A PERTURBATION APPROACH TO CONTROL
OF ROTATIONAL/TRANSLATIONAL MANEUVERS
OF FLEXIBLE SPACE VEHICLES

by

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Thesis submitted to the Graduate Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

in

Engineering Mechanics

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April, 1985

Blacksburg, Virginia
OPEN LOOP MANEUVERS OF A FLEXIBLE SPACECRAFT WITH TRANSLATIONAL/ROTATIONAL COUPLING

by

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(ABSTRACT)

An open loop control law is applied to a flexible spacecraft that admits translational, as well as rotational and flexural motion. The translational degrees of freedom are coupled to the rotation and deformation through the active controls applied to the structure. The objective of any maneuver is to control the attitude of the craft as well as to dissipate any vibrations of the structure.

Depending on the type of maneuver specified, the equations of motion may be divided into two distinct optimal control problems. Single-axis rotational maneuvers (with small flexural deformations) constitute a strictly linear problem. The solution of the resulting two point boundary value problem is accomplished through the use of matrix exponential functions. Maneuvers which involve the translational degrees of freedom, are described by nonlinear equations. The solution method presented is an algorithm that generates successive approximations similar to quasi-linearization. A perturbed linear optimal control problem is solved for each approximation. Examples are presented which illustrate the effectiveness of the solution methods for both types of maneuvers.
ACKNOWLEDGEMENTS

The author wishes to express his most sincere appreciation to his advisor, Dr. John L. Junkins, whose thoughtful guidance and patient understanding has helped the author adjust to the rigors of graduate study. Without Dr. Junkins' insight and enthusiasm, graduate school would be more difficult and much less enjoyable.

Thanks also to Dr. L. G. Kraige and Dr. S. L. Hendricks for their help and support and for serving on the graduate committee. Deserving of thanks too, is Dr. M. P. Kamat for bringing to my attention the numerical integration method utilized in this thesis.

The author is also most grateful to his parents for their everpresent and unyielding support.

Finally, the author extends his appreciation to for skillfully typing this thesis, her assistance in preparing many of the figures, and her determined patience with the author throughout this process.
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CHAPTER 1

INTRODUCTION

Active control of large flexible spacecraft has been the subject of much research by various individuals and organizations (Ref. 9-11). The control methods developed from these studies will provide for more efficient and cost effective utilization of space technology. By actively controlling the configuration and attitude of a spacecraft, the craft itself may be made smaller and less massive or allow the incorporation of more "functional capability" per unit mass to be designed into the spacecraft.

In examining the recent literature, one can observe the chronological development of models toward more complicated structures. Perhaps the simplest model and control combination is that examined by Farrenkopf (Ref. 1) in which open loop controls are applied to perform rotational maneuvers of a spacecraft that consists of two rigid appendages that are pinned to a rigid hub with springs providing a restoring force about the pinned joints. Using the same model but a different performance measure, Markley (Ref. 2) also investigated open loop control laws. A more sophisticated model in which the two appendages are flexible beams that are rigidly attached to the hub has also been examined for rotational maneuvers with open loop controls, including investigations into the controllability of the structure and the controller-observer spillover into unmodeled modes (Ref. 3). Both
open loop and feedback controls were considered by Breakwell (Ref. 7), however, only rotational maneuvers were simulated.

One of the most thoroughly studied of all of the "academic spacecraft" is a variation of the model used by Alfriend and Longman (Ref. 3) in which another pair of appendages is included so that all four appendages are coplanar and equally distributed around the rigid hub (Ref. 4-6). Open loop controls have been applied to the linearized equations of motion as well as equations containing certain kinematic nonlinearities. In each case, however, simplifying assumptions were made that not only allowed the equations to be linearized, but also constrained the appendage deformational motions to a specific "class" of linear problems. For example, the appendages were restricted to antisymmetric deformations of equal magnitude, thereby constraining the mass center of the body to remain coincident with the hub center.

More generalized representations for flexible spacecraft have been used to rationalize solution methods, but most simulations are performed with greatly simplified structures. More often than not, the simulations include only the rotational rigid body mode. While some authors admit translational coordinates in the general formulation, they then constrain the motion to the rotational/flexural degrees of freedom before the presentation of specific examples. Others simply ignore the complicating degrees of freedom by assuming the associated motions to be small (Ref. 8).

The purpose of this thesis is to examine the optimal open loop control of a flexible spacecraft without imposing the restrictive con-
strains that preclude translational motion and translational/rotational coupling. The assumption that only small flexural motions and small angular velocities are admissible will be retained in order that the kinematic equations may be linearized. The spacecraft that is simulated in this thesis is a variation of the design used by Turner (Ref. 4), Turner & Junkins (Ref. 5), and Chun (Ref. 6).

Chapter 2 develops the equations of motion of the spacecraft. First, a cantilever (hub—fixed) version of the appendage equation of motion is found by considering each appendage to be a thin prismatic beam and solving the corresponding partial differential equation with the solution given as an infinite series. Upon finding the hub—fixed eigenfunctions, these are then used as admissible functions to model the appendage deformation in the presence of general hub motion. Lagrange's equations are then used to determine the spacecraft equations of motion. Both linear and nonlinear equations are developed, and the degree of the nonlinearities is discussed. The particular class of geometric nonlinearities arise due to translational/rotational/vibrational coupling.

In Chapter 3, the linear optimal control problem is formulated using a modal coordinate transformation and modal state space representation. The resulting two point boundary value problem (TPBVP) is solved using matrix exponentials (Ref. 12). Although this thesis deals with a specific spacecraft model, the solution method presented in Chapter 3 is applicable to any second order, linear, constant coefficient ordinary differential equations. Specific examples demonstrating the
effectiveness of the reorientation and vibration control of the spacecraft are presented.

Chapter 4 presents a general approach to solving nonlinear optimal control problems. The solution method parallels the approach used in Chapter 3 but requires additional numerical techniques. One of the numerical methods used, although generally applicable to many problems, is modified to produce a specialized method for solving the specific problem at hand. The results of numerical simulations of the nonlinear optimal control problem are included in the chapter.

Chapter 5 presents a summary and conclusion of the work contained in this thesis along with recommendations for further study in this area.
CHAPTER 2

EQUATIONS OF MOTION

2.1 Introduction

To derive the equations of motion of the flexible spacecraft, we first consider the motion of an individual appendage modeled as a thin beam and including a tip mass with non-zero mass moment of inertia. The partial differential equation describing the transverse vibration of a thin prismatic beam is solved by the separation of variables technique. This produces an infinite family of eigenfunctions. The hub-fixed eigenfunctions are used as admissible functions to discretize the equations of motion for the assembled structure (i.e. hub plus four appendages).

By truncating the series of admissible functions to a finite number of terms, expressions for the kinetic and potential energy may be found; the generalized coordinates are the time varying coefficients of the admissible functions. Using a Lagrangian approach, the equations of motion of the spacecraft are then derived. The form of the spacecraft equations of motion are such that for some maneuvers, the equations are linear while for other maneuvers, the equations have nonlinear terms.

2.2 Transverse Vibration of a Thin Prismatic Beam

Consider a thin prismatic beam with one end fixed and a rigid mass attached to the free end (see Fig. 1) and let the mass center of the tip
mass be coincident with the end of the beam. Furthermore, consider the
deformation of the beam to be small (linear motion) and perpendicular to
the major axis of the beam (neglecting any shortening of the distance
between points on the beam, due to the deformations). The deflection of
a point on the beam may then be described by a mathematical function of
one position coordinate \(x\) and time \(t\). Let this function be

\[ y = y(x,t) \]

The partial differential equation describing the motion of such a system
can be shown to be (Ref. 13)

\[ EI \frac{\partial^4 y(x,t)}{\partial x^4} + \rho A \frac{\partial^2 y(x,t)}{\partial t^2} = 0 \quad (2-1) \]

with boundary conditions

\[ y(x,t) \big|_{x=0} = 0 \quad (2-2a) \]

\[ \frac{\partial y(x,t)}{\partial x} \bigg|_{x=0} = 0 \quad (2-2b) \]

\[ \Sigma \text{ moments: } -\, EI \frac{\partial^2 y(x,t)}{\partial x^2} \bigg|_{x=L} = I_T \alpha(x,t) \bigg|_{x=L} \quad (2-2c) \]

\[ \Sigma \text{ forces: } EI \frac{\partial^3 y(x,t)}{\partial x^3} \bigg|_{x=L} = M_T \alpha(x,t) \bigg|_{x=L} \quad (2-2d) \]

where

\[ \alpha(x,t) = \text{angular acceleration} = \frac{d^2}{dt^2} \left( \frac{\partial y(x,t)}{\partial x} \right) \quad (2-2e) \]
\[ a(x,t) = \text{linear acceleration} = \frac{d^2y(x,t)}{dt^2} \] (2-2f)

\begin{align*}
\text{EI} & \quad \text{is the bending stiffness of the beam} \\
\rho A & \quad \text{is the mass per unit length of the beam} \\
I_T & \quad \text{is the mass moment of inertia about a transverse axis through the mass center of the tip mass} \\
M_T & \quad \text{is the mass of the tip mass} \\
L & \quad \text{is the length of the beam}
\end{align*}

Using the method of separation of variables, the function \( y(x,t) \) can be written as an infinite series of the form

\[ y(x,t) = \sum_{n=1}^{\infty} \xi_n(t) \psi_n(x) \] (2-3)

where

\[ \psi_n(x) \text{ is an as yet undetermined function} \]
\[ \xi_n(t) \text{ is an always finite function of time.} \]

For free-vibration, we look for harmonic motion with frequency \( \omega_n \)

\[ \xi_n(t) = k_{1n} \sin \omega_n t + k_{2n} \cos \omega_n t \]

\[ \gamma_n = \frac{\rho A \omega_n^2}{EI} \] (2-4)

and substituting Eqs. (2-3) and (2-4) into Eq. (2-1), the motion of the beam for the nontrivial case \( (\xi_n \neq 0) \) is
\[
\frac{a^4 \psi_n(x)}{a^4} - \gamma_n^4 \psi_n(x) = 0 \quad (2-5)
\]

The solution of Eq. (2-5) can be shown to be the eigenfunctions

\[
\psi_n(x) = C_{1n} \sin \gamma_n x + C_{2n} \cos \gamma_n x + C_{3n} \sinh \gamma_n x + C_{4n} \cosh \gamma_n x \quad (2-6)
\]

The boundary conditions are found by substituting Eqs. (2-3) and (2-4) into Eq. (2-2) to produce

\[
\psi_n(0) = 0 \quad (2-7a)
\]

\[
\psi_n'(0) = 0 \quad (2-7b)
\]

\[
\psi_n''(L) = \frac{\gamma_n I_A \psi_n'(L)}{\rho A} \quad (2-7c)
\]

\[
\psi_n'''(L) = \frac{-\gamma_n M_A \psi_n(L)}{\rho A} \quad (2-7d)
\]

where

\[
(\cdot)' = \frac{d}{dx} (\cdot)
\]

Surveying the solution and boundary conditions, it is clear that there are five unknowns (\(C_{1n}, C_{2n}, C_{3n}, C_{4n}, \gamma_n\)) and only four boundary conditions (equations). Therefore, only four of the unknowns may be determined as functions of the fifth parameter and, as usual, an
arbitrary scaling is present in the system eigenfunctions. On applying Eq. (2-7) to the solution, Eq. (2-6), the space dependent function, \( \psi_n(x) \), may be shown to be (see Appendix A)

\[
K \psi_n(x) = B_n (\sin \gamma_n x - \sinh \gamma_n x) + \cosh \gamma_n x - \cos \gamma_n x \tag{2-8}
\]

where

\[
B_n = \frac{\cos \gamma_n L + \cosh \gamma_n L - \frac{\gamma_n^3 I_T}{A} (\sin \gamma_n L + \sinh \gamma_n L)}{\sin \gamma_n L + \sinh \gamma_n L + \frac{\gamma_n^3 I_T}{A} (\cos \gamma_n L - \cosh \gamma_n L)} \tag{2-9}
\]

\( K \) is an arbitrary constant

and \( \gamma_n \) is the solution of the transcendental equation

\[
\cos \gamma_n L \cosh \gamma_n L + 1 + \frac{\gamma_n M_T}{A} (\cos \gamma_n L \sinh \gamma_n L - \sin \gamma_n L \cosh \gamma_n L) - \frac{\gamma_n^3 I_T}{A} (\cos \gamma_n L \sinh \gamma_n L + \sin \gamma_n L \cosh \gamma_n L)
- \frac{\gamma_n^4 I_T M_T}{(A)^2} (\cos \gamma_n L \cosh \gamma_n L - 1) = 0 \tag{2-10}
\]

By letting the arbitrary constant \( (K) \) in Eq. (2-8) be combined with the as yet undetermined time-varying coordinates, the equation of motion for a single appendage is
\[ y(x,t) = \sum_{n=1}^{\infty} q_n(t) \psi_n(x) \]  

(2-11)

where

- \( q_n(t) \) is a set of undetermined coordinates
- \( \psi_n(x) = B_n (\sin \gamma_n x - \sinh \gamma_n x) + \cosh \gamma_n x - \cos \gamma_n x \)
- \( B_n \) is given by Eq. (2-9)
- \( \gamma_n \) is given by Eq. (2-10)

### 2.3 Lagrange's Equations

The spacecraft model examined in this thesis is composed of a rigid hub about which four coplanar appendages are equally spaced (see Fig. 2). Each appendage, as considered in Sec. 2.2, is a beam of constant cross-sectional area, rigidly supported by the hub with a mass attached to the free end. The motion of the spacecraft is unrestricted in two dimensions such that each appendage may deform independently of all others, and the body may translate and/or rotate about its mass center. The control inputs that may be imparted to the structure are of a specific type and location. Torque controllers are located at the hub center (\( u_0 \)) and the midspan of each appendage (\( u_i, i = 1,2,3,4 \)). Two translational controllers, \( F_x \) and \( F_y \), are located at the hub center and oriented such that \( F_x \) is aligned with the undeformed axis of appendage 1 and \( F_y \) is aligned with the undeformed axis of appendage 2.

Equation (2-11) is then used as a set of admissible functions to describe the motion of each appendage. Since such an infinite order
representation produces an unmanageable situation computationally, the series is truncated to N terms. Therefore, the motion of the spacecraft may be described, at least approximately, by the finite set of coordinates:

\[ x_C \] mass center translation in the inertially fixed "x direction"

\[ y_C \] mass center translation in the inertially fixed "y direction"

\[ \theta \] rotation of the body about the mass center

\[ q_{ij} \] the motion of the \( i^{th} \) appendage where \( i=1,2,3,4 \) and \( j=1,2,3,\ldots,N \)

Of course the "substructure" eigenfunctions \( \psi_n(x) \) will not generally be eigenfunctions of the assembled structure, due to coupling with a generally moving hub. However, in this case, a subset of the assembled structure's eigenfunctions (modes) involve zero hub motion and therefore, we will find a subset of the system eigenfunctions are identically the substructure eigenfunctions. Assuming that the deformation of the appendages is small and that the rate of rotation of the spacecraft is small, the kinetic and potential energy of the spacecraft may be given by (see Appendix B)

\[
T = \frac{1}{2} Mx_C^2 + \frac{1}{2} My_C^2 + \frac{1}{2} J\dot{\theta}^2
\]
\[ + \delta \sum_{i=1}^{4} \sum_{j=1}^{N} \left( \rho \tilde{A}_j + M_T (r + L) \tilde{B}_j + I_T \tilde{F}_j \right) \dot{q}_{ij} \]

\[ + \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{N} \sum_{k=1}^{N} \left( \rho \tilde{E}_{jk} + M_T \tilde{B}_j \tilde{B}_k + I_T \tilde{F}_j \tilde{F}_k - \rho \tilde{C}_j \tilde{C}_k \right) \dot{q}_{ij} \dot{q}_{ik} \]

\[ + M \sum_{j=1}^{N} \sum_{k=1}^{N} \tilde{C}_j \tilde{C}_k \left( \dot{q}_{4j} \dot{q}_{2k} + \dot{q}_{1j} \dot{q}_{3k} \right) \tag{2-12} \]

\[ V = \frac{1}{2} EI \sum_{i=1}^{4} \sum_{j=1}^{N} \sum_{k=1}^{N} \tilde{G}_{jk} \tilde{q}_{ij} \tilde{q}_{ik} \tag{2-13} \]

where

\[ \tilde{B}_j = \psi_j(L_1) \]

\[ \tilde{C}_j = - \frac{1}{M} \left( M_T \tilde{B}_j + \rho A \int_0^{L_1} \psi_j(x) \, dx \right) \]

\[ \tilde{D}_j = \int_0^{L_1} (r + x) \psi_j(x) \, dx \]

\[ \tilde{E}_{jk} = \int_0^{L_1} \psi_j(x) \psi_k(x) \, dx \]

\[ \tilde{F}_j = \psi_j'(L_1) \]
\[ G_{jk} = \int_0^L \psi_j''(x) \psi_k''(x) \, dx \]

\[ (') = \frac{d}{dt} ( \ ) \]

\[ (\ )' = \frac{d}{dx} ( \ ) \]

\( M \) is the total mass of the spacecraft

\( J \) is the moment of inertia of the spacecraft about the center of the hub

\( L_i \) is the length of the \( i \)th appendage

Please refer to Appendix C for the closed form expressions of the coefficients \( B_j \) through \( G_{jk} \).

