On the Robust Stabilization of a Linear

Time-Varying Uncertain System

by

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(ABSTRACT)

In recent years the problem of designing a feedback control law to stabilize a linear, time-varying uncertain system has received considerable attention. However, the problem is usually limited to the case of systems which satisfy the "matching" assumptions. Moreover, the question of the existence of a linear stabilizing control, if a nonlinear stabilizing control exists, is still unanswered. In the present work an attempt is made to design a stabilizing, linear, feedback control law for a specific second-order, linear system which contains time-varying uncertainties into both the state and input matrices and does not satisfy the matching conditions. For specific values of the uncertainty bounds this system is quadratically stabilizable but not quadratically stabilizable via linear control.

In the present work, different techniques have been used to design the stabilizing control law and to maximize the resulted uncertainty bounds. The Elemental Perturubation Bound method

provided maximum relative bounds of magnitude 0.21 and a Matched and Mismatched decomposition of the uncertainties resulted in relative bounds of 0.024. For the case of "slowly" time-varying uncertain parameters, a Lyapunov second-method approach revealed that uncertainties of arbitrary magnitude can be stabilized, provided that the rate of variation of the uncertainties is "sufficiently small", for example, $1.8 \cdot 10^{-4}$ for relative uncertainty bounds equal to 0.1. When the uncertainties are piecewise, constant, periodic functions then using Floquet theory we found that there always exist feedback gains to stabilize the system, but the period and the relative shift of the origin of the uncertainties have significant effect on the stability region. Finally, the Small Gain Theorem provided relative bounds of magnitude 0.707 and the Necessary and Sufficient Conditions for Quadratic Stabilizability revealed a relative bound of magnitude 0.8.

The above results give a measure of the effectiveness and the conservativeness of the methods applied to this specific problem but they were not able to answer the question of the existence of linear stabilizing control for the case where the system is not quadratically stabilizable via linear control.

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K	Feedback Gain Matrix
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s	Uncertain Parameter Vector in Input Matrix
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V	Quadratic Lyapunov Function
x	State Vector

x_o	Initial Condition
γ_1, γ_2	Uncertain Parameters
γ_1^-, γ_2^-	Lower Bounds of the Uncertain Parameters
γ_1^+, γ_2^+	Upper Bounds of the Uncertain Parameters
γ_{1m}, γ_{2m}	Mean Values of the Uncertain Parameters
$\Delta\gamma_1, \Delta\gamma_2$	Perturbations of the Uncertain Parameters
$\overline{\Delta\gamma_1}, \overline{\Delta\gamma_2}$	Bounds of the Perturbations
ϕ	State Transition Matrix

1. INTRODUCTION

Motivation and Objectives

Stability is one of the most important aspects in the analysis and synthesis of dynamical systems. In recent years considerable attention has been focused on the design of a feedback control law to stabilize an uncertain dynamical system. These systems are described by differential equations which contain uncertain parameters, and only the bounds concerning the parameter variations are assumed to be known. The property that the system remain stable for all admissible variations has been termed robust stability.

There are two types of uncertain parameters which can be considered, namely, time-invariant and time-varying. In the case of a linear time-invariant system in the presence of time-invariant uncertain parameters, stabilizability can be determined by testing for the negativity of the real parts of the eigenvalues of the system matrix. The time-varying case is still not completely resolved, because of the lack of necessary and sufficiency criteria for stability in time-varying systems.

Attempts to analyze the robust stability problem has been directed towards both frequency domain and time domain techniques. The frequency domain analysis uses the circle criterion or the singular value decomposition, while the time domain analysis uses the Lyapunov approach.

Literature Review

The literature on robust stabilization of linear time-varying systems primarily deals with systems which satisfy the so-called matching conditions [1-4]. These matching conditions essentially restrict the locations and/or sizes of the uncertain parameters within the system matrices and constitute a sufficient condition for a given uncertain system to be stabilizable. That is, the matching conditions are unduly restrictive and it has been shown [4,5] that there exist many uncertain linear systems which are stabilizable but fail to satisfy the matching conditions. Consequently, recent research efforts has been directed towards developing feedback control laws which will stabilize a general class of linear time-varying systems without any assumption to satisfy the matching conditions. Petersen and Hollot [9] have proposed a Riccati equation technique to produce a linear constant feedback control law to stabilize a linear system with time-varying uncertainties in the state matrix. An extension of this technique to systems with uncertainty in both the state and input matrices was provided by Schmitendorf [10,11]. Yedavalli [12-15] developed an Elemental Perturbation Bound analysis to determine upper bounds on the time-varying perturbations of an asymptotically stable linear system to maintain stability. The above techniques are based on Lyapunov stability theory and thus the resulted bounds are conservative.

The notion of quadratic stabilizability plays an important role in the robust stabilization problem. A quadratically stabilizable uncertain linear system admits a quadratic Lyapunov function which is independent of the uncertainty bounding set. Barmish [18] and Petersen [19] have given necessary and sufficient conditions for quadratic stabilizability. Also, sufficient conditions which are easy to check are given by various authors, for example, Barmish [7], Petersen [20,21] and Zhou and Khargonekar [17,22,23].

Petersen [24] has shown that quadratic stabilizability via nonlinear control does not imply quadratic stabilizability via linear control. Using a second-order linear uncertain system which contains time-varying uncertainties into both the state and input matrices, he demonstrated that for this particular system, a nonlinear control is necessary to achieve stability with a quadratic Lyapunov function. However, it may be possible to achieve stability via linear control, if one is prepared to go beyond the class of quadratic stabilizability. That is, the question of the existence of a linear stabilizing control of a linear uncertain system which is stabilizable by a nonlinear control is still unanswered.

In the present work, our objective will be to examine the robust stabilization problem of the system used by Petersen, applying several different techniques. Moreover, we will try to answer, if possible, the question "Does a nonlinear stabilizing control imply the existence of a linear stabilizing control?"

In Chapter 2 we develop the basic definitions of matching conditions and quadratic stabilizability and we formulate the robust stability problem of the uncertain system we will examine. In Chapters 2 to 8 we present and apply different techniques which provide linear stabilizing feedback control laws. Chapter 9 gives discussion of the results and conclusions.

2. PROBLEM FORMULATION

Basic Assumptions

A linear dynamical system with time-varying uncertainties is described by differential equations of the form:

$$\dot{x}(t) = A(r(t))\ x(t) + B(s(t))\ \ u(t)\ ,\quad t > o \qquad (2.1)$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the control vector, $r(t) \in \Omega \subset R^k$, $s(t) \in \Psi \subset R^l$ are the vectors of the uncertain parameters and $A(\cdot)$, $B(\cdot)$ are nxn and nxm dimensional matrix functions respectively. It is assumed that the matrices $A(\cdot)$ and $B(\cdot)$ depend continuously on their arguments, the bounding sets Ω and Ψ are compact and the uncertainties $r(\cdot)$: $[0,\infty) \rightarrow \Omega$ and $s(\cdot) : [0,\infty) \rightarrow \Psi$ are Lebesgue measurable functions.

The robust stabilization problem is to find a state feedback control law $u(\cdot) = -K\ x(\cdot)$ where K is a constant mxn matrix, such that the closed loop system:

$$\dot{x}(t) = \{A(r(t)) - B(s(t))\ K\}\ x(t) \tag{2.2}$$

is asymptotically stable for all admissible uncertainties $r(t) \in \Omega$ and $s(t) \in \Psi$.

Asymptotic stability will be considered in the sense of Lyapunov. That is, for any $\epsilon > 0$ and time t_o there exist a $\delta = \delta(\epsilon, t_o) > 0$ such that if the initial condition $x(t_o) = x_o$ satisfies $\| x_o \| < \delta$ then $\| x(t) \| < \epsilon$ for all $t > t_o$ and in addition $x(t) \to 0$ as $t \to \infty$.

Quadratic Stabilizability

The system (2.1) is said to be quadratically stabilizable if there exist a continuous feedback control $p(\cdot)$: $R^n \to R^m$ with $p(0) = 0$, an nxn positive-definite symmetric matrix P and a constant $\alpha > 0$ such that the following condition is satisfied: Given any admissible uncertainties $r(\cdot)$ and $s(\cdot)$, the Lyapunov derivative corresponding to the closed loop system with control $u(\cdot) = p(x(\cdot))$ satisfies the inequality:

$$L(x, t) = x^T \{A(r(t))^T P + P\ A(r(t))\}\ x$$

$$+ 2\ x^T\ P\ B(s(t))\ p(x) \leq - \alpha\ \|x\| \tag{2.3}$$

for all pairs $(x,t) \in R^n \times [o,\infty)$

As it is well known [48, 49], if the inequality above holds, the closed loop system

$$\dot{x}(t) = A(r(t))\, x(t) + B(s(t))\, p(x(t))$$

admits a Lyapunov function $V(x) = x^T P x$ having trajectory derivative $L(x,t)$ and the equilibrium point $x = 0$ is uniformly asymptotically stable for any given admissible uncertainties $r(t)$ and $s(t)$.

The system (2.1) is said to be quadratically stabilizable via linear control if it is quadratically stabilizable and, furthermore, the stabilizing control law can be chosen to be in the form $u(\cdot) = p(x(\cdot)) = K\, x(\cdot)$ where K is an mxn real constant matrix.

Matching Conditions

If it is possible to decompose the matrices $A(r)$ and $B(s)$ as follows:

$$A(r) = A_o + B_o\, D(r) \tag{2.4}$$

$$B(s) = B_o + B_o\, E(s) \tag{2.5}$$

for all admissible uncertainties $r(\cdot)$ and $s(\cdot)$, where (A_o, B_o) is a stabilizable pair of constant matrices and $D(\cdot)$ and $E(\cdot)$ are continuous matrix functions with $\| E(s) \| < 1$ for all $s(t) \in \Psi$, then we say that the linear system (2.1) satisfies the matching conditions.

The matching conditions provide sufficient conditions for quadratic stabilizability [1,2].

Statement of Problem

In the present work the problem of robust stabilization of the following second-order, linear, time-varying uncertain system is examined

$$\dot{x}(t) = \begin{bmatrix} \gamma_1(t) & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -\gamma_2(t) \end{bmatrix} u(t) \qquad (2.6)$$

where $x(t) \in R^2$, $u(t) \in R$ and the uncertain parameters $\gamma_1(t) \in R$ and $\gamma_2(t) \in R$ satisfy:

$$\gamma_1^- \leq \gamma_1(t) \leq \gamma_1^+ \qquad (2.7)$$

$$\gamma_2^- \leq \gamma_2(t) \leq \gamma_2^+ \qquad (2.8)$$

for all $t \geq 0$. The bounds γ_1^-, γ_1^+, γ_2^- and γ_2^+ of the uncertainties are assumed to be known and fixed. The importance of this particular uncertain system lies in the fact that because of the locations of the uncertainties in the A and B matrices, the matching conditions are not satisfied.

Given the linearity of this system one might conjecture that if there exist a quadratically stabilizing nonlinear state feedback, then also exist a linear quadratically stabilizing state feedback control law. Petersen has shown that the above conjecture is false. In his famous counterexample [24] he proved that when

$$\gamma_1^- = 1.0 \ , \quad \gamma_1^+ = 12.0 \ , \quad \gamma_2^- = 0.5 \ , \quad \gamma_2^+ = 5.0 \qquad (2.9)$$

the uncertain system (2.6) is quadratically stabilizable via nonlinear control, but not quadratically stabilizable via linear control. However, the question of the existence of a linear stabilizing control for this system is still open if the restriction of quadratic stabilizability is abandoned.

Our effort will be to find a linear feedback control law $u = - K \cdot x$, where $K = [k_1 \ , \ k_2]$, to stabilize the uncertain system (2.6) for given bounds of the uncertainties $\gamma_1(t)$ and $\gamma_2(t)$ and we would like to answer, if possible, the question of linear stabilizability of this system. The different techniques which will be used, will be analysed and compared.

3. ELEMENTAL PERTURBATION BOUND METHOD

Introduction

One of the earliest methods providing explicit bounds on the time-varying perturbations of a linear time invariant system to maintain stability was proposed by Patel and Toda [16]. These bounds are easy to compute numerically but they are based on sufficient conditions for stability and consequently tend to be conservative. Moreover, they assumed that every element of the system matrix may be perturbed independently and they didn't use the structural information about the location of the perturbations into the system to reduce the conservatism.

Improved methods providing less conservative bounds were presented by Yedavalli [12-15]. Yedavalli developed an Elemental Perturbation Bound (EPB) method to analyze robust stability in linear state space models utilizing the structure of the uncertainty. This method assumes that the perturbation model structure is known and also the bounds on the individual elements of the perturbation matrix are known.

Formulation of EPB Method

Consider the following linear dynamic system

$$\dot{x}(t) = A(t)\ x(t) = [A_o + \epsilon(t)]\ x(t), \quad x(0) = x_o \tag{3.1}$$

where x is the n-dimensional state vector, A_o is an nxn nominally asymptotically stable matrix and $\epsilon(t)$ is an nxn time-varying perturbation matrix.

