REDISTRIBUTION OF BENDING MOMENT

IN

CONTINOUS STRUCTURES OF REINFORCED CONCRETE

by

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I. INTRODUCTION

Working stresses in concrete result in strains or deformations which can be considered in two parts: (1) elastic deformations which are instantaneous and proportional to the applied stresses and (2) inelastic or creep deformations that occur continuously and at a decreasing rate over a finite period of time. In concrete, the creep time may be up to five years or longer, and the total amount of inelastic deformation will usually be proportional to and about two to six times the amount of elastic deformation.[1]

Because creep in steel at working stresses is essentially zero,[2] reinforced concrete experiences a non-homogeneous and essentially linear pattern of creep strain across any stressed section in the compression zone. In indeterminate structures of reinforced concrete this creep pattern may result in the redistribution of stresses, movement of points of inflection and changes in the bending moment values at any section.

This phenomenon is examined analytically in this thesis. Moment-curvature relations are developed and the action of continuous reinforced concrete beams under load is predicted.

II. THEORIES OF CREEP

A. Water and Cement in Concrete*

When water is added to a concrete mix it begins to dissolve the surface of the cement particles, forming first, a colloid and then a series of solutions. As solution progresses, heat is evolved causing the colloid to form a gel of calcium silicate which acts as a binding agent between the aggregate particles; that is, the concrete starts to set. Meanwhile, more water attacks the cement particles and forms a mixture of colloid, gel, solutions and free water which has not yet reached the surface of the cement particles. This free water may be squeezed out of the concrete while casting or lost by evaportion during setting. New colloid, as it is formed, turns into gel very rapidly. When the mixture begins to dry out the solutions evaporate and crystals of calcium sulphoaluminate and calcium hydroxide are formed in the minute crevices or micropores which are present in the gel structure.

The water content of concrete can be subdivided into three main types: (a) that firmly held or chemically-combined, as in $Ca\ (OH)_2$; (b) that loosely held such as colloidal water; and (c) that mechanically held as free water. The proportion of free water, initially high, is reduced

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* For a more extensive study of this topic, the reader is referred to References 3 and 4.

considerably by losses in evaporation and hydration of cement, while the proportion of loosely-held water in colloid or gel is progressively reduced by the conversion of the gel to a more stable crystalline form. As a result, the concrete hardens with age and the gradual crystallization causes a corresponding change in the physical properties.

B. Theories of Creep

A number of different theories[5] explaining the creep in concrete have been put forward, but the more generally accepted are the plastic theory, the viscous theory and the seepage theory.

1. Plastic Theory

It has been suggested by Vogt[6] that the creep of concrete may be of the nature of crystalline flow, i.e. a result of slipping along planes within the crystal lattice. This would be similar to plastic flow of a metal. Vogt observed that in some respects the mode of deformations of concrete is similar to that of cast iron and other brittle materials. Plastic flow of metal is a non-elastic deformation which occurs only when a particular stress is exceeded. Some experiments have led Bingham and Reiner[7] to suggest that mortar acts as a plastic solid with a yield point of 65 psi.

2. Viscous Theory

The viscous theory assumes the creep in concrete to be of the form of viscous flow with movement of particles one over another as in the flow of liquid, asphalt, glue or wet clay. According to Thomas[8] concrete consists of two parts: (a) cementitious material, which behaves in this viscous manner when loaded, and (b) enclosed aggregate particles which do not themselves flow under load but are caused to move and react to the viscous flow of the cementitious material. When concrete is loaded, the flow of the cementitious material is resisted by the presence of the aggregate particles. As a result of this resistance, some force is transferred from the cementitious material to the aggregate particles. Since the creep of concrete is proportional to the applied stress, the rate of creep will be progressively reduced as the load is transferred from the viscous material to the inert aggregate particles.

3. Seepage Theory

According to the seepage theory,[9] creep in concrete is due to the loss of liquid from a mass composed of a liquid and a finely divided solid. Such colloidal seepage is common in rigid gels. According to Laynam's theory,[4] the colloidal particles are held in a liquid medium (water) by the equilibrium of a number of forces such as crystal-forming tendencies, surface tension, solid to liquid attraction, and electrical attraction or repulsion. When a colloidal solution turns into a gel, a network of solid phase material permeated by a liquid phase material is formed. Despite its rigid state, the gel can hold an enormous amount of free water which can be lost by evaporation or repelled by pressure. Wetting of the set cement paste results in the gel taking up liquid and swelling; conversely, on drying, expulsion of water and contraction takes place.

4. Other Theories

Among the other theories explaining creep in concrete that have been suggested, there is the Mechanical Deformation Theory by Freyssinet[10,11] and the Elastic After-Effect Theory by Maney.[12]

5. Analysis of the Theories

The three most acceptable theories of creep in concrete have been described. There are many good reasons why each of these theories may be accepted but also, there are a number of phenomena which cast doubts on the validity of each of the theories.

The fact that creep occurs even when moisture movement from the concrete to the surrounding medium is prevented indicates that seepage cannot account for the major portion of creep. From the viscous theory it is known that the rate of creep depends on the nature of the cement but not on the nature of the aggregate. Experimental data[13] show that the creep of neat cement can be as much as 15 times as great as the creep of a 1:6 (cement to aggregate) concrete and twice that of a 1:3 (cement to sand) mortar under similar conditions. If the aggregate were completely inert the creep of neat cement would be decreased numerically in proportion to the aggregate-cement ratio. Thus, in fact the aggregate has a considerable influence on the magnitude of creep. There is no evidence of slipping along planes in the lattice structure of the crystalline products of hydration of cement. Protest's[14] test on concrete members subjected to alternating loading have shown that after a number of alternations, concrete ceases to creep and behaves like a perfectly elastic material. No permanent lateral deformation occurs, which is in marked contrast to the plastic flow of metals. Thus crystalline flow can not be wholly responsible for creep.

Actually, creep of concrete under sustained stress cannot be attributed to a single cause but rather to a combination of causes. At the present stage, it is impossible to tell definitely what proportion of creep is due to each of the influencing factors. Moreover, it is probable that these proportions depend on other conditions as well as stress to which the concrete is subjected. It appears, however, that viscous flow is responsible for a latge portion of the creep.

C. <u>Magnitude and Rate of Creep in Concrete</u>

Results of recent laboratory tests[15] point to the conclusion that within the range of design stresses, creep strain or unit creep is proportional to the stress. The creep-stress relation for a constant sustained load may be written in the form: $\epsilon_c = \sigma \cdot f(t)$ where ϵ_c is creep strain, σ is stress and $f(t)$ is a function of time.

From test results published by Glanville[16] and by Rose[17] it appears that a linear relation exists between stress and creep strain which is exact for stresses from 0 to 650 psi. and approximate for stresses from 650 to 1500 psi.

Creep strain varies while the stress remains constant. Typical creep-time curves[18] are shown below:

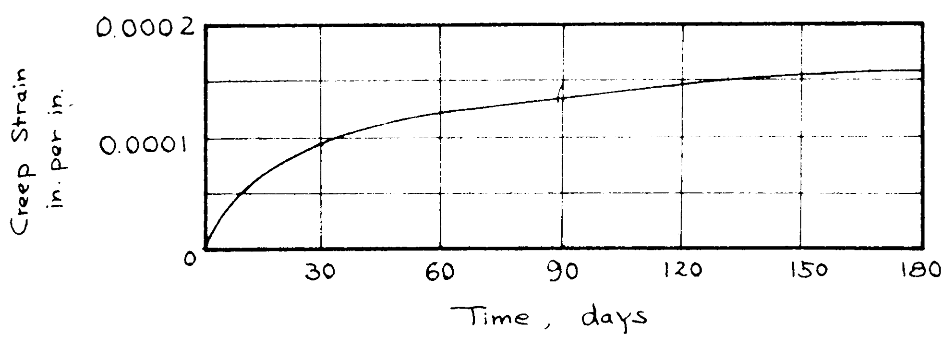

Fig. 1. Creep-Time Curve

Various types of creep-time equations have been proposed to predict the experimental results and take account of the various parameters affecting creep. Probably the most outstanding contribution is a hyperbolic form proposed by Rose:

Assuming $$f(t) = \frac{mt}{n_1' + t}$$

then $$\epsilon_c = \frac{mt}{n_1' + t} \cdot \sigma$$

where t is time and m and n_1' are constants.

When t becomes very large, the quotient approaches unity, and the limiting value of creep strain is then $\epsilon_c = m\sigma$

Now, setting t equal to n', $\epsilon_c = \frac{m}{2}\sigma$ whence it follows that 50 percent of ultimate creep strain is obtained at the time $t = n'$. Values of m and n can be determined from test data.

In the preceding discussion, the results are based only on the consideration that the measured stress is applied instantaneously and maintained constant. A solution for the case of gradual loading over a finite period of time ordinarily can be obtained by a process of summation since the effects of component stresses can be superimposed as long as the total stress does not exceed the limits of the existance of the linear relation between stress and creep strain.