To derive the equations of motion, a Lagrangian approach is used; Lagrange's Equations are [Ref. 21]

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial q_{i}} \right) - \frac{\partial T}{\partial \dot{q}_{i}} + \frac{\partial V}{\partial q_{i}} = Q_{i} \quad i=1,2,3,\ldots,m \tag{2-14a}
\]

where

\[
Z = [x_c, y_c, \theta; q_{11} \ldots q_{1N}; q_{21} \ldots q_{2N}; q_{31} \ldots q_{3N}; q_{41} \ldots q_{4N}]^T \tag{2-14b}
\]

\( m=3 + 4N \)

\( Q_i \) is the \( i \)th generalized force to be derived later in this chapter.
Substituting Eqs. (2-12) and (2-13) into Eq. (2-14a) and computing the appropriate derivatives yields the translational equations of motion:

\[ \ddot{M}_{xc} = Q_1 \]  
\[ \ddot{M}_{yc} = Q_2 \]  

The rotational equation of motion:

\[ J\dot{\omega} + \sum_{i=1}^{4} \sum_{j=1}^{N} [M_{eq}]_{ij} \ddot{q}_{ij} = Q_3 \]  

and the flexural equations of motion of each of four appendages:

\[ \ddot{\theta}[M_{eq}]_{k} + \sum_{j=1}^{N} [M_{qq}]_{jk} \ddot{q}_{1j} + \sum_{j=1}^{N} [M_{qq}]_{jk} \ddot{q}_{3j} + \sum_{j=1}^{N} [K_{qq}]_{jk} q_{1j} = Q_{k+3} \quad k=1,2,3,\ldots,N \]  

\[ \ddot{\theta}[M_{eq}]_{k} + \sum_{j=1}^{N} [M_{qq}]_{jk} \ddot{q}_{2j} + \sum_{j=1}^{N} [M_{qq}]_{jk} \ddot{q}_{4j} + \sum_{j=1}^{N} [K_{qq}]_{jk} q_{2j} = Q_{k+3+N} \quad k=1,2,3,\ldots,N \]  

\[ \ddot{\theta}[M_{eq}]_{k} + \sum_{j=1}^{N} [M_{qq}]_{jk} \ddot{q}_{3j} + \sum_{j=1}^{N} [M_{qq}]_{jk} \ddot{q}_{1j} + \sum_{j=1}^{N} [K_{qq}]_{jk} q_{3j} = Q_{k+3+2N} \quad k=1,2,3,\ldots,N \]
\[
\ddot{\mathbf{M}_{eq}}_k + \sum_{j=1}^{N} \mathbf{M}_{qq}^T_{jk} \ddot{\mathbf{q}}_{4j} + \sum_{j=1}^{N} \mathbf{M}_{qq}^T_{jk} \ddot{\mathbf{q}}_{2j} + \sum_{j=1}^{N} \mathbf{K}_{qq}^T_{jk} \mathbf{q}_{4j} = \mathbf{0}_{k+3+3N} \quad k=1,2,3,...,N
\] (2-15g)

where the elements of the mass and stiffness matrices are:

\[
\mathbf{M}_{eq}^T_k = \rho \mathbf{A} \mathbf{D}_k + \mathbf{M}_T (r + L_i) \mathbf{B}_k + \mathbf{I}_T \mathbf{F}_k
\]

\[
\mathbf{M}_{qq}^T_{jk} = \rho \mathbf{A} \mathbf{E}_{jk} + \mathbf{M}_T \mathbf{B}_j \mathbf{B}_k + \mathbf{I}_T \mathbf{F}_j \mathbf{F}_k - \mathbf{M}_C \mathbf{C}_k
\]

\[
\mathbf{M}_{qq}^T_{jk} = \mathbf{M}_C \mathbf{C}_k
\]

\[
\mathbf{K}_{qq}^T_{jk} = \mathbf{E} \mathbf{I} \mathbf{G}_{jk}
\]

To complete the governing equations, the generalized forces must be calculated. Let the vector, \( \mathbf{R}_p \), locate the position of the hub center relative to the inertial reference frame \( \{\mathbf{N}\} \), and the vector, \( \mathbf{R}_i \), locate the position of a mass element of the \( i \)th appendage relative to the hub center in the body fixed reference frame \( \{\mathbf{b}\} \) (see Fig. 3). The body-fixed unit vectors project onto the inertial unit vectors as

\[
\hat{b}_1 = \cos \theta \hat{N}_1 + \sin \theta \hat{N}_2
\]

\[
\hat{b}_2 = -\sin \theta \hat{N}_1 + \cos \theta \hat{N}_2
\]
\[ \hat{b}_3 = \hat{N}_3 \] 

(2-16)

It can be shown (Ref. 14) that the generalized forces associated with such a configuration are given by

\[
Q_k = F \cdot (\frac{\partial R}{\partial z_k})_N + L_p \cdot \frac{\partial \omega}{\partial \dot{z}_k} + \frac{\partial \omega}{\partial \ddot{z}_k} \cdot \sum_{i=1}^{4} \int L_i \frac{\partial \gamma_i(x,t)}{\partial z_k} \cdot df \]

\[ + \sum_{i=1}^{4} \int L_i \frac{\partial \gamma_i(x,t)}{\partial \dot{z}_k} \cdot df \quad k=1,2,3,...,m \] 

(2-17)

where

\[ \omega = \dot{\theta} \hat{N}_3 \]

\[ F \] is the resultant force applied at point \( p \)

\[ L_p \] is the resultant torque applied at point \( p \)

\[ df \] is the differential force applied to a differential mass

\[ L_i \] indicates that the integration is performed over the appendage length

\( (\cdot)_N \) denotes differentiation with respect to the inertial frame

\( (\cdot)^{\circ} = \frac{d}{dt} (\cdot) \) with respect to the body fixed frame

Defining the vector quantities given in Eq. (2-17), the vectors \( F \) and \( L_p \) follow, from the description of the spacecraft and its control actuators, as
The vector, \( \mathbf{R}_p \), requires a more in depth analysis, but the vector may be shown to be (see Appendix B)

\[
\mathbf{R}_p = x_c \hat{\mathbf{N}}_1 + y_c \hat{\mathbf{N}}_2 + \sum_{i=1}^{N} \tilde{C}_i (q_{4i} - q_{2i}) \hat{\mathbf{b}}_1 \\
+ \sum_{i=1}^{N} \tilde{C}_i (q_{1i} - q_{3i}) \hat{\mathbf{b}}_2
\]  

(2-19)

The translational coordinates \((x_c, y_c)\) are given in the inertial reference frame. As we shall see, this coordinate system leads to nonlinear translational/rotational coupling terms. It should be noted that if the translational coordinates were given in the body fixed coordinate system, the nonlinear terms would disappear, resulting in linear coupling terms. The nonlinear system is presented in this thesis to demonstrate the effectiveness of the perturbation approach to a spacecraft of the "order" typically encountered in real optimal control problems. We represent the control torques applied to an appendage as a force couple of "strength" \(f\) and separation distance \(2\Delta x\), applied about the point \(x_f\) (see Fig. 4). Therefore, the applied force may be written as

\[
d\mathbf{f} = f \delta(x - x_f - \Delta x) - f \delta(x - x_f + \Delta x)
\]  

(2-20)

where \(\delta\) is the Dirac Delta function. Although the direction of the
force, $f$, is immaterial, if it is assumed that the force pair is always perpendicular to the undeformed axis of the appendage, then the cross product indicated in the third term of Eq. (2-17) will always be zero. Similarly, the dot product indicated in the last term will always be equal to the product of the "signed magnitudes" of the two vectors. Recalling the definitions of $y_i(x,t)$ and $z$, it is clear that

$$y_i(x,t) = \sum_{j=1}^{N} \dot{q}_{ij}(t) \psi_j(x)$$

and

$$\frac{\partial y_i(x,t)}{\partial z_k} = \begin{cases} 0 & k = 1, 2, 3 \\ \psi_j(x) & k = 3+j, 3+j+N, 3+j+2N, 3+j+3N \end{cases} \quad (2-21)$$

where $j = 1, 2, 3, ..., N$.

By substituting Eqs. (2-16) and (2-18) through (2-21) into Eq. (2-17), and performing the differentiation required, the generalized forces are

$$Q_1 = F_x \cos \theta - F_y \sin \theta \quad (2-22a)$$

$$Q_2 = F_x \sin \theta + F_y \cos \theta \quad (2-22b)$$

$$Q_3 = -F_x \sum_{i=1}^{N} C_i (q_{1i} - q_{3i}) + F_y \sum_{i=1}^{N} C_i (q_{4i} - q_{2i}) + \sum_{i=0}^{4} u_i \quad (2-22c)$$
\[ Q_{k+3} = C_k F_y + [f_1 \psi_k(x_f + \Delta x) - f_1 \psi_k(x_f - \Delta x)] \]  
(2-22d)

\[ Q_{k+3+N} = -C_k F_x + [f_2 \psi_k(x_f + \Delta x) - f_2 \psi_k(x_f - \Delta x)] \]  
(2-22e)

\[ Q_{k+3+2N} = -C_k F_y + [f_3 \psi_k(x_f + \Delta x) - f_3 \psi_k(x_f - \Delta x)] \]  
(2-22f)

\[ Q_{k+3+3N} = C_k F_x + [f_4 \psi_k(x_f + \Delta x) - f_4 \psi_k(x_f - \Delta x)] \]  
(2-22g)

where

\[ k = 1, 2, 3, ..., N \]

\[ f_i \] is the magnitude of the force applied to the \( i \)th appendage.

In Eqs. (2-22d) through (2-22g), the bracketed term may be expanded in a first order Taylor's series about \( \psi_k(x_f) \) to produce

\[ [f_i \psi_k(x_f + \Delta x) - f_i \psi_k(x_f - \Delta x)] = 2Axf \left. \frac{d\psi_k(x)}{dx} \right|_{x=x_f} \]  
\[ i=1,2,3,4 \]  
(2-23)

Equation (2-23) is an approximation of the generalized force due to a force pair applied symmetrically about the point \( x_f \). However, as \( \Delta x \) is allowed to approach zero and \( f \) approach infinity such that the product \( 2Axf \) remains constant and equal to the magnitude of the applied torque, then the expression is exact. Thus, the bracketed expressions in Eqs. (2-22) become

\[ [f_i \psi_k(x_f + \Delta x) - f_i \psi_k(x_f - \Delta x)] = u_i \psi_k'(x_f) \]  
\[ i=1,2,3,4 \]

\[ k=1,2,3, ..., N \]  
(2-24)
Upon substituting Eqs. (2-22) and (2-24) into Eq. (2-15), the completed equations of motion are

\[ M_{x_C} = F_x \cos \theta - F_y \sin \theta \quad (2-25a) \]

\[ M_{y_C} = F_x \sin \theta + F_y \cos \theta \quad (2-25b) \]

\[ J_{\theta} + \sum_{i=1}^{4} \sum_{j=1}^{N} [M_{\theta q}]_j q_{ij} = - F_x \sum_{j=1}^{N} \tilde{c}_j (q_{1j} - q_{3j}) \]

\[ + F_y \sum_{j=1}^{N} \tilde{c}_j (q_{4j} - q_{2j}) + \sum_{j=0}^{4} u_j \quad (2-25c) \]

\[ \ddot{[M_{\theta q}]}_k + \sum_{j=1}^{N} \{ [M_{qq}]_{jk} \ddot{q}_{1j} + [M_{aq}]_{jk} \ddot{q}_{3j} \}
\]

\[ + [K_{qq}]_{jk} q_{1j} = \ddot{c}_k F_y + \ddot{h}_k u_1 \quad (2-25d) \]

\[ \ddot{[M_{\theta q}]}_k + \sum_{j=1}^{N} \{ [M_{qq}]_{jk} \ddot{q}_{2j} + [M_{aq}]_{jk} \ddot{q}_{4j} \}
\]

\[ + [K_{qq}]_{jk} q_{2j} = - \ddot{c}_k F_x + \ddot{h}_k u_2 \quad (2-25e) \]

\[ \ddot{[M_{\theta q}]}_k + \sum_{j=1}^{N} \{ [M_{qq}]_{jk} \ddot{q}_{3j} + [M_{aq}]_{jk} \ddot{q}_{1j} \}
\]

\[ + [K_{qq}]_{jk} q_{3j} = - \ddot{c}_k F_y + \ddot{h}_k u_3 \quad (2-25f) \]
\[ \ddot{\mathbf{M}_{eq}} + \sum_{j=1}^{N} (\mathbf{M}_{qq})_{jk} \ddot{q}_{4j} + (\mathbf{M}_{qq})_{jk} \ddot{q}_{2j} + (\mathbf{K}_{qq})_{jk} q_{4j} = \mathbf{C}_k F_x + \ddot{H}_k u_4 \quad (2-25g) \]

where

\[ \ddot{H}_k = \mathbf{v}_k'(x_f) \]

\[ x_f = \frac{1}{2} L_1 \]

2.4 Linear and Nonlinear Spacecraft Equations of Motion

The equations of motion of the spacecraft, Eq. (2-25), are composed of \( m \) coupled, second order ordinary differential equations (ODE's). Specifically the rigid body equations are nonlinear coupled ODE's, with the nonlinearities appearing in the forcing terms, whereas the flexural equations are linear ODE's. To be even more precise, the nonlinearities involve only the translational controllers, \( F_x \) and \( F_y \).

Since the nonlinearities in the equations of motion involve only the translational controllers, there are, in essence, two possible optimal control problems. When the translational controllers are not used in a specific maneuver, the resulting problem is linear and when the translational controllers are used, the resulting problem is nonlinear. Clearly, single axis, small deformation rotational maneuvers are linear, whereas maneuvers involving translation as well as translation and rotation are nonlinear problems. The solution method
for determining the optimal controls will, therefore, vary depending on the type of maneuver encountered.

For the nonlinear problem, if the assumption is made that only small angles of rotation will be encountered, in addition to the aforementioned assumptions of small angular velocity and flexural motions, then the three nonlinear equations may be approximated as

\[ M_c \ddot{x} = F_x - F_y \theta \quad (2-26a) \]

\[ M_c \ddot{y} = F_x \theta + F_y \quad (2-26b) \]

\[ J_\theta + \sum_{i=1}^{4} \sum_{j=1}^{N} [M_\theta q_j] \ddot{q}_{ij} = -F_x \sum_{j=1}^{N} C_j (q_{ij} - q_{3j}) \]

\[ + F_y \sum_{j=1}^{N} C_j (q_{4j} - q_{2j}) + \sum_{j=0}^{4} u_j \quad (2-26c) \]

where, for small \( \theta \) we make use of \( \cos \theta = 1 \), \( \sin \theta = \theta \). The equations above are still nonlinear in the unknown variables (configuration coordinates and controls), but the nonlinear trigonometric functions are no longer present. Given that the purpose of the optimal control problem is to solve for a particular set of controls, and that the nonlinearities encountered in Eqs. (2-26) are small for small \( \theta \) and \( q_{ij} \), we seek to solve a nonlinear small angle maneuver as a perturbed linear small angle optimal control problem. That is, by representing the solution as a pedestrian or straightforward expansion, the nonlinear problem is reduced to a series of linear equations with each equation containing a known disturbance (i.e. a known forcing function). In
Chapter 4, the solution of the nonlinear optimal control problem is addressed in detail using this perturbation method.

The method overviewed above is, of course, an approximation of the true nonlinear optimal control problem. The degree of the nonlinearities encountered in the equations of motion will determine the effectiveness of such a method. Highly nonlinear systems could not be solved to within acceptable limits with such a process. From Eqs. (2-26a) and (2-26b), the degree of nonlinearity is determined solely by the magnitude of the variable, $\theta$. Obviously, the smaller the magnitude of $\theta$, the more weakly nonlinear the equations become. For Eq. (2-26c), the degree of nonlinearity is determined by the product of the flexural coordinates, $q_{ij}$, and the coefficients, $C_j$. The variables, $q_{ij}$, are small due to the small motion assumption but the magnitude of $C_j$ is configuration dependent. Substituting the structural parameters of the spacecraft (see Table 1) into the expression for $C_j$, the largest magnitude of the set of coefficients is found to be 0.0101 for the present case. Then for small angles and for translational control magnitudes of comparable order as compared to each other and the torque magnitudes, Eqs. (2-26) are weakly nonlinear and the proposed method of solving these equations is found to produce acceptable results.

For the parameters used in this model, it is clear that the coefficients, $C_j$, are small. However, the "behavior" of these coefficients with respect to each parameter must be examined to determine whether the coefficients remain small for a significant variation of the system design parameters. By varying each parameter,
while holding all others constant, the functional dependence of $C_j$ with respect to a single parameter may be demonstrated (see Fig. 5). The coefficients all increase for an increase in the parameters $M_T$, $\rho A$, and $L$, however, the magnitude of the largest coefficient is less than 0.02 even when one of the given parameters is doubled. The coefficients remain relatively constant for variations in the parameter $I_T$, and decrease rapidly as the hub mass, $M_H$, is increased. Thus, the coefficients do indeed remain small for significant variations of the system parameters about the nominal values given in Table 1.

It is interesting to note that had the translation controllers been oriented in such a fashion that $F_x$ was always in the inertial $\hat{N}_1$ direction and $F_y$ oriented in the $\hat{N}_2$ direction, the first two equations of motion would have been linear and the remaining $(m-2)$ equations would have taken on nonlinear terms involving $F_x$ or $F_y$, $C_j$, and $q_{ij}$ in each of the nonlinear terms. The presence of both $q_{ij}$ and $C_j$ in each term would guarantee a very weakly nonlinear system. Therefore, if the translational controllers were aligned with the inertial reference frame, the proposed solution method could be expected to more closely approximate the problem. However, the "more difficult" nonlinear problem, using Eqs. (2-25) is presented in this thesis. Since many other sources of weak nonlinearity may arise in applications; we wished to illustrate one effective way to modify the linear optimal control problem to efficiently account for such effects.
CHAPTER 3

LINEAR OPTIMAL CONTROL PROBLEM

3.1 Introduction

In the preceding chapter, the equations of motion were found to be nonlinear, coupled, second-order ODE's. For single axis rotational maneuvers with small deformation, however, the system reduces to a set of linear, constant coefficient, coupled ODE's. This chapter presents an open loop formulation and solution process for the linear optimal control problem.

