\frac{\partial u}{\partial y}\right)^2\right]_{max} dx\, dy\, dz$$

$$- \frac{mA_2}{2} \int_0^l \left[\dot{w}_{max}\right]^2 dx = 0 \tag{12}$$

Now noting that $\int_{-b}^{+b} dy = 2\int_0^b dy$ and $\int_{-\frac{t}{2}}^{+\frac{t}{2}} dz = t$ and assuming a periodic motion of the beam which varies in time with a circular frequency ω, this equation becomes

$$\Phi = \int_0^l \left[\frac{EI_1}{2}\left(\frac{\partial\psi}{\partial x}\right)^2 + \frac{A_sG}{2}\left(\frac{\partial w}{\partial x} - \psi\right)^2 - \frac{m(A_1 + A_2)}{2}\omega^2 w^2 - \frac{mI_1}{2}\omega^2 \psi^2\right] dx$$

$$+ \int_0^b \int_0^l \left[2tE\left(\frac{\partial u}{\partial x} - \frac{h}{2}\frac{\partial\psi}{\partial x}\right)^2 + 2tG\left(\frac{\partial u}{\partial y}\right)^2 - 2mt\omega^2 \left(u - \frac{h}{2}\psi\right)^2\right] dx\, dy = 0$$

(13)

Equation (13) must have a stationary value for any arbitrary small variation of any of the three deformations u, w, or ψ. Then we can write the variation of the total energy expression Φ as

$$\delta\Phi = 0 \tag{14}$$

for arbitrary δu, δw, and $\delta\psi$. These three variations will yield three simultaneous differential equations in the three variables. These equations can then be written

$$\int_0^l \left[\frac{A_sG}{2}\, 2\left(\frac{\partial w}{\partial x} - \psi\right)\delta\left(\frac{\partial w}{\partial x}\right) - 2\,\frac{m(A_1 + A_2)}{2}\omega^2\, w\delta w\right] dx = 0 \tag{15}$$

$$\int_0^b \int_0^l \left[2tE(2)\left(\frac{\partial u}{\partial x} - \frac{h}{2}\frac{\partial\psi}{\partial x}\right)\delta\left(\frac{\partial u}{\partial x}\right) + 2tG(2)\frac{\partial u}{\partial y}\,\delta\left(\frac{\partial u}{\partial y}\right) - 2mt\omega^2\,(2)\left(u - \frac{h}{2}\psi\right)\delta u\right] dy\, dx$$

$$= 0 \tag{16}$$

and

$$\int_0^l \left[EI_1 \frac{\partial\psi}{\partial x} \delta\left(\frac{\partial\psi}{\partial x}\right) - A_sG\left(\frac{\partial w}{\partial x} - \psi\right)\delta\psi - mI_1\omega^2 \psi\delta\psi \right] dx - \int_0^b \int_0^l \left[2tE(2) \right.$$

$$\left. \cdot \left(\frac{\partial u}{\partial x} - \frac{h}{2}\frac{\partial\psi}{\partial x}\right)\frac{h}{2}\, \delta\left(\frac{\partial\psi}{\partial x}\right) - 2mt\omega^2 \left(u - \frac{h}{2}\psi\right)\frac{2h}{2}\, \delta\psi \right] dx\, dy = 0 \qquad (17)$$

Now since the processes of variation and differentiation can be interchanged in the calculus of variations, and, by integrating the above expressions by parts as often as necessary, the differential equations can be obtained. An example of this variational procedure (for the first term in eq. (15)) follows:

$$\int_0^l A_sG\left(\frac{\partial w}{\partial x} - \psi\right)\delta\left(\frac{\partial w}{\partial x}\right) dx = A_sG \int_0^l \left(\frac{\partial w}{\partial x} - \psi\right)\frac{\partial(\delta w)}{\partial x}\, dx$$

which is now of the form

$$\int_a^b \phi\, d\Gamma = \phi\Gamma \Big|_a^b - \int_a^b \Gamma\, d\phi$$

where

$$\phi = A_sG\left(\frac{\partial w}{\partial x} - \psi\right)$$

and

$$\Gamma = \delta w$$

Thus our original integral becomes

$$A_sG\left(\frac{\partial w}{\partial x} - \psi\right)\delta w \Big|_0^l - A_sG \int_0^l \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial\psi}{\partial x}\right)\delta w\, dx$$

The first of these two terms appears as the boundary condition for w, while the second term remains under the integral and eventually becomes a part of the δw differential equation. Using this procedure all the differential equations are

$$\int_0^l \left[-A_sG\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \psi}{\partial x}\right) - m(A_1 + A_2)\omega^2 w\right]\delta w \, dx = 0 \tag{18-a}$$

$$\int_0^b \int_0^l \left[4tE\left(\frac{\partial^2 u}{\partial x^2} - \frac{h}{2}\frac{\partial^2 \psi}{\partial x^2}\right) + 4tG\frac{\partial^2 u}{\partial y^2} + 4tm\omega^2\left(u - \frac{h}{2}\psi\right)\right]\delta u \, dx \, dy = 0 \tag{19-a}$$

and

$$\int_0^b \int_0^l \left[2tEh\left(\frac{\partial^2 u}{\partial x^2} - \frac{h}{2}\frac{\partial^2 \psi}{\partial x^2}\right) + 2mth\omega^2\left(u - \frac{h}{2}\psi\right)\right]\delta\psi \, dx \, dy$$

$$-\int_0^l \left[EI_1\frac{\partial^2 \psi}{\partial x^2} + A_sG\left(\frac{\partial w}{\partial x} - \psi\right) + mI_1\omega^2\psi\right]\delta\psi \, dx = 0 \tag{20-a}$$

with the associated boundary conditions appearing as

$$A_sG\left(\frac{\partial w}{\partial x} - \psi\right)\delta w \Big|_x = 0 \tag{18-b}$$

$$4tG\left(\frac{\partial u}{\partial y}\right)\delta u \Big|_y = 0 \quad \text{and} \quad 4tE\left(\frac{\partial u}{\partial x} - \frac{h}{2}\frac{\partial \psi}{\partial x}\right)\delta u \Big|_x = 0 \tag{19-b}$$

and

$$\left[E(I_1 + tbh^2)\frac{\partial \psi}{\partial x} - 2tEh\int_0^b \frac{\partial u}{\partial x}dy\right]\delta\psi \Big|_x = 0 \tag{20-b}$$

The boundary condition, equation (18-b), corresponds either to zero shearing force or zero deflection of the web; the boundary condition, equation (19-b), corresponds to either zero shearing stress or zero deflection in the flange in either the x or y directions; and the boundary condition, equation (20-b), corresponds either to zero bending moment or else the total slope of the cross section is zero. The last case could also be written that the bending slope of the beam equals the shear slope. Since each integrand of these differential equations is multiplied by an arbitrary variation of the displacement, the equations can be identically satisfied when the integrands are zero. Thus the differential equations are

$$A_sG\left(\frac{d^2w}{dx^2} - \frac{d\psi}{dx}\right) + m(A_1 + A_2)\omega^2 w = 0 \tag{21}$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{h}{2}\frac{d^2\psi}{dx^2} + \frac{G}{E}\frac{\partial^2 u}{\partial y^2} + \frac{m\omega^2}{E}\left(u - \frac{h}{2}\psi\right) = 0 \tag{22}$$

$$E(I_1 + tbh^2)\frac{d^2\psi}{dx^2} + A_sG\left(\frac{dw}{dx} - \psi\right) + m(I_1 + tbh^2)\omega^2 \psi - \int_0^b \left[2tEh\frac{\partial^2 u}{\partial x^2} + 2mth\omega^2 u\right]dy = 0 \tag{23}$$

These equations are the three simultaneous partial integro-differential equations that completely define the present problem. On inspection it may be noted (for future reference) that ψ appears as a derivative of odd order when w appears as an even order derivative and w appears as an odd order derivative when u and ψ appear as even order derivatives.

The reduction of this theory to more elementary theories can be seen in several instances. If the flange width, b, becomes zero then equations (21) and (23) reduce to

$$A_sG\left(\frac{d^2w}{dx^2} - \frac{d\psi}{dx}\right) + mA_1\omega^2 w = 0 \tag{24}$$

$$EI_1 \frac{d^2\psi}{dx^2} + A_sG\left(\frac{dw}{dx} - \psi\right) + mI_1\omega^2 \psi = 0$$

which are the equations for natural vibration of a uniform beam including the effects of transverse shear and rotational inertia as determined by Timoshenko (ref. 1). Also these identical equations appear when the deflection u is set equal to zero, since u appears only in the integral portion of the ψ equation and the condition that u vanish identically demands that the integral vanish identically also.

