ORTHOGONAL STATISTICS INVOLVING THE THIRD AND FOURTH SAMPLE MOMENTS FOR NEGATIVE BINOMIAL DISTRIBUTION

by

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TABLE OF CONTENTS

Chapter		Page
I	INTRODUCTION	4
	(a) Orthogonal Polynomials.	4
	(b) Orthogonal Statistics	6
II	THE NEGATIVE BINOMIAL DISTRIBUTION	9
III	ORTHOGONAL STATISTICS FOR PARAMETERS OF THE NEGATIVE BINOMIAL DISTRIBUTION	14
	(a) General Method of Obtaining Orthogonal Polynomials	14
	(b) Relationships Between Q's and Sample Moments for Negative Binomial Distribution	15
	(c) The Use of Recurrence Relationships . .	17
	(d) Tables of Expected Values of Q-Products.	22
	(e) An Example of the Use of the Tables of Expected Q-Products	24
IV	SUMMARY.	26
ACKNOWLEDGEMENTS		63
REFERENCES		64
VITA		65

TABLE OF CONTENTS (cont.)

Chapter	Page
APPENDIX A	30
APPENDIX B (1)	42
APPENDIX B (2)	56
APPENDIX C	59

CHAPTER 1

INTRODUCTION

The purpose of this thesis is to extend the development of tables of orthogonal statistics for the negative binomial distribution. Shenton and Myers [11] have introduced the concept of orthogonal statistics for securing the first few moments of moment estimators and used this development to investigate small sample properties of moment estimators involving the first two sample moments. The purpose of this work is to extend this development to treat statistics involving the first four sample moments.

(a) Orthogonal Polynomials

Let $F(x)$ be a distribution function with finite moments of all order. Then there exists a set of polynomials $q_r(x)$ such that the following orthogonality condition hold:

$$\int_{-\infty}^{\infty} q_r(x) q_s(x) dF(x) = \phi_r \qquad \text{for } r = s$$

$$= 0 \qquad \text{for } r \neq s$$

where $\phi_r > 0$, $\phi_o = 1$, $q_o(x) = 1$ and the coefficient of x^r in $q_r(x)$ is unity. The polynomials will be infinite in number if the set of points of increase of F is infinite, and finite in number if F has only a finite number of points of increase (Cramer [4]). These orthogonal polynomials are well known for many of the classical distributions. For example, the first few polynomials for the standard normal distribution (often called the Hermite polynomials (Cramer [4] are:

$$H_0(x) = 1$$

$$H_1(x) = x$$

$$H_2(x) = x^2 - 1$$

$$H_3(x) = x^3 - 3x$$

etc.

The orthogonal polynomials for the case of a Poisson variate with parameter m can be found from the generating equation:

$$q_r(x) = e^{-m\Delta} x^{(r)}$$

where Δ is an advancing difference operator (i.e., $\Delta x^{(r)} = rx^{(r-1)}$) and $x^{(r)}$ is the factorial term $x(x-1)(x-2)....(x-r+1)$. These polynomials are called the Charlier's polynomials (Szego [13]); the first few of these are:

$$q_0 = 1$$

$$q_1 = x - m$$

$$q_2 = x^2 - x - 2xm + m^2$$

$$q_3 = x(x-1)(x-2)-3mx(x-1)+3xm^2-m^3$$

Certain recurrence relationships exist from which these polynomials can be generated for other distributions. The most general expression for say, the rth polynomial $q_r(x)$ can be written as

$$q_r(x) = x^r+c_{r-1}^{(r)}x^{r-1}+c_{r-2}^{(r)}x^{r-2}+\ldots+c_1^{(r)}x+c_0^{(r)} \tag{1-1}$$

where $c_l^{(r)}$ = coefficient of x^l in $q_r(x)$

Other useful relationships, particularly for products of the q's and methods for finding the polynomials in general are given in Chapter III.

(b) Orthogonal Statistics

Let $(x_1,x_2,\ldots.x_n)$ be a sample of independent observation from a distribution for which the orthogonal polynomials $q_r(x)$ exist. The rth orthogonal statistic (Shenton and Myers [11]) is defined as

$$Q_r = \sum_{j=1}^{n} q_r(x_j)/n \quad . \tag{1-2}$$

As a result of the orthogonality property of the q's, we can write

$$\begin{aligned} E(Q_r) &= 0 && (r=1,2,\ldots) \\ E(Q_rQ_s) &= \delta_{r,s}\,\phi_r/n && (r,s = 1,2,\ldots) \end{aligned} \tag{1-3}$$

where $\delta_{r,s} = 0$ for $r \neq s$

$= 1$ for $r = s$

It will be noted from (1-1) and from the basic definition of the q's that Q_r can be written as a linear function of $m_1', m_2', \ldots m_r'$, where m_k' is the kth crude sample moment i.e.,

$$m_k' = \sum_{j=1}^{n} x_j^k/n \quad .$$

Thus of course m_r' can be written as a linear function of $Q_1, Q_2, \ldots Q_r$. (The expression relating the m's and Q's in general and for the specific case of the negative binomial distribution will be given in Chapter III).

If we desire to obtain the first few moments of a moment estimator $t(m_1' m_2' m_3' \ldots m_k')$ and if we expand this statistic (or say the pth power of the statistic if we desire to obtain the pth moment) in the form

$$\begin{aligned} t &= a_{00\ldots0} + a_{100\ldots0}Q_1 + a_{010\ldots0}Q_2 + \ldots \\ &+ a_{110\ldots0}Q_1Q_2 + a_{1010\ldots0}Q_1Q_3 + \ldots \\ &+ a_{\alpha\beta\ldots\xi}Q_1^{\alpha}Q_2^{\beta}\ldots Q_k^{\xi} + \ldots \end{aligned}$$

then the corresponding crude moment can be obtained by taking expectations term by term of the powers and products of the Q's, and these expected Q-products can be expanded in ascending powers of $1/n$ with the coefficients being the expected values of products of the q's. Associated with these orthogonal statistics are the following tables which can be found in Shenton and Myers [11];

(i) Expected Q-products in terms of expected q-products to order ten through terms in n^{-5} (in general).

(ii) Expected Q-products to order eight through terms in n^{-4} for the negative binomial distribution in terms of the population parameters for products involving Q_1 and Q_2.

Because of the difficulties involved in obtaining the maximum likelihood estimators for negative binomial parameters (Bowman [3]), much attention has been drawn to the investigation of moment estimators.

Tables are presented in this thesis which are an extension of those described by (ii) above to products involving Q_1, Q_2, Q_3 and Q_4. These tables will then aid in the determination of the bias, variance, etc. of moment estimators involving the first four sample moments.

CHAPTER 2

THE NEGATIVE BINOMIAL DISTRIBUTION

The negative binomial distribution is one of the most widely used of the two parameter discrete distributions, having applications in biological data, health and accident statistics, psychology, and other fields. A literature review may be found in Bartko [2].

There is a similarity of the negative binomial distribution to the positive binomial distribution. The probability generating function for the latter is $(pt + q)^n$. If we replace p by -p and n by -n, we arrive at the generating function of negative binomial distribution with mean np and variance np(1+p). Note that the variance exceeds the mean for the negative binomial distribution.

There are various parametric forms for the negative binomial distribution. We shall consider primarily one form, namely that due to Anscombe [1] with probability function:

$$\binom{\alpha+x-1}{x}\frac{\alpha^{\alpha}\lambda^{x}}{(\alpha+\lambda)^{\alpha+x}} \qquad \begin{array}{l} x = 0,1,2,\ldots \\ \alpha > 0 \ , \ \lambda > 0 \end{array}$$

with parameters λ and α. (Here the combinatorial part is taken to be unity when x = 0).

The mean and variance for this particular form are λ and $\lambda + \lambda^2/\alpha$ respectively. Important and useful analytic properties include the cumulant generating function $-\alpha\ln[1-(\alpha+\lambda)(e^{it}-1)/\lambda]$ and the factorial moment generating function $(1-\lambda t/\alpha)^{-\alpha}$. Other parametric forms include that due to Fisher [7] with parameters $p = \lambda/\alpha$ and $k = \alpha$, more recently a form given by Evans [5] with parameters $a = \lambda/\alpha$ (same as p in the Fisher form) and $m = \lambda$, and the Pascal form [2] with probability function

$$\binom{x+r-1}{x} p^r q^x \quad , \qquad x = 0,1,2,\ldots. \quad r \text{ an integer, } p+q=1$$

There are various models which give rise to the negative binomial distribution. For example Feller [6] describes what he calls "Apparent Contagion" i.e., the compounding of the Poisson distribution. If the Poisson parameter m varies and is considered to have a statistical distribution described by a Gamma frequency function given by

$$f(m) = \frac{B^k}{\Gamma(k)} m^{k-1} e^{-Bm}$$

$$B > 0,\ m > 0,\ k > 0.$$

then the frequency generating function becomes

$$\int_0^{\infty} e^{m(t-1)} f(m)dm = [1+(1-t)/B]^{-k} \qquad (B+t>1)$$

The resulting probability function is

$$P_r(X = x) = \frac{k(k+1)\ldots.(k+x-1)}{x} \left(\frac{B}{1+\beta}\right)^k \left(\frac{1}{1+\beta}\right)^x$$

If we set $k = \alpha$ and $\beta = \alpha/\lambda$ we arrive at Anscombe's form of the negative binomial distribution.

The Poisson Distribution is regarded as a limiting form of the negative binomial distribution. In the Pascal form, if we let $p \to 1$, $q \to 0$ and λ be fixed such that $q = \lambda/r$, then the moment generating function of the negative binomial distribution, $p^r/(1-qe^{\theta})^r$, becomes the moment generating function of the Poisson Distribution, $e^{\lambda(e^{\theta}-1)}$, when $r \to \infty$. The negative binomial probability function, then, becomes that of the Poisson with parameter λ.

Other known distributional forms result [2] under certain physical models and limiting conditions.

Estimation of Parameters

We shall limit ourselves to the problems of estimation of the parameters for the form of Anscombe. The maximum likelihood estimators are given by the equations,

$$\lambda^* = m_1$$

$$N\{\ln(1+m_1/\alpha^*)\} = \frac{n_1+\dots+n_R}{\alpha^*} + \frac{n_2+\dots+n_R}{\alpha^*+1} + \dots$$

$$+ \dots + \frac{n_R}{\alpha \; +R-1} \quad ,$$

where n_i is the observed frequency of x_i, R is the maximum value of i such that $i = 0,1,2,\dots R$. $N = \sum_{i=0}^{R} n_i$.

There are iterative methods of solving (2-1). Bowman [3] has investigated some sampling properties of the maximum likelihood estimator of α (Actually this work was not restricted to the Anscombe form; the maximum likelihood estimators for the form of Fisher and Evans were also considered).

Due to the difficulty in solving the likelihood equation, the method of moments is usually used for the estimation of parameters. Equating the first two moments yields the following estimators

$$\hat{\lambda} = m_1'$$
$$\hat{\alpha} = m_1'^2/(m_2 - m_1') \quad .$$

Fisher [7] gives a range of the parameters for which the asymptotic efficiency exceeds 90%. Shenton and Myers [11] investigated the small sample properties of $\hat{\alpha}$ (first four moments through terms in n^{-4} and covariance determinant through terms in n^{-3}) and showed that in the range of the parameter space in which the asymptotic efficiency is high, the expansion for the variance becomes "explosive", i.e., higher order terms are large in comparison to the n^{-1} terms, even when we consider terms through n^{-4}. This suggests that even in the range of high asymptotic efficiency, the merit of $\hat{\alpha}$ is questionable and it might be instructive to consider the investigation of estimators involving moments of higher order, say m_3 or m_4.

Other estimators have been suggested e.g., use of mean and proportion of zeros, inverse moment, and geometric moment. An estimator combining the method of moments and that of using the proportion of zeros has also been mentioned [5] particularly for the parametric form of Evans.

CHAPTER III

ORTHOGONAL STATISTICS FOR PARAMETERS OF THE NEGATIVE BINOMIAL DISTRIBUTION

(a) General Method of Obtaining Orthogonal Polynomials

Let f(x) be a frequency or probability function with $E(x) = \mu$ and all moments existing. Let $X = x - \mu$. The q's are obtained in the following manner (see Kendall and Stuart [8])

$$q_1(x) = -\begin{vmatrix} 1 & X \\ 1 & \mu_1 \end{vmatrix} \quad .$$

$$q_2(x) = \begin{vmatrix} 1 & X & X^2 \\ 1 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \end{vmatrix} \div \mu_2 \quad ,$$

where μ_j = jth central moment for the distribution of x with $\mu_1 = 0$.