If the load is applied at a time $t = \tau$, then;

$$\epsilon_c = \frac{m(t-\tau)}{n' + (t-\tau)} \cdot \sigma$$

If the stress is applied in two separate increments so that the stress increment $\Delta\sigma_1$ becomes instantly effective at the time $t = \tau_1$, and the stress increment $\Delta\sigma_2$ becomes effective at the time $t = \tau_2$, the final stress is equal to: $\sigma_0 = \Delta\sigma_1 + \Delta\sigma_2$

Since creep constants generally vary with the age of the concrete at the time of load application, it becomes necessary to use two sets of constants, and:

$$\epsilon_c = \frac{m_1(t-\tau_1)}{n'_1+(t-\tau_1)}\Delta\sigma_1 + \frac{m_2(t-\tau_2)}{n'_2+(t-\tau_2)}\Delta\sigma_2 \qquad t \geq \tau_2 > \tau_1$$

If a large number P of stress increments $\Delta\sigma$ are applied at regular time intervals, the creep-time relation is expressed in the form:

$$\epsilon_c = \sum_{x=1}^{x=P} m_x \frac{t-\tau_x}{n'_x+(t-\tau_x)} \frac{\Delta\sigma_x}{\Delta\tau_x} \cdot \Delta\tau_x \qquad t \geq \tau_P$$

If creep constants are available the creep strain can be computed with a high degree of accuracy.

III. MOMENT - CURVATURE RELATIONS

A. General

Structural elements subjected to bending moments experience a change in curvature. The change in curvature of non-curved elastic elements due to bending moment alone is the familiar relation for curvature:

$$\phi' = \frac{1}{\text{Radius of Curvature}} = \frac{1}{\rho} = \frac{M}{EI}$$

which is derived in the next section.

A geometric analysis of any structure subjected to a variation of moment along the lengths of its members will result in the determination of curvatures and deflections for that structure. Many techniques for the analysis of structure have been developed on these fundamental concepts.*

In the case of reinforced concrete structures, only very approximate values for curvature and deflections can be obtained by elastic analysis because of the subsequent creep of concrete under load. In the case of homogeneous materials (or where homogeneity is assumed),

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* See any text on Indeterminate Structural Analysis, for example: Kinney - INDETERMINATE STRUCTURAL ANALYSIS; Norris & Wilbur - ELEMENTARY STRUCTURAL ANALYSIS.

where creep strains are directly proportional to elastic strains, and particularly in statically determinate structures, the solution for curvature and deflections can be obtained by multiplying the elastic solution by some creep factor. In the case of non-homogeneous materials and statically indeterminate structures the problem becomes more involved.

In attempting to reach the objectives of this thesis investigation, a preliminary analysis is made for the effect of creep on the resulting moment-curvature relation for a reinforced concrete section of a beam-column element. Section D-1 following, deals with a concentric load (axial); Section D-2 deals with an eccentric load (axial plus bending); and then sections D-3 and D-4 deal with moment loads on simply and doubly reinforced concrete beam elements.

B. Symbols and Notations

A_c Area of Concrete

A_s Area of the Tension Steel

A_s' Area of Compression Steel

b Width of the Beam or Column

C Creep Coefficient

C' Distance from the Surface of the Section to the Center Gravity of the Section

d Distance from the Compression Surface of the Section to the Center Gravity of the Tension Steels

d_2 Distance from the Compression Surface of the Section to the Center Gravity of the Compression Steels

ϵ Elastic Strain

ϵ_c Elastic Strain of the Concrete

ϵ_s Elastic Strain of the Steel

ϵ_c' Total Strain of the Concrete after Creep

ϵ_s' Strain of Steel after Creep of the Concrete

E Modulus of Elasticity

E_c Modulus of Elasticity of Concrete

E_s Modulus of Elasticity of Steel

f_c' Ultimate strength of Concrete

f_s	Allowable Stress in Steel
I	Moment of Inertia
jd	Internal Moment Arm
$kd, k'd$	Distance from the Compression Surface to the Neutral Axis of the Section
M	Bending Moment
n	Ratio of Es to Ec
p	Proportion of Tensile Steel - $\frac{A_s}{bd}$
p'	Proportion of Compression Steel - $\frac{A'_s}{bd}$
P_c	Total Compressive Force in Concrete before Creep
P'_c	Total Compressive Force in Concrete after Creep
P_E	Externally Applied Load
P_{s1}	Total Force in Tensile Steel before Creep
P'_{s1}	Total Force in Tensile Steel after Creep
P_{s2}	Total Force in Compression Steel before Creep
P'_{s2}	Total Force in Compression Steel after Creep
t	Thickness or Depth of a Beam or Column
w	Uniform Distributed Loads
σ	Stress
σ_c	Stress in Concrete before Creep
σ'_c	Stress in Concrete after Creep

σ_s	Stress in Steel before Creep
σ_{s1}	Stress in Tensile Steel before Creep
σ'_{s1}	Stress in Tensile Steel after Creep
σ_{s2}	Stress in Compression Steel before Creep
σ'_{s2}	Stress in Compression Steel after Creep
ϕ'	Curvature

C. Elastic Analysis[19]

A beam, initially straight, is acted upon by bending moments and as a result experiences deflections and a change in shape. The relationship between the change in shape of the beam and the applied bending moment will be developed. The following theory of bending is based on the assumptions that

(1) Plane sections remain plane and normal to the longitudinal fiber stresses after bending.

(2) The material is homogeneous and obeys Hooke's law.

(3) The beam is initially straight and prismatic.

(4) The moduli of elasticity in tension and compression are equal.

Under the acting bending moment M, the element mnop changes its shape as Fig. 2.

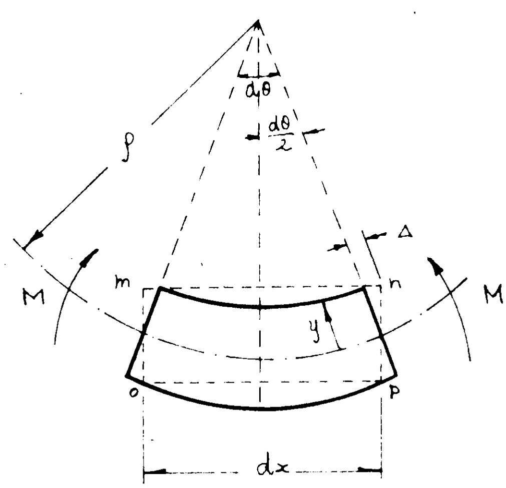


Fig. 2

Using the usual notation (see Page 17, 18 and 19), from the material properties it follows that:

$$\epsilon = \frac{\sigma}{E} \tag{1}$$

$$\sigma = \frac{My}{I} \tag{2}$$

and from Fig. 2 and the given assumptions:

$$\epsilon = \frac{\Delta}{dx/2} \tag{3}$$

$$\frac{\Delta}{y} = \frac{dx/2}{\rho} \tag{4}$$

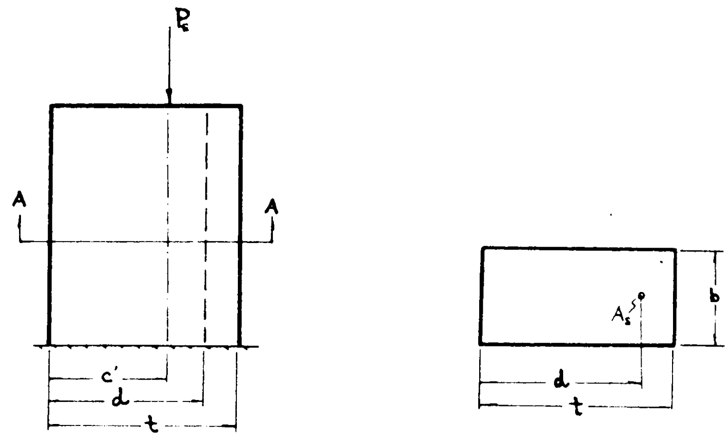
$$\therefore \frac{1}{\rho} = \frac{\Delta}{dx/2} \cdot \frac{1}{y} = \epsilon \cdot \frac{1}{y} \tag{5}$$

Combining equations (2) and (5):

$$\frac{1}{\rho} = \frac{\sigma}{Ey} = \frac{M}{EI} \tag{6}$$

D. Creep Analysis

1. Rectangular Column with Unsymmetrical Reinforcement Subjected to Axial Load.

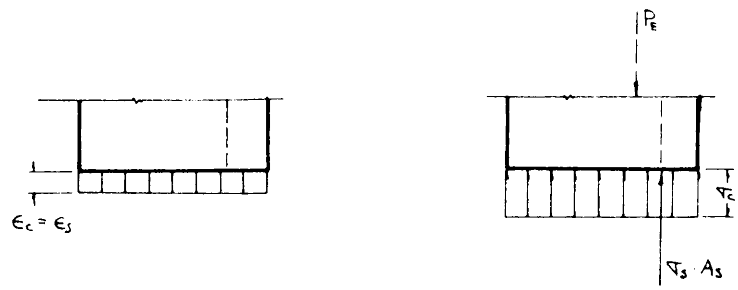


(a)

Sec. A-A
(b)

Fig. 3

The force P_E acts at the centroid of the transformed section. Before creep takes place, the materials of the section act elastically. The strain and stress distribution diagrams are as shown in Fig. 4.