When w and ψ are set equal to zero, equation (22) becomes

$$E \frac{\partial^2 u}{\partial x^2} + G \frac{\partial^2 u}{\partial y^2} + m\omega^2 u = 0 \tag{25}$$

and thus can be seen to be an analogy to the vibrational equation for a thin membrane. (See ref. 8.) Therefore, all cases of simplification reduce to equations that are known to be correct within their own assumptions.

The three equations (21), (22), and (23) also can be combined into a single equation in u. From equation (21)

$$\frac{d\psi}{dx} = \frac{d^2w}{dx^2} + \frac{m(A_1 + A_2)\omega^2 w}{A_sG} \tag{26}$$

Using this expression to define ψ in terms of w alone, the remaining two equations in u and w can be written

$$- \frac{bh}{2}\left[\frac{d^4w}{dx^4} + \left(\frac{m(A1 + A2)}{A_sG} + \frac{m}{E}\right)\omega^2 \frac{d^2w}{dx^2} + \omega^4 \frac{m^2}{E}\left(\frac{A1 + A2}{A_sG}\right)w\right] + \int_0^b \left[\frac{\partial^3u}{\partial x^3} + \frac{G}{E}\frac{\partial^3u}{\partial y^2\,\partial x}\right.$$

$$\left. + \frac{m\omega^2}{E}u\right]dy = 0 \tag{27}$$

and

$$E(I_1 + tbh^2)\left[\frac{d^4w}{dx^4} + \left(\frac{m(A_1 + A_2)}{A_sG} + \frac{m}{E}\right)\omega^2 \frac{d^2w}{dx^2} + \frac{m^2\,\omega^4\,(A_1 + A_2)}{A_sG}w\right]$$

$$- m\omega^2\,(A_1 + A_2)w = 0 \tag{28}$$

Multiplying equation (27) by $\frac{2E}{bh}(I_1 + tbh^2)$ and adding equation (28) to it, the terms in w all drop out except one term, and the equation becomes

$$w = \int_0^b \left[\frac{2E(I_1 + tbh^2)}{bhm\omega^2\,(A_1 + A_2)}\left(\frac{\partial^3u}{\partial x^3} + \frac{G}{E}\frac{\partial^3u}{\partial y^2\,\partial x} + \frac{m\omega^2}{E}u\right) - \frac{2tEh}{m\omega^2\,(A_1 + A_2)}\left(\frac{\partial^3u}{\partial x^3}\right.\right.$$

$$\left.\left. + \frac{2m\omega^2}{E}\frac{\partial u}{\partial x}\right)\right]dy \tag{29}$$

Upon substitution of equation (29) into equation (27) the single equation in u is obtained as

$$\int_0^b \Bigg[-I_1 \frac{\partial^7 u}{\partial x^7} - \left\{ I_1 \left(\frac{m\omega^2 (A_1 + A_2)}{A_s G} + \frac{m\omega^2}{E} \right) - tbh^2 \frac{m\omega^2}{E} \right\} \frac{\partial^5 u}{\partial x^5}$$

$$- (I_1 + tbh^2) \frac{G}{E} \frac{\partial^7 u}{\partial x^5 \, \partial y^2} - \frac{G}{E} (I_1 + tbh^2) \left(\frac{m\omega^2 (A_1 + A_2)}{A_s G} + \frac{m\omega^2}{E} \right) \frac{\partial^5 u}{\partial x^3 \, \partial y^2}$$

$$- (I_1 + tbh^2) \frac{m\omega^2}{E} \frac{\partial^4 u}{\partial x^4} + \left\{ tbh^2 \left(\frac{m^2 \omega^4}{E^2} \right) \left(1 + \frac{E(A_1 + A_2)}{A_s G} \right) + \frac{m\omega^2}{E} (A_1 + A_2) \right.$$

$$\left. - I_1 \frac{m^2 \omega^4}{E A_s G} (A_1 + A_2) \right\} \frac{\partial^3 u}{\partial x^3} + \frac{G}{E} \left\{ \frac{m\omega^2 (A_1 + A_2)}{A_s G} - (I_1 + tbh^2) \frac{m^2 \omega^4}{EG} \left(\frac{(A_1 + A_2)}{A_s} \right) \right\} \frac{\partial^3 u}{\partial x \, \partial y^2}$$

$$- (I_1 + tbh^2) \left(\frac{m\omega^2 (A_1 + A_2)}{A_s G} + \frac{m\omega^2}{E} \right) \frac{m\omega^2}{E} \frac{\partial^2 u}{\partial x^2} + \frac{tbh^2 \, m^3 \omega^6 (A_1 + A_2)}{E^2 A_s G} \frac{\partial u}{\partial x}$$

$$+ \left\{ \frac{m^2 \omega^4 (A_1 + A_2)}{E \, A_s G} - (I_1 + tbh^2) \frac{m^3 \omega^6 (A_1 + A_2)}{E^2 A_s G} \right\} u \Bigg] dy = 0 \qquad (30)$$

However, owing to the high order of this equation, it appears impractical to attack this equation directly. For this reason a return to the previous energy expressions and differential equations will be employed for the example to follow.

X. ILLUSTRATIVE EXAMPLE

A. Assumed deflections

In order that an indication of the relative size of these effects might be obtained, the equations of this thesis must be solved for some specific boundary conditions and comparisons must be made between the resulting frequencies. As an example, the realistic case of a centrally simply supported web was selected. Relative flange bending has been neglected in this example for the sake of clarity. The boundary conditions needed for this case are zero transverse deflections at the ends, zero shearing stress in the flange at the ends, zero moment at the ends of the beam, zero shear stress at the outer edges of the flanges, and finally zero shear lag deflection at the center of the flanges. This last condition of zero deflection is a consequence of the symmetry of the section considered and also the fact that the u displacement is the additional deflection of shear lag over and above that of simple beam bending.

Referring to the derived differential equations (21), (22), and (23), it can be seen that when w is found as an odd-order derivative, both u and ψ appear as even-order derivatives in x and vice versa. Also from the boundary conditions (18-b), (19-b), and (20-b) it can be seen that a series of sine functions for w and a series of cosine functions for ψ and the x variation of u can be satisfactorily substituted in the boundary conditions and satisfy them.

Then a good choice for the deflections appears to be

$$w = \sum_N a_N \sin \frac{N\pi x}{l}, \quad \psi = \sum_N b_N \cos \frac{N\pi x}{l}, \text{ and } \quad u = \sum_N Y_N(y) \cos \frac{N\pi x}{l} \tag{31}$$

The boundary conditions on x are all satisfied, since

$$\left(\frac{\partial u}{\partial x} - \frac{h}{2}\frac{d\psi}{dx}\right) = 0 \quad w = 0 \quad \text{for} \quad x = 0 \quad \text{and} \quad x = l$$

and

$$E(I_1 + tbh^2)\frac{d\psi}{dx} - 2tEh \int_0^b \frac{\partial u}{\partial x}\, dy = 0 \quad \text{for} \quad x = 0 \quad \text{and} \quad x = l$$

but

$$u(x,o) = \frac{\partial u}{\partial y}(x,b) = 0$$

still remain to be satisfied.

These assumptions for w and ψ are the same functions that appear as the exact solution for this problem while neglecting shear lag effects. The assumed functions for the u displacements are governed by the fact that in the x direction u must be of the same character as ψ and opposite to w in regards to even or odd derivatives but in the y direction the boundary conditions and differential equation in $Y_N(y)$ must still be satisfied. The differential equation in $Y_N(y)$ can be found by substituting the assumed deflection for u into equation (22) together with the assumed deflection for ψ, which yields

$$\frac{d^2 Y_N}{dy^2} + \frac{E}{G}\left[\frac{m\omega^2}{E} - \frac{N^2 \pi^2}{l^2}\right]Y_N = \frac{E}{G}\frac{h}{2}\frac{b_N}{c_N}\left[\frac{\omega^2 m}{E} - \frac{N^2 \pi^2}{l^2}\right] \tag{32}$$

The homogeneous solution is

$$Y_H = A \sin\sqrt{\frac{E}{G}\left(\frac{m\omega^2}{E} - \frac{N^2 \pi^2}{l^2}\right)}y + B \cos\sqrt{\frac{E}{G}\left(\frac{m\omega^2}{E} - \frac{N^2 \pi^2}{l^2}\right)}y$$

and the particular solution is

$$Y_P = \frac{h}{2}\frac{b_N}{c_N}$$

so that

$$Y_N = Y_H + Y_P = \frac{h}{2}\frac{b_N}{c_N} + A \sin\sqrt{\frac{E}{G}\left(\frac{m\omega^2}{E} - \frac{N^2 \pi^2}{l^2}\right)}y + B \cos\sqrt{\frac{E}{G}\left(\frac{m\omega^2}{E} - \frac{N^2 \pi^2}{l^2}\right)}y \tag{33}$$

with the boundary conditions

$$Y(o) = 0 \quad \frac{dY(o)}{dy} = 0$$

Thus

$$B = -\frac{h}{2}\frac{b_N}{c_N} \quad \text{and} \quad A = -\frac{h}{2}\frac{b_N}{c_N} \tan\sqrt{\frac{E}{G}\left(\frac{m\omega^2}{E} - \frac{N^2 \pi^2}{l^2}\right)}b$$

or

$$Y_N(y) = -\frac{h}{2}\frac{b_N}{c_N}\left[\tan\sqrt{\frac{E}{G}\left(\frac{m\omega^2}{E} - \frac{N^2 \pi^2}{l^2}\right)}b \sin\sqrt{\frac{E}{G}\left(\frac{m\omega^2}{E} - \frac{N^2 \pi^2}{l^2}\right)}y - 1 \right.$$

$$\left. + \cos\sqrt{\frac{E}{G}\left(\frac{m\omega^2}{E} - \frac{N^2 \pi^2}{l^2}\right)}y\right] \tag{34}$$

Thus a function for $Y(y)$ has now been found to satisfy exactly the y variation of the u deflection.