Let

$$\max_{t \in [t_o, \infty)} |\epsilon_{ij}(t)| \hat{=} \epsilon_{ij} \qquad i,j = 1,\ldots,n \tag{3.2}$$

and

$$\max_{ij} \epsilon_{ij} \hat{=} \epsilon \tag{3.3}$$

that is, ϵ is the magnitude of the maximum deviation expected in the entries of A_o. Define the nxm matrix $U = [u_{ij}]$ as

$$u_{ij} = \frac{\epsilon_{ij}}{\epsilon} \qquad i,j = 1,\ldots,n \tag{3.4}$$

Thus $u_{ij} = 0$ if the perturbation in the i,j element of A_o is known to be zero (that is the i,j element of E is zero). Similarly, $u_{ij} = 1$ if the perturbation in the i,j element of A_o is not explicitly known, which corresponds to the worst-case situation. Hence, it is

$$0 \le u_{ij} \le 1 \tag{3.5}$$

The following theorem is due to Yedavalli [12].

Theorem 3.1: The system (3.1) is asymptotically stable if

$$\epsilon < \frac{1}{\sigma_{max}[\,|P|U]_s} \hat{=} \mu_Y \tag{3.6}$$

where P is the solution of the Lyapunov matrix equation

$$A_o^T P + P A_o + 2 I = 0 \tag{3.7}$$

and $|P|$ denotes the matrix formed by taking the absolute value of every element of P and $[\,|P|U]_s$ denotes the symmetric part of the matrix $|P| \cdot U$, i.e.,

$$[\,|P| \cdot U]_s \hat{=} \frac{|P| \cdot U + (|P| \cdot U)^T}{2} \tag{3.8}$$

$\sigma_{max}(\cdot)$ represents the largest singular value of a matrix.

The proof of this theorem is based on the construction of a quadratic Lyapunov function and the inequality properties of the largest singular value of a matrix [17].

Application of EPB Method

Consider the second-order uncertain linear time-varying system (2.6) where the control law is a linear constant full state feedback $u(\cdot) = -[k_1\ k_2]\ x(\cdot)$. The resulting closed loop system can be easily written in the form (3.1) with

$$A_o = \begin{bmatrix} \gamma_{1m} - k_1 & -k_2 \\ \gamma_{2m}\, k_1 & \gamma_{2m}\, k_2 \end{bmatrix} \tag{3.9}$$

$$\epsilon(t) = \begin{bmatrix} \Delta\gamma_1(t) & 0 \\ \Delta\gamma_2(t)\cdot k_1 & \Delta\gamma_2(t) k_2 \end{bmatrix} \tag{3.10}$$

where

$$\gamma_{1m} = \frac{\gamma_1^{+} + \gamma_1^{-}}{2} \qquad \gamma_{2m} = \frac{\gamma_2^{+} + \gamma_2^{-}}{2} \tag{3.11}$$

$$\Delta\gamma_1(t) = \gamma_1(t) - \gamma_{1m} \qquad \Delta\gamma_2(t) = \gamma_2(t) - \gamma_{2m} \tag{3.12}$$

and the following constraints have to be satisfied

$$|\Delta\gamma_1(t)| \leq \gamma_{1m} - \gamma_1^{-} = \gamma_1^{+} - \gamma_{1m} \hat{=} \overline{\Delta\gamma_1} \tag{3.13}$$

$$|\Delta\gamma_2(t)| \le \gamma_{2m} - \gamma_2^- = \gamma_2^+ - \gamma_{2m} \hat{=} \overline{\Delta\gamma_2} \tag{3.14}$$

The necessary and sufficient conditions for the asymptotic stability of the nominal system matrix A_o are the following

$$k_1 - \gamma_{1m} - \gamma_{2m} k_2 > 0 \tag{3.15}$$

and

$$\gamma_{1m} \ \gamma_{2m} k_2 > 0 \tag{3.16}$$

The matrix U which takes into account the structural information about the perturbation matrix has the form:

$$U = \begin{bmatrix} \dfrac{\overline{\Delta\gamma_1}}{\epsilon} & 0 \\ \dfrac{\overline{\Delta\gamma_2}\cdot|k_1|}{\epsilon} & \dfrac{\overline{\Delta\gamma_2}\cdot|k_2|}{\epsilon} \end{bmatrix} \tag{3.16}$$

where

$$\epsilon \ \hat{=} \max \{\overline{\Delta\gamma_1}, \overline{\Delta\gamma_2}\cdot |k_1| , \overline{\Delta\gamma_2} \cdot |k_2|\} \tag{3.17}$$

Our problem is to find the maximum uncertain bounds $\overline{\Delta\gamma_1}$ and $\overline{\Delta\gamma_2}$ that guarantee stability, i.e., that the conditions of Theorem 3.1 are satisfied for some values of the constant feedback control gains k_1 and k_2. To simplify the problem let's assume that

$$\gamma_{1m} = \gamma_{2m} = \gamma_m > 0 \tag{3.18}$$

and

$$\overline{\Delta\gamma}_1 = \overline{\Delta\gamma}_2 = \overline{\Delta\gamma} \tag{3.19}$$

i.e., the uncertain parameters have the same mean value and the same perturbation bounds. Then

$$U = \begin{bmatrix} \frac{1}{k_1} & 0 \\ 1 & \frac{k_2}{k_1} \end{bmatrix} \tag{3.20}$$

and the sufficient condition for stability is given by

$$\overline{\Delta\gamma} < \frac{\mu_y}{k_1} \tag{3.21}$$

where μ_y is given by (3.6). To find the maximum bound that guarantees stability we have to maximize the right hand side of eq. (3.21) over all feedback gains k_1 and k_2 subject to the constraints (3.15) and (3.16). This problem was solved numerically doing a step by step increment on k_1 and k_2 to determine the region of global maximum of μ_y/k_1, and then using a standard maximization algorithm.

Results

The results we obtained are shown in the following figures. Figure 3.1 shows the maximum uncertain bound $\overline{\Delta\gamma}$ provided by EPB method as a function of the mean value of the uncertainties γ_m. Figure 3.2 shows the relative uncertain bound $\overline{\Delta\gamma}/\gamma_m$ as a function of γ_m and Figures 3.3 and 3.4 shows the values of the constant feedback gains k_1 and k_2 that provide the maximum bounds. We can see that the maximum of $\overline{\Delta\gamma}$ is approximately 0.0316 and appears at γ_m equal to 0.7, and also that as γ_m increases, k_1 goes linearly to infinity and k_2 goes to 1.0. The behaviour of the maximum uncertain bound which decreases as γ_m increases is unexpected and reveals the conservative nature of these results. This conservativeness is based on the fact that Theorem (3.1) provides a sufficient condition for stability and the above behavior indicates that the maximum bounds computed using EPB method are much lower than what the system can tolerate without losing stability.

