STABILITY THEORY OF DIFFERENTIAL EQUATIONS

by

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TABLE OF CONTENTS

I
INTRODUCTION

The problem of determining the number, N, of zeros of a polynomial, P(z), which lie in a given region has been of interest to mathematicians for a considerable period of time. The problem was solved by A. L. Cauchy (1789-1857) by means of his formula:

$$N = \frac{1}{2\pi i}\int_C \frac{P'(z)}{P(z)}dz$$

where C is the rectifiable boundary curve of the region and where it must be assumed $P(z) \neq 0$ on C. Cauchy also developed the notion of the index of a rational function which may be used in determining the number of zeros in any given half plane.

In most of the applied sciences the problem of stability of a system is of utmost importance. Stability is connected with the number of zeros in a half plane in the following manner. Let us consider a system, mechanical, electrical or otherwise and let us describe this system by a set of ordinary linear differential equations. This is possible if certain assumptions are made about the components of the system. This set of equations may now be reduced to a polynomial which is called the characteristic equation of the system. It turns out that the system will be stable if all the roots of the characteristic equation have negative real parts.

Working in the field of dynamics, E. J. Routh in 1877 developed a method based on Cauchy's work for determining whether the characteristic equation of a system has roots with negative real parts. Hurwitz in 1895 independently derived the same conditions in a more elegant form. In 1945 Wall, familiar with the work of both men, deducted the results of Hurwitz by an entirely different method, namely, by means of a continued fraction expansion. Most British texts on advanced dynamics refer to Routh, but never to Hurwitz. The German authors refer to Hurwitz, and not to Routh. In this country the Routh-Hurwitz criterion has received a good deal of attention recently from mathematicians and engineers.

The purpose of this thesis is to present the mathematical theory needed to develop the stability criteria which are in use today. The first section of this thesis discusses the refinements made on the Routh-Hurwitz criteria since its original publication. Also included in this section is the development of the criteria by residue theory. The following section covers the work of H. Nyquist and presents the necessary theorems needed in his graphical analysis of stability. A discussion in connection with the stability of closed cycle control systems is presented to illustrate the use of the Nyquist criterion.

The last section of the thesis discusses the method of phase plane analysis in determining the stability of non-linear equations. The material for this section is mostly from a course in non-linear mechanics given at Virginia Polytechnic Institute.

II
REVIEW OF LITERATURE

Routh, E.J. ADVANCED RIGID DYNAMICS

In this book Routh discusses necessary and sufficient conditions for a polynomial with real coefficients to have zeros with negative real parts. The final form of his main theorem is as follows: Let

$$f(z) = a_o z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n$$

where a_o is assumed to be positive.

Construct a sequence of "test functions" as follows: Let $A_o = a_o$, and derive A_1 from A_o, A_2 from A_1, and so on, by writing the lower elements for the upper elements in the columns of the following table.

a_o	a_1	a_2	a_3	etc.
a_1	$a_2 - \frac{a_o a_3}{a_1}$	a_3	$a_4 - \frac{a_o a_5}{a_1}$	etc.

A zero is to be written whenever the suffix is greater than the degree of the equation. Then necessary and sufficient conditions for $f(z)$ to have no roots with positive real parts is that all the terms in the test function be positive. As an example, consider the case when $n = 3$, then the test functions become:

$$A_o = a_o$$

$$A_1 = a_1$$

$$A_2 = a_2 - \frac{a_o a_3}{a_1}$$

$$A_3 = a_3 - \frac{a_1 a_4}{a_2 - \frac{a_o a_3}{a_1}} = \frac{a_1 a_2 a_3 - a_o a_3^2 - a_1^2 a_4}{a_1 a_2 - a_o a_3}$$

$$A_4 = a_4$$

Note that in obtaining A_3, we replaced a_2 by a_3, a_o by a_1, a_1 by $a_2 - \frac{a_o a_3}{a_1}$ and a_3 by $a_4 - \frac{a_o a_5}{a_1}$ but $a_5 = 0$ so that a_3 was replaced by a_4 only.

Wall, H.S. "Polynomials Whose Zeros Have Negative Real Parts"

In this paper Wall gives necessary and sufficient conditions for a polynomial, f(z), with real coefficients to have zeros with negative real parts by use of continued fraction expansion. Another important result from this paper is necessary and sufficient conditions for f(z) and f(-z) to have a zero in common.

Frank, E. "On The Zeros Of Polynomials With Complex Coefficients"

This paper extends the results of Wall to polynomials with complex coefficients. Frank proves essentially the same theorems as Wall although she gives alternate forms for necessary and sufficient conditions for a polynomial to have zeros with negative real parts.

Marden, M. THE GEOMETRY OF THE ZEROS

This book contains a complete discussion of polynomials having complex coefficients with negative real parts. Marden discusses these polynomials in regard to Sturm sequences, determinant sequences, and Cauchy's indices.

III
STABILITY CRITERIA BY THE METHOD OF CONTINUED FRACTION EXPANSION

Consider a system of second order linear ordinary differential equations given by

$$
\begin{aligned}
& a_{11}\ddot{Y}_1 + \cdots + a_{1n}\ddot{Y}_n + b_{11}\dot{Y}_1 + \cdots \\
& \qquad + b_{1n}\dot{Y}_n + c_{11}Y_1 + \cdots + c_{1n}Y_n = 0 \\
(3.1) \qquad & a_{21}\ddot{Y}_1 + \cdots + a_{2n}\ddot{Y}_n + b_{21}\dot{Y}_1 + \cdots \\
& \qquad + b_{2n}\dot{Y}_n + c_{21}Y_1 + \cdots + c_{2n}Y_n = 0 \\
& \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \\
& a_{n1}\ddot{Y}_1 + \cdots + a_{nn}\ddot{Y}_n + b_{n1}\dot{Y}_1 + \cdots \\
& \qquad + b_{nn}\dot{Y}_n + c_{n1}Y_1 + \cdots + c_{nn}Y_n = 0
\end{aligned}
$$

where the Y_i's are functions of time, t, satisfying certain initial conditions and the a_{ij}'s, b_{ij}'s and the c_{ij}'s are constants.

The system in (3.1) can be written more compactly as follows:

$$(3.2) \quad \sum_{j=1}^{n} a_{ij}\ddot{Y}_j(t) + \sum_{j=1}^{n} b_{ij}\dot{Y}_j(t) + \sum_{j=1}^{n} c_{ij}Y_j(t) = 0$$

or if

$$A = (a_{ij}), \quad B = (b_{ij}), \quad C = (c_{ij})$$

and

$$Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \\ \vdots \\ Y_n(t) \end{bmatrix}$$

then (3.1) can be written as the single matric equation

$$(3.3) \qquad A\ddot{Y}(t) + B\dot{Y}(t) + CY(t) = 0.$$

Letting $L\left[Y(t)\right]$ represent the Laplace transform of $Y(t)$, we obtain

$$L\left[Y(t)\right] = \begin{bmatrix} L\left[Y_1(t)\right] \\ L\left[Y_2(t)\right] \\ \vdots \\ L\left[Y_n(t)\right] \end{bmatrix} = \begin{bmatrix} y_1(p) \\ y_2(p) \\ \vdots \\ y_n(p) \end{bmatrix} = y(p)$$

Upon taking the Laplace transform of Equation (3.3), we obtain

$$(Ap^2 + Bp + C)y(p) = (Ap + B)Y(0) + A\dot{Y}(0)$$

where $Y(0)$ and $\dot{Y}(0)$ are the values of $Y(t)$ and $\dot{Y}(t)$ at $t = 0$, respectively.

Therefore,

$$y(p) = (Ap^2 + Bp + C)^{-1} \left[(Ap + B)Y(0) + A\dot{Y}(0)\right].$$

From matrix theory we know that

$$(Ap^2 + Bp + C)^{-1} = \frac{(A_{ij}p^2 + B_{ij}p + C_{ij})'}{|Ap^2 + Bp + C|}$$

where $(A_{ij}p^2 + B_{ij}p + C_{ij})'$ represents the adjoint of the matrix $(Ap^2 + Bp + C)$. Letting

$$(Ap + B)Y(0) + A\dot{Y}(0) = C_1p + C_2,$$

where C_1 and C_2 are matrices, we may write

$$y(p) = \frac{(A_{ij}p^2 + B_{ij}p + C_{ij})' \ (C_1p + C_2)}{|Ap^2 + Bp + C|}.$$

We note that the denominator is a determinant of order n, which when expanded yields a polynomial of degree 2n. By taking the inverse Laplace transform of both sides, we obtain the solution of $Y(t)$ in the form

$$Y_i(t) = \sum_{i=1}^{2n} d_i e^{p_i t}$$

where the d_i's are constants and the p_i's are the zeros of the polynomial obtained by expanding the determinant

$$\left| Ap^2 + Bp + C \right|.$$

Location of the zeros of this polynomial in the complex p plane determines whether the original system of equations is stable or unstable.

Definition: A solution is stable if and only if the $\lim_{t \to \infty} Y_i(t)$, $(i = 1,2,\cdots,n)$ exists and is zero.

If all the roots lie in the left half plane, then they all have negative real parts. Hence we see that the modulus of $Y_i(t)$ will be a decaying time function, and thus the system is stable. If one or more roots of the polynomial lie in the right half plane, then these roots will have positive real parts. Thus the modulus of $Y_i(t)$ will be an increasing time function and thus unstable. A root on the imaginary axis represents a borderline case between stability and instability. Unless the root is simple, instability results. Consider the case that $p = i\beta$ is a double root of $Y_i(t)$, then $Y_i(t)$ will have a factor of the form

$$A_o e^{i\beta t} + A_1 t e^{i\beta t}$$

whose absolute value increases with time denoting instability.

An interesting method of determining the number of zeros with positive and negative real parts was developed by Wall by the use of continued fraction expansions. Wall developed the theory in the case of a polynomial with real coefficients, and this theory was extended to the case of a polynomial with complex coefficients by Frank.

We now present the two main theorems from the papers of Frank and Wall. Theorem 3.1 of this paper is due to Frank and gives necessary and sufficient conditions for the existence of a continued fraction expansion. Theorem 3.2 is due to Wall and gives a method for determining the number of zeros in each half plane if the continued fraction expansion exists. Included after Theorem 3.1 is a discussion on Cauchy's indices and three lemmas which are needed in the development of Theorem 3.2.

THEOREM 3.1 Let

$$f_o(z) = \alpha_{oo}z^n + \alpha_{o1}z^{n-1} + \cdots + \alpha_{on}$$

and

$$f_1(z) = \alpha_{11}z^{n-1} + \alpha_{12}z^{n-2} + \cdots + \alpha_{1n}$$

be two polynomials of degree n, and n-1, respectively, with complex coefficients. Then the quotient

$$\frac{f_1(z)}{f_o(z)}$$

can be expressed in the form

$$(3.5) \qquad \frac{f_1(z)}{f_o(z)} = \cfrac{1}{c_1 z + s_1 + \cfrac{1}{c_2 z + s_2 + \cdots + \cfrac{1}{c_n z + s_n}}}$$

where the c_i's are constant, $c_i \neq 0$ if and only if $D_p \neq 0$, $p = 0,1,2,\cdots,n$ where $D_o = \alpha_{oo}$ and $D_1, D_2, \cdots, D_n$ are the first n principal minors of odd order (blocked off by lines) in the array:

$$\begin{array}{ccccccc}
\alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} & \cdots \\
\cline{1-1}
\alpha_{oo} & \alpha_{o1} & \alpha_{o2} & \alpha_{o3} & \alpha_{o4} & \alpha_{o5} & \cdots \\
0 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \cdots \\
\cline{1-3}
0 & \alpha_{oo} & \alpha_{o1} & \alpha_{o2} & \alpha_{o3} & \alpha_{o4} & \cdots \\
0 & 0 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \cdots \\
\cline{1-5}
0 & 0 & \alpha_{oo} & \alpha_{o1} & \alpha_{o2} & \alpha_{o3} & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

where $\alpha_{op} = \alpha_{1p} = 0$ if $p > n$.

Before proving the theorem, consider the following discussion. The problem of finding a continued fraction expansion is equivalent to finding $f_p(z)$ of degree $n - p$, $p = 2,3,\cdots,n-1$ which is connected with f_o and f_1 by the recurrence relations

$$
(3.6)\qquad \begin{aligned} f_{p-1} &= (c_p z + s_p)f_p + f_{p+1} \\ f_{n+1} &= 0,\ f_n = \alpha_{nn} \neq 0 \end{aligned}
$$

where the c_p's $\neq 0$ and α_{nn} is a particular constant determined by the coefficients of f_o and f_1. The expansion exists if and only if the euclidean algorithm for the highest common factor of two polynomials when applied to f_o and f_1 gives a system of type (3.6).