(a) Strain Diagram

(b) Stress Diagram

Fig. 4

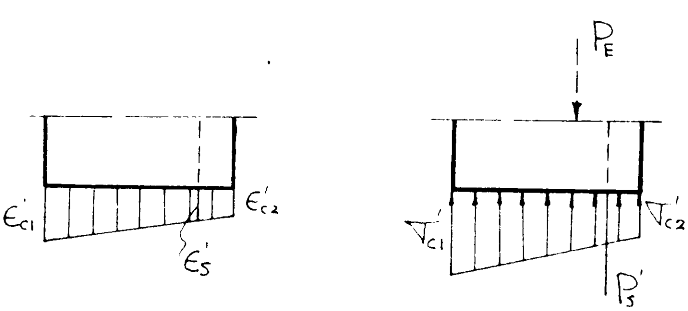
Using the transformed area method, the reactive forces of the concrete and the steel are:

$$P_c = \frac{P_E A_c}{A_c + n A_s}$$

$$P_s = \frac{n P_E A_s}{A_c + n A_s}$$

After a time, creep in the concrete causes the steel to experience further straining. The concrete is relieved of some of its force. And the stress in the steel increases. The strain in the steel must be consistent with the strain in the concrete and the sum of the reactive forces of the steel and concrete must be equal to the applied load.

The strain and stress distribution diagrams of the section at the final stage of equilibrium are as shown in Fig. 5.



(a) Strain Diagram (b) Stress Diagram

Fig. 5

From Fig. 5 and the known properties of the materials, following equations can be established:

$$\epsilon'_{c1} = C \frac{\sigma'_{c1}}{E_c} \qquad (7)$$

$$\epsilon'_{c2} = C \frac{\sigma'_{c2}}{E_c} \qquad (8)$$

Where C is the coefficient of creep.

$$\epsilon_s' = \frac{\sigma_s'}{E_s} \qquad (9)$$

From Fig. 5 (a):

$$\epsilon_s' = \frac{(t-d)\,\epsilon_{c1}' + d\,\epsilon_{c2}'}{t} \qquad (10)$$

The equilibrium condition gives:

$$P_E = P_s' + P_c' \qquad (11)$$

$$P_s' = \sigma_s' \cdot A_s \qquad (12)$$

$$P_c' = bt\ (\sigma_{c2}' + \tfrac{1}{2}(\sigma_{c1}' - \sigma_{c2}')] \qquad (13)$$

And $$P_E \cdot C' = \frac{1}{2} bt^2 \sigma_{c2}' + \frac{1}{6} bt^2 (\sigma_{c1}' - \sigma_{c2}') + P_s' \cdot d \qquad (14)$$

Solving these equations,

$$\sigma_{c1}' = \frac{P_E \{12\,cnd\,A_s(c'-d) + 2bt^3(3c'-2t)\}}{\{12\,cnbt^2 d\,A_s - b^2t^4 - 4cnbt\,A_s(t^2+3d^2)\}} \qquad (15)$$

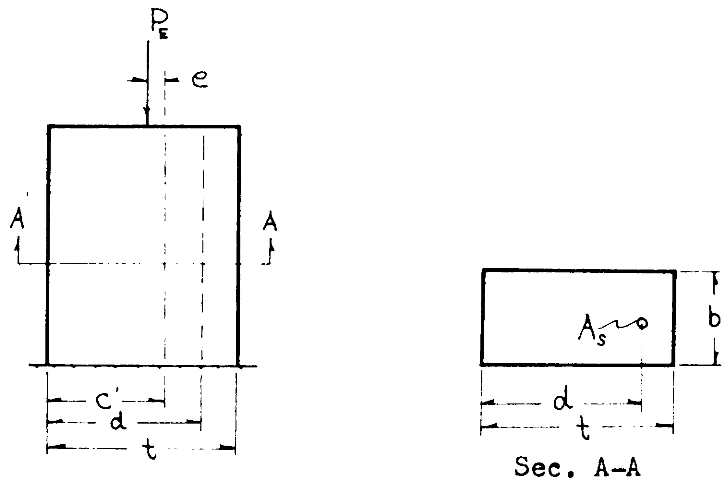
$$\sigma_{c2}' = \frac{2t\,P_E - [2cn\,A_s(t-d) + bt^2]\,\sigma_{c1}'}{2cn\,A_s d + bt^2} \qquad (16)$$

and

$$\sigma_s' = \frac{2cnd\,P_E - Cnbt\,(2d-t)\,\sigma_{c1}'}{2cn\,A_s d + bt^2} \qquad (17)$$

and the stress distribution can be plotted.

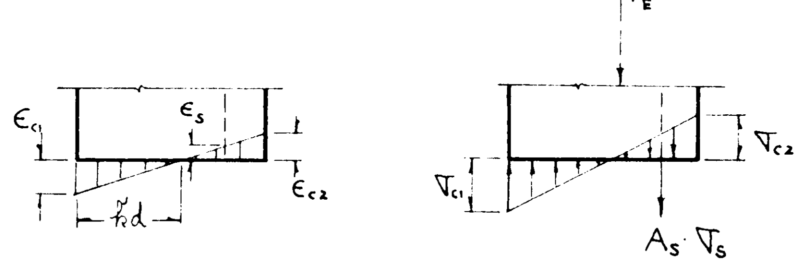
2. Rectangular Column with Unsymmetrical Reinforcement under Eccentric Load.



(a) (b)

Fig. 6

The force P_E acts at a small distance e from the centroid of the transformed section. Before creep takes place, the strain and stress distribution diagrams are as shown in Fig. 7.



(a) Strain Diagram (b) Stress Diagram

Fig.7

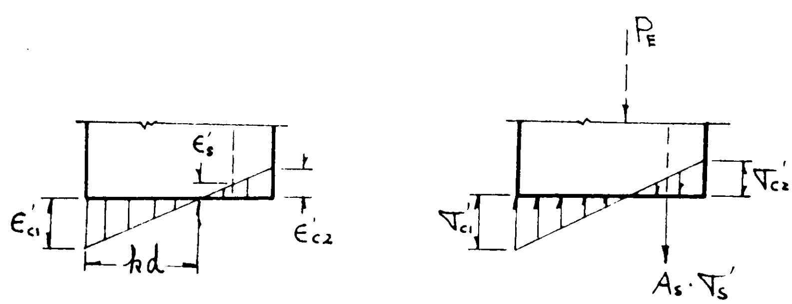
The equilibrium condition gives:

$$\sigma_{c1} = \frac{P_E}{A_c + nA_s} + \frac{P_E \cdot e \cdot kd}{I} \tag{18}$$

$$\sigma_{c2} = \frac{P_E}{A_c + nA_s} \mp \frac{P_E \cdot e \cdot (t - kd)}{I} \tag{19}$$

and $$\sigma_s = n \left[\sigma_{c2} - \frac{t-d}{t} (\sigma_{c2} + \sigma_{c1}) \right] \tag{20}$$

After the creep of the concrete takes place, the strain and stress distribution diagrams will be as shown in Fig. 8.



(a) Strain Diagram (b) Stress Diagram

Fig. 8

The following equations can be established by the properties of materials, equilibrium and the geometry of Fig. 8 (a).

$$\epsilon_{c1}' = C \frac{\sigma_{c1}'}{E_c} \tag{21}$$

$$\epsilon_{c2}' = C \frac{\sigma_{c2}'}{E_c} \tag{22}$$

$$\epsilon_s' = \frac{\sigma_s'}{E_s} \tag{23}$$

$$\epsilon_s' = \epsilon_{c2}' - \frac{t-d}{t}(\epsilon_{c1}' + \epsilon_{c2}') \tag{24}$$

$$\frac{kd}{t} = \frac{\epsilon_{c1}'}{\epsilon_{c1}' + \epsilon_{c2}'} \tag{25}$$

$$P_s' = \sigma_s' \cdot A_s \tag{26}$$

$$P_c' = \frac{1}{2} b \cdot kd \cdot \sigma_{c1}' - \frac{1}{2} b (t - kd) \sigma_{c2}' \tag{27}$$

$$P_E = P_c' - P_s' \tag{28}$$

$$P_E (c' - e) + P_s' d + \frac{\sigma_{c2}'}{2} b (t - kd)\left(kd + \frac{2(t-kd)}{3}\right) - \frac{1}{6} b (kd)^2 \sigma_{c1}' = 0 \tag{29}$$

Combining equations (21) through (29),

$$\sigma_{c1}' = \frac{2kd\, P_E}{\left[(kd)(2bt + 2cnA_s) - (bt^2 + 2cndA_s)\right]}$$

$$\sigma_{c2}' = \frac{2(t - kd)\, P_E}{\left[(kd)(2bt + 2cnA_s) - (bt^2 + 2cndA_s)\right]}$$

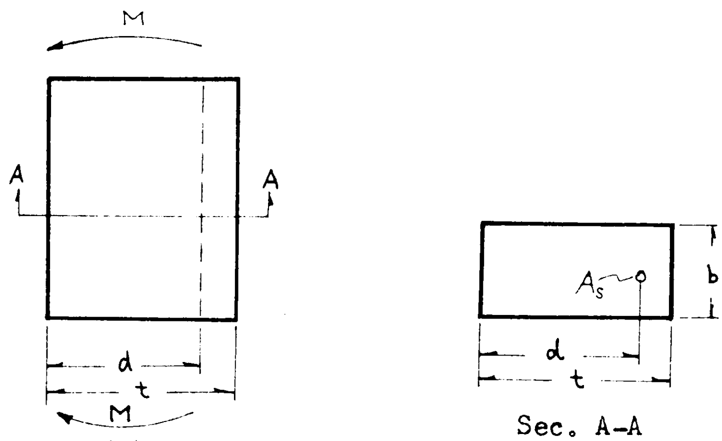
and
$$\sigma_s' = \frac{2cn(-kd + d)\, P_E}{\left[(kd)(2bt + 2cnA_s) - (bt^2 + 2cndA_s)\right]}$$

where

$$kd = \frac{6cndA_s(c' - e - d) + bt^2(3c' - 3e - 2t)}{6cndA_s(c' - e - d) + 3t(2bc' - 2be - bt)}$$

and stress distribution can be plotted.

