A GEOMETRIC ANALYSIS OF MODEL REDUCTION OF LINEAR SYSTEMS

by

Michael T. DiRenzo

Thesis submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

in

Electrical Engineering

APPROVED:

Douglas K. Lindner, Chairman

Hugh F. VanLandingham

William T. Baumann

June, 1989

Blacksburg, Virginia

A GEOMETRIC ANALYSIS OF MODEL REDUCTION OF LINEAR SYSTEMS

by

Michael T. DiRenzo
Committee Chairman: Douglas K. Lindner
Electrical Engineering

(ABSTRACT)

In this thesis we study the model reduction problem in terms of the geometric concepts of linear system theory. By appropriate selection of reducing subspaces, useful lower-order system models can be achieved. The reducing subspaces can be chosen as parts of a system which are "most" and "least" controllable or observable; retaining, of course, the most controllable/observable subspace for model reduction. We review results showing how several measures of controllability and observability can provide this information. Balanced, Jordan canonical form, and dual GHR representations are shown to be state space realizations which naturally identify the reducing subspaces based on these measures. Several results unifying these methods are given.

In another approach, we show that the reducing subspaces can be chosen such that after completing model reduction, a number of Markov parameters and time moments of the full system are retained by the reduced order model. We show how the dual GHR can be used as a tool which identifies these subspaces and state space realizations which naturally display them. Along these lines, a connection between model reduction in the state space and second-order systems is established, particulary the reduction of structures via the Lanczos algorithm.

Acknowledgements

I would like to thank my mom and dad for the encouragement they have given me over the last two years and allowing me the opportunity to go to graduate school for my Master's degree. Along with my sister, , the support of my family has been more important to me than they know. Also, I extend a special thanks to the members of my advisory committee, Dr. VanLandingham and Dr. Baumann, for their help. Finally, I would like to thank my advisor, Dr. Lindner. His direction and patience in the completion of this work, as well as his investment in time, have been extraordinary. I sincerely appreciate working under his guidance; doubtlessly, it is an experience that will benefit me for the rest of my life.

Table of Contents

List of Illustrations

1.0 Introduction

In the design of a control system, the first step is development of a mathematical model for describing the behavior of the system to be controlled. This process inevitably comes down to the resolution of two conflicting factors—accuracy and simplicity [1]. Accuracy of complex models invites good control designs, however, the design of high-order control systems is often analytically difficult, sometimes impossible, and usually expensive. On the other hand, using a simplified model alleviates these problems to some degree, but at risk of producing inaccurate control designs and consequently, inadequate system performance. Clearly there is substantial interest in finding a compromise between accuracy and simplicity—a good reduced order model.

During the past twenty years, much has appeared in the literature discussing the model reduction problem. Underlying many of the simplification techniques is aggregation, which can be considered as searching for a reduced set of state variables giving a good, but coarser, description of the system. Commonly, these methods are presented on either an algebraic or geometric level. Davison's dominant mode method [2], for example, can be considered algebraically as a

truncation of terms in a partial fraction expansion of the system's transfer function. In this thesis, we discuss the model reduction problem from the geometric viewpoint.

Geometric model reduction can be thought of as an orthogonal projection of a vector space $\mathfrak{X}$, describing the full order system, onto a smaller vector space $\mathfrak{Y}$. The space $\mathfrak{Y}$, describing the system in simplified terms, is some subspace of $\mathfrak{X}$ known as a reducing subspace. Obviously, the reducing subspaces are of central importance in geometric model reduction. A key theme throughout this thesis is the proper selection of these subspaces. Measures of controllability and observability are proposed as one criterion for the selection of reducing subspaces. Model reduction, then, proceeds by projecting the state space onto the subspaces of the system which are most controllable and observable, discarding almost uncontrollable and unobservable ones. The other variable involved in geometric model reduction is the definition of the projector from $\mathfrak{X}$ to $\mathfrak{Y}$. If the coordinate basis chosen to represent the full system is selected such that the reducing subspaces are spanned by the natural basis, then the orthogonal projection of the system onto these subspaces is just a truncation of the state space description. Along these lines, several aggregational methods that we consider in this thesis are balancing [3], dominant modes [2], and dual generalized Hessenberg representation (dual GHR) [4].

A system is said to be balanced if it is represented by a (nearly unique) basis in which the controllability and observability Gramians are equal and diagonal. Moore [3] shows that the Gramians can be viewed as mappings of the unit hypersphere into hyperellipsoids. The ellipsoids are precisely the sets of states reachable by injecting unit (L_2) norm energy and observable by extracting unit (L_2) norm energy. The principal axes of these ellipsoids have lengths called

second-order modes and provide measures of relative controllability and observability in the system. Because of the orientation of these structures along the natural basis vectors in the balanced basis, it is a convenient setting for geometric model reduction based on these measures.

Modal aggregation methods rely on determining the system modes which are most important in the behavior of the system, retaining them, and deleting the others. Measures of modal controllability and observability [5] are given by angles between the left eigenvectors and the input map, B; the right eigenvectors and the output map, C^T, respectively. The residues of a system also measure this property. These measures generalize the result that a mode is uncontrollable (unobservable) if its corresponding left (right) eigenvector is orthogonal to B (C^T). Because of the eigenstructure orientation along the natural basis in the Jordan canonical form, it is the coordinate basis of choice for reduction based on these measures.

The dual GHR is a canonical state space representation of a single-input single-output system which imposes the natural basis on controllability and unobservability structures of a system: specifically, for controllability the range space of B and subspaces containing it; and for unobservability, the null space of C and subsets. Using a metric to determine the distance between these structures and A-invariant subspaces yields another set of controllability and observability measures [4]. These measures generalize the well known result that a system is unobservable (uncontrollable) if an A-invariant subspace is contained in the null space C (range space of B).

The dual GHR is originally due to Kalman [6] in the context of partial realizations derived from a sequence of Markov parameters. Thus model reduction via the dual GHR can be viewed as selecting reducing subspaces to

match Markov parameters. Indeed, selecting reducing subspaces in this way for model reduction gives an alternative criterion to measures of controllability and observability.

In this thesis we discuss, relate, and build on these ideas. Balancing, dominant modes, and dual GHR reduction methods (along with the measures of controllability and observability on which they are based) are presented in detail.

Fundamentally different, the model reduction methods can be compared through the reducing structures involved. Lindner and his colleagues [7] developed several relationships between dual GHR and modal techniques through geometrical arguments. Here, from measures supplied by the Gramians, we define weakly controllable and observable subspaces which indicate possible reduced order models by truncating balanced realizations. Geometric arguments lead to several results relating dual GHR and balancing.

The dual GHR is also presented as model reduction based on matching Markov parameters of the full system. In a variation of this idea we develop several results showing how to find reduced order models which match a number of Markov parameters and time moments of the full system. The results are reminiscent of the generalized Cauer continued fraction expansions [8]. A connection between these results and the reduction of second-order system models, specifically model reduction of structures via the Lanczos algorithm, is established.

Chapter 2 introduces the model reduction problem which we consider in this thesis. In Chapter 3, we study the Gramian and balancing. The dominant mode method and modal measures of controllability and observability is examined in Chapter 4. Chapter 5 introduces and discusses the dual GHR, continued fraction methods of model reduction, and the connection to the Lanczos algorithm for

structural system models. In Chapter 6, results are given which provide some unification of the methods which are based on controllability and observability measures. Frequency domain aspects of model reduction are considered. Finally, Chapter 7 gives the conclusions and presents some possible future research directions in this area of mathematical systems theory.

1.1 Notation and Basics

The following notation is used in this thesis. The fields of n-dimensional real vectors and complex vectors are denoted R^n and $\mathbb{C}^n$, respectively. The space of real m x n matrices is denoted $R^{m \times n}$. Script capital letters denote vector spaces and Roman capital letters denote matrices or maps. If $A : \mathfrak{X} \rightarrow \mathfrak{Y}$ then $\mathcal{N}[A]$ is the null space (kernel) of A and $\mathcal{I}m[A]$ is the image space (range space) of A. We describe a vector space, $\mathfrak{X}$, with a spanning set (basis) on R^n by,

$$\mathfrak{X} = \mathrm{sp}\left[\begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array} \right] = \left\{ y \in \mathfrak{X} \mid y = a_1 x_1 + \cdots + a_n x_n \right\}, \qquad (1.1.1)$$

and a_i are some constants. The natural basis of R^n is given by,

$$\mathrm{sp}\left[\begin{array}{cccc} e_1 & e_2 & \cdots & e_n \end{array} \right], \text{ where } e_i = \left[\begin{array}{c} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{array} \right], \qquad (1.1.2)$$

and the 1 appears in the i^{th} row. The dimension of a vector space, $\mathfrak{X}$, is denoted $\dim(\mathfrak{X})$. The orthogonal complement of a vector space, $\mathfrak{X}$, is denoted, $\mathfrak{X}^{\perp}$. The direct sum, Σ, of two vector subspaces, $\mathfrak{X}$ and $\mathfrak{Y}$ is written as $\Sigma = \mathfrak{X} \oplus \mathfrak{Y}$.

The boundary of a set, S, is denoted by $\partial(S)$. The set of eigenvalues of a matrix, A, is denoted by $\lambda(A) = \{\lambda_i\}$. The set of singular values of a matrix, A, is denoted by $\sigma(A) = \{\sigma_i\}$. The eigenvalue of largest magnitude and singular value of largest magnitude of A are denoted $\overline{\lambda}(A)$ and $\overline{\sigma}(A)$, respectively. The eigenvalue and singular value of A with smallest magnitude are denoted $\underline{\lambda}(A)$ and $\underline{\sigma}(A)$, respectively.

Definition 1.1.1: [9] Let A: $\mathbb{C}^n \rightarrow \mathbb{C}^n$ be a linear transformation, then a subspace $\mathcal{M} \subset \mathbb{C}^n$ is ***A-invariant*** if $Ax \in \mathcal{M}$ for every $x \in \mathcal{M}$ □

Definition 1.1.2: Let A: $\mathbb{C}^n \rightarrow \mathbb{C}^n$, and let the eigenvalues of A be distinct. If $A = P\Lambda Q^T$, where Λ is the diagonal matrix of the eigenvalues of A and $P^{-1} = Q^T$, where

$$P = \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix}, \text{ and } Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}, \qquad (1.1.3)$$

then p_i is a ***right eigenvector*** of A and q_i is a ***left eigenvector*** of A. □

Definition 1.1.2 means that $Ap_i = \lambda_i p_i$ and $A^T q_i = \lambda_i q_i$. Also, a set of right eigenvectors of A forms an A-invariant subspace; and a set of left eigenvectors of A forms an A^T-invariant subspace.

Several norms are used in this thesis. The norm of a vector, $\|x\|$, is given by the Euclidean vector norm,

$$\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}. \tag{1.1.4}$$

The spectral norm of a matrix, $\|A\|$, is given by,

$$\|A\| = \sup_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|} = \bar{\sigma}(A). \tag{1.1.5}$$

The Frobenius norm of a matrix, $\|A\|_F$, is given by,

$$\|A\|_F = \sqrt{\sum_{i,j} a_{ij}^2}\ , \tag{1.1.6}$$

and a_{ij} denotes the ij^{th} element in A. The L_2 norm of f(t) over the interval $t \in [0, t_0]$, $\|f(t)\|_{L_2}$, is given by,

$$\|f(t)\|_{L_2} = \Big(\int_0^{t_0} \|f(t)\|^2 dt\Big)^{1/2}. \tag{1.1.7}$$

2.0 Preliminaries

2.1 A Model Reduction Framework

In this thesis we explore the problem of approximating the system,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{2.1.1}$$

where $x \in R^n$, $u \in R$, and $y \in R$, by a simplified set of first-order, ordinary differential equations,

$$\begin{aligned} \dot{x}_a &= Fx_a + Gu, \\ y_r &= Hx_a, \end{aligned} \tag{2.1.2}$$

where $x_a \in \mathfrak{S}_a = R^p$, $y_r \in R$, and $p < n$. This is the *model reduction problem.* In this section we introduce a geometric framework for model reduction, where later we relate this procedure to model simplification by truncating balanced, Jordan

canonical form, and dual GHR realizations.

From a geometric point of view, the specific approach to model reduction that we will study in this thesis can be described in three steps [4].

Step 1: Let $\mathfrak{X} = R^n$ and define an inner product such that the vector space R^n is an inner product space. □

Step 1 is equivalent to specifying a coordinate basis for the state space of the system.

Step 2: Decompose R^n into two subspaces, $\mathfrak{X}_a$ and $\mathfrak{X}_r$, such that $\mathfrak{X}_a$ and $\mathfrak{X}_r$ form a direct decomposition of R^n and are orthogonal, $\mathfrak{X}_a \oplus \mathfrak{X}_r = R^n$ and $\mathfrak{X}_a \perp \mathfrak{X}_r$. The subspaces, $\mathfrak{X}_a$ and $\mathfrak{X}_r$, are called ***reducing subspaces.*** □

Step 3: Define the orthogonal projection $P : \mathfrak{X} \rightarrow \mathfrak{X}_a$, and obtain the reduced order model (2.1.2) from the orthogonal projections of $A : \mathfrak{X} \rightarrow \mathfrak{X}$, $B : R \rightarrow \mathfrak{X}$, and $C : \mathfrak{X} \rightarrow R$, onto the subspace $\mathfrak{X}_a$ with P as the projector. □

In step 3, if the reducing subspaces are chosen so that they are spanned by the natural basis vectors then the orthogonal projections of A,B, and C onto $\mathfrak{X}_a$ are simply appropriate truncations of the state-space description.

Example 2.1.1: Consider the following system for geometric model reduction,

$$\dot{x} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u, \tag{2.1.3}$$

$$y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x.$$

Let the natural basis given by (2.1.3) satisfy step 1 above. Choosing the reducing subspaces, $\mathfrak{S}_a = \text{sp } [e_1]$ and $\mathfrak{S}_r = \text{sp } [e_2\ e_3]$, completes step 2. Thus, in accordance with step 3, a reduced order model is obtained by truncating (2.1.3) such that the simplified model is given by,

$$\dot{x}_a = \alpha x_a + u, \tag{2.1.4}$$

$$y_r = x_a. \qquad \square$$

In Example 2.1.1, the reducing subspaces were chosen to delete x_2, an unobservable state, and x_3, an uncontrollable state. Removing uncontrollable or unobservable subspaces from a system model is logically an appropriate initial step in the model reduction process, since the effect of these subspaces is not seen in the input/output behavior of the system. Often after deleting these subspaces, however, we are still left with a high order system model. At this point, it is not clear how to best choose the reducing subspaces to further simplify the system model. In this thesis, we propose using measures of controllability and observability as one way to assist in the selection of reducing subspaces for model reduction.

2.2 Controllability and Observability

The ideas of controllability and observability of linear systems are well known.

Definition 2.2.1: [10] The system (2.1.1) is (completely state) ***controllable*** (at time t_0) if there exists a finite time t_1, $t_1 > t_0$, such that for any state, x_0, at time t_0, and any state x_1, there exists an input $u(t)$ over the time interval $t \in [t_0, t_1]$ that will transfer the state x_0 to the state x_1 at time t_1. □

Definition 2.2.2: [10] The system (2.1.1) is (completely state) ***observable*** (at time t_0) if there exists a finite time t_1, $t_1 > t_0$, such that for any state, x_0, at time t_0, the knowledge of the input $u(t)$ and the output $y(t)$ over the time interval $t \in [t_0, t_1]$ is sufficient to determine the state x_0. □

One way in which system controllability or observability can be determined is by applying the following tests,

Proposition 2.2.3: Given the system (2.1.1), let $\mathcal{V}_l$ be the space of left eigenvectors of A. Then (2.1.1) is completely controllable if for $q_i \in \mathcal{V}_l$

$$q_i^T \left[B \quad \lambda_i I - A \right] \neq 0, \text{ for all i.} \tag{2.2.1}$$

Furthermore, let $\mathcal{V}_l$ be an A^T-invariant subspace of (2.1.1). Then (2.1.1) is uncontrollable if $\mathcal{V}_l \subset \mathcal{N}[B^T]$. □

Proposition 2.2.4: Given the system (2.1.1), let $\mathcal{V}_r$ be the space of right eigenvectors of A. Then (2.1.1) is completely observable if for $p_i \in \mathcal{V}_r$

$$\begin{bmatrix} C \\ \lambda_i I\text{-}A \end{bmatrix} p_i \neq 0, \text{ for all i.} \tag{2.2.2}$$

Furthermore, let $\mathcal{V}_r$ be an A-invariant subspace of (2.1.1). Then (2.1.1) is unobservable if $\mathcal{V}_r \subset \mathcal{N}[C]$. □

Definition 2.2.1 imposes no restriction on the magnitude of the input required to drive the state vector. Likewise, Definition 2.2.2 puts no constraint on the output required for observing the state vector. It is intuitive that some systems would be more controllable or more observable than others. By asking that the input and the output be constrained to some condition, controllability and observability can be quantified, yielding *measures of controllability and observability.* Similarly, in generalizing the tests given by Proposition 2.2.3 and 2.2.4 to cases where (2.2.1) and (2.2.2) are non-zero, but small; and where (A^T) A invariant subspaces are not contained in, but are near, the null space of (B^T) C leads to another set of controllability and observability measures.

2.3 Chapter Summary

In this chapter, model reduction was formally introduced in the geometric context as a two-fold problem. First the reducing subspaces must be determined. Controllability and observability were defined and suggested as one possible criteria for selecting the reducing subspaces. Secondly, it was noted that if the basis chosen to represent the system displays the reducing subspaces along the natural basis vectors, then the orthogonal projection of the full order system state space onto the reducing subspaces is a truncation of the state matrix.

3.0 Balancing

As mentioned in Chapter 2, after removing the uncontrollable and unobservable subspaces of system, one alternative in continuing model reduction is to determine that part of the system which is least controllable and observable and delete it. In this chapter, we show how the controllability and observability Gramians can be employed to determine the least controllable and observable subspaces of a linear system. Using the resulting measures of controllability and observability motivates model reduction by balancing [3]. Balancing is the procedure which finds a state space representation where the controllability and observability Gramians are equal and diagonal. Model reduction proceeds by deleting the least controllable/observable subspaces from the model. Here we discuss the motivation for and analytical development of a balanced system, introduce geometric ideas of weak controllability and observability, and explore some of the properties of model reduction via balancing.

3.1 Controllability Gramian

Moore [3] shows that the geometry of the controllability Gramian can be used to identify the set of states reachable from the origin by a unit (L_2) norm energy input. The result can be interpreted as an ellipsoid in the state space, which provides a natural way of assigning measures of controllability to each direction in the state space. We provide proof of this result below, after some preliminary definitions.

Consider the single-input single-output system described by (2.1.1).

Definition 3.1.1: The *controllability Gramian at time* t_0, $W_c^2(t_0)$, is defined by

$$W_c^2(t_0) = \int_0^{t_0} e^{A\tau} B B^T e^{A^T \tau} d\tau. \qquad \square$$

The controllability Gramian is a symmetric, positive semi-definite matrix. Let $\lambda(W_c^2(t_0)) = \{\sigma_{ci}^2\}$ with corresponding eigenvectors $v_{c1}, \ldots, v_{cn}$. By definition,

$$\sigma_{ci}^2 = v_{ci}^T W_c^2 v_{ci}. \tag{3.1.1}$$

From (3.1.1) it follows that

$$\Sigma_c^2 = V_c^T W_c^2 V_c, \tag{3.1.2}$$

where,

$$\Sigma_c^2 = \begin{bmatrix} \sigma_{c1}^2 & & & \\ & \sigma_{c2}^2 & & \\ & & \ddots & \\ & & & \sigma_{cn}^2 \end{bmatrix} \text{ and, } V_c = \begin{bmatrix} v_{c1} & v_{c2} & \cdots & v_{cn} \end{bmatrix}. \qquad (3.1.3)$$

Definition 3.1.2: [3] The quantity, σ_{ci}, is defined as the ***i^{th} principal component magnitude of the controllability Gramian.*** □

Definition 3.1.3: [3] The vector, v_{ci}, is defined as the ***i^{th} principal component vector of the controllability Gramian.*** □

It is well known that the solution to (2.1.1) where x(0)=0 is given by,

$$x(t_0) = \int_0^{t_0} e^{A(t_0-\tau)}Bu(\tau)\, d\tau. \qquad (3.1.4)$$

Definition 3.1.4: The set of ***reachable states under a unit energy norm input at time t_0***, $S_c(t_0)$, is defined as

$$S_c(t_0) = \left\{ x(t_0) \mid x(t_0) = \int_0^{t_0} e^{A(t_0-\tau)}Bu(\tau)\, d\tau \;\; ; \;\; \left(\int_0^{t_0} \|u(\tau)\|^2\, d\tau\right)^{1/2} \leq 1 \right\}. \; \square$$

We will show that the boundary of the set S_c forms an ellipsoid. Our approach is to define a set, using principal components of the controllability Gramian, known to be an ellipsoid; and then show that this set corresponds to the boundary of S_c. This result is proven in Proposition 3.1.7, but first we state several needed intermediate results.

Theorem 3.1.5: [11] There exists a u(t), $0 \leq t \leq t_0$, which drives the state vector, x(t) in (2.1.1) from the origin at t=0 to the state $\hat{x}$ at $t=t_0$ if and only if $\hat{x}$ is in the range space of the controllability Gramian. Moreover, if q is any solution of $W_c^2\, q = \hat{x}$ then,

$$u(t) = -B^T e^{A^T(t_0-t)} q, \tag{3.1.5}$$

is one control which drives x(t) to $\hat{x}$ at $t=t_0$. □

Theorem 3.1.6: [10] Let $\hat{u}(t)$ be any control that drives (2.1.1) to $\hat{x}$ at time t_0. Let u(t) be defined by (3.1.5). Then,

$$\int_0^{t_0} \|\hat{u}(t)\|^2\, dt \geq \int_0^{t_0} \|u(t)\|^2\, dt. \qquad \square$$

Proposition 3.1.7: Let $\hat{S}_c = \left\{ x \mid x = V_c \Sigma_c \rho,\ \|\rho\| = 1 \right\}$. Then, $\hat{S}_c = \partial(S_c)$.

Proof: $\hat{S}_c$ is an ellipsoid in the state space with semi-axes equal to $\sigma_{ci} v_{ci}$. First we show that $\hat{S}_c \subset S_c$. For every $\hat{x} \in \hat{S}_c$, there exists a vector ρ, $\|\rho\| = 1$, such that

$$\hat{x} = V_c \Sigma_c \rho. \tag{3.1.6}$$

From (3.1.6), simple manipulations yield,

$$\hat{x} = V_c \Sigma_c^2\, \Sigma_c^{-1} \rho$$

$$\hat{x} = V_c \Sigma_c^2\, V_c^T\, V_c \Sigma_c^{-1} \rho$$

$$\hat{x} = W_c^2 \, V_c \Sigma_c^{-1} \rho \tag{3.1.7}$$

Equation (3.1.7) says that $q = V_c \Sigma_c^{-1} \rho$ satisfies $\hat{x} = W_c^2 q$. Thus by application of Theorem 3.1.5,

$$u(t) = -B^T e^{A^T(t_0 - t)} V_c \Sigma_c^{-1} \rho, \tag{3.1.8}$$

drives $x(t)$ to $\hat{x}$ at time $t=t_0$. Now, by showing

$$\int_0^{t_0} \| u(\tau) \|^2 \, d\tau = 1 \tag{3.1.9}$$

proves that $\hat{S}_c$ is contained in S_c. To this end,

$$\int_0^{t_0} \|u(\tau)\|^2 \, d\tau = \int_0^{t_0} u^T(\tau) \, u(\tau) \, d\tau$$

$$= \int_0^{t_0} [-B^T e^{A^T(t_0 - \tau)} V_c \Sigma_c^{-1} \rho]^T \, [-B^T e^{A^T(t_0 - \tau)} V_c \Sigma_c^{-1} \rho] \, d\tau$$

$$= \rho^T \Sigma_c^{-1} V_c^T \left(\int_0^{t_0} e^{A(t_0 - \tau)} B B^T e^{A^T(t_0 - \tau)} \, d\tau \right) V_c \Sigma_c^{-1} \rho$$

$$= \rho^T \Sigma_c^{-1} V_c^T \; W_c^2 \; V_c \Sigma_c^{-1} \rho$$

$$= \rho^T \Sigma_c^{-1} V_c^T \; V_c \Sigma_c^2 \; V_c^T \; V_c \Sigma_c^{-1} \rho$$

$$= \rho^T \rho = 1. \qquad (3.1.10)$$

From Theorem 3.1.6, u(t) is also the minimum norm input which drives x to $\hat{x}$ at time t_0, thus $\partial(S_c) \subset \hat{S}_c$. Hence, $\hat{S}_c$ must correspond to the boundary of S_c, implying that the region reachable in the state space with unit (L_2) energy norm inputs is bounded by an ellipsoid with semi-axes $\sigma_{ci} v_{ci}$. □

The physical interpretation of the Gramian as an ellipsoid in the state space provides a natural way of assigning measures of controllability to each direction in the state space. All states on the surface of the ellipsoid are equally controllable, however, some states are farther from the origin than others. We associate the direction where states lie ***farthest*** from the origin with the ***most controllable*** direction, because a unit (L_2) norm energy input can drive the state the "farthest" from the origin in that direction. Conversely, the direction where states lie ***closest*** to the origin is the ***least controllable*** direction. In the limiting case where the input drives the state zero distance away from the origin, the system is uncontrollable, and the controllability ellipsoid loses dimension.

Remark 3.1.8: Typically, we are concerned with $W_c^2(t_0)$ as $t_0 \to \infty$. It is well known that a necessary and sufficient condition for existence of the integral defining the controllability Gramian at $t_0 = \infty$, (Definition 3.1.1) is that the real part of all eigenvalues of the A matrix be strictly less than zero. The Gramian,

then, is given by the unique solution of the Lyapunov equation,

$$AW_c^2(\infty) + W_c^2(\infty)A^T = -BB^T. \tag{3.1.11}$$

As notation, unless otherwise stated, $W_c^2(\infty) = W_c^2$. □

Example 3.1.9: Consider the following system,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0.2 \\ 0.4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u. \tag{3.1.12}$$

The controllability Gramian for (3.1.12) is,

$$W_c^2 = \begin{bmatrix} 0.5868 & 0.4340 \\ 0.4340 & 0.3368 \end{bmatrix}. \tag{3.1.13}$$

An eigenvalue decomposition of (3.1.13) is,

$$W_c^2 = \begin{bmatrix} 0.7990 & -0.6014 \\ 0.6014 & 0.7990 \end{bmatrix}\begin{bmatrix} 0.9135 & \\ & 0.0101 \end{bmatrix}\begin{bmatrix} 0.7990 & 0.6014 \\ -0.6014 & 0.7990 \end{bmatrix}. \tag{3.1.14}$$

To interpret this example, the semi-axes for the controllability ellipsoid are defined by the vectors, $v_{c1}^T = [0.7990\ 0.6014]^T$ with corresponding axis length $\sigma_{c1} = \sqrt{0.9135}$ and $v_{c2}^T = [-.6014\ .7990]^T$ with corresponding axis length $\sigma_{c2} = \sqrt{0.0101}$. Clearly v_{c1} defines the more controllable direction. See Figure 1 for a graphical interpretation of this structure. □

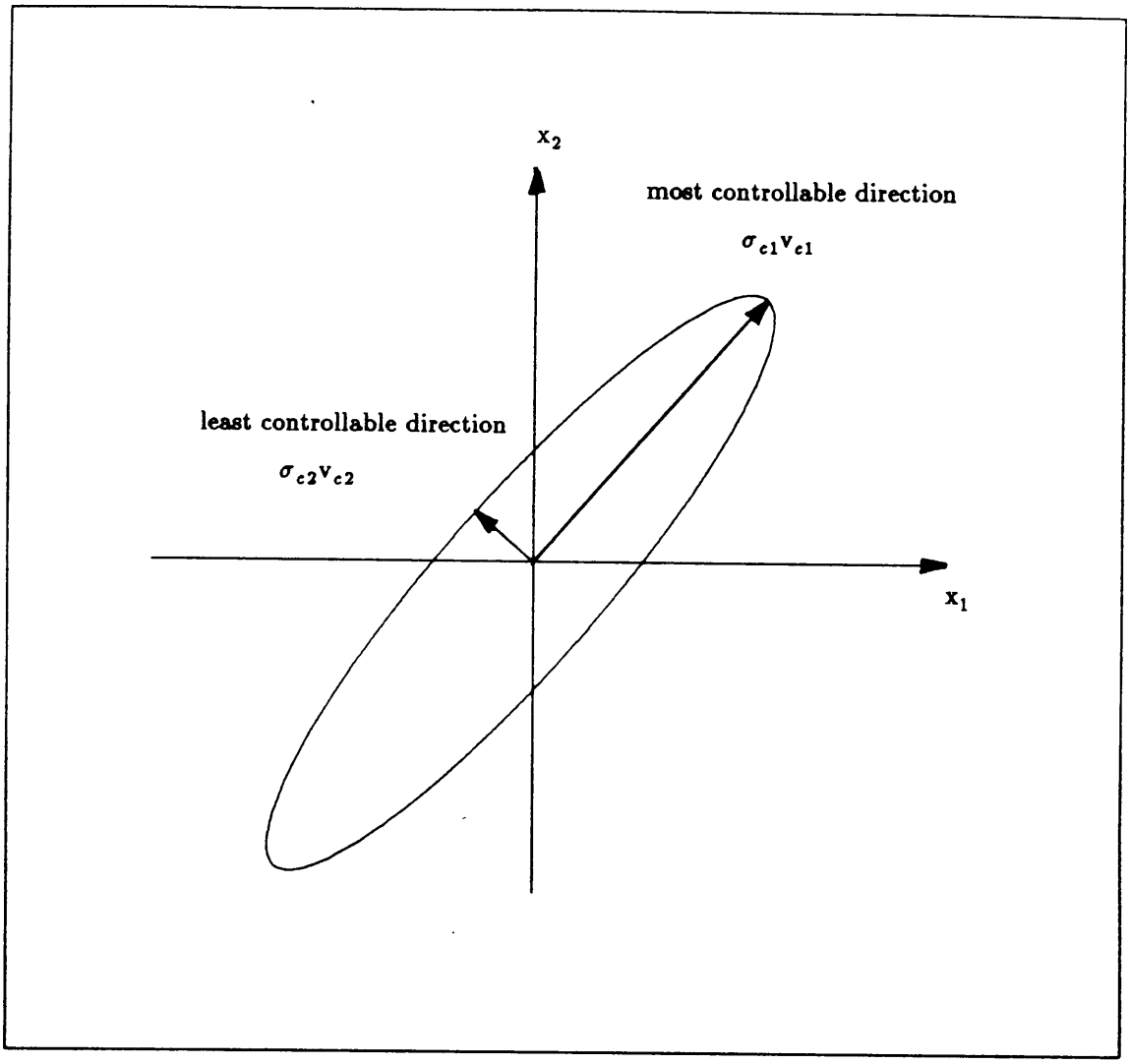

Figure 1: Controllability ellipsoid for Example 3.1.9.