Examining the long division process in the algorithm, we see that the numbers which contribute to the final result are only those contained in Table 1. We see from Table 1 that the continued fraction expansion exists if and only if $\alpha_{oo},\alpha_{11},\alpha_{22},\cdots,\alpha_{nn}$ are different from 0. When it exists, we have

$$
c_p = \frac{\alpha_{p-1,p-1}}{\alpha_{pp}},\ s_p = \frac{\eta_{pp}}{\alpha_{pp}} \quad p = 1,2,\cdots,n.
$$

Now the proof follows:

Proof: Suppose first that the expansion (3.5) exists, so that the numbers α_{pp}, $(p = 0,1,2,\cdots,n)$ defined by the table are not zero. Consider the determinant D_p of order

$$
\begin{array}{llll}
\alpha_{oo} & \alpha_{o1} & \alpha_{o2} & \\
\alpha_{11} & \alpha_{12} & \alpha_{13} & \\
\eta_{11} = \dfrac{\alpha_{11}\alpha_{o1} - \alpha_{oo}\alpha_{12}}{\alpha_{11}}; & \eta_{12} = \dfrac{\alpha_{11}\alpha_{o2} - \alpha_{oo}\alpha_{13}}{\alpha_{11}}; & \eta_{13} = \dfrac{\alpha_{11}\alpha_{o3} - \alpha_{oo}\alpha_{14}}{\alpha_{11}} & \cdots \\
\alpha_{22} = \dfrac{\alpha_{11}\eta_{12} - \eta_{11}\alpha_{12}}{\alpha_{11}}; & \alpha_{23} = \dfrac{\alpha_{11}\eta_{13} - \eta_{11}\alpha_{13}}{\alpha_{11}}; & \alpha_{24} = \dfrac{\alpha_{11}\eta_{14} - \eta_{11}\alpha_{14}}{\alpha_{11}} & \cdots \\
\eta_{22} = \dfrac{\alpha_{22}\alpha_{12} - \alpha_{11}\alpha_{23}}{\alpha_{22}}; & \eta_{23} = \dfrac{\alpha_{22}\alpha_{13} - \alpha_{11}\alpha_{24}}{\alpha_{22}}; & \eta_{24} = \dfrac{\alpha_{22}\alpha_{14} - \alpha_{11}\alpha_{25}}{\alpha_{22}} & \cdots \\
\alpha_{33} = \dfrac{\alpha_{22}\eta_{23} - \eta_{22}\alpha_{23}}{\alpha_{22}}; & \alpha_{34} = \dfrac{\alpha_{22}\eta_{24} - \eta_{22}\alpha_{24}}{\alpha_{22}}; & \alpha_{35} = \dfrac{\alpha_{22}\eta_{25} - \eta_{22}\alpha_{25}}{\alpha_{22}} & \cdots \\
\cdots & \cdots & \cdots & \cdots
\end{array}
$$

Table 1.

Numbers obtained in division of $\dfrac{f_1(z)}{f_o(z)}$

$2p-1$, $2 \leq p \leq n$. If we subtract α_{oo}/α_{11} times the $(2k-1)^{th}$ row from the $2k^{th}$ row ($k = 1,2,\cdots,p-1$), we find that

$$(3.7) \qquad D_p = \alpha_{11} \begin{vmatrix} \eta_{11} & \eta_{12} & \eta_{13} & \cdots \\ \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots \\ 0 & \eta_{11} & \eta_{12} & \cdots \\ 0 & \alpha_{11} & \alpha_{12} & \cdots \\ 0 & 0 & \eta_{11} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

where the new determinant is of order $2p-2$. Upon subtracting η_{11}/α_{11} times the $2k^{th}$ row from the $(2k-1)^{th}$ row for $k = 1,2,\cdots,p-1$ and making use of our table, we obtain

$$(3.8) \qquad D_p = (-1)^{p-1}\alpha_{11}^2 D_{p-1}^{(1)}, \quad p = 2,3,\cdots,n$$

where $D_r^{(k)}$ denotes the determinant D_r with both the subscripts of all its elements increased by k. From (3.8) we then find

$$(3.9) \qquad D_p = (-1)^{p(p-1)/2}\alpha_{11}^2\alpha_{22}^2 \cdots \alpha_{p-1,p-1}^2\alpha_{pp}$$

$$p = 2,3,\cdots,n.$$

Since $\alpha_{pp} \neq 0$, $p = 0,1,2,\cdots,n$ it follows from (3.9) that $D_p \neq 0$.

Conversely suppose that $D_p \neq 0$, then $\alpha_{oo} \neq 0$, $\alpha_{11} \neq 0$ since by definition $D_o = \alpha_{oo}$, $D_1 = \alpha_{11}$. Since $\alpha_{11} \neq 0$, then (3.9) holds for $p = 2$, so that $D_2 = -\alpha_{11}^2\alpha_{22} \neq 0$ or $\alpha_{22} \neq 0$. (Note (3.9) was derived from η_{11}/α_{11}, that is, since $\alpha_{11} \neq 0$, η_{11} exists and thus α_{22} exists). This guarantees that (3.9) holds for $p = 3$ so that

$$D_3 = -\alpha_{11}^2\alpha_{22}^2\alpha_{33} \neq 0.$$

Continuing this argument, we finally arrive at $\alpha_{nn} \neq 0$, and thus the theorem is proved.

CAUCHY INDICES. Let

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n \tag{3.10}$$

be a polynomial with real coefficients having no zeros on the imaginary axis. Multiplying $f(z)$ by i^n and substituting $-iz$ for z, we obtain

$$i^n f(-iz) = U(z) + iV(z) \tag{3.11}$$

where

$$U(z) = z^n - a_2 z^{n-2} + a_4 z^{n-4} - \cdots \tag{3.12}$$

and

$$V(z) = a_1 z^{n-1} - a_3 z^{n-3} + a_5 z^{n-5} - \cdots . \tag{3.13}$$

This transformation rotates the plane by $\pi/2$ radians. Let us set $i^n f(-iz) = P(z)$ for convenience, then on the x axis

$$(3.14) \qquad \arg P(x) = \arc\tan \tau(x), \quad \tau(x) = \frac{V(x)}{U(x)}.$$

Let $x_1, x_2, \cdots, x_k$ denote the distinct zeros of $U(x)$ and let them be such that

$$x_1 < x_2 < \cdots < x_k.$$

Since $f(z)$ has no zeros on the imaginary axis, by hypothesis $P(z)$ will have no zeros on the real axis. Therefore, no zero of $U(x)$ will be a zero of $V(x)$. From the graph of the arc tan $\tau(x)$ we may infer that the change $\Delta_i \arg P(x)$ as z varies from x_i to x_{i+1} will according to Equation (3.14) have the value

$$\Delta_i \arg P(z) = -\pi \quad \text{if} \quad \tau(x_i + \varepsilon) > 0 \quad \text{and} \quad \tau(x_{i+1} - \varepsilon) < 0$$

$$\Delta_i \arg P(z) = +\pi \quad \text{if} \quad \tau(x_i + \varepsilon) < 0 \quad \text{and} \quad \tau(x_{i+1} - \varepsilon) > 0$$

$$\Delta_i \arg P(z) = 0 \quad \text{if} \quad \tau(x_i + \varepsilon)\tau(x_{i+1} - \varepsilon) > 0$$

where $\varepsilon > 0$ and arbitrarily small.

Hence for $i = 1, 2, \cdots, k-1$

$$(3.15) \qquad \Delta_i \arg P(z) = \frac{\pi}{2}\left[\operatorname{sg} \tau(x_{i+1} - \varepsilon) - \operatorname{sg} \tau(x_i + \varepsilon)\right].$$

We must also consider the changes $\Delta_o \arg P(z)$ and $\Delta_k \arg P(z)$ as $z = x$ varies from $-\infty$ to x_1 and from x_k to $+\infty$, respectively. Since $U(x)$ is of degree higher than $V(x)$, $\tau(x) \longrightarrow 0$ as $x \longrightarrow \pm\infty$. Since x_1 and x_k are the smallest and largest zeros of $\tau(x)$, the change in argument of $P(x)$, as $z = x$ varies from $-\infty$ to x_1 or from x_k to $+\infty$, will be $\pm \pi/2$; the sign being determined by the following relations:

(3.16a) $\qquad \Delta_o \arg P(z) = \pi/2 \text{ sg } \tau(x_1 - \varepsilon)$

(3.16b) $\qquad \Delta_k \arg P(z) = -\pi/2 \text{ sg } \tau(x_k + \varepsilon).$

Combining Equations (3.15), (3.16a) and (3.16b), we may compute the change $\Delta_L \arg P(x)$ as x varies from $-\infty$ to $+\infty$. This change is

$$\Delta_L \arg P(x) = \frac{\pi}{2}\left\{\sum_{i=1}^{k-1}\left[\text{sg } \tau(x_{i+1} - \varepsilon) - \text{sg } \tau(x_i + \varepsilon)\right] + \text{sg } \tau(x_1 - \varepsilon) - \text{sg } \tau(x_k + \varepsilon)\right\},$$

that is,

$$\Delta_L \arg P(z) = \pi \sum_{i=1}^{k}\left[\frac{\text{sg } \tau(x_i - \varepsilon) - \text{sg } \tau(x_i + \varepsilon)}{2}\right].$$

The quantity inside the brackets is defined to be the

Cauchy index of $\tau(x)$ at the point $x = x_i$. It may have the values of -1, 0, or $+1$ according as $\tau(x_i - \varepsilon) < 0$ and $\tau(x_i + \varepsilon) > 0$, or $\tau(x_i - \varepsilon)\tau(x_i + \varepsilon) > 0$ or $\tau(x_i - \varepsilon) > 0$ and $\tau(x_i + \varepsilon) < 0$.

LEMMA 3.1 Let $f(z)$ be a polynomial of degree n with real coefficients, such that $f(z)$ has no zeros on the imaginary axis. If $\Delta\theta$ represents the change in polar angle of $f(z)$ as the complex variable z goes from $+i\infty$ to $-i\infty$, then

$$\frac{\Delta\theta}{\pi} = P_R - P_L$$

where P_R and P_L represent the number of zeros of $f(z)$ to the right and to the left of the imaginary axis, respectively.

Proof: Expressing $f(z)$ in polar form, we obtain

$$f(z) = re^{i\theta},\ r > 0.$$

Let us assume $f(z)$ has only one zero located at z_1 in the right half plane. We can then write:

$$f(z) = (z - z_1) = re^{i\theta}.$$

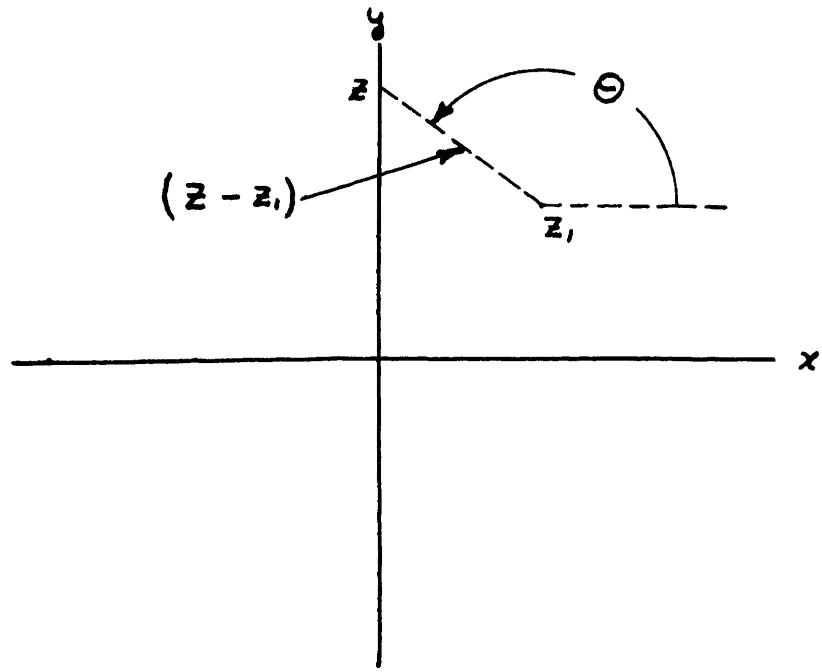


Fig. 1

Taking a point z on the imaginary axis f(z) can be regarded as the vector from z_1 to z. Referring to Fig. 1, we see that as z ranges from $+i\infty$ to $-i\infty$, the net increase in θ is $\Delta\theta = +\pi$, thus $\Delta\theta/\pi = +1$. If z were in the left hand plane, $\Delta\theta = -\pi$ and $\Delta\theta/\pi = -1$.

If f(z) has zeros of multiplicity $q_1, q_2, \cdots, q_k$, we can write it as follows:

$$f(z) = (z - z_1)^{q_1}(z - z_2)^{q_2}\cdots(z - z_k)^{q_k}$$

where $q_1 + q_2 + \cdots + q_k = n$.