B. Derivation of frequency equation

The deflections shown in the preceding section will now be combined with the differential equations (21), (22), and (23) to yield the frequency equation. These deflections could be substituted into the energy expressions and the energy variation with respect to the constants could be set equal to zero. This would determine a set of three simultaneous homogeneous algebraic equations. The determinant of these coefficients would be required to have a zero value in order that a nontrivial solution exist, and this determinant would yield the frequency equation. However, it appears much more advantageous in this case to substitute the deflection relationships into the derived differential equations and solve the determinant equation from this homogeneous set of equations. This method will be used here.

Substitution of the deflection relationships into equation (21) yields the equation

$$\sum_N A_s G \frac{N\pi}{l} b_N \sin \frac{N\pi x}{l} = \sum_N \left[A_s G \left(\frac{N\pi}{l}\right)^2 - m(A_1 + A_2)\omega^2 \right] a_N \sin \frac{N\pi x}{l}$$

Upon equating the coefficients of similar terms on each side of this equation it is seen that

$$b_N - \frac{A_s G \left(\frac{N\pi}{l}\right)^2 - m(A_1 + A_2)\omega^2}{A_s G \left(\frac{N\pi}{l}\right)} a_N = 0 \tag{35}$$

Now equation (22) has been satisfied exactly by the y variation of the deflection u, so that it need not be rewritten again. Finally, substitution of the deflection assumptions into equation (23) will yield the final equation needed. Upon equating coefficients of similar terms on each side of this equation it becomes

$$\left[-E(I_1 + tbh^2)\left(\frac{N^2\pi^2}{l^2}\right) - A_sG + m(I_1 + tbh^2)\omega^2 + \left\{2tEh\,\frac{N^2\pi^2}{l^2}\right.\right.$$

$$\left.\left.-2mth\omega^2\right\}\frac{c_N}{b_N}\int_0^b Y_N(y)dy\right]b_N + A_sG\,\frac{N\pi}{l}\,a_N = 0 \tag{36}$$

Thus equations (35) and (36) furnish two homogeneous equations in a_N and b_N which yield a frequency equation. Now the term $\frac{c_N}{b_N}\int_0^b Y_N(y)dy$ may be evaluated by direct integration to yield

$$\frac{c_N}{b_N}\int_0^b Y_N(y)dy = -\frac{hb}{2}\left[\frac{\tan\sqrt{\frac{E}{G}\left(\frac{m\omega^2}{E} - \frac{N^2\pi^2}{l^2}\right)}\,b}{\sqrt{\frac{E}{G}\left(\frac{m\omega^2}{E} - \frac{N^2\pi^2}{l^2}\right)}\,b} - 1\right]$$

Then with the condition for a nontrivial solution it is seen that the coefficients of a_N and b_N form a determinant whose value is zero. The frequency equation then becomes

$$\left[A_sG\,\frac{N^2\,\pi^2}{l^2} - m(A_1 + A_2)\omega^2\right]\left[-E(I_1 + tbh^2)\frac{N^2\,\pi^2}{l^2} - A_sG + m(I_1 + tbh^2)\omega^2\right.$$

$$\left. - \frac{hb}{2}\left\{2tEh\,\frac{N^2\,\pi^2}{l^2} - 2mth\omega^2\right\}\left\{\frac{\tan\sqrt{\frac{E}{G}\left(\frac{m\omega^2}{E} - \frac{N^2\,\pi^2}{l^2}\right)}b}{\sqrt{\frac{E}{G}\left(\frac{m\omega^2}{E} - \frac{N^2\,\pi^2}{l^2}\right)}b} - 1\right\}\right] + \left(A_sG\,\frac{N\pi}{l}\right)^2 = 0$$

Now it may be seen that the moment of inertia of the flange (I_2) can be approximated very closely by

$$I_2 \approx Ad^2 = 2(2bt)\left(\frac{h}{2}\right)^2 = tbh^2$$

Upon introduction of certain parameters, the frequency equation can then be written

$$N^2\,\pi^2 - k_B^2k_{RI}^2 - \frac{k_B^2}{N^2\,\pi^2 - k_B^2k_s^2} = \frac{I_2}{I_T}\left(k_B^2k_{RI}^2 - N^2\,\pi^2\right)\left(\frac{\tan K\sqrt{k_B^2k_{RI}^2 - N^2\,\pi^2}}{K\sqrt{k_B^2k_{RI}^2 - N^2\,\pi^2}} - 1\right) \qquad (37)$$

where

$$k_B^2 = \frac{A_Tm\omega^2\,l^4}{EI_T}$$

$$k_{RI}^2 = \left(\frac{r}{l}\right)^2 = \frac{I_T}{A_Tl^2}$$

$$k_s^2 = \frac{EI_T}{A_s G l^2}$$

$$K^2 = \frac{E}{G}\frac{b^2}{l^2}$$

with $A_T = A_1 + A_2$ and $I_T = I_1 + I_2$. A discussion of these parameters and their significance will be given in the following section.

If Timoshenko's beam equation is written in terms of these same parameters, the frequency equation is seen to be the somewhat similar equation

$$N^2 \pi^2 - k_B^2 k_{RI}^2 - \frac{k_B^2}{N^2 \pi^2 - k_B^2 k_s^2} = 0 \tag{38}$$

and in the case of elementary theory the frequency can be written

$$k_{B_e}^2 = N^4 \pi^4 \quad \text{or} \quad \omega_e = \frac{N^2 \pi^2}{l^2}\sqrt{\frac{EI_T}{mA_T}} \tag{39}$$

C. Definition of parameters

In the preceding section the frequency equation for a simply supported beam was derived and expressed in terms of four parameters. These parameters are

$$k_B^2 = \frac{A_T m \omega^2 l^4}{EI_T}$$

$$k_{RI}^2 = \left(\frac{r}{l}\right)^2 = \frac{I_T}{A_T l^2}$$

$$k_s^2 = \frac{EI_T}{A_s G l^2}$$

$$K^2 = \frac{E}{G}\frac{b^2}{l^2}$$

The parameter k_B can be easily recognized as the frequency parameter since it alone contains the circular frequency ω. The parameter k_{RI} gives a measure of the resistance of the beam to rotation and is defined as the rotary inertia parameter. It may be noted that this parameter varies directly as the radius of gyration. The parameter k_s varies inversely as the shear stiffness A_sG, and is thus defined as the shear stiffness parameter. This inverse relationship means that the larger the shear stiffness of a beam the smaller is the effect of transverse shear distortion. The last parameter is the parameter K which is directly proportional to the aspect ratio $\left(\frac{b}{l}\right)$ of a half-flange. It is this parameter that introduces the effect of shear lag. The introduction of shear lag can have a marked effect on the higher modes.

D. Discussion of results

The two frequency equations (eqs. (37) and (38) were solved for the frequency parameter, k_B, over a wide range of values of the parameters k_s, k_{RI}, and k. The values of k_B thus obtained are shown in table 1 for the three lowest modes of a beam. The frequencies are seen to be reduced by as much as 60 percent by the combined effect of shear lag, rotary inertia, and transverse shear as in the third mode. The effect of shear lag alone is seen to be over 35 percent in one instance.