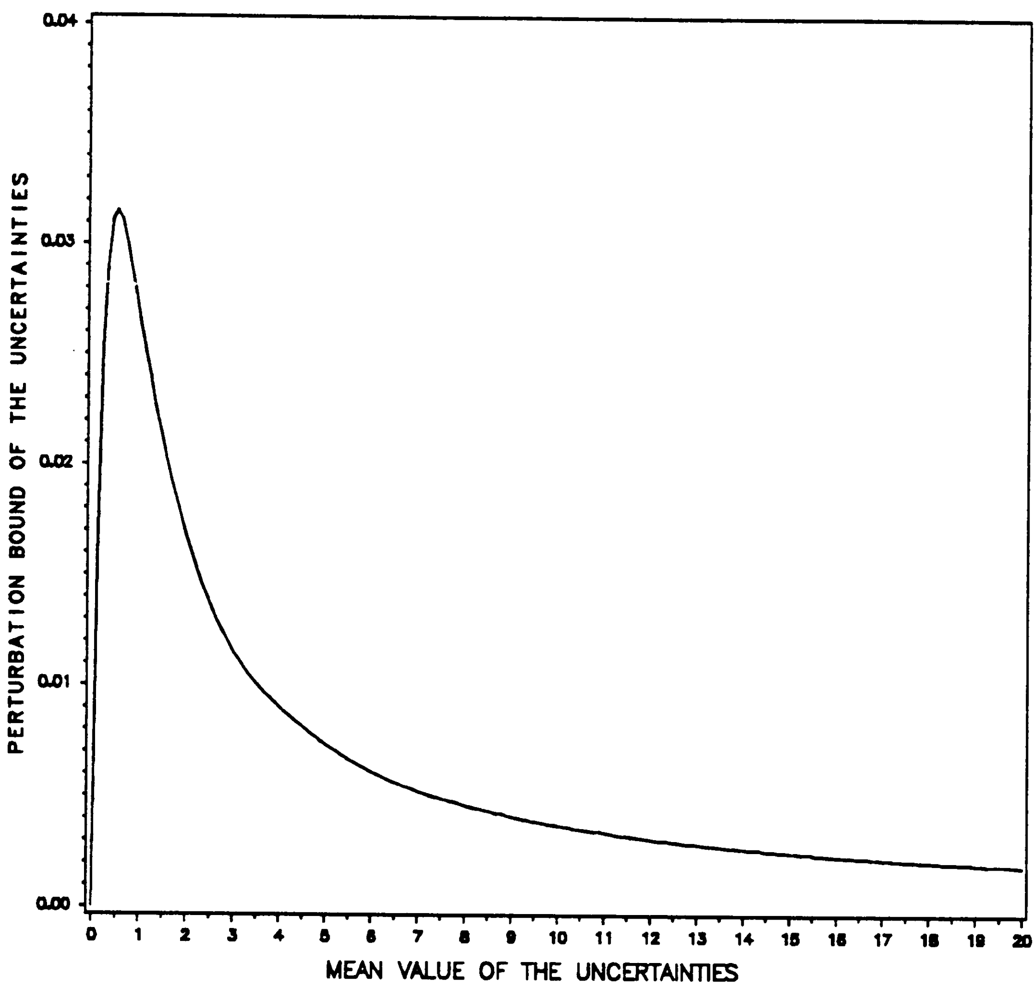

Fig. 3.1: Perturbation bound of the uncertainties

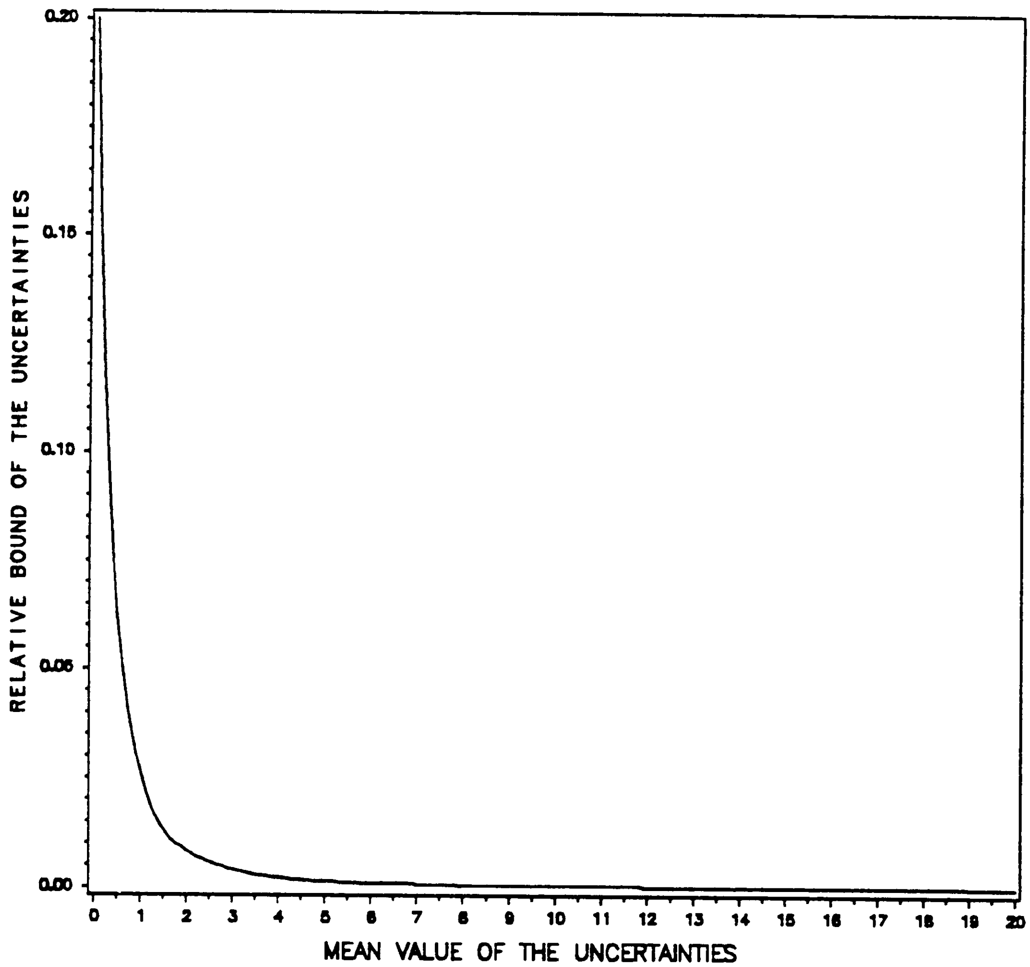

Fig. 3.2: Relative perturbation bound of the uncertainties

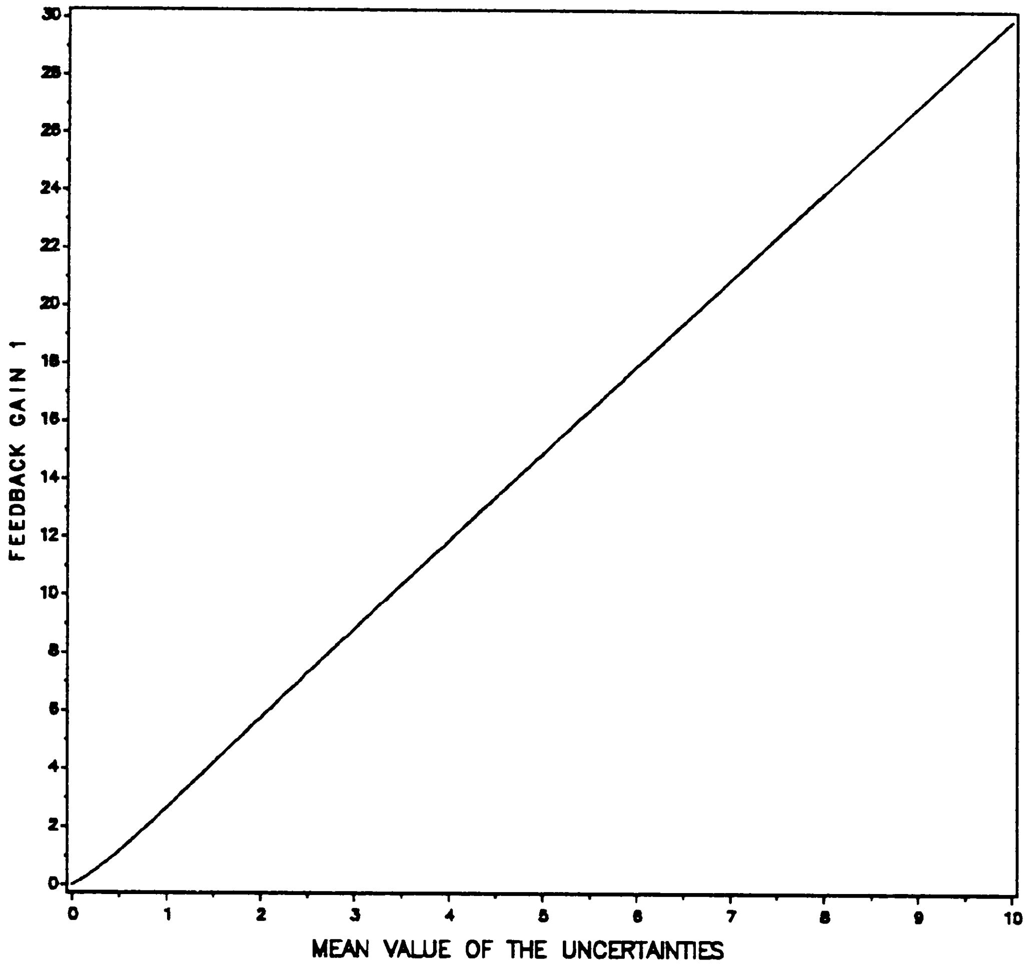

Fig. 3.3: Feedback gain 1

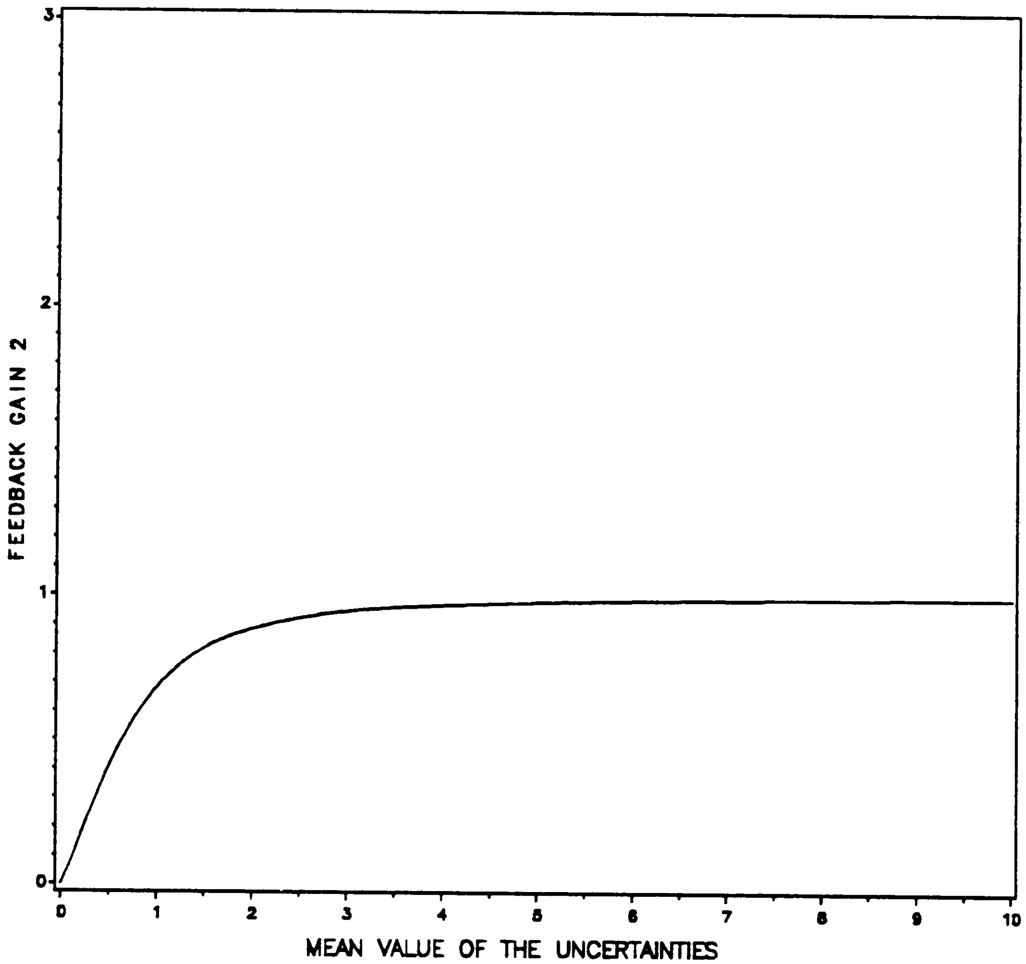

Fig. 3.4: Feedback gain 2

4. SLOWLY TIME-VARYING UNCERTAIN PARAMETERS

Introduction

It is well known that stability of a time-varying linear system

$$\dot{x}(t) = A(t) \cdot x(t) \qquad (4.1)$$

is not ensured by having the eigenvalues of A(t) negative for all values of time t [25,26]. But as one intuitively expects and as Rosenbrock proved [27] the system (4.1) is asymptotically stable if in addition the rate of change of the elements of the system matrix A is sufficiently slow. In other words, "slowly" time-varying parametric disturbances cannot cause instability in a time-varying system if the system matrix is stable for all values of time. The difficulty remains on setting bounds to the permitted rate of variation of A and we will try to develop such bounds here for the particular system given by (2.6).

Stability Theorems for Slowly Varying Parameters

Consider the system (4.1) where $t \rightarrow A(t)$ is an nxn continuously differentiable, matrix-valued function. The following theorem is from [29].

Theorem 4.1: The system (4.1) is asymptotically stable if the eigenvalues of A(t) have negative real parts for all times t and also

$$2 \cdot \| \dot{A}(t) \| \cdot \| P(t) \|^2 < 1 \tag{4.2}$$

for all t where P(t) is the unique solution of the matrix Lyapunov equation

$$A^T(t)\, P(t) + P(t)\, A(t) = -\, I \tag{4.3}$$

$\dot{A}(t)$ denotes the time derivative of the matrix A(t) and $\|\cdot\|$ denotes the matrix norm defined by the absolute sum of all elements. The first part of the theorem requires that for any fixed time the system with the associated constant matrix would be an asymptotically stable system and the second part requires the time derivative of the parameters to be sufficiently small. The proof of the theorem is basically a Lyapunov second-method approach.

The interesting thing about this theorem is that theoretically a uncertain system with perturbations of any magnitude can be stabilized provided that the rate of change of the perturbations is small enough. A major drawback is that the theorem handles the perturbations in an "unstructured" form, i.e., it doesn't use the information about the specific locations and sizes of the uncertainties entered in the system matrix. Therefore the results are highly conservative and as we will see later the bounds on the derivatives of the uncertainties are extremely tight.

Application of Theorem for Slowly Varying Parameters

The system matrix of the second-order uncertain system (2.6) under the control law $u(\cdot) = - [k_1 \; k_2] \, x$ is:

$$A(t) = \begin{bmatrix} \gamma_1(t) - k_1 & - k_2 \\ \gamma_2(t) \cdot k_1 & \gamma_2(t) \cdot k_2 \end{bmatrix} \tag{4.4}$$

Let's assume that the uncertain parameters satisfy

$$0 < \gamma_1^- \leq \gamma_1(t) \leq \gamma_1^+ \tag{4.5}$$

$$0 < \gamma_2^- \leq \gamma_2(t) \leq \gamma_2^+ \tag{4.6}$$

and the derivatives of the uncertain parameters are such that

$$| \; \dot{\gamma}_1(t) \; | \leq d_1 \tag{4.7}$$

and

$$| \; \dot{\gamma}_2(t) \; | \leq d_2 \tag{4.8}$$

Therefore we have

$$\dot{A}(t) = \begin{bmatrix} \dot{\gamma}_1(t) & 0 \\ \dot{\gamma}_2(t)k_1 & \dot{\gamma}_2(t)\ k_2 \end{bmatrix} \tag{4.9}$$

and

$$||\dot{A}(t)|| = |\dot{\gamma}_1(t)| + |\dot{\gamma}_2(t)k_1| + |\dot{\gamma}_2(t)k_2| \tag{4.10}$$

So

$$||\dot{A}(t)|| \leq d_1 + d_2(k_1 + k_2) \tag{4.11}$$

Solving the Lyapunov equation (4.2) and using the inequalities (4.5) and (4.6) we have that

$$||P(t)|| \leq \frac{(k_1^2+k_2^2)(1+\gamma_2^+)^2 - 2k_1\gamma_1^-(1+\gamma_2^-)+\gamma_1^+(\gamma_1^+ + 2k_2\gamma_2^+)}{2\gamma_1^-\gamma_2^-k_2(k_1 - \gamma_1^+ - \gamma_2^+ k_2)} \tag{4.12}$$

Therefore if

$$[d_1+d_2(k_1+k_2)]\ \frac{\{(k_1^2+k_2^2)\cdot(1+\gamma_2^+)^2 - 2k_1\gamma_1^-(1+\gamma_2^-)+\gamma_1^+(\gamma_1^+ + 2k_2\gamma_2^+)\}^2}{2\ \gamma_1^{-2}\ \gamma_2^{-2}\ k_2^2\ (k_1 - \gamma_1^+ - \gamma_2^+k_2)^2} < 1 \tag{4.13}$$

and in addition $A(t)$ has eigenvalues with negative real parts for all values of t, i.e., if

$$k_1 - \gamma_1^+ - \gamma_2^+\ k_2 > 0 \tag{4.14}$$

and

$$k_2 > 0 \tag{4.15}$$

then the conditions of Theorem (4.1) are satisfied and the system is asymptotically stable.

To simplify the expressions let's consider the case where the uncertain parameters have the same bounds, i.e.,

$$\gamma_1^- = \gamma_2^- = \gamma^- \tag{4.16}$$

$$\gamma_1^- = \gamma_2^- = \gamma^- \tag{4.17}$$

and

$$d_1 = d_2 = d \tag{4.18}$$

Then we can see that the stability of the system is guaranteed for uncertain perturbation of any magnitude provided that the rate of variation of the uncertainties satisfy

$$d < \frac{2\gamma^{-4} k_2^2 (k_1 - \gamma^+ - \gamma^+ k_2)^2}{(1+k_1+k_2)[(k_1^2+k_2^2)(1+\gamma^+)^2 - 2k_1\gamma^-(1+\gamma^-)+\gamma^{+2}(1+2k_2)]^2} \tag{4.19}$$

To find the largest allowed bound in the derivatives of the uncertain parameters we have to maximize the right hand side of (4.19) over all feedback gains k_1 and k_2 subject to the constraints (4.14) and (4.15). The maximization procedure was performed numerically varying k_1 and k_2 to determine the region of global maximum and then using a standard maximization algorithm.

Results

Let's define, as before, the mean value of the uncertainties

$$\gamma_m = \frac{\gamma^+ + \gamma^-}{2} \tag{4.20}$$

and the maximum deviation from the mean value

$$\overline{\Delta\gamma} = \frac{\gamma^+ - \gamma^-}{2} \tag{4.21}$$

In the following figures 4.1 to 4.9 we have plotted the maximum bound of the derivatives of the uncertainties d as a function of γ_m for a specific relative bound $\overline{\Delta\gamma}/\gamma_m$ and also the feedback gains k_1 and k_2 where this maximum occurs. We can see that the maximum value of d occurs at γ_m near the value of 0.6 and this maximum is approximately $1.8 \cdot 10^{-4}$ for $\overline{\Delta\gamma}/\gamma_m$ equal to 0.1, $7.5 \cdot 10^{-5}$ for $\overline{\Delta\gamma}/\gamma_m$ equal to 0.2 and $4 \cdot 10^{-6}$ for $\overline{\Delta\gamma}/\gamma_m$ equal to 0.5. As γ_m increases, k_1 goes linearly to infinity and k_2 tends asymptotically to a value near 1.9. As we expected this maximum bound on the derivative is decreasing as $\overline{\Delta\gamma}/\gamma_m$ increases but generally is extremely small and consequently can have very limited practical interest.