Expressing each factor in polar form, we obtain

$$f(z) = r_1^{q_1} r_2^{q_2} r_3^{q_3} \cdots r_k^{q_k} e^{i(q_1\theta_1 + q_2\theta_2 + \cdots q_k\theta_k)}.$$

Then as z traces the path from $+i\infty$ to $-i\infty$, the increase in θ will be

$$\Delta\theta = q_1\Delta\theta_1 + q_2\Delta\theta_2 + \cdots + q_k\Delta\theta_k$$

where each $\Delta\theta_i$ will have a value of π or $-\pi$ depending upon whether the corresponding zero is in the right or left hand plane. Thus we can represent $\Delta\theta$ as:

$$\Delta\theta = \pi(P_R - P_L)$$

where P_R and P_L are defined in the hypothesis. Thus

$$\frac{\Delta\theta}{\pi} = P_R - P_L.$$

LEMMA 3.2 Let

$$P(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$

where P(z) is a polynomial having real coefficients, and let

$$Q(z) = a_1 z^{n-1} + a_3 z^{n-3} + a_5 z^{n-5} + \cdots$$

be the polynomial obtained from P(z) by dropping out the first, third, fifth, ..., terms. If the quotient Q(z)/P(z) has a continued fraction expansion of the form

$$\frac{Q(z)}{P(z)} = \cfrac{1}{c_1 z + 1 + \cfrac{1}{c_2 z + \cdots + \cfrac{1}{c_n z}}}$$

where $c_p \neq 0$, $(p = 1,2,\cdots,n)$ then Q(z)/P(z) is irreducible.

Proof: Let

$$b_o = \frac{1}{c_1}, \quad b_p = \frac{1}{c_p c_{p+1}}$$

so that the continued fraction expansion is of the form:

$$\frac{Q(z)}{P(z)} = \cfrac{b_o}{z + b_o + \cfrac{b_1}{z + \cfrac{b_2}{z + \cdots + \cfrac{b_{n-1}}{z}}}} .$$

The advantage to this is that the coefficients of the highest powers of z in the successive denominators listed are equal to unity.

$$P_1(z) = z + b_o$$

$$P_2(z) = z^2 + b_o z + b_1$$

$$P_3(z) = z^3 + b_o z^2 + z(b_1 + b_2) + b_o b_2$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$P_n(z) = P(z)$$

The corresponding numerators are:

$$Q_1(z) = b_o$$

$$Q_2(z) = b_o z$$

$$Q_3(z) = b_o z^2 + b_o b_2$$

$$\cdots\cdots\cdots\cdots$$

$$Q_n(z) = Q(z) .$$

The fractions Q_1/P_1, Q_2/P_2, Q_3/P_3, etc., are called the convergents of the continued fraction. By the properties of convergents we know:

$$Q_m(z)P_{m-1}(z) - Q_{m-1}(z)P_m(z) = (-1)^{m-1} b_o b_1 \cdots b_{m-1}$$

$$m = 1,2,\cdots,n, \ (Q_o = 0, \ P_o = 1).$$

If we put $m = n$ in this formula, we see that $Q(z)$ and $P(z)$ cannot vanish for the same z. Therefore, we conclude that $Q(z)/P(z)$ is irreducible.

LEMMA 3.3 If $P(z)$ and $Q(z)$ are the polynomials defined in Lemma 3.2 and if $Q(z)/P(z)$ is irreducible, then $P(z)$ has no zeros on the imaginary axis.

Proof: Since $Q(z)$ is the polynomial obtained from $P(z)$ by dropping out the first, third, fifth, $\cdots$, terms, we can write

$$Q(z) = \frac{P(z) + P(-z)}{2}$$

or

$$Q(z) = \frac{P(z) - P(-z)}{2}$$

depending upon whether the degree of $P(z)$ is odd or even, respectively. Thus if $z = iy$ is a zero of $P(z)$, it will also be a zero of $P(-z)$ which implies $z = iy$ is a zero of $Q(z)$, contrary to the hypothesis.

THEOREM 3.2 Consider a polynomial having real coefficients

$$P(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n$$

and let

$$Q(z) = a_1 z^{n-1} + a_3 z^{n-3} + a_5 z^{n-5} + \cdots$$

be the polynomial obtained from $P(z)$ by dropping out the first, third, fifth, $\cdots$, terms. If the quotient $Q(z)/P(z)$

has a continued fraction expansion of the form:

$$\frac{Q(z)}{P(z)} = \cfrac{1}{c_1 z + 1 + \cfrac{1}{c_2 z + \cdots + \cfrac{1}{c_n z}}} \tag{3.17}$$

in which $c_p \neq 0$, $(p = 1,2,\cdots,n)$ and if m of the coefficients c_p are negative, then m of the zeros of $P(z)$ have positive real parts and $n - m$ have negative real parts.

Proof: Let

$$R(z) = z^n + a_2 z^{n-2} + a_4 z^{n-4} + \cdots \tag{3.18}$$

so that

$$P(z) = R(z) + Q(z). \tag{3.19}$$

By hypothesis the continued fraction expansion of $Q(z)/P(z)$ exists; therefore, the continued fraction of $Q(z)/R(z)$ exists and is equal to

$$\frac{Q(z)}{R(z)} = \frac{Q(z)}{P(z) - Q(z)} = \frac{1}{\{P(z)/Q(z) - 1\}} \tag{3.20}$$

which when expanded yields

$$\frac{Q(z)}{R(z)} = \cfrac{1}{c_1 z + \cfrac{1}{c_2 z + \cfrac{1}{c_3 z + \cdots + \cfrac{1}{c_n z}}}} . \tag{3.21}$$

By Lemma 3.2 we know that $Q(z)/P(z)$ is irreducible, thus by Lemma 3.3 $P(z)$ has no zeros on the imaginary axis. By Lemma 3.1 we know that the difference in the number of zeros of $P(z)$ in the right and left hand plane is

$$P_R - P_L = \frac{\Delta\theta}{\pi}. \tag{3.22}$$

Substituting $-iz$ for z in $P(z)$, we rotate the plane by $\pi/2$ radians after which the roots which were formerly in the right half plane will now be in the upper half plane and the roots formerly in the left half plane, in the lower half plane. We now have z ranging from $-\infty$ to $+\infty$ over the real axis; the previous relations still being valid.

Now multiplying by a constant, i^n, and substituting $-iz$ for z in the equation of $P(z)$, we get

$$i^n P(-iz) = U(z) + iV(z) \tag{3.23}$$

where

$$U(z) = z^n - a_2 z^{n-2} + a_4 z^{n-4} + \cdots \tag{3.24}$$

and

$$V(z) = a_1 z^{n-1} - a_3 z^{n-3} + a_5 z^{n-5} + \cdots . \tag{3.25}$$

We note that $U(z)$ and $V(z)$ are real for real values of z. By (3.19) we obtain the relation

$$i^n P(-iz) = i^n R(-iz) + i\{i^{n-1} Q(-iz)\} . \tag{3.26}$$

Considering Equations(3.23) and (3.26), we have the relation

$$\frac{V(z)}{U(z)} = \frac{Q(-iz)}{iR(-iz)} . \tag{3.27}$$

Thus the expansion for $V(z)/U(z)$ is

$$\frac{V(z)}{U(z)} = \cfrac{1}{c_1 z - \cfrac{1}{c_2 z - \ddots - \cfrac{1}{c_n z}}} . \tag{3.28}$$

Let $x_1, x_2, \cdots, x_k$ denote the real distinct zeros of $U(z)$. Let the Cauchy index s_p, be such that $s_p = +1, 0$, or -1 according as $V(z)/U(z)$ increases from $-\infty$ to $+\infty$, does not change, or decreases from $+\infty$ to $-\infty$, respectively, as z increases through x_p. We then have

$$\frac{\Delta\theta}{\pi} = \sum_{p=1}^{k} s_p. \tag{3.29}$$

To compute $\Delta\theta/\pi$, let the polynomials $f_o = 1$, $f_1 = c_n z$, $\cdots$, f_n be defined by the recurrence formula

$$f_{p+1} = c_{n-p} z f_p - f_{p-1}, \quad (p = 1,2,\cdots,n-1) \tag{3.30}$$

and define polynomials $F_o = 0$, $F_1 = 1,\cdots,F_n$ by the formula

$$F_{p+1} = c_{n-p} z F_p - F_{p-1}, \quad (p = 1,2,\cdots,n-1). \tag{3.31}$$

On multiplying (3.31) by f_p and (3.30) by $-F_p$ and adding, we get

$$F_{p+1} f_p - F_p f_{p+1} = F_p f_{p-1} - F_{p-1} f_p \tag{3.32}$$

from which we conclude

$$F_{p+1} f_p - F_p f_{p+1} = 1. \tag{3.33}$$

Consider now the sequence

$$f_o, f_1, \cdots, f_n. \tag{3.34}$$

From (3.33) it follows that two successive members of this sequence may not vanish simultaneously. From (3.30) it follows that when f_p vanishes, then f_{p-1} and f_{p+1} have opposite signs ($1 \leq p \leq n-1$). Hence as x increases through a real zero of f_p there is no loss or gain in the number of variations in signs in the sequence (3.34) since the only possibilities are the following:

	f_{p-1}	f_p	f_{p+1}
$z = x_p - \varepsilon$	+	+	-
$z = x_p + \varepsilon$	+	-	-

Hence as z increases through real values from $-\infty$ to $+\infty$, any change in the number of variations must be due to the vanishing of f_n. Moreover, there will be a loss or a gain in the number of variations according as the product $f_{n-1}f_n$ changes from negative to positive or from positive to negative, respectively, as z passes through a zero of f_n. However, $f_{n-1}/f_n \equiv V(z)/U(z)$ and consequently $\Delta\theta/\pi$ is precisely the net loss in the number of variations in signs in the sequence (3.34) as z ranges from $-\infty$ to $+\infty$ through real values.

For large values of z the signs of the polynomials $f_o, f_1, \cdots, f_n$ are determined by the signs of their highest power of z. When z is large and positive, the signs are those of

(3.35)

$$1,\ +c_n,\ +c_{n-1}c_n,\ +c_{n-2}c_{n-1}c_n, \cdots,\ +c_1c_2\cdots c_n$$

while when z is large and negative, the signs are those of

(3.36)

$$1,\ -c_n,\ +c_{n-1}c_n,\ -c_{n-2}c_{n-1}c_n, \cdots,\ (-1)^n c_1c_2\cdots c_n.$$

By hypothesis, m is the number of negative signs in the sequence $c_1, c_2, \cdots, c_n$, thus there are m variations in sign in sequence (3.35)and, therefore, n - m variations in sign in sequence (3.36).

Referring to Equations (3.22) and (3.29), we have

$$P_R - P_L = \frac{\Delta\theta}{\pi} = m - (n-m) = -n + 2m. \tag{3.37}$$

We know that the

$$P_R + P_L = n \tag{3.38}$$

since P(z) is an n^{th} degree polynomial. Therefore, $P_R = m$ and $P_L = n - m$. Since m was the number of negative signs in the sequence, $c_1, c_2, \cdots, c_n$, the theorem is proved.

COROLLARY 3.1 Let P(z) and Q(z) be defined as in Theorem 3.2. Assume that the continued fraction expansion of $P(z)/Q(z)$ exists and is given by (3.17). Then $P(z)$ has all of its roots with negative real parts if and only if all the coefficients c_p are positive.

On the following page is an example illustrating the use of Theorem 3.1. The long division process is used to show the method by which Table 1 was obtained.

Example 1. Let

$$f_o(z) = z^3 + (2 + i)z^2 + (3 + i)z + (2i + 2)$$

and

$$f_1(z) = 2z^2 + iz + 2.$$

To find $f_1(z)/f_o(z)$,

$$2z^2 + iz + 2 \,\big)\, z^3 + (2 + i)z^2 + (3 + i)z + (2i + 2) \,\big(\, \tfrac{1}{2}z$$

$$\underline{z^3 \quad + \tfrac{i}{2}z^2 \quad + z}$$

$$(2 + \tfrac{i}{2})z^2 + (2 + i)z + 2i + 2 \qquad \big(1 + \tfrac{i}{4}$$

$$\underline{2z^2 \quad + \quad iz + \quad 2}$$

$$\tfrac{i}{2}z^2 \qquad +2z + 2i$$

$$\underline{\tfrac{i}{2}z^2 \qquad -z/_4 + \tfrac{i}{2}}$$

$$\tfrac{9}{4}z + {}^3/_2 i$$

$$\tfrac{9}{4}z + \tfrac{3}{2}i \,\big)\, 2z^2 + iz + 2 \,\big(\, \tfrac{8}{9} - \tfrac{4i}{27}$$

$$\underline{2z^2 + \tfrac{4}{3}iz}$$

$$-{}^1/_3 iz + 2$$

$$\underline{-{}^1/_3 iz + {}^2/_9}$$

$$^{16}/_9$$

$$\tfrac{16}{9} \,\big)\, \tfrac{9}{4}z + \tfrac{3}{2}i \,\big(\, \tfrac{81}{64} + \tfrac{27i}{32}$$

$$\underline{\tfrac{9}{4}z + \tfrac{3}{2}i}$$

$$0$$

Constructing Table 1, we now see how the different entries were arrived at by observing the remainders in the previous long division process. We obtain

$\alpha_{00} = 1$ $\quad \alpha_{01} = 2 + i$ $\quad \alpha_{02} = 3 + i$ $\quad \alpha_{03} = 2i + 2$

$\alpha_{11} = 2$ $\quad \alpha_{12} = i$ $\quad \alpha_{13} = 2$

$\eta_{11} = 2 + \frac{i}{2}$ $\quad \eta_{12} = 2 + i$ $\quad \eta_{13} = 2i + 2$

$\alpha_{22} = {}^{9}/_{4}$ $\quad \alpha_{23} = {}^{3i}/_{2}$

$\eta_{22} = -{}^{i}/_{3}$ $\quad \eta_{23} = 2$

$\alpha_{33} = {}^{16}/_{9}$

$\eta_{33} = {}^{3i}/_{2}$.