To begin the solution process, the linear system is first decoupled using a modal space transformation of the system coordinates. Upon choosing a subset of the modes to be controlled, the optimal control problem is formulated, leading to a set of necessary conditions for determining the optimal trajectories. By using a modal state space representation of the necessary conditions, a linear, constant coefficient, first order two-point boundary value problem (TPBVP) is produced.

The solution of the TPBVP is then accomplished using matrix exponentials in a recursive equation. This process allows for the determination of the states and costates at discrete intervals of time for a given maneuver. Example maneuvers, accomplishing reorientation and vibration control, are presented to demonstrate the procedure.
3.2 Modal Coordinate Transformation

If the translational controllers are not used, then Eqs. (2-25) are a set of linear, coupled, constant coefficient, second order ODE's. The equations may then be put into a simple matrix form given as

\[ \ddot{M}z + Kz = Bu \]  

(3-1)

where

\[ M = \begin{bmatrix}
M & 0 & 0 & 0^T & 0^T & 0^T & 0^T \\
0 & M & 0 & 0^T & 0^T & 0^T & 0^T \\
0 & 0 & J & M_{eq}^T & M_{eq}^T & M_{eq}^T & M_{eq}^T \\
0 & 0 & M_{eq} & M_{qq} & 0 & M_{qq} & 0 \\
0 & 0 & M_{eq} & 0 & M_{qq} & 0 & M_{qq} \\
0 & 0 & M_{eq} & 0 & M_{qq} & 0 & M_{qq}
\end{bmatrix} \quad (mxm) \]

\[ K = \begin{bmatrix}
0 & 0 & 0 & 0^T & 0^T & 0^T & 0^T \\
0 & 0 & 0 & 0^T & 0^T & 0^T & 0^T \\
0 & 0 & 0 & 0^T & 0^T & 0^T & 0^T \\
0 & 0 & 0 & K_{qq}^T & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & K_{qq}^T & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & K_{qq}^T & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & K_{qq}^T
\end{bmatrix} \quad (mxm) \]
\[
B = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 \\
0 & H & 0 & 0 & 0 \\
0 & 0 & H & 0 & 0 \\
0 & 0 & 0 & H & 0 \\
0 & 0 & 0 & 0 & H \\
\end{bmatrix}
\quad \text{(mx5)}
\]

\(Z\) is defined in Eq. (2-14b)

\(u\) = \([u_0, u_1, u_2, u_3, u_4]^T\)

\(H\) = \([H_1, H_2, H_3 \ldots H_N]^T\)

\(M_{0q}\) is an Nx1 vector

\(M_{qq}\) is an NxN matrix

\(K_{qq}\) is an NxN matrix

\(m = 3 + 4N\)

Because the equations are coupled, the solution of the system, as given, is a nontrivial exercise for systems of "higher" order. However, by using a linear transformation of the coordinates, a set of decoupled equations can be produced.

To decouple the equations of motion, the unforced system (\(u = 0\)) is assumed to have a solution in exponential form. Such an assumption leads to the eigenvalue problem
\[(K - \sigma_i M)e_i = 0 \quad i = 1,2,3, \ldots, m \quad (3-2)\]

where

$\sigma_i$ is a scalar eigenvalue

$e_i$ is a constant eigenvector

For the system given in Eq. (3-1), there will be $m$ eigenvalues and eigenvectors. The eigenvalues will not necessarily be distinct; since the matrix, $K$, is positive semidefinite. Each eigenvector may be normalized using the mass matrix as a weight matrix, yielding

\[
e_i^T M e_i = 1 \quad i = 1,2,3, \ldots, m \quad (3-3a)
\]

\[
e_i^T K e_i = \sigma_i \quad i = 1,2,3, \ldots, m \quad (3-3b)
\]

Placing the $m$ eigenvalues and eigenvectors into matrix format produces two $mxm$ matrices given to be

\[
\Lambda = \begin{bmatrix}
\sigma_1 & 0 & & \\
& \sigma_2 & & \\
& & \sigma_3 & \\
0 & & & \sigma_m
\end{bmatrix}
\]

\[
E = [e_1 \ e_2 \ \ldots \ e_n] \quad (3-4)
\]

Substituting the definitions in Eq. (3-4) into Eqs. (3-3) results in

\[
E^T M E = I \quad (3-5a)
\]
where $I$ is the identity matrix. The linear (modal) transformation of the variables is defined to be

$$
\ddot{Z} = E \ddot{n}
$$

Substituting Eqs. (3-6) into Eq. (3-1), multiplying through by $E^T$, and using Eqs. (3-5), produces

$$
\ddot{n} + \Lambda n = D u
$$

where

$$
D = E^T B
$$
or in scalar form

$$
\ddot{n}_i + \sigma_i n_i = D_i u
$$

where

$D_i$ is the $i^{th}$ row of the matrix $D$

The eigenvectors, $e_i$, define the "mode shapes" of the structure with a corresponding frequency of vibration, $\sqrt{\sigma_i}$, where a mode shape is a "natural" vibration configuration. For linearly independent
eigenvectors, each mode shape or configuration is independent and cannot be derived from any combination of the other eigenvectors/mode shapes. Therefore, any possible motion of the spacecraft may be described by a linear combination of the mode shapes. For most actual structures, a significant number of the calculated vibration frequencies and modes are sufficiently inaccurate that they are physically meaningless; typically these are the higher frequency modes. A normal practice is to consider at most the lower half of the calculated modes and frequencies to be valid. For the model presently under consideration, the first eleven mode shapes with the corresponding frequencies are shown in Fig. 6.

3.3 Optimal Control Formulation

An optimal trajectory of the system state vector is determined to be the path which minimizes some measure of the "cost" accrued in pursuing that particular trajectory. For a spacecraft, some obvious choices of measurement might be:

(1) quadratic summation of modal coordinates, thereby assigning a higher cost to trajectories with larger modal deformations

(2) quadratic measures of the control inputs, favoring the smallest controls that are capable of accomplishing the maneuver

(3) measures of control rates and/or "accelerations" thus producing smoother control profiles for each trajectory (Ref. 15).
In the optimal control formulations presented in this thesis, performance measures based upon all of the above are included.

Specifically, the problem at hand is to bring the spacecraft to rest at the desired attitude in an undeformed configuration, from a given set of arbitrary initial conditions (consistent with the constraints) in a fixed interval of time. A certain subset, \( n \), of the modes are chosen to be controlled and all of the remaining \( r \) modes (where \( r = m - n \)) will form a set called the residual modes. The measure of how well any maneuver is accomplished is a composite function of the controlled modal coordinates, controlled modal velocities, controls, control rates, and control "accelerations". These criteria may be most easily accomplished mathematically by producing an augmented state space representation of Eq. (3-7). Let the vector \( \mathbf{z} \) be defined as

\[
\mathbf{z} = [\mathbf{n}_C^T \quad \dot{\mathbf{n}}_C^T \quad \mathbf{u}^T \quad \dot{\mathbf{u}}^T]^T
\]  

(3-9)

where

- \( \mathbf{n}_C \) is the \( nx1 \) vector of the controlled modes
- \( \mathbf{u} \) is the \( ncx1 \) vector of the controls

Using the definition in Eq. (3-7) and the equations of motion from Eq. (3-8) the state space form of the equations of motion is

\[
\mathbf{z} = F\mathbf{z} + G\mathbf{u}
\]  

(3-10)

where
The optimal control problem we seek to solve involves minimizing the performance measure

$$ J = \frac{1}{2} \left[ z^T S z \right]_{t=t_f} + \frac{1}{2} \int_{t_0}^{t_f} (z^T Q z + u^T R u) dt $$

(3-11)

where

- $R$ is a positive definite diagonal matrix
- $Q$ is a positive semidefinite weight matrix (where $Q = 0$ is not excluded)
- $S$ is a positive definite weight matrix
\( t_0 \) is the initial time
\( t_f \) is the final time

and subject to the condition that Eq. (3-10) holds true.

The Hamiltonian, a scalar function of the performance index and system equation, is

\[
H = \frac{1}{2} \left( Q Z^T Z + U^T RU \right) + \lambda^T (F Z + GU) \tag{3-12}
\]

with \( \lambda \) defined to be an \( n_s \times 1 \) vector of Lagrange multipliers. The necessary conditions for determining the optimal controls, called the Pontryagin necessary conditions [Ref. 16] are

\[
\dot{Z} = \frac{\partial H}{\partial \lambda},
\]

\[
\dot{\lambda} = -\frac{\partial H}{\partial Z},
\]

\[
0 = \frac{\partial H}{\partial U} \tag{3-13}
\]

with the boundary conditions

\[
\lambda(t_f) = S Z(t_f) \tag{3-14}
\]

Substituting Eq. (3-12) into Eqs. (3-13) gives

\[
\dot{Z} = F Z + GU
\]

\[
\dot{\lambda} = -Q Z - F^T \lambda
\]

\[
0 = RU + G^T \lambda
\]

\[
\lambda(t_f) = S Z(t_f) \tag{3-15}
\]

By again using a state space representation, Eqs. (3-15) can be put into a single first order matrix equation. Similarly, the residual modes may be integrated into the same equation, thus providing a simple method for
checking control spillover into the residual modes. The completed optimal control formulation is

\[ \dot{X} = AX \]  

(3-16)

where

\[ X = [z^T \Lambda_r^T n_r^T \hat{n}_r^T]^T \]

\[ A = \begin{bmatrix}
F & -GR^{-1}G^T & 0 & 0 

-GR & 0 & 0 & 0 

-Q & 0 & 0 & 0 

0 & 0 & 0 & 0 

0 & 0 & D_r & 0 

0 & 0 & 0 & -\Lambda_r 

0 & 0 & 0 & 0 

0 & 0 & 0 & 0
\end{bmatrix} \]

and

\[ n_r \] is the vector of residual modal coordinates

\[ \Lambda_r \] is the diagonal matrix of the residual eigenvalues

\[ D_r \] is the matrix composed of the r rows of D corresponding to the residual modes.

3.4 Solution of the Two Point Boundary Value Problem

Equation (3-16) is a linear, constant coefficient, first order ODE whose solution is known to be

\[ X(t) = e^{A(t-t_0)} X(t_0) \]  

(3-17)

where the exponential matrix is called a state transition matrix. The boundary conditions associated with Eq. (3-17) are not given, in the
present problem, for all \( X(t_0) \). To be precise, both the initial and final conditions are known for the elements of \( Z \) and \( n_r \), but the only condition known for the elements of \( \lambda \) are given by Eq. (3-14). An equation such as (3-17) with split boundary conditions is defined as a Two-Point-Boundary-Value-Problem (TPBVP). Before the elements of \( X \), which includes the modal coordinates and controls, may be found for an arbitrary time, \( t \), the initial conditions of the costates, \( \lambda \), must be determined. A recursive solution can then be generated that will solve for \( X \) at fixed time intervals within the overall interval from \( t_0 \) to \( t_f \).

### 3.4.1 Solution of the Initial Costates

Substituting the final time into Eq. (3-17) produces

\[
X(t_f) = \phi(t_f, t_0)X(t_0)
\]

where

\[
\phi(t_f, t_0) = e^{A(t_f-t_0)}
\]

Partitioning the vectors and state transition matrix indicated in Eq. (3-18) yields

\[
\begin{bmatrix}
Z(t_f) \\
\lambda(t_f) \\
n_r(t_f) \\
\dot{n}_r(t_f)
\end{bmatrix} =
\begin{bmatrix}
\phi_{ZZ}(t_f, t_0) & \phi_{Z\lambda}(t_f, t_0) & \phi_{Zn_r}(t_f, t_0) & \phi_{Z\dot{n}_r}(t_f, t_0) \\
\phi_{\lambda Z}(t_f, t_0) & \phi_{\lambda\lambda}(t_f, t_0) & \phi_{\lambda n_r}(t_f, t_0) & \phi_{\lambda\dot{n}_r}(t_f, t_0) \\
\phi_{n_r Z}(t_f, t_0) & \phi_{n_r\lambda}(t_f, t_0) & \phi_{n_r n_r}(t_f, t_0) & \phi_{n_r\dot{n}_r}(t_f, t_0) \\
\phi_{\dot{n}_r Z}(t_f, t_0) & \phi_{\dot{n}_r\lambda}(t_f, t_0) & \phi_{\dot{n}_r n_r}(t_f, t_0) & \phi_{\dot{n}_r\dot{n}_r}(t_f, t_0)
\end{bmatrix}
\begin{bmatrix}
Z(t_0) \\
\lambda(t_0) \\
n_r(t_0) \\
\dot{n}_r(t_0)
\end{bmatrix}
\]

Carrying out the matrix products indicated in the first two partitioned
rows of Eq. (3-19) produces an equation relating \( \mathcal{Z}(t_f) \) to the initial conditions, and another relating \( \lambda(t_f) \) to the initial conditions. Substituting these two equations into Eq. (3-14) and combining terms gives

\[
\begin{align*}
\phi_{\lambda \lambda} - S\phi_{Z \lambda} \lambda(t_o) &= [S\phi_{Z Z} - \phi_{\lambda Z}]\mathcal{Z}(t_o) + [S\phi_{Z \eta_r} - \phi_{\lambda \eta_r}]\eta_r(t_o) \\
+ [S\phi_{Z \dot{\eta}_r} - \phi_{\lambda \dot{\eta}_r}]\dot{\eta}_r(t_o)
\end{align*}
\]

(3-20)

where the time interval arguments have been discarded (for brevity) from the partitions of the state transition matrix. Note that the only unknown quantity in Eq. (3-20) is the vector, \( \lambda(t) \), which may be found by solving Eq. (3-20) using a Gauss elimination method. By solving for the initial costates Eq. (3-17) may now be used to find the modal and control trajectories for any time within the interval \( t_0 \leq t \leq t_f \).

3.4.2 Recursive Solution of the State Trajectories

Although Eq. (3-17) provides a method for finding the state vector, \( \mathcal{X}(t) \), for any arbitrary time, \( t \), the equation in this form is of questionable value if many evaluations are to be made. Computing the matrix exponential is a nontrivial exercise and may be prohibitively expensive to produce for numerous data points. Therefore, it is advantageous to develop a recursive equation to produce the state vector at integer multiples of a small time interval, \( \Delta t \). Since any time interval may be translated through a change of variables to an interval where the initial time is zero, no generality is lost by assuming that \( t_0 \) is identically zero. The small time interval is defined to be
\[ \Delta t = \frac{t_f}{N} \]  

where \( N \) is an integer, thus discretizing the maneuver time into an integer number of steps. Then, any discrete time may be defined by

\[ t = (k + 1)\Delta t \quad k = 0, 1, 2, \ldots, N - 1 \]  

Substituting Eq. (3-22) into Eq. (3-17), where \( t_o = 0 \), produces

\[ X[(k + 1)\Delta t] = e^{A(k+1)\Delta t} X(0) \]  

Using the properties of exponentials Eq. (3-23) may be written as

\[ X[(k + 1)\Delta t] = e^{A\Delta t} e^{Ak\Delta t} X(0) \]  

Recognizing that

\[ X(k\Delta t) = e^{Ak\Delta t} X(0) \]  

and substituting Eq. (3-25) into Eq. (3-24) yields the recursive formula

\[ X[(k + 1)\Delta t] = e^{A\Delta t} X(k\Delta t) \]

Therefore, to solve the linear optimal control problem and compute \( N \) values of the state vector, \( X \), requires a matrix exponential to be computed twice; once for \( t = t_f \) to solve for the initial costates, and once for \( t = \Delta t \) for use in the recursive equation.

3.5 Simulation Results

Two numerical examples produced by the solution method generated in this chapter are presented in Sections 3.5.1 and 3.5.2. The structural parameters defining the spacecraft are given in Table 1.
3.5.1 Case 1

The first case to be presented is a simple rest to rest maneuver where the spacecraft is rotated through an angle of 90 degrees. The subset of the modes that are to be controlled are modes 3 through 11 with all five torque controllers active. The performance measure is based solely upon the final states and the control second derivatives with a higher penalty on the amplitudes of the appendage controller accelerations. Therefore, the weight matrices are selected as

\[
Q = 0
\]
\[
R = \begin{bmatrix} 0.1 & 0 \\ 0 & 1 \end{bmatrix}
\]
\[
S = 10^{20}I
\]

The extremely large values in the weight matrix, S, places a severe penalty upon trajectories that produce large deviations from the desired final state. That is, a large element of S demands that the corresponding final state be met as rigidly as possible. The results of this simulation are depicted graphically in Figures 7 and 8. In Fig. 7, the modal trajectories of several of the controlled modes are shown. The rigid body rotational coordinate shows a smooth transition from the initial to the final conditions. Not unexpectedly, modes 7 and 11 are involved to a high degree since the corresponding mode shapes show a significant rotation. All of the other modes are unexcited numerically, to at least eight orders of magnitude. The control trajectories (see Fig. 8) are very smooth, with control rates of zero initially and finally for all five controllers, reducing the chances of exciting the
higher frequency modes through control spillover. Note that the magnitude of the hub torquer, $u_0$, is approximately ten times higher than the magnitude of a single appendage controller, resulting from the choice of the weight matrix $R$. This choice was made since the hub controllers are typically designed to be the primary source of control in a maneuver with the appendage controllers serving to suppress the vibrations of the spacecraft.