Thus
$$q_1(x) = X \quad , \tag{3-1}$$

$$q_2(x) = X^2 - X\mu_3/\mu_2 - \mu_2 \quad . \tag{3-2}$$

$$\vdots \qquad\qquad \vdots$$

In general, the rth orthogonal polynomial $q_r(x)$ can be written in determinantal form as follows:

$$q_r(x) = (-1)^r \begin{vmatrix} 1 & x & x^2 & \cdots & x^r \\ 1 & \mu_1 & \mu_2 & \cdots & \mu_r \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{r+1} \\ \vdots & & & & \\ \mu_{r-1} & \mu_r & \mu_{r+1} & \cdots & \mu_{2r-1} \end{vmatrix} \div \begin{vmatrix} 1 & \mu_1 & \mu_2 & \cdots & \mu_{r-1} \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_r \\ \vdots & & & & \\ \mu_{r-1} & \mu_r & \mu_{r+1} & \cdots & \mu_{2r-2} \end{vmatrix}$$

Based on (3-1), (3-2) and the basic definition of Q's given by (1-2), we arrive at

$$Q_1 = m_1' - \mu \quad ,$$

$$Q_2 = m_2' - m_1'(2\mu+\mu_3/\mu_2) - \mu_2'+2\mu^2+\mu\mu_3/\mu_2 \quad .$$

Conversely, then we can write

$$m_1' = Q_1 + \mu \quad ,$$

$$m_2' = \mu_2' + Q_1(2\mu + \mu_3/\mu_2) + Q_2 \quad .$$

(b) <u>Relationships Between Q's and Sample Moments for the Negative Binomial Distribution</u>

Shenton and Wallington [12] give an inverse moment relationship for the case of the parametric form of Anscombe;

$$x^{(r)} = (1 + \lambda/\alpha\Delta)^{\alpha+r-1} q_r(x) \quad ,$$

where $x^{(r)} = x(x-1) \ldots (x-r+1)$. (Here Δ is a symbolic differerential operator, i.e., $\Delta q_r(x) = r q_{r-1}(x)$).

We can use the form above and the definition of the Q's to express the sample factorial moments in terms of the Q's. For example

$$x^{(1)} = (1+\lambda/\alpha\Delta)^{\alpha} q_1(x) = q_1(x)+\lambda\Delta q_1(x)+f(0) \quad ,$$

Summing both sides over the sample $(x_1, x_2, \ldots x_n)$,

$$\sum_{j=1}^{n} x_j^{(1)}/n = \sum_{j=1}^{n} [q_1(x_j)+\lambda]/n \quad ,$$

$$m_{(1)} = Q_1 + \lambda \quad .$$

Using the same procedure we can obtain:

$$m_{(2)} = Q_2+2\lambda(1+1/\alpha)Q_1+\lambda^2(1+1/\alpha) \quad ,$$

$$m_{(3)} = Q_3+3\lambda(1+2/\alpha)Q_2+3\lambda^2(1+2/\alpha)(1+1/\alpha)Q_1$$
$$+ \lambda^3(1+2/\alpha)(1+1/\alpha) \quad .$$

Here $m_{(k)}$ refers to the Kth factorial sample moment, i.e., $\sum_{j=1}^{n} x_j(x_j-1)\ldots(x_j-k+1)/n$.

Conversely, we can write

$$Q_1 = m_{(1)} - \lambda \quad ,$$

$$Q_2 = m_{(2)} - 2\lambda(1+1/\alpha)m_1 + \lambda^2(1+1/\alpha) \quad ,$$

$$Q_3 = m_{(3)} - 3\lambda(1+2/\alpha)m_{(2)} + 3\lambda^2(1+2/\alpha)(1+1/\alpha)m_{(1)}$$
$$- \lambda^3(1+2/\alpha)(1+1/\alpha) \quad .$$

(c) The Use of Recurrence Relationships

Szego [13] gives a general fundamental recurrence relationship for the q's in the form

$$q_r(x) = (x-a_r)q_{r-1} - b_r q_{r-2} \quad , \qquad (3\text{-}3)$$

where $r \geq 2$, $q_0 = 1$ and $q_1 = x - a_0$ in this case. (Note the argument x has been omitted). Myers [9] give the relationships among a_r, b_r for the parameters of negative binomial distribution as follows:

$$a_r = \lambda + (r-1)(2\lambda+\alpha)/\alpha \quad , \qquad (3\text{-}4)$$

$$b_r = \lambda(\lambda+\alpha)(r-1)(\alpha+r-2)/\alpha^2 \quad . \qquad (3\text{-}5)$$

In developing the expected powers and products of the Q's in terms of the parameters λ and α, it is useful to have expressions involving ordinary products and powers of the q's. For example we have the expressions developed by Myers [9]

$$q_1 q_{r-1} = q_r + q_{r-1}(r-1)(\alpha+2\lambda)/\alpha + q_{r-2}(r-1)(\alpha+r-2)\lambda(\lambda+\alpha)/\alpha^2 , \qquad (3\text{-}6)$$

$$q_2q_{r-1} = q_{r+1}+q_r \times 2(r-1)(\alpha+2\lambda)/\alpha+q_{r-1}(r-1)[2(\alpha+r-1)\lambda(\lambda+\alpha)$$

$$+(r-2)(\alpha+2\lambda)^2]/\alpha^2$$

$$+q_{r-2} \times 2(r-1)(r-2)(\alpha+r-2)\lambda(\lambda+\alpha)(\alpha+2\lambda)/\alpha^3$$

$$+q_{r-3}(r-1)(r-2)(\alpha+r-3)(\alpha+r-2)\lambda^2(\lambda+\alpha)^2/\alpha^4 \quad . \qquad (3\text{-}7)$$

All q-products can be written as linear functions of individual q's. Another useful expression developed by Shenton [10] is

$$E(q_r^2) = \phi_r = \lambda^r(\lambda+\alpha)^r(\alpha+r-1)^{(r)}r!/\alpha^{2r} \quad , \qquad (3\text{-}8)$$

where $(\alpha+r-1)^{(r)} = (\alpha+r-1)(\alpha+r-2)\ldots(\alpha+1)\alpha$.

From (3-8) it can be verified that

$$\phi_1 = \lambda(\lambda+\alpha)/\alpha \quad ,$$

$$\phi_2 = 2\lambda^2(\lambda+\alpha)^2(\alpha+1)/\alpha^3 \quad ,$$

$$\phi_3 = 6\lambda^3(\lambda+\alpha)^3(\alpha+1)(\alpha+2)/\alpha^5 \quad .$$

From (3-6), (3-7) and (3-8) we can derive certain expected q-products involving q_1 and q_2. For example if we need, say $E(q_1^2q_2)$ we can set $r=3$ in (3-6), multiply by q_1 take expectations, the result is, very simply

$$E(q_1^2q_2) = 2\lambda(\lambda+\alpha)(\alpha+1)\phi_1/\alpha^2$$

$$= 2\lambda^2(\lambda+\alpha)^2(\alpha+1)/\alpha^3$$

Note how we have taken advantage of the orthogonality property of q's.

At this point let us insert an explanation of notation to be used throughout the remainder of the text. We shall refer to $E(Q_1^{\alpha} Q_2^{\beta} Q_3^{r} Q_4^{\xi})$ as $(1^{\alpha}2^{\beta}3^{r}4^{\xi})$ and $E(q_1^{\alpha}q_2^{\beta}q_3^{r}q_4^{\xi})$ as $[1^{\alpha}2^{\beta}3^{r}4^{\xi}]$ (The same notation will be used in the Appendices).

Since our goal is to extend the tables of orthogonal statistics to involve Q_3 and Q_4, it is important that we develop recursion formulae similar to (3-6) and (3-7) for q_3 and q_4.

If we set r = 3 in (3-4) and then express a_r in terms of a_3, we have

$$a_r = a_3 + (r-3)(2\lambda+\alpha)/\alpha \quad . \tag{3-9}$$

From (3-3), for r = 3 we can write

$$q_3 = (x-a_3)q_2-b_3q_1 \quad ,$$

and thus

$$(x-a_3) = q_3/q_2+b_3q_1/q_2 \quad . \tag{3-10}$$

If we substitute (3-9) into (3-3) and insert the expression in (3-10) for x - a_3, we obtain

$$q_r = [q_3/q_2+b_3q_1/q_2-(r-3)(2\lambda+\alpha)/\alpha]q_{r-1}-b_rq_{r-2} \quad . \tag{3-11}$$

We then transfer the product q_3q_{r-1} to the left-hand side and obtain

$$q_3q_{r-1} = q_2q_r-b_3q_1q_{r-1}+(r-3)(2\lambda+\alpha)q_2q_{r-1}/\alpha+b_rq_2q_{r-2} \quad . \tag{3-12}$$

(3-12) contains q-products which then must be expressed as linear functions of individual q's. We can use (3-5)(3-6) and (3-7) to expand q_2q_4, q_1q_{r-1}, q_2q_{r-1} and $b_rq_2q_{r-2}$. After much simplification the result becomes

$$
\begin{aligned}
q_3q_{r-1} = {} & q_{r+2} + q_{r+1} \times 3(r-1)\eta/\alpha + q_r 3(r-1)[(\alpha+r)\lambda\chi + (r-2)\eta^2]/\alpha^2 \\
& + q_{r-1}(r-1)(r-2)\eta[6(\alpha+r-1)\lambda\chi + (r-3)\eta^2]/\alpha^3 \\
& + q_{r-2} \times 3(r-1)(r-2)(\alpha+r-2)\lambda\chi[(\alpha+r-1)\lambda\chi + (r-3)\eta^2]/\alpha^4 \\
& + q_{r-3} \times 3(r-1)(r-2)(r-3)(\alpha+r-3)(\alpha+r-2)\lambda^2\chi^2\eta/\alpha^5 \\
& + q_{r-4}(r-1)(r-2)(r-3)(\alpha+r-4)(\alpha+r-3)(\alpha+r-2)\lambda^3\chi^3/\alpha^6 ,
\end{aligned}
\tag{3-13}
$$

where $2\lambda + \alpha = \eta$ and $\lambda + \alpha = \chi$.

Using a similar procedure, we obtain for q_4;

$$
\begin{aligned}
q_4q_{r-1} = {} & q_{r+3} + q_{r+2} \times 4(r-1)\eta/\alpha \\
& + q_{r+1} \times 2(r-1)[2(\alpha+r+1)\lambda\chi + 3(r-2)\eta^2]/\alpha^2 \\
& + q_r \times 4(r-1)(r-2)\eta[3(\alpha+r)\lambda\chi + (r-3)\eta^2]/\alpha^3 \\
& + q_{r-1}(r-1)(r-2)\{6(\alpha+r-1)\lambda\chi[(\alpha+r)\lambda\chi + 2(r-3)\eta^2] \\
& + (r-3)(r-4)\eta^4\}/\alpha^4 \\
& + q_{r-2} \times 4(r-1)(r-2)(r-3)(\alpha+r-2)\lambda\chi\eta[3(\alpha+r-1)\lambda\chi + (r-4)\eta^2]/\alpha^5 \\
& + q_{r-3} \times 2(r-1)(r-2)(r-3)(\alpha+r-3)(\alpha+r-2)\lambda^2\chi^2[2(\alpha+r-1)\lambda\chi \\
& + 3(r-4)\eta^2]/\alpha^6 \\
& + q_{r-4} \times 4(r-1)(r-2)(r-3)(r-4)(\alpha+r-4)(\alpha+r-3)(\alpha+r-2)\lambda^2\chi^2\eta/\alpha^7
\end{aligned}
$$

$$+ q_{r-5}(r-1)(r-2)(r-3)(r-4)(\alpha+r-5)(\alpha+r-4)(\alpha+r-3)(\alpha+r-2)$$

$$\lambda^4\chi^4/\alpha^8 \quad , \qquad (3\text{-}14)$$

As an illustration of the use of (3-13) and (3-14), suppose we need to obtain, say [1234].

$$[1234] = E\{q_4 q_3[q_3+q_2 2\eta/\alpha+q_1 2(\alpha+1)\lambda\chi/\alpha^2]\}$$

$$= Eq_4[q_3q_3+q_3q_2 2\eta/\alpha+q_3q_1 2(\alpha+1)\lambda\chi/\alpha^2]$$

$$= Eq_4[\ldots+q_4(9\alpha\lambda\chi+36\lambda\chi+18\eta^2)/\alpha^2+\ldots$$

$$+ q_4 \; 12\eta^2/\alpha^2\ldots+q_4 \; 2(\alpha+1)\lambda\chi/\alpha^2+\ldots] \quad ,$$

(Note that the terms not shown in the expression above will result in zero contribution after multiplying by q_4 and taking expectation).

$$= \phi_4[9\alpha\lambda\chi+36\lambda\chi+18\eta^2+12\eta^2+2\alpha\lambda\chi+2\lambda\chi]/\alpha^2$$

$$= 24(\alpha+1)(\alpha+2)(\alpha+3)\lambda^4\chi^4(11\alpha\lambda\chi+38\lambda\chi+30\eta^2)/\alpha^9 \quad .$$

As another example, suppose we wish to find the term $[14^3]$.

$$[14^3] = E(q_4^2 q_1 q_4)$$

If we let r = 2 in (3-14) and evaluate q_1q_4, we have:

$$[14^3] = E\; q_4^2\{q_5+q_4 4\eta/\alpha+q_3\, 4(\alpha+3)\lambda\chi/\alpha^2\}$$

$$= E\; q_4\{q_4q_5+q_4^2\; 4\eta/\alpha+q_4q_3\; 4(\alpha+3)\lambda\chi/\alpha^2\}$$

we can use the recursion formulae in a similar manner to evaluate q_4q_5, q_4^2, and q_3q_4, the result becomes

$$[14^3] = Eq_4\{\ldots q_4 240[\lambda\chi\eta(\alpha+4)(3\alpha\lambda\chi+15\lambda\chi+2\eta^2)]/\alpha^5\cdots$$

$$\ldots+q_4 96\eta[3\lambda\chi(\alpha+4)(\alpha\lambda\chi+5\lambda\chi+4\eta^2)+\eta^4]/\alpha^5\cdots$$

$$\ldots+q_4 96\lambda\chi\eta(\alpha+3)[3(\alpha+4)\lambda\chi+\eta^2]/\alpha^5+\ldots\}$$

$$= 1152(\alpha+1)(\alpha+2)(\alpha+3)\lambda^4\chi^4\eta\{5\lambda\chi(\alpha+4)(3\alpha\lambda\chi+15\lambda\chi$$

$$+ 2\eta^2)+2[3\lambda\chi(\alpha+4)(\alpha\lambda\chi+5\lambda\chi+4\eta^2)+\eta^4]$$

$$+ 2\lambda\chi(\alpha+3)(3\alpha\lambda\chi+12\lambda\chi+\eta^2)\}/\alpha^{12}$$