3.2 Observability Gramian

In posing the dual problem of controllability, the results of Section 3.1 extend to observability. Instead of seeking an ellipsoid in the state space reachable by unit (L_2) norm energy inputs, we now search for a region in the state space of initial states where unit (L_2) norm energy can be extracted from the system such that the state of the system can be identified in a finite amount of time. Again, we find that the Gramian provides such an interpretation and prove this result below, after several preliminary definitions.

Definition 3.2.1: The ***observability Gramian at time*** t_0, $W_o^2(t_0)$, is given by

$$W_o^2(t_0) = \int_0^{t_0} e^{A^\top \tau} C^\top C e^{A\tau} \, d\tau. \qquad \square$$

The observability Gramian is a symmetric, positive semi-definite matrix. Let $\lambda(W_o^2(t_0)) = \{\sigma_{oi}^2\}$ with corresponding eigenvectors v_{o1}, v_{o2}, . . . , v_{on}. So, by definition

$$\sigma_{oi}^2 = v_{oi}^\top W_o^2 v_{oi}. \tag{3.2.1}$$

To extend (3.2.1) to matrix form,

$$\Sigma_o^2 = V_o^\top W_o^2 V_o, \tag{3.2.2}$$

where,

$$\Sigma_o^2 = \begin{bmatrix} \sigma_{o1}^2 & & & \\ & \sigma_{o2}^2 & & \\ & & \ddots & \\ & & & \sigma_{on}^2 \end{bmatrix} \quad \text{and} \quad V_o = \begin{bmatrix} v_{o1} & v_{o2} & \cdots & v_{on} \end{bmatrix}. \tag{3.2.3}$$

Definition 3.2.2: [3] The quantity, σ_{oi}, is defined as the ***i^{th} principal component magnitude of the observability Gramian.*** □

Definition 3.2.3: [3] The vector, v_{oi}, is defined as the ***i^{th} principal component vector of the observability Gramian.*** □

It is well known that the solution to (2.2.1) for $u(t) = 0$ is,

$$y(t) = Ce^{At}x(0). \tag{3.2.4}$$

Definition 3.2.4: The set of ***observable initial states under the constraint of extracting unit norm energy at time t_0***, $S_o(t_0)$, is given by,

$$S_o(t_0) = \left\{ x(0) \mid \int_0^{t_0} \| y(\tau) \|^2 \, d\tau \leq 1 \right\}. \qquad \square$$

To show that S_o describes an ellipsoid in the state space, again, our approach will be to find a set, defined by the principal components of the observability Gramian known to be an ellipsoid; and then show that it corresponds to the boundary of S_o.

Proposition 3.2.5: Let $\hat{S}_o = \left\{ x(0) \mid x(0) = V_o \Sigma_o^{-1} \rho, \ \|\rho\| = 1 \right\}$. Then $\hat{S}_o = \partial(S_o)$.

Proof: $\hat{S}_o$ is an ellipsoid in the state space with semi-axes equal to $\sigma_{oi}^{-1} v_{oi}$. The boundary of S_o is described by the set of initial states, $x(0)$, that satisfy

$$1 = \int_0^{t_0} \| y(\tau)\|^2 d\tau = \int_0^{t_0} y^T(\tau)\, y(\tau)\, d\tau$$

$$= \int_0^{t_0} [Ce^{A\tau} x(0)]^T \, [Ce^{A\tau} x(0)] \, d\tau$$

$$= x^T(0) \left(\int_0^{t_0} e^{A^T \tau} C^T C e^{A\tau} \, d\tau \right) x(0)$$

$$= x^T(0)\, W_o^2\, x(0). \tag{3.2.5}$$

Substituting (3.2.2) into (3.2.5) yields,

$$1 = \int_0^{t_0} \|y(\tau)\|^2 d\tau = x^T(0)\, V_o \Sigma_o^2 V_o^T \, x(0) \tag{3.2.6}$$

Showing that $x(0) \in \hat{S}_o$ satisfies (3.2.6) is equivalent to showing that $\hat{S}_o$ is the boundary of S_o. For every $x(0) \in \hat{S}_o$, there exists a vector ρ, $\|\rho\| = 1$, such that

$$x(0) = V_o \Sigma_o^{-1} \rho. \tag{3.2.7}$$

Substituting (3.2.7) into (3.2.6) gives,

$$x^T(0)\ V_o\Sigma_o^2V_o^T\ x(0) = \rho^T\Sigma_o^{-1}V_o^TV_o\Sigma_o^2V_o^TV_o\Sigma_o^{-1}\rho$$

$$= \rho^T\rho = 1. \tag{3.2.8}$$

Thus, $x(0) = V_o\Sigma_o^{-1}\rho$ satifies (3.2.6), hence S_o is an ellipsoid in the state space. □

The physical interpretation associated with the observability Gramian is similar to that for the controllability Gramian. The ellipsoid provides a surface such that all points on the ellipsoid are equally observable. The direction where states lie ***closest*** to the origin is the ***most observable*** direction, and the direction where states lie ***farthest*** from the origin is the ***least observable*** direction. This is exactly the opposite interpretation of the controllability Gramian and is a consequence of σ_{oi}^{-1} appearing in the description of the ellipsoid. Intuitively think of this measure as "how far in the state space does the system have to move in order to extract unit energy" (from output measurements). It is easily seen that if the system only travels a short distance in the state space, then it has moved from a more observable direction than if it had followed a longer trajectory, extracting an equal amount of energy.

Example 3.2.6: Consider the following system,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0.2 \\ 0.4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{3.2.9}$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The observability Gramian of (3.2.9) is,

$$W_o^2 = \begin{bmatrix} 0.6667 & 0.4167 \\ 0.4167 & 0.2917 \end{bmatrix}. \tag{3.2.10}$$

An eigenvalue decomposition of (3.2.10) is,

$$W_o^2 = \begin{bmatrix} 0.8398 & -0.5430 \\ 0.5430 & 0.8398 \end{bmatrix} \begin{bmatrix} 0.9361 & \\ & 0.0223 \end{bmatrix} \begin{bmatrix} 0.8398 & 0.5430 \\ -0.5430 & 0.8398 \end{bmatrix}. \tag{3.2.11}$$

The semi-axes for the observability ellipsoid in Example 3.2.6 are defined by the vectors, $v_{o1}^T = [0.8398\ 0.5430]^T$ with corresponding axis length $\sigma_{o1}^{-1} = \sqrt{0.9361}$ and $v_{o2}^T = [-0.5430\ .8398]^T$ with corresponding axis length $\sigma_{o2}^{-1} = \sqrt{0.0223}$. See Figure 2 for the geometric picture of the observability Gramian for this example. □

3.3 Balanced Realizations

Sections 3.1 and 3.2 show that the principal component magnitudes of the controllability and observability Gramians can be used to measure the relative controllability and observability in a linear system. These measures, however, are not invariant to coordinate transformation. Moore [3] offers the following example as illustration of this.

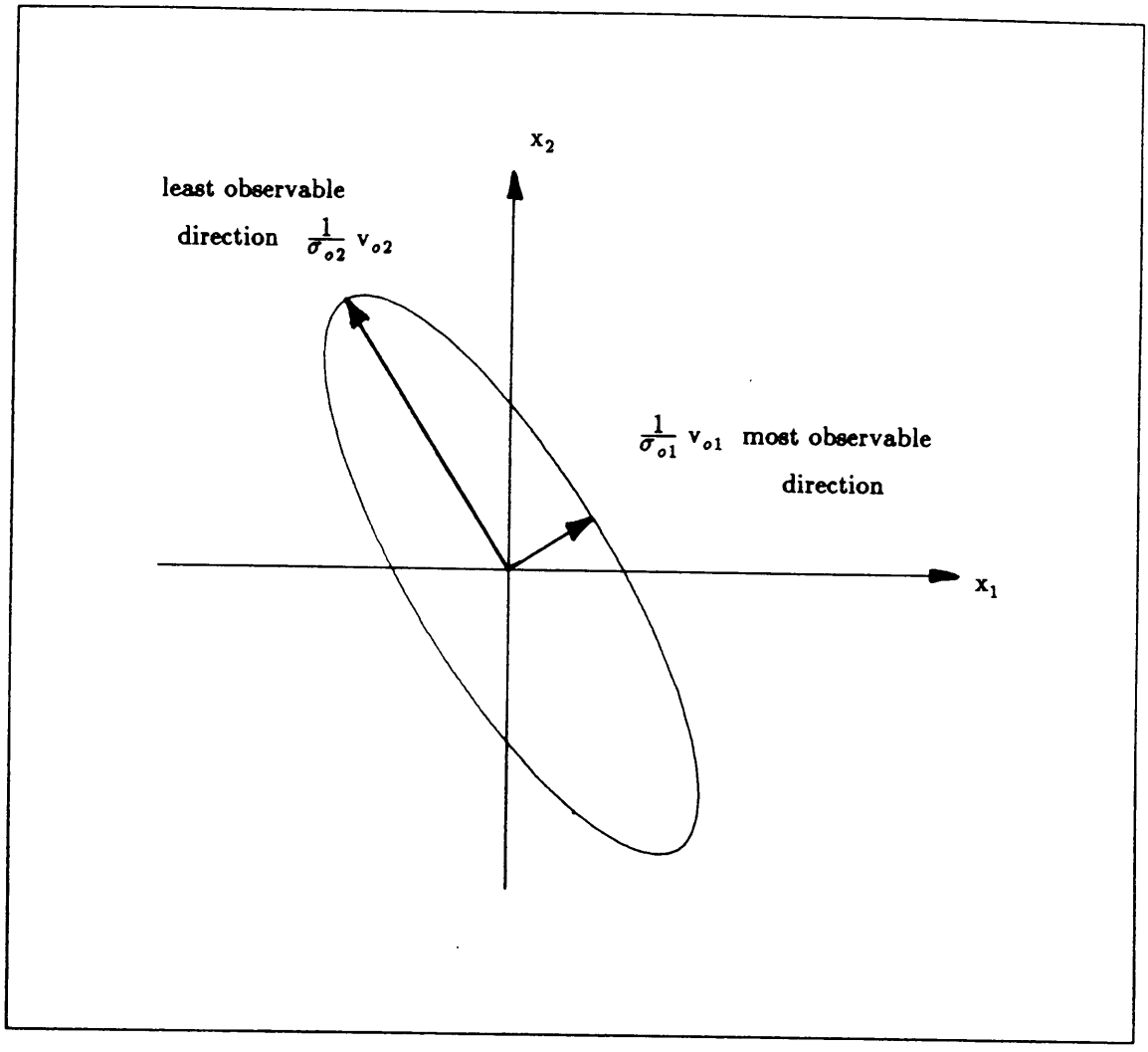

Figure 2: Observability ellipsoid for Example 3.2.6.

Example 3.3.1:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 10^{-6} \\ 10^{6} \end{bmatrix} u(t), \tag{3.3.1}$$

$$y(t) = \begin{bmatrix} 10^{6} & 10^{-6} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The principal component magnitudes of this system are,

$$\begin{aligned} \sigma_{ci} &= \{\ 5 \text{ x } 10^{5},\ 2.36 \text{ x } 10^{-7}\}, \text{ and} \\ \sigma_{oi} &= \{\ 7.07 \text{ x } 10^{5},\ 1.67 \text{ x } 10^{-7}\}. \end{aligned} \tag{3.3.2}$$

Define a transformation matrix, T,

$$T = \begin{bmatrix} 10^{-6} & 0 \\ 0 & 10^{6} \end{bmatrix}, \tag{3.3.3}$$

and perform a state transformation, $x=T\tilde{x}$, on (3.3.1). This yields a state space representation with the exact input/output characteristics of (3.3.1), but internally, radically different. The new system is given by,

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t), \tag{3.3.4}$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix},$$

and has principal component magnitudes,

$$\sigma_{ci} = \{\ 0.855,\ 0.138\ \},\ \text{and} \qquad (3.3.5)$$
$$\sigma_{oi} = \{\ 0.855,\ 0.138\ \}.$$

Although it may be tempting to set almost zero elements of the input and output matrices in (3.3.1) to exactly zero, this example shows that for poorly scaled systems, this is not the right course of action. Examining the principal component magnitudes of (3.3.1), we might be led to believe that one almost uncontrollable mode and one almost unobservable mode exist. To the contrary, (3.3.5) says that both system modes of (3.3.4) are (roughly) equally controllable and observable. This demonstrates a need to study controllability and observability properties of systems concurrently. □

By introducing a measure of the internal scaling, or in some sense the distortion of the Gramian ellipsoids, we gain an indication of which scaling is more desirable to work with.

Definition 3.3.2: [3] The ***condition number with respect to pointwise state control***, μ_c, is

$$\mu_c = \frac{\overline{\sigma}_c}{\underline{\sigma}_c}. \qquad \square$$

Definition 3.3.3: [3] The ***condition number with respect to zero-input state observation***, μ_o, is

$$\mu_o = \frac{\overline{\sigma}_o}{\underline{\sigma}_o}. \qquad \square$$

The condition numbers for (3.3.1) (μ_c=$\mu_o \approx 10^{12}$) are very large compared with

those for (3.3.4) (μ_c=μ_o=6.2) implying the latter realization has better internal scaling. Naturally the question arises, can we find a best internal scaling? By basis transformation we can make the controllability Gramian of the system equal to the identity matrix, thus setting the condition number with respect to control equal to 1. However, it can be shown that this transformation trades away observability in the system—the condition number with respect to observation grows larger. Similarly, we can find a basis such that the observability Gramian is an identity matrix, at the expense of losing controllability. Thus the motivation for balancing. By balancing a system, we transform to a basis which is the best compromise between controllability and observability. Indeed, Moore [3] shows that the max(μ_c, μ_o) is minimized (best conditioned) in a balanced representation.

Consider the SISO system (2.1.1). If we assume that the system is stable, controllable, and observable then the controllability Gramian,

$$W_c^2(t_0) = \int_0^{t_0} e^{A\tau} B B^T e^{A^T \tau} d\tau, \tag{3.3.6}$$

and the observability Gramian,

$$W_o^2(t_0) = \int_0^{t_0} e^{A^T \tau} C^T C e^{A\tau} \, d\tau, \tag{3.3.7}$$

exist and are both nonsingular for $t_0 > 0$. As seen above, the Gramians are not (individually) invariant under basis transformations on the system (2.1.1). Moore [3] shows that there always exists an equivalent system for which the Gramians are diagonal and equal.

Definition 3.3.4: [3] If the system (2.1.1) is asymptotically stable then there exists a state transformation, $x = Tz$, such that the transformed system

$$\dot{z} = A_b z + B_b u, \tag{3.3.8}$$
$$y = C_b z,$$

where $A_b = T^{-1}AT$, $B_b = T^{-1}B$, and $C_b = CT$, satisfies the Lyapunov equations,

$$A_b\Sigma^2 + \Sigma^2 A_b^T = - B_b B_b^T, \tag{3.3.9}$$
$$A_b^T\Sigma^2 + \Sigma^2 A_b = - C_b^T C_b. \tag{3.3.10}$$

The matrix $\Sigma^2 = W_c^2 = W_o^2$ and is diagonal,

$$\Sigma^2 = \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_n^2 \end{bmatrix}. \tag{3.3.11}$$

The system (3.3.8) is said to be ***balanced*** on $[0, \infty)$. □

Definition 3.3.5: [3] The singular values (eigenvalues), σ_i^2, of the controllability and observability Gramians of a balanced system are known as ***second-order modes.*** □

Remark 3.3.6: The second-order modes are coordinate invariant and are given by the singular values of the product $W_c W_o$. An algorithm for determining the

transformation matrix, T, is outlined in the appendix. Glover [12] shows that if the second-order modes are distinct, then T is unique within a similarity transformation of a diagonal matrix with ± 1's as its elements. In general, T is unique up to the point that $T\Sigma = \Sigma T$ and $T^*T = I$. Also, by convention, T is chosen such that the second-order modes appear in non-increasing order ($\sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_n^2$) in a singular value decomposition of the controllability or observability Gramian. We shall see that this ordering is important when considering model reduction. □

Example 3.3.7: Consider the following system.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -50 & -79 & -33 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u, \tag{3.3.12}$$

$$y = \begin{bmatrix} 50 & 15 & 1 & 0 \end{bmatrix} x.$$

The transformation, T, to the balanced basis is given by,

$$T = \begin{bmatrix} .0017 & -.0110 & .0216 & .0295 \\ .0018 & -.0255 & -.0351 & -.0922 \\ .0123 & .1279 & -.2140 & -.1595 \\ -.0523 & .9486 & .9285 & .1431 \end{bmatrix}. \tag{3.3.13}$$

The balanced system is,

$$\dot{x} = \begin{bmatrix} -.5183 & 1.4503 & -.3911 & -.3501 \\ -1.4503 & -2.1954 & 4.7533 & 1.2180 \\ -.3911 & -4.7533 & -.6297 & -1.1963 \\ .3501 & 1.2180 & 1.1963 & -1.6566 \end{bmatrix} x + \begin{bmatrix} .7729 \\ .8047 \\ .3373 \\ -.2523 \end{bmatrix} u, \quad (3.3.14)$$

$$y = \begin{bmatrix} .7729 & -.8047 & .3373 & .2523 \end{bmatrix}.$$

And the second-order modes are,

$$\sigma_i = \{0.5763,\ 0.1475,\ 0.0904,\ 0.0192\}. \quad (3.3.15)$$

□

3.4 Geometry and Reducing Subspaces

Definition 3.4.1: Let the singular value decomposition of the controllability Gramian for an arbitrary system (2.1.1) be given by,

$$W_c^2 = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \Sigma_{c1}^2 & \\ & \Sigma_{c2}^2 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}^T, \quad (3.4.1)$$

where the σ_{ci}^2 are ordered such that $\sigma_{c1}^2 \geq \sigma_{c2}^2 \geq \cdots \geq \sigma_{cn}^2$, $\Sigma_{c1}^2 \in R^{ixi}$ and $\Sigma_{c2}^2 \in R^{(n-i)x(n-i)}$. Let,

$$\mu_{ci} = \frac{\|\Sigma_{c2}^2\|_F}{\|\Sigma_{c1}^2\|_F}, \text{ and } \mathcal{K}_i \equiv \text{sp} \begin{bmatrix} P_{12} \\ P_{22} \end{bmatrix}. \quad (3.4.2)$$

Then $\mathcal{K}_i$ is said to be μ_{ci}-*controllable.* □

Definition 3.4.2: Let the singular value decomposition of the observability Gramian for an arbitrary system (2.1.1) be given by,

$$W_o^2 = \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{21} & \hat{P}_{22} \end{bmatrix} \begin{bmatrix} \Sigma_{o1}^2 & \\ & \Sigma_{o2}^2 \end{bmatrix} \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{21} & \hat{P}_{22} \end{bmatrix}^T, \tag{3.4.3}$$

where the σ_{oi}^2 are ordered such that $\sigma_{o1}^2 \geq \sigma_{o2}^2 \geq \cdots \geq \sigma_{on}^2$, $\Sigma_{o1}^2 \in R^{i \times i}$ and $\Sigma_{o2}^2 \in R^{(n-i)\times(n-i)}$. Let,

$$\mu_{oi} = \frac{\|\Sigma_{o2}^2\|_F}{\|\Sigma_{o1}^2\|_F}, \text{ and } \mathcal{W}_i \equiv \text{sp} \begin{bmatrix} \hat{P}_{12} \\ \hat{P}_{22} \end{bmatrix}. \tag{3.4.4}$$

Then $\mathcal{W}_i$ is said to be ***μ_{oi}-observable.*** □

Theorem 3.4.3: With respect to the balanced coordinate basis (3.3.8),

$$\mathcal{K}_i = \mathcal{W}_i = \sum_{j=i+1}^{n} \text{sp}(e_j). \tag{3.4.5}$$

Furthermore, $\mu_{ci} = \mu_{oi} = \mu_i$. □

The proof of Theorem 3.4.3 follows from applying Definitions 3.4.1 and 3.4.2 to (3.3.8).

Notice that the subspaces $\mathcal{W}_i$ and $\mathcal{W}_i^\perp$ ($\mathcal{K}_i$ and $\mathcal{K}_i^\perp$) provide a set of candidate reduced order models; the subspaces give a direct decomposition of $\mathfrak{S}$ and are orthogonal, $\mathcal{W}_i \oplus \mathcal{W}_i^\perp = R^n$ and $\mathcal{W}_i \perp \mathcal{W}_i^\perp$ by definition of orthogonal complements. For model reduction, choose a set of reducing subspaces,

$$\mathfrak{S}_a = \mathcal{W}_i^{\perp}, \qquad (3.4.6)$$
$$\mathfrak{S}_r = \mathcal{W}_i.$$

Decompose (3.3.8) with respect to (3.4.6), that is, block partition the balanced realization (3.3.8) such that

$$\begin{bmatrix} \dot{x}_a \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_a \\ x_r \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \qquad (3.4.7)$$
$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_a \\ x_r \end{bmatrix},$$

where $x_a \in R^i$, $\dot{x}_r \in R^{n-i}$, $u \in R$, $y \in R$, and all other submatrices are conformally dimensioned. Define the orthogonal projection,

$$P = \begin{bmatrix} I_i & 0 \end{bmatrix}. \qquad (3.4.8)$$

Apply (3.4.8) to (3.4.7) to obtain the reduced order model. This is equivalent to truncating (3.4.7) such that the resulting reduced order model is

$$\dot{x}_a = A_{11}x_a + B_1u, \qquad (3.4.9)$$
$$y = C_1x_a.$$

Remark 3.4.4: Theorem 3.4.3 provides n-1 candidate reduced order models. It is not always clear how to best partition the state matrix for model reduction. Many times a trade-off between model order and approximation error is involved. To help choose among the possible reduced order models we employ the second-

order modes. We will be interested in reducing subspaces where μ_i is small. Such subspaces are said to be ***weakly controllable and observable.*** □

Example 3.4.5: For the system (3.3.14) we can reduce the model order by truncating 3,2, or 1 states from the balanced system deleting the reducing subspace $\mathcal{W}_1$, $\mathcal{W}_2$, or $\mathcal{W}_3$, respectively. In each case, μ_i can be found from Definition 3.4.1 or Definition 3.4.2; μ_1=0.30, μ_2=0.16, and μ_3=0.03. In view of these values we might elect to reduce the system by only 1 state (select reducing subspaces $\mathcal{W}_3^{\perp}$ and $\mathcal{W}_3$) to,

$$\dot{x}_a = \begin{bmatrix} -.5183 & 1.4503 & -.3911 \\ -1.4503 & -2.1954 & 4.7533 \\ .3911 & -4.7533 & -.6297 \end{bmatrix} x_a + \begin{bmatrix} .7729 \\ .8047 \\ .3373 \end{bmatrix} u, \qquad (3.4.10)$$

$$y = \begin{bmatrix} .7729 & -.8047 & .3373 \end{bmatrix} x_a. \qquad \square$$

Remark 3.4.6: Glover [12], among others, has shown that reduced order models obtained by balancing exhibit the properties of stability, controllability, and observabilility. □

3.5 Chapter Summary

In this chapter, we showed that the controllability and observability Gramians characterize the states reachable and states observable under certain input-output constraints of a linear system. The result gives a way of assigning measures of controllability and observability to each direction in the state space. From the natural geometric interpretation of these measures, weakly controllable and weakly observable subspaces were defined. The need to consider controllability and observability concurrently prior to proceeding with model reduction based on these measures was shown. By balancing the system, a coordinate basis is chosen where the weakly controllable/observable subspaces coincide and fall along the natural basis vectors. Thus balancing emerges as a convenient way to perform model reduction by deleting these subspaces.

4.0 Modal Methods

Modal analysis approaches to model reduction attempt to preserve the modes of the system which have the most affect on the system response. Davison [2] introduced a method where the slow eigenvalues of the system are kept to form the reduced order model and the fast eigenvalues are discarded. It can be shown that the slow eigenvalues are not always the dominant modes. In this chapter, we introduce two measures of controllability and observability which can be used to assist in the selection of dominant modes. The modal controllability (observability) measure [5], indicating the ease of exciting (observing) a mode, measures the angle between left (right) eigenvectors and B (C^T). The residues can also be interpreted as a measure of controllability and observability for system modes. We introduce the Jordan canonical form as a coordinate basis for modal reduction, discuss modal reduction methods from a geometric viewpoint, and study some properties of model reduction using modal approaches.

4.1 Modal Measures and Residues

Using the Gramian to assign controllability and observability measures can be classified as an energy approach. These measures generalize Definitions 2.2.1 and 2.2.2 by constraining the allowable input and output with a unit (L_2) norm bound. The controllability measures, for example, rely on how the system will respond to unit (L_2) norm energy inputs. This gives an inherent bias to low frequency poles. Consider the following scalar example.

Example 4.1.1: The system,

$$\dot{x} = ax + bu, \tag{4.1.1}$$

satisfies the Lyapunov equation,

$$2a\sigma^2 = -b^2. \tag{4.1.2}$$

From (4.1.2) it is apparent that σ^2 is inversely proportional to a. Hence, the controllability measure, $|\sigma| \to \infty$ as $|a| \to 0$. □

Modal measures, generalizing Propositions 2.2.3 and 2.2.4, can be considered structural measures. Here the notion that a system is uncontrollable if a left eigenvector of A is orthogonal to B and unobservable if a right eigenvector is orthogonal to C^T is extended to the case where the vectors are almost orthogonal and the system is almost uncontrollable or unobservable.

Definition *4.1.2:* [5] Given (2.1.1). For each eigenvalue, λ_i, and associated left eigenvector, q_i, the ***modal controllability measure*** for the i^{th} mode, $\cos\phi(q_i, B)$, is

$$\cos\phi(q_i, B) = \frac{|q_i^T B|}{\|q_i\| \, \|B\|}. \qquad \square$$

Definition *4.1.3:* [5] Given (2.1.1). For each eigenvalue, λ_i, and associated right eigenvector, p_i, the ***modal observability measure*** for the i^{th} mode, $\cos\phi(p_i, C^T)$, is

$$\cos\phi(p_i, C^T) = \frac{|Cp_i|}{\|p_i\| \, \|C\|}. \qquad \square$$

Physically, the modal controllability measure can be viewed as the cosine of the angle between q_i and B. With respect to (2.2.1), when $\cos(q_i, B) = 0$, q_i is contained in the null space of B^T, and the i^{th} system mode is uncontrollable. When $\cos(q_i, B)$ is small, we say that the i^{th} mode is almost uncontrollable. Along the same lines, the modal observability measure can be interpreted as the cosine of the angle between p_i and C^T. With respect to (2.2.2), when $\cos(p_i, C^T)$ is zero, p_i is contained in the null space of C, the i^{th} mode is unobservable. When this measure is small, the i^{th} mode is almost unobservable.