TABLE 1.- RATIO OF TIMOSHENKO AND EXACT FREQUENCIES

TO ELEMENTARY FREQUENCIES

k_{RI} \ k_s	0	0.05	0.10	0.15	0.20	0.25	0.30
			$K = 0$ $N = 1$ (Timoshenko frequency)				
0	1.0000	0.9879	0.9540	0.9046	0.8468	0.7864	0.7277
.05	.9879	.9765	.9444	.8972	.8414	.7827	.7252
.10	.9540	.9444	.9170	.8757	.8257	.7717	.7176
.15	.9046	.8972	.8757	.8424	.8005	.7535	.7049
.20	.8468	.8414	.8257	.8005	.7675	.7289	.6872
.25	.7864	.7827	.7717	.7535	.7289	.6988	.6648
.30	.7277	.7252	.7176	.7049	.6872	.6648	.6382
			$K = 0.10$ $N = 1$ (Exact frequency)				
0	0.9856	0.9738	0.9414	0.8939	0.8380	0.7794	0.7221
.05	.9742	.9634	.9326	.8862	.8332	.7758	.7197
.10	.9427	.9330	.9067	.8666	.8180	.7653	.7123
.15	.8958	.8886	.8675	.8349	.7938	.7478	.7002
.20	.8408	.8350	.8196	.7947	.7622	.7241	.6830
.25	.7820	.7783	.7673	.7493	.7248	.6950	.6612
.30	.7247	.7222	.7146	.7019	.6842	.6618	.6354
			$K = 0.20$ $N = 1$ (Exact frequency)				
0	0.9477	0.9371	0.9083	0.8653	0.8143	0.7602	0.7068
.05	.9381	.9283	.9007	.8591	.8096	.7570	.7045
.10	.9114	.9029	.8786	.8411	.7966	.7477	.6979
.15	.8711	.8605	.8446	.8141	.7754	.7320	.6869
.20	.8223	.8173	.8021	.7784	.7469	.7105	.6713
.25	.7691	.7656	.7546	.7370	.7130	.6841	.6530
.30	.7156	.7131	.7059	.6937	.6746	.6533	.6273
			$K = 0.30$ $N = 1$ (Exact frequency)				
0	0.8962	0.8875	0.8627	0.8256	0.7809	0.7328	0.6846
.05	.8890	.8806	.8567	.8208	.7773	.7302	.6828
.10	.8681	.8606	.8392	.8066	.7664	.7222	.6770
.15	.8358	.8296	.8116	.7838	.7486	.7089	.6674
.20	.7950	.7902	.7760	.7537	.7245	.6905	.6538
.25	.7492	.7456	.7351	.7180	.6950	.6672	.6362
.30	.7017	.6991	.6916	.6790	.6617	.6401	.6148

$N = 1$ $k_{B_e} = 9.870$

TABLE 1.- RATIO OF TIMOSHENKO AND EXACT FREQUENCIES

TO ELEMENTARY FREQUENCIES - Continued

k_{RI} \ k_S	0	0.05	0.10	0.15	0.20	0.25	0.30
			$K = 0$ $N = 2$ (Timoshenko frequency)				
0	1.0000	0.9540	0.8468	0.7277	0.6227	0.5370	0.4686
.05	.9540	.9170	.8257	.7176	.6180	.5348	.4675
.10	.8468	.8257	.7675	.6872	.6033	.5277	.4639
.15	.7277	.7176	.6872	.6382	.5774	.5147	.4574
.20	.6227	.6180	.6033	.5774	.5398	.4942	.4467
.25	.5370	.5348	.5277	.5147	.4942	.4654	.4305
.30	.4686	.4675	.4639	.4574	.4467	.4305	.4082
			$K = 0.10$ $N = 2$ (Exact frequency)				
0	0.9477	0.9083	0.8143	0.7068	0.6094	0.5284	0.4629
.05	.9115	.8786	.7966	.6979	.6051	.5263	.4618
.10	.8222	.8021	.7470	.6712	.5918	.5198	.4584
.15	.7158	.7057	.6754	.6269	.5683	.5077	.4522
.20	.6171	.6123	.5974	.5711	.5336	.4886	.4422
.25	.5344	.5321	.5248	.5114	.4904	.4616	.4269
.30	.4673	.4661	.4624	.4556	.4447	.4282	.4056
			$K = 0.20$ $N = 2$ (Exact frequency)				
0	0.8419	0.8139	0.7442	0.6595	0.5783	0.5078	0.4488
.05	.8205	.7957	.7324	.6530	.5749	.5060	.4479
.10	.7622	.7448	.6978	.6330	.5643	.5004	.4449
.15	.6832	.6732	.6442	.5992	.5453	.4902	.4394
.20	.6008	.5957	.5802	.5535	.5168	.4743	.4306
.25	.5263	.5238	.5159	.5017	.4800	.4510	.4173
.30	.4632	.4619	.4579	.4505	.4387	.4215	.3986
			$K = 0.30$ $N = 2$ (Exact frequency)				
0	0.7456	0.7260	0.6752	0.6101	0.5441	0.4842	0.4323
.05	.7323	.7143	.6670	.6051	.5413	.4826	.4314
.10	.6947	.6807	.6428	.5899	.5326	.4777	.4286
.15	.6394	.6303	.6040	.5644	.5172	.4689	.4236
.20	.5759	.5707	.5549	.5289	.4945	.4554	.4158
.25	.5130	.5102	.5016	.4865	.4645	.4363	.4044
.30	.4561	.4546	.4501	.4420	.4292	.4113	.3884

$N = 2$ $k_{B_e} = 39.478$

TABLE 1.- RATIO OF TIMOSHENKO AND EXACT FREQUENCIES

TO ELEMENTARY FREQUENCIES - Concluded

k_{RI} \ k_S	0	0.05	0.10	0.15	0.20	0.25	0.30
			K = 0 N = 3 (Timoshenko frequency)				
0	1.0000	0.9111	0.7416	0.5932	0.4836	0.4044	0.3457
.05	.9111	.8516	.7185	.5855	.4820	.4031	.3453
.10	.7416	.7185	.6320	.5601	.4725	.3995	.3435
.15	.5932	.5855	.5601	.5138	.4534	.3919	.3403
.20	.4836	.4820	.4725	.4534	.4215	.3783	.3345
.25	.4044	.4031	.3995	.3919	.3783	.3551	.3240
.30	.3457	.3453	.3435	.3403	.3345	.3240	.3068
			K = 0.10 N = 3 (Exact frequency)				
0	0.8956	0.8256	0.6846	0.5552	0.4565	0.3836	0.3290
.05	.8358	.7838	.6674	.5491	.4542	.3826	.3285
.10	.7017	.6790	.6148	.5286	.4465	.3794	.3271
.15	.5686	.5611	.5355	.4896	.4306	.3730	.3244
.20	.4656	.4628	.4535	.4346	.4027	.3611	.3186
.25	.3895	.3884	.3847	.3773	.3635	.3406	.3101
.30	.3329	.3324	.3308	.3276	.3219	.3117	.2946
			K = 0.20 N = 3 (Exact frequency)				
0	0.7456	0.7035	0.6100	0.5132	0.4323	0.3688	0.3196
.05	.7162	.6811	.5988	.5084	.4302	.3679	.3191
.10	.6394	.6191	.5644	.4929	.4236	.3650	.3177
.15	.5439	.5353	.5083	.4640	.4106	.3592	.3150
.20	.4560	.4528	.4420	.4210	.3886	.3491	.3103
.25	.3856	.3843	.3800	.3714	.3560	.3321	.3024
.30	.3311	.3306	.3287	.3251	.3186	.3071	.2874
			K = 0.30 N = 3 (Exact frequency)				
0	0.6450	0.6172	0.5512	0.4766	0.4097	0.3546	0.3101
.05	.6266	.6022	.5426	.4724	.4078	.3536	.3096
.10	.5770	.5606	.5169	.4593	.4015	.3505	.3081
.15	.5100	.5013	.4758	.4361	.3899	.3449	.3052
.20	.4404	.4366	.4245	.4025	.3714	.3356	.3006
.25	.3787	.3771	.3719	.3618	.3449	.3210	.2932
.30	.3280	.3273	.3251	.3207	.3129	.3002	.2817

N = 3 k_{B_e} = 88.826

The shear lag effect increases with increased flange width but this shear lag effect also decreases with increase in either the transverse shear parameter or the rotary inertia parameter. To afford the reader an idea of the relative importance of these parameters, a plot of table 1 would be more useful. Figure 2 shows the variation of the frequency ratio $\frac{k_B}{k_{B_e}}$ (exact frequency $\div$ elementary frequency) with k_s for various values of k_{RI} and K for the three lowest modes. Thus, given a particular beam, these curves can be used to obtain the frequency ratio for any exact values of k_s and k_{RI} for each of four values of K. Then a simple plot of frequency ratio versus K can be made to obtain the exact value of K for the particular problem. As an example of this procedure, the frequency of the first mode of vibration for a 14WF87 beam will be computed. In this case let

l 110 inches

E 30×10^6 lb/sq in.