(d) Table of Expected Values of Q-products

As was mentioned in Chapter I, the expectation of products of the Q's can be expressed in terms of expansions in powers of $1/n$ with the coefficients being the $[1^\alpha 2^\beta 3^r 4^\xi]$ terms. In fact, in general the expansion (see Myers [9]) will be

$$(r^\alpha s^\beta t^r\ldots) = \sum_{\lambda=a}^{b} (r^\alpha s^\beta t^r\ldots)_\lambda/n^\lambda \qquad ,$$

where $a = \{(\alpha+\beta+r+\ldots+1)/2\}$, $b = -1+\alpha+\beta+r+\ldots$. (Here $\{Z\}$ denotes the integral part of Z). $(r^\alpha s^\beta t^r\ldots)_\lambda$, which represents the coefficient of $n^{-\lambda}$ in the expansion is expressed in terms of the expected q-products. Shenton and Myers [11] give tables for the $(r^\alpha s^\beta t^r u^\xi)$ in terms of the

$[r^{\alpha}s^{\beta}t^{r}u^{\xi}]$ for any four Q's in powers not exceeding ten through terms in n^{-5}. These were used as an aid in constructing tables of $(1^{\alpha}2^{\beta}3^{r}4^{\xi})$ in terms of λ and α, for $\alpha + \beta + r + \xi \leq 8$, through terms in n^{-4}. These tables are shown in Appendices A, B and C. An explanation of the use of the tables is given at the beginning of the appendices. Of course (3-6), (3-7), (3-13) and (3-14) were used to obtain the required $[1^{\alpha}2^{\beta}3^{r}4^{\xi}]$. As a brief illustration of how these tables were constructed, suppose we consider the expansion of $E(Q_3^4)$. From the tables of Shenton and Myers [11], we have the expression for (3^4) as

$$(3^4) = 3[3^2]^2/n^2 + \{[3^4] - 3[3^2]^2\}/n^3 \quad .$$

From the recurrence formulae, $[3^4]$ and $[3^2]$ were derived, resulting in the final expressions,

$$(3^4)_2 = 108\ a^2b^2\lambda^6\chi^6/\alpha^{10} \quad ,$$

$$\begin{aligned}(3^4)_3 = {} & 36ab\lambda^3\chi^3[120cde\lambda^3\chi^3+270cd\lambda^2\chi^2\eta^2 \\ & +54cg^2\lambda\chi+6\eta^2(6c\lambda\chi+\eta^2)^2+18b\lambda\chi(c\lambda\chi+\eta^2)^2 \\ & + 9ab\lambda^2\chi^2\eta^2+3\alpha ab\lambda^3\chi^3]/\alpha^{11} \quad ,\end{aligned}$$

where $a = \alpha + 1$, $b = \alpha + 2$, $c = \alpha + 3$, $d = \alpha + 4$, $e = \alpha + 5$, $f = \alpha + 6$, $g = \alpha + 7$, $\eta = 2\lambda + \alpha$, and $\chi = \lambda + \alpha$.

(e) An Example of the Use of the Tables of Expected Q-Products

Using the tables in the Appendices, one can investigate the expansions of properties such as bias, variance and higher order moments, etc., for moment estimators of the negative binomial parameters involving the first four sample moments. These expansions can be investigated through terms in n^{-4}. Suppose we consider the moment estimators of α and λ found by equating the first moment and the ratio of the 3rd to the 2nd factorial moment, i.e.,

$$\hat{\lambda} = m_1' \quad ,$$

and $$\frac{m_{(3)}}{m_{(2)}} = \frac{\mu_{(3)}}{\mu_{(2)}} = \frac{\lambda^3(1+1/\alpha)(1+2/\alpha)}{\lambda^2(1+1/\alpha)} = \lambda(1+2/\alpha) \quad ,$$

The estimator of α becomes

$$\hat{\alpha} = 2m_1 m_{(2)}/[m_{(3)} - m_1 m_{(2)}] \quad .$$

Suppose we consider, say the bias of $\hat{\alpha}$. The sample factorial moments are easily expressed in terms of the Q's and thus we can expand $\hat{\alpha}$ as

$$\begin{aligned}\hat{\alpha} = {} & \alpha + Q_2(\alpha+3)/\lambda(\alpha+1) + Q_3\alpha/2\lambda^3(\alpha+1) + 2Q_1^2\alpha/\lambda \\ & - Q_2^2\alpha(\alpha+3)^2/\lambda^3(\alpha+1)^2 - Q_3^2\alpha/2\lambda^6(\alpha+1) \\ & + Q_1Q_2[\alpha^3+2\alpha^2-6\lambda(\alpha+3)]/2\lambda^3(\alpha+1) \\ & - Q_1Q_3 3(\alpha+3)/\lambda^2(\alpha+1) - Q_2Q_3\alpha^2(\alpha+3)(\alpha+1)/2\lambda^5(\alpha+1) \\ & + \ldots\ldots \qquad (3\text{-}15)\end{aligned}$$

Note that for the purpose of illustration we are only expanding through powers of 2 in order to find the bias through terms in n^{-1}. In practice, of course, we would want to go to higher order terms.

For example if we desired to express the bias to terms in n^{-2}, we would expand $\hat{\alpha}$ through terms of order 4 in the Q's.

Upon taking expectation of $\hat{\alpha}$ in (3-15), we have,

$$E(\hat{\alpha}) = \alpha + \frac{1}{n}\{2\alpha(1^2)_1/\lambda-(2^2)_1\alpha(\alpha+3)^2/[\lambda^3(\alpha+1)^2]$$

$$-(3^2)_1\alpha/[2\lambda^6(\alpha+1)]+(12)_1(\alpha^3+2\alpha^2-6\alpha\lambda-18\lambda)/[2\lambda^3(\alpha+1)]$$

$$-(13)_1\,3(\alpha+3)/[\lambda^2(\alpha+1)]-(23)_1\alpha^2(\alpha+3)(\alpha+1)/[2\lambda^5(\alpha+1)]\}$$

$$+\ldots,$$

we can substitute expression for $(3^2)_1$, $(12)_1$, $(13)_1$ and $(23)_1$ found in Appendix A. Expressions for $(1^2)_1$ and $(2^2)_1$ are, of course, found in the tables of Shenton and Myers. The result becomes

$$E(\hat{\alpha}) = \alpha + \frac{1}{n}[2\chi-2\chi^2(\alpha+3)^2/\alpha^2\lambda(\alpha+1)-3\chi^2(\alpha+2)/\alpha^4\lambda^3]+\ldots,$$

A similar procedure could be performed for powers of $\hat{\alpha}$ if we desired to find higher order moments.

CHAPTER IV

SUMMARY

This work is an extension of the development of orthogonal statistics in estimation problems for the negative binomial distribution. The method can enable one to investigate the sampling properties of moment estimators involving the first four sample moments.

The stimulation for this work resulted out of curiosity concerning the properties of moment estimators involving high order moments. If we call $q_r(x)$, $(r = 0,1,2,\ldots)$ the set of orthogonal polynomials associated with the negative binomial distribution, then the rth orthogonal statistic, defined for the sample $x_1, x_2, \ldots x_n$ is

$$Q_r(x) = \sum_{j=1}^{n} q_r(x_j)/n$$

Expectations of powers and products of the Q's can be expressed as expansions in powers of $1/n$ with the coefficients being the population parameters. Myers and Shenton [11], who used this method to investigate sampling properties of moment estimators of the negative binomial parameters involving the first two moments, constructed tables of expectations of powers and products of the Q's involving Q_1 and Q_2 through order 8, to terms in n^{-4}. Appendices A, B and C of this thesis give expansions involving Q_1, Q_2, Q_3 and Q_4. Assuming

an estimator, say t (m_1, m_2, m_3, m_4) (or powers of t for the case of variance and higher order moments of t) can be expanded as a linear function of powers and products of the Q's, one can take expectation term by term of this expansion and, through the tables in Appendices A, B and C, obtain an expansion of the desired moment of t through terms in n^{-4}. This technique can be used to investigate such properties as bias, variance and higher order moments.

Tables of $E(Q_1^{\alpha}Q_2^{\beta}Q_3^{r}Q_4^{\xi})$ obtained in [11] in terms of the expectation of products of q's were helpful in this work. However there still remained the problem of going from $E(q_1^{\alpha}q_2^{\beta}\cdots)$ to the population parameters. Certain recurrence formulae (See Chapter III) involving products of the q's were developed to use in this task. Expressions for q_3q_{r-1} and q_4q_{r-1}, $r = 1,2,\ldots$, as linear functions of the individual q's were derived. Then by manipulating these formulae and their products, for appropriate value of r, the necessary expected q-products were found.

A P P E N D I C E S

APPENDIX

Tables of Expected Q-products involving Q_1, Q_2, Q_3 and Q_4 through terms in n^{-4}.

Explanation and Example

Appendix A contains cases involving Q_1, Q_2 and Q_3. Appendix $B_{(1)}$ contains expected Q's involving Q_1, Q_2, Q_3 and Q_4, but only through order 5. Appendix $B_{(2)}$ contains the balance of those involving the first four Q's, i.e., order 6, 7, and 8 but rather than being expressed in terms of the population parameters, they are expressed as functions of the square bracket terms (the expected q-products). Appendix C contains expressions for the expected q-products that are necessary for the complete use of Appendix B_2, i.e., square bracket term necessary in evaluating expected Q-products of order 6, 7 or 8 involving Q_1, Q_2, Q_3 and Q_4. As an example of the use of Appendices B_2 and C, suppose we desire the expansion of $E(Q_2Q_3Q_4^5)$. For the scope of this work, this involves only one terms, namely the coefficient of n^{-4}, i.e., $(234^5)_4$. From Appendix $B_{(2)}$ we see that $(234^4)_4 = 15[4^2]^2[234]$. From Appendix C we note that $[4^2] = 24abc\lambda^4\chi^4/\alpha^7$ and $[234] = 144abc\lambda^4\chi^4\eta/\alpha^8$. By substitution then we can easily see that $(234^5)_4 = 1244160a^3b^3c^3\lambda^{12}\chi^{12}\eta/\alpha^{22}$.

APPENDIX A

* Expected Q-Products Involving Q_1, Q_2 and Q_3

Notation

$\lambda + \alpha = \chi$	$2b\lambda\chi + \eta^2 = j$
$2\lambda + \alpha = \eta$	$c\lambda\chi + \eta^2 = k$
$\alpha + 1 = a$	$6c\lambda\chi + \eta^2 = p$
$\alpha + 2 = b$	$d\lambda\chi + 2\eta^2 = q$
$\alpha + 3 = c$	$2d\lambda\chi + 3\eta^2 = r$
$\alpha + 4 = d$	$3d\lambda\chi + \eta^2 = s$
$\alpha + 5 = e$	$3d\lambda\chi(e\lambda\chi + 4\eta^2) + \eta^4 = t$
$\alpha + 6 = f$	$e\lambda\chi + 3\eta^2 = u$
$\alpha + 7 = g$	$3e\lambda\chi + 2\eta^2 = v$
$\alpha + 8 = h$	$e\lambda\chi(f\lambda\chi + 6\eta^2) + \eta^4 = w$
$\alpha + 9 = i$	$f\lambda\chi + \eta^2 = y$
	$2f\lambda\chi + 9\eta^2 = z$

* Those products involving only Q_1 and Q_2 can be found in tables by Shenton and Myers [11]

<u>Products of Order 2</u>

$$(3^2)_1 = 6ab\lambda^3\chi^3/\alpha^5$$

$$(12)_1,\ (13)_1 \text{ and } (23)_1 = 0$$

<u>Products of Order 3</u>

$$(3^3)_2 = 36abp\lambda^3\chi^3\eta/\alpha^8$$

$$(3^21)_2 = 18ab\lambda^3\chi^3\eta/\alpha^6$$

$$(3^22)_2 = 36abk\lambda^3\chi^3/\alpha^7$$

$$(31^2)_2 = 0$$

$$(32^2)_2 = 24ab\lambda^3\chi^3\eta/\alpha^6$$

$$(321)_2 = 6ab\lambda^3\chi^3/\alpha^5$$

<u>Products of Order 4</u>

$$(3^4)_2 = 108a^2b^2\lambda^6\chi^6/\alpha^{10}$$

$$(3^21^2)_2 \quad 6ab\lambda^4\chi^4/\alpha^6$$

$$(3^22^2)_2 = 12a\ b\lambda^5\chi^5/\alpha^8$$

$(3^31)_2$, $(3^32)_2$, $(3^212)_2$, $(31^22)_2$, $(312^2)_2$, $(31^3)_2$, and $(32^3)_2 = 0$