Therefore, recalling that $c_p = \frac{\alpha_{p-1,p-1}}{\alpha_{pp}}$ and $s_p = \frac{\eta_{pp}}{\alpha_{pp}}$, we obtain

$c_1 = {}^{1}/_{2}$ $\quad c_2 = {}^{8}/_{9}$ $\quad c_3 = {}^{81}/_{64}$

$s_1 = 1 + {}^{i}/_{4}$ $\quad s_2 = -{}^{4i}/_{27}$ $\quad s_3 = {}^{27i}/_{32}$.

The continued fraction expansion is then of the form:

$$\frac{f_1(z)}{f_o(z)} = \cfrac{1}{\frac{1}{2}z + 1 + \frac{i}{4} + \cfrac{1}{\frac{8}{9}z - \frac{4i}{27} + \cfrac{1}{\frac{81}{64}z + \frac{27i}{32}}}} .$$

Although not included in this thesis, Frank proves a theorem which shows that all the zeros of $f_o(z)$ will lie in the left half plane if and only if all the c_p's are positive and all the s_p's are pure imaginary, except for s_1 which must be of the form $1 + b_o$ where b_o is pure imaginary. Using this theorem, we see that the conditions are satisfied in our example, and thus $f_o(z)$ has all roots with negative real parts.

We may now note that Theorem 3.2 is a special case of the above theorem. The purpose of using Wall's theorem rather than Frank's was to illustrate the use of Cauchy's indices, which was the method originally used by Routh in arriving at his conclusions.

IV
HURWITZ STABILITY CRITERION

Definition: A polynomial

$$f(z) = a_n z^n + a_n z^{n-1} + \cdots + a_1 z + a_o = 0$$

with real coefficients is said to be a Hurwitz polynomial if and only if all of the roots of f(z) have negative real parts.

We shall now establish in a variety of forms necessary and sufficient conditions for f(z) to be a Hurwitz polynomial.

THEOREM 4.1 If f(z) is a Hurwitz polynomial, then all the coefficients of f(z) are positive.

Proof: Since f(z) has real coefficients, it may be factored into linear and quadratic factors of the type (z + b) and (z^2 + cz + d) where b, c, and d are real. The linear factors yield the real roots, and the quadratic factors yield the complex. Since f(z) is Hurwitz, all its roots have negative real parts. Therefore, the constants, b, c, and d, must be positive. The product of any number of linear and quadratic factors containing only positive coefficients always yields a polynomial with positive coefficients, which is the desired result.

As a corollary to this theorem, we may note the following:

COROLLARY 4.1 If a polynomial has one or more, but not all, negative coefficients, then at least one root will lie in the right half plane.

THEOREM 4.2 If $f(z)$ is a Hurwitz polynomial, then for the complex variable, z, the following relations hold:

$$(4.1)\qquad \begin{array}{lll} |f(z)| > |f(-z)| & \text{for} & \mathrm{Re}(z) > 0 \\ |f(z)| = |f(-z)| & \text{for} & \mathrm{Re}(z) = 0 \\ |f(z)| < |f(-z)| & \text{for} & \mathrm{Re}(z) < 0 . \end{array}$$

Proof: Let $f(z)$ be written in its factored form:

$$f(z) = a_n(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n),$$

and consider the linear factor $z - \alpha_k$, where α_k is real. If $z = x + iy$, then we have

$$|z - \alpha_k|^2 = (x - \alpha_k)^2 + y^2 .$$

Hence, since by hypothesis $\alpha_k < 0$, we obtain

$$(4.2)\qquad \begin{array}{lll} |z - \alpha_k| > |-z - \alpha_k| & \text{for} & x = \mathrm{Re}(z) > 0 \\ |z - \alpha_k| = |-z - \alpha_k| & \text{for} & x = \mathrm{Re}(z) = 0 \\ |z - \alpha_k| < |-z - \alpha_k| & \text{for} & x = \mathrm{Re}(z) < 0 . \end{array}$$

Since complex zeros occur in conjugate pairs, let us consider $z - \alpha_r$ and $z - \alpha_s$ where α_r and α_s are such that

$$\alpha_r = \gamma_r + i\delta_r, \quad \alpha_s = \gamma_r - i\delta_r, \quad \gamma_r < 0, \quad \delta_r > 0.$$

Then

$$|z - \alpha_r|^2 = (x - \gamma_r)^2 + (y - \delta_r)^2$$
$$|z - \alpha_s|^2 = (x - \gamma_r)^2 + (y + \delta_r)^2$$

and

$$|-z - \alpha_r|^2 = (x + \gamma_r)^2 + (y + \delta_r)^2$$
$$|-z - \alpha_s|^2 = (x + \gamma_r)^2 + (y - \delta_r)^2.$$

However,

$$(x - \gamma_r)^2 > (x + \gamma_r)^2 \quad \text{if} \quad x > 0$$
$$(x - \gamma_r)^2 = (x + \gamma_r)^2 \quad \text{if} \quad x = 0$$
$$(x - \gamma_r)^2 < (x + \gamma_r)^2 \quad \text{if} \quad x < 0.$$

Consequently,

$$|(z - \alpha_r)(z - \alpha_s)| > |(-z - \alpha_r)(-z - \alpha_s)| \quad \text{for} \quad \mathrm{Re}(z) > 0$$
$$|(z - \alpha_r)(z - \alpha_s)| = |(-z - \alpha_r)(-z - \alpha_s)| \quad \text{for} \quad \mathrm{Re}(z) = 0$$
$$|(z - \alpha_r)(z - \alpha_s)| < |(-z - \alpha_r)(-z - \alpha_s)| \quad \text{for} \quad \mathrm{Re}(z) < 0.$$

The inequalities (4.1) now follow easily from (4.2) and the previous inequalities.

An immediate result of the previous theorem is:

COROLLARY 4.2 If $\varphi(z) = f(z)/f(-z)$, where $f(z)$ is a Hurwitz polynomial, then

$$\begin{aligned} |\varphi(z)| &> 1 \quad \text{for} \quad \mathrm{Re}(z) > 0 \\ |\varphi(z)| &= 1 \quad \text{for} \quad \mathrm{Re}(z) = 0 \\ |\varphi(z)| &< 1 \quad \text{for} \quad \mathrm{Re}(z) < 0 . \end{aligned}$$

THEOREM 4.3 If $\tau(z) = \dfrac{\varphi(z) + 1}{\varphi(z) - 1} = \dfrac{f(z) + f(-z)}{f(z) - f(-z)}$,

where $f(z)$ is a Hurwitz polynomial, then

$$\begin{aligned} \mathrm{Re}(\tau) &> 0 \quad \text{for} \quad |\varphi(z)| > 1 \\ \mathrm{Re}(\tau) &= 0 \quad \text{for} \quad |\varphi(z)| = 1 \\ \mathrm{Re}(\tau) &< 0 \quad \text{for} \quad |\varphi(z)| < 1 . \end{aligned}$$

It can be shown that this transformation is a Möbius transformation mapping the interior of the unit circle in the φ plane upon the left half of the τ plane.

From Corollary 4.2 and Theorem 4.3 we may formulate the following:

COROLLARY 4.3 If $\tau(z) = \dfrac{f(z) + f(-z)}{f(z) - f(-z)}$, where $f(z)$ is a Hurwitz polynomial, then

$$\begin{array}{lll} \mathrm{Re}(\tau) > 0 & \text{for} & \mathrm{Re}(z) > 0 \\ \mathrm{Re}(\tau) = 0 & \text{for} & \mathrm{Re}(z) = 0 \\ \mathrm{Re}(\tau) < 0 & \text{for} & \mathrm{Re}(z) < 0 \; . \end{array}$$

COROLLARY 4.4 If $\frac{1}{\tau(z)} = \frac{f(z) - f(-z)}{f(z) + f(-z)}$ then

$$\begin{array}{lll} \mathrm{Re}\left(\frac{1}{\tau(z)}\right) > 0 & \text{for} & \mathrm{Re}(z) > 0 \\ \mathrm{Re}\left(\frac{1}{\tau(z)}\right) = 0 & \text{for} & \mathrm{Re}(z) = 0 \\ \mathrm{Re}\left(\frac{1}{\tau(z)}\right) < 0 & \text{for} & \mathrm{Re}(z) < 0 \; . \end{array}$$

If the polynomial $f(z)$ is written in the form:

$$f(z) = M(z) + N(z)$$

where $M(z)$ and $N(z)$ represent the even and odd powers of z, respectively, then $\tau(z)$ can be written as:

$$\tau(z) = \frac{f(z) + f(-z)}{f(z) - f(-z)} = \frac{M(z)}{N(z)} \; .$$

THEOREM 4.4 If $\tau(z)$ is the quotient of the even and odd parts of a Hurwitz polynomial, then

1. $\tau(z)$ has poles on the imaginary axis only and
2. these poles are simple and have real positive residues.

Proof: Assume that the rational function $\tau(z) = \frac{M(z)}{N(z)}$ has a pole of order s at some point $z = z_k$. The Laurent series expansion of $\tau(z)$ in a neighborhood of z_k is of the form

$$\tau(z) = \frac{b_{-s}}{(z - z_k)^s} + \cdots + \frac{b_{-1}}{(z - z_k)} + b_o + b_1(z - z_k) + \cdots .$$

Therefore, $\tau(z)$ can be written as:

$$\tau(z) = \tau_1(z)\left[1 + \frac{b_{-s+1}}{b_{-s}}(z - z_k) + \cdots + \frac{b_{-1}}{b_{-s}}(z - z_k)^{s-1} + \cdots\right]$$

where

$$\tau_1(z) = \frac{b_{-s}}{(z - z_k)^s} .$$

For z in a sufficiently small δ neighborhood of z_k, we can write:

(4.3) $$\tau(z) = \tau_1(z)\left\{1 + \eta(\delta)\right\}$$

where $\eta \longrightarrow 0$ as $\delta \longrightarrow 0$. In the function $\tau_1(z)$ let

(4.4) $$b_{-s} = de^{i\beta}$$

and

(4.5) $$(z - z_k) = pe^{i\alpha}$$

thus

(4.6) $$\tau_1(z) = \frac{d}{p^s}\, e^{i(\beta - s\alpha)}$$

and

(4.7) $$\mathrm{Re}\{\tau_1(z)\} = \frac{d}{p^s}\cos(s\alpha - \beta).$$

Allowing α to vary from 0 to 2π as z traverses a circle of radius p, $p < \delta$, then $\mathrm{Re}(\tau_1(z))$ changes sign 2s times. By relation (4.3) we see that $\mathrm{Re}(\tau(z))$ must change sign 2s times also. From Corollary 4.3 we must conclude that $\tau(z)$ has no poles in either the right or left hand plane since the $\mathrm{Re}(\tau(z))$ can only change sign when $\mathrm{Re}(z)$ does. Thus $\tau(z)$ has poles on the imaginary axis only.

By Corollary 4.3 we also know that $\mathrm{Re}(\tau(z))$ changes sign 2 times as z traverses a closed contour about a pole z_k, of $\tau(z)$. Thus by relation 4.3, $\tau_1(z)$ must also have this property. For this to be true, we must have $s = 1$ in Equation (4.7). We must also have $\beta = 0$ since there cannot be any lag angle. Thus Equation (4.7) becomes

(4.8) $$\mathrm{Re}\{\tau_1(z)\} = \frac{d}{p}\cos\alpha\ .$$

According to Equation (4.5), $\mathrm{Re}(z) > 0$ corresponds to $-\pi/2 < \alpha < \pi/2$ and $\mathrm{Re}(z) < 0$ corresponds to $\pi/2 < \alpha < 3\pi/2$, whereas for $\mathrm{Re}(z) = 0$, $\alpha = \pm\frac{\pi}{2}$. Thus $s = 1$ implies the poles are simple and $\beta = 0$ implies that they have positive real residues, and thus the theorem is proved.

By Corollary 4.4 we know that $1/\tau(z)$ has the same properties as $\tau(z)$. Therefore, $1/\tau(z)$ must also have simple poles on the imaginary axis and positive real residues; and therefore, both the even and odd parts of $f(z)$ must be polynomials whose zeros are simple and lie on the imaginary axis. Also since poles of $\tau(z)$ and $1/\tau(z)$ which lie at $z = 0$ or at $z = \infty$ must also be simple, it follows that the highest powers of $M(z)$ and $N(z)$, as well as their lowest powers, may not differ by more than one. They obviously differ by at least one since $M(z)$ and $N(z)$ are even and odd, respectively.