Example *4.1.4:* To demonstrate modal measures consider,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0.2 \\ 0.4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \qquad (4.1.3)$$

An eigenvalue decomposition of A is,

$$A = \begin{bmatrix} 1 & -0.1861 \\ 0.3723 & 1 \end{bmatrix} \begin{bmatrix} -0.9255 & \\ & -2.0745 \end{bmatrix} \begin{bmatrix} 0.9352 & 0.1741 \\ -0.3482 & 0.9352 \end{bmatrix}. \qquad (4.1.4)$$

From (4.1.4) the modal controllability and observability measures can be directly computed using Definitions 4.1.2 and 4.1.3.

mode	$\cos\phi(C, p_i)$	$\cos\phi(B, q_i)$
-0.9255	0.8246	0.9094
-2.0745	0.4160	0.5658

(4.1.5)

Figure 3 illustrates the geometric interpretation of modal measures. □

An alternate way of assigning a measure of controllability and observability to each mode of the system is through residues.

Definition 4.1.5: Let G(s) be the transfer function of (2.1.1), $G(s) = C(sI-A)^{-1}B$. If the poles of G(s) are distinct, then the partial fraction expansion of G(s) is given by,

$$G(s) = \sum_{i=1}^{n} \frac{r_i}{(s-\lambda_i)}, \qquad (4.1.6)$$

where r_i are the ***residues*** of G(s). □

The residues of G(s) can also be found directly from the state space description.

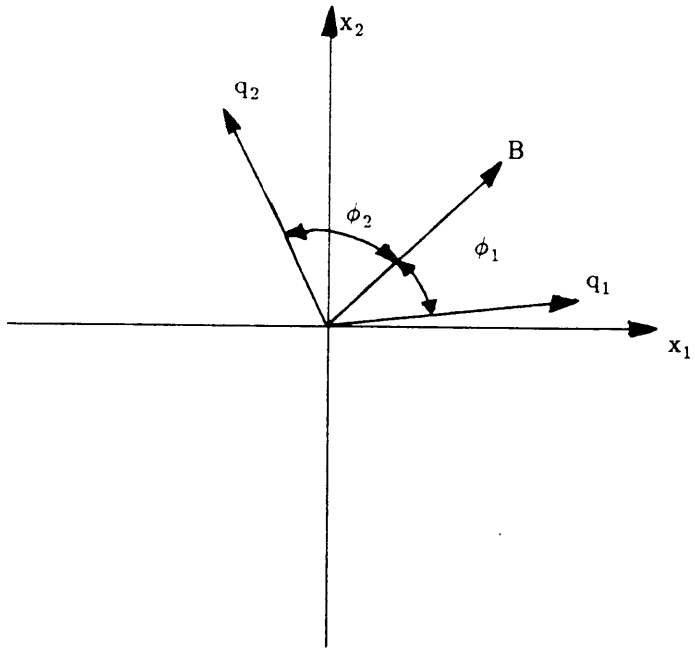

Figure 3a: Modal controllability measures for Example 4.1.4. The subscript 1 corresponds to λ = -0.9255, and the subscript 2 to λ = -2.0745.

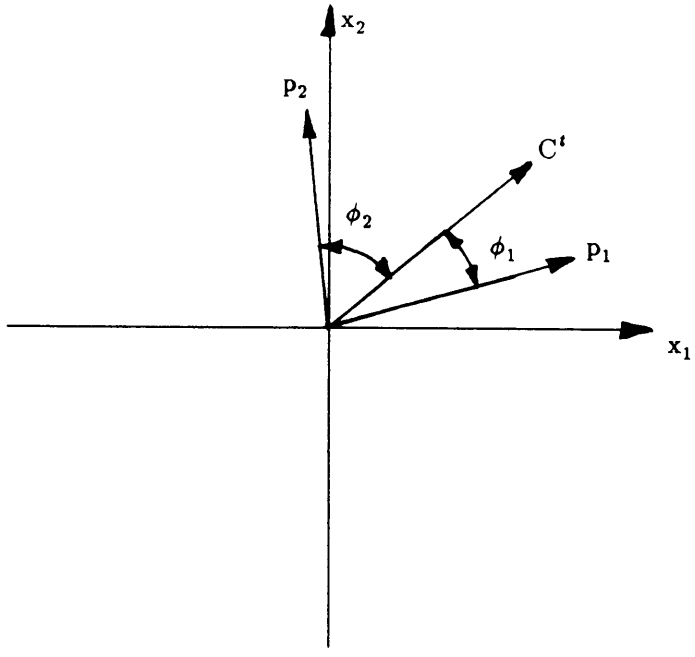

Figure 3b: Modal observability measures for Example 4.1.4.

Lemma 4.1.6: [13] Given the system (2.1.1). Let $\lambda(A)$ be distinct. Then,

$$r_i = Cp_iq_i^TB. \qquad \square$$

The modal measures defined above are not invariant to coordinate transformations. In Chapter 3, we saw the need to consider controllability and observability concurrently when contemplating model reduction based on such measures. Using the product of the modal controllability and observability measures (which is basis invariant) is one way of overcoming this. Lemma 4.1.6 shows that the residues are another. The magnitude of the residue can be viewed as a coordinate invariant, composite measure of the controllability and observability of a mode. Thus the modal measures and the residues carry the same information. The exact relationship is immediately found by applying Definitions 4.1.2 and 4.1.3 to Lemma 4.1.6 [13],

$$|r_i| = \cos\phi(C^T, p_i)\ \|p_i\|\ \|C\|\ \cos\phi(q_i, B)\ \|q_i\|\ \|B\|. \qquad (4.1.7)$$

Example 4.1.7: Consider the system (4.1.3). The transfer function can be written as,

$$G(s) = \frac{0.4778}{s+2.0745} + \frac{1.5222}{s+0.9255}. \qquad (4.1.8)$$

The residues of G(s) are 0.4778 for λ = -2.0745, and 1.5222 for λ = -0.9255. With respect to the residues, the pole at λ = -0.9255 is more controllable and observable than the pole at λ = -2.0745. As expected this agrees with Example 4.1.4. $\square$

4.2 Jordan Canonical Form

In balancing, the state-space representation is chosen such that the reducing subspaces are spanned by natural basis vectors. The reducing subspaces for the dominant mode methods we describe here are subspaces spanned by eigenvectors. Thus, it makes sense that we use the Jordan canonical form.

Consider the system (2.1.1). Let $\lambda(A)$, be distinct and in no particular order and $\{p_i\}$ be a set of right eigenvectors of A. The ***modal matrix***, P, is given by,

$$P = \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix}. \tag{4.2.1}$$

Define a state transformation $x = Pz$. Applying this transformation to (2.2.1) yields,

$$\begin{aligned} \dot{z} &= \Lambda z + P^{-1}Bu, \\ y &= CPz, \end{aligned} \tag{4.2.2}$$

where $\Lambda = P^{-1}AP$ and is given by,

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}. \tag{4.2.3}$$

The system (4.2.2) is said to be in ***Jordan canonical form.***

4.3 Geometry and Reducing Subspaces

Simply put, model reduction proceeds by partitioning (4.2.2) into two subsystems, a dominant and a residual, and then discarding the residual subsystem. The process can be formalized as follows.

Rearrange the state variables of (4.2.2) such that the eigenvalues are ordered based on some criteria from most dominant, λ_1, to least dominant, λ_n. The set of eigenvalues determined most dominant are called ***dominant modes.***

Definition 4.3.1: Let e_j be the j^{th} natural basis vector with respect to (4.2.2). The ***i^{th} dominant subspace***, $\mathcal{V}_i$, is given by,

$$\mathcal{V}_i = \sum_{j=1}^{i} \mathrm{sp}(e_j). \qquad \square$$

Definition 4.3.2: Let e_j be the j^{th} natural basis vector with respect to (4.2.2). The ***i^{th} residual subspace***, $\mathcal{V}_i^{\perp}$, is given by,

$$\mathcal{V}_i^{\perp} = \sum_{j=i+1}^{n} \mathrm{sp}(e_j). \qquad \square$$

The interaction of $\mathcal{V}_i$ and $\mathcal{V}_i^{\perp}$ provides a set of candidate reduced order models. Since $\mathcal{V}_i \perp \mathcal{V}_i^{\perp}$ and $\mathcal{V}_i \oplus \mathcal{V}_i^{\perp} = \mathfrak{X}$ for all i, any of the subspaces $\mathcal{V}_i$ and $\mathcal{V}_i^{\perp}$ are suitable choices for reducing subspaces. To continue, choose among the candidate reducing subspaces and let,

$$\mathfrak{S}_a = \mathfrak{V}_i, \tag{4.3.1}$$
$$\mathfrak{S}_r = \mathfrak{V}_i^{\perp}.$$

Decompose (4.2.2) with respect to (4.3.1) as,

$$\begin{bmatrix} \dot{x}_a \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} \Lambda_a & 0 \\ 0 & \Lambda_r \end{bmatrix} \begin{bmatrix} x_a \\ x_r \end{bmatrix} + \begin{bmatrix} B_a \\ B_r \end{bmatrix} u, \tag{4.3.2}$$
$$y = \begin{bmatrix} C_a & C_r \end{bmatrix} \begin{bmatrix} x_a \\ x_r \end{bmatrix}.$$

The reduced order model is given by,

$$\dot{x}_a = \Lambda_a x_a + B_a u, \tag{4.3.3}$$
$$y_r = C_a x_a,$$

where $x_a \in R^i$, $u \in R$, and $y \in R$.

To show the various ways in which the reducing subspaces (dominant modes) can be chosen, consider the following example.

Example 4.3.3:

$$\dot{x} = \begin{bmatrix} -14 & -43 & -30 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u, \tag{4.3.4}$$

$$y = \begin{bmatrix} 11 & 67 & 74 \end{bmatrix} x.$$

First, transform (4.3.4) to its Jordan canonical form.

$$\dot{z} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -10 \end{bmatrix} + \begin{bmatrix} 0.0556 \\ -0.6429 \\ 1.587 \end{bmatrix} u, \qquad (4.3.5)$$

$$y = \begin{bmatrix} 18 & -3.11 & 5.04 \end{bmatrix} z.$$

The system (4.3.5) is suitable for model reduction by using Davison's orginal method [2]. His criteria is to construct (4.3.2) such that $\overline{\lambda}(\Lambda_r) \ll \underline{\lambda}(\Lambda_a)$. Thus, with respect to (4.3.5), a second-order model would be formed by removing the pole at -10 giving,

$$\dot{z}_r = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} z_r + \begin{bmatrix} 0.0556 \\ -0.6429 \end{bmatrix} u, \qquad (4.3.6)$$

$$y_r = \begin{bmatrix} 18 & -3.11 \end{bmatrix} z_r.$$

The assumption that a fast eigenvalue will contribute less to the overall system response than a slow eigenvalue is not always true, and can lead to a poor reduced order model. As an alternative for selecting the reducing subspaces for (4.3.5), consider the modal measures and magnitude of the residues of (4.3.5), summarized below.

mode,λ_i	$\cos\phi(p_i,C)\cdot\cos\phi(q_i,B)$	$\lvert r_i \rvert$	
-1	0.0308	1	
-3	0.0616	2	(4.3.7)
-10	0.2464	8.	

These measures suggest that we discard the pole at -1. First re-order the state variables of (4.3.5) and then truncate to obtain the alternate second-order model,

$$\dot{\hat{z}}_r = \begin{bmatrix} -10 & 0 \\ 0 & -3 \end{bmatrix} \hat{z}_r + \begin{bmatrix} 1.587 \\ -0.6429 \end{bmatrix} u, \tag{4.3.8}$$

$$\hat{y}_r = \begin{bmatrix} 5.04 & -3.11 \end{bmatrix} \hat{z}_r.$$

The modal measures and residues fail to consider the spectral information in the system and thus, also do not always identify the dominant modes either. Because Davison's method considers only spectral information and the modal measures account for only structure, determination of dominant modes has been proposed [14] by considering the magnitude of the residue normalized by the magnitude of the corresponding pole—a compromise. With respect to (4.3.5), these measures are,

mode	$\lvert r_i \rvert / \lvert \lambda_i \rvert$
-1	1
-3	0.667
-10	0.800.

(4.3.9)

The measures given in (4.3.9) imply even a third different mode, λ = -3, to be deleted from the model. Re-ordering states and following (4.3.1)-(4.3.3) gives,

$$\dot{\tilde{z}}_r = \begin{bmatrix} -1 & 0 \\ 0 & -10 \end{bmatrix} \tilde{z}_r + \begin{bmatrix} 0.0556 \\ 1.587 \end{bmatrix} u, \tag{4.3.10}$$

$$\tilde{y}_r = \begin{bmatrix} 18 & 5.04 \end{bmatrix} \tilde{z}_r. \qquad \square$$

Remark 4.3.4: Because the eigenvalues of the simplified model are essentially selected a priori, reduced order models obtained from modal methods are stable, assuming stability of the full system. □

Remark 4.3.5: The model reduction methods described in this chapter readily extend to the general case where the Jordan canonical form of (2.1.1) is block diagonal. □

4.4 Chapter Summary

A method of assigning a measure of controllability and observability to each mode of a system from the angles between the input and output maps and the system eigenvectors was introduced. It was shown that the residues of the system carry similar information. These measures lead to the selection of dominant modes in a system which in turn determine reducing subspaces for model reduction. The Jordan canonical form is shown to display these subspaces along the natural basis vectors, and emerges as the coordinate basis for modal reduction by appropriately truncating the state-space description of the system.

5.0 Dual GHR & Related Methods

The dual GHR [4] can also be used for model reduction based on determining and then deleting the least controllable and observable portions of the full system. The dual GHR is a canonical state space representation of a single-input, single-output system which specifies, a priori, the natural basis for a set of controllability and unobservability structures of a linear system. In this chapter, we introduce the gap [4] as a measure of controllability and observability for A-invariant subspaces which lie near these structures. These measures can be directly estimated from elements in the state matrix of the dual GHR, indicating the existence of nearly uncontrollable and nearly unobservable subspaces. These subspaces assist in the identification of possible reduced order models. Also, we show that the dual GHR is intimately related to model reduction through Cauer continued fraction expansions [15]. This approach is not motivated by measures of controllability and observability, but by matching Markov parameters and time moments of a reduced system with those of the full system. Here we show how the dual GHR can be used to identify these reduced models. By defining a second set of controllability and unobservability structures, these methods are related to

geometry and reducing subspaces. Finally, we establish a relationship between model reduction of this type and the reduction of second-order systems, via the Lanczos algorithm.

5.1 The Gap Between Subspaces

The gap metric is conceptually similar to the modal measure. Recall a modal measure as the cosine of the angle between two subspaces each spanned by one vector; for instance, modal controllability measures find the angle between a left eigenvector and the range space of B. This physical interpretation, however, breaks down when we consider subspaces spanned by two or more vectors. To handle this problem, we introduce the gap, a measure of distance between subspaces.

Definition 5.1.1: [16] Let $\mathcal{U}$ and $\mathcal{V}$ be subspaces of $\mathbb{C}^n$. The ***gap*** between $\mathcal{U}$ and $\mathcal{V}$, $\tau(\mathcal{U}, \mathcal{V})$, is defined as

$$\tau(\mathcal{U},\mathcal{V}) = \max\left\{ \sup_{\|u\|=1} \inf_{v\in\mathcal{V}} \|v\text{-}u\|,\ \sup_{\|v\|=1} \inf_{u\in\mathcal{U}} \|v\text{-}u\| \right\}. \qquad \square$$

This is a formal definition which is of limited use in our purposes. Again we give a geometric interpretation.

Definition 5.1.2: [16] Let $\mathcal{U}$ and $\mathcal{V}$ be subspaces of $\mathbb{C}^n$ ($\dim(\mathcal{U}) = \dim(\mathcal{V})$) with orthonormal bases U and V, respectively. Let σ_i be the singular values of U^TV. The *canonical angles*, θ_i, between $\mathcal{U}$ and $\mathcal{V}$ are given by,

$$\theta_i = \cos^{-1}\sigma_i. \qquad \square$$

Proposition 5.1.3: [17] Let $\mathfrak{U}$ and $\mathfrak{V}$ be subspaces of R^n. Let θ_i be the canonical angles between $\mathfrak{U}$ and $\mathfrak{V}$. Then,

$$\tau(\mathfrak{U}, \mathfrak{V}) = | \sin \theta_{max} |. \qquad \square$$

So, if all of the canonical angles between two subspaces are small, then we say that the subspaces are close. By examining Proposition 5.1.3, notice that the gap is somewhat of a complementary measure to modal measures, in fact, it is exactly the complement to the modal measure when $\mathfrak{U}$ and $\mathfrak{V}$ are one dimensional subspaces (i.e. U and V are vectors). It may seem intuitive that we interpret controllability and observability in a complementary fashion as well. Recall that the system (2.1.1) was unobservable when the modal observability measure, $\cos\phi(p, C^T) = 0$. Let $\mathfrak{P} = sp[p]$. Then (2.1.1) is unobservable when

$$\tau(\mathfrak{P}, \mathfrak{Im}[C^T]) = 1. \qquad (5.1.1)$$

We now would like to extend these results to multi-dimensional subspaces.

Definition 5.1.4: [4] The ***i^{th} unobservable subspace***, $\mathfrak{L}_i$, is defined by,

$$\mathfrak{L}_i = \bigcap_{j=0}^{i-1} \mathcal{N}[CA^j], \quad \mathfrak{L}_0 = \mathfrak{X}. \qquad \square$$

Note that $\mathfrak{L}_{i+1} \subset \mathfrak{L}_i$. That is, each of these unobservable subspaces is a subset of

the null space of C.

Definition 5.1.5: [4] The i^{th} ***reachable subspace***, $\mathcal{R}_i$, is defined by,

$$\mathcal{R}_i = \sum_{j=0}^{i-1} \mathcal{I}m[A^j B], \quad \mathcal{R}_0 = 0. \qquad \square$$

From Definition 5.1.5, note that $\mathcal{R}_i \subset \mathcal{R}_{i+1}$. Thus, each of the reachable subspaces contains the image space of B.

Example 5.1.6: With respect to (5.1.1), if $\dim(\mathcal{P}) = \dim(\mathcal{N}[C]) = 1$ then (2.1.1) is also unobservable when

$$\tau(\mathcal{P}, \mathcal{N}[C]) = 0. \tag{5.1.2}$$

In general, if $\mathcal{P}$ is any A-invariant subspace of (2.1.1), then (2.1.1) is unobservable if

$$\tau(\mathcal{P}, \mathcal{L}_i) = 0, \text{ for some i.} \tag{5.1.3}$$

$\square$

We see that a system is unobservable if there exists an A-invariant subspace $\mathcal{V}_i$ contained in $\mathcal{N}[C]$, i.e. $\tau(\mathcal{V}_i, \mathcal{L}_i) = 0$. Likewise, a system is uncontrollable if there exists an A^T-invariant subspace $\mathcal{V}_j^{\perp}$ contained in $\mathcal{N}[B^T]$; which implies existence of an A-invariant subspace, $\mathcal{V}_j$, containing $\mathcal{I}m[B]$, i.e. $\tau(\mathcal{V}_j, \mathcal{R}_i) = 0$. In view of Definition 5.1.5 it will be more convenient to think about uncontrollability in these terms. It is worth noting here, that if $\dim(\mathcal{X}) \neq \dim(\mathcal{Y})$ then $\tau(\mathcal{X}, \mathcal{Y}) = 1$. So we must be careful to choose appropriately dimensioned subspaces when applying this measure.

Definition 5.1.8: [4] Let $\mathcal{V}$ be an A-invariant subspace of (2.1.1). Then $\mathcal{V}^{\perp}$ is defined to be ϵ_o-***uncontrollable*** if there exists some j such that $\tau(\mathcal{R}_j, \mathcal{V}) \leq \epsilon_o$. □

Definition 5.1.9: [4] Let $\mathcal{V}$ be an A-invariant subspace of (2.1.1). Then $\mathcal{V}$ is defined to be ϵ_o-***unobservable*** if there exists some j such that $\tau(\mathcal{L}_j, \mathcal{V}) \leq \epsilon_o$. □

Example 5.1.10: For the system (3.3.12), we illustrate the calculations of some observability and controllability measures using the gap metric. For (3.3.12), orthonormal bases for $\mathcal{L}_1$ and $\mathcal{R}_2$ are given by,

$$\mathcal{L}_1 = \text{sp } [L_1] = \mathcal{N}[C] = \text{sp} \begin{bmatrix} 0.0154 & -0.2875 & 0 \\ 0.0154 & 0.9577 & 0 \\ -0.9998 & 0.0103 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{5.1.4}$$

$$\mathcal{R}_2 = \text{sp } [R_2] = \mathcal{I}m[B] \oplus \mathcal{I}m[AB] = \text{sp} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.1961 & -0.9806 \\ -0.9806 & -0.1961 \end{bmatrix}. \tag{5.1.5}$$

From the set of right eigenvectors an A-invariant subspace, $\mathcal{V}_1$, is given by the orthonormal basis,

$$\mathcal{V}_1 = \text{sp}[V_1] = \text{sp} \begin{bmatrix} .1085 & -1.934+j.0311 & .4308-j.1407 \\ -.2169 & .4404-j.0061 & -.6688+j.2474 \\ .4339 & -.6601-j.3071 & -.5261-j.0136 \\ -.8677 & -.4643-j.1481 & -.0420-j.0862 \end{bmatrix}. \tag{5.1.6}$$

The singular values of $L_1^T V_1 = \{1, 1, 0.9714\}$. From Definition 5.1.2, the canonical angles are $\theta_i = \{0, 0, \text{and } 13.7^\circ\}$. So from Proposition 5.1.3, $\tau(\mathcal{L}_1, \mathcal{V}_1) = |\sin \theta_{max}| = \sin 13.7^\circ = 0.2374$.

An orthonormal basis for a second A-invariant subspace, $\mathcal{V}_2$, is given by,

$$\mathcal{V}_2 = \text{sp}[V_2] = \text{sp}\begin{bmatrix} .5 & .513 \\ -.5 & -.326 \\ .5 & -.047 \\ -.5 & .793 \end{bmatrix}. \tag{5.1.7}$$

The singular values of $R_2^T V_2 = \{.9949, .3749\}$. The canonical angles are $\theta_i = \{5.8^\circ, 68^\circ\}$. Then, $\tau(\mathcal{R}_2, \mathcal{V}_2) = |\sin \theta_{max}| = \sin 68^\circ = 0.9271$.

We might say that $\mathcal{V}_1$ is nearly unobservable. $\mathcal{V}_2^\perp$, however, is not nearly uncontrollable. Note that there are a number of A-invariant subspaces that we might have used in this example. When determining system controllability or observability with respect to these measures, all A-invariant subspaces must be checked. □

5.2 Dual GHR Realizations

For model reduction based on the measures of controllability and observability of the previous section, it is desirable to find a coordinate basis such that the reachable subspaces and the unobservable subspaces are spanned by the natural basis vectors. The dual GHR is such a representation. For the SISO system

(2.1.1) there exists a state space transformation x = Tz, such that the transformed system,

$$\dot{z} = Fz + Gu, \qquad (5.2.1)$$
$$y = Hz,$$

where F= $T^{-1}AT$, G = $T^{-1}B$, and H=CT takes on the following canonic form,

$$F = \begin{bmatrix} F_1 & H_2 & 0 & \cdots & 0 \\ G_2 & F_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & H_r \\ 0 & \cdots & 0 & G_r & F_r \end{bmatrix},$$

$$G = \begin{bmatrix} G_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad H^T = \begin{bmatrix} H_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad (5.2.2)$$

and,

$$F_k = \begin{bmatrix} a_{k1} & 1 & & \\ \vdots & & \ddots & \\ \vdots & & & 1 \\ a_{k\sigma_k} & & & \end{bmatrix}, \; G_k = \begin{bmatrix} & \\ \gamma_k & \end{bmatrix}, \; H_k = \begin{bmatrix} & \\ \epsilon_k\gamma_k & \end{bmatrix}. \qquad (5.2.3)$$

This representation is the ***dual GHR*** [4]. All matrix entries left blank are zero entries. The state matrix has a block tridiagonal structure with $F_k \in R^{\sigma_k x \sigma_k}$. The diagonal blocks are in a phase canonical form. The off diagonal blocks have only one non-zero element which always appears in the lower left hand corner. The matrix $G_k \in R^{\sigma_k x \sigma_{k-1}}$ and $H_k \in R^{\sigma_{k-1} x \sigma_k}$. Also, as a convention we have chosen $\gamma_k > 0$ and $\epsilon_k = \pm 1$. To gain familiarity with the dual GHR we offer these examples.

Example 5.2.1: Consider the system (3.3.12). The transformation, T, to the dual GHR is,

$$T = \begin{bmatrix} 0 & 0 & 0 & .0198 \\ 0 & 0 & .0266 & -.0921 \\ 1 & 0 & -.3986 & .3934 \\ -5 & 1 & 4.6506 & -1.2974 \end{bmatrix}, \tag{5.2.4}$$

and the dual GHR of this system is,

$$\dot{x} = \begin{bmatrix} 10.0 & 1 & 0 & 0 \\ -133 & 0 & 37.63 & 0 \\ 37.63 & 0 & -10.339 & -1.345 \\ 0 & 0 & 1.345 & -4.66 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u, \tag{5.2.5}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x. \qquad \square$$

Example 5.2.2: Consider the system given by,

$$G(s) = \frac{(s+79.142)(s+10.619)(s+40.23)(s+0.10876)}{(s+2.203)(s+20.59)(s+70.1)(s+79.143)(s+0.10868)}. \tag{5.2.6}$$

The dual GHR of this system is,

$$\dot{x} = \begin{bmatrix} -42.04 & 30.36 & 0 & 0 & 0 \\ 30.36 & -34.82 & 11.45 & 0 & 0 \\ 0 & 11.45 & -16.05 & 0.95 & 0 \\ 0 & 0 & 0.95 & -79.05 & 2.41 \\ 0 & 0 & 0 & 2.41 & -0.18 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u, \tag{5.2.7}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} x. \qquad \square$$

Remark 5.2.3: The system (5.2.7) shows the generic form of the dual GHR. It occurs when σ_i=1 for all i, so F is a tridiagonal matrix and $H = G^T$ (modulo a sign). When the dual GHR takes this form, we say that the system is ***regular***. □

Remark 5.2.4: The transformation matrix, T, can be constructed from an extension of chained aggregation [18]. The algorithm is outlined in detail [19], and an algorithm for computing the dual GHR using MATLAB is given in the appendix. □

5.3 Geometry and Reducing Subspaces

Recall Definitions 5.1.4 and 5.1.5, the i^{th} reachable and i^{th} unobservable subspaces, respectively. Let e_j be the j^{th} unit basis vector with respect to the dual GHR basis (5.2.1).