- -

$$(3^4)_3 = 36ab\lambda^3\chi^3(120cde\lambda^3\chi^3+270cd\lambda^2\chi^2\eta^2+54cq^2\lambda\chi$$

$$+6p^2\eta^2+18bk^2\lambda\chi+9ab\lambda^2\chi^2\eta^2+3\alpha ab\lambda^3\chi^3)/\alpha^{11}$$

$$(3^3 1)_3 = 108ab\lambda^3\chi^3(2cq\lambda\chi+p\eta^2+6k\lambda\chi)/\alpha^9$$

$$(3^3 2)_3 = 24ab\lambda^3\chi^3\eta(a\lambda\chi+4b\lambda\chi+12c\lambda\chi+8\eta^2)/\alpha^{10}$$

$$(3^2 1^2)_3 = 6ab\lambda^3\chi^3(6k-\alpha\lambda\chi+3\eta)/\alpha^7$$

$$(3^2 2^2)_3 = 12ab\lambda^3\chi^3(18cq\lambda\chi+12p\eta^2+6jk+6a\lambda\chi\eta^2$$

$$+2a\lambda^2\chi^2-\alpha a\lambda^2\chi^2)/\alpha^9$$

$$(3^2 12)_3 = 36ab\lambda^3\chi^3\eta(a\lambda\chi+8c\lambda\chi+3\eta^2)/\alpha^8$$

$$(31^2 2)_3 = 30ab\lambda^3\chi^3\eta/\alpha^6$$

$$(312^2)_3 = 12ab\lambda^3\chi^3(3k+4\eta^2+a\lambda\chi)/\alpha^7$$

$$(31^3)_3 = 6ab\lambda^3\chi^3/\alpha^5$$

$$(32^3)_3 = 24ab\lambda^3\chi^3\eta(a\lambda\chi+4b\lambda\chi+12c\lambda\chi+8\eta^2)/\alpha^8$$

<u>Products of Order 5</u>

$$(3^5)_3 = 2160\ a^2b^2p\lambda^6\chi^6\eta/\alpha^{13}$$

$$(3^4 1)_3 = 648\ a^2b^2\lambda^6\chi^6\eta/\alpha^{11}$$

$$(3^4 2)_3 = 1296\ a^2b^2k\lambda^6\chi^6/\alpha^{12}$$

$$(3^3 1^2)_3 = 36abp\lambda^4\chi^4\eta/\alpha^9$$

$$(3^3 2^2)_3 = 72abp\lambda^5\chi^5\eta\ /\alpha^{11}$$

$$(3^3 12)_3 = 108a^2b^2\lambda^6\chi^6/\alpha^{10}$$

$$(3^2 1^3)_3 = 60ab\lambda^4\chi^4\eta/\alpha^7$$

$$(3^2 1^2 2)_3 = 12ab\lambda^4\chi^4(a\lambda\chi+3k)/\alpha^8$$

$$(3^2 12^2)_3 = 60a^2b\lambda^5\chi^5\eta/\alpha^9$$

$$(3^2 2^3)_3 = 24a^2b\lambda^5\chi^5(j+9k)/\alpha^{10}$$

$$(31^4)_3 = 0$$

$$(31^3 2)_3 = 18ab\lambda^4\chi^4/\alpha^6$$

$$(31^2 2^3)_3 = 24ab\lambda^4\chi^4\eta/\alpha^7$$

$$(312^3)_3 = 36a^2b\lambda^5\chi^5/\alpha^8$$

$$(32^4)_3 = 288a^2b\lambda^5\chi^5\eta/\alpha^9$$

- -

$$\begin{aligned}(3^5)_4 = {} & 216ab\lambda^3\chi^3\eta[\lambda^3\chi^3(1500cdef\lambda\chi+9400cde\eta^2 \\ & + 360cdeq+90bcdk+360cdeq+18abcq \\ & + 20cdep+90bcdk+9ab^2k+18abcq+9ab^2k+abp-9\alpha abp) \\ & + 2\lambda^2\chi^2(1350cdp\eta^4+25cdqu+270cdp\eta^2+270cdqu+54bckq \\ & + 9abp\eta^2+54bckq+18b^2k)+9\lambda\chi(24cq^2s+6cpq^2+6cpq^2+4bk^2p) \\ & + 6p^3\eta^2]/\alpha^{14}\end{aligned}$$

$$
\begin{aligned}
(3^4 1)_4 = {} & 108ab\lambda^3\chi^3\eta[\lambda^3\chi^3(100cde+30bcd \\
& + 4abc+3ab^2-5\alpha ab)+3\lambda^2\chi^2(40cdu \\
& + 90cd\eta^2+12bcq+8bck+4b^2k \\
& + 3ab\eta^2)+6\lambda\chi(8cqs+9cq^2+2cpq+bkp+3bk^2)+6p^2\eta]/\alpha^{12} \\
(3^4 2)_4 = {} & 24ab\lambda^3\chi^3[9\lambda^3\chi^3(50cdef\lambda\chi+200cde\eta^2 \\
& + 600cde\eta^2+10bcdk+24de\eta^2+12ab\eta^2 \\
& + 20cdek+60bcd\eta^2+6ab^2\eta^2+6abcq \\
& + 3ab^2k-6\alpha abk)+27\lambda^2\chi^2\eta^2(100cd\eta^2 \\
& + 120du+24bk+90cdk+3abk+24bcq \\
& + 8b^2k+abp)+54\lambda\chi(24qs\eta^2+6pq\eta^2 \\
& + 9ckq^2+3bk^2+2bkp\eta^2)+54kp^2\eta^2]/\alpha^{13} \\
(3^3 1^2)_4 \quad & 36ab\lambda^3\chi^3\eta[30cd\lambda^2\chi^2+\lambda\chi(42cq+4cp+3bp \\
& + 15bk+3ab-\alpha p)+9p\eta^2]/\alpha^{10} \\
(3^3 2^2)_4 = {} & 72ab\lambda^3\chi^3\eta[2\lambda^3\chi^3(60cde+3abc+20cde \\
& - 2\alpha ab)+3\lambda^2\chi^2(60cdu+12bck+180cd\eta^2 \\
& + 6ab\eta^2+10cdj+ab+8acq+2abk) \\
& + 6\lambda\chi(12qs+3cpq+18cq^2+6bk^2 \\
& + 6cjq+2bjk+ap\eta^2)+12p^2\eta^2 \\
& + 6jkp]/\alpha^{12}
\end{aligned}
$$

$$(3^3 12)_4 = 36ab\lambda^3\chi^3[2\lambda^3\chi^3(10cde-\alpha ab)+3\lambda^2\chi^2(90cd\eta^2$$

$$+\overset{5}{3}ab\eta^2+20cd\eta^2+4acq+2abk)$$

$$+6\lambda\chi(\overset{7}{\cdot}cq^2+3bk^2+12c\cdot\eta^2+4bk\eta^2$$

$$+ap\eta^2)+6p\eta^2(p+2k)]/\alpha^{11}$$

$$(3^2 1^3)_4 = 6ab\lambda^3\chi^3\eta(-10\alpha\lambda\chi+24b\lambda\chi+40c\lambda\chi+27\eta^2)/\alpha^8$$

$$(3^2 1^2 2)_4 = 12ab\lambda^3\chi^3[3\lambda\chi(6cq+4bk+3a\eta^2)+15p\eta^2+12k\eta^2]/\alpha^9$$

$$(3^2 1 2^2)_4 = 12ab\lambda^3\chi^3\eta[6\lambda^2\chi^2(15cd+2ab)+12\lambda\chi(12cq$$

$$+cp+bp+5bk+ak+ab\eta^2)+42p\eta^2+12k\eta^2]/\alpha^{10}$$?

$$(3^2 2^3)_4 = 24ab\lambda^3\chi^3[30cde\lambda^3\chi^3+6\lambda^2\chi^2(90cd\eta^2$$

$$+36ck+3ab\eta^2-\alpha ak)+6\lambda\chi(6cqr$$

$$+36cq\eta^2+3cjq+6cp\eta^2+ap\eta^2$$

$$+12bk\eta^2+2ak\eta^2+aj\eta^2)+36kp\eta^2$$

$$+12jp\eta^2+6j^2k]/\alpha^{11}$$

$$(31^4)_4 = 6ab\lambda^2\chi^2\eta(5\lambda\chi+1)/\alpha^6$$

$$(31^2 2)_4 = 6ab\lambda^3\chi^3[\lambda\chi(2a+3b+4c-3\alpha)+19\eta^2]/\alpha^7$$

$$(31^2 2^2)_4 = 12ab\lambda^3\chi^3\eta[\lambda\chi(5a+10b+25c-2\alpha)+23\eta^2]/\alpha^8$$

$$(312^3)_4 = 24ab\lambda^3\chi^3[\lambda^2\chi^2(3bc-\alpha a)+\lambda\chi(4cr$$

$$+\underset{24}{42}c\eta^2+2cj+5a\eta^2+12b\eta^2)+18k\eta^2+6j\eta^2+j^2]/\alpha^9$$

3

$$(32^4)_4 = 48ab\lambda^3\chi^3\eta[\lambda^2\chi^2(a^2+3ab+6bc+40cd$$

$$- 5\alpha a)+\lambda\chi(aj+3ak+4a\eta^2+24b\eta^2$$

$$+ 6cj+12cr+72c\eta^2+18ck)+36k$$

$$+ 2j^2]/\alpha^{10}$$

Products of Order 6

$$(3^6)_3 = 3240a^3b^3\lambda^9\chi^9/\alpha^{15}$$

$$(3^41^2)_3 = 108a^2b^2\lambda^7\chi^7/\alpha^{11}$$

$$(3^42^2)_3 = 216a^3b^2\lambda^8\chi^8/\alpha^{13}$$

$$(3^21^4)_3 = 18ab\lambda^5\chi^5/\alpha^7$$

$$(3^21^22^2)_3 = 12a^2b\lambda^6\chi^6/\alpha^9$$

$$(3^22^4)_3 = 72a^3b\lambda^7\chi^7/\alpha^{11}$$

$(3^51)_3$, $(3^52)_3$, $(3^412)_3$, $(3^31^22)_3$, $(3^312^2)_3$, $(3^31^3)_3$, $(3^32^3)_3$, $(3^21^32)_3$, $(3^212^3)_3$, $(31^5)_3$, $(31^42)_3$, $(31^32^2)_3$, $(31^22^2)_3$, $(312^4)_3$, and $(32^5)_3 = 0$

$$(3^6)_4 = 3240\, a^2b^2\lambda^6\chi^6[\lambda^3\chi^3(20cde+3\alpha ab)$$

$$+ 9\lambda^2\chi^2\eta^2(30cd+ab)+18\lambda\chi(3cq^2+bk^2)+10p^2\eta^2]/\alpha^{16}$$

$$(3^5 1)_4 = 12960a^2b^2\lambda^6\chi^6(cq\lambda\chi+3k\lambda\chi+p\eta^2)/\alpha^{14}$$

$$(3^5 2)_4 = 6480a^2b^2\lambda^6\chi^6[\lambda^2\chi^2\eta(10cd+a^2)+4\lambda\chi(3cq\eta$$

$$+ bk)+2kp(\eta+1)]/\alpha^{15}$$

$$(3^4 12)_4 = 432a^2b^2\lambda^6\chi^6\eta(3a\lambda\chi+5c\lambda\chi+20\eta^2)/\alpha^{13}$$

$$(3^4 1^2)_4 = 54ab\lambda^4\chi^4[\lambda^3\chi^3(40cde+3\alpha ab)$$

$$+ 18\lambda^2\chi^2(13ab\eta^2+30cd\eta+6abk)$$

$$+ 18\lambda\chi(3cq^2+bk^2)+b\eta^2]/\alpha^{12}$$

$$(3^4 2^2)_4 = 72a^2b\lambda^5\chi^5[\lambda^3\chi^3(20cde+3\alpha ab)$$

$$+ 9\lambda^2\chi^2(30cd\eta^2+5ab\eta^2+12bcq)$$

$$+ 6\lambda\chi(21bk^2+20bp\eta^2+9cq^2+6jk)+6p^2\eta^2]/\alpha^{14}$$

$$(3^3 1^2 2)_4 = 36ab\lambda^4\chi^4\eta[3\lambda^2\chi^2(11ab+10cd+a^2)$$

$$+ 2\lambda\chi(ap+18cq+6bk)+6kp]/\alpha^{11}$$

$$(3^3 12^2)_4 = 72ab\lambda^5\chi^5[3ab\lambda^2\chi^2(a+3c)$$

$$+ 3a\lambda\chi(13b\eta^2+6bk\eta+2cq+6k)+ap\eta(2ab+3p\eta)]/\alpha^{12}$$

$$(3^3 1^3)_4 = 36ab\lambda^4\chi^4[3ab\lambda^2\chi^2+18\lambda\chi(cq+3k)+10p\eta^2]/\alpha^{10}$$

$$(3^3 2^3)_4 = 72a^2b\lambda^5\chi^5[3\lambda^2\chi^2\eta(30cd+3a^2+24bc+8b^2$$

$$+ 2ab)+12\lambda\chi(9bk\eta+9cq\eta+3bk+4b\eta^3)$$

$$+ 2p\eta+18kp]/\alpha^{13}$$

$$(3^2 1^4)_4 = 6ab\lambda^4\chi^4[2\lambda\chi(a-4\alpha)+31\eta^2+36k]/\alpha^8$$