3.5.2 Case 2

In the second simulation, the initial configuration is composed of a large angle rotation, again 90 degrees, combined with initial deflections in each appendage (see Fig. 9). That is, at the time the controls are activated, the spacecraft is not only vibrating, but has also rotated through a finite angle. For this particular case, the weight matrices are chosen such that the performance index is determined by the controller second derivatives and the rigid enforcement of the final states. Furthermore, all controller accelerations are presumed to be of "equal value" in the computation of the optimal trajectories. Under these conditions, the weight matrices are

\[
Q = 0 \\
R = I \\
S = 10^{20}[I]
\]

The modal trajectories, generated in the solution of this case, are presented in Figs. 10 and 11. Examining the results shows that all of the controlled modes are nominally excited to a high degree, as are the
mass center motions as viewed from the rigid hub relative to the body fixed axes. The rotational coordinate, mode 3, shows a smooth transition to the desired final state in a manner reminiscent of the results of Case 1. The primary flexural coordinates, modes 4-7, demonstrate, with the exception of mode 7, a very steady damping of the vibrations throughout the course of the maneuver. The secondary flexural coordinates, modes 8-11, are initially unexcited, as is mode 7, but show significant excursions as the maneuver progresses through time. In modes 8-10, the high frequency vibrations are clearly evident. The mass center vibrations, as viewed from the hub center, are also shown to be damped out by the active controls applied to the spacecraft.

The control trajectories for Case 2 are quite different from those encountered in Case 1 (see Fig. 12). The rigid body torque is similar to Case 1 with a smooth profile and zero magnitudes and "velocities" at the end points. The appendage controllers, however, show a higher frequency oscillation applied over a generally smooth profile. The controllers must "pump" energy out of the system in such a way that the vibrations are dissipated and the rotational maneuver is accomplished. For Case 2, the appendage controller magnitudes are much larger than the hub torque magnitude due primarily to the strict requirement that the vibrations must be eliminated. In examining the modal coordinate histories, it is clear that the maneuver objectives have been accomplished.
CHAPTER 4

NONLINEAR OPTIMAL CONTROL PROBLEM

4.1 Introduction

The rigid body equations of motion for maneuvers that involve translational motion include nonlinearities in the forcing terms. As discussed in Section 2.4, an approximate solution may be found, for small angles of rotation, by using basic perturbation methods to represent the solution with a straightforward expansion. The augmented state space equation will then degenerate into a series of linear equations with known forcing functions applied to each equation. Each equation is then solved for the optimal control trajectory that corresponds to that "order" of the expansion. The complete approximation of the nonlinear system is obtained by combining the series of linear problems in the straightforward expansion. To determine the effectiveness of the approximation, the nonlinear equations can be integrated numerically, and the final conditions can be compared to the desired final state.

4.2 Optimal Control Formulation

Since the optimal control formulation is similar to the method developed in Chapter 3, only the new definitions and a few key equations, included for continuity, are presented in this section.

The equations of motion can be separated into a linear part and a nonlinear term such as
\[ Mz + Kz = Bu + d \]  \hspace{1cm} (4-1)

where

\( M \) is defined in Eq. (3-1)

\( K \) is defined in Eq. (3-1)

\( z \) is defined in Eq. (2-14b)

\[
B = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 \\
0 & c & 0 & H & 0 & 0 \\
-\bar{c} & 0 & 0 & 0 & H & 0 \\
0 & -\bar{c} & 0 & 0 & 0 & H \\
c & 0 & 0 & 0 & 0 & H
\end{bmatrix}
\]

\[
u = [F_x \ F_y \ u_0 \ u_1 \ u_2 \ u_3 \ u_4]^T
\]

\[
d = \begin{cases}
-F_y \theta \\
F_x \theta \\
-N \sum_{j=1}^{N} C_j(q_{1j} - q_{3j}) + F_y \sum_{j=1}^{N} C_j(q_{4j} - q_{2j}) \\
0 \\
\vdots \\
0
\end{cases}
\]

Because the mass and stiffness matrices are identical to the corresponding matrices in Eq. (3-1), the left hand side of Eq. (4-1) may be uncoupled by the same set of eigenvectors and eigenvalues to give

\[
\ddot{\eta} + \Lambda \eta = Du + E^Td
\]  \hspace{1cm} (4-2)
where \( D = ETB \) and the vector, \( u \), have a different numerical composition from the definitions in Chapter 3.

The state space representation of the optimal control problem is established by defining the state vector, \( \dot{z} \), to include the controlled modes, the controlled mode derivatives, the controls, and the control derivatives. The equations of motion may then be written as

\[
\dot{z} = Fz + Gu + \tilde{d}
\]  

(4-3)

where \( \tilde{d} = \begin{bmatrix} 0^T \\ (ET_d)^T \\ 0^T \\ 0^T \end{bmatrix} \) and the other terms retain the same definitions as before. Using the performance index given by Eq. (3-11), the Euler-Lagrange equations are

\[
\begin{align*}
\dot{z} &= Fz + Gu + \tilde{d} \\
\dot{\lambda} &= -Qz - F^T \lambda - \left[ -\frac{\partial}{\partial z} (\tilde{d}) \right]^T \lambda \\
0 &= RU + G^T \lambda \\
\lambda(t_f) &= SZ(t_f)
\end{align*}
\]  

(4-4)

Following the same procedure as before, the augmented state vector, \( x \), is defined to be

\[
x = [z^T, \lambda^T]^T \quad (2n x 1)
\]

Equations (4-4) may then be combined yielding

\[
\dot{x} = Ax + \{NLT\}
\]  

(4-5)

where
\[ A = \begin{bmatrix} F & -GR^{-1}G^T \\ -Q & -F^T \end{bmatrix} \quad (2n \times 2n) \]

\[ \{NLT\} = \left\{ \frac{\partial}{\partial X} \right\} \quad (2n \times 1) \]

We note that the vector \{NLT\} has a specific composition that results in each element of the vector being a quadratic term in the elements of the augmented state vector \(X\). This particular form will be exploited in the straightforward expansion.

To proceed at this point with the perturbation method, we let the solution to Eq. (4-5) be represented by

\[ X = \varepsilon X_1 + \varepsilon^2 X_2 + \varepsilon^3 X_3 + \ldots \quad (4-6) \]

where \(\varepsilon\) is a small parameter that is of order comparable to the amplitude of the motion. Since we have already restricted our maneuvers to small motion, the representation should be accurate and its accuracy will improve as the amplitudes decrease. Substituting Eq. (4-6) into Eq. (4-5) produces

\[ \dot{X}_1 + \varepsilon^2 \ddot{X}_2 + \varepsilon^3 \dddot{X}_3 + O(\varepsilon^4) = A[\varepsilon X_1 + \varepsilon^2 X_2 + \varepsilon^3 X_3 + O(\varepsilon^4)] \]

\[ + \{NLT[\varepsilon X_1 + \varepsilon^2 X_2 + \varepsilon^3 X_3 + O(\varepsilon^4)]\} \quad (4-7) \]

Taking advantage of the quadratic nature of the elements of \{NLT\}, Eq. (4-7) may be fully expanded to
\[
\begin{align*}
\varepsilon \dot{X}_1 + \varepsilon^2 \dot{X}_2 + \varepsilon^3 \dot{X}_3 + O(\varepsilon^4) &= \varepsilon AX_1 + \varepsilon^2 AX_2 + \varepsilon^3 AX_3 + O(\varepsilon^4) \\
&+ \varepsilon^2 \{\text{NLT}(X_1)\} + \varepsilon^3 \{\text{NLT}(X_1, X_2)\} + O(\varepsilon^4) \\
&= \varepsilon X_1 + \varepsilon^2 \{\text{NLT}(X_1)\} + \varepsilon^3 \{\text{NLT}(X_1, X_2)\} + O(\varepsilon^4) \quad (4-8)
\end{align*}
\]

Then upon equating like powers of \( \varepsilon \), the system becomes

\[
\begin{align*}
\dot{X}_1 &= \varepsilon AX_1 \quad (4-9a) \\
\dot{X}_2 &= \varepsilon AX_2 + \{\text{NLT}(X_1)\} \quad (4-9b) \\
\dot{X}_3 &= \varepsilon AX_3 + \{\text{NLT}(X_1, X_2)\} \quad (4-9c)
\end{align*}
\]

Note that the expansion may be carried out to any order but for purposes of illustration we will include only three orders in this thesis.

The boundary conditions associated with the vector \( Z_i \), contained in the augmented vector \( X_i \) (\( i = 1, 2, \ldots \)) in Eqs. (4-9), are found by substituting the boundary conditions into Eq. (4-6) and equating "like" terms.

\[
\begin{align*}
Z_1(t_0) &= Z(t_0) \quad Z_1(t_f) = Z(t_f) \quad (4-10a) \\
Z_2(t_0) &= 0 \quad Z_2(t_f) = 0 \quad (4-10b) \\
Z_3(t_0) &= 0 \quad Z_3(t_f) = 0 \quad (4-10c)
\end{align*}
\]

The solution of Eq. (4-9a) is identical to the method used in Chapter 3. Once the trajectories are known, the disturbance in Eq. (4-9b) is then a known function of time represented by a discrete set of data points. One can see that as each of Eqs. (4-9) is solved, the nonlinear terms in the next order will become known. Furthermore, each
of Eqs. (4-9) is a TPBVP with unknown initial and final conditions for the costates.

4.3 Fourier Series Representation of the Disturbance

The procedure outlined previously is to treat the disturbance vector, represented as a set of data points, as a known function of time applied directly to a linear equation of motion. The proposed solution to Eqs. (4-9b) and (4-9c) will require the disturbance to be represented as a continuous harmonic function of time and not discrete data points. An approximation of the function may be generated by using a least squares fit of the data points to a finite Fourier series of the form

\[ f(t) = b_0 + \sum_{i=1}^{r} a_i \sin i\omega_0 t + b_i \cos i\omega_0 t \]  

(4-11)

where \( \omega_0 = \frac{2\pi}{t_f - t_0} \)

Because a finite Fourier series fails to approximate the end points of a piecewise continuous, periodic function to an acceptable degree of accuracy, the disturbance function is reflected about the axis \( t = t_f \) to produce a continuous "periodic" function. Then substituting the discrete data points into the Fourier series produces the equations

\[ d_i(0) = b_{0i} + b_{1i} + \ldots + b_{ri} \]
\[ d_i(\Delta t) = b_{0i} + a_{1i} \sin \omega_o \Delta t + b_{1i} \cos \omega_o \Delta t + \ldots \]
\[ + a_{ri} \sin r \omega_o \Delta t + b_{ri} \cos r \omega_o \Delta t \]
\[ \vdots \]
\[ d_i(k \Delta t) = b_{0i} + a_{1i} \sin \omega_o k \Delta t + b_{1i} \cos \omega_o k \Delta t + \ldots \]
\[ + a_{ri} \sin r \omega_o k \Delta t + b_{ri} \cos r \omega_o k \Delta t \]
\[ d_i[(k + 1)\Delta t] = d_i[(k - 1)\Delta t] \]
\[ \vdots \]
\[ d_i[(2k - 1)\Delta t] = d_i(\Delta t) \tag{4-12} \]

where the subscript \( i \) designates the \( i \)th element of the vector. These equations may then be put into the matrix form

\[ A \mathbf{C}_i = \mathbf{D}_i \tag{4-13} \]

where

\[
A = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 & \ldots & 0 & 1 \\
1 & s(\tau_1) & c(\tau_1) & s(2\tau_1) & c(2\tau_1) & \ldots & s(r\tau_1) & c(r\tau_1) \\
1 & s(\tau_2) & c(\tau_2) & s(2\tau_2) & c(2\tau_2) & \ldots & s(r\tau_2) & c(r\tau_2) \\
1 & \vdots & & & & & \vdots & & \\
1 & s(\tau_k) & c(\tau_k) & s(2\tau_k) & c(2\tau_k) & \ldots & s(r\tau_k) & c(r\tau_k) \\
1 & s(\tau_{k-1}) & c(\tau_{k-1}) & s(2\tau_{k-1}) & c(2\tau_{k-1}) & \ldots & s(r\tau_{k-1}) & c(r\tau_{k-1}) \\
1 & \vdots & & & & & \vdots & & \\
1 & s(\tau_1) & c(\tau_1) & s(2\tau_1) & c(2\tau_1) & \ldots & s(r\tau_1) & c(r\tau_1)
\end{bmatrix}
\]

\[ \mathbf{C}_i = [b_{0i} \ a_{1i} \ b_{1i} \ a_{2i} \ b_{2i} \ \ldots \ a_{ri} \ b_{ri}]^T \]
$$D_i = [d_i(0) d_i(\Delta t) ... d_i(k\Delta t) d_i((k - 1)\Delta t) ... d_i(\Delta t)]^T$$

$$\tau_i = i\omega_0\Delta t$$

$$c(\ ) = \cos(\ )$$

$$s(\ ) = \sin(\ )$$

and

$A$ is a $(2k) \times (2r + 1)$ matrix

$C_i$ is a $2r + 1$ vector

$D_i$ is a $2k$ vector

The unknown vector, $C_i$, may be given a least square approximation by substituting $A$ and $D_i$ into the equation

$$C_i = (A^T A)^{-1} A^T D_i \quad (4-14)$$

This procedure must be carried out for each nonzero element of the disturbance vector in Eqs. (4-9). However, the matrix $A$ is constant, thus reducing Eq. (4-14) to a simple matrix multiplication once the product $(A^T A)^{-1} A^T$ has been calculated. As the number of sample points become large, it can be verified that $A^T A$ becomes diagonal due to the orthogonality of the sine and cosine functions. Alternatively, for a finite number of sample points symmetrically located about $\tau = \pi$, it can be verified that $A^T A$ is diagonal for this case as well. In either case, one can usually avoid the matrix inverse of Eq. (4.14) and calculate the finite Fourier series coefficients efficiently.

The finite Fourier series may also be conveniently given a matrix exponential format of the form

$$d(t) = Pe^{at}P_0 \quad (4-15)$$
where
\[
\mathbf{p}_0 = [1 \ 0 \ 1 \ 0 \ 1 \ \ldots \ 0 \ 1]^T \quad (2r + 1) \times 1
\]

\[
\mathbf{p} = [\mathbf{p}_1 \ \mathbf{p}_2 \ \mathbf{p}_3 \ \ldots \ \mathbf{p}_n]^T \quad n \times (2r + 1)
\]

\[
\mathbf{p}_i = [b_{0_i} \ \omega a_{1i} \ b_{1i} \ 2\omega a_{2i} \ b_{2i} \ \ldots \ r\omega a_{ri} \ b_{ri}]^T
\]

\[
\mathbf{a} = \begin{bmatrix}
0 \\
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\vdots \\
\alpha_r
\end{bmatrix} \quad (2r + 1) \times (2r + 1)
\]

\[
\alpha_j = \begin{bmatrix}
0 & 1 \\
-(j\omega)^2 & 0
\end{bmatrix} \quad (2 \times 2)
\]

Equation (4-15) provides a concise mathematical formula for the disturbance given as a continuous function of time. The utility of such a representation will become abundantly clear in the following section.

4.4 Solution of the Two Point Boundary Value Problem

Each of Eqs. (4-9b) and (4-9c) is a first order, linear, constant coefficient ODE with a known forcing function. The general form of the solution of an equation of this type is

\[
\mathbf{x}(t) = e^{At}\mathbf{x}_0 + \int_0^t e^{-At}\mathbf{d}(\tau) \ d\tau
\]  

(4-16)
where \( x_0 = x(t_0) \) represents the initial conditions. The integral in Eq. (4-16) is not easily evaluated except when the function \( d(t) \) is given in certain specific forms. One of these forms, and the reason for developing Eq. (4-15), is the exponential. Substituting Eq. (4-15) into Eq. (4-16) yields

\[
X_1(t) = e^{At}[x_0 + \int_0^t e^{-A\tau}P e^{A\tau} d\tau]P_0
\]

(4-17)

where the subscript denotes the order of the solution (i.e. \( i = 2, 3, \ldots \)). Note that \( P \) and \( Q \) will be different numerically for each order, but the subscript was not included for clarity. It can be shown (Ref. 18,19) that an integral involving matrix exponential functions as given in Eq. (4-17) may be evaluated by using matrix exponentials. Let the matrix \( Y_1 \) be defined to be

\[
Y_1 = \begin{bmatrix} A & P \\ 0 & Q \end{bmatrix} (2ns + 2r + 1) \times (2ns + 2r + 1)
\]

(4-18)

The matrix exponential of Eq. (4-18) may then be partitioned as

\[
e^{Y_1 t} = \begin{bmatrix} \phi_1(t) & \phi_2(t) \\ 0 & \phi_3(t) \end{bmatrix}
\]

(4-19)

By the definition of a matrix exponential the partitions may be shown (Ref. 19) to be

\[
\phi_1(t) = e^{At}
\]
\[ \phi_2(t) = e^{At} \int_0^t e^{-A\tau} pe^{At} d\tau \]  

Equations (4.19, 4.20) are referred to as the "Van Loan Identities". Then substituting Eqs. (4-20) into Eq. (4-17) reduces the integral to a product of the partitions of the matrix exponential of Eq. (4-19). Therefore the solution of Eqs. (4-9b) and (4-9c) may be written as

\[ x_1(t) = \phi_1(t)x_0 + \phi_2(t)p_0 \]  

4.4.1 Solution of the Initial Costates

The form of the solution of the TPBVP is given by Eq. (4-21). However, the initial costates are at present unknown. Before Eq. (4-21) will be of any practical use the initial costates must be found. Substituting \( t = t_f \) into the equation and writing the result in partitioned form produces

\[ \begin{bmatrix} z(t_f) \\ \Lambda(t_f) \end{bmatrix} = \phi_1(t_f) \begin{bmatrix} z(t_0) \\ \Lambda(t_0) \end{bmatrix} + \phi_2(t_f)p_0 \]  

It will be helpful to partition the matrix \( \phi_1(t_f) \), and rewrite the last term as a partitioned vector, defined as

\[ \phi_1(t_f) = \begin{bmatrix} \phi_{11}^{11}(t_f) & \phi_{11}^{12}(t_f) \\ \phi_{12}^{21}(t_f) & \phi_{12}^{22}(t_f) \end{bmatrix} \]
Then the matrix equation may be separated into the two coupled equations

\[ Z(t_f) = \phi_{11}(t_f)Z(t_o) + \phi_{12}(t_f)\lambda(t_o) + \psi_1(t_f) \]

\[ \lambda(t_f) = \phi_{21}(t_f)Z(t_o) + \phi_{22}(t_f)\lambda(t_o) + \psi_2(t_f) \]  

(4-24)

Recalling the boundary condition, \( \lambda(t_f) = S\lambda(t_f) \) and the conditions from Eqs. (4-10), the two equations may be combined into one, eliminating \( \lambda(t_f) \), leaving only \( \lambda(t_o) \) as the remaining unknown. Then \( \lambda_o = \lambda(t_o) \) is found by using Gauss' elimination to solve

\[ [\phi_{22}(t_f) - S\phi_{12}(t_f)]\lambda_o = S\psi_1 - \psi_2 \]  

(4-25)

Now that all of the initial conditions are known, Eq. (4-21) may be used to calculate the states and controls for any arbitrary time.