3. Rectangular Beam Section with Simple Reinforcement under Pure Bending.

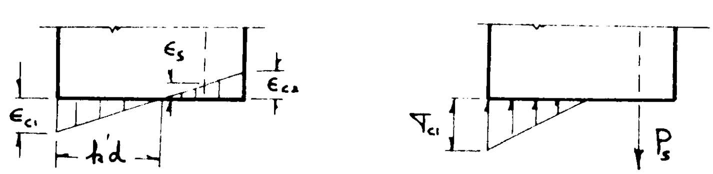


Sec. A-A

(a) (b)

Fig. 9

Equilibrium requires that an external bending moment on any beam must be resisted by internal stresses. A reinforced concrete beam is usually analyzed by assuming the reinforcing steel carries all the tension and the concrete carries all the compression. Concrete carries no tension. Then, before the creep of concrete takes place, the strain and stress distribution diagrams are as shown on Fig. 10.*

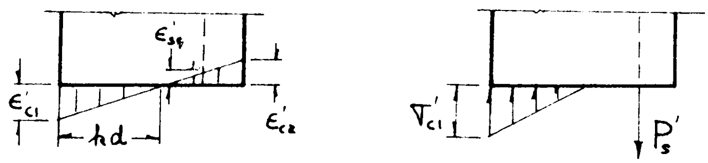


(a) Strain Diagram (b) Stress Diagram

Fig. 10

* The method of elastic analysis can be found in any reinforced concrete text. See for example: Dunham THEORY AND PRACTICE OF REINFORCED CONCRETE, Ferguson REINFORCED CONCRETE FUNDAMENTALS.

After the creep in concrete takes place, the strain and stress distribution diagrams become as in Fig. 11.



(a) Strain Diagram (b) Stress Diagram

Fig. 11

The following equations are established from Fig. 11, the properties of the materials, and equilibrium:

$$\epsilon'_{c1} = C\,\frac{\sigma'_{c1}}{E_c} \tag{33}$$

$$\epsilon'_{s1} = \frac{\sigma'_s}{E_s} \tag{34}$$

$$P'_c = \frac{1}{2}\, b \cdot kd \cdot \sigma'_{c1} \tag{35}$$

$$P'_s = \sigma'_{s1} \cdot A_s \tag{36}$$

$$P'_c = P'_s \tag{37}$$

$$k = \frac{\epsilon'_{c1}}{\epsilon'_{c1} + \epsilon'_{s1}} \tag{38}$$

$$k = 3\left(1 - \frac{jd'}{d}\right) \tag{39}$$

$$\phi' = \frac{\epsilon'_{c1} + \epsilon'_{s1}}{d} \tag{40}$$

$$M = P'_c \cdot jd \tag{41}$$

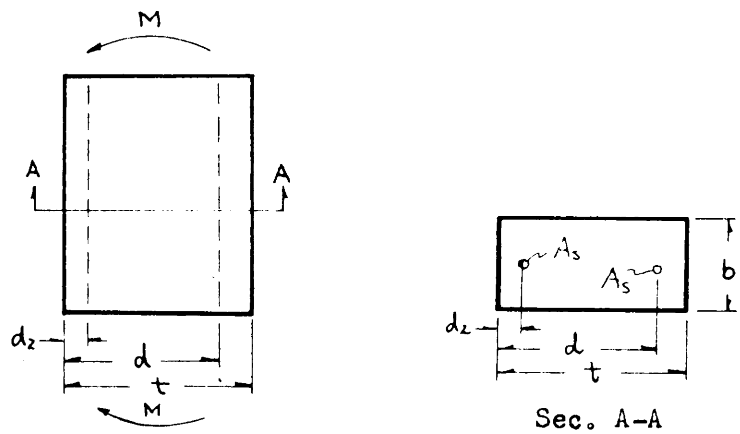
Solving the equations:

$$\phi' = \left[\frac{2cnp + k}{A_s E_s\, kd\,\left(d - \frac{kd}{3}\right)}\right] M \tag{42}$$

where

$$k = \sqrt{(cnp)^2 + 2cnp} - Cnp$$

4. Rectangular Beam Section with Double Reinforcements under Pure Bending Moment.

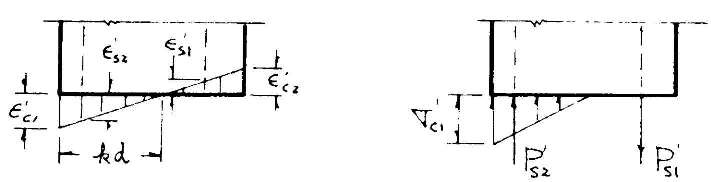


(a)

Sec. A–A
(b)

Fig. 12

After the creep in concrete takes place, the strain and stress distribution diagrams are as shown in Fig. 13.



(a) Strain Diagram

(b) Stress Diagram

Fig. 13

From Fig. 13, properties of materials and equilibrium following equations can be established:

$$\epsilon'_{c1} = C \frac{\sigma'_{c1}}{E_c} \tag{43}$$

$$\epsilon'_{s1} = \frac{\sigma'_{s1}}{E_s} \tag{44}$$

$$\epsilon'_{s2} = \frac{\sigma'_{s2}}{E_s} \tag{45}$$

$$\epsilon'_{s2} = \frac{kd - d_2}{kd} \cdot \epsilon'_{c1} \tag{46}$$

$$P'_c = \frac{1}{2} b \cdot kd \cdot \sigma'_{c1} \tag{47}$$

$$P'_{s1} = \sigma'_{s1} \cdot A_s \tag{48}$$

$$P'_{s2} = \sigma'_{s2} \cdot A'_s \tag{49}$$

$$k = \frac{\epsilon'_{c1}}{\epsilon'_{c1} + \epsilon'_{s1}} \tag{50}$$

$$P'_{s1} = P'_c + P'_{s2} \tag{51}$$

$$\phi' = \frac{\epsilon'_{c1} + \epsilon'_{s1}}{d} \tag{52}$$

$$M = P'_{s2}(d - d_2) + P'_c\left(d - \frac{kd}{3}\right) \tag{53}$$

Solving the equations:

$$\phi' = \frac{2CM}{2CA'_s E_s (d - d_2)(kd - d_2) + E_c \cdot b \cdot (kd)^2 \left(d - \frac{kd}{3}\right)} \tag{54}$$

where

$$k = \sqrt{C^2 n^2 (p + p')^2 + 2Cn\left(p + p'\frac{d_2}{d}\right)} - Cn(p + p')$$

IV. MOMENT - CURVATURE RELATIONS APPLIED TO STATICALLY INDETERMINATE STRUCTURES

A. General

In the analysis of any structure, the change in geometric configuration and subsequent deflections can be determined if the moment-curvature relation and the value of moment are everywhere known. For statically indeterminate structures, the value of moment can be calculated if the moment-curvature relation and sufficient boundary conditions are known.

For structures that exhibit elastic stress-strain properties, determination of the moment of inertia of the cross-section leads to a simple solution for the moment-curvature relation (See Section III C). For homogeneous materials exhibiting known non-linear stress-strain relations, the moment-curvature relations can be calculated with somewhat greater difficulty. In the case of structures of non-homogeneous materials exhibiting non-linear stress-strain relations, the determination of moment-curvature relations is of greater complexity. Such a determination for reinforced concrete beams subject to creep of the concrete is the subject of this thesis.

Whereas the moment-curvature relations derived in Section III (D - 1, 2, 3, 4) cover a wide range of structural elements, they will become useful for analysis or design only when they are expressed in the form of graphs or tables within a working range of parameter values. Section B, following, contains a representative set of such graphs.

B. Graphs

A series of curves have been here developed. Fig. 14 shows the moment-curvature relations for two typical reinforced concrete sections and for two conditions: (1) When the action of the concrete is purely elastic, and (2) when creep in the concrete is considered (the creep coefficient is assumed equal to 3). Figs. 15, 16, 17, 18 show the variation of relative stiffness for two reinforced concrete sections (each for $n = 8$ and $n = 10$) with a variation of the percentage of tension steel and the ratio of compression steel to tension steel. Fig. 19 is a composite of some of the curves from Figs. 15 through 18 to show the variation of relative stiffness with steel percentage for a section with $n=8$ and $n=10$ elastically and as creep is considered.

These graphs have been developed using the equation found in Section III D and in the Appendix.