$$(3^2 1^3 2)_4 = 36ab\lambda^4\chi^4\eta(7a\lambda\chi+25c\lambda\chi+10\eta^2)/\alpha^9$$

$$(3^2 1^2 2^2)_4 = 12ab\lambda^4\chi^4[2a\lambda^2\chi^2(6b+3c-\alpha a)+3\lambda\chi(4a+3a\eta^2+6cq+4\alpha k)$$

$$+ 2(j^2+6p\eta^2+3jk)]/\alpha^{10}$$

$$(3^2 1 2^3)_4 = 24a^2b\lambda^5\chi^5\eta[\lambda\chi(12\alpha+9a+42b+90c+22)+50\eta^2]/\alpha^{11}$$

$$(3^2 2^4)_4 = 48a^2b\lambda^5\chi^5[\lambda^2\chi^2(3bc-\alpha a)+\lambda\chi(84b\eta^2$$

$$+ j^2+20a\eta^2+54cq)+30jk+36p\eta^2]/\alpha^{12}$$

$$(31^5)_4 = 60ab\lambda^4\chi^4/\alpha^6$$

$$(31^4 2)_4 = 204ab\lambda^4\chi^4\eta/\alpha^7$$

$$(31^4 2)_4 = 204ab\lambda^4\chi^4\eta/\alpha^7$$

$$(31^3 2^2)_4 = 12ab\lambda^4\chi^4(10a\lambda\chi+9c\lambda\chi+23\eta^2)/\alpha^8$$

$$(31^2 2^3)_4 = 12ab\lambda^4\chi^4\eta(41a\lambda\chi+8b\lambda\chi+24c\lambda\chi+16\eta^2)/\alpha^9$$

$$(312^4)_4 = 144a^2b\lambda^5\chi^5(a\lambda\chi+2b\lambda\chi+3c\lambda\chi+12\eta^2)/\alpha^{10}$$

$$(32^5)_4 = 480a^2b\lambda^5\chi^5\eta(a\lambda\chi+8b\lambda\chi+12c\lambda\chi+10\eta^2)/\alpha^{11}$$

<u>Products of Order 7</u>

$$(3^7)_4 = 136080a^3b^3p\lambda^9\chi^9\eta/\alpha^{18}$$

$$(3^6 1)_4 = 29160a^3b^3\lambda^9\chi^9\eta/\alpha^{16}$$

$$(3^6 2)_4 = 58320a^3b^3k\lambda^9\chi^9/\alpha^{17}$$

$$(3^5 1^2)_4 = 2160a^2b^2p\lambda^7\chi^7\eta/\alpha^{14}$$

$$(3^5 2^2)_4 = 4320a^3b^2\lambda^8\chi^8\eta(1+3b\lambda\chi)/\alpha^{16}$$

$$(3^5 12)_4 = 3240\ a^3b^3\lambda^9\chi^9/\alpha^{15}$$

$$(3^4 1^3)_4 = 2052\ a^2b^2\lambda^7\chi^7\eta/\alpha^{12}$$

$$(3^4 2^3)_4 = 432a^3b^2\lambda^8\chi^8(18k+j)/\alpha^{15}$$

$$(3^4 1^2 2)_4 = 216a^2b^2\lambda^7\chi^7(a\lambda\chi+6k)/\alpha^{13}$$

$$(3^4 12^2)_4 = 1728a^3b^2\lambda^8\chi^8\eta/\alpha^{14}$$

$$(3^3 1^4)_4 = 108abp\lambda^5\chi^5\eta/\alpha^{10}$$

$$(3^3 1^3 2)_4 = 324a^2b\lambda^7\chi^7/\alpha^{11}$$

$$(3^3 12^3)_4 = 648a^3b^2\lambda^8\chi^8/\alpha^{13}$$

$$(3^3 1^2 2^2)_4 = 72a^2b\lambda^6\chi^6\eta(6c\lambda\chi+6b+\eta^2)/\alpha^{12}$$

$$(3^3 2^4)_4 = 432a^3b^2p\lambda^7\chi^7\eta/\alpha^{14}$$

$$(3^2 1^5)_4 = 330ab\lambda^5\chi^5\eta/\alpha^8$$

$$(3^2 1^4 2)_4 = 36ab\lambda^5\chi^5(2a\lambda\chi+3k)/\alpha^9$$

$$(3^2 1^3 2^2)_4 = 144a^2b\lambda^6\chi^6\eta/\alpha^{10}$$

$$(3^2 1^2 2^3)_4 = 24a^2 b\lambda^6 \chi^6 (3a\lambda\chi + j + 9k)/\alpha^{11}$$

$$(3^2 1 2^4)_4 = 504a^3 b\lambda^7 \chi^7 \eta/\alpha^{12}$$

$$(3^2 2^5)_4 = 240a^3 b\lambda^7 \chi^7 (4j+9k)/\alpha^{13}$$

$$(3 1^6)_4 = 0$$

$$(3 1^5 2)_4 = 90ab\lambda^5 \chi^5/\alpha^7$$

$$(3 1^4 2^2)_4 = 72ab\lambda^5 \chi^5 \eta/\alpha^8$$

$$(3 1^3 2^3)_4 = 108a^2 b\lambda^6 \chi^6/\alpha^9$$

$$(3 1^2 2^4)_4 = 288a^2 b\lambda^6 \chi^6 \eta/\alpha^{10}$$

$$(3 1 2^5)_4 = 360a^3 b\lambda^7 \chi^7/\alpha^{11}$$

$$(3 2^6)_4 = 4320a^3 b\lambda^7 \chi^7 \eta/\alpha^{12}$$

<u>Products of Order 8</u>

$$(3^8)_4 = 136080a^4 b^4 \lambda^{12} \chi^{12}/\alpha^{20}$$

$$(3^6 1^2)_4 = 3240a^3 b^3 \lambda^{10} \chi^{10}/\alpha^{16}$$

$$(3^6 2^2)_4 = 6480a^4 b^3 \lambda^{11} \chi^{11}/\alpha^{18}$$

$$(3^4 1^2 2^2)_4 = 216a^3 b^2 \lambda^9 \chi^9/\alpha^{14}$$

$$(3^4 1^4)_4 = 324a^2 b^2 \lambda^8 \chi^8/\alpha^{12}$$

$$(3^4 2^4)_4 = 1296a^4 b^2 \lambda^{10} \chi^{10}/\alpha^{16}$$

$$(3^2 1^6)_4 = 90a\lambda^6 \chi^6/\alpha^8$$

$$(3^2 2^6)_4 = 720a^4 \lambda^9 \chi^9/\alpha^{14}$$

$$(3^2 2^2 1^4)_4 = 36a^2 b\lambda^7 \chi^7/\alpha^{10}$$

$$(3^2 1^2 2^4)_4 = 72a^3 b\lambda^8 \chi^8/\alpha^{12}$$

APPENDIX $B_{(1)}$

Expected Q-Products Involving Q_1, Q_2, Q_3 and Q_4 Through Order 5

(Note: See Appendix A for notation).

Products of Order 2

$$(4^2)_1 = 24abc\lambda^4\chi^4/\alpha^7$$

Products of Order 3

$$(4^3)_2 = 576abct\lambda^4\chi^4/\alpha^{11}$$

$$(4^2 1)_2 = 96abc\lambda^4\chi^4\eta/\alpha^8$$

$$(4^2 2)_2 = 96abcr\lambda^4\chi^4/\alpha^9$$

$$(4^2 3)_2 = 576abcs\lambda^4\chi^4\eta/\alpha^{10}$$

$$(41^2)_2 = 0$$

$$(42^2)_2 = 24abc\lambda^4\chi^4/\alpha^7$$

$$(412)_2 = 0$$

$$(413)_2 = 24abc\lambda^4\chi^4/\alpha^7$$

$$(423)_2 = 144abc\lambda^4\chi^4\eta/\alpha^8$$

$$(43^2)_3 = 216abcq\lambda^4\chi^4/\alpha^9$$

<u>Products of Order 4</u>

$$(4^4)_2 = 1728a^2b^2c^2\lambda^8\chi^8/\alpha^{14}$$

$$(4^2 1^2)_2 = 24abc\lambda^5\chi^5/\alpha^8$$

$$(4^2 2^2)_2 = 48a^2bc\lambda^6\chi^6/\alpha^{10}$$

$$(4^2 3^2)_2 = 144a^2b^2c\lambda^7\chi^7/\alpha^{12}$$

$(4^3 1)_2$, $(4^3 2)_2$, $(4^3 3)_2$, $(4^2 12)_2$, $(4^2 13)_2$, $(4^2 23)_2$, $(413)_2$, $(423)_2$, $(433)_2$, $(41^2 2)_2$, $(41^2 3)_2$, $(412^2)_2$, $(42^2 3)_2$, $(413^2)_2$, $(423^2)_2$, and $(4123)_2 = 0$

- -

$$(4^4)_3 = 576abc\lambda^4\chi^4[2\lambda^4\chi^4(35defg-\alpha ac)$$

$$+ 16\lambda^3\chi^3\eta(140defg\eta+abc)+8\lambda^2\chi^2(10dez^2+bcr^2)$$

$$+ 96\lambda\chi\eta^2(5dv+cs^2)+24t^2]/\alpha^{15}$$

$$(4^3 1)_3 = 1152abc\lambda^4\chi^4\eta(5dv\lambda\chi+2cs\lambda\chi+2t)/\alpha^{12}$$

$$(4^3 2)_3 = 1152abc\lambda^4\chi^4[\lambda^2\chi^2(5de+bcr)+4\lambda\chi\eta^2(10dv+3cs)+2rt]/\alpha^{13}$$

$$(4^3 3)_3 = 2304abc\lambda^4\chi^4\eta[\lambda^3\chi^3(35def+abc)+3\lambda^2\chi^2(10dez$$

$$+ bcr)+3\lambda\chi(10duv+3cqs)+6st]/\alpha^{14}$$

$$(4^2 1^2)_3 = 192abc\lambda^4\chi^4(d\lambda\chi+2\eta^2)/\alpha^9$$

$$(4^2 2^2)_3 = 192abc\lambda^4\chi^4(2a\lambda\chi\eta^2+jr+3t+12s\eta^2)/\alpha^{11}$$

$$(4^2 3^2)_3 = 432abc\lambda^4\chi^4[\lambda^2\chi^2(10dez+3ab\eta)+4\lambda\chi(30dv\eta$$

$$+ bkr)+12qt+8ps\eta^2]/\alpha^{13}$$

$$(4^2 12)_3 = 192abc\lambda^4\chi^4\eta(a\lambda\chi+11d\lambda\chi+6\eta^2)/\alpha^{10}$$

$$(4^2 13)_3 = 288abc\lambda^4\chi^4(2t+6s\eta^2+br\lambda\chi)/\alpha^{11}$$

$$(4^2 23)_3 = 576abc\lambda^4\chi^4\eta(ab\lambda^2\chi^2+2b\lambda\chi+10dv\lambda\chi+6t+6ks)/\alpha^{12}$$

$$(41^3)_3 = 0$$

$$(42^3)_3 = 48abc\lambda^4\chi^4(2b\lambda\chi+4d\lambda\chi+19\eta^2)/\alpha^9$$

$$(43^3)_3 = 432abc\lambda^4\chi^4\eta[\lambda^2\chi^2(ab+20de)+6\lambda\chi(bk+6du)$$

$$+ 3pq+12qs]/\alpha^{12}$$

$$(41^2 2)_3 = 24abc\lambda^4\chi^4/\alpha^7$$

$$(41^2 3)_3 = 168abc\lambda^4\chi^4\eta/\alpha^8$$

$$(412^2)_3 = 192abc\lambda^4\chi^4\eta/\alpha^8$$

$$(42^2 3)_3 = 96abc\lambda^4\chi^4\eta(a\lambda\chi+3j+9q+6s)/\alpha^{10}$$

$$(413^2)_3 = 144abc\lambda^4\chi^4\eta(3b\lambda\chi+3q+4s)/\alpha^{10}$$

$$(423^2)_3 = 144abc\lambda^4\chi^4[\lambda^2\chi^2(ab+10de)+2\lambda\chi\eta^2(5d$$

$$+ 6b)+9kq+24s\eta^2]/\alpha^{11}$$

$$(4123)_3 = 24abc\lambda^4\chi^4(2a\lambda\chi+9q^2+12\eta^2)/\alpha^9$$

Products of Order 5

$$(4^{5})_3 = 138240a^2b^2c^2t\lambda^8\chi^8/\alpha^{18}$$

$$(4^{4}1)_3 = 13824a^2b^2c^2\lambda^8\chi^8\eta/\alpha^{15}$$

$$(4^{4}2)_3 = 13824a^2b^2c^2r\lambda^8\chi^8/\alpha^{16}$$

$$(4^{4}3)_3 = 82944a^2b^2c^2s\lambda^8\chi^8\eta/\alpha^{17}$$

$$(4^{3}1^{2})_3 = 576abct\lambda^5\chi^5/\alpha^{12}$$

$$(4^{3}2^{2})_3 = 576a^2bc\lambda^6\chi^6(2t+3bc\lambda^2\chi^2)/\alpha^{14}$$

$$(4^{3}3^{2})_3 = 3456a^2b^2c\lambda^7\chi^7(6cq\lambda\chi+t)/\alpha^{16}$$

$$(4^{3}12)_3 = 0$$

$$(4^{3}13)_3 = 576a^2b^2c^2\lambda^8\chi^8/\alpha^{14}$$

$$(4^{3}23)_3 = 3456a^2b^3c^2\lambda^8\chi^8\eta/\alpha^{15}$$

$$(4^{2}1^{3})_3 = 120abc\lambda^5\chi^5\eta/\alpha^{9}$$

$$(4^{2}2^{3})_3 = 96a^2bc\lambda^6\chi^6(j+6r)/\alpha^{12}$$

$$(4^{2}3^{3})_3 = 288a^2b^2c\lambda^7\chi^7\eta(3p+2s)/\alpha^{15}$$

$$(4^{2}1^{2}2)_3 = 48abc\lambda^5\chi^5(a\lambda\chi+2r)/\alpha^{10}$$

$$(4^{2}1^{2}3)_3 = 576abcs\lambda^5\chi^5\eta/\alpha^{11}$$

$$(4^2 1 2^2)_3 = 288 a^2 b c \lambda^6 \chi^6 \eta / \alpha^{11}$$

$$(4^2 2^2 3)_3 = 576 a^2 b c \lambda^6 \chi^6 (2s + b\lambda\chi\eta)/\alpha^{13}$$