LEMMA 4.1 If $\tau(z)$ is the ratio of the even and odd parts of a Hurwitz polynomial, then along the imaginary axis, $\tau(z)/i$ is a non-decreasing function,

Proof: Let $\tau(z)$ be expressed in the form:

$$\tau(z) = U(x,y) + iV(x,y).$$

The results of Corollary 4.3 yield the condition

$$\frac{\partial U}{\partial x} \geq 0 \quad \text{for} \quad x = 0.$$

Since $\tau(z)$ is analytic, except at a finite number of points in the plane, it must satisfy the Cauchy Reimann equations everywhere except at these points. Thus

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}.$$

Therefore,

$$\frac{\partial V}{\partial y} \geqslant 0 \quad \text{for} \quad x = 0,$$

except at the poles of the function where the derivative fails to exist.

Referring to Corollary 4.3, we know that

$$U(x,y) = 0 \quad \text{for} \quad x = 0.$$

Therefore,

$$\frac{\partial \tau(z)}{\partial y} = \frac{i\partial V(x,y)}{\partial y}$$

or

$$\frac{\partial \tau(z)}{i\partial y} \geqslant 0 \quad \text{for} \quad x = 0.$$

The last condition shows that along the imaginary axis, where $\tau(z)$ has pure imaginary values only, $\tau(z)/i$ is a non-decreasing function.

THEOREM 4.5 If $\tau(z)$ is the ratio of the even and odd parts of a Hurwitz polynomial, then the zeros and poles of $\tau(z)$ alternate.

Before proving the theorem, consider:

$$\frac{\tau(z)}{i} = \frac{M(z)}{iN(z)}.$$

As z goes from $-i\infty$ to $+i\infty$ along the imaginary axis, $\tau(z)/i$ will change sign at every zero and pole since the poles and zeros of $\tau(z)$ are simple. As $z \longrightarrow -i\infty$, $\tau(z)$ has either a pole or a zero. If it has a pole there, then $(\tau(z)/i) \longrightarrow -\infty$ as $z \longrightarrow -i\infty$, and if it has a zero as $z \longrightarrow -i\infty$, then $(\tau(z)/i) \longrightarrow 0$. For convenience let us assume that $(\tau(z)/i) \longrightarrow 0$ as $z \longrightarrow -i\infty$. This is the case when the degree of the original polynomial $f(z)$ is odd. We note that if $f(z)$ is of degree n and n is odd, then there will be n poles of $\tau(z)$ and $n+1$ zeros of $\tau(z)$ since $\tau(z) \longrightarrow 0$ as $z \longrightarrow \pm\infty$.

Proof: Let $p_1, p_2, \cdots, p_n$ and $z_1, z_2, \cdots, z_{n+1}$ be the poles and zeros, respectively, of $\tau(z)$, and let them be ordered such that

$$-ip_1 < -ip_2 < \cdots < -ip_n$$

and

$$-iz_1 < -iz_2 < \cdots < -iz_{n+1}.$$

For $-iz_1 < y < -iz_1 + \varepsilon$, $\varepsilon > 0$, $\frac{\tau(iy)}{i} = \frac{M(iy)}{iN(iy)} > 0$. Since $\tau(iy)/i$ is non-decreasing, $iN(iy)$ must approach zero. As iy passes through p_1, $\tau(iy)/i$ becomes negative. Thus $M(iy)$ must approach zero until iy passes through z_2, at which time $(\tau(iy)/i) > 0$. The process of $z = iy$ passing alternately through the zeros and poles of $\tau(z)$ continues along the imaginary axis, and thus the theorem is proved.

If the degree of $f(z)$ is even, then $(\tau(z)/i) \longrightarrow -\infty$ as $z \longrightarrow -i\infty$, and the same reasoning as described above shows the alternation of the zeros and poles in this case.

If we write $M(z)$ and $N(z)$ in their factored forms, $\tau(z)$ becomes

$$\tau(z) = \frac{a_n(z^2 - z_1^2)(z^2 - z_3^2) \cdots (z^2 - z_{2n-1}^2)}{a_{n-1}z(z^2 - z_2^2)(z^2 - z_4^2) \cdots (z^2 - z_{2n-2}^2)}$$

in which it is assumed $f(z)$ is of even degree $2n$. The alternation of zeros and poles along the imaginary axis is expressed by the condition:

$$0 < |z_1| < |z_2| < \cdots < |z_{2n-2}| < |z_{2n-1}| < \infty.$$

which is known as the separation property of the zeros and poles of $\tau(z)$.

By the theory of the complex variable, we know that we can express the residue of $\tau(z)$ at one of its poles z_p by

$$b_p = \lim_{z \to z_p} \left[(z - z_p)\tau(z) \right]$$

which can be written as:

$$\frac{a_n(z_p^2 - z_1^2)(z_p^2 - z_3^2)\cdots(z_p^2 - z_{2n-1}^2)}{a_{n-1}2z^2{}_p(z_p^2 - z_2^2)\cdots(z_p^2 - z_{p-2})(z_p^2 - z_{p+2}^2)\cdots(z_p^2 - z_{2n-2}^2)}.$$

Assuming that a_n/a_{n-1} is positive, it may be seen from the above result that the separation property assures the positiveness of the residues of $\tau(z)$ at all of its poles. The residue of $\tau(z)$ as $z \longrightarrow \infty$ is seen to be a_n/a_{n-1}, whereas the residue at $z = 0$ has the value of

$$\lim_{z \to 0} \{z\tau(z)\}$$

which is positive since all the quantities $-z_j^2$ are positive.

Before presenting the theorem Hurwitz formulated, consider the following: Let

$$f(z) = a_n z^n + \cdots + a_1 z + a_o \tag{4.9}$$

be a polynomial with real coefficients. Set

$$\begin{aligned} 1/2\sigma(z) = {} & a_o a_1 + \mu(a_1 a_2 - a_o a_3)z + a_o a_3 z^2 + \\ & + \mu(a_1 a_4 - a_o a_5)z^3 + \cdots \end{aligned} \tag{4.10}$$

where μ is a real constant. The last term of the expansion is $a_o a_n z^{n-1}$ if the degree of $f(z)$ is odd and $\mu a_1 a_n z^{n-1}$ if the degree of $f(z)$ is even.

It is easily verified by algebraic manipulations that:

$$z\sigma(z) = \left[a_o(1 - \tfrac{\mu}{z}) + a_1\mu\right] f(z) - \left[a_o(1 - \tfrac{\mu}{z}) - a_1\mu\right] f(-z)$$

or

$$z\sigma(z) = A(z)f(z) - B(z)f(-z) \tag{4.11}$$

where

$$A(z) = a_o(1 - \tfrac{\mu}{z}) + a_1\mu \tag{4.12}$$

$$B(z) = a_o(1 - \tfrac{\mu}{z}) - a_1\mu.$$

Suppose that $a_o > 0$, $a_1 > 0$, $\mu > 0$ and let z be a complex number not equal to zero such that

$$\mathrm{Re}(\mu/z) < 1.$$

Then it is easily seen that

$$|A(z)| > |B(z)|. \tag{4.13}$$

Although the roots of $\sigma(z) = 0$ vary with μ, they will be bounded away from zero if μ is bounded since $a_o > 0$ and $a_1 > 0$. Thus if z_p is a root of $\sigma(z) = 0$, then $1/z_p$ will be bounded above. Therefore, we can find $\mu > 0$ such that for all roots z_p of $\sigma(z) = 0$

$$\mathrm{Re}(\mu/z_p) \quad 1.$$

LEMMA 4.2 If $\mathrm{Re}(\mu/z_p) < 1$, for all z_p, where z_p is a zero of $\sigma(z)$, then $\sigma(z)$ is Hurwitz if and only if $f(z)$ is Hurwitz.

Proof: Let $f(z)$ be Hurwitz and assume there exists a zero, η, of $\sigma(z)$ such that

$$\mathrm{Re}(\eta) \geq 0.$$

From Equation (4.11) we obtain

$$A(\eta)f(\eta) = B(\eta)f(-\eta)$$

and from relation (4.13) we have

$$|A(\eta)| > |B(\eta)|$$

since

$$\mathrm{Re}(\mu/\eta) < 1 .$$

Consequently,

$$|f(\eta)| = \left|\frac{B(\eta)}{A(\eta)}\right| |f(-\eta)| < |f(-\eta)|$$

which contradicts Theorem 4.2.

Conversely assume that $\sigma(z)$ is Hurwitz and that $a_o > 0$, $a_1 > 0$. Before proving this part of the lemma, consider the identity:

$$z\sigma(z) = A(z)f(z) - B(z)f(-z).$$

Replacing z by -z, we obtain

$$z\sigma(-z) = B(-z)f(z) - A(-z)f(-z).$$

Solving for f(z), we obtain

$$f(z) = \frac{zA(-z)}{A(z)A(-z)-B(z)B(-z)}\sigma(z) - \frac{zB(z)}{A(z)A(-z)-B(z)B(-z)}\sigma(-z).$$

However,

$$A(z)A(-z) - B(z)B(-z) = 4a_o a_1 \mu$$

and so

$$(4.14) \qquad 4a_o a_1 \mu f(z) = zA(-z)\sigma(z) - zB(z)\sigma(-z).$$

The proof of sufficiency now follows.

Proof: Suppose that $\eta \neq 0$ is a zero of f(z) such that $Re(\eta) \geq 0$. It then follows from (4.14) that

$$(4.15) \qquad A(-\eta)\sigma(\eta) = B(\eta)\sigma(-\eta).$$

Setting $\eta = \frac{1}{c + ib}$, we have

$$A(-\eta) = a_o(1 + c\mu + ib\mu) + a_1\mu$$

$$B(\eta) = a_o(1 - c\mu - ib\mu) - a_1\mu$$

and

$$|A(-\eta)|^2 = \{a_o(1 + c\mu) + a_1\mu\}^2 + a_o^2 b^2 \mu^2$$

$$|B(\eta)|^2 = \{a_o(1 - c\mu) - a_1\mu\}^2 + a_o^2 b^2 \mu^2 .$$

Thus

$$|A(-\eta)|^2 - |B(\eta)|^2 = 4\mu a_o^2 c + 4a_o a_1 \mu > 0$$

since $\frac{c}{c^2 + b^2} = \mathrm{Re}(\eta) \geqslant 0$ and $\mu, a_o, a_1 > 0$. Therefore,

$$|A(-\eta)|^2 > |B(\eta)|^2 .$$

Then from (4.15)

$$|\sigma(\eta)| = \left|\frac{B(\eta)}{A(-\eta)}\right| |\sigma(-\eta)| < |\sigma(-\eta)|$$

which is impossible since $\mathrm{Re}(\eta) \geqslant 0$.

The theorem of Hurwitz is now demonstrated. The proof given here is due to I. Schur.

THEOREM 4.5 Let

$$f(z) = a_n z^n + \cdots + a_1 z + a_o = 0$$

be a polynomial with real coefficients. Then f(z) is a

Hurwitz polynomial if and only if the n determinants,

$$D_1 = a_1, \quad D_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, \qquad D_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}, \cdots$$

$$D_n = \begin{vmatrix} a_1 & a_0 & \cdots & 0 \\ a_3 & a_2 & & 0 \\ \cdot & \cdot & \cdot & \cdot \\ a_{2n-1} & a_{2n-2} & \cdots & a_n \end{vmatrix},$$

(where all coefficients a_r having $r > n$ or $r < 0$ are replaced by 0) are positive, provided, as can legitimately be assumed, $a_o > 0$.

Proof: (a) $D_i > 0$, $(i = 1,2,\cdots,n)$ implies that $f(z)$ is Hurwitz. We shall proceed by induction. In case $n = 2$, the conditions are:

$$a_o > 0, \quad a_1 > 0, \quad \begin{vmatrix} a_1 & a_o \\ 0 & a_2 \end{vmatrix} = a_1 a_2 > 0$$

or $a_o > 0$, $a_1 > 0$, $a_2 > 0$. Thus it is evident that

$$a_2 z^2 + a_1 z + a_o = 0$$

has roots with negative real parts.

Now let us assume that $D_i > 0$, $(i = 1,2,3,\cdots n-1)$ implies that $f(z)$ of degree $n-1$ is Hurwitz. We shall show that $D_i > 0$, $(i = 1,2,\cdots,n)$ implies that $f(z)$ of degree n is Hurwitz thus completing the induction.