4.4.2 Recursive Solution for the State Trajectories

Evaluating Eq. (4-21) for a given time requires the computation of the matrix exponential in Eq.(4-19). The matrix \( Y_1 \) can be of very large order for reasonably simple systems. Therefore, calculating the exponentials for a large number of data points would not only be costly but time consuming as well. A recursive equation that would produce the states at fixed intervals of time through a series of multiplications and additions would circumvent the numerous exponential computations.
To develop the formula, first note that the properties of the exponential function are such that

\[ Y_{1}(k+1)\Delta t = Y_{1}\Delta t Y_{1}k\Delta t \]

or in partitioned form

\[
\begin{bmatrix}
\phi_1[(k + 1)\Delta t] & \phi_2[(k + 1)\Delta t] \\
0 & \phi_3[(k + 1)\Delta t]
\end{bmatrix}
= \begin{bmatrix}
\phi_1(\Delta t) & \phi_2(\Delta t) \\
0 & \phi_3(\Delta t)
\end{bmatrix}
\begin{bmatrix}
\phi_1(k\Delta t) & \phi_2(k\Delta t) \\
0 & \phi_3(k\Delta t)
\end{bmatrix}
\]

(C4-26)

Carrying out the partitioned matrix multiplications gives the three important equations

\[
\begin{align*}
\phi_1[(k + 1)\Delta t] &= \phi_1(\Delta t)\phi_1(k\Delta t) \\
\phi_2[(k + 1)\Delta t] &= \phi_1(\Delta t)\phi_2(k\Delta t) + \phi_2(\Delta t)\phi_3(k\Delta t) \\
\phi_3[(k + 1)\Delta t] &= \phi_3(\Delta t)\phi_3(k\Delta t)
\end{align*}
\]

(C4-27)

where

\[
\begin{align*}
\phi_1(0) &= I \\
\phi_2(0) &= 0 \\
\phi_3(0) &= I
\end{align*}
\]

To calculate the states at an integer multiple of a small time step, \(\Delta t\), substitute \(t = (k + 1)\Delta t\) into Eq. (4-21)

\[
X[(k + 1)\Delta t] = \phi_1[(k + 1)\Delta t]X_0 + \phi_2[(k + 1)\Delta t]P_0
\]

(C4-28)
Upon substituting Eqs. (4-27) into Eq. (4-28), the resulting formula is

\[
X[(k + 1)\Delta t] = \phi_1(\Delta t)\phi_1(k\Delta t)X_0 \\
+ \{\phi_1(\Delta t)\phi_2(k\Delta t) + \phi_2(\Delta t)\phi_3(k\Delta t)\}b_0
\]  

(4-29)

Next, to simplify Eq. (4-29), we define the vectors, \( V_i \), \( i = 1, 2, 3 \), to be

\[
V_1(k\Delta t) = \phi_1(k\Delta t)X_0 \\
V_2(k\Delta t) = \phi_2(k\Delta t)b_0 \\
V_3(k\Delta t) = \phi_3(k\Delta t)b_0
\]

(2r + 1) x 1

(4-30)

Finally, substituting Eq. (4-30) into Eq. (4-29) produces the recurrence equation

\[
X[(k + 1)\Delta t] = V_1((k + 1)\Delta t) + V_2((k + 1)\Delta t)
\]  

(4-31)

where

\[
V_1[(k + 1)\Delta t] = \phi_1(\Delta t)V_1(k\Delta t) \\
V_2[(k + 1)\Delta t] = \phi_1(\Delta t)V_2(k\Delta t) + \phi_2(\Delta t)V_3(k\Delta t) \\
V_3[(k + 1)\Delta t] = \phi_3(\Delta t)V_3(k\Delta t)
\]

and the initial conditions are

\[
V_1(0) = X_0 \\
V_2(0) = 0
\]
Once the matrix exponentials given by Eq. (4-19) are computed for $t = \Delta t$, the repeated application of Eq. (4-31) will produce the states at each step $\Delta t$ in the interval 0 to $t_f$. The results from the state trajectory can then be used to calculate the disturbance in the next higher order.

4.5 Numerical Integration of the Nonlinear Equations of Motion

Once the state trajectories for each of Eqs. (4-9) has been calculated the control trajectories may be isolated and combined using Eq. (4-6). The result of the combination of the different orders is the optimal control trajectory that we seek to perform the required maneuver. To examine the accuracy with which the optimal controls perform the given duty, we numerically integrate the nonlinear equations of motion (in configuration space) using the optimal controls generated by the perturbation method.

The implicit numerical integration method developed in this section is a three-part recurrence relation for second order equations (Ref. 17) and may be applied directly to a constant coefficient equation of the form

$$M\ddot{x} + C\dot{x} + Kx + f = 0$$  \hspace{1cm} (4-32)

The procedure develops a recurrence equation to produce the vector of unknowns for a given time, $t = (j + 1)\Delta t$, based upon the known values of the vector at $t = j\Delta t$ and $t = (j - 1)\Delta t$. Therefore three "nodal" points
are required to generate the recurrence formula. The vector, \( \mathbf{x} \), is
discretized at the three nodes as the sum of a scalar, time dependent
shape function multiplied by the nodal values of \( \mathbf{x} \), given as

\[
\mathbf{x} = r_{j-1}x_{j-1} + r_jx_j + r_{j+1}x_{j+1}
\]  

(4-33)

The shape functions must be at least parabolic since second
derivatives must be found, and are specifically given to be (see Fig.
13)

\[
r_{j+1} = \frac{\varsigma(\varsigma + 1)}{2}
\]

\[
r_j = (1 - \varsigma)(1 + \varsigma)
\]

\[
r_{j-1} = \frac{\varsigma(\varsigma - 1)}{2}
\]  

(4-34)

where

\[
\varsigma = \frac{t}{\Delta t}
\]

Taking the first and second derivatives of the shape functions with
respect to time yields

\[
\dot{r}_{j+1} = \frac{1}{2} + \varsigma
\]

\[
\ddot{r}_{j+1} = \frac{1}{(\Delta t)^2}
\]

\[
\dot{r}_j = -\frac{2\varsigma}{\Delta t}
\]

\[
\ddot{r}_j = -\frac{2}{(\Delta t)^2}
\]

\[
\dot{r}_{j-1} = -\frac{1}{2} + \varsigma
\]

\[
\ddot{r}_{j-1} = \frac{1}{(\Delta t)^2}
\]  

(4-35)

The domain of the expansion is a time interval of \( 2\Delta t \) that extends
from \(-\Delta t\) to \( \Delta t \). By using \( \varsigma \) as a change of variables the domain can be
considered to be the interval from \(-1 \leq \varsigma \leq 1\). Substituting Eqs. (4-
33), (4-34), and (4-35) into Eq. (4-32) and integrating over the domain of \( \zeta \) provides the formula

\[
[M + \mu \Delta t C + \beta \Delta t^2 K] \tilde{x}_{j+1} \\
+ [-2M + (1 - 2\mu) \Delta t C + (\frac{1}{2} - 2\beta + \mu) \Delta t^2 K] \tilde{x}_j \\
+ [M - (1 - \mu) \Delta t C + (\frac{1}{2} + \beta - \mu) \Delta t^2 K] \tilde{x}_{j-1} + \tilde{\epsilon} \Delta t^2 = 0 \quad (4-36)
\]

where

\[
\begin{align*}
\mu &= \frac{\int_{-1}^{1} W_i (\zeta + \frac{1}{2}) d\zeta}{\int_{-1}^{1} W_i d\zeta} \\
\beta &= \frac{\frac{1}{2} \int_{-1}^{1} W_i \zeta (1 + \zeta) d\zeta}{\int_{-1}^{1} W_i d\zeta} \\
\tilde{f} &= \frac{\int_{-1}^{1} W_i f d\zeta}{\int_{-1}^{1} W_i d\zeta} \\
\tilde{\epsilon} &= \frac{1 \int_{-1}^{1} W_i f d\zeta}{\int_{-1}^{1} W_i d\zeta}
\end{align*}
\]

\( W_i \) is a scalar weighting function.

Applying the same expansion to \( \tilde{f} \) as given for \( x \) results in \( \tilde{f} \) defined as

\[
\tilde{f} = \beta \tilde{f}_{j+1} + (\frac{1}{2} - 2\beta + \mu) \tilde{f}_j + (\frac{1}{2} + \beta - \mu) \tilde{f}_{j-1} \quad (4-37)
\]

Using Eqs. (4-36) and (4-37), the vector \( \tilde{x}_{j+1} \) may be found for known values of \( \tilde{x}_j, \tilde{x}_{j-1}, \) and \( \tilde{f} \). This formula is generally applicable to any second order, linear ODE. For generally nonlinear equations, an
iteration would be necessary within each time step. As stated in
Section 2.4, the nonlinear equations of the spacecraft are only
nonlinear in the forcing terms of the three rigid body equations. For a
set of known forcing functions (controls) the nonlinearities take on the
appearance of an additional "stiffness" term. By carefully manipulating
the three nonlinear equations, these pseudo-stiffness terms may be
combined with the stiffness matrix to produce a linear recursive
equation. Therefore instead of iterating a solution in each time step,
the above procedure may be modified to integrate (without iteration) the
particular nonlinear equations for this specific spacecraft model. The
equation that results from this exercise may be shown to be

\[
[M + \Delta t^2(\beta K - T_6)]z_{j+1} - [2M + \Delta t^2(T_5 - (\frac{1}{2} - 2\beta + \mu)K)]z_j
\]

\[
+ [M + \Delta t^2((\frac{1}{2} + \beta - \mu)K - T_4)]z_{j-1}
\]

\[
+ (\frac{1}{2} + \beta - \mu)Bu_{j-1} + (\frac{1}{2} - 2\beta + \mu)Bu_j + \beta Bu_{j+1} = 0
\]  (4-38)

where the derivation of the equation and the definitions of the \(m \times m\)
matrices \(T_4, T_5,\) and \(T_6\) are shown in Appendix D.

Equation (4-38) provides a straightforward formula for finding the
unknown vector \(z_{j+1}\). The corresponding modal coordinates may then be
found by applying the transformation

\[ n = E^TMz \]  (4-39)
4.6 **Simulation Results**

Using the analytical procedure developed in the preceding sections, three numerical examples are presented, illustrating the utility of the method. The first example is a scalar problem with a nonlinear term of the same form as the larger system. This example is used as an introduction to the perturbation method. The remaining two examples are translational/rotational maneuvers of the spacecraft model developed in this thesis.

4.6.1 **Case 3**

To illustrate the perturbation approach without the confusion of a large order system, the first example to be illustrated will be a rest to rest maneuver of a simple scalar system. In order to provide some continuity between this example and the large order model developed in Chapter 2, the scalar equation of motion will have a similar form (including the nonlinear term) given by

\[ \ddot{n} + \Delta n = U + \alpha n U \]

where

\[ n(t_0) = 0.8 \quad \dot{n}(t_0) = 0.0 \quad \alpha = 0.1 \quad \Delta = 0.5 \]

The purpose of the maneuver shall be to drive the system to zero in a time span of two seconds. We shall choose to penalize only the control acceleration and the final states resulting in the weight matrices

\[ Q = 0 \]

\[ R = I \]
\[ S = 10^{20}[I] \]

The presence of the large values in the weight matrix, \( S \), serves to rigidly enforce the final conditions. Following the perturbation procedure developed, the optimal control problem given by Eq. (4-5) is a matrix equation of order 8.

By solving the linearized system (also the first order perturbation expansion), and integrating the nonlinear equation of motion using the optimal control generated by the linearized problem, the final condition is \( n(t_f) = -0.0373 \). However by applying the perturbation approach to third order, and integrating the nonlinear equation of motion using the optimal control produced by this approach, the final condition is \( n(t_f) = -0.000166 \). Clearly the perturbation approach is very effective, producing an improved optimal control that results in a final position error over two orders of magnitude smaller than the error generated by an associated linear optimal control problem.

### 4.6.2 Case 4

This particular case is a rest to rest maneuver of the system described by Eq. (4.1), with initial deflections in the rigid body modes and zero initial conditions for the flexural modes. The spacecraft is to be rotated through an angle of 5° and translated in both the x and y directions a distance of 0.1 ft. Recalling that the analysis is dependent upon the angle being "small", a 5° rotation is certainly a reasonably small angular deflection.
The performance measure is dependent upon the control accelerations and final states, with the final states rigidly enforced. The weight matrices are

\[
Q = 0 \\
R = I \\
S = 10^{20}[I]
\]

The graphical results of this simulation is presented in Figs. 14-16. Although the first seven modes were controlled, mode 4 showed no participation at all whereas mode 5 produced a trajectory similar to mode 6. In Fig. 14, modes 1-5 and 7 are shown. The rigid body modes are smoothly driven towards the final conditions in a manner consistent with the linear model solution. The degree of improvement in the "final conditions" of the translational modes is 21% over the approximate trajectories produced by the linearized model with the expansion carried to third order. That is, by solving the linearized model and integrating the nonlinear equations using the controls from the first order solution results in small but nonzero final conditions for the translational modes. However, by proceeding with the perturbation technique, the final conditions of the perturbed optimal control problem are 21% closer to zero than the final conditions from the linearized solution.

Figure 15 presents the rigid body controls while the appendage controls are shown in Fig. 16. All seven of the controllers exhibit the smooth trajectories with zero magnitude and zero slope at the end points. In addition, the general form of the control trajectories is
typical of what one would expect for an optimal control performing a rest to rest maneuver.

4.6.3 Case 5

The last simulation to be presented is a vibration control problem with initial deflections in the appendages as well as translational and rotational displacements (see Fig. 17). Specifically, the spacecraft must be rotated through 5° and translated through distances of 0.1 ft., in both inertial directions, while dissipating the vibrations. The weight matrices are again

\[ Q = 0 \]
\[ R = I \]
\[ S = 10^{20}[I] \]

The modal trajectories, for the first six modes, are shown in Fig. 18. Clearly, all of the controlled modes are involved in the maneuver, but each trajectory is smoothly controlled with the desired final conditions being produced at the end of the maneuver. In this case, the technique produces a 78-81% improvement in the final conditions over the linear (first order) solution for a third order expansion.

The rigid body controls are given in Fig. 19. Not unexpectedly, the rigid body control trajectories are "typical" of those encountered in Case 4. The appendage control trajectories (see Fig. 20) are also very smooth with zero magnitudes and zero slopes at the end points. Each profile demonstrates large single sided excursions required for the control of the flexural vibrations.
This case in particular illustrates the effectiveness of the solution algorithm in approximating the solution of a nonlinear system through the use of strictly linear optimal control problems.
CHAPTER 5

DISCUSSION AND CONCLUSIONS

A numerical technique for the solution of optimal open loop maneuvers of a spacecraft with translational/rotational coupling has been presented, in which the nonlinear equations of motion were solved using a straightforward perturbation method. Although a specific problem was examined in this thesis, the perturbation approach presented is applicable to a large family of weakly nonlinear problems. The significance of this thesis is that it combines the use of two very powerful techniques (the straightforward expansion method and the matrix exponential algorithm) to generate a novel solution process for nonlinear optimal control problems; providing a non-iterative solution path. The disadvantage of this technique is that large amplitude, multifrequency motions and/or highly nonlinear equations will cause the accuracy of solutions using this method to deteriorate perhaps to an unacceptable degree in a given application.

For the given problem at hand, the numerical procedure developed in this thesis works quite well as demonstrated by the two most challenging cases included in Chapter 4. In particular, Case 5 represents a fairly difficult problem to control since all of the first seven modes are excited. The expected relative improvement produced by the technique should typically range from 10% to 60% for third order expansions. Although the procedure may theoretically be applied to any order, the
extensive numerical manipulations required may lead to unacceptable expense and arithmetic errors.