G 12×10^6 lb/sq in.

r 6.15 inches

b 7.25 inches

I_T 966.9 inches^4

A_s $0.420 \times 14 = 5.88$ inches^2

Now the values of the parameters in this case can be calculated

$$K = \frac{b}{l}\sqrt{\frac{E}{G}} = \frac{7.25}{110}\sqrt{\frac{30}{12}} = 0.1042$$

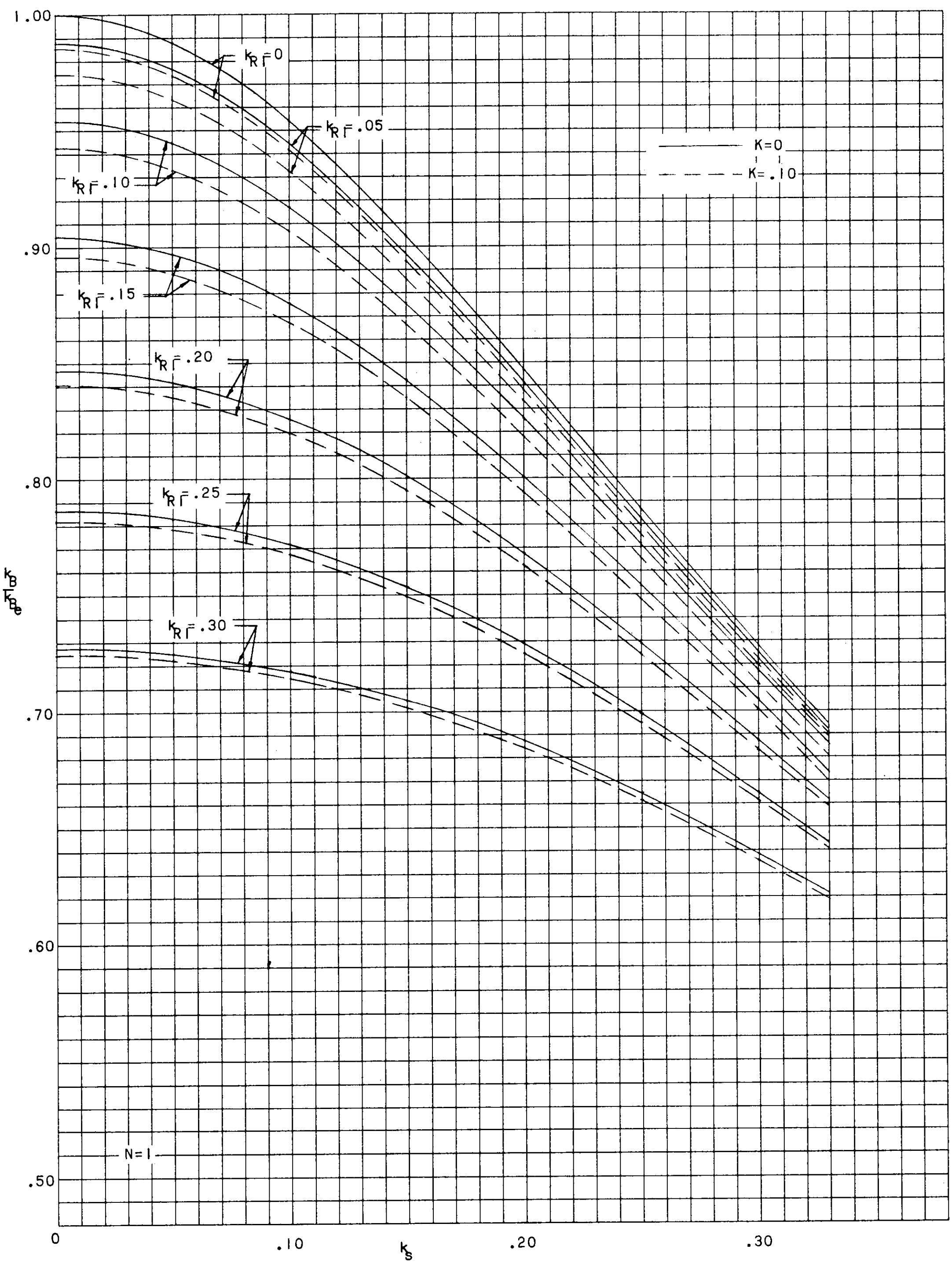

Figure 2.- Variations of the frequency ratio.

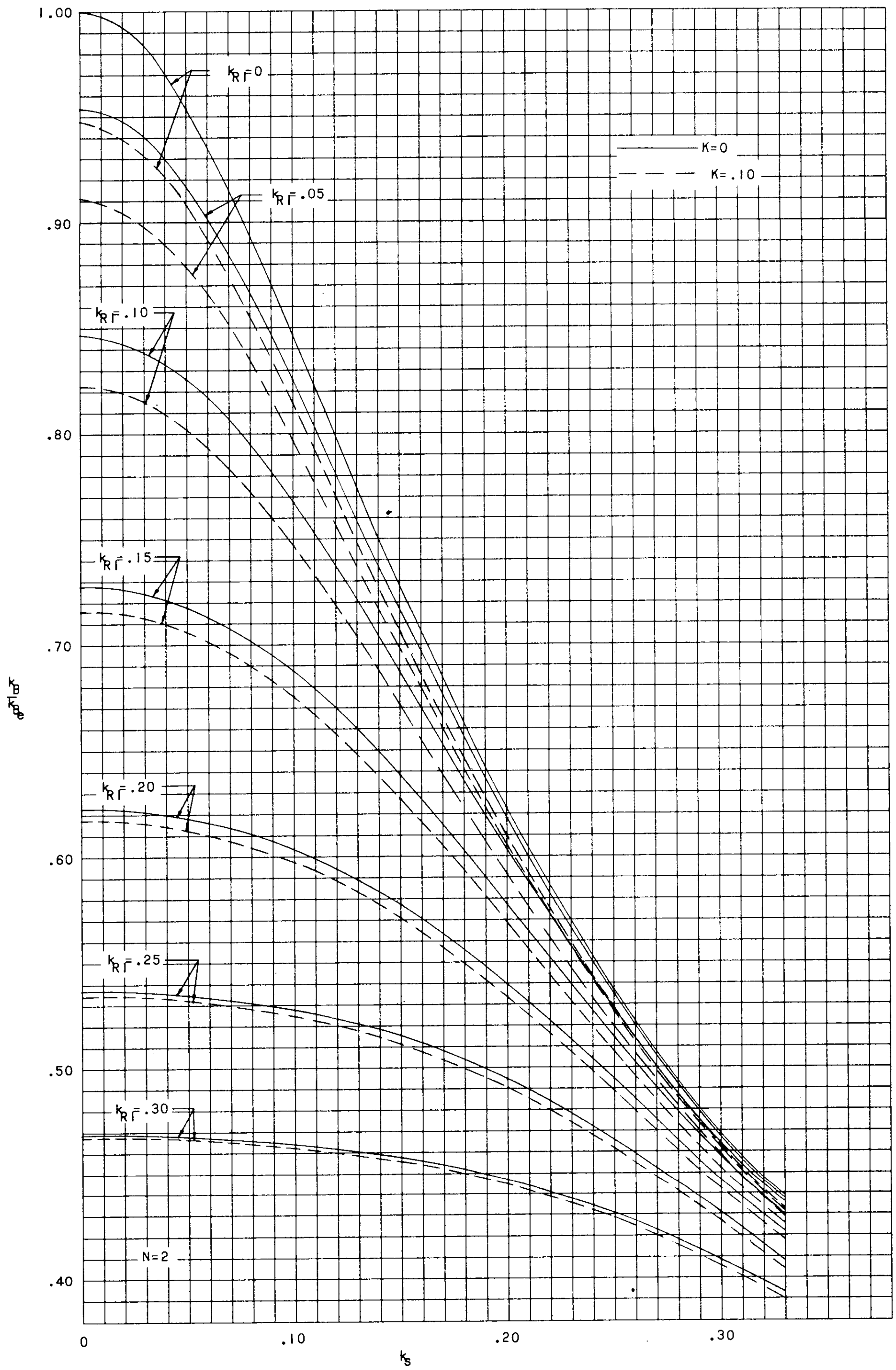

Figure 2.- Continued.