Corresponding to the determinants $D_1, D_2, \cdots$, for $f(z)$, we have for the equation $\sigma(z) = 0$, $\sigma(z)$ as defined by Equation (4.10), the determinants

$$\beta_1 = \left|\mu(a_1a_2 - a_oa_3)\right|, \qquad \beta_2 = \begin{vmatrix} \mu(a_1a_2 - a_oa_3) & a_oa_1 \\ \mu(a_1a_4 - a_oa_5) & a_oa_3 \end{vmatrix},$$

$$\beta_3 = \begin{vmatrix} \mu(a_1a_2 - a_oa_3) & a_oa_1 & 0 \\ \mu(a_1a_4 - a_oa_5) & a_oa_3 & \mu(a_1a_2 - a_oa_3) \\ \mu(a_1a_6 - a_oa_7) & a_oa_5 & \mu(a_1a_4 - a_oa_5) \end{vmatrix}, \cdots .$$

Now

$$\beta_1 = \mu D_2$$

$$\beta_2 = \mu a_o \begin{vmatrix} a_1a_2 - a_oa_3 & a_1 \\ a_1a_4 - a_oa_5 & a_3 \end{vmatrix}$$

$$= \mu a_o a_1^{-1} \begin{vmatrix} a_1 & 0 & 0 \\ a_3 & a_1a_2 - a_oa_3 & a_1 \\ a_5 & a_1a_4 - a_oa_5 & a_3 \end{vmatrix}$$

$$= \mu a_o \begin{vmatrix} a_1 & a_o & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}$$

or

$$\beta_2 = \mu a_o D_3.$$

Similarly, we obtain the following relationships:

$$\beta_{2k-1} = \mu^k a_o^{k-1} a_1^{k-1} D_{2k}$$

$$\beta_{2k} = \mu^k a_o^k a_1^{k-1} D_{2k+1}.$$

We see that every determinant β_k differs from D_{k+1} by a factor which is greater than zero. By hypothesis $a_o > 0$, and

$$D_1 > 0,\ D_2 > 0,\ \cdots,\ D_n > 0;$$

therefore,

$$\beta_1 > 0,\ \beta_2 > 0,\ \cdots,\ \beta_{n-1} > 0.$$

Hence assuming that the criterion holds for equation of degree n-1, $\sigma(z) = 0$ will have all of its roots with negative real parts. Thus Lemma 4.2 $f(z) = 0$ will also have all its roots with negative real parts, that is, $f(z)$ is Hurwitz.

Proof:(b) $f(z)$ Hurwitz implies that $D_i > 0$, $(i = 1,2,\cdots,n)$. Again we shall proceed by induction.

If $f(z)$ is of degree 2, then

$$f(z) = a_o + a_1 z + a_2 z^2.$$

Hence by Theorem 4.1 a_o, a_1, and a_2 are positive. Therefore,

$$D_1 = a_1 > 0, \quad D_2 = \begin{vmatrix} a_1 & a_o \\ 0 & a_2 \end{vmatrix} = a_1 a_2 > 0.$$

Now let us assume that $f(z)$ Hurwitz and of degree n-1 implies that $D_1, D_2, \cdots, D_{n-1}$ are positive. We must now show that $f(z)$ Hurwitz of degree n implies that $D_i > 0$, $i = 1, \cdots, n$. Since $f(z)$ is Hurwitz, it has all its zeros with negative real parts. This implies by Lemma 4.2 that $\sigma(z)$ also has all its zeros with negative real parts. Therefore, it is necessary since $\sigma(z)$ is Hurwitz and of degree n-1 that

$$\beta_1 > 0, \quad \beta_2 > 0, \quad \cdots, \quad \beta_{n-1} > 0.$$

Thus $D_2, D_3, \cdots, D_n$ are consequently > 0. Therefore,

$$D_1 > 0, \quad D_2 > 0, \quad \cdots, \quad D_n > 0$$

if all the roots of $f(z) = 0$ have negative real parts as was to be shown.

Following is an example illustrating Theorem 4.5.

Example: Let the characteristic equation of a system be given by

$$f(z) = 3z^4 + 5z^3 + 11z^2 + 7z + 9.$$

To determine whether the system is stable, form the following determinants:

$$D_1 = 9 \quad D_2 = \begin{vmatrix} 7 & 9 \\ 5 & 11 \end{vmatrix} = 32$$

$$D_3 = \begin{vmatrix} 7 & 9 & 0 \\ 5 & 11 & 7 \\ 0 & 3 & 5 \end{vmatrix} = 13 \quad D_4 = \begin{vmatrix} 7 & 9 & 0 & 0 \\ 5 & 11 & 7 & 9 \\ 0 & 3 & 5 & 11 \\ 0 & 0 & 0 & 3 \end{vmatrix} = 39 .$$

We see that $D_1 > 0$, $D_2 > 0$, $D_3 > 0$, and $D_4 > 0$ and, therefore, by Theorem 4.5 we see that $f(z)$ is Hurwitz.

It is easy to see from our previous example that $D_4 = 3D_3$. In general $D_n = a_n D_{n-1}$ where a_n is the coefficient of the highest power of z. Therefore, in practice it is not necessary to compute D_n, but instead we need only consider the sequence, D_1, D_2, $\cdots$, D_{n-1}, a_n. If this sequence consists of only positive terms, then the corresponding polynomial is Hurwitz.

V
NYQUIST STABILITY CRITERION

The derivation of the Nyquist criterion is based upon a theorem due to Cauchy called the "Principle of the Argument." Following are the definitions and theorems which are needed in connection with this principle. The proofs may be found in almost any book on the theory of the complex variable.

THEOREM 5.1 Suppose that f(z) is holomorphic inside and on a "scrod" (simple closed rectifiable oriented curve) C, except for a finite number of isolated singularities, none of which lie on C. Then

$$\int_C f(z)dt = 2\pi i S(f_i; C_i)$$

where $S(f_i; C_i)$ is the sum of the residues of f(z) in the interior of C.

Definition: A function f(z) is said to be meromorphic if it is the quotient of two entire functions.

Definition: The function f(z) has a zero of order (multiplicity) n at z = a, if f(z) is holomorphic in some neighborhood of a and if there is a function $g(z)$ also holomorphic at a, such that $g(a) \neq 0$ and

$$f(z) = (z - a)^n g(z).$$

THEOREM 5.2 Let C be a "scroc" and let $f(z)$ be a function which is meromorphic in the union of C and its interior, but which has neither zeros nor poles on C. Then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = Z_f - P_f$$

where Z_f, is the number of zeros of $f(z)$ in the interior C, and P_f is the number of poles of $f(z)$ there, zeros and poles being counted with their proper multiplicities.

THEOREM 5.3 (The Principle of the Argument) Let C be a "scroc" and let $f(z)$ be meromorphic in the union of C and its interior but having neither zeros nor poles on C. When z describes C, the argument of $f(z)$ increases by a multiple of 2π, namely,

$$\Delta_C \arg f(z) = 2\pi(Z_f - P_f)$$

where Z_f and P_f have the meaning stated above.

With this result we can now discuss the stability of a system. We know that a system is unstable if the characteristic function of the system contains any zeros in the right half plane. By examining the characteristic function of a system, we can clearly see that

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

has no poles. Consider the closed curve consisting of the straight line joining the points $-iM$, and $+iM$, M real and positive, and the semicircle having the origin as center and passing through the points, $-iM$, M, and $+iM$. For our path of integration we take the contour described above as $M \to \infty$. This contour is shown pictorially in Fig. (2). It is referred to as a Bromwich curve.

If we map the Bromwich curve under f, where we let $f(z) = re^{i\theta}$, then the resulting change in the polar angle θ will be, due to the previous results, equal to 2π times the number of zeros enclosed in this contour, or equivalently the number of zeros with positive real parts.

The characteristic equation $f(z) = 0$ is represented by the origin in the f plane. Thus if the point (0,0) in the f plane is enclosed by the image of C under f, we conclude that a zero with positive real parts lies in C.

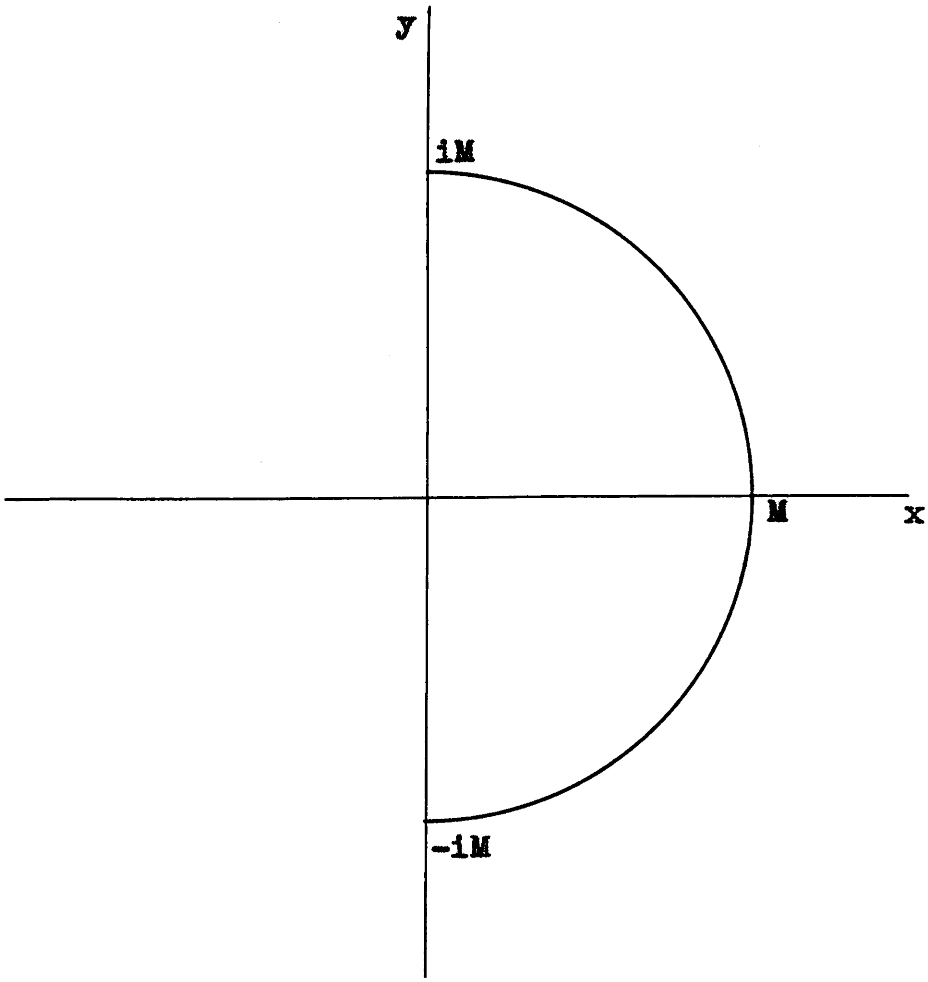


Fig. (2) Bromwich Curve

Thus we may state the conditions for stability as follows:

THEOREM 5.4 Let

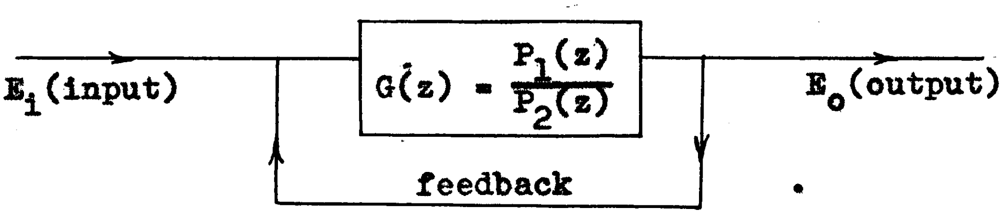
$$f(z) = z^n + \cdots a_{n-1}z + a_n$$

be the characteristic function of a system. Then the system will be stable if and only if the image of the curve in Fig. (2) under f does not enclose the origin.

This is essentially the statement of the Nyquist criterion. In most engineering books and even in Nyquist's original paper, we find that the point (-1,0) instead of (0,0) must lie outside the curve for stability. This is because in the analysis of feedback systems the equation of interest is usually of the form:

$$f(z) = 1 + \frac{P_2(z)}{P_1(z)}$$

where $P_1(z)$ and $P_2(z)$ are polynomials in z. This may easily be seen by considering a feedback system given diagramatically by



.

$G(z)$ is called the transfer function, and $P_1(z)$ and $P_2(z)$ are polynomials. For this system to be stable the over-all transfer function T_o, which is defined to be

$$T_o = \frac{E_o}{E_i} = \frac{G(z)}{1 + G(z)}$$

must have no poles in the right half plane. By substituting the value of $G(z)$ we see that

$$T_o = \frac{P_1(z)}{P_1(z) + P_2(z)} = \frac{1}{1 + \frac{P_2(z)}{P_1(z)}} .$$

Thus the equation of interest is no longer a polynomial but one of the form

$$f(z) = 1 + \frac{P_2(z)}{P_1(z)} .$$

Let

$$f_1(z) = \frac{P_2(z)}{P_1(z)} ,$$

then $f(z) = 0$ when $f_1(z) = -1$; and hence we need only consider the image of the curve in Fig. (2) under f_1 and see the number of times it encircles the point $(-1,0)$.