Future studies of the perturbation approach should include investigation into the effect of "secular terms" that will cause the approach presented in this thesis to lose accuracy as the maneuver time becomes large, as such terms are inherent in the straightforward expansion. Other perturbation techniques, such as the method of multiple scales and variation of parameters, may also be applicable to optimal control problems. Since these methods do not include secular terms, the solutions would remain accurate regardless of the maneuver time.
REFERENCES


66


<table>
<thead>
<tr>
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<td>Appendage Mass</td>
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<td>Tip Mass</td>
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<td>Tip Mass Moment of Inertia</td>
<td>0.0018 slug-ft²</td>
</tr>
<tr>
<td>Appendage Stiffness</td>
<td>11x10⁶ lbf/in²</td>
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APPENDIX A

APPLICATION OF THE BOUNDARY CONDITIONS
FOR THE TRANSVERSE VIBRATION OF A THIN PRISMATIC BEAM

This section presents the detailed algebraic manipulations required to solve for four of the unknowns that describe the space dependent function as derived for the transverse motion of a thin prismatic beam. For clarity, the general solution and boundary conditions are repeated.

\[ \psi_n(x) = C_1 \sin \gamma_n x + C_2 \cos \gamma_n x + C_3 \sinh \gamma_n x + C_4 \cosh \gamma_n x \quad (A-1) \]

\[ \psi_n(0) = 0 \]

\[ \psi_n'(0) = 0 \]

\[ \psi_n''(L) = \frac{\gamma_n}{\rho A} I_T \psi_n'(L) \]

\[ \psi_n'''(L) = \frac{-\gamma_n}{\rho A} M_T \psi_n(L) \quad (A-2) \]

where

\[ (\cdot)' = \frac{d}{dx} (\cdot) \]

\[ I_T = \text{inertia of the tip mass} \]

\[ M_T = \text{mass of the tip mass} \]

The first three derivatives of Eq. (A-1) with respect to x are

\[ \psi_n'(x) = \gamma_n [C_1 \cos \gamma_n x - C_2 \sin \gamma_n x + C_3 \cosh \gamma_n x + C_4 \sinh \gamma_n x] \]
\[
\psi''(x) = \gamma_n^2[-C_{1n}\sin\gamma_n x - C_{2n}\cos\gamma_n x + C_{3n}\sinh\gamma_n x + C_{4n}\cosh\gamma_n x]
\]  
(A-3)

\[
\psi'''(x) = \gamma_n^3[-C_{1n}\cos\gamma_n x + C_{2n}\sin\gamma_n x + C_{3n}\cosh\gamma_n x + C_{4n}\sinh\gamma_n x]
\]

Substituting the identities in Eqs. (A-1) and (A-3) into the corresponding expressions in Eq. (A-2) yields four equations in the five unknowns \(C_{1n}, C_{2n}, C_{3n}, C_{4n}, \gamma_n\).

\[
0 = C_{2n} + C_{4n}
\]  
(A-4a)

\[
0 = \gamma_n C_{1n} + \gamma_n C_{3n}
\]  
(A-4b)

\[
\gamma_n^2[-C_{1n}\sin\gamma_n L - C_{2n}\cos\gamma_n L + C_{3n}\sinh\gamma_n L + C_{4n}\cosh\gamma_n L]
\]

\[
= \frac{\lambda_n^5}{\partial A} [C_{1n}\cos\gamma_n L - C_{2n}\sin\gamma_n L + C_{3n}\cosh\gamma_n L + C_{4n}\sinh\gamma_n L]
\]  
(A-4c)

\[
\gamma_n^3[-C_{1n}\cos\gamma_n L + C_{2n}\sin\gamma_n L + C_{3n}\cosh\gamma_n L + C_{4n}\sinh\gamma_n L]
\]

\[
= -\frac{\gamma_n^4 M}{\partial A} [C_{1n}\sin\gamma_n L + C_{2n}\cos\gamma_n L + C_{3n}\sinh\gamma_n L + C_{4n}\cosh\gamma_n L]
\]  
(A-4d)

Combining the terms that multiply each of the coefficients \(C_{1n}, i = 1,2,3,4\) and placing the equations in matrix form yields

\[
\mathbf{A}\mathbf{C} = \mathbf{0}
\]  
(A-5)

where
For a homogeneous set of equations as shown in Eq. (A-5), there exists a nontrivial solution if and only if the determinant of the "coefficient" matrix is zero. This constraint produces a transcendental equation for the determination of $\gamma_n$, given by

\[
Q = [C_{1n} \ C_{2n} \ C_{3n} \ C_{4n}]^T
\]

\[
A = \begin{bmatrix}
0 & 1 & 0 & 1 \\
\gamma_n & 0 & \gamma_n & 0 \\
\frac{\gamma_n}{\rho A} \cos \gamma_n L & -\frac{\gamma_n}{\rho A} \sin \gamma_n L & \frac{\gamma_n}{\rho A} \cosh \gamma_n L & -\frac{\gamma_n}{\rho A} \sinh \gamma_n L \\
+\gamma_n^2 \sin \gamma_n L & +\gamma_n^2 \cos \gamma_n L & -\gamma_n^2 \sinh \gamma_n L & +\gamma_n^2 \cosh \gamma_n L \\
-\frac{\gamma_n^2}{\rho A} \sin \gamma_n L & -\frac{\gamma_n^2}{\rho A} \cos \gamma_n L & +\frac{\gamma_n^2}{\rho A} \sinh \gamma_n L & -\frac{\gamma_n^2}{\rho A} \cosh \gamma_n L \\
\gamma_n^3 \cos \gamma_n L & +\gamma_n^3 \sin \gamma_n L & -\gamma_n^3 \cosh \gamma_n L & +\gamma_n^3 \sinh \gamma_n L \\
\gamma_n^3 \sin \gamma_n L & \gamma_n^3 \cos \gamma_n L & -\gamma_n^3 \cosh \gamma_n L & +\gamma_n^3 \sinh \gamma_n L \\
\gamma_n^3 \cos \gamma_n L & \gamma_n^3 \sin \gamma_n L & -\gamma_n^3 \cosh \gamma_n L & +\gamma_n^3 \sinh \gamma_n L \\
\gamma_n^3 \sin \gamma_n L & \gamma_n^3 \cos \gamma_n L & -\gamma_n^3 \cosh \gamma_n L & +\gamma_n^3 \sinh \gamma_n L
\end{bmatrix}
\]

\[
\frac{6}{\gamma_n^6} \left( \frac{\gamma_n}{\rho A} \right)^3 \cos \gamma_n L - \gamma_n^2 \sin \gamma_n L \left( \frac{\gamma_n}{\rho A} \right) \cosh \gamma_n L - \gamma_n^2 \sinh \gamma_n L \left( \frac{\gamma_n}{\rho A} \right) \sinh \gamma_n L
\]

\[
- \left( \frac{\gamma_n}{\rho A} \right)^3 \sin \gamma_n L - \left( \frac{\gamma_n}{\rho A} \right) \cosh \gamma_n L \left( \frac{\gamma_n}{\rho A} \right) \sinh \gamma_n L - \left( \frac{\gamma_n}{\rho A} \right) \sin \gamma_n L + \cos \gamma_n L
\]

\[
- \gamma_n^3 \left( \frac{\gamma_n}{\rho A} \right) \sin \gamma_n L - \left( \frac{\gamma_n}{\rho A} \right) \cosh \gamma_n L \left( \frac{\gamma_n}{\rho A} \right) \sinh \gamma_n L - \gamma_n^3 \left( \frac{\gamma_n}{\rho A} \right) \sin \gamma_n L + \cos \gamma_n L
\]

\[
- \gamma_n^3 \left( \frac{\gamma_n}{\rho A} \right) \sin \gamma_n L - \gamma_n^3 \left( \frac{\gamma_n}{\rho A} \right) \cosh \gamma_n L \left( \frac{\gamma_n}{\rho A} \right) \sinh \gamma_n L
\]
The solution to Eq. (A-6) is found by setting the bracketed term equal to zero since we are not interested in the trivial solution where $\gamma_n = 0$, for all $n$. Performing the indicated products and combining the coefficients of like powers of $\gamma_n$ simplifies the equation to

\[
\frac{2\gamma_n^4}{(\partial A)^2} \left(-\cos \gamma_n \cosh \gamma_n + 1\right) + \frac{2\gamma_n^3}{\partial A} \left(-\cos \gamma_n \sinh \gamma_n - \sin \gamma_n \cosh \gamma_n\right) + \frac{2\gamma_n^2}{\partial A} \left(-\sin \gamma_n \cosh \gamma_n + \cos \gamma_n \sinh \gamma_n\right) + 2(1 + \cos \gamma_n \cosh \gamma_n) = 0 \quad (A-7)
\]

Dividing both sides of the equation by two and rearranging gives (also Eq. (2-10))

\[
\cos \gamma_n \cosh \gamma_n + 1 + \frac{\gamma_n^M}{\partial A} \left(\cos \gamma_n \sinh \gamma_n - \sin \gamma_n \cosh \gamma_n\right)
\]
Now that the determinant of the coefficient matrix in Eq. (A-5) is guaranteed to be zero for all \( \gamma_n \) defined by Eq. (A-8), there will be an infinite number of possible solutions of the matrix formula Eq. (A-5). Therefore, we can parameterize this infinite set of solutions by solving for three of the remaining four coefficients \( (C_{in}, i = 1, 2, 3, 4) \) in terms of the "fourth" coefficient. Note that only three of Eqs. (A-4) are needed as the system now contains one redundant equation. By allowing this last coefficient to take on any arbitrary value, the entire spectrum of solutions can be represented in a concise form.

Since any three of Eqs. (A-4) may be used we shall use the first three, due to the simple form of Eqs. (A-4a) and (A-4b), and solve for each of the constants \( C_{1n}, C_{2n}, \) and \( C_{3n} \) in terms of \( C_{4n} \). Rearranging Eqs. (A-4a) and (A-4b) produces

\[
C_{2n} = -C_{4n} \\
\gamma_n C_{3n} = -\gamma_n C_{1n}
\]

Substituting Eq. (A-9) into Eq. (A-4c) produces one equation relating \( C_{1n} \) and \( C_{4n} \).
\[
y_n^2(-C_{1n}\sin\gamma_n L + C_{4n}\cos\gamma_n L - C_{1n}\sinh\gamma_n L + C_{4n}\cosh\gamma_n L)
\]

\[
\frac{5nI}{\rho A} [C_{1n}\cos\gamma_n L + C_{4n}\sin\gamma_n L - C_{1n}\cosh\gamma_n L + C_{4n}\sinh\gamma_n L] \quad (A-10)
\]

Combining the terms that multiply the coefficients \(C_{1n}\) and \(C_{4n}\) produces

\[
C_{1n}[-\gamma_n^2(\sin\gamma_n L + \sinh\gamma_n L) + \frac{5nI}{\rho A} (\cosh\gamma_n L - \cos\gamma_n L)]
\]

\[
= C_{4n}[-\gamma_n^2(\cos\gamma_n L + \cosh\gamma_n L) + \frac{5nI}{\rho A} (\sin\gamma_n L + \sinh\gamma_n L)] \quad (A-11)
\]

and solving for \(C_{1n}\) in terms of \(C_{4n}\) gives

\[
C_{1n} = \frac{C_{4n}[\cos\gamma_n L + \cosh\gamma_n L - \frac{\gamma_n^2}{\rho A} (\sin\gamma_n L + \sinh\gamma_n L)]}{\sin\gamma_n L + \sinh\gamma_n L + \frac{\gamma_n^2}{\rho A} (\cos\gamma_n L - \cosh\gamma_n L)} \quad (A-12)
\]

Upon substituting Eqs. (A-12) and (A-9) into Eq. (A-1), the solution for the function \(\psi_n(x)\) becomes

\[
\psi_n(x) = C_{4n} B_{\gamma_n} \sin\gamma_n x - C_{4n} B_{\gamma_n} \cos\gamma_n x - C_{4n} B_{\gamma_n} \sinh\gamma_n x
\]

\[+ C_{4n} B_{\gamma_n} \cosh\gamma_n x \quad (A-13)
\]

where

\[
B_n = \frac{\cos\gamma_n L + \cosh\gamma_n L - \frac{\gamma_n^2}{\rho A} (\sin\gamma_n L + \sinh\gamma_n L)}{\sin\gamma_n L + \sinh\gamma_n L + \frac{\gamma_n^2}{\rho A} (\cos\gamma_n L - \cosh\gamma_n L)} \quad (A-14)
\]

Finally let

\[
C_{4n} = \frac{1}{k}
\]
so that Eq. (A-13) may be written as

\[ K\psi_n(x) = B_n (\sin\gamma_n x - \sinh\gamma_n x) + \cosh\gamma_n x - \cos\gamma_n x \]  \hspace{1cm} (A-15)

This completes the evaluation of the space dependent functions and the boundary conditions. Note that Eq. (A-14) is identical to Eq. (2-9) and Eq. (A-15) is also Eq. (2-8).
To develop the mathematical representation of the spacecraft, consider a right handed inertial reference frame designated by the unit vectors \( \hat{N}_1, \hat{N}_2, \) and \( \hat{N}_3 \) with the spacecraft oriented such that the plane of the appendages is the plane formed by the \( \hat{N}_1 \) and \( \hat{N}_2 \) vectors (see Fig. B1). Let a right handed body fixed reference frame be denoted by the unit vectors \( \hat{b}_1, \hat{b}_2, \) and \( \hat{b}_3 \) with the origin located at the hub center. The orientation of the body fixed frame is such that the vector, \( \hat{b}_1, \) is in the direction of the undeformed axis of appendage 1, and \( \hat{b}_2 \) is in the direction of the undeformed axis of appendage 2. For the sake of simplicity, all four appendages are assumed to be identical thin rectangular beams with constant cross-sectional area and identical masses attached to the free end. The motion of each appendage, however, is completely independent of the motion of the other appendages. Furthermore, the flexural deformations of all of the appendages are assumed to be small and perpendicular to the undeformed axis.

Let the vector, \( \mathbf{R}_C \), denote the position of the mass center of the spacecraft relative to the inertial frame, and the vector, \( \mathbf{R}_H \), denote the position of the hub center relative to the mass center. Similarly, let the vector, \( \mathbf{R}_i \), denote the position of a differential mass element on the \( i^{th} \) deformed appendage relative to the hub center.
Using the definition of the mass center of a body, the location of the mass center is found by

\[
\begin{align*}
\bar{M_R} &= \bar{M_H}(R_C + R_H) + \sum_{i=1}^{4} \int_{M_i} (R_C + R_H + R_i) \, dm \\
&\quad + \sum_{i=1}^{4} M_T (R_C + R_H + R_i) \bigg|_{x=L_i} 
\end{align*}
\]

where

- \( M \) is the total mass of the spacecraft
- \( M_H \) is the mass of the rigid hub
- \( M_i \) is the mass of the \( i \)th appendage
- \( M_T \) is the mass attached to the free end of an appendage
- \( L_i \) is the length of the \( i \)th appendage

Since the vectors, \( R_C \) and \( R_H \), are independent of a particular mass element, then

\[
\int_{M_i} (R_C + R_H) \, dm = (R_C + R_H)M_i
\]  

(B-2)

Noting that the total mass of the spacecraft is defined as

\[
M = M_H + 4M_T + \sum_{i=1}^{4} M_i
\]  

(B-3)

then substituting Eqs. (B-2) and (B-3) into Eq. (B-1) yields

\[
\bar{M_R} = \bar{M_R} + \bar{M_H} + \sum_{i=1}^{4} \int_{M_i} R_i \, dm + \sum_{i=1}^{4} M_T R_i \bigg|_{x=L_i}
\]  

(B-4)
Equation (B-4) provides a definition of the vector, $R_H$, in terms of the configuration coordinates (contained implicitly in the vector $R_i$), given as

$$R_H = -\frac{1}{M} \sum_{i=1}^{4} \left( \int_{M_i}^{R_i} \frac{1}{dm} + M_i R_i \right)_{x=L_i} \quad (B-5)$$

The deflection of a single appendage (see Appendix A) is approximated by truncating an infinite series describing the motion, thereby producing a discretized system of "reasonable" order. The deformation of a mass element, perpendicular to the undeformed axis of the $i^{th}$ appendage is

$$y_i(x,t) = \sum_{j=1}^{N} q_{ij}(t) \psi_j(x) \quad (B-6)$$

where

$$\psi_j(x) = B_j (\sin \gamma_j x - \sinh \gamma_j x) + \cosh \gamma_j x - \cos \gamma_j x$$

$B_j$ is defined in Eq. (2-9)

$\gamma_j$ is defined in Eq. (2-10)

$N$ is the number of terms in the finite series.