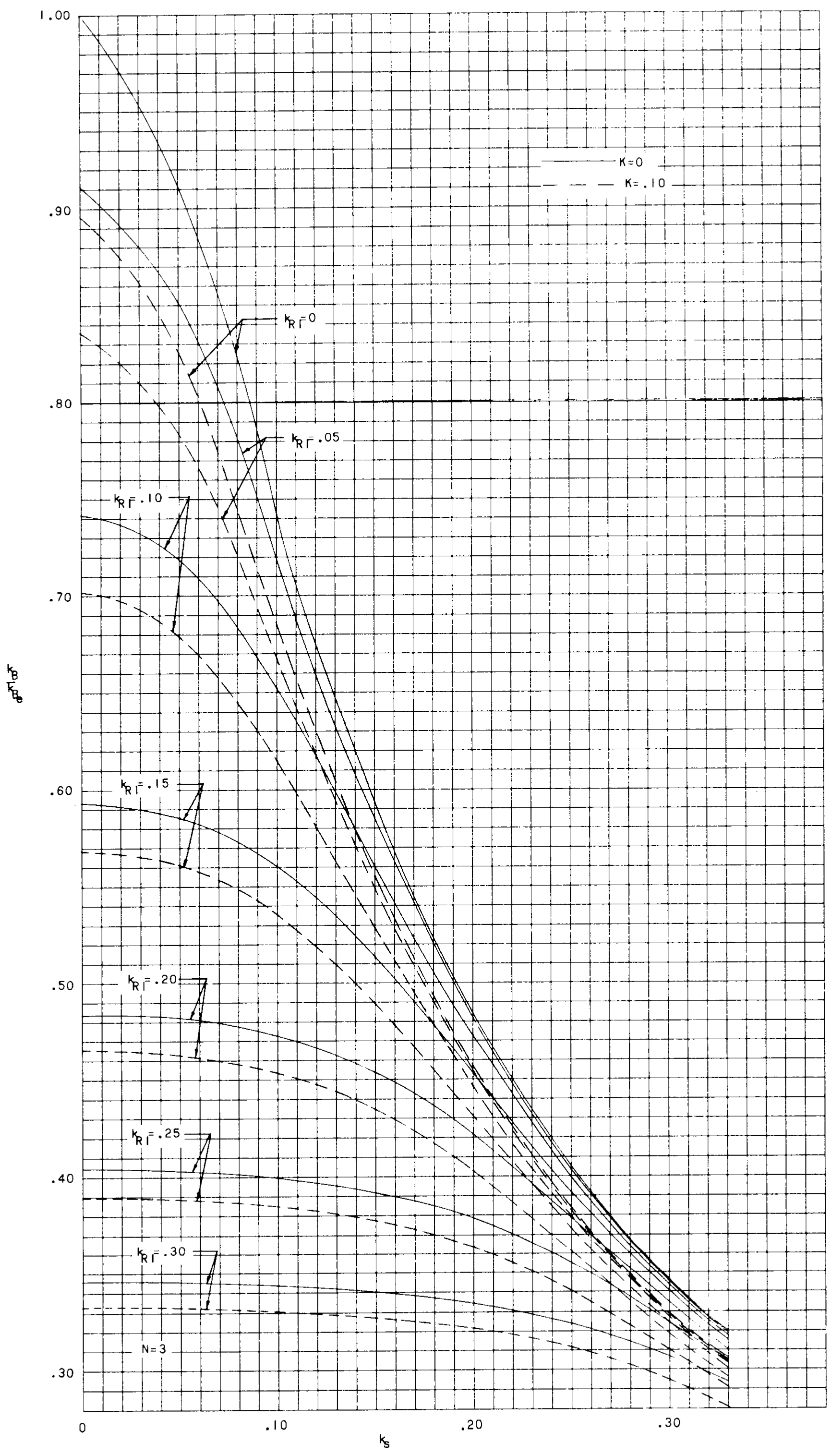

Figure 2.- Continued.

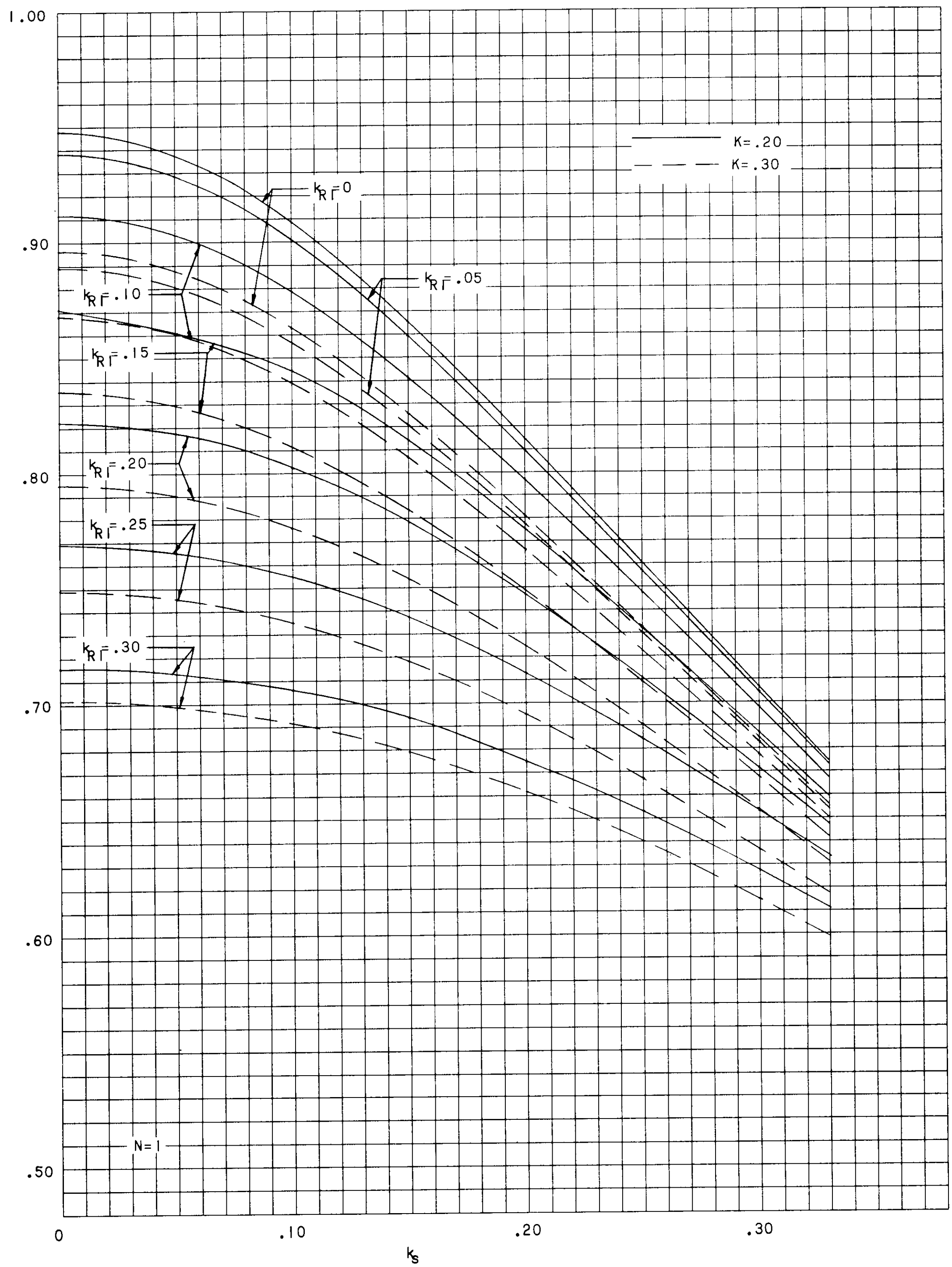

Figure 2.- Continued.