Let us consider the feedback system just discussed where the transfer function G is defined to be

$$G(z) = \frac{1}{k}\frac{P_1(z)}{P_2(z)},$$

k being a parameter. The over-all transfer function is thus given by

(5.1) $$T_o = \frac{1}{1 + k\frac{P_2(z)}{P_1(z)}} .$$

Define the symbol

(5.2) $$(RF) = \frac{P_2(z)}{P_1(z)} .$$

Then the denominator of Equation (5.1) can be written as

(5.3) $$1 + k(RF).$$

We may obtain the map of the curve in Fig. (2) under the function defined in Equation (5.3) by considering the image of the curve under the function $k\frac{P_2(z)}{P_1(z)}$ and then translating the origin to (-1,0). Each new value of k requires a new plot of function (5.3). This results in a family of similar curves, some of which enclose the point (-1,0) and some which do not, describing instability or stability, respectively.

Function (5.3) may be written in the form

$$\frac{1}{k} + (RF).$$

In this form the curve in Fig. (2) is mapped under the function (RF) with unity as the coefficient. The points corresponding to $1/k$ are plotted on the negative real axis. Nyquist's criterion now refers to the point $(-1/k,0)$ rather than $(-1,0)$, thus reducing the task to one map of the curve in Fig. (2) under (RF) rather than several for different values of k.

The Nyquist stability criterion was originally derived for a system similar to the one in the previous example. This type of system is known as a single loop feedback system. In case the system contains several feedback loops, it is necessary to calculate the over-all transfer function for each loop and to record the algebraic number of counter clockwise encirclements of the critical points. Stability of the over-all system is subsequently decided by the sum of these encirclements.

Following is an example which uses the Nyquist criterion to examine an equation found in closed cycle control systems.

Nyquist Criterion For Closed Cycle Control Systems

Closed cycle control systems exhibit a time lag between the input and output signals. This is partly due to inertial and damping forces present in the system and partly in the inherent delay in transmission of the control system.

Assume the feedback to be proportional to the 1^{st} derivative of the displacement and to be delayed by a constant time τ. Assume also linearity of the system and viscous damping.

Let:

t = independent variable, time

τ = constant time lag

θ = dependent variable

I, R, K = constant parameters of system

S = magnification of feedback signal.

The differential equation of the freely oscillating, output system of a closed cycle control device with viscous damping is given by

$$I\ddot{\theta}(t) + R\dot{\theta}(t) + K\theta(t) = 0 \tag{5.4}$$

where I, R, K are real and positive.

Introducing a feedback term proportional to $\dot{\theta}(t - \tau)$, we obtain

(5.5) $$I\ddot{\theta}(t) + R\dot{\theta}(t) + K\theta(t) = -S\dot{\theta}(t - \tau).$$

Let $F(p) = \mathcal{L}\{\theta(t)\}$ where $\mathcal{L}$ = Laplace transform. Taking the transform of Equation (5.5), we obtain

(5.6) $$(p^2 I + pR + K + Spe^{-\tau p})F(p) = pI\theta(0) + I\dot{\theta}(0) + R\theta(0)$$

where $\theta(0)$, $\dot{\theta}(0)$ are the values of $\theta(t)$ and $\dot{\theta}(t)$ at $t = 0$.

Letting $L(p) = pI\theta(0) + I\dot{\theta}(0) + R\theta(0)$

$Y(p) = p^2 I + pR + K,$

we get

(5.7) $$F(p) = \frac{L(p)}{Y(p) + Spe^{-p\tau}} = \frac{L(p)}{Y(p)} \frac{1}{1 + \dfrac{Spe^{-p\tau}}{p^2 I + pR + K}} .$$

Considering $Y(p)$ we see that it is the transform of a system described by the differential equation (5.4). The solution of this equation is known to be stable since I, R, and K are positive. Therefore, $Y(p)$ will have no zeros in the right half of the p plane.

It remains to examine the solution of

$$1 + \frac{Spe^{-p\tau}}{p^2 I + pR + K} = 0.$$

In order to do this we use Theorem 5.4 which may be restated as follows:

THEOREM 5.4b Let $f(p)$ be holomorphic in a simply connected domain D and let B be a "scroc" whose closure is contained in D, where $f(p) \neq 0$ for p on B. Then if p traverses B in a counter clockwise direction, $f(p)$ will traverse a closed curve in the f plane, and the number of zeros of $f(p)$ in D is equal to the number of times the contour in the f plane encircles the origin.

For B we choose a Bromwich curve. We shall assume that no zeros of $f(p)$ lie on the imaginary axis. Zeros on the imaginary axis occur only when a critical combination of constants of the system occur.

Rewriting the second term of Equation (5.8), we obtain, by setting the result equal to $f(p)$,

$$f(p) = \frac{Ape^{-p\tau}}{ap^2 + bp + 1} \tag{5.9a}$$

where

$$A = \frac{S}{K}, \quad a = \frac{I}{K}, \quad b = \frac{R}{K}. \tag{5.9b}$$

Considering (5.9) we see that along the contour when p = ±

$$\lim_{\alpha \to \pm \infty} f(i\alpha) = \lim_{\alpha \to \pm\infty} \frac{Ai\alpha e^{-i\alpha\tau}}{-a\alpha^2 + bi\alpha + 1}$$

therefore,

$$\lim_{\alpha \to \pm\infty} \left| f(i\alpha) \right| < \lim_{\alpha \to \pm\infty} \left| \frac{A\alpha}{a\alpha^2} \right| = 0.$$

Also consider the case as p traverses the infinite semicircle. Letting

$$p = re^{i\theta} = r(\cos \theta + i \sin \theta)$$

and substituting in Equation (5.9a), we obtain

$$f(re^{i\theta}) = \frac{Are^{i\theta}e^{-r\tau(\cos \theta + i \sin \theta)}}{ar^2e^{2i\theta} + bre^{i\theta} + 1} .$$

Then for any fixed θ in the range $\pi/2 > \theta > -\pi/2$

$$\left| f(re^{i\theta}) \right| \longrightarrow \frac{Ar^{-1}}{a} e^{-r\tau \cos \theta} \longrightarrow 0$$

since r and τ are positive. Therefore, we limit our study to the straight line contour along the imaginary axis such that $-\infty < ip < \infty$.

Letting $p = i\alpha$ in (5.9a) and simplifying, we get

$$\text{(5.10a)} \qquad f(i\alpha) = \frac{\alpha A e^{-i(\alpha\tau - \varphi)}}{\left\{(1 - \alpha^2 a)^2 + (b\alpha)^2\right\}^{1/2}}$$

where

$$\text{(5.10b)} \qquad \tan \varphi = \frac{-a\alpha^2 + 1}{b\alpha}.$$

A graph of Equation (5.10a) for $-\infty < \alpha \leq 0$ is shown in Fig. (3) which may be found at the end of this section. As can be seen from Fig. (3), it is a spiral starting at the origin when $\alpha = -\infty$ and returning to the origin when $\alpha = 0$. As α increases from 0 to $+\infty$, the graph follows the second branch of the spiral which is symmetrical to the first branch with respect to the real axis.

Equation (5.8) can be rewritten in the form:

$$f(p) + 1 = 0. \tag{5.11}$$

Now as p varies along the imaginary axis between $-i\infty$ and $+i\infty$, the contour B encloses all of the roots in the right half of the p plane, and the number of times f(p) encloses the point (-1,0) will be equal to the number of these roots. Because of the symmetry of the two branches, unstable roots will occur in conjugate pairs. Therefore, we need only to investigate the behavior of one of the branches of f(p).

By direct examination it follows that if $b > A$ in Equation (5.10a), then $|f(i\alpha)| < 1$; therefore, the point (-1,0) is not enclosed and the system cannot possibly become unstable. Reference to Equation (5.9b) shows that this implies that the system will always be stable if $R > S$.

Assuming that $S > R$ ($S = R$ is trivial for all practical purposes), set

$$(5.12) \qquad \frac{A\alpha}{\{(1 - \alpha^2 a)^2 + (b\alpha)^2\}^{1/2}} = \pm 1,$$

and solving for α^2 we get

(5.13)

$$\alpha^2 = \frac{-(b^2 - 2a - A^2) \pm \sqrt{(b^2 - 2a - A^2)^2 - 4a^2}}{2a^2} \quad . \; . \; . \; .$$

Thus there are in general four real values of α (two for each branch of the curve) for which $|f(i\alpha)| = 1$. The conditions for existence of these values are:

$$b^2 - 2a - A^2 < 0$$

and

$$(b^2 - 2a - A^2)^2 - 4a^2 > 0.$$

If these are combined with the aid of Equation (5.9b), we get $S > R$. Therefore, a danger of instability exists whenever $S > R$. Since the magnitude of the radius vector of the spiral, as given by the left side of Equation (5.12), is a continuous function of α, we may differentiate it in order to determine its extreme values. It turns out to have only one maximum for each branch which occurs at

$$\alpha = \pm(1/a)^{1/2}.$$

Thus on each branch the magnitude of the radius vector of the spiral attains the value of 1 twice, say, at α_1 and α_2, and a maximum once. It follows that the magnitude of the radius vector will be greater than 1 between α_1 and α_2 and will be less than 1 everywhere else. Along the positive branch ($\alpha \geq 0$) let

$$\begin{aligned} \beta_1 &= -\alpha_1\tau + \varphi_1 \\ \beta_2 &= -\alpha_2\tau + \varphi_2 \end{aligned} \tag{5.14a}$$

and along the negative branch ($\alpha \leq 0$)

$$\begin{aligned} \beta_1 &= -\alpha_1\tau + \varphi_1 - \pi \\ \beta_2 &= -\alpha_2\tau + \varphi_2 - \pi \end{aligned} \tag{5.14b}$$

(the subtraction of π is made necessary by the factor α in front of Equation (5.10a)). This form of Equation (5.14b) makes the radius vector always positive.

Note from Equation (5.10b) that as α increases along each branch, the phase angle in Equations (5.14) decreases monotonically along each branch from $\pi/2$ to $-\pi/2$. Hence if $\alpha_1 > \alpha_2$ (on either branch), we have $-\alpha_2 > -\alpha_1$ and $\varphi_2 > \varphi_1$. Consequently, if $\alpha_1 > \alpha_2$, then $\beta_2 > \beta_1$ or, in other words, the angle β shown in Fig. (4) is a monotone decreasing function of α.

Now we know from the foregoing that as α_1 increases to α_2, the magnitude of the radius remains greater than 1. Then $\beta_1 > \beta_2$ and the shaded sector in Fig. (4), obtained by a rotation from the smaller to the larger value of β, will enclose all of the values of the radius vector which are greater than 1.

The stability criterion is reduced to the requirement that the shaded sector on Fig. (4) does not include the negative real axis. This can be stated as follows:

The system will be stable if no integer n exists such that

$$\beta_1 \gtrless (2n + 1)\pi \gtrless \beta_2. \tag{5.15}$$

If $|\beta_1 - \beta_2| > 2\pi$, the system becomes unstable, since there is a complete turn of the spiral with the magnitude of the radius vector greater than 1. If $|\beta_1 - \beta_2| < 2\pi$, condition (5.15) can be satisfied within a certain range of values of the time lag τ which will now be determined. The extreme case is shown in Fig. (5). As seen from Fig. (5) if $\beta_2 > \beta_1$, we must have for stability

$$\begin{gathered} \beta_2 < (2n + 1)\pi \\ \beta_1 > (2n + 1)\pi - 2\pi = (2n - 1)\pi \,. \end{gathered} \tag{5.16}$$

Using Equation (5.14a) and keeping in mind that $\tau > 0$, we obtain after simple transformations the following range in τ within which the system will be stable:

$$(5.17) \qquad \frac{\varphi_2 + (2n - 1)\pi}{\alpha_2} < \tau < \frac{\varphi_1 + (2n + 1)\pi}{\alpha_1}$$

where $\alpha_1 > \alpha_2$ and n are allowed integral positive values.

Examination of formula (5.17) shows that in general for any given system (fixed α_1, α_2, and φ_1, φ_2) τ is allowed a band spectrum of values for increasing n with the band width given by formula (5.17). It should be noted, however, that the spectrum will terminate at some value of n after which

$$(5.18) \qquad \frac{\varphi_2 + (2n - 1)\pi}{\alpha_2} > \frac{\varphi_1 + (2n + 1)\pi}{\alpha_1}$$

This is due to the fact that $\beta_2 - \beta_1$ increases with increasing τ, and condition (5.18) corresponds to $\beta_2 - \beta_1$, exceeding 2π.

In the original paper by Ansoff, from where this material was obtained, the cases when the feedback is proportional to the n^{th} derivative of the displacement, $n \neq 1$ are also considered. The results are that if $n = 2$ or $n = 0$, then there exists conditions such that the time lag τ is allowed a band spectrum of values and within this band spectrum the system is stable. For the case when $n > 2$, the system is unstable.