Theorem 5.3.1: [20] For a n^{th} order system represented by (5.2.1),

$$\mathcal{L}_i = \sum_{j=i+1}^{n} \mathrm{sp}(e_j), \tag{5.3.1a}$$

For $m = 1, \ldots, r$

For $k = 0, \ldots, \sigma_m - 1$

$$\mathcal{R}_i = \mathcal{R}_{m+k} = \mathcal{R}_{\rho_{(m-1)}} + \sum_{j=\rho_m - k}^{\rho_m} \mathrm{sp}(e_j), \tag{5.3.1b}$$

where $\rho_i = \sum_{j=1}^{i} \sigma_j$, $\rho_0 = 0$. □

Remark 5.3.2: The subspaces $\mathcal{L}_i$ characterize the output structure of the system (the null space of C and subsets) and the subspaces $\mathcal{R}_i$ similarly characterize the input structure of the system (the range space of B and subspaces containing it). We can specify other bases in which either the output structure or the input structure of the system is displayed by the state space representation. Theorem 5.3.1, however, implies that the dual GHR simultaneously identifies both sets of subspaces. Thus, the dual GHR has a unique property of identifying the input-output interaction in terms of the internal state. □

Example 5.3.3: As an example of Theorem 5.3.1 consider the system in dual GHR form,

$$\dot{x} = \begin{bmatrix} \alpha_1 & 1 & & & & & \\ \alpha_2 & 0 & \epsilon_2\gamma_2 & & & & \\ \gamma_2 & & \beta_1 & \epsilon_3\gamma_3 & & & \\ & & & \delta_1 & 1 & 0 & \\ & & & \delta_2 & 0 & 1 & \\ & & \gamma_3 & \delta_3 & 0 & 0 & \end{bmatrix} x + \begin{bmatrix} \gamma_1 \\ \\ \\ \\ \\ \\ \end{bmatrix} u, \qquad (5.3.2)$$

$$y = \begin{bmatrix} \epsilon_1\gamma_1 & 0 & & & & & \end{bmatrix} x.$$

For (5.3.2), σ_1=2, σ_2=1, σ_3=3; ρ_1=2, ρ_2=3, ρ_3=6. By application of Theorem 5.3.1 we have,

$$\begin{aligned} \mathcal{L}_1 &= \mathrm{sp}\{e_2, \ldots, e_6\} & \mathcal{R}_1 &= \mathrm{sp}\{e_2\} \\ \mathcal{L}_2 &= \mathrm{sp}\{e_3, \ldots, e_6\} & \mathcal{R}_2 &= \mathrm{sp}\{e_1, e_2\} \\ \mathcal{L}_3 &= \mathrm{sp}\{e_4, e_5, e_6\} & \mathcal{R}_3 &= \mathrm{sp}\{e_1, e_2, e_3\} \\ \mathcal{L}_4 &= \mathrm{sp}\{e_5, e_6\} & \mathcal{R}_4 &= \mathrm{sp}\{e_1, e_2, e_3, e_6\} \\ \mathcal{L}_5 &= \mathrm{sp}\{e_6\} & \mathcal{R}_5 &= \mathrm{sp}\{e_1, e_2, e_3, e_5, e_6\} \\ \mathcal{L}_6 &= 0 & \mathcal{R}_6 &= \mathfrak{X} \end{aligned} \qquad (5.3.3)$$

□

The interaction of the subspaces $\mathcal{L}_i$ and $\mathcal{R}_i$ also provides a set of candidate reduced order models, however $\mathcal{L}_i$ and $\mathcal{R}_i$ must be chosen so that $\mathcal{L}_i \perp \mathcal{R}_i$ and $\mathcal{L}_i \oplus \mathcal{R}_i = \mathfrak{X}$. Thus, not all the subspaces $\mathcal{L}_i$, $\mathcal{R}_i$ are reducing subspaces. In particular, the subspaces must be chosen so that none of the diagonal blocks is split. We should note that for regular systems, all subspaces $\mathcal{L}_i$ and $\mathcal{R}_i$ provide a direct decomposition of $\mathfrak{X}$ and may be chosen as reducing subspaces. In Example 5.3.3, we see that the direct decompositions of $\mathfrak{X}$ are $\mathcal{L}_2 \oplus \mathcal{R}_2$ and $\mathcal{L}_3 \oplus \mathcal{R}_3$ which

says that only two candidate reduced order models exist. The reduction process can be formalized as follows [4].

Define the subspace $\mathcal{M}_i$ by,

$$\begin{aligned} \mathcal{R}_{\rho_{i-1}} + \mathcal{M}_i + \mathcal{L}_{\rho_i} &= \mathrm{R}^n, \\ \mathcal{R}_{\rho_{i-1}} \perp \mathcal{M}_i \perp \mathcal{L}_{\rho_i}, \; i &= 1, \ldots, \mathrm{r}. \end{aligned} \tag{5.3.4}$$

Proposition 5.3.4: Suppose $\mathcal{M}_i$ are defined as in (5.3.4). Then with respect to the basis defined by the dual GHR, we have

$$\mathcal{M}_i = \sum_{j=\rho_{i-1}+1}^{\rho_i} \mathrm{sp}(\mathrm{e}_j) \qquad i = 1, \ldots, \mathrm{r}. \qquad \square$$

The decomposing subspaces $\mathcal{M}_i$ suggest a set of candidate reduced order models. One possible model can be constructed as follows. Choose the reducing subspaces by,

$$\begin{aligned} \mathfrak{S}_a &= \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_i = \mathcal{R}_{\rho_i}, \\ \mathfrak{S}_r &= \mathcal{M}_{i+1} \oplus \cdots \oplus \mathcal{M}_r = \mathcal{L}_{\rho_i}. \end{aligned} \tag{5.3.5}$$

Decompose (5.2.1) with respect to (5.3.5) as,

$$\begin{bmatrix} \dot{\mathrm{x}}_a \\ \dot{\mathrm{x}}_r \end{bmatrix} = \begin{bmatrix} \mathrm{F}^i & \mathrm{H}^{i+1} \\ \mathrm{G}^{i+1} & \mathrm{E}^{i+1} \end{bmatrix} \begin{bmatrix} \mathrm{x}_a \\ \mathrm{x}_r \end{bmatrix} + \begin{bmatrix} \mathrm{G}^1 \\ 0 \end{bmatrix} \mathrm{u}, \tag{5.3.6}$$

$$\mathrm{y} = \begin{bmatrix} \mathrm{H}^1 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{x}_a \\ \mathrm{x}_r \end{bmatrix},$$

where,

$$F^i = \begin{bmatrix} F_1 & H_2 & & \\ G_2 & \ddots & \ddots & \\ & \ddots & \ddots & H_i \\ & & G_i & F_i \end{bmatrix}, \; G^{i+1} = \begin{bmatrix} & G_{i+1} \\ & \end{bmatrix},$$

$$H^{i+1} = \begin{bmatrix} & \\ H_{i+1} & \end{bmatrix}, \text{ and } E^{i+1} = \begin{bmatrix} F_{i+1} H_{i+2} & & \\ G_{i+2} & \ddots & \ddots & \\ & \ddots & \ddots & H_n \\ & & G_n & F_n \end{bmatrix}. \tag{5.3.7}$$

The reduced order model is given by,

$$\begin{aligned} \dot{x}_a &= F^i x_a + G^1 u, \\ y_r &= H^1 x_a, \end{aligned} \tag{5.3.8}$$

where $x_a \in R^{\rho_i}$, $u \in R$, and $y \in R$.

5.4 Near Uncontrollability/Unobservability

As the ideas of weak observability and weak controllability are used for selection of reducing subspaces in balancing, we introduce near uncontrollability and near unobservability as criteria for selecting the reducing subspaces for the dual GHR. With regard to Definitions 5.1.8 and 5.1.9, we are interested in subspaces where ϵ_o is small. In this case, we say that the subspaces are nearly uncontrollable and nearly unobservable, respectively.

Although the gap can be computed from Definition 5.1.2 and Proposition 5.1.3, it is useful to estimate the gap as follows. Suppose that the natural basis yields a basis representation for $\mathcal{R}_i$ and $\mathcal{L}_i$. (This is exactly the representation implied in the dual GHR basis). Then, in matrix form,

$$\mathcal{R}_i = \mathrm{sp}\begin{bmatrix} I_{\rho_i} \\ 0 \end{bmatrix} = \mathrm{sp}\,[R_i], \quad \mathcal{L}_i = \mathrm{sp}\begin{bmatrix} 0 \\ I_{n-\rho_i} \end{bmatrix} = \mathrm{sp}\,[L_i]. \tag{5.4.1}$$

Let a second subspace, $\mathcal{V}$, and its orthogonal complement, $\mathcal{V}^\perp$, be spanned by the orthonormal basis,

$$\mathcal{V}^\perp = \mathrm{sp}[V_c], \quad \mathcal{V} = \mathrm{sp}[V]. \tag{5.4.2}$$

One way to form (5.4.2) is,

$$[V_c\ V] = \begin{bmatrix} I_{\rho_i} & P \\ -P^T & I_{n-\rho_i} \end{bmatrix} \begin{bmatrix} (I+PP^T)^{-1/2} & 0 \\ 0 & (I+P^TP)^{-1/2} \end{bmatrix}. \tag{5.4.3}$$

Any orthonormal set of vectors can be written in the form of (5.4.3).

Now, we want to compute the canonical angles between $\mathcal{L}_i$ and $\mathcal{V}$. From (5.4.3) an orthonormal basis spanning $\mathcal{V}$ is,

$$\mathcal{V} = \mathrm{sp}\begin{bmatrix} P(I+P^TP)^{-1/2} \\ (I+P^TP)^{-1/2} \end{bmatrix}. \tag{5.4.4}$$

To compute canonical angles, first form the product,

$$L_i^TV = (I + P^TP)^{-1/2}. \tag{5.4.5}$$

Let P have singular values, σ_i. Then, the canonical angles between $\mathcal{L}_i$ and $\mathcal{V}$ are,

$$\theta_i = \cos^{-1}(1 + \sigma_i^2)^{-1/2}. \tag{5.4.6}$$

Solving for σ_i,

$$\cos\,\theta_i = (1 + \sigma_i^2)^{-1/2} \tag{5.4.7}$$

$$\cos^2\theta_i = \frac{1}{1 + \sigma_i^2} \tag{5.4.8}$$

$$\cos^2\,\theta_i + \sigma_i^2\,\cos^2\theta_i = 1 \tag{5.4.9}$$

$$\sigma_i^2\,\cos^2\theta_i = \sin^2\theta_i \tag{5.4.10}$$

$$\sigma_i = \tan\,\theta_i \tag{5.4.11}$$

Since $\tan\,\theta \geq \sin\,\theta$ for all θ and $\|\mathrm{P}\| = \overline{\sigma}(\mathrm{P}) = \sigma_{i,max}$, it follows that,

$$\tau(\mathcal{L}_i, \mathcal{V}) = |\sin\,\theta(\mathcal{L}_i, \mathcal{V})| \leq |\tan\,\theta(\mathcal{L}_i, \mathcal{V})| = \|\mathrm{P}\|. \tag{5.4.12}$$

Proposition 5.4.1: [17] Given (2.1.1), if there exists an A-invariant subspace $\mathcal{V}$ of the form (5.4.4) then,

$$\epsilon_o = \tau(\mathcal{L}_i, \mathcal{V}) \leq \|\mathrm{P}\| = \frac{\epsilon_o}{\sqrt{(1-\,\epsilon_o^2)}}. \qquad \square$$

Definition 5.4.2: [17] Consider F^i and E^{i+1} as defined in (5.3.6) and P as defined in (5.4.3). Let T be a linear operator in P such that,

$$T(P) = F^i P - P E^{i+1}. \tag{5.4.13}$$

The eigenvalues of T are non-zero iff F^i and E^{i+1} have no common eigenvalues. The ***separation***, δ, of F^i and E^{i+1} is given by,

$$\delta = \begin{cases} \|T^{-1}\|^{-1}, & 0 \notin \lambda(T) \\ 0\ , & 0 \in \lambda(T) \end{cases} \qquad \square$$

Theorem 5.4.3: [17] Given the system (2.2.1) and that the system is separable ($\delta > 0$). If

$$\frac{\|H^{i+1}\|\ \|G^{i+1}\|}{\delta^2} = \frac{\gamma_{i+1}^2}{\delta^2} < \frac{1}{4} \tag{5.4.14}$$

then there exists a matrix P_j such that $\|P_j\| \leq \frac{2\gamma_{i+1}}{\delta}$ j = 1,2 where,

$$\mathcal{V}_1 = \text{sp}\begin{bmatrix} P_1 \\ I \end{bmatrix}, \text{ and } \mathcal{V}_2 = \text{sp}\begin{bmatrix} I \\ P_2 \end{bmatrix}, \tag{5.4.15}$$

are A-invariant subspaces. $\square$

Remark 5.4.4: We search for reducing subspaces, $\mathcal{L}_j$ and $\mathcal{R}_j$, such that $\tau(\mathcal{L}_j, \mathcal{V}_2) < \epsilon_o$ and $\tau(\mathcal{R}_j, \mathcal{V}_1) < \epsilon_o$ where ϵ_o is small. The results of this section formalize that procedure. Theorem 5.4.3 defines conditions on the existence of an appropriately dimensioned A-invariant subspace. Combining this result with (5.4.12) yields the inequality,

$$\frac{2\gamma_{i+1}}{\delta} \geq \|P\| \geq \tau(\mathcal{L}_i, \mathcal{V}). \tag{5.4.16}$$

From (5.4.16) we can see that as $\gamma_{i+1} \rightarrow 0$, $\tau(\mathcal{L}_i, \mathcal{V}) \rightarrow 0$. Thus, by inspection of

the state matrix of the dual GHR, (5.2.1), if we find a small super(sub)-diagonal element, γ_{i+1}, then we should expect to find an A-inviariant subspace, $\mathcal{V}$, such that $\tau(\mathcal{L}_i, \mathcal{V})$ is small. □

The exact computation of the subsystem separation, δ, is not well understood. Thus, in (5.4.16) the gap between $\mathcal{L}_i$ and $\mathcal{V}$ may not be small when γ_{i+1} is small. Consequently, we do not propose (5.4.16) as a vehicle for computing a bound for $\tau(\mathcal{L}_i, \mathcal{V})$. Equation (5.4.16) simply provides insight into the selection of the reducing subspaces as mentioned above. Furthermore, as noted in Example 5.1.10, for a large system, there are many A-invariant subspaces to consider. As the calculation quickly becomes an involved task, estimating $\tau(\mathcal{L}_i, \mathcal{V})$ via (5.4.16) and the state matrix of the dual GHR becomes convenient.

Example 5.4.5: Given the system (5.2.5), already in dual GHR form, we wish to find an approximate reduced order model. From inspection of the state matrix, we see that γ_4= -1.34 is "small" compared with the other matrix elements. Thus, we should look for an A-invariant subspace, $\mathcal{V}$, near $\mathcal{L}_3$. Computing the gaps between all one-dimensional A-invariant subspaces and $\mathcal{L}_3$ we find that the tightest bound for $\tau(\mathcal{L}_3, \mathcal{V}) = 0.9866$. Apparently (in reference to Remark 5.4.4) γ_4 is not small enough and/or δ is not large enough. Thus, we have no motivation for model reduction. Note the difference between this example (dual GHR) and Example 3.4.5 (balancing) where we found small second-order modes prompting us to simplify the model. □

Example 5.4.6: Consider the following system (in dual GHR form) for model reduction.

$$\dot{x} = \begin{bmatrix} -17.21 & 1 & 0 & 0 \\ 26.44 & 0 & -17.13 & 0 \\ 17.13 & 0 & -10.00 & -.187 \\ 0 & 0 & .187 & -1.30 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u, \qquad (5.4.17)$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x.$$

Again, we see from examination of the state matrix of (5.4.17) that γ_4=-.187 is "small" and we should look for an A-invariant subspace close to $\mathcal{L}_3$.

$$\mathcal{L}_3 = \text{sp}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ and, } \mathcal{V} = \text{sp}\begin{bmatrix} -.0272 \\ -.4325 \\ -.0751 \\ 1 \end{bmatrix}. \qquad (5.4.18)$$

From (5.4.18) we can calculate $\tau(\mathcal{L}_3, \mathcal{V}) = 0.4025$. This is a relatively small gap, thus we might elect to reduce the system by deleting $\mathcal{L}_3$ giving,

$$\dot{x}_a = \begin{bmatrix} -17.21 & 1 & 0 \\ 26.44 & 0 & -17.13 \\ 17.13 & 0 & -10.00 \end{bmatrix} x_a + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u, \qquad (5.4.19)$$

$$y_r = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x_a.$$

Note that the gap can be estimated from Proposition 5.4.1, $\tau(\mathcal{L}_3,\mathcal{V}) \leq \|P\| = 0.4398$. □

Remark 5.4.7: Reduced order models obtained from the dual GHR are controllable and observable, however, stability cannot be guaranteed. □

5.5 Continued Fractions, Markov Parameters, & Moments

Continued fraction expansions (CFE's) have played an important role in the partial realization problem and the model reduction problem. The methods can be described as expanding the transfer function of a system into a certain continued fraction in which each successive quotient has decreasing importance in characterizing system behavior. In this section, we show how several continued fraction approaches are related to the dual GHR. First, we review some results from partial realization theory.

Definition 5.5.1: The ***Markov parameters***, J_i, of a system $G(s) = C(sI\text{-}A)^{-1}B$ are given by,

$$J_i = CA^{i-1}B \quad \text{for } i = 1,2, \ldots \qquad \square$$

Definition 5.5.2: The ***time moments***, Y_i, of a system $G(s) = C(sI\text{-}A)^{-1}B$ are given by,

$$Y_i = CA^{-i}B \text{ for } i = 1,2, \ldots \qquad \square$$

Given a finite sequence of Markov parameters, $J = \{J_1, J_2, \ldots, J_N\}$, the (scalar) ***partial realization problem*** is to determine a system,

$$\dot{x}_n = A_n x_n + B_n u, \qquad (5.5.1)$$
$$y = C_n x_n,$$

where $x_n \in R^n$, $u \in R$, $y \in R$, and n is as small as possible. Let Σ_n, denote the partial realization given by (5.5.1). Rissanen [21] showed that every system may be realized sequentially in such a way that successive realizations $\Sigma_0, \Sigma_1, \ldots, \Sigma_n$ each contain all previous realizations as a subsystem. The dual GHR is such a method [6]. Partial realizations of this type are called ***nested.***

Example 5.5.3: Consider the system (5.3.6) already in dual GHR form. For each i, $0 \leq i \leq r$, a partial realization, Σ_i, is given by

$$\dot{x}_a = F^i x_a + G^1 u, \qquad (5.5.2)$$
$$y_a = H^1 x_a,$$

where $x_a \in R^{\rho_i}$, $u \in R$, and $y \in R$. The set $\{\Sigma_0, \ldots, \Sigma_r\}$ is a set of nested partial realizations of (5.3.6). □

Remark 5.5.4: It can be shown [22] that a system, known to be n^{th} order, is determined uniquely by the finite sequence of Markov parameters, $J = \{J_1, \ldots, J_{2n}\}$. Furthermore, nested partial realizations of r^{th} order are uniquely determined by the first 2r of these Markov parameters. Thus, reduced order models obtained by truncating the dual GHR of a system match Markov parameters of the full system. □

The partial realization problem can be extended to determining a system from a finite sequence of both Markov parameters and time moments. Systems known to be n^{th} order are uniquely determined by any combination of 2n Markov parameters and time moments.

Markov parameters and time moments are closely tied to a family of continued fractions known as ***Cauer CFE's.*** The Cauer first form [15], matches Markov parameters with a Taylor series expansion about $s=\infty$. Similarly, Cauer second form [15] matches time moments with a Taylor series about s=0. By mixing these methods, variations are obtained which match both time moments and Markov parameters [8]. Consider the system (2.1.1). The transfer function of (2.1.1) can be written

$$G(s) = \sum_{i=1}^{\infty} J_i s^{-i}, \tag{5.5.3}$$

or assuming the state matrix is non-singular,

$$G(s) = - \sum_{i=1}^{\infty} Y_i s^{i-1}. \tag{5.5.4}$$

The principle underlying model reduction using the Cauer CFE's is to approximate G(s) with a simplified model, $\hat{G}(s)$, that matches the first several terms of (5.5.3) (Cauer first form), (5.5.4) (Cauer second form), or both (mixed forms). In general, the n^{th} order Cauer continued fraction is found by retaining m terms related to Markov parameters and 2n-m ($m \leq 2n$) terms related to time moments. The form is,

$$G(s)=[sh_1+[h_2+[sh_3+\cdots[h_m+k_1+s[k_2+s[k_3+\cdots+s[k_{2n-m}]^{-1}\cdots]^{-1}]^{-1}]^{-1}]^{-1}\cdots]^{-1}]^{-1}]^{-1} \tag{5.5.5}$$

where h_i (a function of J_i) and k_i (a function of Y_i) can be calculated from a generalized Routh algorithm. This algorithm is outlined in the appendix. Specifically, retaining 2n terms related to Markov parameters, for an n^{th} order system, yields the Cauer first form,

$$G(s) = \cfrac{1}{sh_1 + \cfrac{1}{h_2 + \cfrac{1}{sh_3 + \cdots + \cfrac{1}{h_{2n}}}}}. \tag{5.5.6}$$

An r^{th} order reduced model ($r < n$) is obtained from (5.5.6) by truncating after 2r quotients. For example, the second-order reduced model of G(s) via Cauer first form is,

$$\hat{G}(s) = \cfrac{1}{sh_1 + \cfrac{1}{h_2 + \cfrac{1}{sh_3 + \cfrac{1}{h_4}}}}. \tag{5.5.7}$$

Keeping 2n terms related to time moments for an n^{th} order system yields the Cauer second form,

$$G(s) = \cfrac{1}{k_1 + \cfrac{s}{k_2 + \cfrac{s}{k_3 + \cdots + \cfrac{s}{k_{2n}}}}}. \tag{5.5.8}$$

The r^{th} order reduced model ($r < n$) is found by truncating (5.5.8) after 2r quotients.

Remark 5.5.5: Any of the mixed Cauer CFE's can be obtained similarly from (5.5.5). Since it is often desirable to work exclusively in the state space, there is considerable interest in relating the Cauer CFE's to canonical state space realizations. Canonical forms exist for the Cauer first form and Cauer second form [15]. Except for Hwang [23], who has developed the canonical form for a special case of the generalized Cauer form where equal numbers of Markov parameters and time moments are matched (known as Cauer third form), no work has been done in determining the general Cauer CFE forms directly from an arbitrary state space description. □

We now state the relationship between the dual GHR and continued fractions.

Theorem 5.5.6: [6] Every pair of polynomials {n(s), d(s)} with degree n(s) < degree d(s), d(s) monic, has a continued fraction representation of the form,

$$\frac{n(s)}{d(s)} = \cfrac{\psi_1}{p_1(s) - \cfrac{\psi_1\psi_2}{p_2(s) - \cdots - \cfrac{\psi_{n-1}\psi_n}{p_n(s)}}} \tag{5.5.9}$$

where all $\psi_i \neq 0$ and all $p_i(s)$ are monic. □

Proposition 5.5.7: [6] The dual GHR is the canonical state space representation of (5.5.9) where

$$\begin{aligned} \epsilon_1\gamma_1^2 &= \psi_1 \\ \epsilon_2\gamma_2^2 &= \psi_1\psi_2 \\ &\vdots \\ \epsilon_n\gamma_n^2 &= \psi_{n-1}\psi_n \end{aligned} \tag{5.5.10}$$

and $p_i(s) = \det(sI - F_i)$. □

Example 5.5.8: Consider the system (5.3.1) given in dual GHR form. From Theorem 5.5.6 and Proposition 5.5.7, we can write the transfer function G(s) as,

$$G(s) = \cfrac{\epsilon_1\gamma_1^2}{(s^2 - \alpha_1 s - \alpha_2) - \cfrac{\epsilon_2\gamma_2^2}{(s - \beta_1) - \cfrac{\epsilon_3\gamma_3^2}{(s^3 - \delta_1 s^2 - \delta_2 s - \delta_3)}}} \tag{5.5.11}$$

□

Knowing that model reduction by both the dual GHR and Cauer first form match Markov parameters, at this point, we ask how the continued fraction (5.5.9) relates to the Cauer CFE's.

Proposition 5.5.9: If a Cauer first form expansion, $G_c(s)$, of (2.1.1) exists, and $\mathfrak{L}_r$ and $\mathfrak{R}_r$ are reducing subspaces of (2.1.1), then model reduction by retaining 2r quotients of $G_c(s)$ yields the same reduced order system as truncating the dual GHR of (2.2.1) to r^{th} order.

Proof: Let Σ_r be the reduced order model obtained by truncating the dual GHR. The reduced system, Σ_r, is also dual GHR, thus canonical, unique and matches 2r Markov parameters of (2.1.1). The reduced model obtained from Cauer first form matches 2r Markov parameters also. Thus, they must be equivalent. □

Conjecture: If the dual GHR of (2.1.1) is regular, then the Cauer first form continued fraction expansion exists and the coefficients are given (with respect to the elements of the dual GHR) by,

$$h_{2i-1} = \frac{q_{i-1}^2}{\prod_{n=1}^{i} \epsilon_n \gamma_n^2} \quad , \quad h_{2i} = \frac{-\prod_{n=1}^{i} \epsilon_n \gamma_n^2}{q_{i-1}\, q_i} \quad , \quad i = 1,\ldots, r, \tag{5.5.12}$$

where $r \leq n$, $q_i = q_{i-1}a_{ii} - q_{i-2}\epsilon_i\gamma_i^2$, $q_{-1} = 0$, and $q_0 = 1$. □

Remark 5.5.10: A necessary condition for existence of the Cauer first form CFE is that only one infinite zero exists in the system transfer function. Theorem 5.5.6 says that the dual GHR will always exist. Thus, the Cauer first form may be considered as a special case of the dual GHR. □

The dual GHR can also be used as a vehicle to obtain reduced order models which match both Markov parameters and time moments of the full system. In showing this, we make considerable use of computing the dual GHR form of triples other than (A, B, C). By the statement, compute the dual GHR of (A^{-1}, B, C) to form a new system (A_g^{-1}, B_g, C_g), we mean—determine the state transformation such that the matrices A_g^{-1}, B_g, and C_g take on the form of F, G, and H, respectively, in (5.2.2). Analogous to σ_k as the dimension of the k^{th} diagonal block in F, let the dimension of the k^{th} diagonal block of A_g^{-1} be ν_k.

Proposition 5.5.11: Given the system (2.1.1), let A^{-1} exist. If the dual GHR of (A^{-1}, B, C) is regular, then an r^{th} order reduced model matching 2 Markov parameter and 2r-2 ($2r \leq n$) time moments exists and may be determined from the following procedure:

1. Compute the dual GHR of the system (A^{-1}, B, C) to obtain a new state space triple (A_g^{-1}, B_g, C_g).

2. Partition the triple (A_g, B_g, C_g) as

$$\begin{bmatrix} \dot{x}_a \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} \hat{A}_g & * \\ * & * \end{bmatrix} \begin{bmatrix} x_a \\ x_r \end{bmatrix} + \begin{bmatrix} \hat{B}_g \\ 0 \end{bmatrix} u, \qquad (5.5.13)$$

$$y = \begin{bmatrix} \hat{C}_g & 0 \end{bmatrix} \begin{bmatrix} x_a \\ x_r \end{bmatrix},$$

where $x_a \in R^r$, $x_r \in R^{n-r}$, $u \in R$, and $y \in R$.

3. Obtain the r^{th} order reduced model by truncating the states associated with x_r,

$$\dot{x}_a = \hat{A}_g x_a + \hat{B}_g u, \qquad (5.5.14)$$

$$y = \hat{C}_g.$$

Proof: Without loss of generality, let (2.1.1) be given such that (A^{-1},B,C) is in dual GHR form. Partition (A^{-1},B,C) as,

$$A^{-1} = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}, \; B = \begin{bmatrix} \hat{B}_g \\ 0 \end{bmatrix}, \; C^t = \begin{bmatrix} \hat{C}_g^t \\ 0 \end{bmatrix}, \qquad (5.5.15)$$

where $M_1 \in R^{r \times r}$, $M_4 \in R^{(n-r)\times(n-r)}$, and all other submatrices are appropriately dimensioned. Since (5.5.15) is also in dual GHR form,

$$\hat{C}_g M_1^i \hat{B}_g = CA^{-i}B, \text{ for } i = 1, \ldots, 2r\text{-}1. \tag{5.5.16}$$

Let A be given by,

$$A = \begin{bmatrix} \hat{A}_g & * \\ * & * \end{bmatrix}. \tag{5.5.17}$$

From matrix inversion lemmas,

$$\hat{A}_g = (M_1 - M_2 M_4^{-1} M_3)^{-1}. \tag{5.5.18}$$

Since M_2 is a matrix consisting of all zeros except for the lower left entry, and M_3 is all zeros except for the upper right element, we can write the inverse of (5.5.18) in the form,

$$\hat{A}_g^{-1} = (M_1\text{-}X), \tag{5.5.19}$$

where $X \in R^{r \times r}$ with all zeros except for the lower right element. It can be shown by simple induction that for a tridiagonal matrix, A, and a matrix X with all zeros except for the lower right element that,

$$(A\text{-}X)^i = A^i + \begin{bmatrix} 0 & & & * \\ & & * & * \\ & * & * & * \\ * & * & * & * \end{bmatrix}, \tag{5.5.20}$$

where the $*$'s appear along and below the $(2r\text{-}i)^{th}$ anti-diagonal. Thus,

$$\hat{C}_g(\hat{A}_g^{-1})^i\hat{B}_g = \hat{C}_g(M_1\text{-}X)^i\hat{B}_g = \hat{C}_gM_1\hat{B}_g, \text{ for } i < 2r\text{ -}1. \tag{5.5.21}$$

From (5.5.15) and (5.5.17) it follows that,

$$\begin{aligned} \hat{C}_g\hat{B}_g &= CB \\ \hat{C}_g\hat{A}_g\hat{B}_g &= CAB, \end{aligned} \tag{5.5.22}$$

and from (5.5.16) and (5.5.21),

$$\hat{C}_g\hat{A}_g^{-i}\hat{B}_g = CA^{-i}B, \text{ for } i = 1 \text{ to } 2r\text{-}2. \tag{5.5.23}$$

□

By defining alternate sets of reachable and unobservable subspaces, we can attach geometry to the model reduction method given by Proposition 5.5.11.