The vector, $R_i$, may then be written in terms of its components in the body fixed frame, producing

$$R_1 = (r + x) \hat{b}_1 + y_1(x,t) \hat{b}_2$$

$$R_2 = -y_2(x,t) \hat{b}_1 + (r + x) \hat{b}_2$$
\[ R_3 = -(r + x)\hat{b}_1 - y_3(x,t)\hat{b}_2 \]

\[ R_4 = y_4(x,t)\hat{b}_1 - (r + x)\hat{b}_2 \]  \hspace{1cm} (B-7)

where

- \( r \) is the radius of the hub
- \( x \) is the position of a mass element along the undeformed appendage axis as measured from the hub end of the appendage

To describe the motion of the mass center of the body relative to the inertial frame, two independent coordinates are required. Let these coordinates be:

- \( x_c \) for motion in the \( \hat{N}_1 \) direction
- \( y_c \) for motion in the \( \hat{N}_2 \) direction

Therefore the vector, \( R_c \), may be written as

\[ R_c = x_c\hat{N}_1 + y_c\hat{N}_2 \]  \hspace{1cm} (B-8)

Because the body fixed frame is a single axis rotation about the \( \hat{N}_3 \) axis, the two coordinate frames are related by

\[
\begin{bmatrix}
\hat{b}_1 \\
\hat{b}_2 \\
\hat{b}_3 \\
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\hat{N}_1 \\
\hat{N}_2 \\
\hat{N}_3 \\
\end{bmatrix}
\]  \hspace{1cm} (B-9)

whereas the inverted form of Eq. (B-9) is

\[
\begin{bmatrix}
\hat{N}_1 \\
\hat{N}_2 \\
\hat{N}_3 \\
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\hat{b}_1 \\
\hat{b}_2 \\
\hat{b}_3 \\
\end{bmatrix}
\]  \hspace{1cm} (B-10)
Turning to the derivation of the kinetic energy of the rigid hub, the total kinetic energy of the hub is the sum of the translational and rotational kinetic energies. The vector locating the hub center is

$$\mathbf{r}_h = \mathbf{R}_c + \mathbf{R}_H$$  \hspace{1cm} (B-11)

Recalling, for the moment, Eq. (8-5), it is clear that the vector, \( \mathbf{R}_H \), is most easily written in terms of the body fixed unit vectors since the vector, \( \mathbf{R}_i \), has a simple form when represented in the body fixed frame. Therefore, the time derivative of Eq. (8-11) yields

$$\dot{\mathbf{r}}_h = \dot{\mathbf{R}}_c + \dot{\mathbf{R}}_H + \mathbf{\omega} \times \mathbf{R}_H$$  \hspace{1cm} (B-12)

where

$$\left( \cdot \right) = \frac{d}{dt} \left( \cdot \right) \text{ in the inertial frame}$$

$$\left( \circ \right) = \frac{d}{dt} \left( \cdot \right) \text{ in the body fixed frame}$$

$$\mathbf{\omega} = \begin{pmatrix} 0 \\ \dot{\omega} \end{pmatrix} \text{ is the angular velocity of the body fixed frame relative to the inertial frame}$$

and where use was made of the identity

$$\frac{d}{dt} \left( \cdot \right)_N = \frac{d}{dt} \left( \cdot \right)_B + \omega_N \times \left( \cdot \right)$$

The total kinetic energy of the hub is then given by

$$T_h = \frac{1}{2} M_h \dot{\mathbf{r}}_h \cdot \dot{\mathbf{r}}_h + \frac{1}{2} I_H \dot{\theta}^2$$  \hspace{1cm} (B-13)

where

- \( I_H \) is the rotational inertia of the hub about the \( b_3 \) axis
A similar process is used to determine the kinetic energy of the appendages. The vector locating a differential mass element on the \( i \)th appendage relative to the inertial origin is

\[
\mathbf{r}_1 = \mathbf{R}_c + \mathbf{R}_h + \mathbf{R}_i \tag{B-14}
\]

Taking the derivative of Eq. (B-14) with respect to time gives

\[
\dot{\mathbf{r}}_1 = \dot{\mathbf{R}}_c + \dot{\mathbf{R}}_h + \omega \times \mathbf{R}_h + \dot{\mathbf{R}}_i + \omega \times \mathbf{R}_i \tag{B-15}
\]

Since Eq. (B-15) represents the velocity of a differential mass element, the kinetic energy of the appendage will be the sum of the energies of all the mass elements. Therefore the kinetic energy of all of the appendages is

\[
T_a = \frac{1}{2} \sum_{i=1}^{4} \int M_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \, dm \tag{B-16}
\]

Finally, the expression for the total kinetic energy of a tip mass must be found. The translational velocity of a tip mass is equal to the velocity of the tip of the appendage to which it is attached. The angular velocity of a tip mass is the time derivative of the angle through which the mass rotates. Because of the assumption that all appendage motions are small, the angle of a tip mass relative to the inertial frame may be approximated to first order by

\[
v_i = \theta + \frac{\partial y_i(x,t)}{\partial x} \bigg|_{x=L} \tag{B-17}
\]

Therefore the angular velocity of the \( i \)th tip mass is
\[
\dot{v}_1 = \dot{\theta} + \frac{\partial y_i(x,t)}{\partial x}\bigg|_{x=L}
\]  

(B-18)

where

\[
y_i(x,t) = \sum_{j=1}^{N} \hat{q}_{ij}(t)\psi_j(x)
\]

The total kinetic energy of all of the tip masses is then

\[
T_m = \sum_{i=1}^{4} \frac{1}{2} M_T \dot{r}_i \cdot \dot{r}_i \bigg|_{x=L} + \frac{1}{2} I_T (\dot{\theta} + \frac{\partial y_i(x,t)}{\partial x})^2 \bigg|_{x=L}
\]  

(B-19)

The total kinetic energy of the spacecraft is then the sum of Eqs. (B-13), (B-16) and (B-19) given as

\[
T = \frac{1}{2} M_H \dot{r}_h \cdot \dot{r}_h + \frac{1}{2} I_H \dot{\theta}^2
\]

\[
+ \frac{1}{2} \sum_{i=1}^{4} \left[ \int_{M_i} \dot{r}_i \cdot \dot{r}_i \, dm + M_T \dot{r}_i \cdot \dot{r}_i \bigg|_{x=L} + I_T (\dot{\theta} + \frac{\partial y_i(x,t)}{\partial x})^2 \bigg|_{x=L} \right]
\]  

(B-20)

To evaluate Eq. (B-20), not only are the definitions given in Eqs. (B-5) through (B-10) used quite extensively, but one encounters in the process the following integrals.

(a) \[ \int_{M_i} (r + x) \, dm \]

(b) \[ \int_{M_1} (r + x)^2 \, dm \]
(c) \[ \int_{M_1} \psi_j(x) \, dm \]

(d) \[ \int_{M_1} (r + x) \psi_j(x) \, dm \]

(e) \[ \int_{M_1} \psi_j(x) \psi_k(x) \, dm \] \hspace{1cm} (B-21)

For the particular model presented in this thesis, the appendages are considered to be identical rectangular beams with constant cross sectional area. Therefore, a differential mass element is

\[ dm = \rho A \, dx \] \hspace{1cm} (B-22)

and

\[ M_1 = \int_{M_1} dm = \rho A L \]

where

\[ \rho A \] is the mass per unit length of the beam.

Now any integral over the mass of an appendage may be written as

\[ \int_{M_1} \left( \right) \, dm = \rho A \int_0^L \left( \right) \, dx \] \hspace{1cm} (B-23)

By using Eq. (B-23), the integrals given in Eqs. (B-5) and (B-21) may then be completed.

The algebraic manipulations involved in evaluating Eq. (B-20), using the information supplied in Eqs. (B-5) through (B-10), and (B-21) through (B-23), is straightforward but rather lengthy and will not be
included in this thesis. Many of the terms produced in Eq. (8-20) are
cubic and quartic in various combinations of the variables \( \dot{\theta}, q_{ij}, \)
and \( \dot{q}_{ij} \). Therefore, if we assume that the flexural deformations are
small and that the rotation rate is also small, then these higher order
terms may be disregarded thus producing a more concise kinetic energy
equation of second order. To complete the tasks indicated above, \( R_H \) can
be computed by substituting Eqs. (8-6), (8-7), and (8-23) into Eq. (8-5)
to yield

\[
R_H = \sum_{i=1}^{N} C_i (q_{4i} - q_{2i}) \dot{b}_1 + \sum_{i=1}^{N} C_i (q_{1i} - q_{3i}) \dot{b}_2 \tag{B-24}
\]

Then by substituting Eqs. (8-6) through (8-10) and (8-21) through (8-24)
into Eq. (8-20), the kinetic energy equation is

\[
T = \frac{1}{2} Mx^2 + \frac{1}{2} My^2 + \frac{1}{2} J\dot{\theta}^2
\]

\[
+ \dot{\theta} \sum_{i=1}^{4} \sum_{j=1}^{N} (pAD_j + M_T (r + L) \dot{b}_j + I_T F_j) \dot{q}_{ij}
\]

\[
+ \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{N} \sum_{k=1}^{N} (pAE_{jk} + M_T B_{jk} + I_T F_j F_k - \tilde{M}_c \tilde{C}_c) \dot{q}_{ij} \dot{q}_{ik}
\]

\[
+ M \sum_{j=1}^{N} \sum_{k=1}^{N} C_{jk} C_{jk} (\dot{q}_{4j} \dot{q}_{2k} + \dot{q}_{1j} \dot{q}_{3k}) \tag{B-25}
\]

where

\[
J = I_H + 4I_T + 4M_T (r + L)^2 + 4pA(r^2L + rL^2 + \frac{L^3}{3})
\]
The only source of any potential energy contributed to the spacecraft is due to the strain energy of the flexible members as a result of elastic deformations. Therefore the total potential energy may be shown to be

\begin{equation}
V = \frac{1}{2} \sum_{i=1}^{4} E I \int_{0}^{L} \left( \frac{\partial^2 y_i(x,t)}{\partial x^2} \right)^2 dx
\end{equation}

(B-26)

where \( EI \) is the bending stiffness of the beam. Substituting Eq. (B-6) into Eq. (B-26) produces

\begin{equation}
V = \frac{1}{2} \sum_{i=1}^{4} E I \int_{0}^{L} \sum_{j=1}^{N} \sum_{k=1}^{N} q_{ij}(t) q_{ik}(t) \psi_j''(x) \psi_k''(x) dx
\end{equation}

(B-27)

Interchanging the summation and integral operations and defining the
The coefficient \( G_{jk} \) is defined as:

\[
G_{jk} = \int_0^L \psi_j^*(x) \psi_k^*(x) dx
\]

The potential energy expression becomes:

\[
V = \frac{1}{2} EI \sum_{i=1}^4 \sum_{j=1}^N \sum_{k=1}^N G_{jk} q_i q_j q_k
\]  

(B-28)
Figure B1  Spacecraft Vector Notations and Orientations
APPENDIX C

CLOSED FORM EXPRESSIONS OF THE ENERGY COEFFICIENTS

The computation of the kinetic and potential energy expressions, and hence the mass and stiffness matrices, requires the evaluation of several expressions involving the space dependent function describing the appendage motion. Some of these expressions are integral formulae involving the function or its derivatives. The space dependent function is

\[ \psi_j(x) = B_j (\sin \gamma_j x - \sinh \gamma_j x) + \cosh \gamma_j x - \cos \gamma_j x \]  

where

\[ B_j = \frac{\cos \gamma_j L + \cosh \gamma_j L - \frac{\gamma_j^3 I_T}{\partial A} (\sin \gamma_j L + \sinh \gamma_j L)}{\sin \gamma_j L + \sinh \gamma_j L + \frac{\gamma_j^3 I_T}{\partial A} (\cosh \gamma_j L - \cosh \gamma_j L)} \]

and \( \gamma_j \) is the solution set of the equation

\[ \cos \gamma_j L \cosh \gamma_j L + 1 + \frac{\gamma_j^4 M I_T}{\partial A} (\cos \gamma_j L \sinh \gamma_j L - \sin \gamma_j L \cosh \gamma_j L) \]

\[ - \frac{\gamma_j^3 I_T}{\partial A} (\cos \gamma_j L \sinh \gamma_j L + \sin \gamma_j L \cosh \gamma_j L) \]

\[ - \frac{\gamma_j^4 M I_T}{(\partial A)^2} (\cos \gamma_j L \cosh \gamma_j L - 1) = 0 \]

with the definitions

\( I_T \) is the rotational inertia of a tip mass
MT is the mass of the tip mass

ρA is the mass per unit length of the beam

L is the length of the beam

To evaluate the expressions requires, in some cases, a considerable amount of algebraic manipulations, however, the methods are straightforward. Substituting Eq. (C-1) into the expressions and carrying out the indicated operations produces

\[ \tilde{B}_j = \psi_j(L) \]

\[ = B_j(\sin \gamma_j L - \sinh \gamma_j L) + \cosh \gamma_j L - \cos \gamma_j L \] (C-2)

\[ \tilde{C}_j = -\frac{1}{M} (M \tilde{B}_j + \rho A \int_0^L \psi_j(x)dx) \]

\[ = -\frac{1}{M} \{M \{B_j(\sin \gamma_j L - \sinh \gamma_j L) + \cosh \gamma_j L - \cos \gamma_j L\} \]

\[ + \rho A \{B_j(2 - \cos \gamma_j L - \cosh \gamma_j L) + \sinh \gamma_j L - \sin \gamma_j L\}\} \] (C-3)

\[ \tilde{D}_j = \int_0^L (r + x)\psi_j(x)dx \]

\[ = \frac{1}{2 \gamma_j^2} \{B_j(\sin \gamma_j L + \sinh \gamma_j L) - \cosh \gamma_j L - \cos \gamma_j L + 2\} \]

\[ + \frac{r+L}{\gamma_j} \{-B_j(\cos \gamma_j L + \cosh \gamma_j L) + \sinh \gamma_j L - \sin \gamma_j L\} \]

\[ + \frac{2B_j r}{\gamma_j} \] (C-4)

where \( r \) is the radius of the hub
\[ E_{jk} = \int_0^L \psi_j(x)\psi_k(x)dx \]

(a) For \( j \neq k \)

\[
E_{jk} = B_jB_k \left[ \frac{\sin(\gamma_j - \gamma_k)L}{2(\gamma_j - \gamma_k)} - \frac{\sin(\gamma_j + \gamma_k)L}{2(\gamma_j + \gamma_k)} \right] \\
+ \frac{\gamma_j (\cos \gamma_k \sinh \gamma_j L - \sin \gamma_j \cosh \gamma_k L)}{\gamma_j^2 + \gamma_k^2} \\
+ \frac{\gamma_k (\cos \gamma_j \sinh \gamma_k L - \sin \gamma_k \cosh \gamma_j L)}{\gamma_j^2 + \gamma_k^2} \\
+ \frac{\sinh(\gamma_j + \gamma_k)L}{2(\gamma_j + \gamma_k)} - \frac{\sinh(\gamma_j - \gamma_k)L}{2(\gamma_j - \gamma_k)} \\
+ B_j \left[ \frac{\gamma_j (\sin \gamma_k \sinh \gamma_j L + \sin \gamma_j \sinh \gamma_k L)}{\gamma_j^2 + \gamma_k^2} \right] \\
+ \frac{\gamma_j (\cos \gamma_k \cosh \gamma_j L - \cos \gamma_j \cosh \gamma_k L)}{\gamma_j^2 + \gamma_k^2} \\
+ \frac{\cos(\gamma_j - \gamma_k)L}{2(\gamma_j - \gamma_k)} + \frac{\cos(\gamma_j + \gamma_k)L}{2(\gamma_j + \gamma_k)} \\
- \frac{\cosh(\gamma_j + \gamma_k)L}{2(\gamma_j + \gamma_k)} - \frac{\cosh(\gamma_j - \gamma_k)L}{2(\gamma_j - \gamma_k)} \\
+ B_k \left[ \frac{\gamma_j (\sin \gamma_k \sinh \gamma_j L + \sin \gamma_j \sinh \gamma_k L)}{\gamma_j^2 + \gamma_k^2} \right] \\
+ \frac{\gamma_k (\cos \gamma_j \cosh \gamma_k L - \cos \gamma_k \cosh \gamma_j L)}{\gamma_j^2 + \gamma_k^2} \]

\[ (a) \quad \text{For } j \neq k \]
\[
- \frac{\cosh(\gamma_j + \gamma_k) L}{2(\gamma_j + \gamma_k)} - \frac{\cosh(\gamma_k - \gamma_j) L}{2(\gamma_k - \gamma_j)} \\
+ \frac{\cos(\gamma_k - \gamma_j) L}{2(\gamma_k - \gamma_j)} + \frac{\cos(\gamma_j + \gamma_k) L}{2(\gamma_j + \gamma_k)} \\
+ \frac{\sinh(\gamma_j + \gamma_k) L}{2(\gamma_j + \gamma_k)} + \frac{\sinh(\gamma_j - \gamma_k) L}{2(\gamma_j - \gamma_k)} \\
\frac{\gamma_j (\cos \gamma_k L \sinh \gamma_j L + \sin \gamma_j L \cosh \gamma_k L)}{\gamma_j^2 + \gamma_k^2} \\
- \frac{\gamma_k (\sin \gamma_k L \cosh \gamma_j L + \cos \gamma_j L \sinh \gamma_k L)}{\gamma_j^2 + \gamma_k^2} \\
+ \frac{\sin(\gamma_j - \gamma_k) L}{2(\gamma_j - \gamma_k)} + \frac{\sin(\gamma_j + \gamma_k) L}{2(\gamma_j + \gamma_k)} \\
\text{(C-5a)}
\]

(b) For \( j = k \)

\[
E_{jj} = B_j^2 \left[ \frac{\sinh^2 \gamma_j L}{4 \gamma_j} \right] - \frac{\cos \gamma_j L \sin \gamma_j L}{2 \gamma_j} \left( \cos \gamma_j L \sinh \gamma_j L - \sin \gamma_j L \cosh \gamma_j L \right) \cdot \frac{1}{\gamma_j} \\
+ 2B_j \left[ \frac{\sin \gamma_j L \sinh \gamma_j L}{\gamma_j} \right] - \frac{\sin^2 \gamma_j L}{2 \gamma_j} - \frac{\sinh^2 \gamma_j L}{2 \gamma_j} \\
+ \frac{\sinh 2 \gamma_j L}{2 \gamma_j} + L + \frac{\sin \gamma_j L \cosh \gamma_j L}{2 \gamma_j} \\
- \frac{\cos \gamma_j L \sinh \gamma_j L + \sin \gamma_j L \cosh \gamma_j L}{\gamma_j} \\
\text{(C-5b)}
\]
\[ \tilde{F}_j = \frac{d}{dx} [\psi_j(x)] \bigg|_{x=L} \]
\[ = \frac{d}{dx} [\psi_j(x)] ]_{x=L} \]
\[ = \gamma_j [B_j (\cos \gamma_j L - \cosh \gamma_j L) + \sinh \gamma_j L + \sin \gamma_j L] \]
\[ (C-6) \]

\[ \tilde{G}_{jk} = \int_0^L \frac{d^2}{dx^2} [\psi_j(x)] \frac{d^2}{dx^2} [\psi_k(x)] dx \]