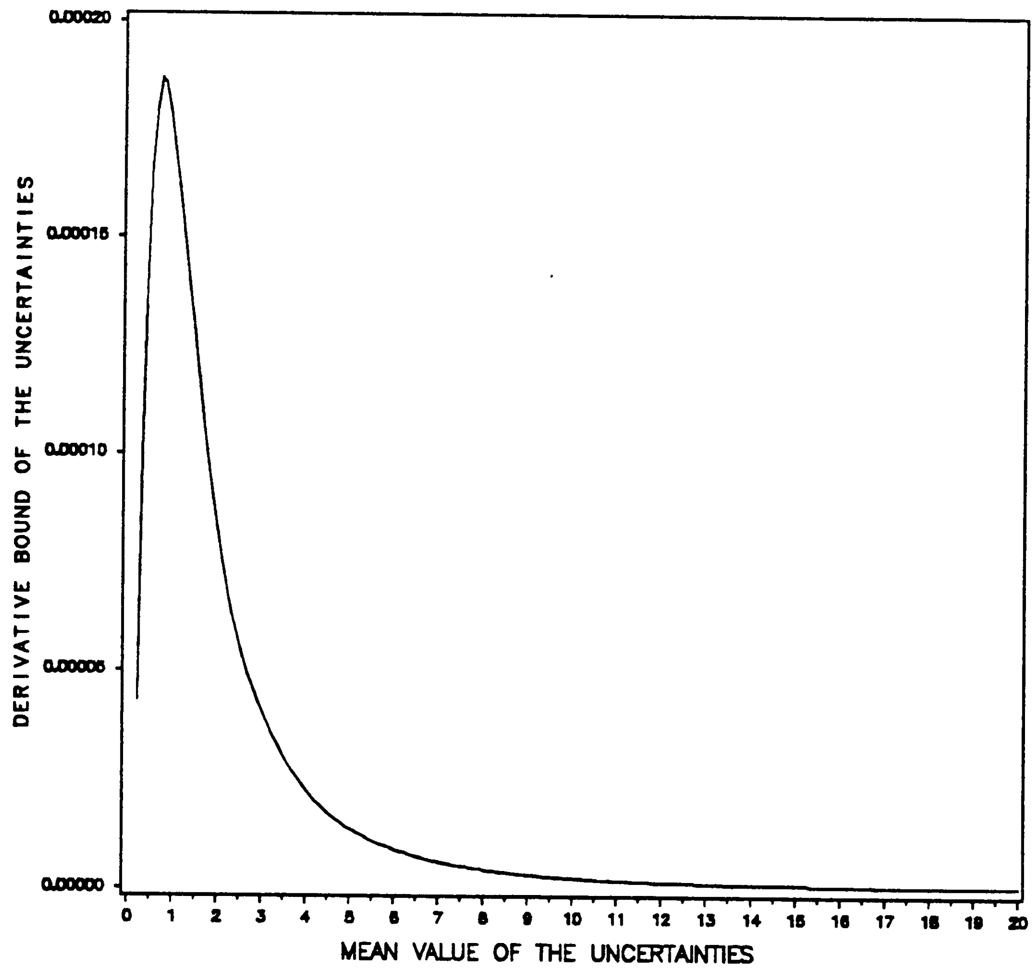

Fig. 4.1: Derivative bound of the uncertainties for $\overline{\Delta\gamma}/\gamma_m = 0.1$

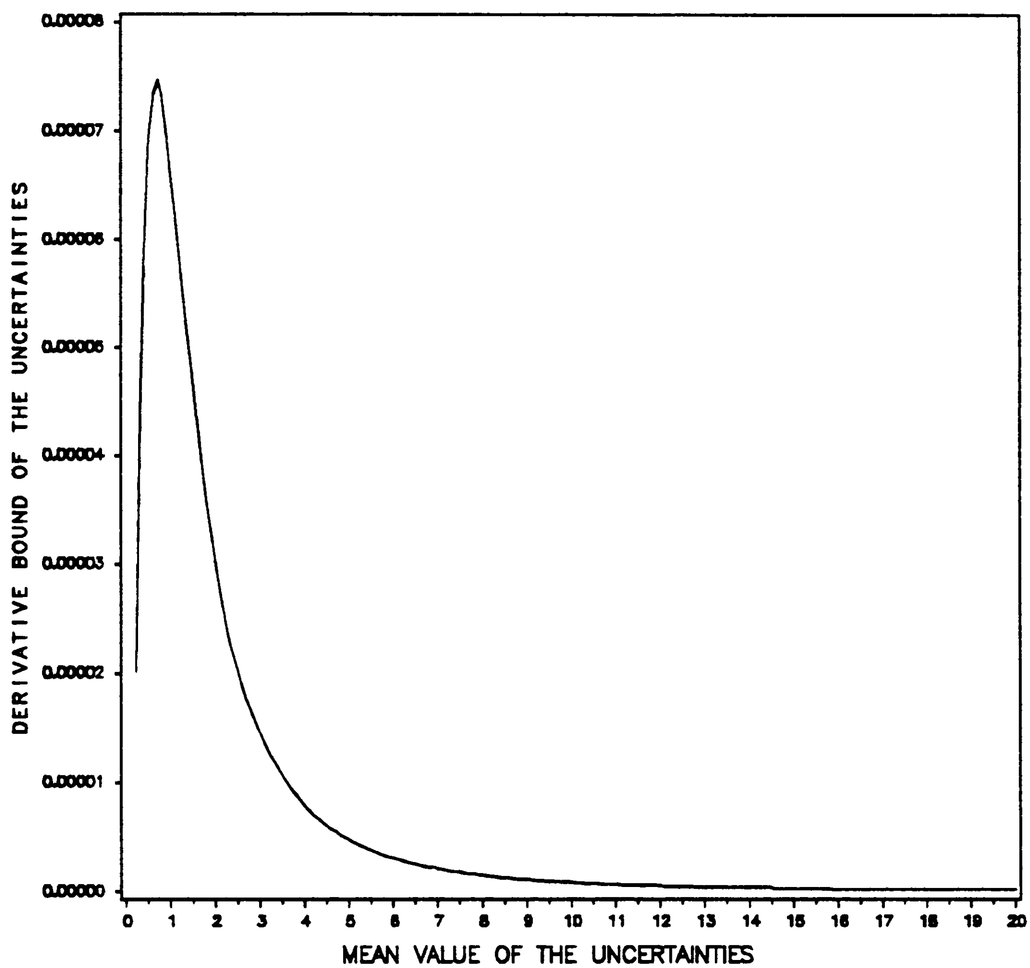

Fig. 4.2: Derivative bound of the uncertainties for $\overline{\Delta\gamma}/\gamma_m = 0.2$

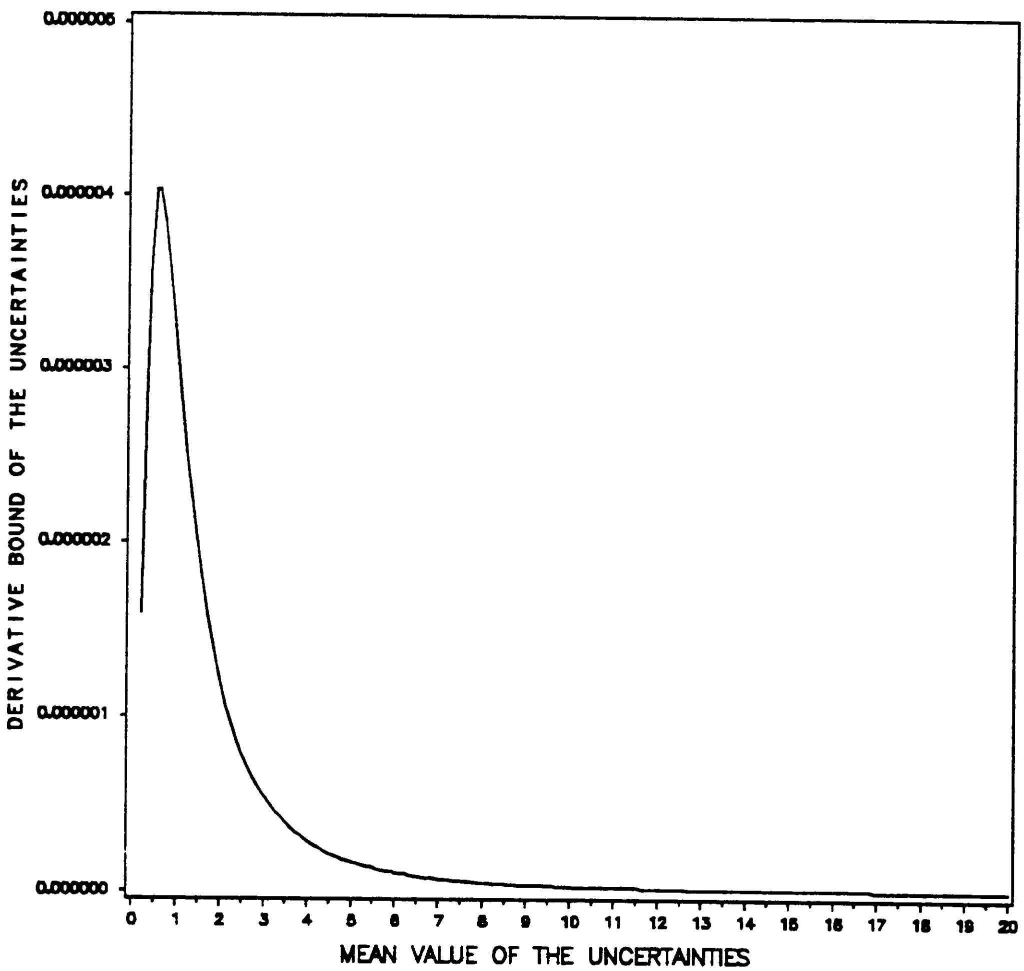

Fig. 4.3: Derivative bound of the uncertainties for $\overline{\Delta\gamma}/\gamma_m = 0.5$