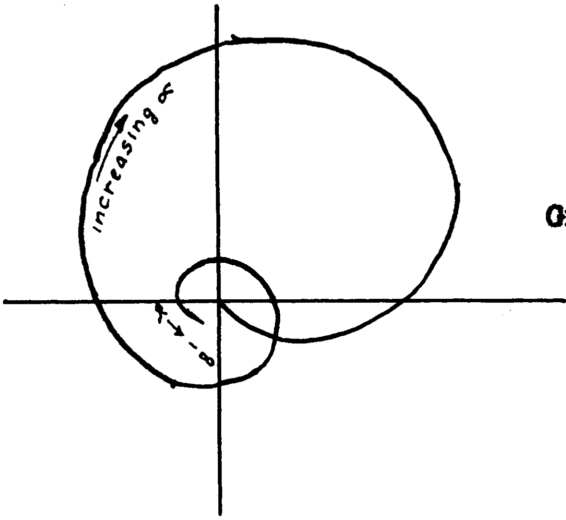


Fig. (3)
Graph of Equation (5.10a)
for $-\infty < \alpha \leq 0$

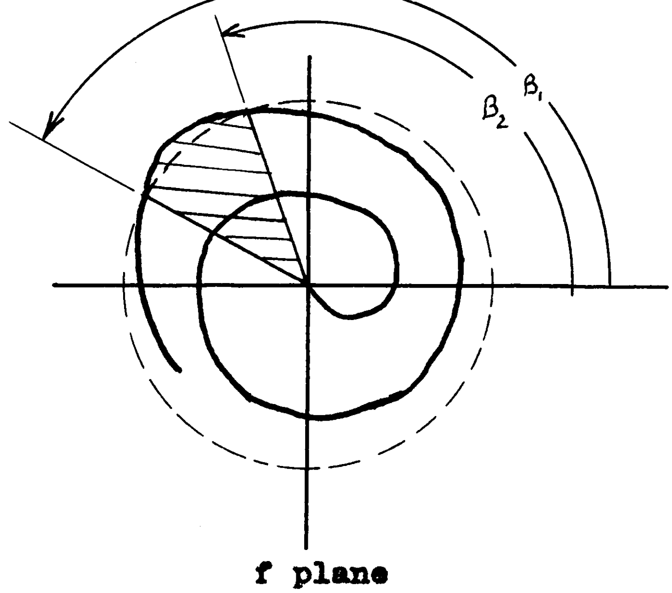


Fig. (4)
A Stable Configuration

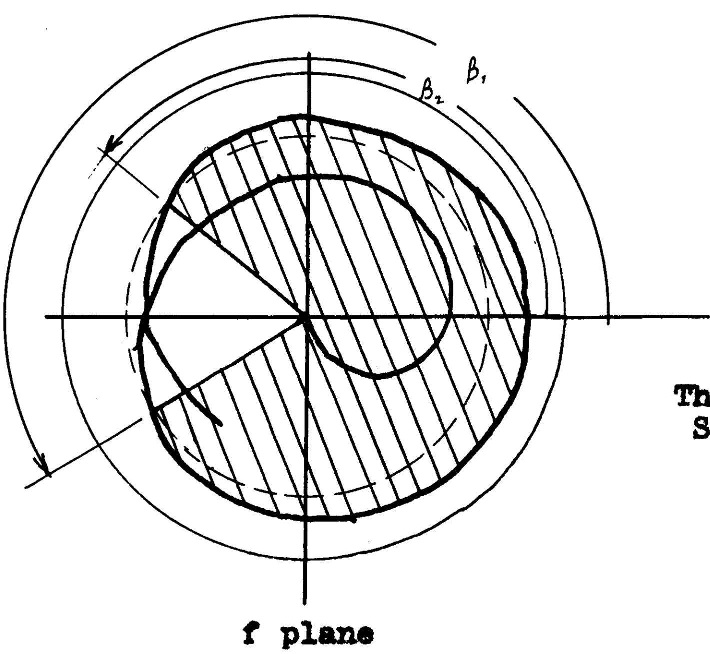


Fig. (5)
The Extreme Case For A
Stable Configuration

VI
STABILITY CRITERION FOR NON-LINEAR EQUATIONS

Although non-linear equations are very important due to their occurence in the physical world, the mathematical theory needed in handling these equations is still imperfect.

To handle problems in which non-linear equations arise, it has been necessary to describe the properties of the equation even though a solution in closed form could not be attained.

Let us consider a second order differential equation:

$$(6.1) \qquad \ddot{x} + \varphi(\dot{x}) + f(x) = 0$$

where the differentiation is with respect to time. This may be rewritten in the form:

$$(6.2) \qquad \frac{d\dot{x}}{dx}\frac{dx}{dt} + \varphi(\dot{x}) + f(x) = 0.$$

By setting $v = \dot{x} = \frac{dx}{dt}$ and solving, we have

$$(6.3) \qquad \frac{dv}{dx} = \frac{-\varphi(\dot{x}) - f(x)}{v}$$

which is a special case of the differential equation,

$$\frac{dv}{dx} = \frac{Q(x,v)}{P(x,v)}. \tag{6.4}$$

Integration of Equation (6.4) yields v as a function of x. If we plot these curves on a plane in which x and v are rectangular coordinates, then the plane is called the phase plane. The curves are called trajectories. A point on the curve is referred to as a system point.

Much work has been done in connection with the phase plane, and an insight of the behavior of non-linear systems has thus been obtained.

Definition: An equilibrium point of Equation (6.4) is a point (x_i, v_i) such that $Q(x_i, v_i) = 0$ and $P(x_i, v_i) = 0$.

In studying linear systems by phase plane analysis, we would see that there is only one equilibrium point. Consider a linear equation of the form:

$$\ddot{x} + m\dot{x} + nx = 0$$

then

$$\frac{dv}{dx} = \frac{-mv - nx}{v}$$

and the only equilibrium point of this system is (0,0). In the analysis of non-linear equations more than one equilibrium point may exist. Since this is the case, we can no longer speak of an equation being stable, but instead must consider the stability of the equation in a neighborhood of each equilibrium point. It turns out that the definition of stability given in Section 1 of this paper is too restrictive in the case of non-linear systems. We must, therefore, introduce new definitions of stability which will serve to describe the system.

Definition: An equilibrium point (x_o, v_o) is said to be asymptotically stable if for each $\varepsilon > 0$, there exists a $\delta > 0$, such that every solution passing through a point (x_1, v_1) within distance δ of (x_o, v_o) at time t_1 stays within an ε distance for $t > t_1$ and approaches (x_o, v_o) as $t \longrightarrow +\infty$.

Definition: The equilibrium point (x_o, v_o) is said to be neutrally stable if it satisfies the conditions for an asymptotically stable equilibrium point except for the requirement that the solutions approach (x_o, v_o) as $t \longrightarrow +\infty$.

The equilibrium points can be considered as the intersections of the curves $Q = 0$ and $P = 0$. In relatively simple cases there are three types of equilibrium points, a focus, a node, and a saddle point. These configurations are given in Fig. (6), (7), (8), respectively. The direction of the arrows in each figure represents the direction of the system point with increasing time. If, as time increases, the system point approaches the equilibrium point, the system is stable. If the system point goes away from the equilibrium point, the system is unstable. In the case of a saddle point, the system is always unstable in any neighborhood of this point.

In all of the above cases the trajectories in the x,v plane were not closed. Closed trajectories occur when periodic oscillations of the system are present. When this phenomenon occurs, we call the resulting curves limit cycles.

Definition: A closed cycle trajectory C is a limit cycle of a system if there exists a non-closed trajectory C_1 such that given $\varepsilon > 0$ there exists a time t_o such that for $t > t_o$ or $t < t_o$ the distance from the system point on C_1 to C is less than ε.

The question of stability of limit cycles depends upon how the non-closed trajectory approaches the limit cycle.

The chart below shows the type of stability we may have.

	$C_1 \rightarrow C$ from inside as $t \rightarrow +\infty$	$C_1 \rightarrow C$ from inside as $t \rightarrow -\infty$
$C_1 \rightarrow C$ from outside as $t \rightarrow +\infty$	stable	semi-stable
$C_1 \rightarrow C$ from outside as $t \rightarrow -\infty$	semi-stable	unstable

An example follows to illustrate the above principles. Let

$$\frac{dv}{dx} = \frac{x + v(x^2 + v^2 - 1)}{-v + x\,(x^2 + v^2 - 1)}.$$

Writing $\frac{dv}{dx}$ in parametric form, we have

$$\frac{dv}{dt} = x + v(x^2 + v^2 - 1)$$

$$\frac{dx}{dt} = -v + x(x^2 + v^2 - 1).$$

Transforming to polar coordinates where $x = r \cos \varphi$ and $v = r \sin \varphi$, we obtain

$$\dot{r} = r(r^2 - 1) \tag{6.5}$$

$$\dot{\varphi} = -1. \tag{6.6}$$

Integrating Equation (6.5), we find

$$r = \frac{1}{\sqrt{1 - Ae^{2t}}}$$

where

$$A = \frac{r_o^2 - 1}{r_o^2}$$

and where r_o is the value of r at $t = 0$.

There are two cases to consider, $r_o < 1$, and $r_o > 1$. If

$r_o < 1$, then

$$r = \frac{1}{\sqrt{1 + |A|\, e^{2t}}}$$

since A is negative. Now as $t \rightarrow \infty$, $r \rightarrow 0$. Thus the origin is a limit cycle. Letting $t \rightarrow -\infty$, we see that $r \rightarrow 1$. Thus there is a limit cycle at $r = 1$.

Now consider the case when $r_o > 1$, then

$$r = \frac{1}{\sqrt{1 - |A|\, e^{2t}}}$$

as $t \rightarrow -\infty$, $r \rightarrow 1$.

From our chart we see that $r = 1$ is an unstable limit cycle since as $t \rightarrow -\infty$ the inside and outside curves both approach the limit cycle. The point 0 is a stable limit cycle. There is no inside or outside approach to this limit cycle.

Phase plane analysis is one of several methods used to study non-linear equations. Some of the other more prominent ones are the method of Van Der Pol, which was modified by Krylov and Bogoliubov; Duffing's method; and the perturbation method due to Poincaré. In the execution of the above methods, many assumptions are made about the parameters of the system. Although many problems may be solved by these methods, the non-linear equations need be rather simple so that the approximate solutions obtained are within certain bounds of the true solutions.

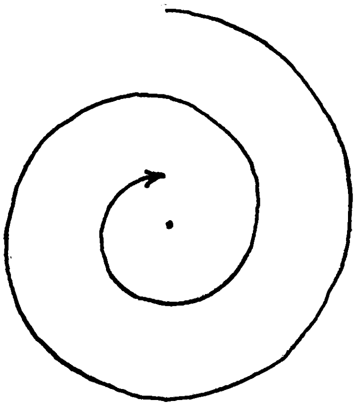


Fig. (6)

Focus- - Stable Configuration

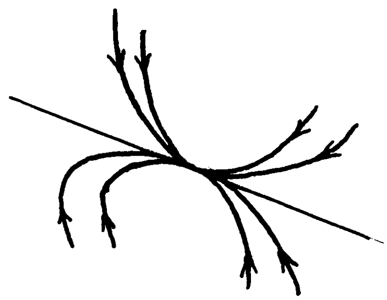


Fig. (7)
Node - Stable Configuration

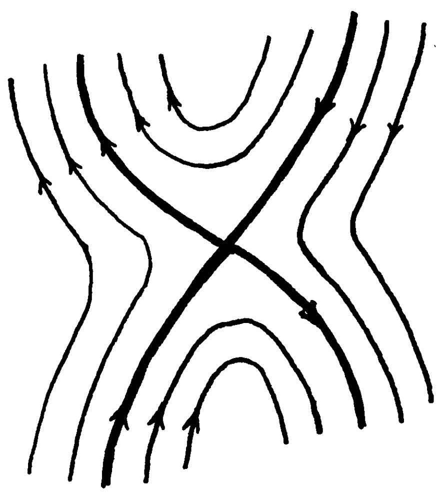


Fig. (8)
Saddle Point
Unstable Configuration

VII
CONCLUSIONS

The methods of continued fraction expansion and the forming of determinant sequences may be used to determine the stability of a system only when the coefficients of the characteristic equation are known. In physical applications the exact coefficients are not usually known, and thus the Nyquist criterion for determining the stability of a system is more useful. We note that the existence of parameters is allowed in the system when the stability is investigated by the Nyquist method.

In the field of non-linear analysis it is obvious that much work is needed. Today many mathematicians are attacking these problems, but as of yet, only certain types of equations have been studied.

VIII
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IX
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ABSTRACT

The problem of determining the stability of a set of linear differential equations has been of interest to mathematicians and engineers for a considerable length of time.

The problem is attacked by obtaining the characteristic equation of the original set of equations and determining the stability of this equation.

The stability of the characteristic equation is first considered in terms of a continued fraction expansion. Necessary and sufficient conditions are given for the characteristic equation to be stable.

The stability of the equation is then determined by means of a determinant sequence, which was the manner originally presented by A. Hurwitz in 1895.

The Nyquist criterion, which is a graphical method for determining whether the equation is stable, is then presented.

An example is given for each of the above methods to illustrate the procedure used in determining whether the equation is stable or unstable. Also included is a brief analysis of stability for non-linear equations.