Definition 5.5.12: The ***i^{th} inverse reachable subspace***, $\mathcal{S}_i$, is given by,

$$\mathcal{S}_i = \sum_{j=1}^{i} \mathcal{I}m[A^{-j}B], \ \mathcal{S}_0 = 0. \qquad \square$$

Definition 5.5.13: The ***i^{th} inverse unobservable subspace***, $\mathcal{T}_i$, is given by,

$$\mathcal{T}_i = \bigcap_{j=1}^{i} \mathcal{N}[CA^{-j}], \ \mathcal{T}_0 = \mathfrak{X}. \qquad \square$$

Theorem 5.5.14: For the basis defined by (5.5.13), if ν_i is the dimension of the i^{th} diagonal block of A_g^{-1} then,

$$\mathcal{L}_1 \cap \mathcal{T}_{i-1} = \sum_{j=i+1}^{n} \text{sp } [e_j], \tag{5.5.24}$$

and, for m = 1, ... , r

for p = 0, ... , ν_m-1

$$\mathcal{R}_1 \cap \mathcal{S}_{i-1} = \mathcal{R}_1 \cap \mathcal{S}_{m+p-1} = \mathcal{M}_{\kappa_{m-1}} + \sum_{j=\kappa m-p}^{\kappa m} \text{sp } [e_j]. \tag{5.5.25}$$

where $\kappa_i = \sum_{j=1}^{i} \nu_j$, $\kappa_0 = 0$; and $\mathcal{M}_{\kappa m} = \mathcal{R}_1 \oplus \mathcal{S}_{\kappa m-1}$, $\mathcal{M}_0 = 0$.

Proof: We give the proof for (5.5.24). Equation (5.5.25) is proven similarly. The basis (5.5.13) is of the form,

$$\dot{x} = \begin{bmatrix} F_1 & H_2 & & \\ G_2 & F_2 & \ddots & \\ & \ddots & \ddots & H_r \\ & & G_r & F_r \end{bmatrix}^{-1} x + \begin{bmatrix} G_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u, \tag{5.5.26}$$

$$y = \begin{bmatrix} H_1 & 0 & \cdots & 0 \end{bmatrix} x,$$

with F_i, G_i, H_i defined by (5.2.2). For i = 1, $\mathcal{L}_1 \cap \mathcal{T}_0 = \mathcal{N}[C]$. Thus, we must determine the space, $\mathcal{M}$, such that for all $x \in \mathcal{M}$, Cx = 0. From (5.5.26),

$$\begin{bmatrix} \gamma_1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0, \tag{5.5.27}$$

implies that x_1 must be zero. Hence, $\mathcal{N}[C] = sp[e_2, \ldots, e_n]$, which verifies (5.5.24). For i = 2, $\mathcal{L}_1 \cap \mathcal{T}_1 = \mathcal{N}[C] \cap \mathcal{N}[CA^{-1}]$ requires that we find the space, $\mathcal{M}$, such that for all $x \in \mathcal{M}$, that $Cx = 0$ and $CA^{-1}x = 0$. The case Cx=0 implied that $x_1 = 0$. With this in mind, the following equation must be satisfied

$$\begin{bmatrix} a_{11}\gamma_1 & \gamma_1\gamma_2 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0, \qquad (5.5.28)$$

implying that x_2 must also be zero. Thus, $\mathcal{N}[C] \cap \mathcal{N}[CA^{-1}] = sp[e_3, \ldots, e_n]$, again verifying (5.5.24). Following this procedure, the proof can be completed by induction. □

If $(\mathcal{L}_1 \cap \mathcal{T}_{i-1}) \perp (\mathcal{R}_1 \oplus \mathcal{S}_{i-1})$ and $(\mathcal{L}_1 \cap \mathcal{T}_{i-1}) \oplus (\mathcal{R}_1 \oplus \mathcal{S}_{i-1}) = \mathfrak{X}$, then these sets of subspaces provide a candidate set of reduced order models. Indeed, these are the reducing subspaces used in the model reduction procedure described by Propositions 5.5.11. As with the dual GHR, when $A_{\mathcal{S}}^{-1}$ is block tridiagonal the blocks cannot be split for model reduction. In light of this, Proposition 5.5.11 can be extended to the general case.

Corollary 5.5.15: Given the system (2.1.1), let A^{-1} exist. Then if $\mathcal{L}_1 \cap \mathcal{T}_{r-1}$ and $\mathcal{R}_1 \oplus \mathcal{S}_{r-1}$ are reducing subspaces for this system, then an r^{th} order reduced model matching 2 Markov parameters and 2r-2 ($2r \leq n$) time moments of the full system exists and is found by the procedure outlined in Proposition 5.5.11. □

Along these lines, we give other methods of model reduction matching certain numbers of Markov parameters and moments.

Proposition 5.5.16: Given (2.1.1), let A^{-1} exist. If $\mathcal{L}_1 \cap \mathcal{T}_{r-1}$ and $\mathcal{R}_1 \oplus \mathcal{S}_{r-1}$ are reducing subspaces for (2.1.1), then an r^{th} order reduced model matching 1 Markov parameter and 2r-1 ($2r \leq n$) time moments of the full system exists and may be found from the following procedure:

1. Compute the dual GHR of the system (A^{-1}, B, C) to form the new triple (A_g^{-1}, B_g, C_g).

2. Partition (A_g^{-1}, B_g, C_g) as,

$$\begin{bmatrix} \dot{x}_a \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} \hat{A}_g^{-1} & * \\ * & * \end{bmatrix} \begin{bmatrix} x_a \\ x_r \end{bmatrix} + \begin{bmatrix} \hat{B}_g \\ * \end{bmatrix} u, \qquad (5.5.29)$$

$$y = \begin{bmatrix} \hat{C}_g & * \end{bmatrix} \begin{bmatrix} x_a \\ x_r \end{bmatrix},$$

where $x_a \in R^r$, $x_r \in R^{n-r}$, $u \in R$, and $y \in R$.

3. Obtain the r^{th} order reduced model,

$$\dot{x}_a = \hat{A}_g x_a + \hat{B}_g u, \qquad (5.5.30)$$

$$y = \hat{C}_g,$$

Proof: From Remark 5.5.4, the system (5.5.29) satisfies,

$$\begin{aligned} \hat{C}_g\hat{B}_g = CB &= J_1 \\ \hat{C}_g\hat{A}_g^{-1}\hat{B}_g = CA^{-1}B &= Y_1 \\ \vdots &\vdots \\ \hat{C}_g\hat{A}_g^{-(2r-1)}\hat{B}_g = CA^{-(2r-1)}B &= Y_{(2r-1)} \end{aligned} \tag{5.5.31}$$

□

Remark 5.5.17: The reducing subspaces used in both Propositions 5.5.11 and 5.5.16 are the same. The difference is that in model reduction via Proposition 5.5.17, A_g^{-1} is projected onto $\Re_1 \oplus \Im_{r-1}$. More familiarly, in Proposition 5.5.11, A_g is projected. □

Proposition 5.5.18: Given (2.1.1), an r^{th} order reduced model matching 1 time moment and 2r-1 Markov parameters may be found from an arbitrary state space description (A, B, C) by the following procedure:

1. Compute the dual GHR of the system (A, B, C) to obtain a new state space triple, (A_g, B_g, C_g).

2. Partition the triple (A_g^{-1}, B_g, C_g) as

$$\begin{bmatrix} \dot{x}_a \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} \hat{A}_g^{-1} & * \\ * & * \end{bmatrix} \begin{bmatrix} x_a \\ x_r \end{bmatrix} + \begin{bmatrix} \hat{B}_g \\ 0 \end{bmatrix} u, \tag{5.5.32}$$

$$y = \begin{bmatrix} \hat{C}_g & 0 \end{bmatrix} \begin{bmatrix} x_a \\ x_r \end{bmatrix},$$

where $x_a \in R^r$, $x_r \in R^{n-r}$, $u \in R$, and $y \in R$.

3. Obtain the r^{th} order reduced model by truncating the states associated with x_r,

$$\dot{x}_a = \hat{A}_g x_a + \hat{B}_g u, \qquad (5.5.33)$$
$$y = \hat{C}_g. \qquad \square$$

Proposition 5.5.19: Given (2.1.1), an r^{th} order reduced model matching 2r time moments may be found from an arbitrary state space description (A, B, C) by the following procedure:

1. Compute the dual GHR of the system $(A^{-1}, A^{-1}B, CA^{-1})$ to obtain a new state space triple, $(A_g^{-1}, A_g^{-1}B_g, C_gA_g^{-1})$.

2. Partition the triple (A_g, B_g, C_g) as

$$\begin{bmatrix} \dot{x}_a \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} \hat{A}_g & * \\ * & * \end{bmatrix} \begin{bmatrix} x_a \\ x_r \end{bmatrix} + \begin{bmatrix} \hat{B}_g \\ * \end{bmatrix} u, \qquad (5.5.34)$$
$$y = \begin{bmatrix} \hat{C}_g & * \end{bmatrix} \begin{bmatrix} x_a \\ x_r \end{bmatrix},$$

where $x_a \in R^r$, $x_r \in R^{n-r}$, $u \in R$, and $y \in R$.

3. Obtain the r^{th} order reduced model by truncating the states associated with x_r,

$$\dot{x}_a = \hat{A}_g x_a + \hat{B}_g u, \qquad (5.5.35)$$
$$y = \hat{C}_g. \qquad \square$$

Propositions 5.5.18 and 5.5.19 are proven similarly to Propositions 5.5.11 and 5.5.16.

Theorem 5.5.20: For the basis defined by (5.5.34), if ν_i is the dimension of the i^{th} diagonal block of A_g^{-1} then

$$\mathcal{T}_i = \sum_{j=i+1}^{n} \text{sp } [e_j], \tag{5.5.36}$$

and, for m = 1, ... , r

for p = 0, ... , ν_m-1

$$\mathcal{S}_i = \mathcal{S}_{m+p} = \mathcal{S}_{\kappa m-1} + \sum_{j=\kappa m-p}^{\kappa m} \text{sp } [e_j]. \tag{5.5.37}$$

where $\kappa_i = \sum_{j=1}^{i} \nu_j$, $\kappa_0 = 0$. □

Theorem 5.5.20 can be shown by following the proof of Theorem 5.5.14.

Remark 5.5.21: Note that model reduction to i^{th} order by choosing the reducing subspaces $\mathfrak{B}_r=\mathcal{L}_i$, $\mathfrak{B}_a=\mathcal{R}_i$ and then projecting A_g (A_g^{-1}) onto $\mathcal{R}_i$ retains 2i Markov parameters (2i-1 Markov parameters and 1 moment). Likewise, model reduction to i^{th} order choosing reducing subspaces as $\mathfrak{B}_r=(\mathcal{L}_1 \cap \mathcal{T}_{i-1})$, $\mathfrak{B}_a=(\mathcal{R}_1 \oplus \mathcal{S}_{i-1})$ and then projecting A_g (A_g^{-1}) retains 2 Markov parameters and 2i-2 time moments (1 Markov parameter and 2i-1 moments). Apparently, there is a relationship between the interaction of these subspaces, Markov parameters, and time moments. This relationship has not been fully exploited. □

5.6 Model Reduction of Structures

To this point, we have been concerned with model reduction of linear systems modelled as a set of first-order, ordinary differential equations (2.1.1). In this section we will consider an important class of second-order system models which arise in the dynamic analysis of structures. Although these systems can be represented in the state space, they are often described by,

$$M\ddot{x}(t) + Kx(t) = f(t), \tag{5.6.1}$$

where M and K are the n x n mass and stiffness matrices, respectively; f is the external load on the structure; and x is the displacement vector of generalized coordinates describing the response of the structure. Also, here we assume the usual case when M and K are positive definite, symmetric matrices.

The model (5.6.1) is usually derived from a finite element analysis where it is not uncommon that such an analysis yields a several hundred degree of freedom model. It is generally accepted in the literature [24], [25] that the best way to the reduce (5.6.1) is by using enough of low frequency modes to satisfactorily express the response of the structure. This procedure is called modal superposition and is based on the introduction of a displacement transformation to modal coordinates, u,

$$x(t) = Yu(t), \tag{5.6.2}$$

where Y is an n x m ($m \leq n$) matrix chosen as the eigenvectors corresponding to

the lowest frequency modes of the system. Substituting (5.6.2) into (5.6.1) and pre-multiplying by Y^T yields,

$$\hat{M}\ddot{u}(t) + \hat{K}u(t) = f(t), \tag{5.6.3}$$

where $\hat{M} = Y^T M Y$, $\hat{K} = Y^T K Y$, and $\hat{f} = Y^T f$.

Due to orthogonality properties of modal coordinates, this transformation leads to an uncoupled set of linear, second-order differential equations. That is, $\hat{M}$ and $\hat{K}$ are diagonal matrices. The independent equations can then be solved separately and the total response of the structure can be obtained by superposition. It is important to note that Y can be chosen as any set of orthogonal ***Ritz vectors***, not necessarily the eigenvectors of the system, however, if arbitrary Ritz vectors are used then the mass and stiffness matrices of the reduced model will not be diagonal.

As mentioned in [24]-[27], this method of model reduction has several problems. The main drawback with the use of the eigenvectors in the mode superposition method is that the solution to the eigenvalue problem is computationally expensive for large systems. Also it has not been proven that modal coordinates (i.e. choosing the Ritz vectors as the eigenvectors) will give the best results, in terms of describing the structure, with the fewest degrees of freedom. Nour-Omid and Clough [24] propose the use of Lanczos vectors for extracting the low frequencies of (5.6.1). The idea is that the Lanczos algorithm [28], an efficient method of finding the eigenvalues and eigenvectors of a system, is applied to a special set of Ritz vectors known as a Krylov sequence. The resulting

orthonormal set of vectors is then applied to (5.6.1) as in modal superposition. We outline the procedure below.

Definition 5.6.1: [26] Let A: $\mathfrak{X} \rightarrow \mathfrak{X}$ and $\phi \in \mathfrak{X}$. A ***Krylov subspace*** of order j, $\mathcal{P}_j$, is a j-dimensional vector space given by,

$$\mathcal{P}_j = \text{sp}\left[\begin{array}{cccc} \phi & A\phi & A^2\phi & \cdots A^{j-1}\phi \end{array}\right] \qquad \square$$

Definition 5.6.2: [26] An undamped n-degree of freedom structure is completely ***disturbable*** if

$$D_N = \left[\begin{array}{cccc} K^{-1}f & K^{-1}MK^{-1}f & \cdots & (K^{-1}M)^{N-1}K^{-1}f \end{array}\right] \qquad (5.6.4)$$

has rank N. $\square$

Disturbability and controllability are synonymous concepts. In structural dynamics, disturbability is the property that each mode of the system can be excited by some input disturbance. The subspace $\mathcal{D}_i = \text{sp}[D_i]$ is clearly a Krylov subspace. Model reduction of (5.6.1) from N-degrees of freedom to i-degrees of freedom proceeds by projecting the mass and stiffness matrices onto $\mathcal{D}_i$. The projector used is determined by applying the Lanczos algorithm to the set of vectors, D_i. The result is an M-orthonormal set of basis vectors, Q, of $\mathcal{D}_i$. That is, $\mathcal{D}_i = \text{sp}[D_i] = \text{sp}[Q]$ and $Q^tMQ = I$. The Lanczos algorithm is described in the appendix.

In transforming the equations of motion (5.6.1) to a reduced form the

transformation,

$$x(t) = Qz(t), \tag{5.6.5}$$

is not directly applied to (5.6.1) as we may have expected. The orthogonality properties of the Lanczos vectors can be taken advantage of to find a more useful form for the simplified model. Pre-multiplying (5.6.1) by MK^{-1} gives,

$$MK^{-1}M\ddot{x}(t) + Mx(t) = MK^{-1}f(t). \tag{5.6.6}$$

Nour-Omid and Clough [24] show that the solution of (5.6.6) is identical to the solution of (5.6.1). By transforming the displacement vector, x, to Lanczos coordinates, z, as in (5.6.5) and pre-multiplying (5.6.6) by Q^t yields,

$$Q^TMK^{-1}MQ\ddot{z}(t) + z(t) = Q^TMK^{-1}f(t). \tag{5.6.7}$$

For convenience, rewrite (5.6.7) as

$$T\ddot{u}(t) + u(t) = g(t), \tag{5.6.8}$$

where $T = Q^TMK^{-1}MQ$ and $g = Q^TMK^{-1}f$.

Proposition 5.6.3: [27] If $Q = [q_1 \cdots q_n]$ is a matrix of Lanczos vectors and A is a positive definite, symmetric matrix then,

$$T = Q^TAQ \tag{5.6.9}$$

is the projection of A onto the subspace sp[Q] and is tridiagonal. □

This is the so-called Lanczos phenomenon [27]. Thus in (5.6.8) T is tridiagonal. In view of the results of the previous section, we would like to resolve the relationship of model reduction by projecting onto a space spanned by Lanczos vectors with model reduction using other reducing subspaces. First, represent (5.6.1) in the state space by

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}f \end{bmatrix} u. \tag{5.6.10}$$

From (5.6.10), it is apparent that model reduction of (5.6.10) may be viewed, without loss of information, as model reduction of (2.1.1) where $A = M^{-1}K$, $B = M^{-1}f$, and C is arbitrary—for now. At this point, we make two observations. First, the $\mathfrak{D}_i$ is exactly the i^{th} inverse reachable subspace $\mathfrak{s}_i$. Second, the Lanczos procedure yields a tridiagonal mass matrix which orthogonalize the vectors spanning this subspace. Consider the state space system

$$\begin{aligned} \dot{x} &= M^{-1}Kx + M^{-1}fu, \\ y &= Cx, \end{aligned} \tag{5.6.11}$$

where $x \in R^n$, $u \in R$, and $y \in R$. In Proposition 5.5.19, we show how (5.6.11) can transformed to a basis where truncating the state matrix is actually an orthogonal projection onto the subspace $\mathfrak{s}_i$, for some i. This transformation, of course, depends on the selection of C. We will show that C can be chosen such that the result is the same as from the Lanczos method.

Proposition 5.6.4: Given (5.6.11) and (5.6.1) where M and K are symmetric and positive definite, the extended chained aggregation algorithm and the Lanczos algorithm will yield the same result, $T_g = Q$, if in (5.6.11) C is chosen such that $C = f^T$.

Proof: Let T_g be the transformation (via extended chained aggregation) from the triple $(K^{-1}M, K^{-1}f, CK^{-1}M)$ to,

$$\dot{z} = T_g^{-1}K^{-1}MT_g z + T_g^{-1}K^{-1}fu, \tag{5.6.12}$$
$$y = CK^{-1}MT_g z.$$

Let Q be the transformation (via the Lanczos algorithm) to the Lanczos coordinates, then $Q^TMK^{-1}Q$ is a tridiagonal, symmetric matrix. We choose C such that the state matrix (5.6.12) is also tridiagonal, symmetric, and

$$T_g^{-1}K^{-1}MT_g = Q^TMK^{-1}MQ. \tag{5.6.13}$$

From (5.6.13),

$$T_g^{-1} = Q^TM, \tag{5.6.14a}$$
$$T_g = Q. \tag{5.6.14b}$$

From (5.6.12) we know,

$$CK^{-1}MT_g = T_g^{-1}K^{-1}f. \tag{5.6.15}$$

Substituting (5.6.14) into (5.6.15) gives,

$$CK^{-1}MQ = Q^{T}MK^{-1}f. \tag{5.6.16}$$

From (5.6.16), it is clear that $C = f^{T}$. □

Proposition 5.6.4 says that the Lanczos algorithm is related to the method of Proposition 5.5.19. Hence, the Lanczos method can be viewed as model reduction by projecting onto $\mathcal{S}_i$, matching time moments.

5.7 Chapter Summary

The gap between A-invariant subspaces and the generalized reachable and unobservable subspaces of the system are given as another measure of controllability and observability in a system. A-invariant subspaces lying close to these structures define nearly uncontrollable or unobservable subspaces. It was shown how existence of nearly uncontrollable or unobservable subspaces can be directly estimated from the elements in the state matrix of the dual GHR. The dual GHR was also shown to display the generalized reachable and unosbservable subspaces along the natural basis vectors making it a natural setting for model reduction based on these measures.

It was shown that the dual GHR can be considered as model reduction by matching Markov parameters of the full system, leading to development of simplification methods which match both Markov parameters and time moments of the full system. Geometry is attached to this method by determing the reducing subspaces involved in reduction of this type. The results are shown to be similar to transfer function methods known as the Cauer continued fraction

expansions. Furthermore, it was shown how a popular method in the model reduction of structures, the Lanczos method, is fundamentally related to these model reduction schemes.

6.0 A Comparative Study

Balancing, modal methods, and dual GHR are three methods of model reduction which can be used to produce useful simplifed models of a linear system. In this chapter we investigate how these methods relate to each other. Specifically, we compare them in terms of relationships between the measures of controllability and observability used in each approach. Each reduction technique has been presented in terms of its reducing subspaces and the controllability and observability measures given as guidelines for the selection of these subspaces. Balancing removes weakly observable/controllable subspaces. In dual GHR near unobservability/uncontrollability plays a role in deleting subspaces, and in modal methods the modal measures or residues determine which A-invariant subspace is deleted. To relate modal methods to the dual GHR we review results relating modal measures to near unobservability. To relate balancing to the dual GHR, in this chapter we ask, are nearly unobservable subspaces also weakly observable? Also, the relationships of controllability and observability measures to the zeros of the transfer function is studied. Finally, we compare the reduction approaches as they effect the frequency response of the system.

6.1 Near Unobservability and Modal Measures

We introduced model reduction via the dual GHR basis as determining an A-invariant subspace near $\mathcal{L}_i$, for some i, and then deleting $\mathcal{L}_i$. Modal approaches suggest removing the A-invariant subspace instead. Clearly a relationship exists between the methods. Here, we compare the methods on the level of controllability and observability measures.

Lindner, Babendreier, and Hamdan [13] have shown that the gap between A-invariant subspaces and the unobservable (reachable) subspaces bound the modal measures.

Theorem 6.1.1: [13] Let $\mathcal{V}_o$ and $\mathcal{V}_c$ be subspaces of $\mathbb{C}^n$ such that $p_i \in \mathcal{V}_o$ and $q_i \perp \mathcal{V}_c$. Then

$$\cos\phi(C^T, p_i) \leq \tau(\mathcal{V}_o, \mathcal{L}_h) \text{ and } \cos\phi(q_i, B) \leq \tau(\mathcal{V}_c, \mathcal{R}_h), \qquad (6.1.1)$$

for all $i = 1, \ldots, n$ and $h = 1, \ldots, n$. □

Previously we noted that the modal measures and residues, both structural measures of controllability and observability, essentially carry the same information.

$$|r_i| = \cos\phi(C^T, p_i)\ \|p_i\|\ \|C\|\ \cos\phi(q_i, B)\ \|q_i\|\ \|B\|. \qquad (4.1.7)$$

By applying Theorem 6.1.1 to (4.1.7), a bound on the residues of G(s) is obtained.

$$|r_i| \leq \tau(\mathcal{L}_i, \mathcal{V}_o)\ \tau(\mathcal{R}_i, \mathcal{V}_c)\ \|p_i\|\ \|q_i\|\ \|B\|\ \|C\|. \tag{6.1.2}$$

Finally, this result can be related to the dual GHR.

Theorem 6.1.2: [13] Let (2.1.1) be represented by its dual GHR. Assume that Theorem 5.4.3 holds. Then,

$$|r_i| \leq 4\gamma_1^2 \|p_i\|\ \|q_i\| \frac{\gamma_{i+1}^2}{\delta^2}. \tag{6.1.3}$$

The proof of Theorem 6.1.2 follows from applying Theorem 5.4.3 to (6.1.2). □

Remark 6.1.3: Equation (6.1.3) tells us that if the gaps between $\mathcal{L}_h$ and $\mathcal{V}_o$; $\mathcal{R}_h$ and $\mathcal{V}_c$ are small, then a system will exhibit small residues. Theorem 6.1.2 says that the state matrix of the dual GHR of such a system may indicate this by a small (sub)super-diagonal element, γ_{h+1}. In this case, model reduction by dominant modes and dual GHR should be similar. □

Example 6.1.4: Consider the system (5.2.7). For $\mathcal{V}$ spanned by the right eigenvectors associated with λ=79.143 and λ=0.10868, $\tau(\mathcal{L}_3, \mathcal{V}) = 0.0225$. The residues associated with these modes are of the order of 10^{-5}. So, in view of Remark 6.1.3, model reduction by dual GHR, deleting $\mathcal{L}_3$, and by dominant modes, deleting $\mathcal{V}$, should be similar. Model reduction of this system to third order by truncating the dual GHR gives,

$$G_1(s) = \frac{(s+40.23)\ (s+10.63)}{(s+91.5)\ (s+0.708+j\ 5.86)\ (s+0.708-j\ 5.86)}, \tag{6.1.4}$$

and by the dominant mode method yields,

$$G_2(s) = \frac{(s+40.29)\ (s+10.61)}{(s+91.47)\ (s+0.708+j\ 5.85)\ (s+0.708-j\ 5.85)}. \qquad (6.1.5)$$

Inspection of the poles and zeros (6.1.4) and (6.1.5) reveals virtually no difference between the reduced order models. □

6.2 Weak Observability and Near Unobservability

In this section, we relate balancing and dual GHR methods. Recall Definitions 3.4.2 and 5.1.9. A subspace, $\mathcal{W}_i$, is weakly observable if it is spanned by the n-i principal component vectors corresponding to small second-order modes of the system. An A-invariant subspace is nearly unobservable if it lies near $\mathcal{L}_i$, for some i. If the gap between $\mathcal{L}_i$ and $\mathcal{W}_i$ is small, then model reduction via these reducing subspaces should be similar.

In pursuit of formalizing this statement, first we develop a relationship between the system eigenvectors and principal component vectors. DeCarlo and Wicks [29] have shown that the geometry of the controllability Gramian can be related to the system eigenstructure. Their result can be interpreted as a parallelepiped in the state space which bounds, and is tangent to, the controllability ellipsoid described in Section 3.1. We review their results next.

Consider the single-input single-output system (2.1.1). Let $\{p_i\}$ be a set of *normalized* right eigenvectors of A and $\{q_i\}$ a set of left eigenvectors of A, the

reciprocal basis of $\{p_i\}$. Decompose B into modal components,

$$B = p_1\hat{b}_1 + p_2\hat{b}_2 + \cdots + p_n\hat{b}_n, \tag{6.2.1}$$

where $\hat{b}_i = q_i^T B$. This decomposition allows the effect of an input on the state to be considered on an individual mode basis. The bounding region (parallelepiped) is determined by measuring this quantity for each mode (eigenvector) and then applying superposition. We can think of $\hat{b}_i$ as the projection of B onto the eigenvector p_i with q_i as the projector. With respect to (6.2.1), it is simple to show that the solution to (2.1.1) with x(0)=0 can be written as,

$$x(t_0) = \sum_{i=1}^{n} \int_0^{t_0} e^{\lambda_i(t_0-t)} p_i \, \hat{b}_i \, u(t) \, dt. \tag{6.2.2}$$

Define the quantity,

$$\xi_i \equiv \hat{b}_i \int_0^{t_0} e^{\lambda_i(t_0-t)} u(t) \, dt. \tag{6.2.3}$$

Substituting (6.2.3) into (6.2.2) gives,

$$x(t_0) = \sum_{i=1}^{n} \xi_i \, p_i. \tag{6.2.4}$$

The quantity, ξ_i is the distance that the state travels along p_i. Recall $S_c(t_0)$ from Definition 3.1.4. To determine where the surface of $S_c(t_0)$ lies in relationship to a given eigenvector, p_i, it is necessary to find u(t) ($\|u(t)\|_{L_2}=1$, $t \in [0, t_0]$) which maximizes ξ_i. This bounds the distance that the state may travel along p_i.