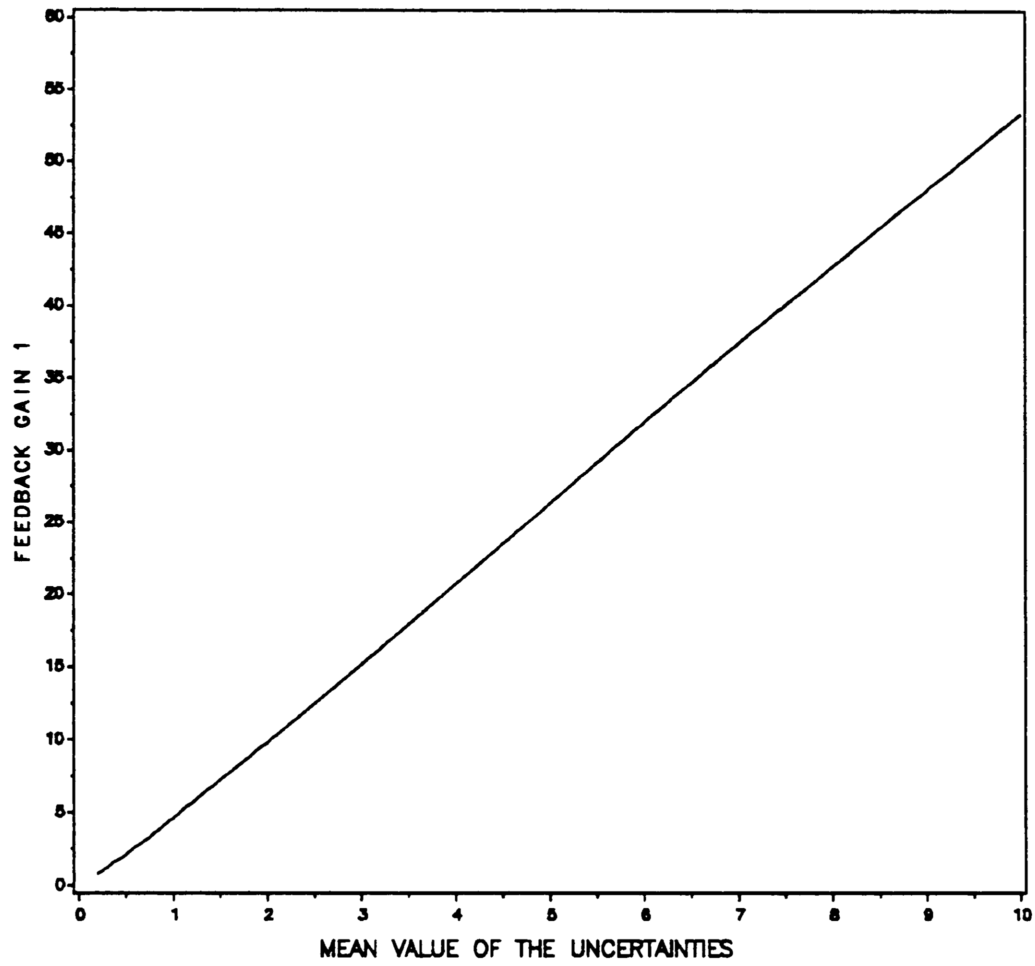

Fig. 4.4: Feedback Gain 1 for $\overline{\Delta\gamma}/\gamma_m = 0.1$

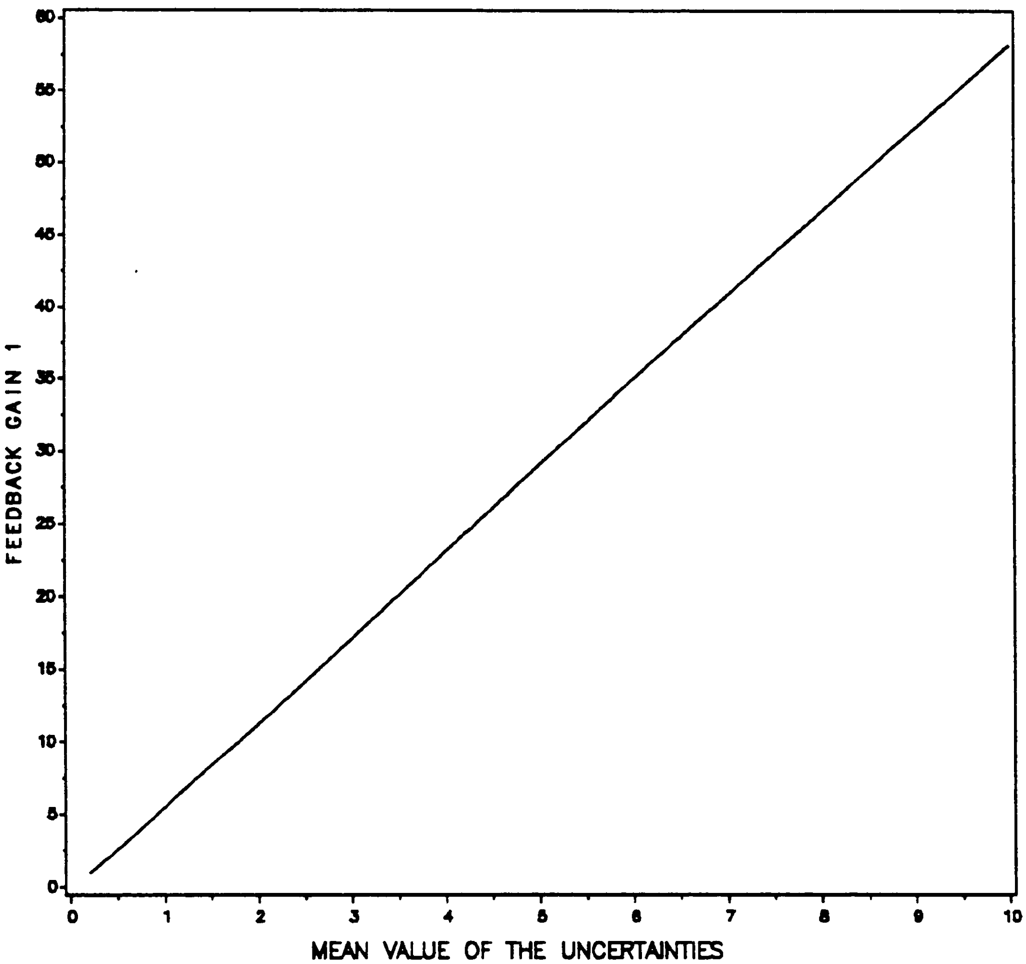

Fig. 4.5: Feedback Gain 1 for $\overline{\Delta\gamma}/\gamma_m = 0.2$

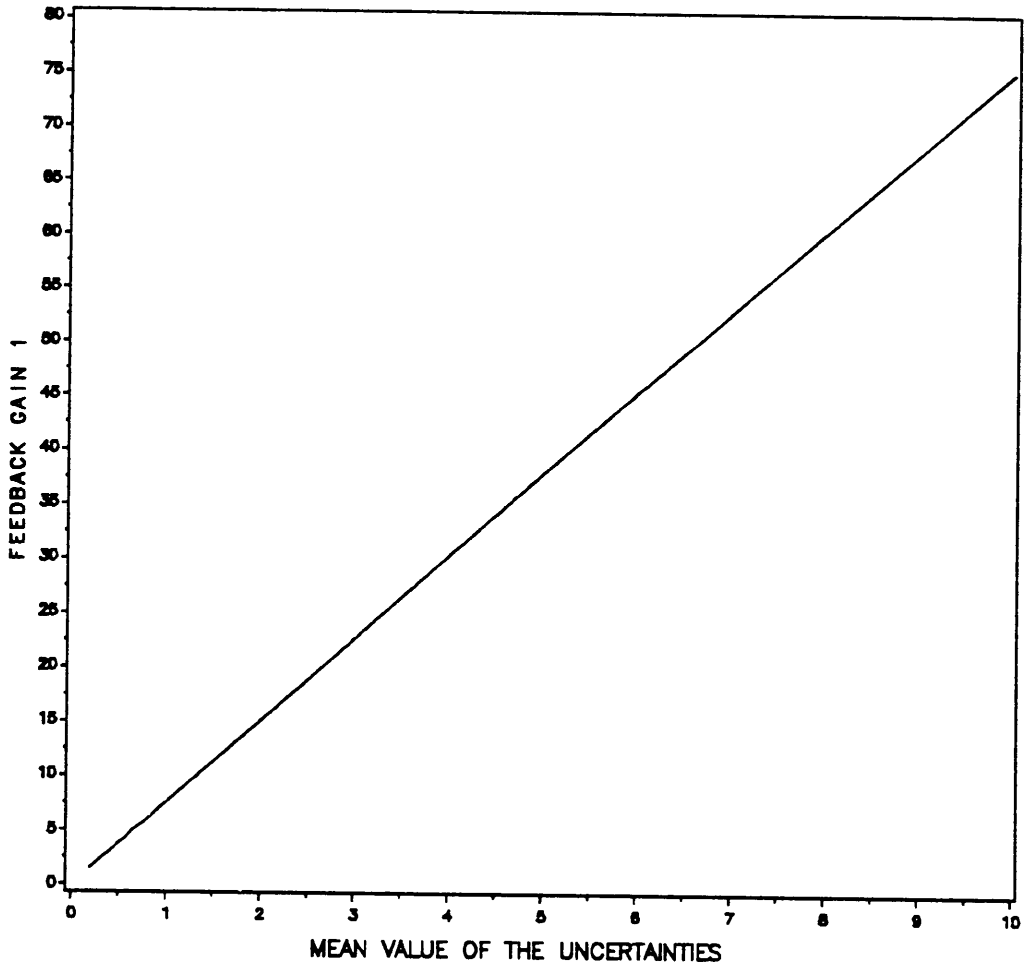

Fig. 4.6: Feedback Gain 1 for $\overline{\Delta\gamma}/\gamma_m = 0.5$

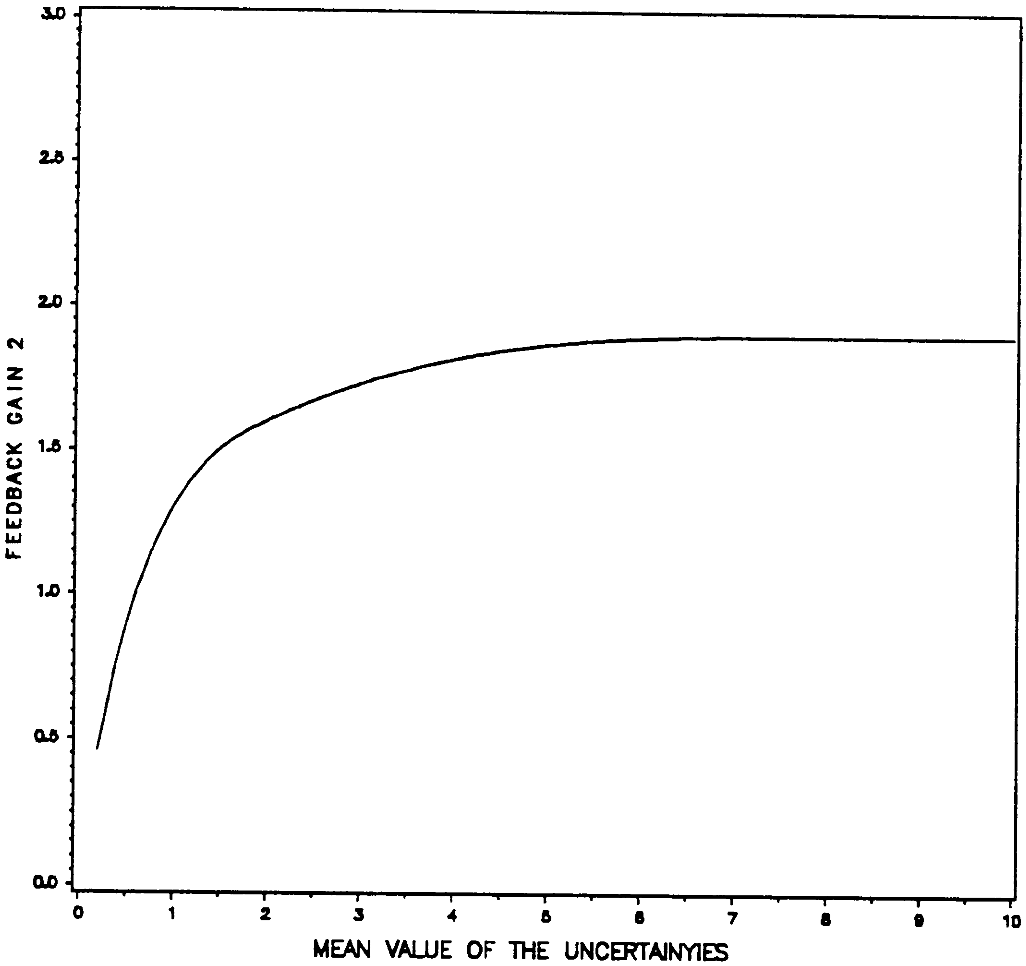

Fig. 4.7: Feedback Gain 2 for $\overline{\Delta\gamma}/\gamma_m = 0.1$

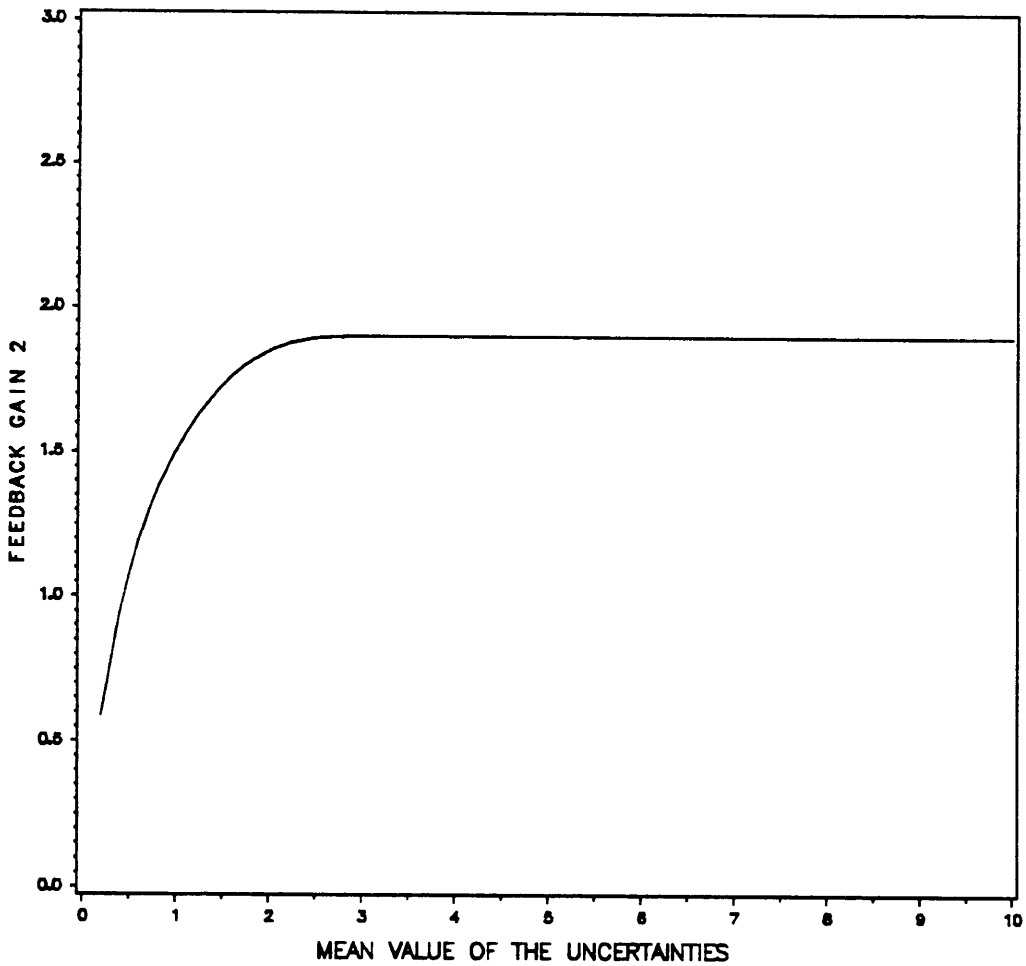

Fig. 4.8: Feedback Gain 2 for $\overline{\Delta\gamma}/\gamma_m = 0.2$

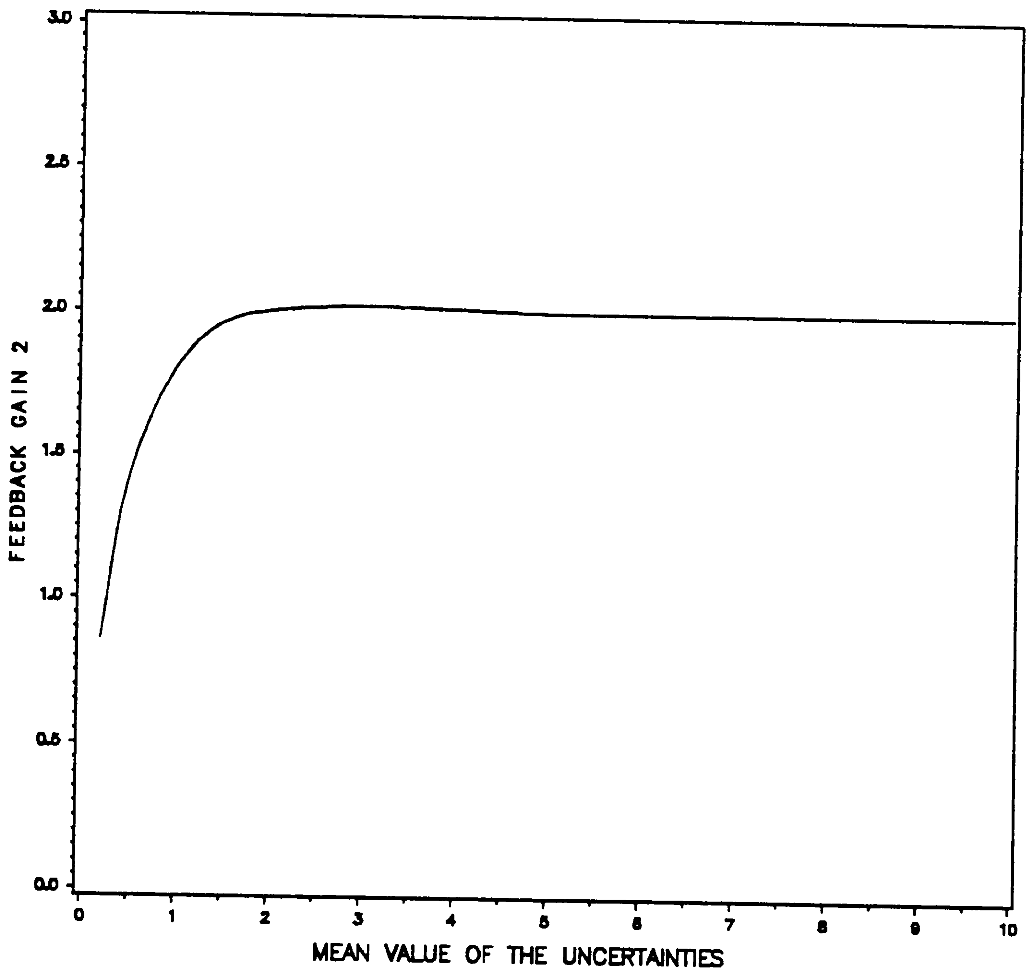

Fig. 4.9: Feedback Gain 2 for $\overline{\Delta\gamma}/\gamma_m$ = 0.5

5. MISMATCHED UNCERTAIN SYSTEMS

Introduction

In recent years feedback stabilizing controllers have been proposed that guarantee stability for uncertain systems satisfying the matching conditions (2.4) and (2.5). Under the matching condition assumption, which impose restrictions regarding the structure and the location of the uncertainty, arbitrary large uncertainties can be tolerated as long as no control bound is prescribed. To study the stabilization problem of systems which does not satisfy the matching conditions Barmish and Leitmann [21] introduced a certain decomposition of the uncertainty into two parts - the matched part which satisfy the matching conditions and a mismatched part. A certain measure of mismatch M can be defined such that a stabilizing control exists as long as M does not exceed some critical measure of mismatch M*. We shall use a new notion, the ultimate boundedness which will define the desired system behaviour.

Ultimate Boundedness Theorem

Consider the uncertain dynamical system

$$\dot{x}(t) = [A + \Delta A(r(t)]\ x(t) + [B + \Delta B(s(t)]u(t) \qquad (5.1)$$

where A and B are the nominal system matrices and $\Delta A(r)$ and $\Delta B(s)$ are matrices which represent the system uncertainty and the input uncertainty respectively. Also we impose the additional assumption that the pair (A,B) is stabilizable.

We now introduce the following definitions. A solution $x: [t_o,\infty) \rightarrow R^n$, $x(t_o) = x_o$ of (5.1) is said to be uniformly bounded if given any $s \in (0,\infty)$ there exist a $d(s) < \infty$ such that $||x_o|| \leq s$ implies that $||x(t)|| \leq d(s)$ for all $t \in [t_o,\infty)$. A solution $x:[t_o,\infty) \rightarrow R^n$, $x(t_o) = x_o$ of (5.1) is said to be uniformly ultimate bounded if given any $\bar{s} \geq \underline{s}$ and any $s \in (0,\infty)$, there exist a $T(\bar{s},s) \rightarrow [0,\infty)$ such that $||x_o|| \leq s$ implies $||x(t)|| \leq \bar{s}$ for all $t \geq t_o + T(\bar{s},s)$. Note that the notion of ultimate boundedness is weaker than that of asymptotic stability and guarantees that the states eventually end up and remain within some prescribed region.

The uncertain linear system (5.1) is said to be decomposable if there exist continuous matrix functions $\Delta A_m(\cdot)$, $\Delta A(\cdot)$, $\Delta B_m(\cdot)$ and $\Delta B(\cdot)$ such that the matrix functions $\Delta A(r)$ and $\Delta B(r)$ can be decomposed as:

$$\Delta A(r) = \Delta A_m(r) + \Delta\tilde{A}(r) \tag{5.2}$$

$$\Delta B(s) = \Delta B_m(s) + \Delta\tilde{B}(s) \tag{5.3}$$

for all uncertainties $r(t) \in \Omega$ and $s(t) \in \Psi$, and in addition the reduced dynamical system

$$\dot{x}_m(t) = [A + \Delta A_m(r(t))]x_m(t) + [B + \Delta B_m(s(t))]u_m(t) \tag{5.4}$$

admits a feedback control

$$u_m = K_m x_m + p_m(x_m) \tag{5.5}$$

(where K_m is a constant mxn matrix and $p(\cdot) : R^n \to R^m$) which renders every solution of (5.4) uniformly bounded. The matrix functions $\Delta A_m(r)$ and $\Delta B_m(s)$ are the matched portion of the uncertainty which can be controlled by existing techniques and $\Delta A(r)$ and $\Delta B(s)$ can be viewed as residual uncertainty. Obviously the decomposition above is not unique and a reasonable possibility is to choose $\Delta A_m(\cdot)$ and $\Delta B_m(\cdot)$ to satisfy the matching conditions. Therefore, in this case, there are matrix functions D(r) and E(s) such that

$$\Delta A_m(r) = B \cdot D(r) \tag{5.6}$$

$$\Delta B_m(s) = B \cdot E(s) \tag{5.7}$$

where E(s) is chosen so that

$$||E(s)|| < 1 \tag{5.8}$$

for all $s(t) \in \Psi$.

Consider the linear time-invariant control law

$$p(x) = - K x - \gamma_o B^T P x \tag{5.9}$$

where K is such that $\bar{A} = A - BK$ is asymptotically stable, P is the solution of the Lyapunov equation

$$P \bar{A} + \bar{A}^T P + Q = 0 \tag{5.10}$$

for a given constant, positive-definite symmetric nxn matrix Q and γ_o is a constant such that

$$\gamma_o \geq \frac{1}{2(1 - \rho_E)} \cdot \frac{(\rho_D + \rho_{EK})^2}{C_1 \lambda_{min}(Q)} \tag{5.11}$$