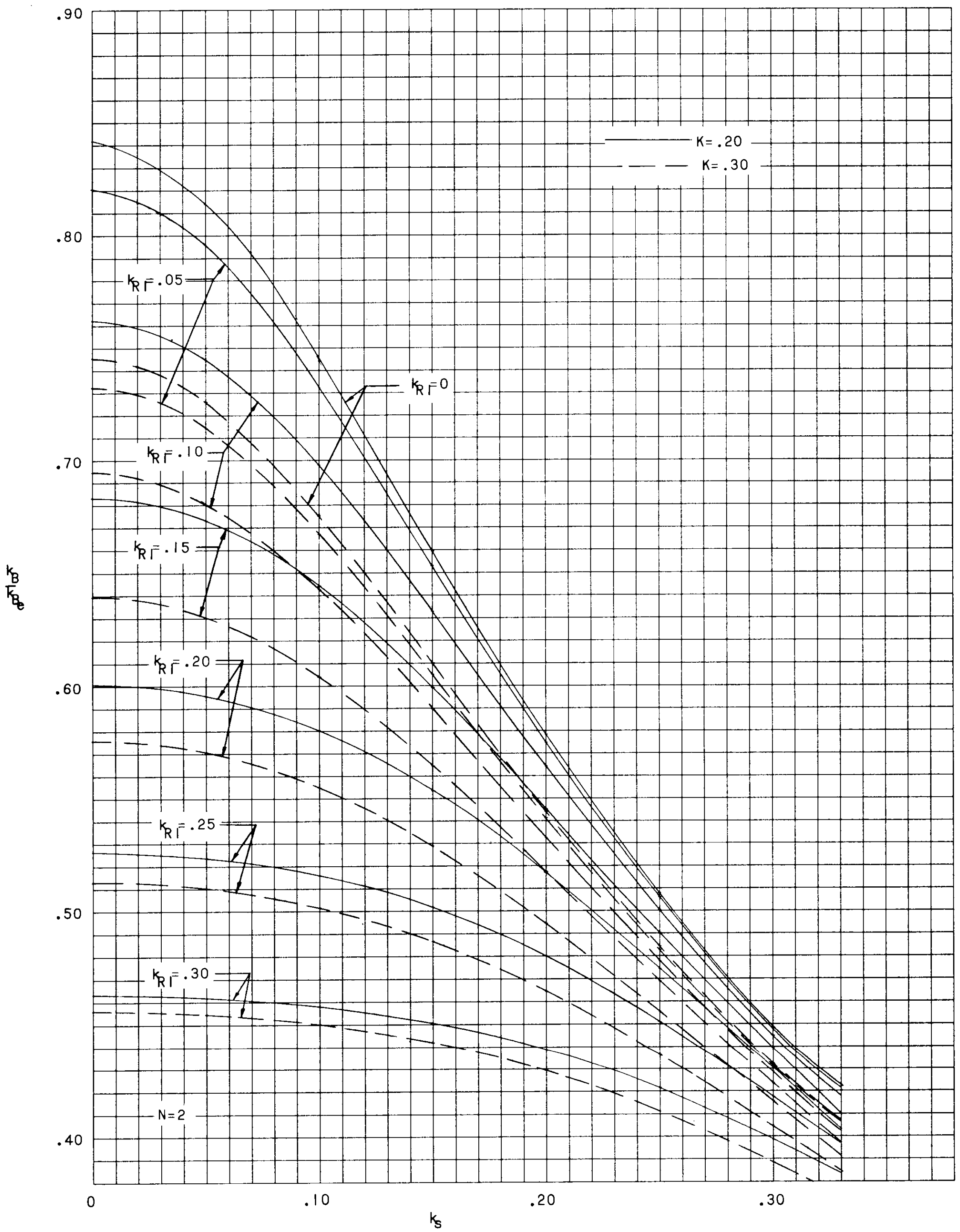

Figure 2.- Continued.

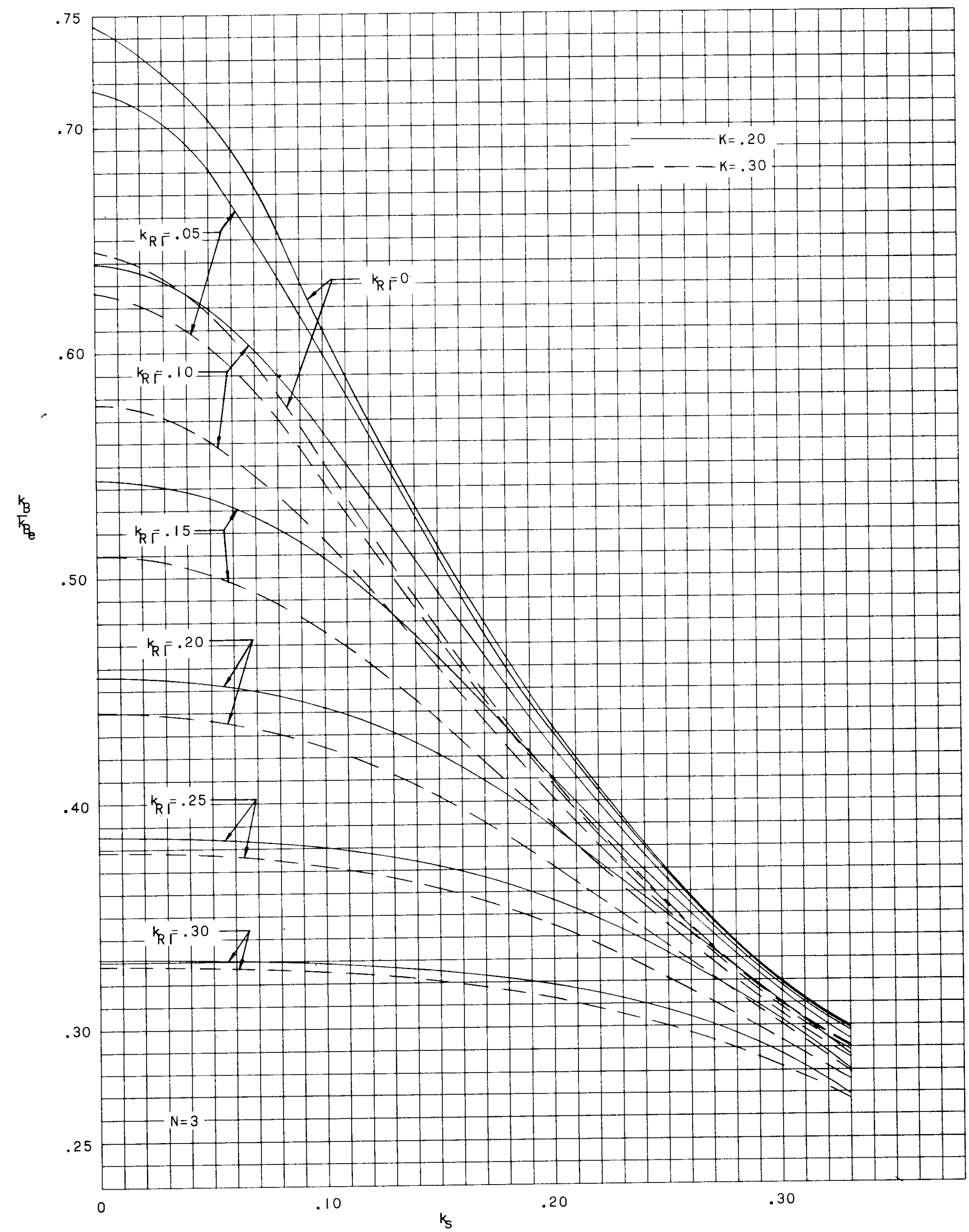

Figure 2.- Concluded.

$$k_{RI} = \frac{r}{l} = \frac{6.15}{110} = 0.0559$$

and

$$k_s = \frac{1}{l}\sqrt{\frac{EI_T}{A_s G}} = \frac{1}{110}\sqrt{\frac{30 \times 966.9}{12 \times 5.88}} = 0.1843$$

Now from figure 2 the following results can be obtained directly.

$N = 1$	$k_s = 0.184$	$k_{RI} = 0.056$	$K = 0$	$\frac{k_B}{k_{B_e}} = 0.858$
			$= .10$	$= .848$
			$= .20$	$= .823$
			$= .30$	$= .791$

These values can be plotted as shown in figure 3 and the lowest frequency in this case is 84.8 percent of elementary frequency. It can be seen that the effect of shear lag in this particular case is approximately 1.0 percent. The frequencies for the second and third modes (similarly determined) are shown to be reduced to 63.2 percent and 48.1 percent, respectively, with shear lag accounting for 1.3 percent and 3.0 percent, respectively.

Thus it appears that upon checking many commercially available wide-flange sections the effects of shear lag are relatively negligible. However, in cases where a very wide-flanged structure is specifically designed for a particular application, the effects of shear lag may become significantly important especially in those situations where the transverse shear effects are relatively small. Also, for higher modes shear lag becomes increasingly important.