$$(4^2 1 3^2)_3 = 1008 a^2 b^2 c \lambda^7 \chi^7 \eta / \alpha^{13}$$

$$(4^2 2 3^2)_3 = 288 a^2 b^2 c \lambda^7 \chi^7 (2r + 3k)/\alpha^{14}$$

$$(4^2 1 2 3)_3 = 144 a^2 b^2 c \lambda^7 \chi^7 / \alpha^{12}$$

$$(4 1^4)_3 = 0$$

$$(4 2^4)_3 = 144 a^2 b c \lambda^6 \chi^6 / \alpha^{10}$$

$$(4 3^4)_3 = 1296 a^2 b^2 c q \lambda^7 \chi^7 / \alpha^{14}$$

$$(4 1^3 2)_3 = 0$$

$$(4 1^3 3)_3 = 72 a b c \lambda^5 \chi^5 / \alpha^8$$

$$(4 1 2^3)_3 = 0$$

$$(4 2^3 3)_3 = 864 a^2 b c \lambda^6 \chi^6 \eta / \alpha^{11}$$

$$(4 1 3^3)_3 = 432 a^2 b^2 c \lambda^7 \chi^7 / \alpha^{12}$$

$$(4 2 3^3)_3 = 2592 a^2 b^2 c \lambda^7 \chi^7 \eta / \alpha^{13}$$

$$(4 1^2 2^2)_3 = 24 a b c \lambda^5 \chi^5 / \alpha^8$$

$$(4 1^2 3^2)_3 = 216 a b c q \lambda^5 \chi^5 / \alpha^{10}$$

$$(4 2^2 3^2)_3 = 72 a^2 b c \lambda^6 \chi^6 (2b\lambda\chi + 3q)/\alpha^{12}$$

$$(41^2 23)_3 = 144abc\lambda^5\chi^5\eta/\alpha^9$$

$$(412^2 3)_3 = 48a^2bc\lambda^6\chi^6/\alpha^{10}$$

$$(4123^2)_3 = 0$$

$$(4^5)_4 = 9216abc\lambda^4\chi^4\ \{10defg\lambda^4\chi^4[147h\lambda\chi(i\lambda\chi+12\eta^2)$$

$$+\ 5791\eta^4+705h\lambda\chi\eta^2+5040e\lambda\chi\eta^2+840z(2h\lambda\chi$$

$$+\ 15\eta^2+63d\lambda\chi(e\lambda\chi+4\eta^2)]+40def\lambda^3\chi^3\eta^2[882q\lambda\chi(h\lambda\chi$$

$$+\ 10\eta^2)+3108\eta^4+210z(3g\lambda\chi$$

$$+\ 4\eta^2)+420V(g\lambda\chi+6\eta^2)+507d\lambda\chi(e\lambda\chi$$

$$+\ 4\eta^2]+15de\lambda^2\chi^2[60z^2(fg\lambda^2\chi^2+8f\lambda\chi\eta^2$$

$$+\ 2\eta^4)+480vyz\eta^2+16tz^2+4656csz\lambda\chi\eta^2$$

$$+\ 3bcrz\lambda^2\chi^2+480Vyz\eta^2]+12d\lambda\chi\eta^2[V^2w+12tV^2$$

$$+\ 582suV\lambda\chi+4bcrV\lambda^2\chi^2+abcV\lambda^3\chi^3]+3t[12t^2$$

$$+\ 8bcr^2\lambda^2\chi^2+4656cs^2\lambda\chi\eta^2+16abc\lambda^3\chi^3\eta^2]$$

$$+\ 8c^2s\lambda^2\chi^2\eta^2[2342qs+441br\lambda\chi+288ab\lambda^2\chi^2]$$

$$+\ 12b^2c^2r^2\lambda^4\chi^4\}/\alpha^{19}-135480a^2b^2c^2t\lambda^8\chi^8/\alpha^{18}$$

$$(4^4 1)_4 = 2304abc\lambda^4\chi^4\eta\{5\lambda^3\chi^3[224defg\lambda\chi$$

$$+\ 1288def\eta^2+28def\lambda\chi-\alpha abc\lambda\chi]$$

$$+\ 40dez\lambda^2\chi^2[15y+2z+3c\lambda\chi]$$

$$+\ 60dV\lambda\chi[5w+8V\eta^2+2cu\lambda\chi]$$

$$+ 12t[5dV\lambda\chi+2t+2cs\lambda\chi]$$

$$+ 12cs\lambda\chi[deu\lambda\chi+2s\eta^2+3cq\lambda\chi]$$

$$+ 4bcr\lambda^2\chi^2[5d\lambda\chi+2r+3c\lambda\chi]$$

$$+ abc\lambda^3\chi^3[5d\lambda\chi+16\eta^2+4c\lambda\chi]\}/\alpha^{16}$$

$$(4^42)_4 = 1152abc\lambda^4\chi^4\{140defg[3k\lambda\chi+84\eta^2+d\lambda\chi]$$

$$+ 40def\lambda^3\chi^3\eta^2[39g\lambda\chi+892\eta^2+56r+21c\lambda\chi]$$

$$+ 2dez\lambda^2\chi^2[225(fg\lambda^2\chi^2+8f\lambda\chi\eta^2+2\eta^2)+30t$$

$$+ 2400y\eta^2+40rz+36c\lambda\chi\eta^2+15bc\lambda^2\chi^2]$$

$$+ 40dV\lambda\chi\eta^2[90ey\lambda\chi+65w+12rV+18u+12t$$

$$+ 6bc\lambda^2\chi^2]+24cs\lambda\chi\eta^2[15de\lambda^2\chi^2+20du\lambda\chi$$

$$+ 4rs+9cq\lambda\chi+3bc\lambda^2\chi^2+6t]+bcr\lambda^2\chi^2[15de\lambda^2\chi^2$$

$$+ 160d\lambda\chi\eta^2+8r+12t+72c\lambda\chi\eta^2$$

$$+ 6bc\lambda^2\chi^2]+8abc\lambda^3\chi^3\eta^2[5d\lambda\chi+2r+3c\lambda\chi]$$

$$+ 24rt^2-11\alpha abcr\lambda^4\chi^4\}/\alpha^{17}$$

$$(4^43)_4 = 13824abc\lambda^4\chi^4\eta\{20cde\lambda^3\chi^3[84gh\lambda^2\chi^2$$

$$+ 210g\lambda\chi\eta^2+140\eta^4+10z(3g\lambda\chi+4\eta^2)$$

$$+ 10V(g\lambda\chi+6\eta^2)+4t+cs\lambda\chi]+30de\lambda^2\chi^2[$$

$$+ 14fg\lambda^2\chi^2(2h\lambda\chi+15\eta^2)+280f\lambda\chi\eta^2(3g\lambda\chi$$

$$+ 4\eta^2)+30z(fg\lambda^2\chi^2+8f\lambda\chi\eta^2+2\eta^4)$$

$$+ 240Vy\eta^2+4tz+24cV\lambda\chi\eta^2+bcV\lambda^2\chi^2$$

$$+ 6cqz\lambda X+7cfg\lambda^2X^2+bcz]+10du\lambda X[\lambda^3X^3(21efg+abc)$$

$$+ +4\lambda^2X^2(16ef\eta^2+bcr)$$

$$+ 12\lambda X(10eyz+csu)+12V(5w+t)]$$

$$+ 70defs\lambda^3X^3(g\lambda X+32\eta^2)+8s\lambda^2X^2(10dez^2$$

$$+ bcr^2)+96s\lambda X\eta^2(5dV^2+cs^2)+6cq\lambda X[30duV\lambda X$$

$$+ 3st+9bcqs\lambda X+3bcr\lambda^2X^2$$

$$+ abc\lambda^3X^3]+6bc\lambda^2X^2[bcr\lambda^2X^2+12cs\lambda X\eta^2$$

$$+ 40dV\lambda X\eta^2+2rt]+2abc\lambda^3X^3[5dV\lambda X$$

$$+ 2t+2cs\lambda X]+24st^2-5\alpha abc\lambda^4X^4\}/\alpha^{18}$$

$$(4^31^2)_4 = 1152abc\lambda^4X^4[\lambda^2X^2(5dez+bcr)+\lambda X\eta^2(45dV$$

$$+ 14cs)+2t(r+\eta^2)]/\alpha^{13}$$

$$(4^31^2)_4 = 1152abc\lambda^4X^4\{\lambda^4X^4(35defg-\alpha abc)$$

$$+ 40de\lambda^2X^2(35f\lambda X\eta^2+z^2)+8\eta^2[3\lambda^2X^2(10dez+bcr)$$

$$+ 3\lambda X(10duV+10dV+3cqs+2cs^2)+3st]$$

$$+ 2\lambda^2X^2(2bcr^2+5dejz+bcjr)+8j\lambda X\eta^2(10dV+3cs)$$

$$+ 4a\lambda X\eta^2[5dV\lambda X+2t+2cs\lambda X+4bc\lambda X\eta]+4t(3t+r)\}/\alpha^{15}$$

$$(4^33^2)_4 = 1728abc\lambda^4X^4\{20de\lambda^2X^2[14fg(2h\lambda X+15\eta^2)$$

$$+ 280f\lambda X\eta^2(3g\lambda X+4\eta^2)+3f\lambda X(64\eta^3+21g\lambda X$$

$$+ 112q\eta^2+10gz\lambda X+80z\eta^2)+60z\eta^4$$

$$+ 240Vy\eta^2+14f\lambda X\eta^2+12pz\eta^2+4tz$$

$$+\ 24cs\lambda\chi\eta^2+bcr\lambda^2\chi^2]+60d\lambda\chi\eta[120eyz\lambda\chi$$

$$+\ 60Vw+12tu+12csu\lambda\chi+4bc\lambda^2\chi^2+abc\lambda^3\chi^3$$

$$+\ 24qV^2\eta]+3q\lambda^4\chi^4(70defg-3\alpha abc)$$

$$+\ 24cq\lambda\chi(12s^2\eta^2+br^2)+8p\eta^2(30duV\lambda\chi$$

$$+\ 3st+9cqs\lambda\chi+3bcr\lambda^2\chi^2+abc\lambda^3\chi^3)$$

$$+\ 12bn\lambda\chi[\lambda^2\chi^2(5dez+bcr)+4\lambda\chi\eta^2(10dV$$

$$+\ 3cs)+2rt]+12ab\lambda^2\chi^2\eta^2(5dV\lambda\chi$$

$$+\ 4cq\lambda\chi+2cs\lambda\chi+2t)+72qt^2\}/\alpha^{17}$$

$$(4^3 12)_4 = 2304abc\lambda^4\chi^4\eta[\lambda^3\chi^3(35def+abc)+\lambda^2\chi^2(2acs$$

$$+\ 4bcr+35dez+5adV)+\lambda\chi(30duV$$

$$+\ 9cqs+40dV\eta^2+12cs\eta^2+2at)+3st]/\alpha^{14}$$

$$(4^3 13)_4 = 1152abc\lambda^4\chi^4\{35defg\lambda^4\chi^4+14\lambda^3\chi^3\eta^2(95def$$

$$+\ abc)+5dez\lambda^2\chi^2(8z+36\eta^2+3b\lambda\chi)$$

$$+\ 60du\lambda\chi\eta^2(4u+3V)+6cs\lambda\chi\eta^2(8s+9q)$$

$$+\ 4c\lambda^2\chi^2\eta^2(b+54r)+3b\lambda\chi(40dV\lambda\chi\eta^2$$

$$+\ 2rt+12s\lambda\chi\eta^2+bcr\lambda^2\chi^2)+12t^2+18st\eta^2+14abc\lambda^3\chi^3\eta^2\ /\alpha^{15}$$

$$(4^3 23)_4 = 1152abc\lambda^4\chi^4\eta\{10d\lambda\chi[2ef\lambda^2\chi^2(16g\lambda\chi$$