(a) For \( j \neq k \)
\[ \tilde{G}_{jk} = \gamma_j \gamma_k \left[ B_j B_k \left( \frac{\sin (\gamma_j - \gamma_k) L}{2(\gamma_j - \gamma_k)} - \frac{\sin (\gamma_j + \gamma_k) L}{2(\gamma_j + \gamma_k)} \right) \right. \]
\[ + \frac{\gamma_k (\sin \gamma_j L \cosh \gamma_k L - \cos \gamma_k L \sinh \gamma_j L)}{\gamma_j^2 + \gamma_k^2} \]
\[ + \frac{\gamma_j (\sin \gamma_k L \cosh \gamma_j L - \cos \gamma_j L \sinh \gamma_k L)}{\gamma_j^2 + \gamma_k^2} \]
\[ + \frac{\sinh (\gamma_j + \gamma_k) L}{2(\gamma_j + \gamma_k)} - \frac{\sinh (\gamma_j - \gamma_k) L}{2(\gamma_j - \gamma_k)} \]
\[ + \frac{\cos (\gamma_j - \gamma_k) L}{2(\gamma_j - \gamma_k)} + \frac{\cos (\gamma_j + \gamma_k) L}{2(\gamma_j + \gamma_k)} \]
\[ + B_j \left[ \frac{\cos (\gamma_j - \gamma_k) L}{2(\gamma_j - \gamma_k)} + \frac{\cos (\gamma_j + \gamma_k) L}{2(\gamma_j + \gamma_k)} \right. \]
\[ + \frac{\gamma_j (\cos \gamma_j L \cosh \gamma_k L - \cos \gamma_k L \cosh \gamma_j L)}{\gamma_j^2 + \gamma_k^2} \]
\[ + \frac{\gamma_k (\sin \gamma_j L \sinh \gamma_k L + \sin \gamma_k L \sinh \gamma_j L)}{\gamma_j^2 + \gamma_k^2} \]
\[ - \frac{\cosh (\gamma_j + \gamma_k) L}{2(\gamma_j + \gamma_k)} - \frac{\cosh (\gamma_j - \gamma_k) L}{2(\gamma_j - \gamma_k)} \]
\[ \begin{align*}
&+ B_k \left( \frac{\cos(\gamma_k - \gamma_j) L}{2(\gamma_k - \gamma_j)} + \frac{\cos(\gamma_j + \gamma_k) L}{2(\gamma_j + \gamma_k)} \right) - \\
&\quad \frac{\gamma_k (\cos \gamma_k L \cosh \gamma_j L - \cos \gamma_j L \cosh \gamma_k L)}{\gamma_j^2 + \gamma_k^2} + \\
&\quad \frac{\gamma_j (\sin \gamma_k L \sinh \gamma_j L + \sin \gamma_j L \sinh \gamma_k L)}{\gamma_j^2 + \gamma_k^2} - \\
&\quad \frac{\cosh (\gamma_j + \gamma_k) L}{2(\gamma_j + \gamma_k)} - \frac{\cosh (\gamma_k - \gamma_j) L}{2(\gamma_k - \gamma_j)} + \\
&\quad \frac{\sinh (\gamma_j + \gamma_k) L}{2(\gamma_j + \gamma_k)} + \frac{\sinh (\gamma_j - \gamma_k) L}{2(\gamma_j - \gamma_k)} + \\
&\quad \frac{\gamma_j (\cos \gamma_k L \sinh \gamma_j L + \sin \gamma_j L \cosh \gamma_k L)}{\gamma_j^2 + \gamma_k^2} - \\
&\quad \frac{\gamma_k (\sin \gamma_k L \cosh \gamma_j L + \cos \gamma_j L \sinh \gamma_k L)}{\gamma_j^2 + \gamma_k^2} + \\
&\quad \frac{\sin (\gamma_j + \gamma_k) L}{2(\gamma_j + \gamma_k)} + \frac{\sin (\gamma_j - \gamma_k) L}{2(\gamma_j - \gamma_k)} \right)
\end{align*} \]

(C-7a)

(b) For \( j = k \)

\[ G_{jj} = \gamma_j^4 B_j^2 \left( \frac{\sinh^2 \gamma_j L}{4 \gamma_j} - \frac{\cos \gamma_j L \sin \gamma_j L}{2 \gamma_j} \right) + \\
\quad \frac{\sin \gamma_j L \cosh \gamma_j L - \cos \gamma_j L \sinh \gamma_j L}{\gamma_j} - \\
\quad 2B_j \left( \frac{\sin^2 \gamma_j L}{2 \gamma_j} + \frac{\sinh^2 \gamma_j L}{2 \gamma_j} + \frac{\sin \gamma_j L \sinh \gamma_j L}{\gamma_j} \right) \]
\[
\sinh(2\gamma_j L) + \frac{\cos \gamma_j L \sinh \gamma_j L + \sin \gamma_j L \cosh \gamma_j L}{\gamma_j} + L + \frac{\sin \gamma_j L \cos \gamma_j L}{2\gamma_j}
\]  

(C-7b)

In order to verify the solutions given above for the coefficients defined by integral equations, Eqs. (C-3), (C-4), (C-5), and (C-7), each closed form solution was compared to a numerically integrated solution.
APPENDIX D

DEVELOPMENT OF THE NUMERICAL INTEGRATION
RECURRENCE EQUATION

The numerical integration method presented in Section 4.3 is generally applicable to any second order equation. For most nonlinear equations an iteration procedure would be required for each step in the integration. However, the system equations for this particular spacecraft model contain nonlinearities involving only the translational controllers in the three rigid body equations. These terms take on the character of stiffness terms for known control trajectories. Therefore, it will be possible to combine these pseudo-stiffness terms with the stiffness matrix to produce a linear recurrence formula where the augmented stiffness matrix will then be numerically different at each time step.

From Section 4.3, the expansion of the system variables about three consecutive nodal points yields

\[
[M + \beta \Delta t^2 K]z_{j+1} + [-2M + (\frac{1}{2} + \beta - \mu)\Delta t^2 K]z_j + [M + (\frac{1}{2} + \beta - \mu)\Delta t^2 K]z_{j-1} - \tilde{f}\Delta t^2 = 0
\]  
\[\text{where}\]
\[\tilde{f} = \frac{1}{\int_{-1}^{1} W_i f d\rho}\]

\[f = \frac{1}{\int_{-1}^{1} W_i d\rho}\]  

(D-1)

(D-2)
and the quantities, $a$, $u$, and $W_f$ are defined as before. Applying the same expansion to the forcing terms gives

$$f = \tau_{j-1} f_{j-1} + \tau_j f_j + \tau_{j+1} f_{j+1} \quad (D-3)$$

The first three elements of each of the force vectors in Eq. (D-3) are nonlinear and functions of the system coordinates. In order to write the elements of the force vectors in a concise form, we define

$$v_1 = [0 \ 0 \ 1 \ 0 \ 0 \ 0 \ \ldots \ 0]^T \ (m \times 1)$$

$$v_2 = [0 \ 0 \ 0 \ -\tilde{C}^T \ 0^T \ \tilde{C}^T \ 0]^T \ (m \times 1)$$

$$v_3 = [0 \ 0 \ 0 \ 0^T \ -\tilde{C}^T \ 0 \ \tilde{C}^T]^T \ (m \times 1) \quad (D-4)$$

Since the shape functions, $r$, are scalars, the expansion of the nonlinear force vector may be given element by element as

$$f_1 = F_x - F_y v_1 \tilde{z}$$

$$f_2 = F_y + F_x v_1 \tilde{z}$$

$$f_3 = F_x v_2 \tilde{z} + F_y v_3 \tilde{z} + \tilde{u}_T$$

$$f_i = B_i \tilde{u} \quad i = 4, 5, 6, \ldots, m \quad (D-5)$$

where

$$(-) = \tau_{j-1}(\ )_{j-1} + \tau_j(\ )_j + \tau_{j+1}(\ )_{j+1}$$

$$u_T = [u_0 \ u_1 \ u_2 \ u_3 \ u_4]^T$$
\[ \mathbf{u} = [F_x \ F_y \ u_0 \ u_1 \ u_2 \ u_3 \ u_4]^T \]

\( B_T \) is a truncated matrix composed of the last 4N rows of the matrix \( B \)

Substituting Eqs. (D-5) into Eq. (D-2), the following scalar integrals are encountered.

\[
\begin{align*}
(1) & \quad g_1 = \frac{\int_{-1}^{1} W_i \gamma_{j-1} \, d\zeta}{W_i} \\
(2) & \quad g_2 = \frac{\int_{-1}^{1} W_i \gamma_j \, d\zeta}{W_i} \\
(3) & \quad g_3 = \frac{\int_{-1}^{1} W_i \gamma_{j+1} \, d\zeta}{W_i} \\
(4) & \quad g_4 = \frac{\int_{-1}^{1} W_i \gamma_j^2 \, d\zeta}{W_i} \\
(5) & \quad g_5 = \frac{\int_{-1}^{1} W_i \gamma_{j-1} \gamma_j \, d\zeta}{W_i} \\
(6) & \quad g_6 = \frac{\int_{-1}^{1} W_i \gamma_{j-1} \gamma_{j+1} \, d\zeta}{W_i} \\
(7) & \quad g_7 = \frac{\int_{-1}^{1} W_i \gamma_j^2 \, d\zeta}{W_i} \\
(8) & \quad g_8 = \frac{\int_{-1}^{1} W_i \gamma_j \gamma_{j+1} \, d\zeta}{W_i}
\end{align*}
\]
\[ g_9 = \frac{1}{-1} \int_{-1}^{1} W_i \Gamma_{j+1} d\zeta \]

\[ \hat{W}_i = \int_{-1}^{1} W_i d\zeta \]

From the definitions of the scalar shape functions, the completed integrals are

\[ g_1 = \frac{1}{2} + \beta - \mu \]
\[ g_2 = \frac{1}{2} - 2\beta + \mu \]
\[ g_3 = \beta \]
\[ g_4 = \frac{1}{2} \beta - \frac{1}{4} \mu + \frac{1}{8} - \frac{1}{4} \delta \]
\[ g_5 = \beta - \mu + \frac{1}{2} - \frac{1}{2} \delta + \frac{1}{2} \varepsilon \]
\[ g_6 = -\frac{1}{2} \beta + \frac{1}{4} \mu - \frac{1}{8} + \frac{1}{4} \delta \]
\[ g_7 = -4\beta + 2\mu + \delta \]
\[ g_8 = \beta - \frac{1}{2} \varepsilon - \frac{1}{2} \delta \]
\[ g_9 = \frac{1}{2} \beta - \frac{1}{4} \mu + \frac{1}{8} + \frac{1}{4} \delta + \frac{1}{2} \varepsilon \]

(D-6)

where

\[ \mu = \frac{1}{-1} \int_{-1}^{1} W_i (\zeta + \frac{1}{2}) d\zeta \]
The coefficients, $\beta$, $\mu$, $\epsilon$, and $\delta$ are given for various weighting functions in Fig. D1. Then by using Eqs. (D-5) and (D-6), the elements of the expanded forcing function, $\tilde{f}$, are

\[
\tilde{f}_1 = g_1 F x_{j-1} + g_2 F x_j + g_3 F x_{j+1} - (g_4 F y_{j-1} + g_5 F y_j + g_6 F y_{j+1}) v_{1z,j-1}^T - (g_5 F y_{j-1} + g_7 F y_j + g_8 F y_{j+1}) v_{1z,j}^T
\]

\[
+ (g_6 F y_{j-1} + g_7 F y_j + g_9 F y_{j+1}) v_{1z,j+1}^T
\]

\[
\tilde{f}_2 = g_1 F y_{j-1} + g_2 F y_j + g_3 F y_{j+1} + (g_4 F x_{j-1} + g_5 F x_j + g_6 F x_{j+1}) v_{1z,j-1}^T + (g_5 F x_{j-1} + g_7 F x_j + g_8 F x_{j+1}) v_{1z,j}^T
\]

\[
+ (g_6 F x_{j-1} + g_7 F x_j + g_9 F x_{j+1}) v_{1z,j+1}^T
\]
\[ f_3 = (g_4 F_{x_{j-1}} + g_5 F_{x_j} + g_6 F_{x_{j+1}})\frac{y_1}{2} z_{j-1} + (g_5 F_{x_{j-1}}
\] + g_7 F_{x_j} + g_8 F_{x_{j+1}})y_{2z_j} + (g_6 F_{x_{j-1}} + g_8 F_{x_j}
\] + g_9 F_{x_{j+1}})y_{2z_{j+1}} + (g_4 F_{y_{j-1}} + g_5 F_{y_j} + g_6 F_{y_{j+1}})\frac{y_1}{3} z_{j-1}
\] + (g_5 F_{y_{j-1}} + g_7 F_{y_j} + g_8 F_{y_{j+1}})y_{3z_j}
\] + (g_6 F_{y_{j-1}} + g_8 F_{y_j} + g_9 F_{y_{j+1}})y_{3z_{j+1}} + g_1 u_{T_{j-1}} + g_2 u_{T_j} + g_3 u_{T_{j+1}}
\]

\[ f_i = g_1 B_{x_{j-1}} + g_2 B_{x_j} + g_3 B_{x_{j+1}} \quad i = 4, 5, 6, \ldots, m \quad (D-7) \]

By examining the composition of the components of the forcing term, it is clear that we can divide this vector into four separate vectors. One vector will be independent of any \( z \) and the others will depend on the nodal points \( z_{j-1}, z_j, \) and \( z_{j+1} \). These vectors may then be combined with the other terms in Eq. (D-1) to produce a recursive formula eliminating the need for an iteration at each time step.

The vector of forcing terms that are independent of all \( z \) is
Looking closely at Eq. (D-8) and recalling the description of the control influence matrix of the linear part of the problem, Eq. (4-1), we see that this vector is simply

\[ \tilde{F}_o = B(g_{1}u_{j-1} + g_{2}u_{j} + g_{3}u_{j+1}) \]  

(D-9)

The vector of forcing terms that are dependent upon the system vector \( z_{j-1} \) may be written as

\[ \tilde{F}_1 = T_4 z_{j-1} \]  

(D-10)

where

\[
T_4 = \begin{bmatrix}
- (g_4 F_{x_{j-1}} + g_5 F_{x_{j}} + g_6 F_{x_{j+1}}) v_1^T \\
( g_4 F_{y_{j-1}} + g_5 F_{y_{j}} + g_6 F_{y_{j+1}}) v_2^T \\
( g_4 F_{x_{j-1}} + g_5 F_{x_{j}} + g_6 F_{x_{j+1}}) v_2^T \\
+ ( g_4 F_{y_{j-1}} + g_5 F_{y_{j}} + g_6 F_{y_{j+1}}) v_3^T \\
0
\end{bmatrix}
\]

(mxmx)

Similarly the force vector that is dependent upon, \( z_{j} \), is found to be
\[ \mathbf{F}_2 = T_5 \mathbf{z}_j \]  

where

\[
T_5 = \begin{bmatrix}
-(g_5 F_y j - 1 + g_7 F_y j + g_8 F_y j + 1) v_1^T \\
(g_5 F_x j - 1 + g_7 F_x j + g_8 F_x j + 1) v_1^T \\
(g_5 F_x j - 1 + g_7 F_x j + g_8 F_x j + 1) v_2^T \\
0
\end{bmatrix}
\]

Continuing the procedure, the force vector that is dependent on the unknowns, \( z_{j+1} \), is found to be

\[ \mathbf{F}_3 = T_6 \mathbf{z}_{j+1} \]

where

\[
T_6 = \begin{bmatrix}
-(g_6 F_y j - 1 + g_8 F_y j + g_9 F_y j + 1) v_1^T \\
(g_6 F_x j - 1 + g_8 F_x j + g_9 F_x j + 1) v_1^T \\
(g_6 F_x j - 1 + g_8 F_x j + g_9 F_x j + 1) v_2^T \\
0
\end{bmatrix}
\]

Therefore the expanded force vector is
\[
\tilde{f} = \tilde{F}_0 + \tilde{F}_1 + \tilde{F}_2 + \tilde{F}_3 
\]  \hspace{1cm} \text{(D-13)}

Then substituting Eqs. (D-9) through (D-12) into Eq. (D-1), and substituting for the constants \( g_1, g_2, \) and \( g_3 \) yields the recursive formula for the integration of the nonlinear equations (also Eq. (4-38)).

\[
[M + \Delta t^2 (a K - T_6)] z_{j+1} \\
- [2M + \Delta t^2 (T_5 - (\frac{1}{2} - 2\beta + \mu) K)] z_j \\
+ [M + \Delta t^2 ((\frac{1}{2} + \beta - \mu) K - T_4)] z_{j-1} \\
+ (\frac{1}{2} + \beta - \mu) B u_{j-1} + (\frac{1}{2} - 2\beta + \mu) B u_j \\
+ a B u_{j+1} = 0 
\]  \hspace{1cm} \text{(D-14)}
Figure D1 Weight Functions and Parameters for the Numerical Integration Method
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