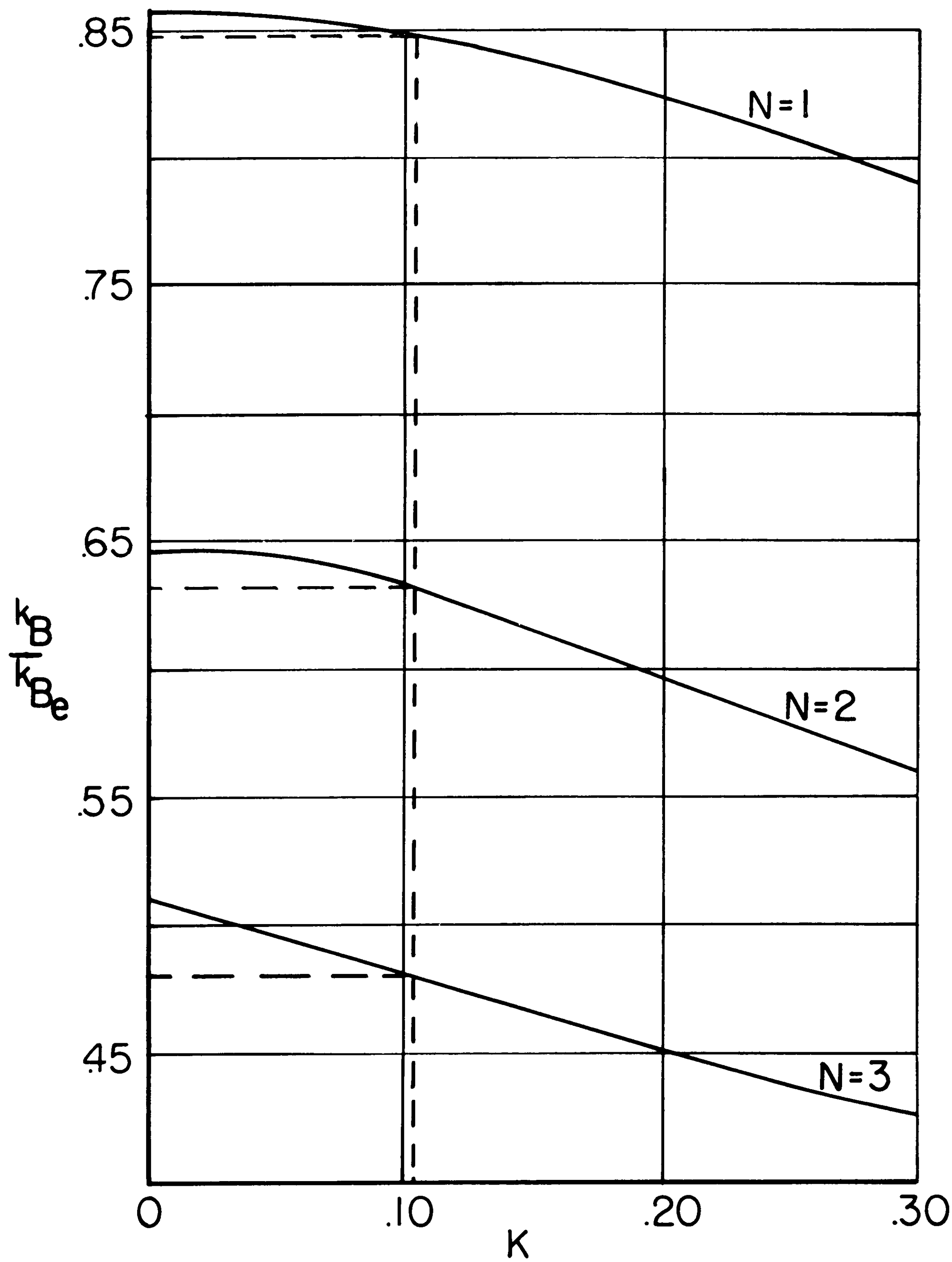

Figure 3.- Frequency determination for illustrative example.

XI. CONCLUSIONS

The effect of shear lag has been included in the equations for a vibrating beam. The significance of the effect has been determined for a wide range of parameters and a particular illustration of the use of these equations has been given.

In addition to the shear lag effect, the influences of transverse shear and rotary inertia have also been investigated. Consequently, the information available herein will permit a designer to include all such effects on frequencies within the range of parameters investigated. The large frequency reductions found indicate that these effects must be included for reasonable results.

In addition to the simply supported beam problem actually solved in closed form, it should be pointed out that solutions for beams with other types of supports may be obtained from the basic energy equations developed herein. The problem of relative flange bending can also be attacked (at least approximately) by employing either the energy equations or a direct attack on the differential equations using approximate boundary conditions.

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XIII. APPENDIX A

RELATIVE FLANGE BENDING

The investigation of the bending of the flanges relative to the web is an important point to consider when a structure with exceptionally wide flanges is vibrating. Assuming $\bar{w}$ is the additional deflection in the flange due to its bending, it can be seen that the energies for the web will remain the same. However, the flange energies become

$$T_2 = 2mt \int_0^b \int_0^l \left[\left(\frac{\partial u}{\partial t} - \frac{h}{2}\frac{\partial \psi}{\partial t}\right)^2 + \left(\frac{\partial \bar{w}}{\partial t} + \frac{\partial w}{\partial t}\right)^2\right] dA$$

and

$$V_2 = \frac{1}{2} \iiint \left[\epsilon_x \sigma_x + \epsilon_y \sigma_y + \tau_{xy}\gamma_{xy}\right] dxdydz \tag{A-1}$$

where

$$\epsilon_x = \left(\frac{\partial u}{\partial x} - \frac{h}{2}\frac{\partial \psi}{\partial x}\right) - z\,\frac{\partial^2 \bar{w}}{\partial x^2}$$

$$\epsilon_y = -z\,\frac{\partial^2 \bar{w}}{\partial y^2}$$

$$\sigma_x = \frac{E}{1-\mu^2}\left(\epsilon_x + \mu\,\epsilon_y\right)$$

$$\sigma_y = \frac{E}{1-\mu^2}\left(\epsilon_y + \mu\,\epsilon_x\right)$$

$$\tau_{xy} = G\gamma_{xy}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} - 2z\,\frac{\partial^2 \bar{w}}{\partial x\,\partial y}$$

$$G = \frac{E}{2(1+\mu)}$$

Substitution of these quantities into the above energy expressions amounts to adding the following integral to the original energy expressions.

$$\int_0^b \int_0^l \left[-2mt \left\{ 2 \frac{\partial w}{\partial t} \frac{\partial \bar{w}}{\partial t} + \frac{\partial \bar{w}}{\partial t}^2 \right\} + \frac{4Et^3}{2(1 - \mu^2)12} \left\{ \left(\frac{\partial^2 \bar{w}}{\partial x^2}\right)^2 \right.\right.$$

$$\left.\left. + \left(\frac{\partial^2 \bar{w}}{\partial y^2}\right)^2 + 2\mu \frac{\partial^2 \bar{w}}{\partial x^2} \frac{\partial^2 \bar{w}}{\partial y^2} + \frac{4(1 - \mu)}{2} \left(\frac{\partial^2 \bar{w}}{\partial x\, \partial y}\right)^2 \right\} \right] dxdy$$

It can be seen that, after using the variational procedure, the only change in the previous equations is the addition of $\int_0^b 4mt \frac{\partial^2 \bar{w}}{\partial t^2} dy$ to the w equation. All other equations and boundary conditions remain the same. By applying the variational procedure and integrating by parts an equation in $\bar{w}$ and the appropriate boundary conditions for $\bar{w}$ are obtained. This equation is

$$\frac{\partial^4 \bar{w}}{\partial x^4} + 2 \frac{\partial^4 \bar{w}}{\partial x^2\, \partial y^2} + \frac{\partial^4 \bar{w}}{\partial y^4} + \frac{mt}{D}\left(\frac{\partial^2 \bar{w}}{\partial t^2} + \frac{\partial^2 w}{\partial t^2}\right)$$

or

$$\nabla^4 \bar{w} = - \frac{mt}{D}\left(\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 \bar{w}}{\partial t^2}\right) \qquad \text{(A-2)}$$

with the boundary conditions

$$\left(\frac{\partial^2 \bar{w}}{\partial x^2} + \mu \frac{\partial^2 \bar{w}}{\partial y^2}\right)\delta\bar{w} \Big|_x = 0 \qquad \left(\frac{\partial^3 \bar{w}}{\partial x^3} + (2 - \mu) \frac{\partial^3 \bar{w}}{\partial x\, \partial y^2}\right)\delta\left(\frac{\partial \bar{w}}{\partial x}\right)\Big|_x = 0$$

$$\left(\frac{\partial^2 \bar{w}}{\partial y^2} + \mu \frac{\partial^2 \bar{w}}{\partial x^2}\right)\delta\bar{w} \Big|_y = 0 \qquad \left(\frac{\partial^3 \bar{w}}{\partial y^3} + (2 - \mu) \frac{\partial^3 w}{\partial x^2\, \partial y}\right)\delta\left(\frac{\partial \bar{w}}{\partial y}\right)\Big|_y = 0 \qquad \text{(A-3)}$$

Inspection of the $\bar{w}$ equation shows it to be the same form as the equation of forced vibration of a plate, and the boundary conditions are those appropriate to plate theory. Thus the effect of relative flange bending appears only as a plate equation with mass coupling between the relative flange bending $(\bar{w})$ and the beam deflection (w). The inclusion of this single effect then completes the picture of structural vibration of wide-flanged structures. It appears wise to note that in the case of the illustrative example herein, the $\bar{\ }$ equation becomes the forced vibration of a cantilever-free-free plate. An exact analytical solution for this problem is not available at present.

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EFFECTS OF SHEAR DEFORMATIONS ON THE VIBRATIONAL FREQUENCIES OF WIDE-FLANGED STRUCTURES

by

Deene J. Weidman

ABSTRACT

The well-known Timoshenko beam equations (which include transverse shear deformation and rotary inertia effects) are extended for a wide-flanged structure to include the additional shear lag deformation of the flanges; thus, cross-sections of the beam are allowed to distort instead of remaining plane sections. The effect of relative flange bending (bending of the flanges relative to the web) is also included and the integro-differential equations appropriate to the problem are derived. The frequency equation is given in closed form (neglecting the relative flange bending) and solutions for various values of the nondimensional parameters are given. A reduction of the elementary frequency by as much as 40 percent in the first mode is shown.