$$+\ 21k+378\eta^2)+12ez\lambda\chi(2z+3k+10y)$$

$$+\ 144V^2\eta^2+36kuV+60Vw+12tV+12csu\lambda\chi$$

$$+\ 4bcr\lambda^2\chi^2+abc\lambda^3\chi^3]+36t(2t+ks)$$

$$+ 36cs\lambda\chi(8s\eta^2+3kq)+12b\lambda\chi[\lambda^2\chi^2(bcr$$

$$+ 5dez)+\lambda\chi(40dV\eta^2+12cs\eta^2+2cr^2+3ckr)$$

$$+ 2rt]+6ab\lambda^2\chi^2[\lambda\chi(5dV+2cs+8c\eta^2$$

$$+ 2ck)+2t]\}/\alpha^{16}$$

$$(4^21^3)_4 = 96abc\lambda^4\chi^4\eta(2a\lambda\chi+3r+6s+\eta^2)/\alpha^{10}$$

$$(4^22^3)_4 = 384abc\lambda^4\chi^4\{\lambda^2\chi^2(15dez+3bcr-2\alpha ar)+2\lambda\chi\eta^2(a$$

$$+ 6b+90dV)+3t(2b\lambda\chi+4d\lambda\chi+19\eta^2)+6s\eta^2(a\lambda\chi$$

$$+ 4b\lambda\chi+12c\lambda\chi+8\eta^2)+2a\lambda\chi\eta^2(a\lambda\chi+5b\lambda\chi+\eta^2)+rj^2\}/\alpha^{13}$$

$$(4^23^3)_4 = 576abc\lambda^4\chi^4\eta\{1890defg\lambda^4\chi^4+1260def\lambda^3\chi^3(d\lambda\chi$$

$$+ 2q\lambda\chi+27\eta^2)+180dV\lambda\chi[5e\lambda\chi(f\lambda\chi+4\eta^2)$$

$$+ 54\eta^2(2c\lambda\chi+\eta^2)+6qu+3p\eta^2+bk\lambda\chi]$$

$$+ 90dez\lambda^2\chi^2(35f\lambda\chi+98\eta^2+12d\lambda\chi+4c\lambda\chi)$$

$$+ 6\lambda^2\chi^2(150det+20cdes+270s\eta^2+9abs\eta^2)$$

$$+ 108\lambda\chi(15dt\eta^2+3cq^2s+k^2s)+216qst$$

$$+ 54pqt+36p^2s\eta^2+6\alpha abs\lambda^3\chi^3$$

$$+ 9bk\lambda\chi[3\lambda^2\chi^2(ab+8cd)+4\lambda\chi(6c\eta^2+bk)$$

$$+ 2kp+12t]+18ab\lambda^2\chi^2(2cd\lambda^2\chi^2+10c\lambda\chi\eta^2$$

$$+ \eta^4+t)\}/\alpha^{16}-288a^2b^2c\lambda^7\chi^7\eta(ep+2s)\}/\alpha^{16}$$

$$(4^21^22)_4 = 192abc\lambda^4\chi^4[9de\lambda^2\chi^2+\lambda\chi(99d\eta^2+2br+3a\eta^2)$$

$$+ 24\eta^4+2r\eta^2]/\alpha^{11}$$

$$(4^2 1^2 3)_4 = 288abc\lambda^4\chi^4\eta[2ab\lambda^2\chi^2+5\lambda\chi(br+4dV)$$

$$+\ 14t+12ks+6s\eta^2]/\alpha^{12}$$

$$(4^2 1 2^2)_4 = 96abc\lambda^4\chi^4\eta[4\lambda\chi(2ab\lambda\chi+ar+5br+6cs$$

$$=\ 15dV+2a\eta^2)+12s(j+6\eta^2)+4r\eta^2]/\alpha^{12}$$

$$(4^2 2^2 3)_4 = 576abc\lambda^4\chi^4\eta[20de\lambda^2\chi^2(7f\lambda\chi+8z)$$

$$+\ 20dV\lambda\chi(6e\lambda\chi+37\eta^2+2b\lambda\chi)$$

$$+\ 4\lambda\chi(at+27dt+4t+9cds\lambda\chi+3as\eta^2$$

$$+\ 54cs\eta^2)+12(8t\eta^2+2s\eta^4+jk)$$

$$+\ 2br\lambda\chi(12c\lambda\chi+4b\lambda\chi+a\lambda\chi+8\eta^2)$$

$$+\ 2ab\lambda^2\chi^2(2c\lambda\chi+b\lambda\chi+7\eta^2)\}/\alpha^{14}$$

$$(4^2 1 3^2)_4 = 144abc\lambda^4\chi^4\eta[40de\lambda^2\chi^2(14f\lambda\chi+15z)$$

$$+\ 120dV\lambda\chi(2e\lambda\chi+21\eta^2+3d\lambda\chi)+12t(27d\lambda\chi$$

$$+\ 26\eta^2+2c\lambda\chi)+72qs+kp\eta^2+12r\lambda\chi(ab\lambda\chi$$

$$+\ 8bc\lambda\chi+3b\eta^2)+4ab\lambda^2\chi^2(6c\lambda\chi+9\eta^2)]/\alpha^{14}$$

$$(4^2 2 3^2)_4 = 288abc\lambda^4\chi^4\{140def\lambda^3\chi^3(g\lambda\chi+42\eta^2)$$

$$+\ 60dez\lambda^2\chi^2(4f\lambda\chi+40\eta^2+3q)$$

$$+\ 120dV\lambda\chi\eta^2(10e\lambda\chi+15u+12q+60\lambda\chi+\eta^2)$$

$$+\ 12\lambda^2\chi^2(5det+30cds\eta^2+3abs\eta^2)$$

$$+\ 36\lambda\chi(20dt\eta^2+bkt+12cqs\eta^2+4bks\eta^2)$$

$$+ 72(qrt+pt\eta^2+2kps\eta^2)$$

$$+ 12br\lambda\chi(3cq\lambda\chi+2p\eta^2+jk+\lambda\chi\eta^2)$$

$$+ 12ab\lambda^2\chi^2\eta^2(p+2k+a\lambda\chi)\}/\alpha^{15}$$

$$(4^2123)_4 = 576abc\lambda^4\chi^4\{\lambda^2\chi^2(abr+5ab\eta^2+10dez)$$

$$+ \lambda\chi(110dV\eta^2+2at+9dt+6as\eta^3$$

$$+ 48cs\eta^2+3br+4br\eta^2)+30t\eta^2+18s\eta^4\}/\alpha^{13}$$

$$(41^4)_4 = 24abc\lambda^7\chi^7/\alpha^7$$

$$(42^4)_4 = 48abc\lambda^4\chi^4[\lambda^2\chi^2(15de+6bc-2\alpha a)$$

$$+ 4\lambda\chi\eta^2(60d+6b+a)+4r(4d\lambda\chi+19\eta^2$$

$$+ 2b\lambda\chi)+12\eta^2(12c\lambda\chi+8\eta^2+4b\lambda\chi+a\lambda\chi\eta)+2j^2]/\alpha^{11}$$

$$(43^4)_4 = 432abc\lambda^4\chi^4\{105def\lambda^3\chi^3(d\lambda\chi+2q\lambda\chi+27\eta^2)$$

$$+ 180de\lambda^2\chi^2\eta^2[\lambda\chi(35f+12d+4c)+98\eta^2]$$

$$+ 60du\lambda\chi[5e\lambda\chi(f\lambda\chi+4\eta^2)+54\eta^2(2e\lambda\chi+\eta^2)$$

$$+ 6qu+3kp\eta^2]+72s\eta^2[\lambda^2\chi^2(ab+50de)$$

$$+ 6\lambda\chi(bk+15d\eta^2)+3q(4s+p)]$$

$$+ 3q[10cd\lambda^2\chi^2(2e\lambda\chi+27\eta^2)+9\lambda\chi(ab\lambda\chi\eta$$

$$+ 6cq^2+2bk)+6p^2\eta^2]+18b\lambda\chi\eta^2[\lambda^2\chi^2(ab$$

$$+ 10cd)+4\lambda\chi(3cq+bk)+2kp]$$

$$+ 6ab\lambda^2\chi^2(2cq\lambda\chi+p\eta^2+bk\lambda\chi)\}/\alpha^{15}$$

$$(41^32)_4 = 216abc\lambda^4\chi^4\eta/\alpha^8$$

$$(41^3 3)_4 = 24abc\lambda^4\chi^4[2\lambda\chi+9q\eta+19\eta^2]/\alpha^9$$

$$(412^3)_4 = 96abc\lambda^4\chi^4\eta[\lambda\chi(102a+101)+46\eta^2]/\alpha^{10}$$

$$\begin{aligned}(42^3 3)_4 = {} & 96abc\lambda^4\chi^4\eta[2\lambda^2\chi^2(45de+9bc+a^2+4ab-3\alpha a) \\ & + \lambda\chi(180du+24bs+48ds+9aq+36bq \\ & + 108cq+72b\eta^2+12a\eta^2+2a\eta^2)+228s\eta^2 \\ & + 72q\eta^2+6j^2]/\alpha^{12}\end{aligned}$$

$$\begin{aligned}(413^3)_4 = {} & 144ace\lambda^4\chi^4[15d\lambda\chi(5ef\lambda^2\chi^2+168e\lambda\chi\eta^2 \\ & + 18c\lambda\chi\eta^2+129\eta^4+6qu+bk\lambda\chi) \\ & + 12\eta^2(12qs+3pq+6bk\lambda\chi+ab\lambda^2\chi^2) \\ & + 2\lambda^3\chi^3(10cde-\alpha ab)+9\lambda^2\chi^2\eta^2(30cd+ab) \\ & + 18\lambda\chi(3cq^2+bk^2)+6p^2\eta^2]/\alpha^{13}\end{aligned}$$

$$\begin{aligned}(423^3)_4 = {} & 432abc\lambda^4\chi^4\eta\{15de\lambda^2\chi^2[\lambda\chi(35f+12d+4c) \\ & + 98\eta^2]+40d\lambda\chi[5ef\lambda^2\chi^2+2\lambda\chi\eta^2(64e+9c) \\ & + 57\eta^4+6qu+bk\lambda\chi]+4r[\lambda^2\chi^2(ab+50de) \\ & + 6\lambda\chi(15d\eta^2+bk)+3q(4s+p) \\ & + 2[9\lambda^2\chi^2\eta^2(ab+30cd)+18\lambda\chi(3cq^2+bk^2) \\ & + 6p^2\eta^2]+3b\lambda\chi[\lambda^2\chi^2(ab+10cd-\alpha ab) \\ & + 4\lambda\chi(3cq+bk)+2kp]\}/\alpha^{14}\end{aligned}$$

$$(41^2 2^2)_4 = 24abc\lambda^4\chi^4[4\lambda\chi(b+2d)+58\eta^2]/\alpha^9$$

$$(41^2 3^2)_4 = 72abc\lambda^4\chi^4[5d\lambda^2\chi^2(3d+2e)+\lambda\chi(8c\eta^2$$
$$+ 213d\eta^2+12cq-3\alpha q)+2\eta^2(52\eta^2+3p)]/\alpha^{11}$$

$$(42^2 3^2)_4 = 72abc\lambda^4\chi^4\{5de\lambda^2\chi^2(7f\lambda\chi+192\eta^2)$$
$$+ 20du\lambda\chi(2b\lambda\chi+6e\lambda\chi+37\eta^2)$$
$$+ 16s\eta^2[\lambda\chi(a+27d+4)+24\eta^2]$$
$$+ 3q[18cd\lambda^2\chi^2+6\lambda\chi\eta^2(a+18c)+12\eta^2$$
$$+ 6jk]+12\lambda\chi\eta^2[\lambda\chi(a+4b+12c)+8\eta^2]$$
$$+ 2ab\lambda^2\chi^2[\lambda\chi(2c+6-\alpha)+7\eta^2]\}/\alpha^{13}$$

$$(41^2 23)_4 = 72abc\lambda^4\chi^4\eta[\lambda\chi(2a+8b+45d)+58\eta^2]/\alpha^{10}$$

$$(412^2 3)_4 = 48abc\lambda^4\chi^4[5d\lambda\chi(2b\lambda\chi+6e\lambda\chi+37\eta^2)$$
$$+ 2\lambda\chi\eta^2(16+7a+54c+108d)$$
$$+ 6(3cd\lambda^2\chi^2+jk+34\eta^4)]/\alpha^{11}$$

$$(4123^2)_4 = 144abc\lambda^4\chi^4\eta[75de\lambda^2\chi^2+20d\lambda\chi(2e\lambda\chi$$
$$+ 21\eta^2+3d\lambda\chi)+2\lambda\chi(27dr+2cr+18cq$$
$$+ 18bk)+2\eta^2(26r+9p)+3\lambda\chi(ab\lambda\chi$$
$$+ 8bc\lambda\chi+36\eta^2)]/\alpha^{12}$$

APPENDIX $B_{(2)}$

* Expected Q-Products of Order 6, 7 and 8

(Note, see Appendix C for the evaluation of necessary expected q-products. All expected Q-products not shown are 0).