Theorem 6.2.1: [29] The quantity, ξ_i, is maximized over the set $\|u_i(t)\|_{L_2}=1$, $t\in[0, t_0]$, by

$$u_i(t) = K_i^{-1}\hat{b}_i e^{\overline{\lambda}_i(t_0-t)}, \tag{6.2.5}$$

where $K_i = |\hat{b}_i| \sqrt{\frac{e^{2Re(\lambda_i)t_o} - 1}{2Re(\lambda_i)}}$. Furthermore, the maximum is given by,

$$\xi_{i,max} = K_i. \qquad \square$$

As notation, $\xi_{i,max} = \xi_{i,i}$.

The vertices defined by the 2^n vectors, $\pm\xi_{11}p_1\pm\xi_{22}p_2\pm \cdots \pm\xi_{nn}p_n$, describe a set which forms a parallelepiped in the state space. The set characterizes the maximum distance the state can travel from the origin in the direction of any eigenvector, under the condition that $\|u(t)\|_{L_2} = 1$. The set $S_c(t_0)$ is contained by and tangent to this parallelepiped. To determine the exact relation between the eigenvectors and the principal component vectors we require more information. Specifically, we need to identify the points of tangency. This is done by taking into account the modal coupling. That is, in general, when u(t) is chosen to drive the state vector a maximum distance ($\xi_{i,i}$) in the direction of an eigenvector (p_i) the trajectory will not be exclusively along p_i. The state vector will also move somewhat in the direction of the other eigenvectors. To measure this property, define

$$\xi_{i,k} \equiv \hat{b}_i \int_0^{t_0} e^{\lambda_i(t_0-t)} u_k(t)\, dt. \tag{6.2.6}$$

The quantity, $\xi_{i,k}$ is the distance that the state vector moves in the direction of

the i^{th} eigenvector, provided the input, $u_k(t)$, which maximizes the distance that the state vector moves in the direction of the k^{th} eigenvector is applied to the system. Evaluating (6.2.6) with respect to (6.2.5) gives,

$$\xi_{i,k} = \hat{b}_i \frac{\hat{b}_k}{|\hat{b}_k|} \frac{\sqrt{|2\ \mathrm{Re}[\lambda_k]|}}{\lambda_i + \overline{\lambda}_k} \frac{e^{(\lambda_i+\overline{\lambda}_k)t_0} - 1}{\sqrt{|e^{2\mathrm{Re}[\lambda_k]t_0} - 1\ |}}. \tag{6.2.7}$$

See Figure 4 for a graphical view as to how the quantities $\xi_{i,i}$ and $\xi_{i,k}$ relate to the geometry of the Gramian and eigenstructure.

Theorem 6.2.2: [29] Let $W_c^2(t_0) = W_c(t_0)W_c^T(t_0)$ be the controllability Gramian of (2.1.1). Then $W_c(t_0)$ has the following decomposition:

$$W_c(t_0) = P\Xi\mathcal{Z}^{1/2}. \tag{6.2.8}$$

$$\text{where } P = \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix}, \ \Xi = \begin{bmatrix} \xi_{11} & 0 & \cdots & 0 \\ 0 & \xi_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi_{nn} \end{bmatrix}, \tag{6.2.9}$$

$$\mathcal{Z} = \begin{bmatrix} \zeta_{11} & \zeta_{12} & \cdots & \zeta_{1n} \\ \zeta_{21} & \zeta_{22} & \cdots & \zeta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{n1} & \zeta_{n2} & \cdots & \zeta_{nn} \end{bmatrix}, \ \zeta_{ij} \equiv \frac{\xi_{ij}}{\xi_{ii}}. \qquad \square$$

To take the result of Theorem 6.2.2 a step further, we develop a relationship between the Gramian, modal measures, and eigenvalues.

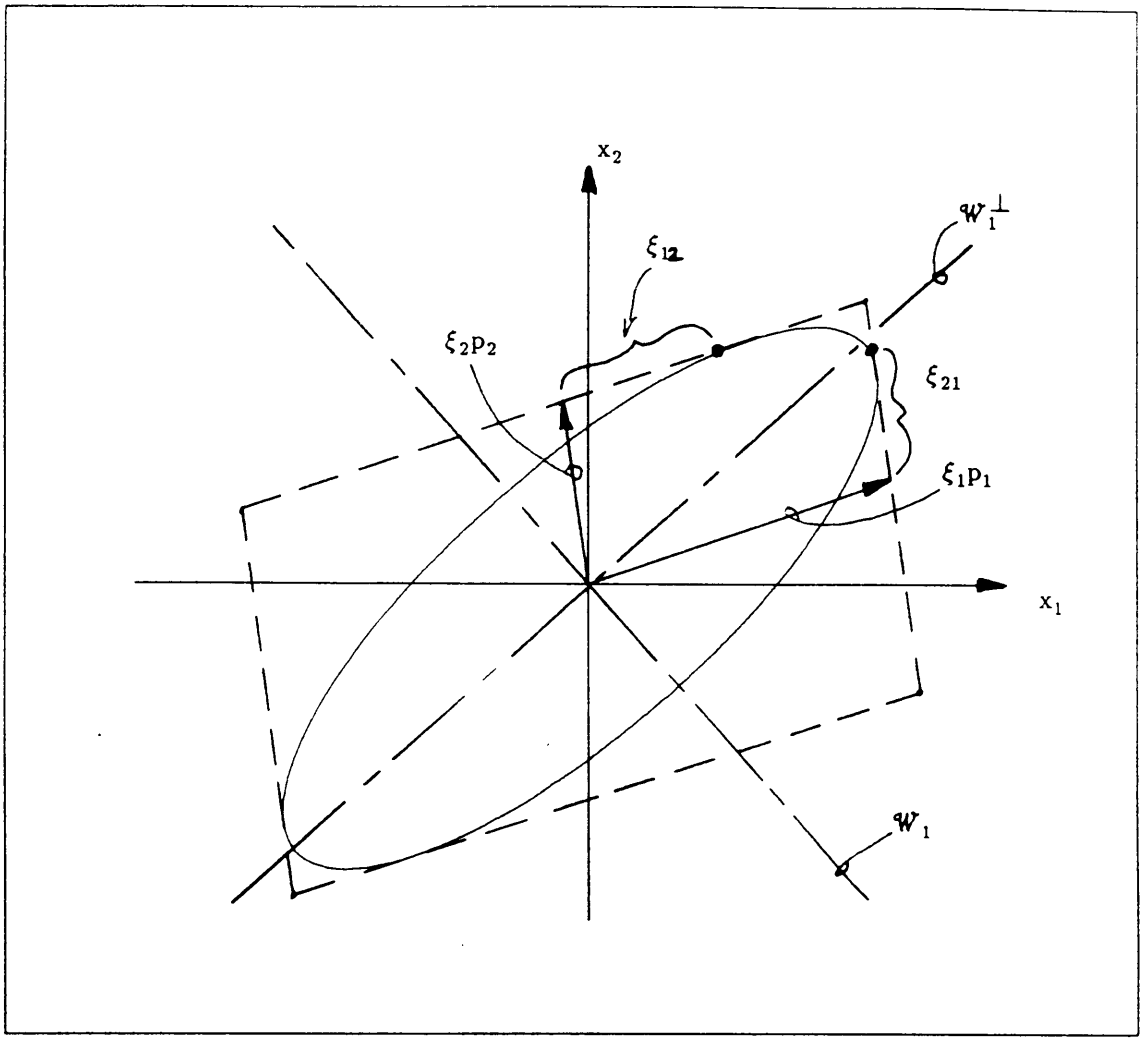

Figure 4: Relationship of the controllability (Gramian) ellipsoid to the system eigenvectors.

Definition 6.2.3: The ***signature for the i^{th} modal controllability measure***, f_{ci}, of (2.1.1) is given by,

$$f_{ci} = \text{sgn}(q_i^T B). \qquad \square$$

Definition 6.2.4: The system (2.1.1) is said to be ***orthogonally symmetric*** if there exists a state transformation, $x = Tz$, such that,

$$\hat{A} = T^T A T = \hat{A}^T \text{ and } T^T B = CT. \tag{6.2.10}$$

$\square$

Remark 6.2.5: In this section we give a geometric proof relating measures from the dual GHR basis to measures from the balanced basis. It is necessary that the two bases are related by orthonormal transformation to preserve angles, lengths, etc... . Thus, only symmetric systems are considered in the remainder of this section. $\square$

Theorem 6.2.6: Let $W_c^2 = W_c^2(\infty)$ be the controllability Gramian of (2.1.1). If (2.1.1) is an asymptotically stable, symmetric system, then W_c^2 has the following decomposition:

$$W_c^2 = \|B\|^2 \, P M_c S M_c P^T, \tag{6.2.11}$$

where M_c is a diagonal matrix consisiting of signed modal measures,

$$M_c = \begin{bmatrix} f_{c1}\cos\phi(q_1, B) & & \\ & \ddots & \\ & & f_{cn}\cos\phi(q_n, B) \end{bmatrix}. \tag{6.2.12}$$

S is an nxn matrix whose ijth element is given by,

$$s_{ij} = \frac{1}{|\lambda_i + \overline{\lambda}_j|}. \tag{6.2.13}$$

Proof: Begin with the result from Theorem 6.2.2, $W_c = P\Xi\mathbb{Z}^{1/2}$. It follows that,

$$W_c^2 = P\Xi\mathbb{Z}\Xi P^T. \tag{6.2.14}$$

The ijth term of $\Xi\mathbb{Z}\Xi$ is $\xi_{ii}\zeta_{ij}\xi_{jj}$, which from (6.2.7) and (6.2.9), letting $t_0 \rightarrow \infty$, can be written as,

$$\xi_{ii}\zeta_{ij}\xi_{ii} = -2|\hat{b}_i| \sqrt{\frac{-1}{2\ \mathrm{Re}[\lambda_i]}}\ \frac{\hat{b}_i\hat{b}_j}{|\hat{b}_i|\ |\hat{b}_j|}\ \frac{\sqrt{\mathrm{Re}[\lambda_i]\ \mathrm{Re}[\lambda_j]}}{\lambda_i + \overline{\lambda}_j}\ |\hat{b}_j| \sqrt{\frac{-1}{2\ \mathrm{Re}[\lambda_j]}},$$

and after simplifying rewritten,

$$\xi_{ii}\zeta_{ij}\xi_{ii} = \frac{\hat{b}_i\hat{b}_j}{|\lambda_i + \overline{\lambda}_j|}. \tag{6.2.15}$$

From (6.2.1), $\hat{b}_i = q_i^T B$. Substituting this into (6.2.15) and applying Definitions 4.1.2 and 6.2.3 gives,

$$\xi_{ii}\zeta_{ij}\xi_{ii} = \frac{\|q_i\|\ \|B\|\ f_{ci}\cos\phi(q_i,\ B)\ \|q_j\|\ \|B\|\ f_{cj}\cos\phi(q_j,\ B)}{|\lambda_i + \overline{\lambda}_j|}. \tag{6.2.16}$$

Since we have assumed that (2.1.1) is a symmetric system, $\|q_i\| = 1$ for all i. Thus,

$$\xi_{ii}\zeta_{ij}\xi_{ii} = \|B\|^2\ \frac{f_{ci}\cos\phi(q_i,\ B)\ f_{cj}\cos\phi(q_j,\ B)}{|\lambda_i + \overline{\lambda}_j|}. \tag{6.2.17}$$

Write (6.2.17) in matrix form,

$$\Xi\mathfrak{Z}\Xi = \|B\|^2 \, M_c S M_c. \tag{6.2.18}$$

Substitute (6.2.18) into (6.2.14) to complete the proof. □

The dual result of Theorem 6.2.6 is,

Theorem 6.2.7: Let $W_o^2 = W_o^2(\infty)$ be the observability Gramian of (2.1.1). If (2.1.1) is an asymptotically stable, symmetric system, then W_o^2 has the following decomposition:

$$W_o^2 = \|C\|^2 \, Q M_o S M_o Q^T, \tag{6.2.19}$$

where M_o is a diagonal matrix consisting of signed modal observability measures, with signatures defined similar to Definition 6.2.3. □

Lemma 6.2.8: [5] If the system (2.1.1) is balanced, then each mode of the system is as controllable as it is observable with respect to the modal controllability and modal observability measures, i.e.;

$$\cos\phi(p_i, C^T) = \cos\phi(q_i, B), \quad \text{for all i.}$$ □

Corollary 6.2.9: Given that (2.1.1) is symmetric and balanced, then $W_c^2 = W_o^2 = \Sigma^2$ and $M_c = M_o = M$. Thus,

$$\Sigma^2 = \|B\|^2 \, PMSMP^T, \tag{6.2.20}$$

or equivalently,

$$\Sigma^2 = \|C\|^2 \, QMSMQ^T. \tag{6.2.21}$$

□

Corollary 6.2.9 gives a relationship between the second-order modes, the structural properties of the system (modal measures), and the spectral information of the system (eigenvalues). The interactions are complex, however, we make the following observations to add to our intuition.

- $\sigma_i \to 0$ as $\cos\phi_j \to 0$ for some i and some j
- $\sigma_i \to 0$ as $|\mathrm{Re}[\lambda_j]| \to \infty$ for some i and some j
- $\sigma_i \to \infty$ as $|\mathrm{Re}[\lambda_j]| \to 0$ for some i and some j

These statements can be shown from (6.2.20) or (6.2.21).

Unfortunately, the information contained in a second-order mode does not necessarily yield information about the structure or the spectra of the system. For example, if σ_i is small, then it is not clear whether a system modal measure is small or that an eigenvalue of the system is fast, or even both. Conversely, a slow mode may have a small modal measure which implies virtually nothing about second-order modes. It is these complications which make the problem of relating near unobservability to weak observability difficult. By using the geometry associated with these measures, we are able to shed some light on the problem.

We will show that for symmetric systems, near unobservability implies weak

observability of a certain degree. That is, for some A-invariant subspace $\mathcal{V}$, if $\tau(\mathcal{L}_i, \mathcal{V}) < \epsilon$ then $\tau(\mathcal{L}_i, \mathcal{W}_i)$ is less than some bound, η. We state this result formally in Theorem 6.2.15; several preliminary results, however, are needed.

First, express the gap between an A-invariant subspace and $\mathcal{W}_i$ in terms of the system eigenstructure.

Lemma 6.2.10: Let the system (2.2.1) be asymptotically stable, symmetric, and balanced. Let $P=[p_1\ p_2\ \cdots\ p_n]$ be a set of normalized right eigenvectors. Let P be partitioned,

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{6.2.22}$$

where $P_{11} \in R^{i \times i}$, $P_{22} \in R^{(n-i)\times(n-i)}$, P_{12} and P_{21} are conformally dimensioned. Let $\mathcal{V}_i$ be the A-invariant subspace given by,

$$\mathcal{V}_i = \mathrm{sp}\begin{bmatrix} P_{12} \\ P_{22} \end{bmatrix}, \tag{6.2.23}$$

and $\mathcal{W}_i$ be the appropriately dimensioned μ_i-observable subspace,

$$\mathcal{W}_i = \mathrm{sp}\begin{bmatrix} 0 \\ I_{n-i} \end{bmatrix}. \tag{6.2.24}$$

Then $\tau(\mathcal{V}_i, \mathcal{W}_i) = \|P_{12}\|$. □

Next, bound the gap between $\mathcal{L}_i$ and $\mathcal{W}_i$.

Lemma 6.2.11: Let $\mathcal{V}_i$ be defined as in (6.2.23) and let $\tau(\mathcal{L}_i, \mathcal{V}_i) < \epsilon$. Then

$$\tau(\mathcal{L}_i, \mathcal{W}_i) \leq \epsilon + \|P_{12}\|. \tag{6.2.25}$$

Proof: From the triangle gap inequality,

$$\tau(\mathcal{L}_i, \mathcal{W}_i) \leq \tau(\mathcal{L}_i, \mathcal{V}_i) + \tau(\mathcal{V}_i, \mathcal{W}_i). \tag{6.2.26}$$

Apply Lemma 6.2.10 to the right hand side of (6.2.26). □

Although Lemma 6.2.11 bounds $\tau(\mathcal{L}_i, \mathcal{W}_i)$, it is not a complete result. At this point, $\tau(\mathcal{L}_i, \mathcal{V}_i) < \epsilon$ yields no information on $\|P_{12}\|$. Thus, we continue by exploring the relationship between $\tau(\mathcal{L}_i, \mathcal{V}_i)$ and $\|P_{12}\|$.

Lemma 6.2.12: Let (2.2.1) be asymptotically stable, symmetric, and balanced. Let P be given by (6.2.22). Also, let M be the diagonal matrix of signed modal measures as in (6.2.20) and decompose M conformally with P,

$$M = \begin{bmatrix} M_1 & \\ & M_2 \end{bmatrix}. \tag{6.2.27}$$

Then $\|P_{12}\| \rightarrow 0$ if and only if $\|M_2\| \rightarrow 0$.

Proof: (i) Show if $\|P_{12}\| \rightarrow 0$ then $\|M_2\| \rightarrow 0$. Parameterize (6.2.20) as follows,

$$\frac{1}{\|B\|^2}\begin{bmatrix} \Sigma_1^2 & \\ & \Sigma_2^2 \end{bmatrix} = \begin{bmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} M_1 & \\ & M_2 \end{bmatrix}\begin{bmatrix} S_{11} & S_{12} \\ S_{12}^t & S_{22} \end{bmatrix}\begin{bmatrix} M_1 & \\ & M_2 \end{bmatrix}\begin{bmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{21} & P_{22} \end{bmatrix}^T. \tag{6.2.28}$$

Note that for a symmetric system, $\|P_{21}\| = \|P_{12}\|$. The parameterization (6.2.28) allows $\|P_{12}\| \rightarrow 0$ by letting $\epsilon \rightarrow 0$. Performing the matrix multiplications on (6.2.28) yields,

$$\frac{1}{\|B\|^2}\Sigma_1^2 = P_{11}M_1S_{11}M_1P_{11}^T+\epsilon P_{12}M_2S_{12}^TM_1P_{11}^T+\epsilon P_{11}M_1S_{12}M_2P_{12}^T+\epsilon^2 P_{12}M_2S_{22}M_2P_{12}^T \tag{6.2.29a}$$

$$0 = \epsilon P_{11}M_1S_{11}M_1P_{12}+\epsilon^2 P_{12}M_2S_{12}^TM_1P_{12}^T+P_{11}M_1S_{12}M_2P_{22}^T+\epsilon P_{12}^TM_2S_{22}M_2P_{22}^T \tag{6.2.29b}$$

$$\frac{1}{\|B\|^2}\Sigma_2^2 = \epsilon^2 P_{21}M_1S_{11}M_1P_{21}^T+\epsilon P_{22}M_2S_{12}^TM_1P_{21}^T+\epsilon P_{21}M_1S_{12}M_2P_{22}^T+P_{22}M_2S_{22}M_2P_{22}^T \tag{6.2.29c}$$

Let $\|P_{12}\| \rightarrow 0$ by setting ϵ to zero. Equation (6.2.29) reduces to,

$$\frac{1}{\|B\|^2}\Sigma_1^2 = P_{11}M_1S_{11}M_1P_{11}^T \tag{6.2.30a}$$

$$0 = P_{11}M_1S_{12}M_2P_{22}^T \tag{6.2.30b}$$

$$\frac{1}{\|B\|^2}\Sigma_2^2 = P_{22}M_2S_{22}M_2P_{22}^T \tag{6.2.30c}$$

In (6.2.30b), P_{11} and P_{22} are invertible, thus pre-multiplying (6.2.30b) by P_{11}^{-1} and post-multiplying P_{22}^{-T} gives,

$$0 = M_1S_{12}M_2. \tag{6.2.31}$$

Writing out (6.2.30) in more detail,

$$0 = \begin{bmatrix} m_1 & & \\ & \ddots & \\ & & m_i \end{bmatrix}\begin{bmatrix} s_{1,i+1} & \cdots & s_{1n} \\ \cdot & \ddots & \cdot \\ s_{i,i+1} & \cdots & s_{in} \end{bmatrix}\begin{bmatrix} m_{i+1} & & \\ & \ddots & \\ & & m_n \end{bmatrix} \tag{6.2.32}$$

The quantity, s_{ij}, is non-zero for all i and j. Close inspection of (6.2.32) implies that either, $m_j = 0$ for j=1, ... ,i or $m_j = 0$ for j=i+1, ... , n. Since, by convention, the second-order modes are chosen such that all elements of Σ_1^2 are greater than or equal to all elements of Σ_2^2 it is apparent that $m_j = 0$ for j=i+1 to n must be true. Thus, $M_2 = 0$ and $\|M_2\| = 0$.

(ii) The proof of the converse is similar, beginning with a parameterization of M_2 in (6.2.28), instead of a parameterization of P_{12}. Note that this result applies to the Frobenius norm of M_2 and P_{12} as well. □

Remark 6.2.13: From Lemma 6.2.12 it is true that,

$$\lim_{\|M_2\| \to 0} \|P_{12}\| = 0. \tag{6.2.33}$$

So, by the definition of a limit, given an $\epsilon > 0$ there exists an η such that if $\|M_2\| < \epsilon$, then $0 < \|P_{12}\| < \eta$. □

Finally, we relate the modal measures to the gap with the following theorem.

Lemma 6.2.14: Let M be partitioned as in Lemma 6.2.12, where $M_1 \in R^{i \times i}$ and $M_2 \in R^{(n-i)\times(n-i)}$. Let $\mathcal{V} = \text{sp}\,[p_{i+1} \cdots p_n]$ so that $\dim(\mathcal{V}) = \dim(\mathcal{L}_i)$. Then,

$$\tau(\mathcal{L}_i, \mathcal{V}) \geq \|M_2\|. \tag{6.2.34}$$

Proof: $\|M_2\| = \max\{\cos\phi(C^T, p_j)\}$, j=i+1,...,n. The result follows directly by applying Theorem 6.1.1. □

Theorem 6.2.15: Let the system (2.2.1) be symmetric. Let $\mathcal{V}$ be an A-invariant subspace, $\mathcal{V}$= sp $[p_{i+1} \cdots p_n]$ as defined in Lemma 6.2.10. If $\tau(\mathcal{L}_i, \mathcal{V}) < \epsilon$, then there exists an η such that $\tau(\mathcal{L}_i, \mathcal{W}_i) < \eta$.

Proof: From the triangle gap inequality,

$$\tau(\mathcal{L}_i, \mathcal{W}_i) \leq \tau(\mathcal{L}_i, \mathcal{V}) + \tau(\mathcal{V}, \mathcal{W}_i) \tag{6.2.35}$$

Applying Lemma 6.2.10 to (6.2.35) gives,

$$\tau(\mathcal{L}_i, \mathcal{W}_i) \leq \tau(\mathcal{L}_i, \mathcal{V}) + \|P_{12}\| \tag{6.2.36}$$

From Lemma 6.2.14 $\tau(\mathcal{L}_i, \mathcal{V}) \geq \|M_2\|$, and from the hypothesis, this implies that $\|M_2\| < \epsilon$. Corollary 6.2.13 tells us that there exists a $\hat{\eta}$ such that $0 \leq \|P_{12}\| \leq \hat{\eta}$. Choosing $\eta = \epsilon + \hat{\eta}$ completes the proof. □

The interpretation of Theorem 6.2.15 is that for symmetric systems, a nearly unobservable subspace (ϵ small) implies the existence of a weakly observable subspace "near" such a subspace (η small). The result can be related to the dual GHR by strengthening the result from Lemma 6.2.11, in light of Theorem 6.2.15.

Corollary 6.2.16: Let the system (2.2.1) be symmetric and assume that Theorem 5.4.3 holds. If the dual GHR of (2.2.1) has a super(sub)-diagonal element, γ_{i+1}, then

$$\tau(\mathcal{L}_i, \mathcal{W}_i) \leq \frac{2\gamma_{i+1}}{\delta} + \|P_{12}\| \tag{6.2.37}$$

With respect to Corollary 6.2.13, if $\frac{2\gamma_{i+1}}{\delta} < \epsilon$ then there exists an $\hat{\eta}$ such that $0 < \|P_{12}\| < \hat{\eta}$. □

The proof follows from Lemma 6.2.11 and Theorem 6.2.15.

Corollary 6.2.16 says that if the dual GHR has a small super(sub)-diagonal element, then a nearly unobservable and a μ_i-observable subspace should be close to each other. To make this result complete, we must have some information on μ_i. Our approach is to bound μ_i in terms of (6.2.20).

Theorem 6.2.17: Let the system (2.2.1) be symmetric and balanced. Then

$$\mu_i \leq \frac{2\|B\|^2}{\|B_1\|^2} (\|P_{21}M_1S_{11}M_1P_{21}^T\|_F + 2\|P_{22}M_2S_{12}^TM_1P_{21}^T\|_F + \|P_{22}M_2S_{22}M_2P_{22}^T\|_F) \cdot (\|P_{11}\Lambda_1P_{11}^T + P_{12}\Lambda_2P_{12}{}^T\|_F). \tag{6.2.38}$$

Proof: We need an upper bound on $\|\Sigma_2^2\|_F$ and a lower bound on $\|\Sigma_1^2\|_F$. To find an upper bound of $\|\Sigma_2^2\|_F$, rewrite (6.2.29c) with ϵ=1 and take the Frobenius norm of both sides.

$$\|\Sigma_2^2\|_F = \|B\|^2\|P_{21}M_1S_{11}M_1P_{21}^T + P_{22}M_2S_{12}^TM_1P_{21}^T + P_{21}M_1S_{12}M_2P_{22}^T + P_{22}M_2S_{22}M_2P_{22}^T\|_F \tag{6.2.39}$$

Applying the triangle inequality to the right hand side of (6.2.39) yields,

$$\|\Sigma_2^2\|_F \le \|B\|^2\Big\{ \|P_{21}M_1S_{11}M_1P_{21}^T\|_F + 2\|P_{22}M_2S_{12}^TM_1P_{21}^T\|_F + \|P_{22}M_2S_{22}M_2P_{22}^T\|_F \Big\} \tag{6.2.40}$$

A lower bound of $\|\Sigma_1^2\|_F$ can be found from the Lyapunov equation of a balanced, symmetric system,

$$A\Sigma^2 + \Sigma^2A = -C^TC = -BB^T. \tag{6.2.41}$$

Block partition (6.2.41) such that,

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \Sigma^2=\begin{bmatrix} \Sigma_1^2 & \\ & \Sigma_2^2 \end{bmatrix}, B=\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{6.2.42}$$

A series of matrix multiplications of (6.2.41) yields,

$$A_{11}\Sigma_1^2 + \Sigma_1^2A_{11} = -B_1B_1^T. \tag{6.2.43}$$

Taking the norm of both sides of (6.2.43) and simplifying gives,

$$\|\Sigma_1^2\|_F \ge \frac{\|B_1\|^2}{2\|A_{11}\|_F} = \frac{\|B_1\|^2}{2\|P_{11}\Lambda_1P_{11}^T+P_{12}\Lambda_2P_{12}{}^T\|_F} \tag{6.2.44}$$

where $A_{11}= P_{11}\Lambda_1P_{11}^T+P_{12}\Lambda_2P_{12}{}^T$, is found from the Jordan decomposition of (2.2.1). Complete the proof by taking the ratio of (6.2.40) to (6.2.44). □

Remark 6.2.18: If a nearly unobservable subspace exists, i.e. $\tau(\mathcal{L}_i, \mathcal{V})\to 0$, implying $\|M_2\|_F\to 0$ (Lemma 6.2.14) and $\|P_{12}\|_F\to 0$ (Theorem 6.2.12), then Theorem 6.2.17 says that $\mu_i\to 0$. Roughly, we can say that the gap between $\mathcal{L}_i$

and an A-invariant subspace bounds the allowable gap between $\mathcal{L}_i$ and $\mathcal{W}_i$. This in turn controls the allowable magnitude of μ_i. □

Remark 6.2.19: The converse to Theorem 6.2.15 may not be true. A μ_i-observable subspace may be weakly observable due to spectra, thus the gap between $\mathcal{L}_i$ and an A-invariant subspace may not be small. □

Remark 6.2.20: The bounds given by Theorem 6.2.17 and Theorem 6.2.15 are based on the triangle inequality, and thus are conservative. A (sub)super-diagonal element of the dual GHR, γ_{i+1}, for example, could be rather large and the $\tau(\mathcal{L}_i,\mathcal{W}_i)$ could still be very small. □

Example 6.2.21: To illustrate some results of this section, consider the system (5.2.7). Notice that $\gamma_4 = 0.95$ is small compared to the other matrix elements. Thus we should expect to find an A-invariant subspace near $\mathcal{L}_3$. In fact, for $\mathcal{V}$ spanned by the right eigenvectors corresponding to the eigenvalues 79.143 and 0.10868, $\tau(\mathcal{L}_3, \mathcal{V})$= .0225. From Theorem 6.2.15 we should find $\tau(\mathcal{L}_3, \mathcal{W}_3)$ to be small as well. Let x= Tz be the transformation from the dual GHR basis to the balanced basis. □

$$T = \begin{bmatrix} .6544 & -.7227 & .2222 & .0035 & -.0003 \\ .6438 & .3785 & -.6648 & -.0160 & .0022 \\ .3965 & .5781 & .7122 & .0324 & -.0157 \\ .0048 & .0077 & .0127 & -.0303 & 1.0914 \\ .0048 & .0099 & .0341 & -.9987 & -.0336 \end{bmatrix} \tag{6.2.45}$$

Decompose the resulting balanced system into its Jordan form yielding,

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \Lambda_1 & \\ & \Lambda_2 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}^T \tag{6.2.46}$$

From this decomposition, $\|P_{12}\| = 0.0404$. A bound on $\tau(\mathcal{L}_3, \mathcal{W}_3)$ can be computed from Lemma 6.2.11,

$$\tau(\mathcal{L}_3, \mathcal{W}_3) \leq 0.0225 + 0.0404 = 0.0629. \tag{6.2.47}$$

The actual value is $\tau(\mathcal{L}_3, \mathcal{W}_3) = 0.0393$ (which can be computed from T). This verifies Theorem 6.2.15. Remark 6.2.18 asserts that μ_3 should also be small. From Theorem 6.2.17, we obtain a bound on μ_3, $\mu_3 \leq 0.0119$. The actual value is $\mu_3 = 0.000664$. □

6.3 Relationship to Zeros

In the last two sections, we have shown that residues, near unobservability (uncontrollability), and weak observability (controllability) can be related through geometry. We would like to extend these ideas to the system transfer function by generalizing the result—if the system is unobservable (uncontrollable) then the transfer function exhibits a pole-zero cancellation. In previous work, Lindner and his colleagues [7] have shown, through the dual GHR, that near unobservability (uncontrollability) corresponds to almost pole-zero cancellations. Likewise, they have argued that small residues and almost pole-zero cancellations are equivalent concepts. The results are summarized below.