Here

$$\rho_E \hat{=} \max_s || E(s) || \tag{5.12}$$

$$\rho_D \hat{=} \max_r || D(r) || \tag{5.13}$$

$$\rho_{EK} \hat{=} \max_r || E(s) \cdot K || \tag{5.14}$$

and C_1 must be chosen by the designer such that $0 \leq C_1 < 0$ and $C_1 \neq 0$ whenever $\rho_D + \rho_{EK} \neq 0$. Define the mismatch threshold M* as

$$M^{*} \mathrel{\hat{=}} (1-C_1) \frac{\lambda_{min}(Q)}{2 \cdot \lambda_{max}(P)} \tag{5.15}$$

and the measure of mismatch $\tilde{M}$ as

$$\tilde{M} = \max_{r} ||\Delta\tilde{A}(r)|| + \max_{s} ||\Delta\tilde{B}(s)K|| + \gamma_o ||B^T P|| \max ||\Delta\tilde{B}(s)|| \tag{5.16}$$

The following theorem was proved by Chen and and Leitmann [31].

Theorem 5.1: If $\tilde{M} < M^*$ then the feedback control law (5.9) ensures uniform ultimate boundedness of every solution $x(\cdot)$ of the uncertain linear system (5.1).

The problem of maximizing M^* with respect to the choice of Q was solved by Patel and Toda [16]; namely, $Q = I$ in (5.10) maximizes the ratio $\lambda_{min}(Q)/\lambda_{max}(P)$ for any given $\bar{A}$.

Application of Theorem for Mismatched Uncertain Systems

Consider the second-order uncertain system defined by (2.5). Taking

$$\Delta A(t) = \begin{bmatrix} \Delta\gamma_1(t) & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Delta B(t) = \begin{bmatrix} 0 \\ -\Delta\gamma_2(t) \end{bmatrix} \tag{5.16}$$

where we have introduced

$$\gamma_1(t) = \gamma_{1m} + \Delta\gamma_1(t) \ , \quad \gamma_{1m} = \frac{\gamma_1^+ + \gamma_1^-}{2} \tag{5.17}$$

$$\gamma_2(t) = \gamma_{2m} + \Delta\gamma_2(t) \ , \quad \gamma_{2m} = \frac{\gamma_2^+ + \gamma_2^-}{2} \tag{5.18}$$

we can carry out the decomposition suggested in (5.6) - (5.7) and obtain

$$D(t) = [0 \quad 0] \quad , \quad E(t) = \frac{\Delta\gamma_2(t)}{\gamma_{2m}} \tag{5.19}$$

$$\Delta\tilde{A}(t) = \begin{bmatrix} \Delta\gamma_1(t) & 0 \\ 0 & 0 \end{bmatrix} \quad \Delta\tilde{B}(t) = \begin{bmatrix} \dfrac{\Delta\gamma_2(t)}{\gamma_{2m}} \\ 0 \end{bmatrix} \tag{5.20}$$

The asymptotic stability of the matrix $\bar{A} = A - BK$ is equivalent to

$$k_1 - \gamma_{1m} - \gamma_{2m} \cdot k_2 > 0 \tag{5.21}$$

$$k_2 > 0 \tag{5.22}$$

and we can easily find that the quantities (5.12) - (5.14) are

$$\rho_E = \frac{\overline{\Delta\gamma_2}}{\gamma_{2m}} \tag{5.23}$$

$$\rho_D = 0 \tag{5.24}$$

$$\rho_{EK} = \frac{\overline{\Delta\gamma_2}}{\gamma_{2m}} \ (k_1^2 + k_2^2)^{1/2} \tag{5.25}$$

Therefore the constant γ_o defined by (5.11) has to satisfy

$$\gamma_o \geq \frac{1}{2\left(1 - \frac{\overline{\Delta\gamma_2}}{\gamma_{2m}}\right) C_1} \cdot \frac{\overline{\Delta\gamma_2}^2}{\gamma_{2m}^2} \cdot (k_1^2 + k_2^2) \tag{5.26}$$

The mismatch threshold M* and the measure of mismatch $\tilde{M}$ are given by the expressions

$$M^* = (1 - C_1) \cdot \frac{1}{2 \cdot \lambda_{max}(P)} \tag{5.27}$$

and

$$\tilde{M} = \overline{\Delta\gamma_1} + \frac{\overline{\Delta\gamma_2}}{\gamma_{2m}} (k_1 + k_2)^{1/2}$$

$$+ \gamma_o \cdot \frac{\overline{\Delta\gamma_2}}{\gamma_{2m}} \sqrt{(P_{11} - \gamma_{2m} P_{12})^2 + (P_{12} - \gamma_{2m} P_{22})^2} \tag{5.28}$$

where $P = [P_{ij}]$ is the solution of the Lyapunov equation (5.10) for $Q = I$.

Our problem is to maximize the uncertain bounds $\overline{\Delta\gamma_1}$ and $\overline{\Delta\gamma_2}$ such that $\tilde{M} < M^*$ and (5.21) and (5.22) are satisfied. We solved this problem numerically for the simplified case where

$$\gamma_{1m} = \gamma_{2m} = \gamma_m \tag{5.29}$$

and

$$\overline{\Delta\gamma_1} = \overline{\Delta\gamma_2} = \overline{\Delta\gamma} \tag{5.30}$$

The method we used to solve it is an interval division method where we specify an initial interval of $\overline{\Delta\gamma}$ where $\tilde{M} < M^*$. Then we continuously divide the interval by two and keeping the part where $\tilde{M} < M^*$ till the desired accuracy of $\overline{\Delta\gamma}$ is achieved. The procedure is repeated for different values of C_1 such that the maximum bound for the specific mean value of the uncertainty γ_m is found. Then, the stabilizing control law can be found using (5.9).

Results

In Figure 5.1 is plotted the bound of the uncertainties $\overline{\Delta\gamma}$ as a function of the mean value of the uncertainties γ_m. In Figure 5.2 is plotted the relative bound $\overline{\Delta\gamma}/\gamma_m$ as a function of γ_m. The maximum value of $\overline{\Delta\gamma}$ is approximately $1.5 \cdot 10^{-2}$ and occurs at γ_m near the value of 1.0. The feedback gains k_1 and k_2 are plotted in Figures 5.3 and 5.4 and they have the same behaviour as in the previous methods, i.e., k_1 is increasing almost linearly as γ_m increases and k_2 tends to a value near 1.0.