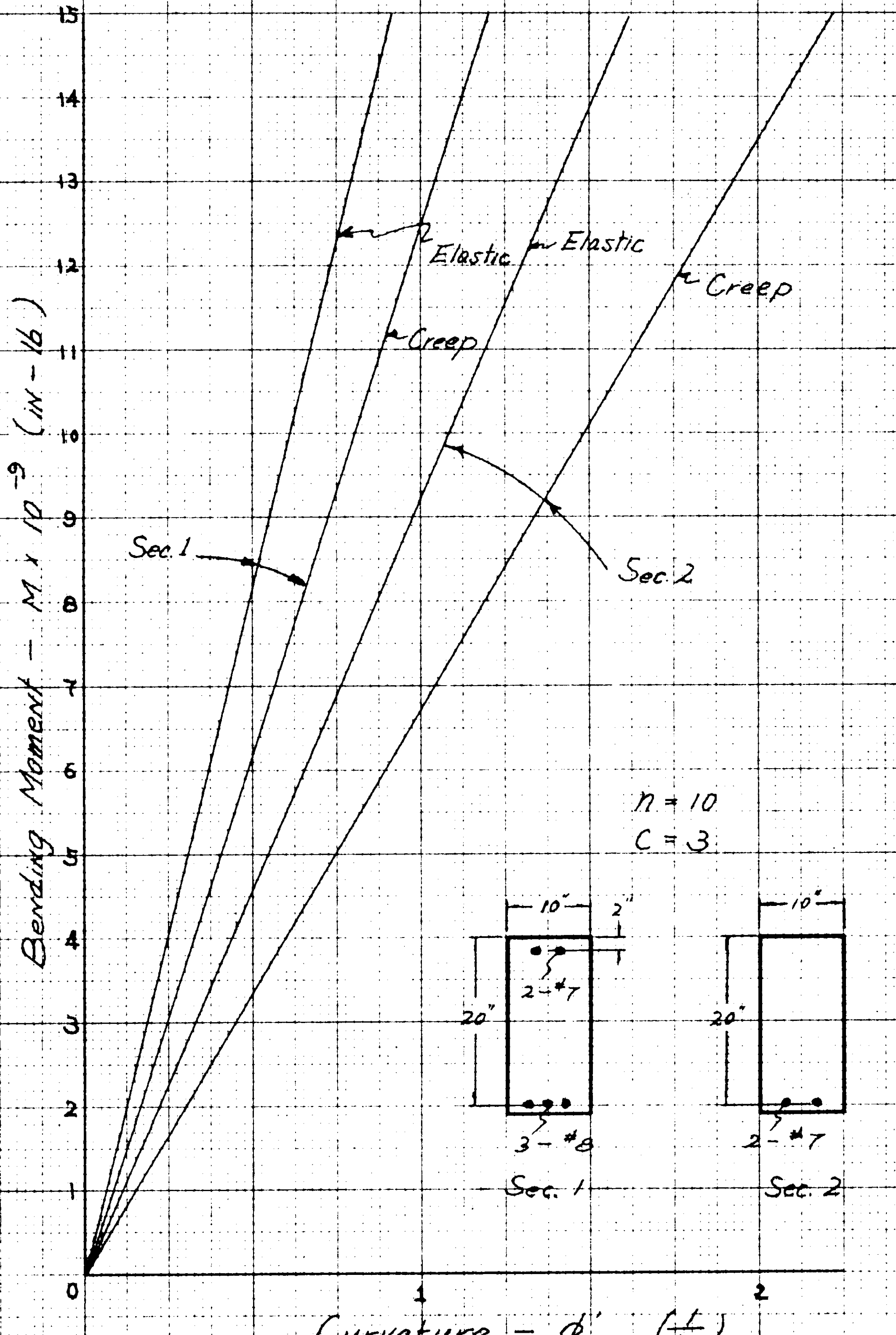


Fig. 14 MOMENT - CURVATURE RELATION

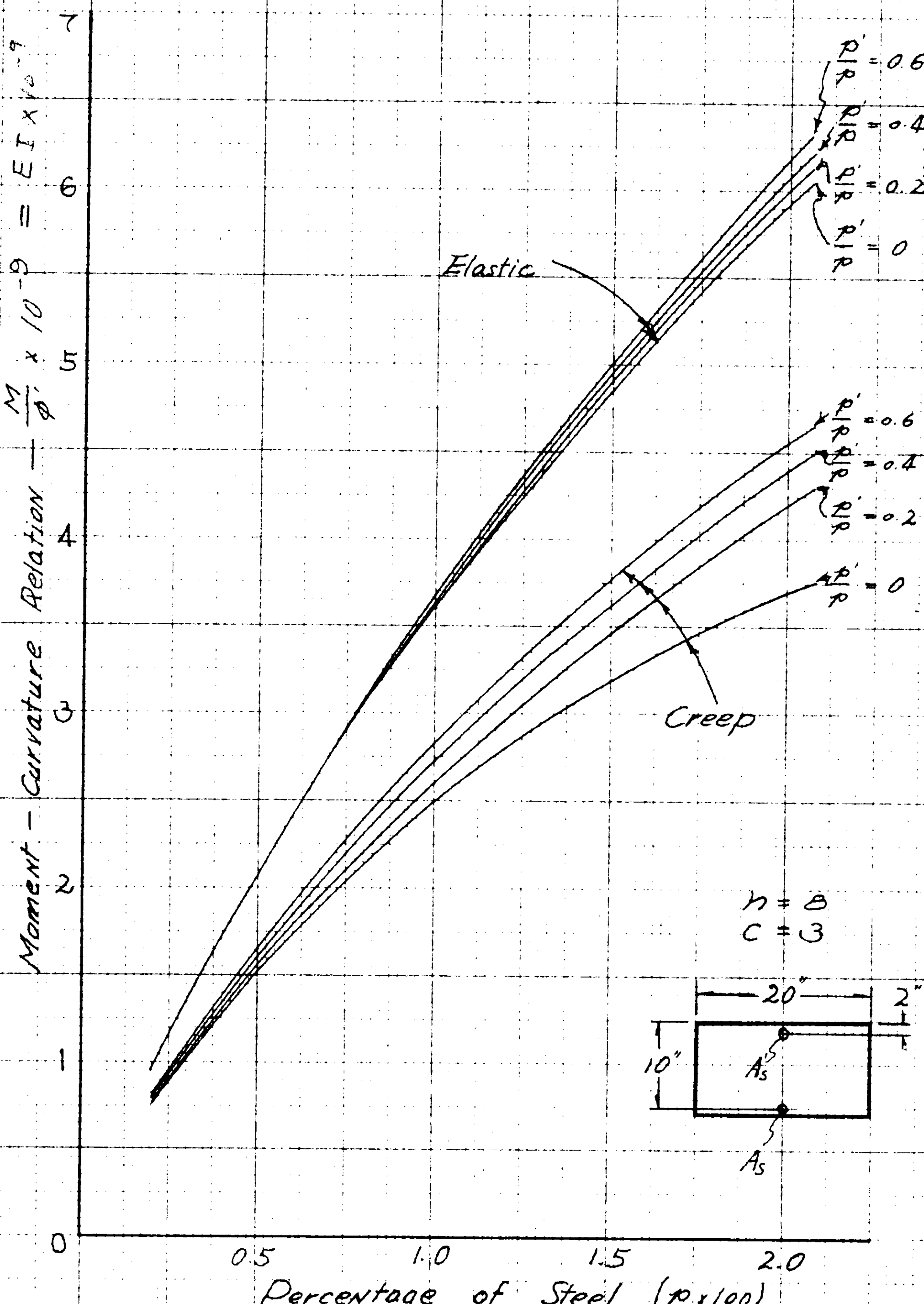


Fig. 15 RELATIVE STIFFNESS OF R/C BEAM

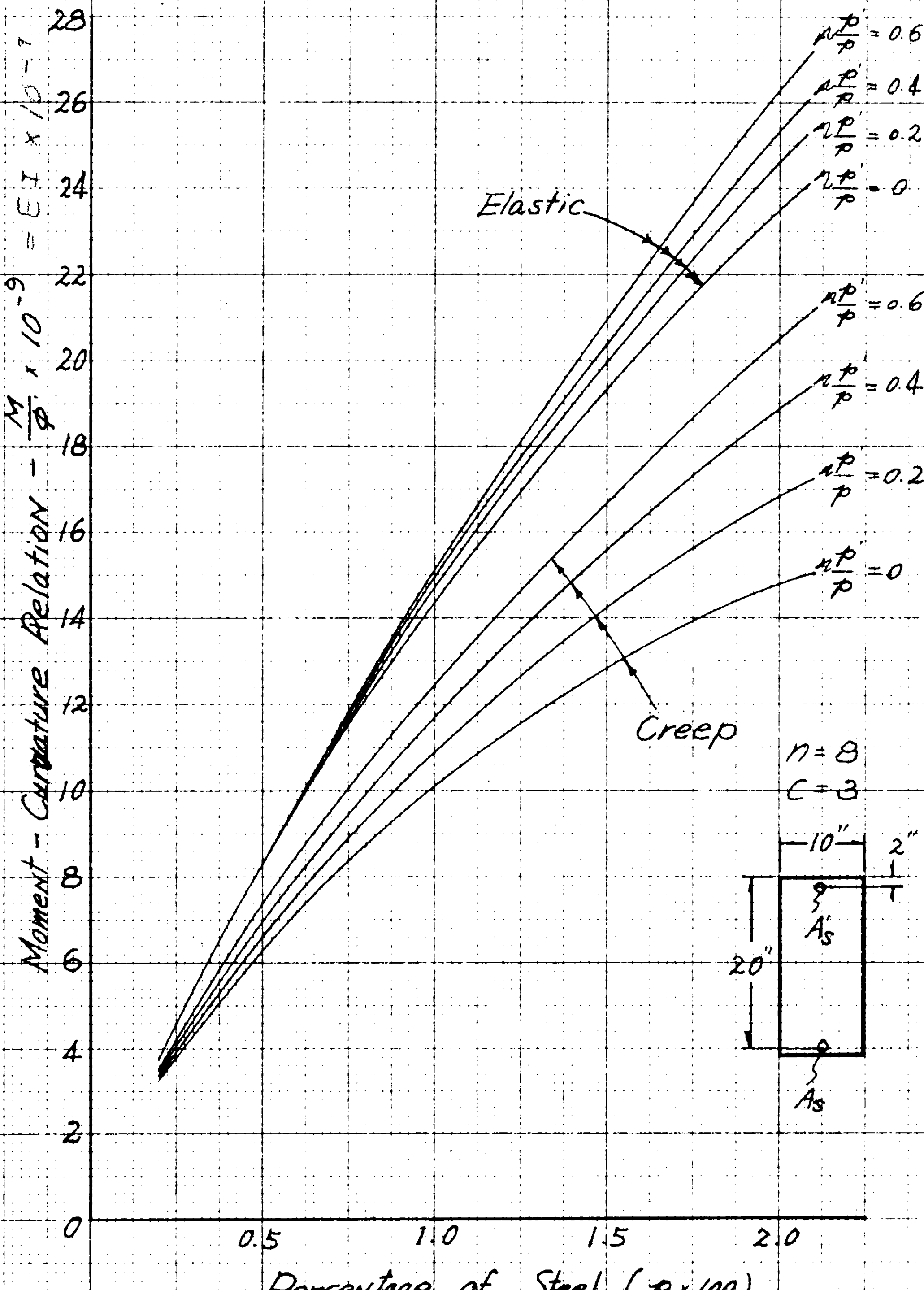


Fig. 16 RELATIVE STIFFNESS OF R/C BEAM

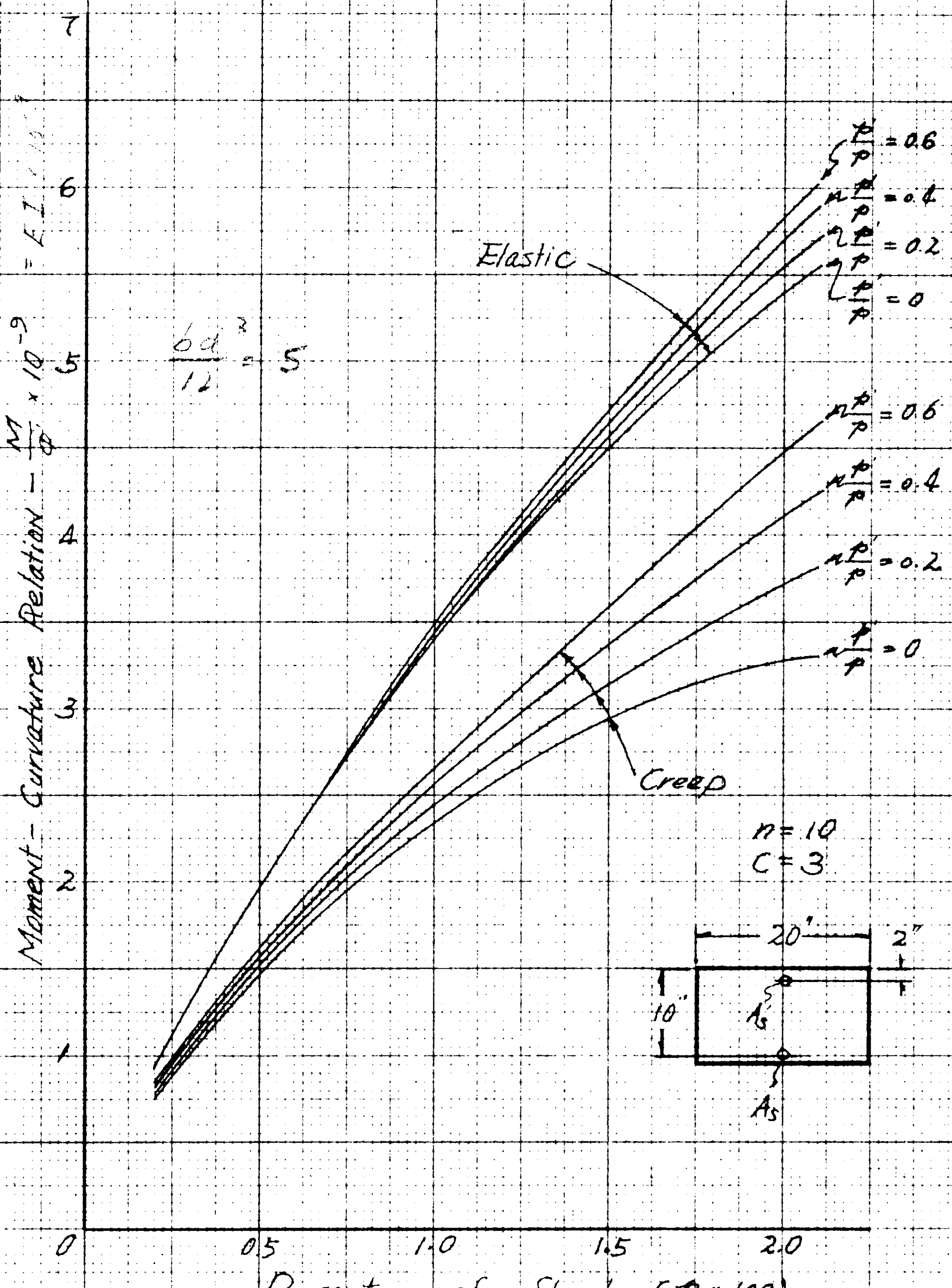


Fig. 17 RELATIVE STIFFNESS OF R/C BEAMS

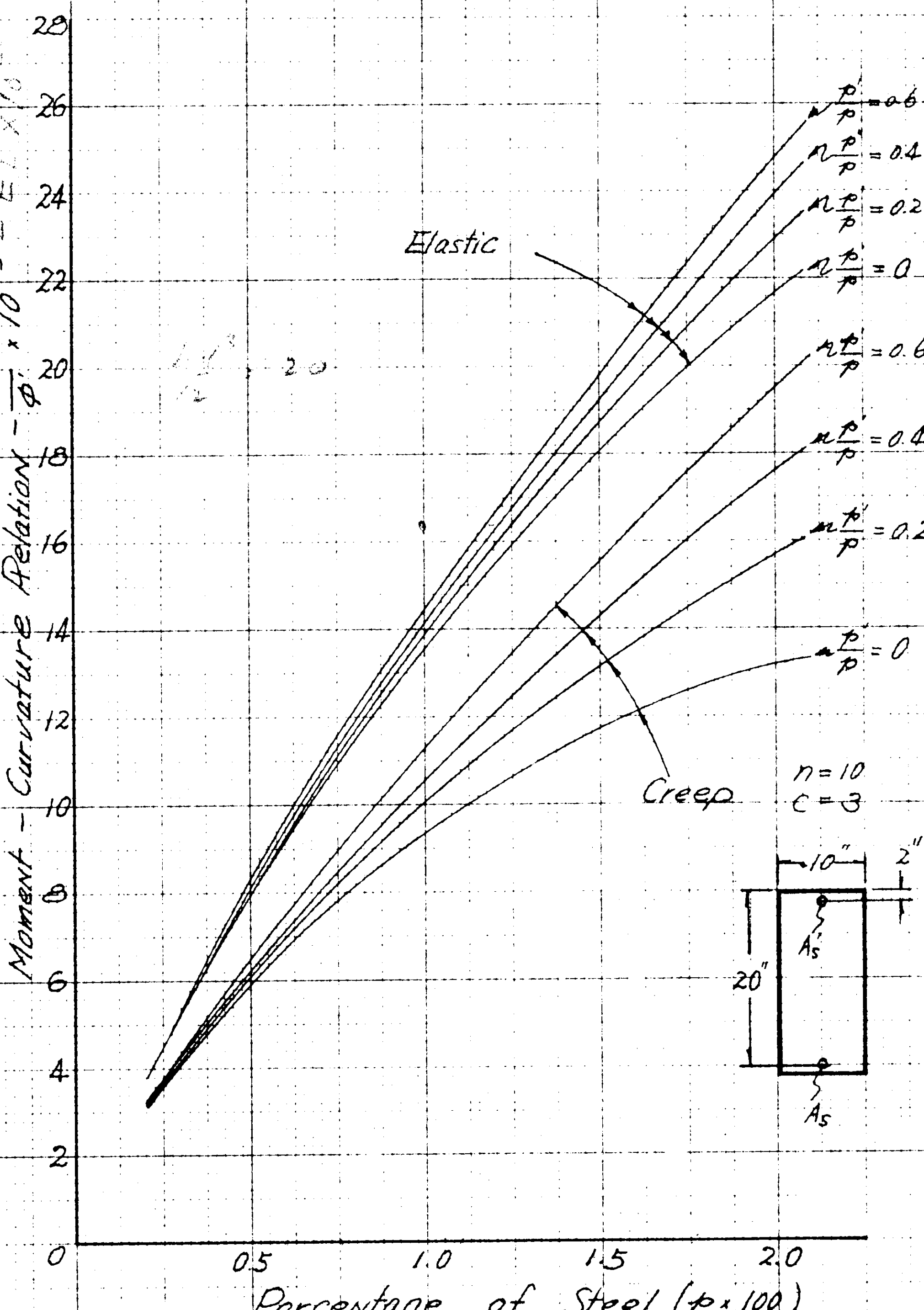


Fig. 18 RELATIVE STIFFNESS OF R/C BEAM

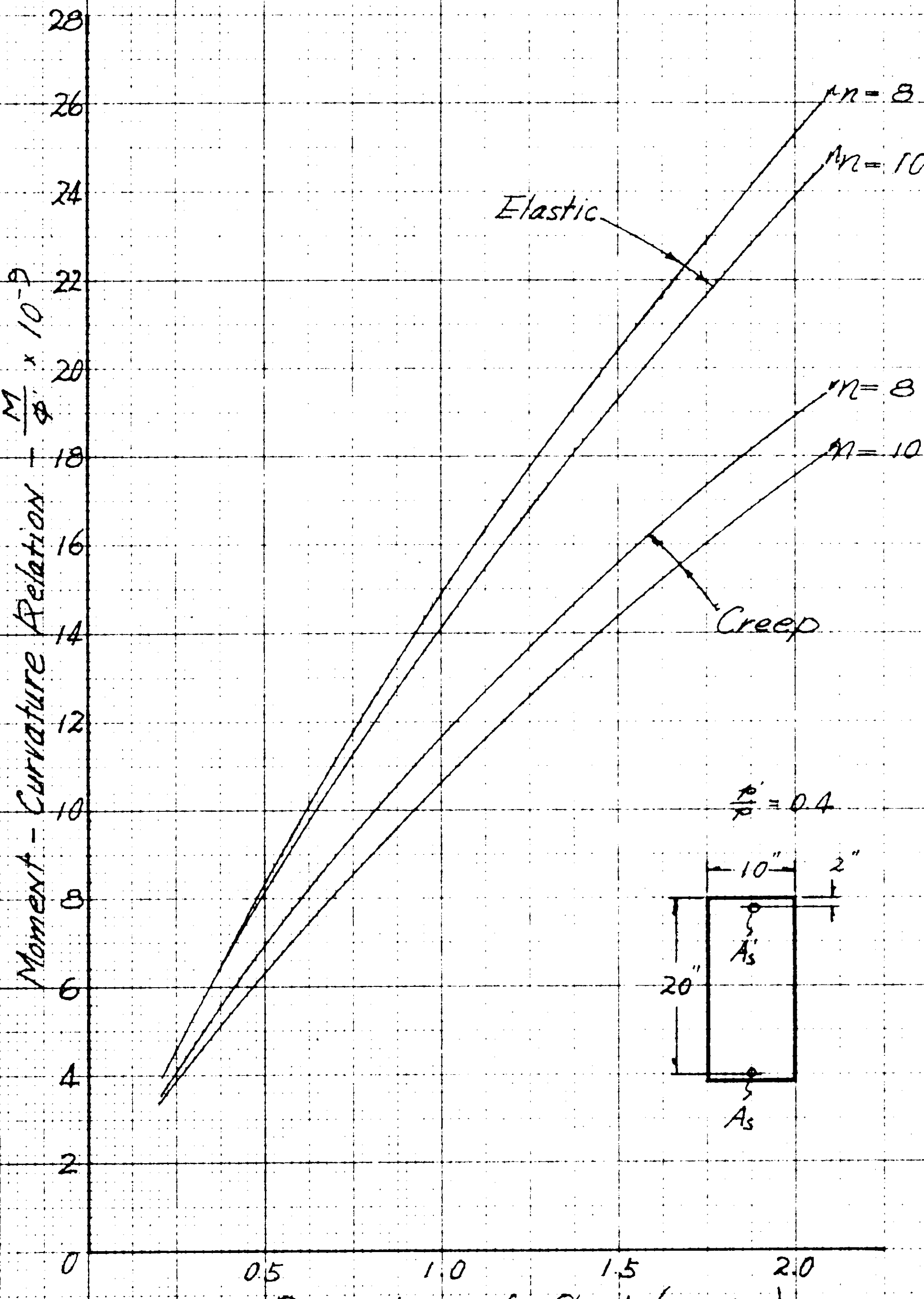
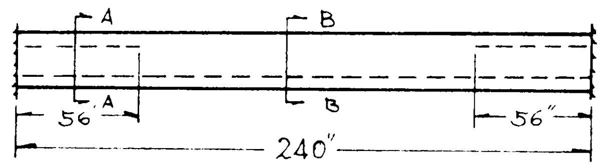


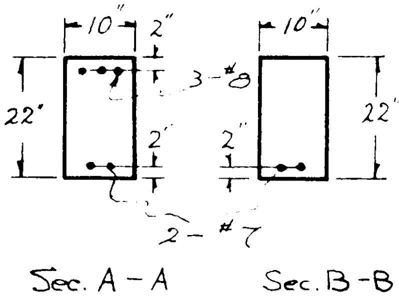
Fig 19 RELATIVE STIFFNESS OF R/C BEAM

C. Example