Products of Order 6

$(r^6)_3 = 15[r^2]^3$, $(r^4s^2)_3 = 3[r^2]^2[s^2]$, $(r^2s^2t^2)_3 = [r^2][s^2][t^2]$

- -

$(r^6)_4 = 10[r^3]^2+15[r^2][r^4]-45[r^2]^3$

$(r^5s)_4 = 10[r^3][r^2s]+10[r^3s][r^2]$

$(r^4st)_4 = 6[r^2s][r^2t]+4[r^3][rst]+6[r^2][r^2st]$

$(r^4s^2)_4 = 6[r^2s]^2+4[r^3][rs^2]+[r^4][s^2]+6[r^2][r^2s^2]-9[r^2]\ [s^2]$

$(r^3stu)_4 = 3[rtu][r^2s]+3[rsu][r^2t]+3[rst][r^2u]+[r^3][stu]$

$+ 3[r^2][rstu]$

$(r^3s^3)_4 = 9[rs^2][r^2s]+[r^3][s^3]+3[s^2][r^3s]+3[r^2][rs^3]$

$(r^2s^2t^2)_4 = 4[rst]^2+2[r^2s][st^2]+2[rt^2][rs^2]+2[r^2t][s^2t]$

$+ [r^2t^2][s^2]+[r^2s^2][t^2]+[s^2t^2][r^2]-3[r^2][s^2][t^2]$

* This Appendix is taken from Shenton and Myers [11]

$(r \neq s \neq t)$.

$$(r^2s^2tu)_4 = 4[rst][rsu]+2[r^2s][stu]+2[rs^2][rtu]$$

$$+ [r^2u][s^2t]+[r^2t][s^2u]+[r^2tu][s^2]+[r^2][s^2tu]$$

$$(r^3s^2t)_4 = 3[rs^2][r^2t]+6[r^2s][rst]+[r^3][s^2t]+[s^2][r^3t]$$

$$+ 3[r^2][rs^2t]$$

<u>Products of Order 7</u>

$$(r^7)_4 = 105[r^2]^2[r^3]$$

$$(r^6s)_4 = 45[r^2]^2[r^2s]$$

$$(r^5st)_4 = 15[r^2]^2[rst]$$

$$(r^5s^2)_4 = 10[r^2][s^2][r^3]+15[r^2]^2[rs^2]$$

$$(r^4stu)_4 = 3[r^2]^2[stu]$$

$$(r^4s^2t)_4 = 6[r^2][s^2][r^2t]+3[r^2]^2[s^2t]$$

$$(r^4s^3)_4 = 18[r^2][s^2][r^2s]+3[4^2]^2[s^3]$$

$$(r^3s^2tu)_4 = 3[r^2][s^2][rtu]$$

$$(r^3s^3t)_4 = 9[r^2][s^2][rst]$$

$$(r^3s^2t^2)_4 = [r^3][s^2][t^2]+3[r^2][s^2][rt^2]+3[r^2][t^2][rs^2]$$

$$(r^2s^2t^2u)_4 = [r^2u][s^2][t^2]+[t^2u][r^2][s^2]+[s^2u][t^2][r^2]$$

Products of Order 8

$$(r^8)_4 = 105[r^2]^4$$

$$(r^4s^2t^2)_4 = 3[r^2]^2[s^2][t^2]$$

$$(r^2s^2t^2u^2)_4 = [r^2][s^2][t^2][u^2]$$

$$(r^6s^2)_4 = 15[r^2]^3[s^2]$$

$$(r^4s^4)_4 = 9[r^2]^2[s^2]^2$$

APPENDIX C

Supplement to Appendix $B_{(2)}$

(Note: See Appendix A for notation)

$[1^2] = \lambda\chi/\alpha$

$[2^2] = 2a\lambda^2\chi^2/\alpha^3$

$[3^2] = 6ab\lambda^3\chi^3/\alpha^5$

$[4^2] = 24abc\lambda^4\chi^4/\alpha^7$

- -

$[1^3] = \lambda\chi\eta/\alpha^2$

$[2^3] = 4aj\lambda^2\chi^2/\alpha^5$

$[3^3] = 36abp\lambda^3\chi^3\eta/\alpha^8$

$[4^3] = 576abct\lambda^4\chi^4/\alpha^{11}$

$[1^22] = 2a\lambda^2\chi^2/\alpha^3$

$[12^2] = 4a\lambda^2\chi^2\eta/\alpha^4$

$[1^23] = 0$

$[13^2] = 18ab\lambda^3\chi^3\eta/\alpha^6$

$[1^24] = 0$

$[14^2] = 96abc\lambda^4\chi^4\eta/\alpha^8$

$[2^23] = 24ab\lambda^3\chi^3\eta/\alpha^6$

$[23^2] = 36abk\lambda^3\chi^3/\alpha^7$

$[2^24] = 24abc\lambda^4\chi^4/\alpha^7$

$[24^2] = 96abcr\lambda^4\chi^4/\alpha^9$

$[3^24] = 216abcq\lambda^4\chi^4/\alpha^9$

$[34^2] = 576abcs\lambda^4\chi^4\eta/\alpha^{10}$

$[123] = 6ab\lambda^4\chi^4\eta/\alpha^7$

$[124] = 0$

$[134] = 24abc\lambda^4\chi^4/\alpha^7$

$[234] = 144abc\lambda^4\chi^4\eta/\alpha^8$

- -

$$[1^4] = \lambda\chi(2a\lambda\chi+\alpha\lambda\chi+\eta^2)/\alpha^3$$

$$[2^4] = 4a\lambda^3\chi^3(\alpha a\lambda\chi+4a\eta^2+6bc\lambda\chi+24b\eta^2+2j^2)/\alpha^7$$

$$[3^4] = 36ab\lambda^3\chi^3[2\lambda^3\chi^3(10cde+3\alpha ab)+9\lambda^2\chi^2\eta^2(ab+30cd)$$
$$+ 18\lambda\chi(3cq^2+bk^2)+6p^2\eta^2]/\alpha^{11}$$

$$[4^4] = 576abc\lambda^4\chi^4[\lambda^4\chi^4(70defg+\alpha ab)+16\lambda^3\chi^3\eta^2(abc$$
$$+ 140def)+8\lambda^2\chi^2(10dez^2+bcr^2)+96\eta^2(5dV^2+cs^2)$$
$$+ 24t^2]/\alpha^{15}$$

$$[1^32] = 6a\lambda^2\chi^2\eta/\alpha^4$$

$$[1^33] = 6ab\lambda^3\chi^3/\alpha^5$$

$$[1^34] = 0$$

$$[2^31] = 8a\lambda^2\chi^2\eta(6\alpha\lambda\chi+11\lambda\chi+\eta^2)/\alpha^6$$

$$[2^33] = 24ab\lambda^3\chi^3\eta(a\lambda\chi+4b\lambda\chi+12c\lambda\chi+8\eta^2)/\alpha^8$$

$$[2^34] = 48abc\lambda^4\chi^4(2b\lambda\chi+4d\lambda\chi+19\eta^2)/\alpha^9$$

$$[3^31] = 108ab\lambda^3\chi^3(2cq\lambda\chi+6k\lambda\chi+p\eta^2)/\alpha^9$$

$$[3^32] = 108ab\lambda^3\chi^3\eta[\lambda^2\chi^2(a^2+10cd)+4\lambda\chi(bk+3cq)+2kp]/\alpha^{10}$$

$$[3^34] = 432abc\lambda^4\chi^4\eta[\lambda^2\chi^2(ab+20de)+6\lambda\chi(5du+bk)$$
$$+ 3q(p+4s)]/\alpha^{12}$$

$$[4^31] = 1152abc\lambda^4\chi^4\eta(2cs\lambda\chi+5dV+2t)/\alpha^{12}$$

$$[4^32] = 1152abc\lambda^4\chi^4[\lambda^2\chi^2(bcr+50dez)+4\lambda\chi\eta^2(10dV+3cs)$$
$$+ 2rt]/\alpha^{13}$$

$$[4^3 3] = 2304abc\lambda^4\chi^4\eta[\lambda^3\chi^3(abc+35def)+3\lambda^2\chi^2(bcr$$
$$+ 10dez)+3\lambda\chi(10duV+3cqs)+6st]/\alpha^{14}$$

$$[1^2 2^2] = 2\lambda\chi(\alpha a\lambda^2\chi^2+2a\lambda\chi\eta^2+2j^2)/\alpha^5$$

$$[1^2 3^2] = 18ab\lambda^3\chi^3(2c\lambda\chi+3\eta^2)/\alpha^7$$

$$[1^2 4^2] = 24abc\lambda^4\chi^4(\alpha\lambda\chi+8q)/\alpha^9$$

$$[2^2 3^2] = 12ab\lambda^3\chi^3(\alpha a\lambda^2\chi^2+18cq\lambda\chi+6a\lambda\chi\eta^2+12p\eta^2+6jk)/\alpha^9$$

$$[2^2 4^2] = 48abc\lambda^4\chi^4(\alpha a\lambda^2\chi^2+8a\lambda\chi\eta^2+12t+48s\eta^2+4jr)/\alpha^{11}$$

$$[3^2 4^2] = 144abc\lambda^4\chi^4[\alpha ab\lambda^3\chi^3+4\lambda^2\chi^2(10dez+3ab\eta)$$
$$+ 12\lambda\chi(30dV\eta+bkr)+36qt+24ps\eta^2]/\alpha^{13}$$

$$[1^2 23] = 30ab\lambda^3\chi^3\eta/\alpha^6$$

$$[1^2 24] = 24abc\lambda^4\chi^4/\alpha^7$$

$$[1^2 34] = 168abc\lambda^4\chi^4\eta/\alpha^8$$

$$[2^2 13] = 12ab\lambda^3\chi^3(a\lambda\chi+3c\lambda\chi+7\eta^2)/\alpha^7$$

$$[2^2 14] = 192abc\lambda^4\chi^4\eta/\alpha^8$$

$$[2^2 34] = 96abc\lambda^4\chi^4\eta[\lambda\chi(a+6b+27d)+27\eta^2]/\alpha^{10}$$

$$[3^2 12] = 36ab\lambda^3\chi^3\eta(a\lambda\chi+8c\lambda\chi+3\eta^2)/\alpha^8$$

$$[3^2 14] = 144abc\lambda^4\chi^4\eta(3b\lambda\chi+15d\lambda\chi+10\eta^2)/\alpha^{10}$$

$$[3^2 24] = 144abc\lambda^4\chi^4(ab\lambda^2\chi^2+10du\lambda\chi+12b\lambda\chi\eta^2$$
$$+ 24s\eta^2+9kq)/\alpha^{11}$$

$$[4^2 12] = 192abc\lambda^4\chi^4\eta(a\lambda\chi+11d\lambda\chi+6\eta^2)/\alpha^{10}$$

$$[4^2 13] = 288abc\lambda^4\chi^4[2d\lambda^2\chi^2(b+3e)+3\lambda\chi\eta^2(b+14d) + 8\eta^2]/\alpha^{11}$$

$$[4^2 23] = 192abc\lambda^4\chi^4\eta(3ab\lambda^2\chi^2+30dV\lambda\chi+6br\lambda\chi + 18t+18sk)/\alpha^{12}$$

$$[4123] = 24abc\lambda^4\chi^4(2a\lambda\chi+9d\lambda\chi+30\eta^2)/\alpha^9$$

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REFERENCES

[1] Anscombe, F. J. (1950) Sampling theory of the negative binomial and logarithmic series distribution. Biometrika, 37, 358 - 362.

[2] Bartko, T. J. (1961) The negative binomial distribution. A review of properties and applications. The Virginia Journal of Science, 12(1), 18 - 37.

[3] Bowman, K. (1963) Higher moments of maximum likelihood estimators. Thesis for Ph.D. degree, Virginia Polytechnic Institute.

[4] Cramer, H. (1946) Mathematical Methods of Statistics. Princeton: Princeton University Press.

[5] Evans, D. A. (1953) Experimental evidence concerning contagious distributions in ecology. Biometrika, 40. 186 - 211.

[6] Feller, W. (1943) On a general class of contagious distributions. Ann. Math. Stat. 14, 389 - 400

[7] Fisher, R. A. (1941) The negative binomial distribution. Annals of Eugenics, 11, 182 - 187

[8] Kendall, M. G. and Stuart, A. (1958) The Advanced Theory of Statistics. Charles Griffin. London.

[9] Myers, R. H. (1963) Orthogonal statistics and some sampling properties of moment estimators for the Negative Binomial distribution. Thesis for Ph.D. degree. Virginia Polytechnic Institute.

[10] Shenton, L. R. (1958) Moment estimators and maximum likelihood. Biometrika, 45, 411

[11] Shenton, L. R. and Myers, R. H. Orthogonal Statistics. To be published in "Proceedings of the International Symposium On the Classical and Contagious Discrete Distribution.

[12] Shenton, L. R. and Wallington, P. A. (1962) The bias of moment estimators with an application to the negative binomial distribution. Biometrika, 49, 193 - 204.

[13] Szego, G. (1939) Orthogonal Polynomials. American Mathematical Society, New York.

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ABSTRACT

This thesis is an extension of the development of orthogonal statistics which can be used to investigate sampling properties of moment estimators. This work is particularized for estimators of parameters of the negative binomial distribution.

If we call $q_r(x)$ $(r=0, 1, 2, \ldots)$ the set of orthogonal polynomials associated with the negative binomial distribution, then the rth orthogonal statistics is defined as

$$Q_r = \sum_{j=1}^{n} q_r(x_j)/n \quad .$$

The thesis contains tables of the expansions of expected powers and products of the Q's in terms of the population parameters for the parametric form due to Anscombe [1], through terms in n^{-4}. These products and powers involve Q_1, Q_2, Q_3 and Q_4. Since the Q's are expressed as linear functions of the sample moments, one can expand a moment estimator t (or power of t) as a linear function of the Q's, and then through the use of the table of expected Q-products, evaluate such properties as bias, variance, and higher order moments of t, as functions of the population parameters. A brief example of the use of these tables is given.