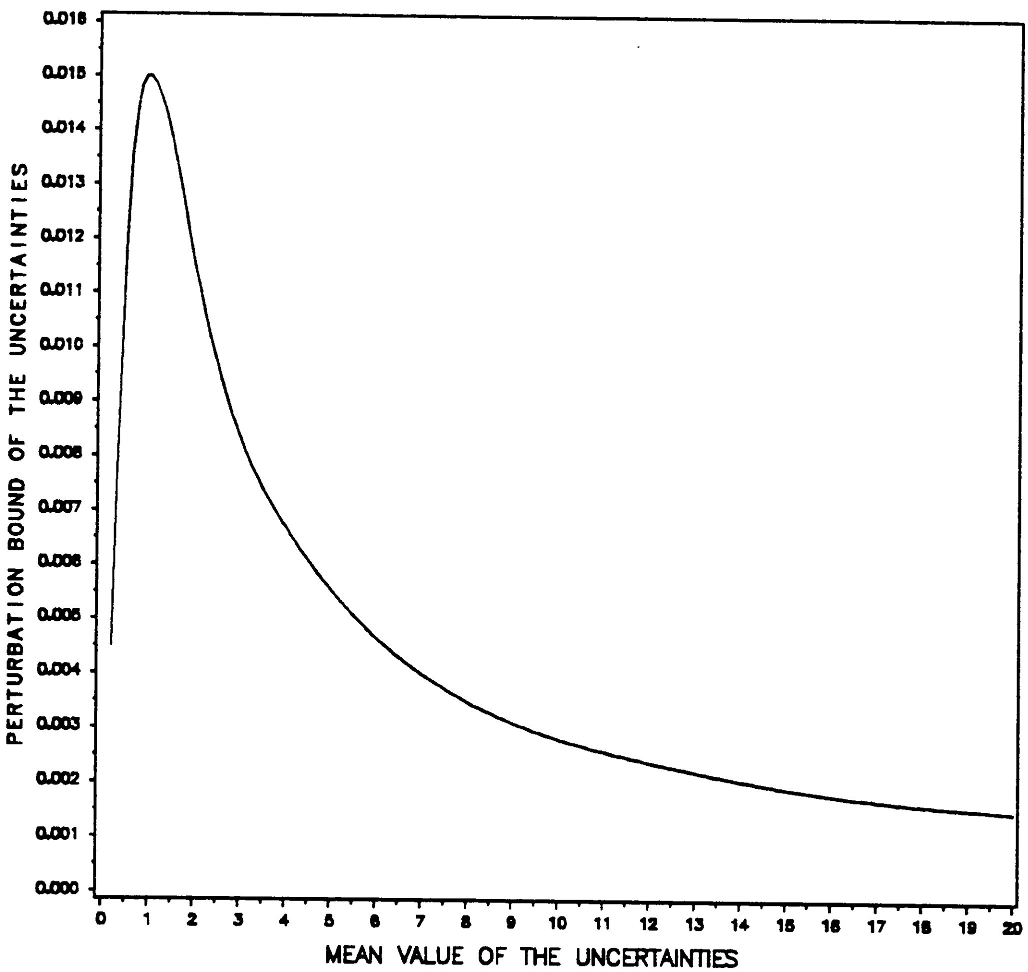

Fig. 5.1: Perturbation bound of the uncertainties

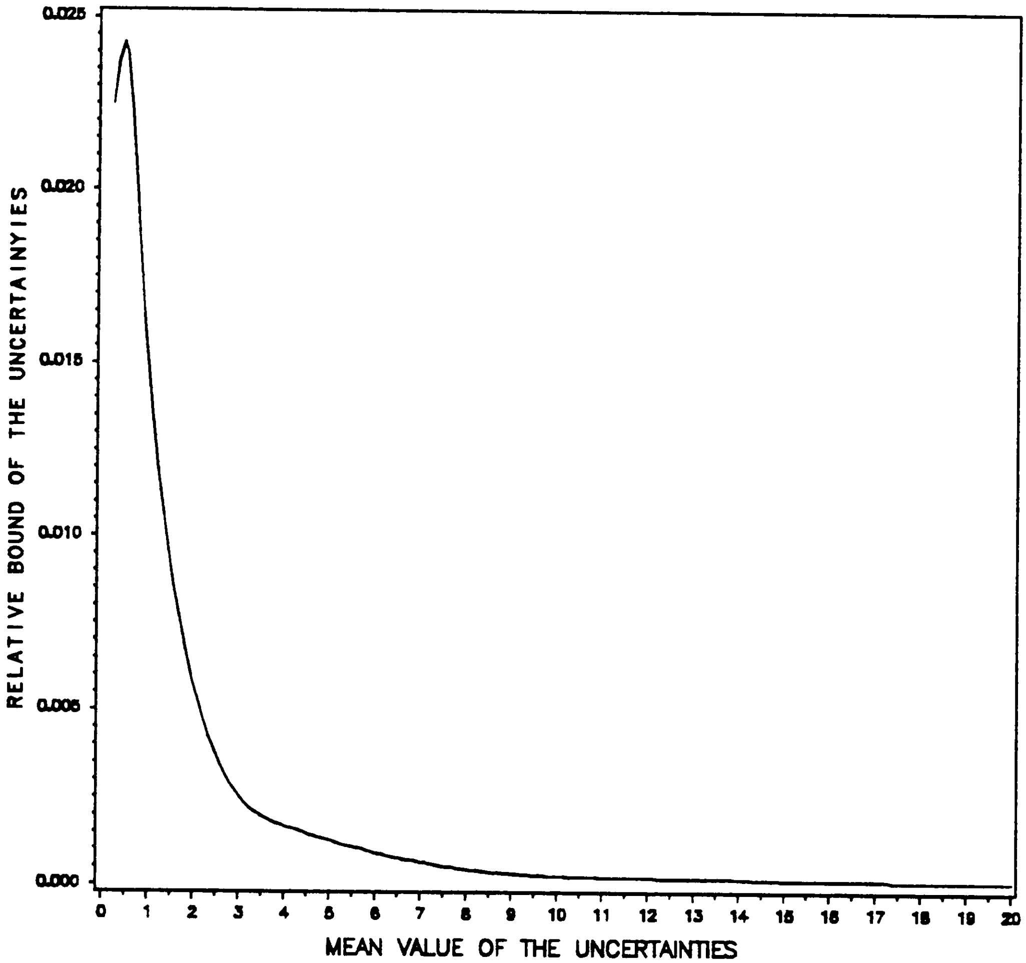

Fig. 5.2: Relative perturbation bound of the uncertainties

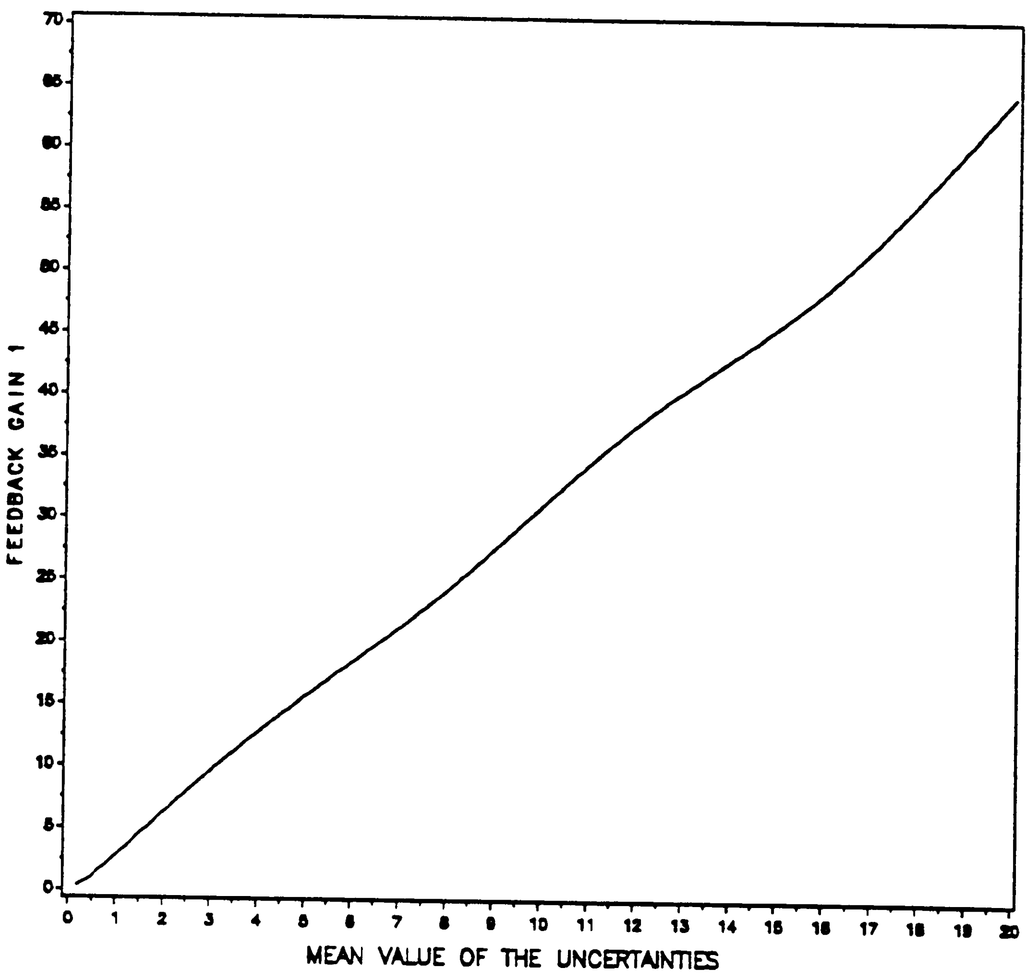

Fig. 5.3: Feedback gain 1

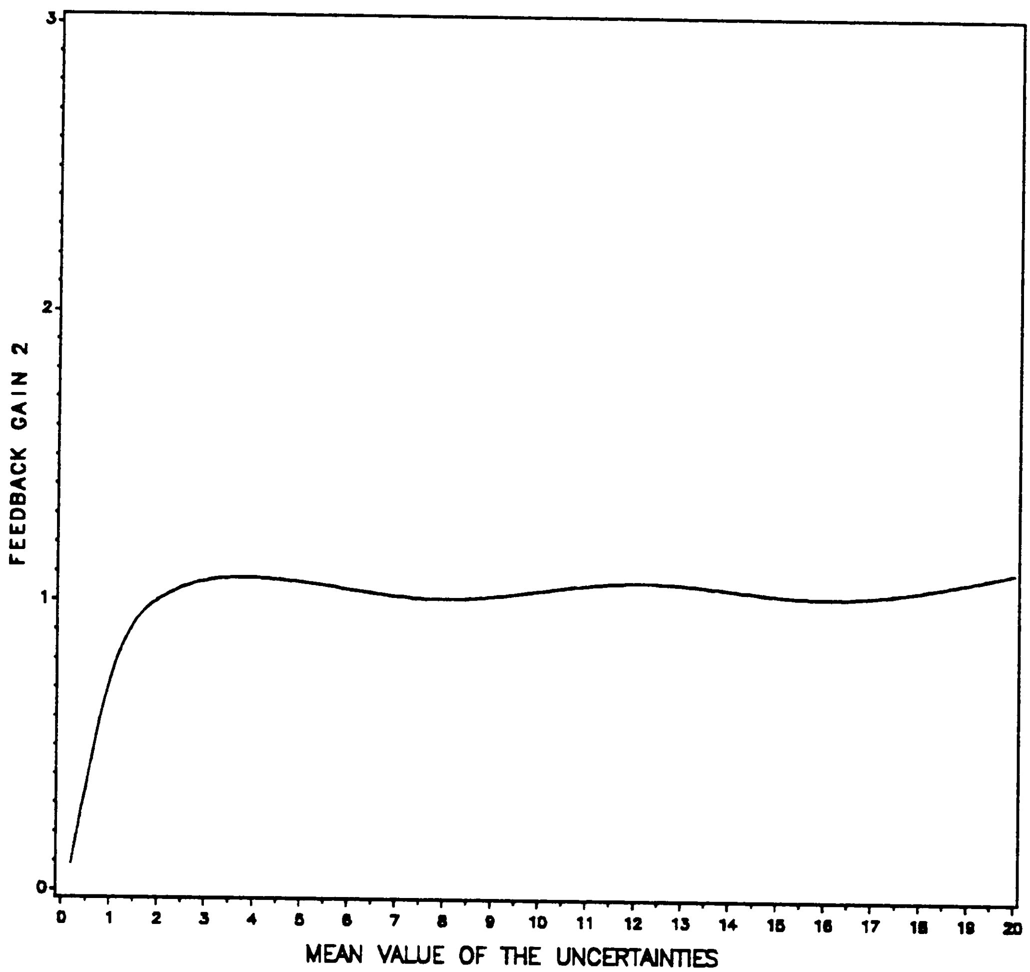

Fig. 5.4: Feedback gain 2

6. PIECEWISE CONSTANT PERIODIC UNCERTAINTY

Introduction

A special class of linear time-varying systems is that comprised of systems with parameters that vary periodically. These systems are characterized by linear differential equation with periodic coefficients and, because of the periodicity of the coefficients, stability study of these periodic systems is considerably more straightforward than that of the general time varying systems. Floquet theory provides the basis for determining the stability characteristics of periodic systems. According to the Floquet theory, a necessary and sufficient condition for stability of a periodic system is that all the eigenvalues of the discrete transition matrix lie within the unit circle. In the special case where the coefficients of the periodic system are piecewise constant functions of time then analytic expression for the eigenvalues of the state transition matrix can be obtained.