Theorem 6.3.1: [4] Let the system (2.2.1) be represented by its dual GHR and

partitioned as,

$$\dot{x} = \begin{bmatrix} F^i & & \\ & \gamma_{i+1} & \\ \gamma_{i+1} & & E^{i+1} \end{bmatrix} x + \begin{bmatrix} G_1 \\ \\ 0 \end{bmatrix} u, \qquad (6.3.1)$$

$$y = \begin{bmatrix} H_1 & 0 \end{bmatrix} x,$$

where $F \in R^{ixi}$ and $x \in R^n$. If $\gamma_{i+1} \rightarrow 0$, then the system exhibits n-i pole zero cancellations. □

Basically, Theorem 6.3.1 is formal proof of the statement that if γ_{i+1} is small then there will be n-i almost pole-zero cancellations in the system transfer function. By applying Theorem 6.3.1 to Theorem 6.1.2, the relationship to residues follows.

Theorem 6.3.2: [4] Let the system (6.1.1) be represented by its dual GHR and assume Theorem 5.4.3 holds. Then if $\gamma_{i+1} \rightarrow 0$,

$$|r_j| \leq 4\gamma_1^2 \|p_j\| \, \|q_j\| \, \frac{\gamma_{i+1}^2}{\delta^2}, \qquad (6.3.2)$$

and $|r_j| \rightarrow 0$ for j= 1,. . ., n-i. □

Note that exactly n-i eigenvectors will satisfy this argument. So Theorem 6.3.2 says that if γ_{i+1} is small, then there will be n-i small residues. From Theorem 6.3.1, such a system will also have n-i almost pole-zero cancellations in

its transfer function.

In view of the relationship between near unobservability and weak observability described in Section 6.2, the question that we would like to address here is, if the system transfer function has almost pole-zero cancellations, what does this imply about the system's second-order modes? The following example is useful for illustrating the problem.

Example 6.3.3: Consider the symmetric system, with normalized input and output vectors.

$$\dot{x} = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} x + \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} u, \qquad (6.3.3)$$

$$y = \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} x$$

For a geometric interpretation of the interaction of the subspaces involved in this example, see Figure 5. By varying θ, $\tau(\mathcal{L}_1, \mathcal{V}_i)$ we would like to observe the effect of this on ϕ, $\tau(\mathcal{W}_1, \mathcal{V}_i)$, and the second-order modes.

From the Lyapunov equation $AW^2 + W^2A = -BB^T = -C^TC$, and by recognizing that for this example that the singular value decomposition of the Gramian can be written as

$$W^2 = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \\ & \sigma_2^2 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \qquad (6.3.4)$$

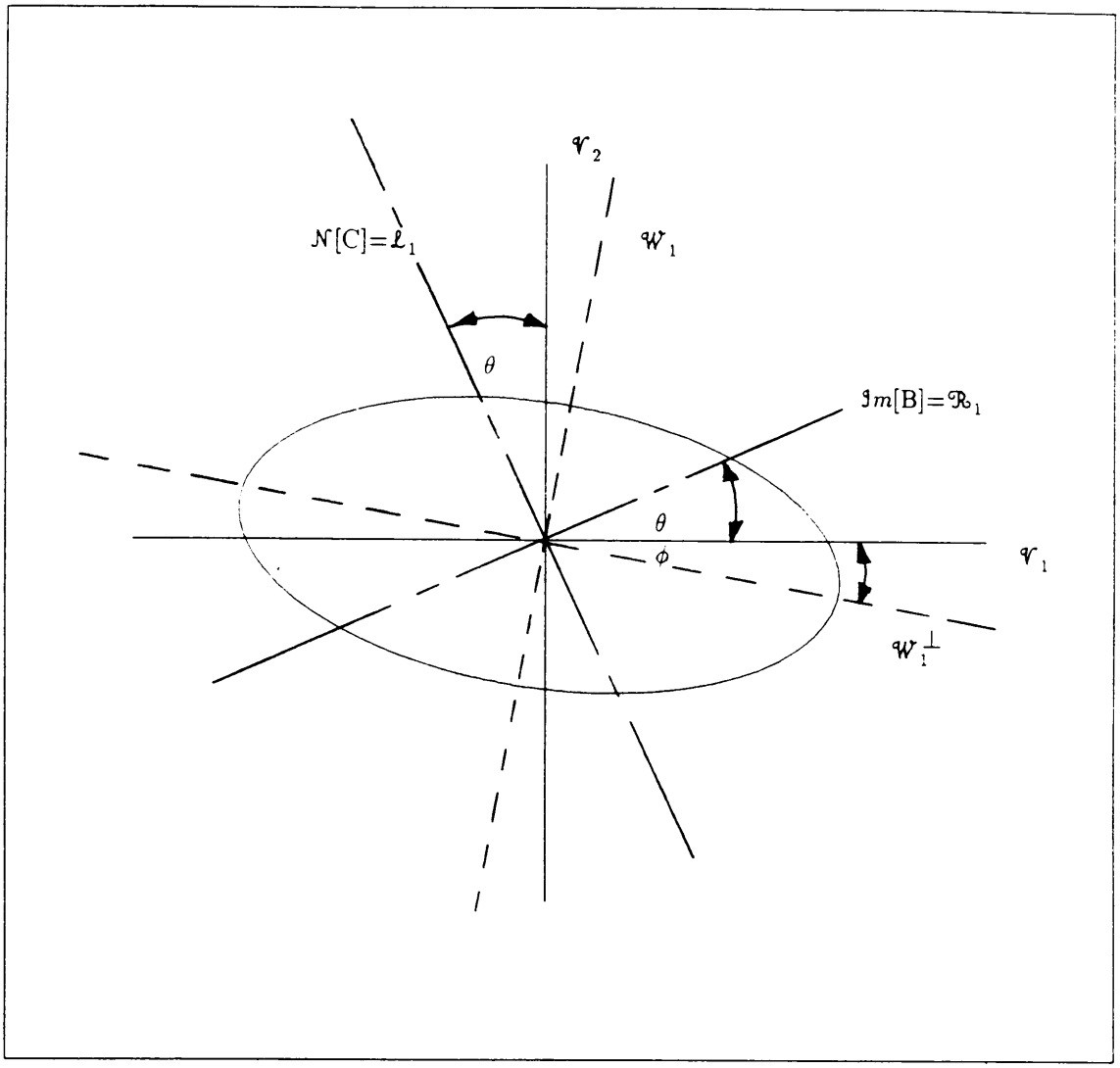

Figure 5: The geometry of Example 6.3.3.

we obtain the following set of scalar equations,

$$2\lambda_1(\sigma_1^2\cos^2\phi + \sigma_2^2\sin^2\phi) = -\cos^2\theta, \tag{6.3.5}$$

$$\cos\phi\ \sin\phi\ (\sigma_1^2 - \sigma_2^2) = -\cos\theta\ \sin\theta, \tag{6.3.6}$$

$$2\lambda_2(\sigma_1^2\sin^2\phi + \sigma_2^2\cos^2\phi) = -\sin^2\theta. \tag{6.3.7}$$

Because the system is symmetric, W^2 can be thought of as an ellipsoid in the state space which is the orthogonal rotation of the familiar Gramian ellipsoid of balanced system. Thus, the singular values of W^2 are the second-order modes of the system. By convention assume $\sigma_1^2 > \sigma_2^2$. We can force the transfer function of the system (6.3.3) to have an almost pole-zero cancellation by letting $|\sin\theta| < \epsilon$, where we are free to choose $\epsilon > 0$, arbitrarily small. From (6.3.6),

$$|\cos\phi\ \sin\phi|\ (\sigma_1^2 - \sigma_2^2) = |\cos\theta\ \sin\theta| < \epsilon \tag{6.3.8}$$

simplifying,

$$|\sin(2\phi)| < \frac{2\epsilon}{(\sigma_1^2 - \sigma_2^2)} \tag{6.3.9}$$

Roughly, (6.3.9) says that if there is an almost pole-zero cancellation in the system transfer function, then ϕ, $\tau(\mathcal{W}_1, \mathcal{V}_i)$ will be small. From (6.3.7),

$$|2\lambda_2|\ |(\sigma_1^2\sin^2\phi + \sigma_2^2\cos^2\phi)| < \epsilon^2 \tag{6.3.10}$$

and by algebraic manipulation,

$$\sigma_2^2 \; < \; \left(\frac{\epsilon^2}{|2\lambda_2|} - \sigma_1^2 \sin^2\phi \right) \frac{1}{\cos^2\phi}. \tag{6.3.11}$$

Since from (6.3.9) we know that ϕ is small, an approximate bound to (6.3.11) is,

$$\sigma_2^2 \; < \; \frac{\epsilon^2}{|2\lambda_2|}. \tag{6.3.12}$$

By similar algebraic manipulation of (6.3.5), we obtain this approximate inequality,

$$\sigma_1^2 \; > \; \frac{1-\epsilon^2}{|2\lambda_1|}. \tag{6.3.13}$$

From (6.3.12) and (6.3.13),

$$(\sigma_1^2 - \sigma_2^2) > \frac{1}{|2\lambda_1|} - \epsilon^2 \left(\frac{1}{|2\lambda_1|} + \frac{1}{|2\lambda_2|} \right). \tag{6.3.14}$$

Equation (6.3.14) tells us that if there is an almost pole-zero cancellation, ϵ small, then $\sigma_1^2 \gg \sigma_2^2$, implying μ_i is small, if λ_1 (the retained mode) is slow. This information is contained, in more general terms, in the bound given by Theorem 6.2.17.

$$\mu_i \leq \frac{2\|B\|^2}{\|B_1\|^2} \left(\|P_{21}M_1S_{11}M_1P_{21}^T\|_F + 2\|P_{22}M_2S_{12}^TM_1P_{21}^T\|_F + \|P_{22}M_2S_{22}M_2P_{22}^T\|_F \right) \cdot \left(\|P_{11}\Lambda_1P_{11}^T + P_{12}\Lambda_2{P_{12}}^T\|_F \right). \tag{6.2.38}$$

The first term of the right hand side of (6.2.38) is essentially a constant. Since almost pole-zero cancellations imply $\|M_2\|_F \rightarrow 0$, and consequently $\|P_{12}\|_F \rightarrow 0$. The

bound of μ_i, then, is largely influenced by $\|\Lambda_1\|_F$, the spectra of the retained modes. Clearly for μ_i small, the slow modes should be retained.

6.4 Classical Comparisons

Evaluating reduced order models by how well its frequency response matches that of the full order model is a well established method of comparison in the literature. A reasonable question to ask then, is how is the frequency response of the full model affected by balancing, dual GHR, dominant mode, and Markov parameter/time moment matching techniques. Since the quality of a reduced order model (or any model) is largely dependent on application, it is not our intent here to quantify or define a good or a bad reduced model. We simply point out, qualitatively, what is going on in the frequency domain when using the model reduction techniques described in this thesis.

Glover [12] has proven a very useful result for model reduction via balancing in his work on optimal Hankel-norm approximations.

Theorem 6.4.1: [12] Let G(s) be a stable, rational transfer function with second-order modes $\sigma_1 > \sigma_2 > \cdots > \sigma_n$. Let $\hat{G}(s)$ be the reduced order model obtained by truncating the balanced realization of G(s) to the first k states. Then

$$\|G(j\omega) - \hat{G}(j\omega)\|_{L\infty} \leq 2 \sum_{i=k+1}^{n} \sigma_i \qquad (6.4.1)$$

where $\|H(j\omega)\|_{L\infty} = \sup_{\omega} |H(j\omega)|$. □

Remark 6.4.2: Theorem 6.4.1 also implies that if the bound given by the L_∞ norm is tight, then $\hat{G}(j\omega)$ will be a very good approximation of $G(j\omega)$ in the pass band of the system. Thus, reduced models obtained via balancing should well approximate the full system in the pass band of the system. □

From (5.5.2), the transfer function of a system can be written out in terms of Markov parameters,

$$G(s) = J_1 s^{-1} + J_2 s^{-2} + J_3 s^{-3} + \cdots . \tag{6.4.2}$$

As $s \rightarrow \infty$, the first several terms of (6.4.2) dominate this description of G(s). Thus, we should expect reduction techniques which match Markov parameters to yield good high frequency approximations. Similarly, (5.5.3) can be written out in terms of time moments,

$$G(s) = - Y_1 - Y_2 s - Y_3 s^2 - \cdots , \tag{6.4.3}$$

and as $s \rightarrow 0$ the first few terms of (6.4.3) dominate. Thus, reduction techniques preserving time moments should give a good low frequency approximation.

Remark 6.4.3: The dual GHR and the Cauer first form continued fraction expansion both produce reduced order models which match a number of Markov parameters. Thus, we should expect a good approximation of $G(j\omega)$ at high frequencies. □

Remark 6.4.4: The mixed Cauer forms described in Section 5.5 match both Markov parameters and time moments. A fair approximation of $G(j\omega)$ should

result over all frequencies. □

Remark *6.4.5:* For the dominant mode method, we expect a good approximation of $G(j\omega)$ at and near the frequencies of the retained modes. □

In support of these statements, consider the following example.

Example *6.4.6:* [7] The system below models the dynamics of a hoop column antenna.

$$\dot{x} = \begin{bmatrix} -.013 & -6.76 & & & & & \\ 6.76 & -.001 & -1.599 & & & & \\ & 1.599 & -.023 & -6.148 & & & \\ & & 6.148 & .01 & -1.477 & & \\ & & & 1.477 & -.001 & -.605 & \\ & & & & .605 & -.019 & -1.249 \\ & & & & & 1.249 & .009 \end{bmatrix} x + \begin{bmatrix} 0.01 \\ \\ \\ \\ \\ \\ \end{bmatrix} u,$$

$$y = \begin{bmatrix} 0.01 & & & & & & \end{bmatrix} x. \qquad (6.4.4)$$

Fourth order models were obtained using several of the methods described in this thesis. The resulting transfer functions of the simplified models are,

Balancing:

$$G_1(s) = \frac{.0001(.997s^3+.0130s^2+42.28s+.0185)}{s^4+.0258s^3+88.26s^2+1.101s+1833.6} \qquad (6.4.5)$$

Dual GHR (8 Markov parameters):

$$G_2(s) = \frac{.0001(s^3+.0140s^2+40.35s+.0122)}{s^4+.027s^3+86.05s^2+1.131s+1727.3} \tag{6.4.6}$$

1 Markov parameter/7 time moments:

$$G_3(s) = \frac{.0001(s^3+92.105s^2+.1447s+9.7739)}{s^4+125.6s^3+44.2s^2+3921.4s+0.8} \tag{6.4.7}$$

Dominant mode:

$$G_4(s) = \frac{.0001(.9968s^3+.0125s^2+42.41s+.0068)}{(s^2+.0112s+33.47)(s^2+.0146s+54.8)}. \tag{6.4.8}$$

The frequency response of each method is given in Figures 6-9, compared with the full order model, and Table 1 displays some specific frequency data each system. As expected, $G_3(s)$ which matches time moments, gives the best match to G(s) at frequencies up to about $s = j\omega = 1$ (see Figure 8). At middle frequencies, balancing (Figure 6) gives a good match and $G_4(s)$ (Figure 9) is the best match for the s = 5.79 and 7.40, which were poles of the full system. Finally, at high frequencies, the Dual GHR (Figure 7), matching Markov parameters, matches the full system response most closely. The mixed method, $G_3(s)$ (Figure 8), which only marginally approximates G(s) at mid-frequencies, closely approximates G(s) are high frequencies. □

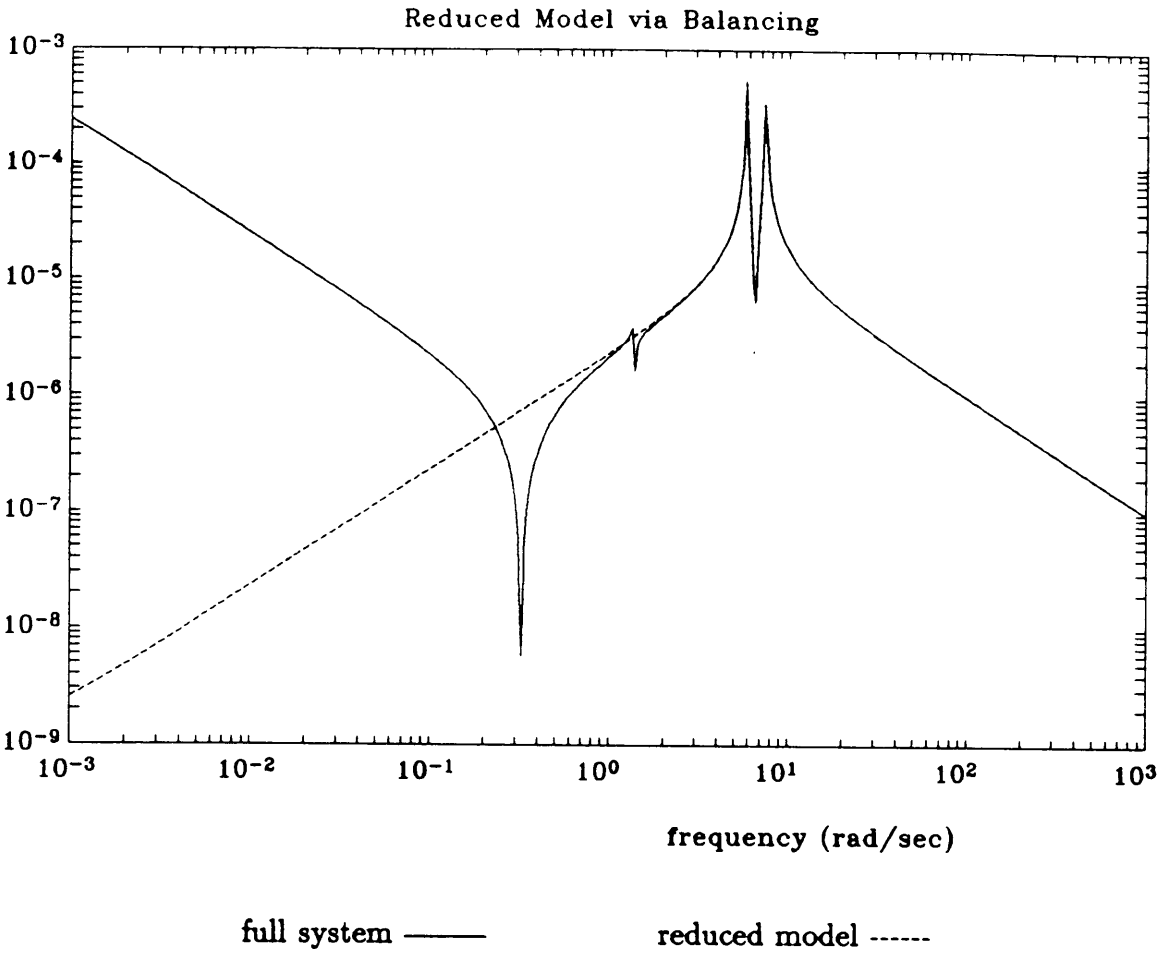

Figure 6: Frequency response comparison of the full system in Example 6.4.6 to a 4^{th} order reduced model found by balancing.

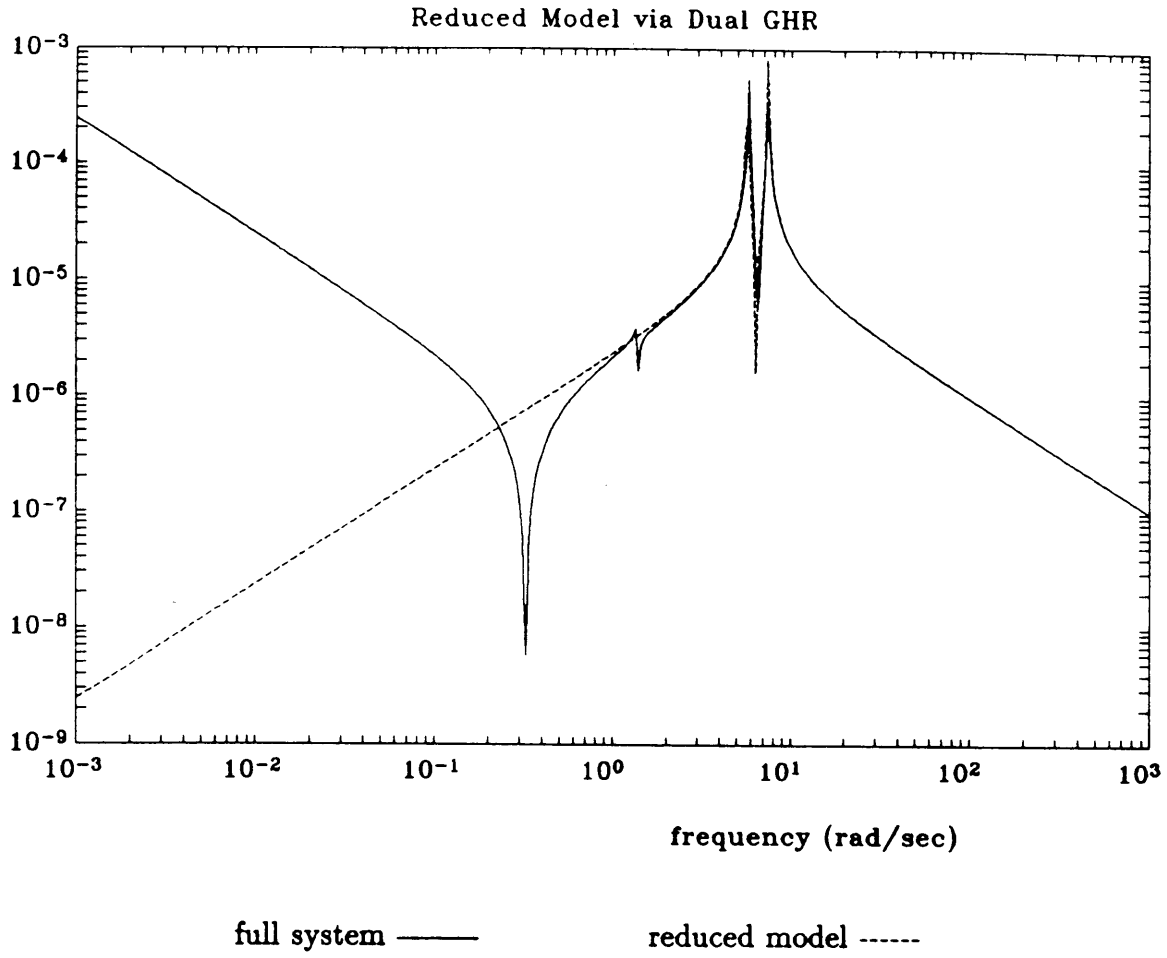

Figure 7: Frequency response comparison of the full system in Example 6.4.6 to a 4^{th} order reduced model found by dual GHR (matching 8 Markov parameters).

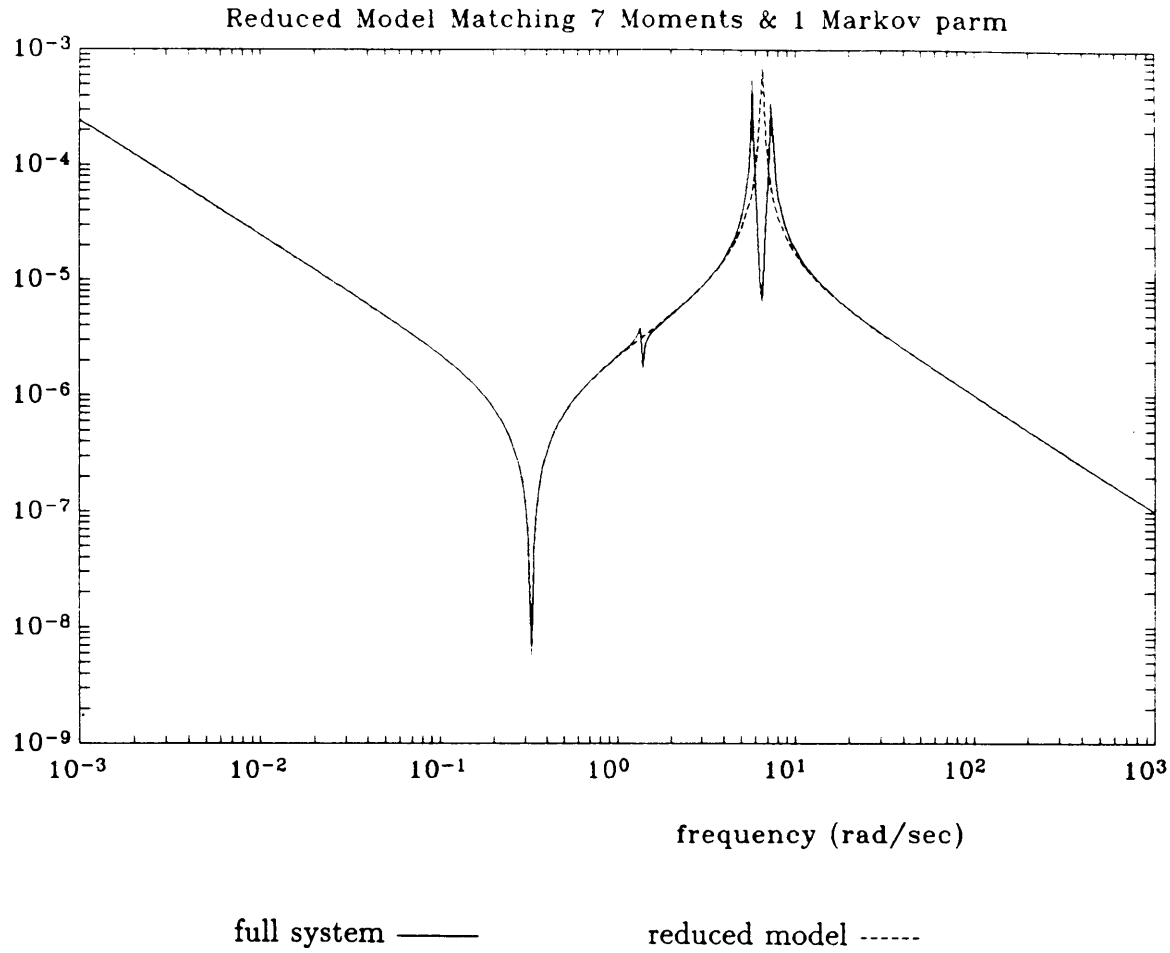

Figure 8: Frequency response comparison of the full system in Example 6.4.6 to a 4^{th} order reduced model found by matching 1 Markov parameter and 7 time moments.