The use of the developed curves is illustrated in the following example of a fixed-end reinforced concrete beam with reinforcement, dimension and loading as shown in Fig. 20. Two analyses are given - one for the beam as an elastic structure and one for the beam with the effects of creep included. The coefficient of creep is assumed equal to 3.

The elastic analysis used here takes into account the difference in the moments of inertia at the center and at the ends of the beam. This is desirable for a better comparison between the elastic and creep analyses of the final values of moments. Norminally it is recommended that a weighted average of moment of inertia be used for simplicity.[20]





$f_s = 20{,}000$ psi $\qquad n = 10$

$f'_c = 3{,}000$ psi $\qquad C = 3$

$w = w_{D.L.} + w_{L.L.} = 2\ ^{K}/_{ft}$

Fig. 20

Calculation:

	Sec. A-A	Sec. B-B
A_s sq. in.	2.37	1.20
A'_s sq. in.	1.20	0
p	0.01185	0.006
p'	0.006	0
$\frac{p'}{p}$	0.506	0

From the graph Fig. 18, page 39

$\frac{M}{\phi}$ (Elastic)	16.4×10^9	9.2×10^9
$\frac{M}{\phi}$ (Creep)	12.5×10^9	6.75×10^9

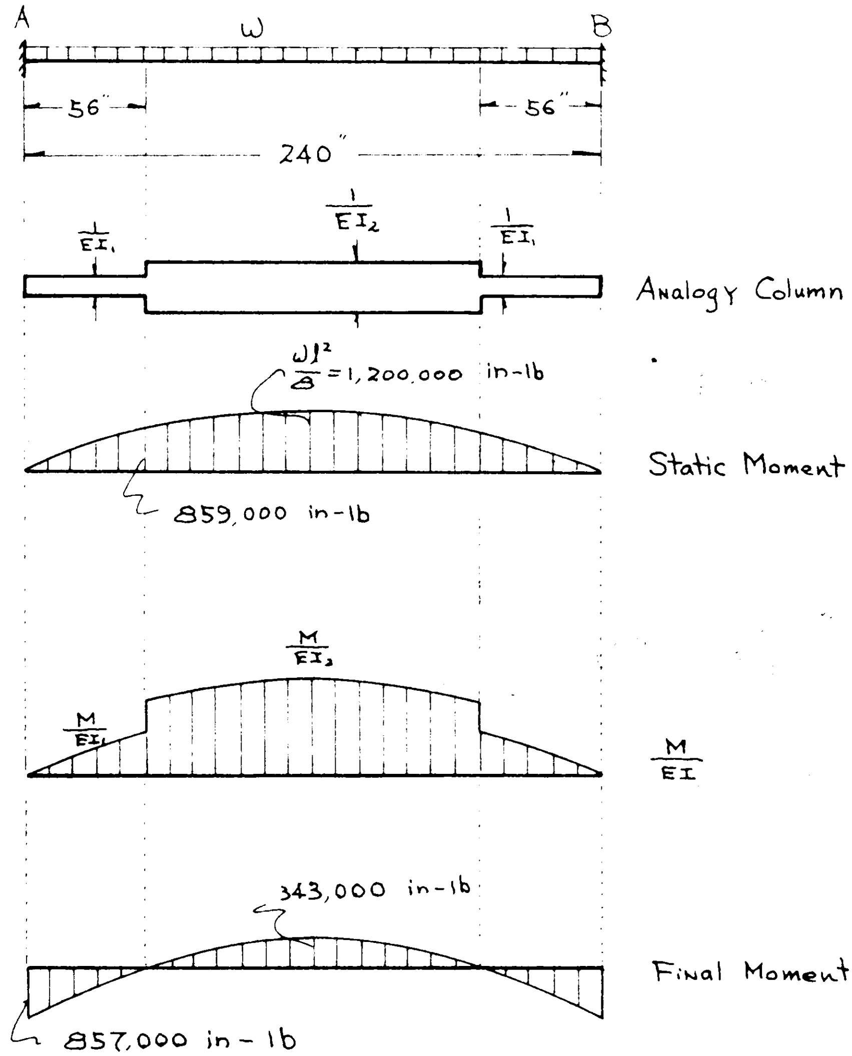


Fig. 21

For the elastic analysis using column analogy, and with reference to Fig. 21:

$$\frac{E I_1}{E I_2} = \frac{16.4}{9.2} = 1.78$$

$$A = \frac{1}{EI_1}[2 \times 56 + 1.78 \times 128] = \frac{1}{EI_1}[340]$$

$$P = \frac{1}{EI_1}[291,400,000]$$

$$\therefore M_A = M_B = 857,000 \text{ in-lb}$$

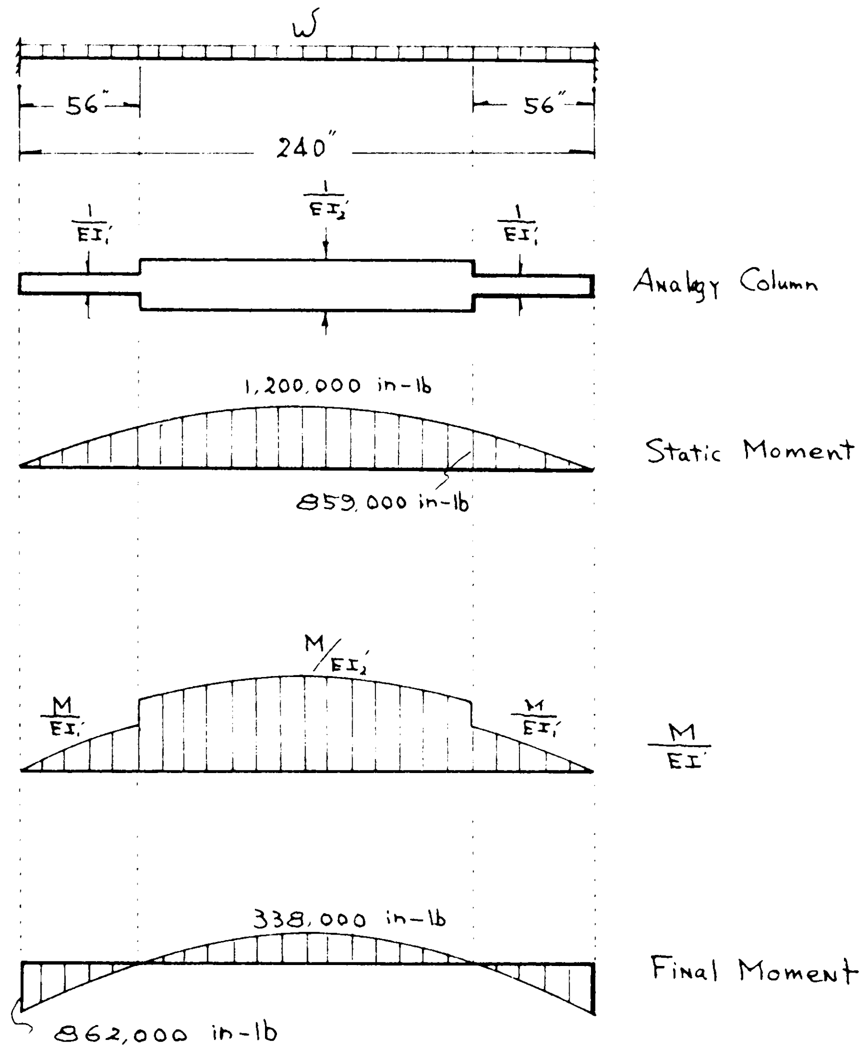


Fig. 22

For the creep analysis using column analogy and with reference to Fig. 22:

$$\frac{EI_1'}{EI_2'} = \frac{12.5}{6.75} = 1.85$$

$$A = \frac{1}{EI_1'}\,[349]$$

$$P = \frac{1}{EI_1'}\,[300{,}400{,}000]$$

$$\therefore\ M_A = M_B = 862{,}000 \text{ in-lb}$$

V. DISCUSSION AND CONCLUSION

Generalized moment-curvature relations of reinforced concrete members for various cases have been developed. For a limited range of parameters, graphs have been prepared. With these graphs, and within the range of the parameters given, it is possible to solve statically indeterminate structures of reinforced concrete with accuracy taking account of the effects of creep in the concrete.

The calculated stiffness of an element of reinforced concrete is smaller when the creep of the concrete is considered. In an indeterminate structure, the decrease in stiffness is not uniform throughout the length of the structure due to the variation in reinforcing steel. The deflections of the structure may be changed by the decrease in stiffness and the distribution of moments may be changed where the decrease in stiffness is not uniform.

In the design of statically indeterminate structures of reinforced concrete a change in the distribution of moments means that some moment values increase while others decrease. Unless the moment values are known, or the changes referred to above are known to be insignificant, the structure as designed may not be a safe one.

The equations developed and the graphs presented in this thesis represent a step in the direction of determining moment-curvature relations for reinforced concrete structures where creep is considered. Analysis of complete structures to determine the effects of creep in one member on the distribution of moments in another might be the next step to pursue.

Certainly the graphs presented in this thesis represent a limited range of parameters; this range could be profitably extended.

With sufficient graphs of the type presented and with sufficient known boundary conditions, any of the usual forms of structural analysis can be used to determine moments and deflections in a statically indeterminate structure of reinforced concrete taking account of creep in the concrete.

The following observations and conclusions can be drawn from the results.

1. The stiffness of a structural member of reinforced concrete is less when creep is taken into account.

2. Its stiffness is increased by increasing both tensil and compressive steels.

3. The effect of tensile steel on the stiffness is greater than the effect of compressive steel.

4. The effect of compressive steel on the stiffness is greater when creep is taken into account.

5. Stiffness is greater for structures where the E of the concrete is greater (i.e. where n is smaller).

6. The calculation of stiffness by the equations developed appeared to be extremely sensitive to slight variations in values of k.

7. For a given cross-sectional area of concrete, as b/d decreases, the stiffness increases. Note that M/ϕ' scales on Figs. 15 and 17 are four times as great as on Figs. 16 and 18.

8. In the illustrated problem it is noted that the maximum moment (negative) is increased by .6% and the positive moment is decreased by 1.4%. These appears to be relatively insignificant values, but this does not warrant a conclusion that all such computations will result in similar insignificant variations. Many more structural situations must be investigated to study this problem.

9. A comparison of the results of the illustrated problem with the results based on an assumed uniform value of moment of inertia indicate a variation of moments up to 15%.

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The author also wishes to express his appreciation to his parents who are largely responsible for whatever accomplishments he may have obtained.

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IX. APPENDIX

Elastic Analysis

1. Moment of inertia of a reinforced concrete section with tension steel only.

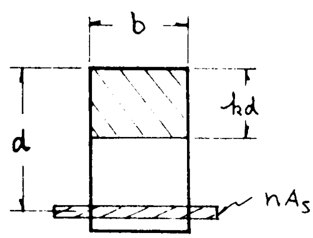

Fig. 23

$$p = \frac{A_s}{bd} \qquad n = \frac{E_s}{E_c}$$

$$k = \sqrt{2pn + (pn)^2} - np$$

$$I = \frac{b(kd)^3}{3} + nA_s (d - kd)^2$$

2. Moment of inertia of a reinforced concrete section with both tension and compression steels.

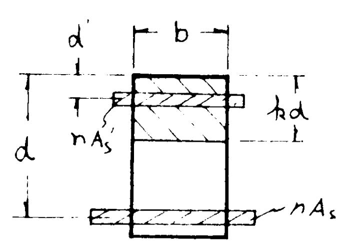

Fig. 24

$$p' = \frac{A_s'}{bd}$$

$$k = \sqrt{2n\left(p'\frac{d'}{d} + p\right) + n^2(p' + p)^2} - n(p' + p)$$

$$I = \frac{b(kd)^3}{3} + nA_s (d - kd)^2 + nA_s' (kd - d')^2$$

ABSTRACT

Because creep in steel at working stresses is essentially zero, reinforced concrete experiences a non-homogeneous pattern of creep strain across any stress section. In statically indeterminate structures of reinforced concrete this creep pattern results in the redistribution of stresses, movement of points of inflection and changes in the bending moment values.

Generalized moment-curvature relations of reinforced concrete members for various cases have been developed. For a limited range of parameters, graphs have been prepared. With these graphs, and within the range of the parameters given, it is possible to solve statically indeterminate structures of reinforced concrete with accuracy taking into ~~the~~ account of the effects of creep in the concrete. An analysis of the effect of various parameters on the stiffness of reinforced concrete members is also given.