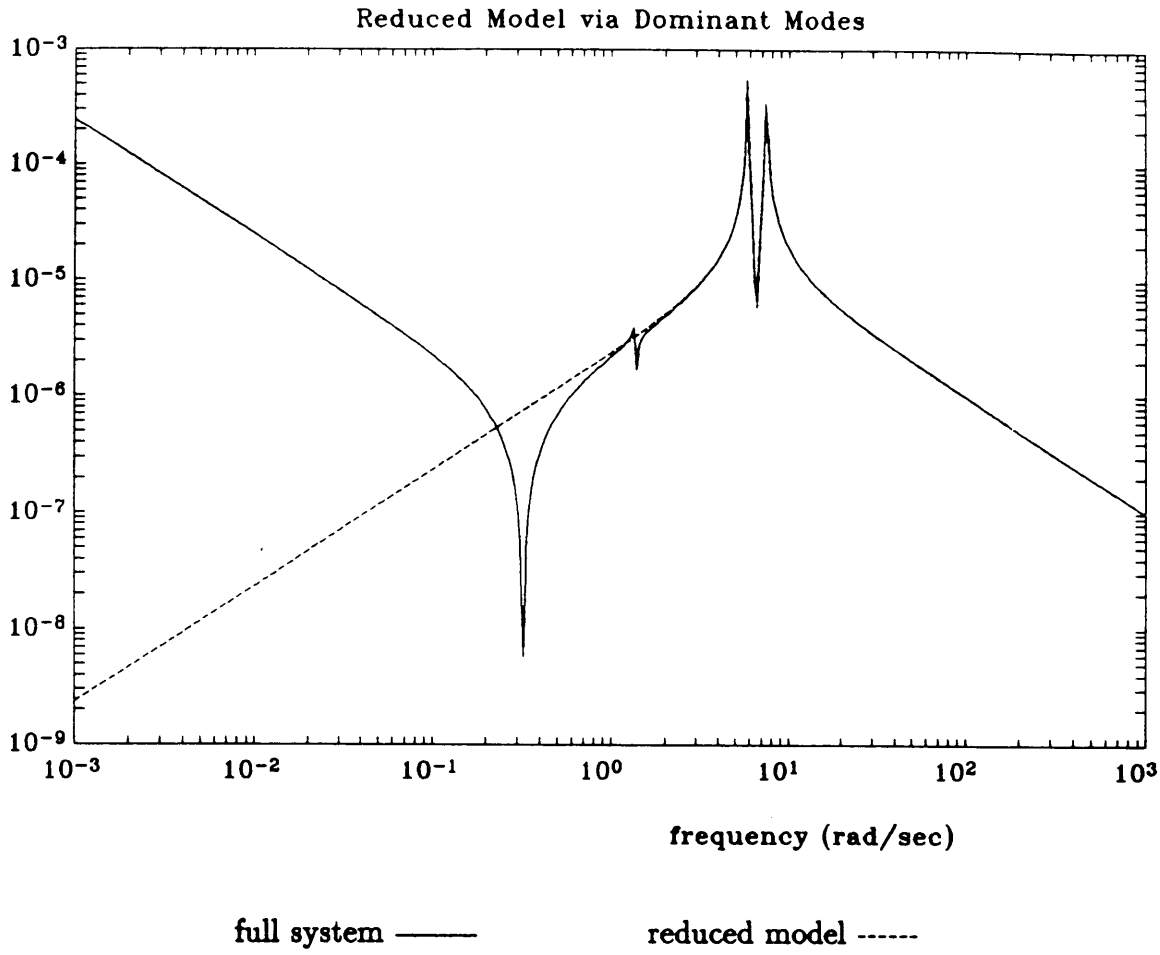

Figure 9: Frequency response comparison of the full system in Example 6.4.6 to a 4^{th} order reduced model found by dominant modes.

Table 1: Selected frequency data for Example 6.4.6. The shaded box corresponds to the best match with $G(\omega)$, the full order system.

freq, ω	$\|G(\omega)\|$ ($\angle G(\omega)$)	$\|G_1(\omega)\|$ ($\angle G_1(\omega)$)	$\|G_2(\omega)\|$ ($\angle G_2(\omega)$)	$\|G_3(\omega)\|$ ($\angle G_3(\omega)$)	$\|G_4(\omega)\|$ ($\angle G_4(\omega)$)
0	$1.254\text{x}10^{-3}$ (0°)	$1.000\text{x}10^{-9}$ (0°)	$7.060\text{x}10^{-10}$ (0°)	$1.220\text{x}10^{-3}$ (0°)	$3.720\text{x}10^{-10}$ (-180°)
0.32	$2.731\text{x}10^{-8}$ (-87.9°)	$7.432\text{x}10^{-7}$ (-270.1°)	$7.494\text{x}10^{-7}$ (-270.1°)	$2.739\text{x}10^{-8}$ (-87.9°)	$7.397\text{x}10^{-7}$ (-270.0°)
1	$2.188\text{x}10^{-6}$ (-270.1°)	$2.375\text{x}10^{-6}$ (-270.0°)	$2.396\text{x}10^{-6}$ (-270.0°)	$2.169\text{x}10^{-6}$ (-270.0°)	$2.3639\text{x}10^{-6}$ (-270.0°)
5.79	$2.855\text{x}10^{-3}$ (-39.8°)	$2.708\text{x}10^{-3}$ (-44.7°)	$1.176\text{x}10^{-4}$ (-87.1°)	$1.801\text{x}10^{-4}$ (-74.4°)	$2.780\text{x}10^{-3}$ (-41.6°)
7.40	$3.852\text{x}10^{-3}$ (-346.6°)	$3.641\text{x}10^{-3}$ (-340.0°)	$7.565\text{x}10^{-4}$ (-79.9°)	$2.308\text{x}10^{-5}$ (-86.9°)	$3.821\text{x}10^{-3}$ (-344.9°)
10	$1.911\text{x}10^{-5}$ (-89.9°)	$1.902\text{x}10^{-5}$ (-89.9°)	$1.091\text{x}10^{-5}$ (-89.9°)	$1.069\text{x}10^{-5}$ (-87.5°)	$1.098\text{x}10^{-5}$ (-89.8°)
100	$1.005\text{x}10^{-5}$ (-90.0°)	$1.002\text{x}10^{-5}$ (-90.0°)	$1.005\text{x}10^{-5}$ (-90.0°)	$8.499\text{x}10^{-7}$ (-81.1°)	$1.001\text{x}10^{-6}$ (-90.0°)
1000	$1.000\text{x}10^{-7}$ (-90.0°)	$9.970\text{x}10^{-8}$ (-90.0°)	$1.000\text{x}10^{-7}$ (-90.0°)	$9.964\text{x}10^{-8}$ (-88.1°)	$9.969\text{x}10^{-8}$ (-90.0°)

6.5 Chapter Summary

Balancing, modal methods, and dual GHR are compared in terms of the geometry of their reducing subspaces. Results relating dual GHR to modal methods are reviewed. Several results relating dual GHR to balancing, for orthogonally symmetric systems are given. The results are also considered in terms of their relationships to the zeros of the system. All model reduction methods discussed in this thesis are then compared as they effect the frequency response of the system.

7.0 Conclusions

We have presented three ways of selecting reducing subspaces for geometric model reduction based on measures of controllability and observability. The controllability and observability Gramians can be used to define weakly controllable subspaces, modal measures or residues lead to dominant and residual A-invariant subspaces, and small gaps between A-invariant subspaces and reachable or unobservable spaces motivates nearly unobservable and uncontrollable subspaces. Since these subspaces lie along the natural basis vectors in balancing, Jordan canonical form, and the dual GHR, the orthogonal projections onto the reducing subspaces is a truncation of the state-space realizations.

The geometry of the reducing subspaces enables the comparison of the different measures of controllability and observability, and consequently comparison of the model reduction approaches. Obviously, if parts of a system are uncontrollable or unobservable, then the measures of controllability and observability, in each case, will be zero; and each model reduction method would identify the same reduced order model. Intuitively, when the measures of

controllability and observability are non-zero, but small, each method should identify a similar reduced order model. This intuition, as is often the case, is not at all trivial to prove. In light of previous results unifying dual GHR with modal methods by showing that residual A-invariant subspaces are also nearly unobservable and uncontrollable, in this thesis, we have shown that for symmetric systems a nearly unobservable subspace is also weakly observable. The converse, however, is not true. This relates balancing to dual GHR. Also, we have shown the analogy of these results with almost pole-zero cancellations. The results provide theoretical justification for unifying the model reduction methods based on measures of controllability and observability.

As an alternative to measures of controllability and observability, we showed that the reducing subspaces can be chosen in such a way to match a certain number of Markov parameters and time moments of the full system. Several results were given showing how a coordinate basis may be found where these reducing subspaces lie along the natural basis, thus facilitating model reduction by matrix truncation. A connection was made between these ideas and the Lanczos algorithm — popularly used in the model reduction of structures.

Finally, we examined the model reduction methods in terms of frequency characteristics of the system and gave some qualitative guidelines as to how the frequency domain and frequency response is affected by various subspace projections.

The results in this thesis suggest several points of departure for further study. First, the relationship between balancing and dual GHR or modal methods, for general systems, remains an unsatisfied question. Another possible research topic would be the extension of these results to the multiple-input multiple-output case. Both balancing and modal techniques have been extended to the multivariable

case. Presently, the dual GHR has not been generalized to this case. In view of the relationship of the dual GHR to the Cauer first form, which has been extended to transfer function matrices, it may be possible that such a generalized canonical form exists. Finally, further theoretical development along the lines of selecting the reducing subspaces to match any desired mix of Markov parameters time moments would produce potentially useful results for model reduction.

List of References

[1] Jamshidi, M., Large-scale systems: modeling and control. New York: Elsevier Science, 1983.

[2] Davison, E. J., "A method for simplifying linear dynamic systems," IEEE Transactions on Automatic Control, vol. AC-12, pp. 119-121, 1966.

[3] Moore, B. C., "Principal component analysis in linear systems: controllability, observability and model reduction," IEEE Transactions on Automatic Control, vol. AC-26, pp. 17-32, 1981.

[4] Lindner, D. K., "Near aggregation, the dual GHR, and pole zero cancellation," Int. Journal of Control, vol. 48, pp. 705-727, 1988.

[5] Hamdan, A. M. A. and A. Nayfeh, "Modal controllability and observability measures of balanced representations," 27^{th} Conference on Decision and Control, 1988.

[6] Kalman, R. E., "On partial realizations, transfer functions, and canonical forms," Acta Polytechnica Scandinavica, Ma 31, pp. 9-32, 1979.

[7] Lindner, D. K., J. Babendreier and A. M. A. Hamdan, "The dual GHR, zeros, and residues, System and Control Letters, vol. 10, pp. 245-250, 1988.

[8] Lai, J. S. and J. C. Hung, "Practical model reduction methods," IEEE Transactions on Industrial Electronics, vol. IE-34, pp. 70-77, 1987.

[9] Gohberg, I., P. Lancaster and L. Rodman, Invariant subspaces of matrices with applications. New York: Wiley, 1986.

[10] Chen, C. T., Linear systems theory and design. New York: Holt, Rinehart, and Winston, 1970.

[11] Brockett, R. W., Finite dimensional linear systems. New York: Wiley, 1970.

[12] Glover, K., "All optimal Hankel-norm approximations of linear multivariable systems and the L^{∞}-error bounds," Int. Journal of Control, vol. 39, pp. 1115-1193, 1984.

[13] Lindner, D. K., J. Babendreier and A. M. A. Hamdan, "Measures of controllability and observability and residues," IEEE Transactions on Automatic Control, vol. AC-34, pp. 648-650, 1989.

[14] Athans, M., P. Kapasouris, E. Kappos, and H. Spang, "Linear quadratic gaussian with loop transfer recovery methodology for the F-100 engine," J. of Guidance, Control, and Dynamics, vol. 9, pp. 45-52, 1986.

[15] Chen, C. F. and L. S. Shieh, "A novel approach to linear model simplification," Int. Journal of Control, vol. 8, pp. 561-570, 1968.

[16] Lindner, D. K. and W. R. Perkins, "The generalized Hessenberg representation and near aggregation," Automatica, vol. 24, pp. 267-270, 1988.

[17] Stewart, G. W., "Error and perturbation bounds for subspaces associated with certain eigenvalue problems," SIAM Review, vol. 15, pp. 727-764, 1973.

[18] Tse, E. C., J. Medanic and W. R. Perkins, "Generalized Hessenberg transformation for reduced order modeling of large scale systems," Int. Journal of Control, vol. 27, pp. 493-512, 1978.

[19] Lindner, D. K. and J. Babendreier, "The dual GHR," internal technical report, department of Electrical Engineering, Virginia Tech, Blacksburg, VA, 1986.

[20] Lindner, D. K., W. R. Perkins and J. Medanic, "Chained aggregation and three-control component design: a geometric analysis," Int. Journal of Control, vol. 35, pp. 621-636, 1982.

[21] Rissanen, J., "Recursive identification of linear systems," SIAM Journal of Control, vol. 9, pp. 420-430, 1971.

[22] Gragg, W. B. and A. Lindquist, "On the partial realization problem," Linear Algebra and its Applications, vol. 50, pp. 277-319, 1983.

[23] Hwang, C., "On Cauer third continued fraction expansion method for the simplification of large system dynamics," Int. Journal of Control, vol. 37, pp. 599-614, 1983.

[24] Nour-Omid, B. and R. W. Clough, "Dynamic analysis of structure using Lanczos coordinates," Earthquake Engineering and Structural Dynamics, vol. 12, pp. 565-577, 1984.

[25] Wilson, E. L., M. W. Yuan and J. M. Dickens, "Dynamic analysis by direct superposition of Ritz vectors," Earthquake Engineering and Structural Dynamics, vol. 10, pp. 813-821, 1982.

[26] Craig, R. R. and A. L. Hale, "The block-Krylov component synthesis method for structural model reduction," J. of Guidance, Control, and Dynamics, to appear.

[27] Cullum, J. and R. A. Willoughby, "The Lanczos phenomenon—an interpretation based upon conjugate gradient optimization," Linear Algebra and its Applications, vol. 29, pp. 63-90, 1980.

[28] Lanczos, C., "An iteration method for the solution of the eigenvalue problem of linear differential and integral operators," J. of Research of the National Bureau of Standards, vol. 45, pp. 255-282, 1950.

[29] Wicks, M. A. and R. A. DeCarlo, "An energy approach to controllability," 27^{th} Conference on Decision and Control, 1988.

Appendix

A.1 Transformation to Balanced Coordinates

The transformation, T, from a system represented in an arbitrary basis (2.1.1) to the balanced basis (3.3.8) can be determined from the following algorithm.

1. Compute the controllability Gramian, W_c^2, of (2.1.1). If (2.1.1) is asymptotically stable, then W_c^2 is found from the unique solution of the Lyapunov equation,

$$AW_c^2 + W_c^2A^T = - BB^T. \tag{A.1.1}$$

2. Determine the singular value decomposition of W_c^2,

$$W_c^2 = V_c\Sigma_c^2V_c^T. \tag{A.1.2}$$

3. Define an intermediate transformation, P_1,

$$P_1 \equiv V_c \Sigma_c. \tag{A.1.3}$$

4. Apply (A.1.3) to (2.1.1) to obtain the intermediate system,

$$\begin{aligned} \tilde{x} &= \tilde{A}\tilde{x} + \tilde{B}u, \\ y &= \tilde{C}\tilde{x}, \end{aligned} \tag{A.1.4}$$

where $\tilde{A} = P_1^{-1}AP_1$, $\tilde{B} = P_1^{-1}B$, and $\tilde{C} = CP_1$. Note that in this basis, the controllability Gramian, $\tilde{W}_c^2$, is given by,

$$\tilde{W}_c^2 = P_1^{-1}W_c^2P_1^{-T} = I. \tag{A.1.5}$$

5. Compute the observability Gramian, $\tilde{W}_o^2$, of (A.1.4).

6. Determine the singular value decomposition of $\tilde{W}_o^2$,

$$\tilde{W}_o^2 = \tilde{V}_o\tilde{\Sigma}_o^2\tilde{V}_o^T. \tag{A.1.6}$$

7. Define a second intermediate transformation,

$$P_2 \equiv \tilde{V}_o{\tilde{\Sigma}_o}^{-1/2}. \tag{A.1.7}$$

8. Apply (A.1.7) to (A.1.4) giving,

$$\begin{aligned} \dot{\hat{x}} &= A_b\hat{x} + B_bu, \\ y &= C_bx. \end{aligned} \tag{A.1.8}$$

With respect to (A.1.8),

$$\hat{W}_o^2 = P_2^T \tilde{W}_o^2 P_2 = \tilde{\Sigma}_o, \tag{A.1.9a}$$

$$\hat{W}_c^2 = P_2^{-1} I P_2^{-T} = \tilde{\Sigma}_o. \tag{A.1.9b}$$

Equation (A.1.9) says that both Gramians are equal and diagonal. Thus (A.1.8) is balanced.

9. The transformation, T, is given by,

$$T = P_1 P_2. \tag{A.1.10}$$

Numerical examples on balancing in this thesis were computed using MATLAB.

A.2 Extended Chained Aggregation

The extended chained aggregation algorithm is used to determine the dual GHR form of a single-input single-output system. A detailed outline is given in [19]. Below, we give a brief summary and an algorithm for computing the dual GHR using MATLAB.

Consider (2.1.1). After one step of extended chained aggregation, the resulting system is,

$$\dot{x}_1 = \begin{bmatrix} F_1 & \hat{C}_1 \\ \hat{B}_1 & \hat{A}_1 \end{bmatrix} x_1 + \begin{bmatrix} G_1 \\ 0 \end{bmatrix} u, \qquad (A.2.1)$$

$$y = \begin{bmatrix} H_1 & 0 \end{bmatrix} x,$$

where F_1, G_1, and H_1 are given by (5.2.3). Recursive application of extended chained aggregation on the subsystem $(\hat{A}_i, \hat{B}_i, \hat{C}_i)$ yields the dual GHR. One way to program this algorithm is as follows.

```
function[A,B,C,Trans]=dualghr(F,G,H)

% (A, B, C) is the current subsystem on which extended chained aggregation is
%  performed.
A=F;
B=G;
C=H;
% determine the order of the full system.
[m,n]=size(F);
%  initialized the transformation matrix as the identity.
Trans=eye(m);
%  specify a zero element.
eps=1e-08;
i=1;
k=0;
while i < m+1
%  determine order of the current subsystem.
[n,n]=size(A);
```

```
% initialize transformation matrix for the current subsystem.
Tt=eye(n);
% determine v such that Cv = [c1 0 ··· 0].
[u,s,v]=svd(C);
%
% chained aggregation transformation.
%
A=v'*A*v;
B=v'*B;
C=C*v;
Tt=Tt*v;
%
% continued chained aggregation until bi ≠ 0.
%
if abs(B(1)) < eps
    while abs(B(1+k)) < eps
        k=k+1;
        h=A(k,k+1:n);
        [u,s,v]=svd(h);
        T=eye(n);
        T(1+k:n,1+k:n)=v;
        A=T'*A*T;
        B=T'*B;
        C=C*T;
        Tt=Tt*T;
    end;
%
% eliminate Br by pivoting on bi.
%
    if k+1<n
    S=eye(n);
    x=B(k+2:n)/B(1+k);
    S(k+2:n,1+k)=x;
    A=inv(S)*A*S;
    B=inv(S)*B;
    C=C*S;
    Tt=Tt*S;
%
% eliminate Ar by pivoting on c2.
%
  for j=k:-1:1
        Ar=A(j+2:n,1+j);
        P=eye(n);
        x=Ar/A(j,1+j);
        P(j+2:n,j)=x;
        A=inv(P)*A*P;
        B=inv(P)*B;
        C=C*P;
        Tt=Tt*P;
    end;
    end;
%
```

```
% transform block to phase canonical form.
%
    for j=1+k:-1:2
        P=eye(n);
        for row = j:1+k
            P(row,j-1)=A(row,j)/A(j-1,j);
        end;
        A=inv(P)*A*P;
        B=inv(P)*B;
        C=C*P;
        Tt=Tt*P;
    end;
%
% determine scaling transformation to balance the input and output elements
% b_i and c_1.
%
    x=1;
    for j=1:k
        x=x*A(j,j+1);
    end;
    alpha(1+k)=sqrt(abs(B(1+k)/(x*C(1))));
    if B(1+k)<0
        alpha(1+k)=-alpha(1+k);
    end;
    for j=k:-1:1
        alpha(j) = A(j,j+1)*alpha(j+1);
    end;
    N=eye(n);
    for j=1:1+k
        N(j,j)=alpha(j);
    end;
    A=inv(N)*A*N;
    B=inv(N)*B;
    C=C*N;
    Tt=Tt*N;
else
%
%  If b_1 ≠ 0 then jump straight to here.
%  eliminate B_r by pivoting on b_1.
%
    S=eye(n);
  if n > 1
    x=B(2:n)/B(1);
    S(2:n,1)=x;
    A=inv(S)*A*S;
    B=inv(S)*B;
    C=C*S;
    Tt=Tt*S;
  end;
%
% scale b_1 and c_1.
%
```

```
    alpha = sqrt(abs(B(1)/C(1)));
    if B(1)<0
        alpha = -alpha;
    end;
    N=eye(n);
    N(1,1)=alpha;
    A=inv(N)*A*N;
    B=inv(N)*B;
    C=C*N;
    Tt=Tt*N;
end;
%
% update transformation matrix
%
T1=eye(m);
T1(i:m,i:m)=Tt;
Trans = Trans*T1;
%
% determine new subsystem for recursion.
%
C=A(1+k,2+k:n);
B=A(2+k:n,1);
A=A(k+2:n,k+2:n);
i=i+k+1;
k=0;
end;
%
%  calculate Dual GHR form.
%
A=inv(Trans)*F*Trans;
B=inv(Trans)*G;
C=H*Trans;
```

A.3 Generalized Routh Algorithm

Let the system (2.1.1) be represented by its transfer function,

$$G(s) = \frac{a_{20}s^{n-1}+a_{21}s^{n-2}+ \cdots + a_{2,n-1}}{a_{10}s^{n}+a_{11}s^{n-1}+ \cdots +a_{1n}}. \qquad (A.3.1)$$

To determine an r^{th} order system based on the general Cauer continued fraction expansion, we match m Markov parameters and 2r-m time moments. The form is

given by (5.5.5). The coefficients of the expansion, h_i and k_i, can be found from the following procedure.

1. Form the generalized Routh array.

$$\begin{array}{ccccccc} a_{10} & a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1n} \\ a_{20} & a_{21} & a_{22} & \cdots & a_{2,n-1} \end{array} \qquad \text{(A.3.2)}$$

2. Complete m rows of (A.3.2), giving,

$$\begin{array}{ccccccc} a_{10} & a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1n} \\ a_{20} & a_{21} & a_{22} & \cdots & a_{2,n-1} \\ a_{30} & a_{31} & a_{32} & \cdots \\ \vdots \\ a_{2+m,0} & \cdots\cdots & a_{2+m,n-\frac{m}{2}+1} \end{array} \qquad \text{(A.3.3)}$$

where $a_{ij} = \dfrac{(a_{i-1,0})(a_{i-2,j+1}) - (a_{i-2,0})(a_{i-1,j+1})}{a_{i-2,0}}$.

3. From the bottom two rows of (A.3.3), form a second Routh array.

$$\begin{array}{ccccccc} b_{10} & b_{11} & b_{12} & \cdots & b_{1,n-1} & b_{1n} \\ b_{20} & b_{21} & b_{22} & \cdots & b_{2,n-1} \end{array} \qquad \text{(A.3.4)}$$

where $b_{ij} = a_{m+i,n-j-\frac{(m+i)}{2}}$.

4. Complete 2r-m rows of (A.3.4) exactly as in step 2.

5. The coefficients of the continued fraction expansion are,

$$h_i = \frac{a_{i,0}}{a_{i+1,0}}, \text{ and } k_i = \frac{b_{i,0}}{b_{i+1,0}}. \tag{A.3.5}$$

An algorithm for computing these coefficients using MATLAB is

```
function[h,k]=handk(num,den,r,m)
n=length(den);
l=length(num);
for i=1:n
  a(1,i)=den(i);
end;
n=n-1;
for i=1:l
  a(2,i)=num(i);
end;
for i=3:m+2
  h(i-2)=a(i-2,1)/a(i-1,1)
  for j=1:n-(i/2)+2
    a(i,j)=a(i-2,j+1)-h(i-2)*a(i-1,j+1);
  end;
end;
  for j=1:n-(m+1)/2+2
    b(1,j)=a(m+1,n-j+3-(m+1)/2);
  end;
  for j=1:n-(m+2)/2+1
    b(2,j)=a(m+2,n-j+2-(m+2)/2);
  end;
end;
for i=3:2*r-m+2
  k(i-2)=b(i-2,1)/b(i-1,1);
  for j=1:n-(m+i)/2+1
    b(i,j)=b(i-2,j+1)-k(i-2)*b(i-1,j+1);
  end;
end;
```

A.4 Lanczos Algorithm

In this section, we give the Lanczos algorithm as described in [24]. Assume that the first j Lanczos vectors, $\{q_1, q_2, \cdots, q_j\}$ have been found, and the analysis of the j+1 vector will be performed. The resulting vectors all satisfy the condition $q_i^T M q_j = \delta_{ij}$ where δ_{ij} is the Kronecker delta. To calculate q_{j+1}, a preliminary vector $\overline{r}_j$ is first calculated from the previous vector, q_j.

$$\overline{r}_j = K^{-1} M q_j. \tag{A.4.1}$$

Now, in general it may be assumed that this preliminary vector contains components from each of the preceeding vectors. Thus,

$$\overline{r}_j = r_j + \alpha_j q_j + \beta_j q_{j-1} + \gamma_j q_{j-2} + \cdots \tag{A.4.2}$$

Multiplying both sides of (A.4.2) by $q_j^T M$ gives,

$$q_j^T M \overline{r}_j = q_j^T M r_j + \alpha_j q_j^T M q_j + \beta_j q_j^T M q_{j-1} + \gamma_j q_j^T M q_{j-2} + \cdots \tag{A.4.3}$$

Here, the first term on the right hand side vanishes and all terms beyond the second are zero due to M-orthogonality. Thus, (A.4.3) reduces to,

$$\alpha_j = q_j^T M \overline{r}_j. \tag{A.4.4}$$

The amplitude of q_{j-1} contained in $\overline{r}_j$ may be found similarly by multiplying (A.4.2) by $q_{j-1}^T M$. In this case all terms except the third vanish by orthogonality,

and the coefficient of β_j is unity, so

$$\beta_j = \mathrm{q}_{j-1}^{\top} \mathrm{M} \overline{\mathrm{r}}_j. \tag{A.4.5}$$

But, using (A.4.1) to eliminate $\overline{\mathrm{r}}_j$ gives $\beta_j = \mathrm{q}_{j-1}^{\top} \mathrm{M} \mathrm{K}^{-1} \mathrm{M} \mathrm{q}_j$ and applying the transpose of (A.4.1) to the $\mathrm{q}_{j-1}^{\top}$ vector gives,

$$\beta_j = \overline{\mathrm{r}}_{j-1}^{\top} \mathrm{M} \mathrm{q}_j. \tag{A.4.6}$$

Finally, expanding $\overline{\mathrm{r}}_{j-1}$ in terms of r_{j-1} and the preceding Lanczos vectors, as in (A.4.2), (A.4.6) becomes,

$$\beta_j = \mathrm{q}_j^{\top} \mathrm{M} \mathrm{r}_{j-1} + \alpha_{j-1} \mathrm{q}_j^{\top} \mathrm{M} \mathrm{q}_{j-1} + \beta_{j-1} \mathrm{q}_j^{\top} \mathrm{M} \mathrm{q}_{j-2} + \cdots \tag{A.4.7}$$

All terms, except the first, in the right hand side of (A.4.7) vanish. Since

$$\mathrm{q}_j = \frac{1}{\beta_j} \mathrm{r}_{j-1} \tag{A.4.8}$$

From (A.4.7)and (A.4.8), the value of β_j is found by,

$$\beta_j = \sqrt{\mathrm{r}_{j-1}^{\top} \mathrm{M} \mathrm{r}_{j-1}}. \tag{A.4.9}$$

The next Lanczos vector, q_{j+1}, is given by

$$\mathrm{q}_{j+1} = \frac{1}{\beta_{j+1}} \mathrm{r}_j. \tag{A.4.10}$$

This gives a recursive procedure for calculating the Lanczos vectors.

The vita has been removed from the scanned document