If the uncertainties in a linear time-varying uncertain system are piecewise-constant periodic functions then the system is periodic with piecewise-constant coefficients, provided that the uncertainties enter linearly into the state and input matrices. In this case it is interesting to examine the necessary and sufficient conditions for the asymptotic stability of the system, using the tools of Floquet theory.

Floquet Theory

Consider the linear system

$$\dot{x}(t) = A(t) \cdot x(t) \;, \quad x(0) = x_o \tag{6.1}$$

where A(t) is continuous and periodic with period T, i.e.,

$$A(t + T) = A(T) \tag{6.2}$$

Let $\phi(t,\tau)$ the state transition matrix of the system (6.1) over the interval (t,τ), which satisfy

$$\frac{\partial}{\partial t} \phi(t,\tau) = A(t) \cdot \Phi(t,\tau) \tag{6.3}$$

$$\phi(t,t) = I \tag{6.4}$$

The discrete transition matrix of the periodic system (6.1) - (6.2) is defined as

$$C = \phi(T,0) \tag{6.5}$$

and describes the system behaviour over one full period of time. The eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix C are called characteristic multipliers associated with the periodic matrix A.

The following theorem is from [35].

Theorem 6.1: The periodic system (6.1) - (6.2) is asymptotically stable if and only if all the characteristic multipliers $\lambda_1, \ldots,$ λ_n lie within the unit circle, i.e., if and only if $|\lambda_i| < 1$, i = 1,...,n.

For second order systems it's easy to check [37] that the necessary and sufficient condition of Theorem 6.1 is equivalent to the conditions

$$\mathrm{trace}(C) < \det(C) + 1 \tag{6.6}$$

and

$$\det(C) \leq 1 \tag{6.7}$$

Generally, as is well known [36], the discrete transition matrix C is very difficult or impossible to be determined. However, in the special case where A(t) is piecewise constant, i.e., A(t) is a constant matrix on a finite number of subintervals of (0,T) then analytic expressions for C can easily be obtained using the transition property of the state transition matrix. The procedure is the following. Consider the interval of the first period (0,T) and the m subintervals (t_{k-1}, t_k), $k=1,\dots,m$ such that $A(t) = A_k$ on (t_{k-1}, t_k). Then

$$C = \prod_{k=1}^{m} \exp\,(A_k\, T_k) \tag{6.8}$$

where

$$T_k = t_k - t_{k-1}, \quad t_o = 0 \text{ and } t_m = T.$$

Application of Floquet Theory for Piecewise Constant Uncertainties

Consider the linear time-varying uncertain system (2.6) under the feedback control law $u(x) = -\,[k_1\ k_2]\ x$.

$$\dot{x}(t) = \begin{bmatrix} \gamma_1(t) - k_1 & -k_2 \\ \gamma_2(t)\ k_1 & \gamma_2(t) \cdot k_2 \end{bmatrix} x(t) \tag{6.9}$$

where

$$\gamma_1^{-} \le \gamma_1(t) \le \gamma_1^{+} \tag{6.10}$$

and

$$\gamma_2^{-} \le \gamma_2(t) \le \gamma_2^{+} \tag{6.11}$$

Assume that the uncertainties $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ are piecewise constant periodic functions with the same period T, i.e., let

$$\gamma_1(t) = \begin{cases} \gamma_1^{+} & 0 \le t \le \frac{T}{2} \\ \gamma_1^{-} & \frac{T}{2} \le t \le T \end{cases}, \quad \gamma_1(t + T) = \gamma_1(t) \tag{6.12}$$

and

$$\gamma_2(t) = \begin{cases} \gamma_2^+ & 0 \le t \le \frac{T}{2} \\ \gamma_2^- & \frac{T}{2} \le t \le T \end{cases}, \quad \gamma_2(t + T) = \gamma_2(t) \quad (6.13)$$

In this case the system (6.9) is piecewise constant periodic and the discrete transition matrix C is given by

$$C = \exp\,(A_1 \cdot T/2)\ \exp\,(A_2 \cdot T/2) \quad (6.14)$$

where

$$A_1 = \begin{bmatrix} \gamma_1^+ - k_1 & - k_2 \\ \gamma_2^+ k_1 & \gamma_2^+ k_2 \end{bmatrix} \quad (6.15)$$

and

$$A_2 = \begin{bmatrix} \gamma_1^- - k_1 & - k_2 \\ \gamma_2^- k_1 & \gamma_2^- k_2 \end{bmatrix} \quad (6.16)$$

Therefore checking the necessary and sufficient conditions for stability (6.6) and (6.7), values of k_1 and k_2 on which stability or instability occurs can be determined and consequently, stability diagrams on the $k_1 - k_2$ plane can be plotted.

The previous case where $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ where assumed to be periodic, piecewise constant functions with the same period was the simplest possible.

In the most general case

$$\gamma_1(t) = \begin{cases} \gamma_1^+ & 0 \le t \le \frac{T_1}{2} \\ \gamma_1^- & \frac{T_1}{2} \le t \le T_1 \end{cases}, \quad \gamma_1(t + T_1) = \gamma_1(t) \tag{6.17}$$

and

$$\gamma_2(t) = \begin{cases} \gamma_2^+ & \Delta T \le t \le \frac{T_2}{2} + \Delta T \\ \gamma_2^- & \frac{T_2}{2} + \Delta T \le t \le T_2 + \Delta T \end{cases}, \quad \gamma_2(t + T_2) = \gamma_2(t) \tag{6.18}$$

where $0 \le \Delta T \le T_2$ and $\frac{T_1}{T_2} = \frac{n_1}{n_2}$, $n_1, n_2 \in N$.

In this case the system (6.9) is piecewise constant periodic with period $T = T_1 \cdot n_2 = T_2 \cdot n_1$, and the discrete transition matrix has the form

$$C = \prod_{i=1}^{m} \exp (A_i \cdot \Delta T_i) \tag{6.19}$$

and A_i has the form

$$A_i = \begin{bmatrix} \gamma_1^{\pm} - k_1 & - k_2 \\ \gamma_2^{\pm} \; k_1 & \gamma_2^{\pm} \; k_2 \end{bmatrix} \tag{6.20}$$

where $\gamma_1^{\pm}$ takes one of the values γ_1^+ or γ_1^- and $\gamma_2^{\pm}$ takes one of the values γ_2^+ or γ_2^- and $\Delta t_i = t_i - t_{i-1}$, $i = 1,\ldots,m$ is the ith time interval where $A(t) = A_i$ is constant. Similarly, checking the validity of the conditions (6.6) and (6.7) for different values of k_1 and k_2 we can plot stability diagrams on the $k_1 - k_2$ plane. We have to note that the stability diagrams depend on the selected value of the period T where we are testing the stability of the system.

Results

The procedure we followed to plot the stability diagrams of the system (6.9) was the following. We selected particular values for the bounds of the uncertainties γ_1^+, γ_1^-, γ_2^+ and γ_1^-, the period of the system T, the ratio of the periods of the uncertain functions n_1 / n_2 and the "shift" of the period of the one uncertain function with respect to the other ΔT. Varying the values of the feedback gains k_1 and k_2 we obtained the pairs (k_1, k_2) which correspond to a stable closed loop system and finally we plotted the regions of stability on the k_1 - k_2 plane.

In Figures 6.1 - 6.24 we have plotted stability diagrams for the special system of the form (6.9) which has

$$\gamma_1^- = 1.0 \;, \quad \gamma_1^+ = 12.0 \;, \quad \gamma_2^- = 0.5 \;, \quad \gamma_2^+ = 5.0 \;. \qquad (6.21)$$

As it was proved in [24], this particular uncertain system is not quadratically stabilizable and therefore it is interesting to examine the stabilizability of this system for different values of T, n_1/n_2 and ΔT.

From the stability diagrams, important conclusions can be deduced.

i) At high frequencies of the periodic functions γ_1 and γ_2 or equivalently, at low values of the period T ($T < 10^{-1}$), the system (6.5) is stable, when the system

$$\dot{x}(t) = \begin{bmatrix} \gamma_{1m} - k_1 & - k_2 \\ \gamma_{2m} \cdot k_1 & \gamma_{2m} \cdot k_2 \end{bmatrix} \cdot x(t) \qquad (6.22)$$

is stable, where

$$\gamma_{1m} = \frac{\gamma_1^{+} + \gamma_1^{-}}{2}, \quad \gamma_{2m} = \frac{\gamma_2^{+} - \gamma_2^{-}}{2} \qquad (6.23)$$

The conclusion is in agreement with the well known result that parametric disturbances of high frequency will not cause instability if the system matrix is a stability matrix for all values of time.

ii) At low frequencies or high values of the period ($T > 10^2$), the region of stability depends upon the shifting ΔT of the one periodic uncertainty with respect to the other. The smallest region of stability corresponds to the case where γ_1 and γ_2 have the same period and no relative shift ($\Delta T = 0$) and this is equal to the region of stability for the case where γ_1 and γ_2 are constants with values γ_1^{+} and γ_2^{+} respectively. The biggest region of stability corresponds to the case where γ_1 and γ_2 have the same period but a phase shift ΔT equal to T/2, i.e., when the maximum values of the one uncertain function occurs at the minimum values of the other uncertain function.

iii) At middle frequencies ($10^{-1} < T < 10^{2}$) of the periodic system generally corresponds the smallest region of stability. Specifically, the smallest region occurs when γ_1 and γ_2 have the same period and there is a shift ΔT of the origin of the one periodic function with respect to the other and this region depends upon the relative magnitude of the two periodic functions. For the system (6.9) and the uncertainty bounds (6.21) this minimum region of stability occurs when ΔT is equal to -0.2 T. Similar were the results for different values of the uncertain bounds γ_1^+, γ_1^-, γ_2^+ and γ_2^-.

In conclusion we can say that when γ_1 and γ_2 are piecewise constant periodic functions then there always exist constant gains k_1 and k_2 which stabilize the system (6.9). The stability region highly depends on the period of the system and generally the smallest region of stability occurs in the middle periods.

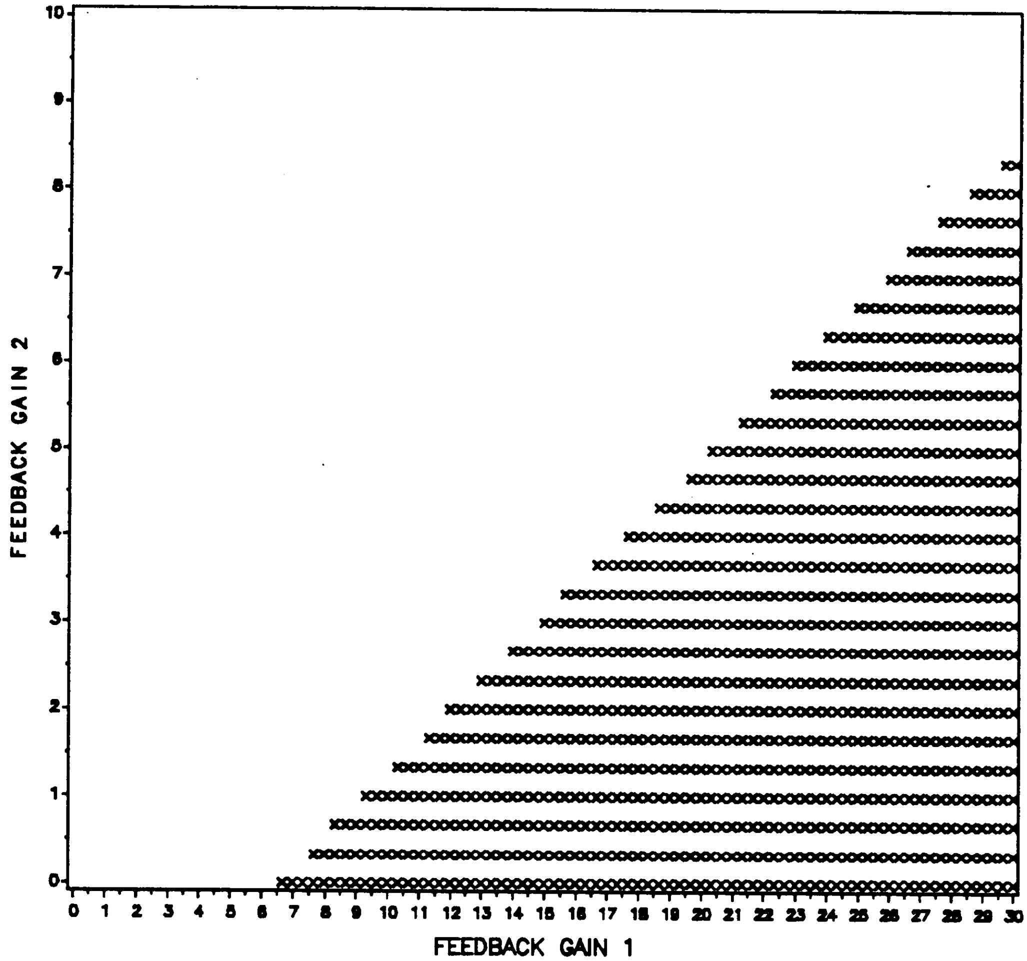

Fig. 6.1: Stability region for $T = 10^{-1}$ and $\Delta T = 0$

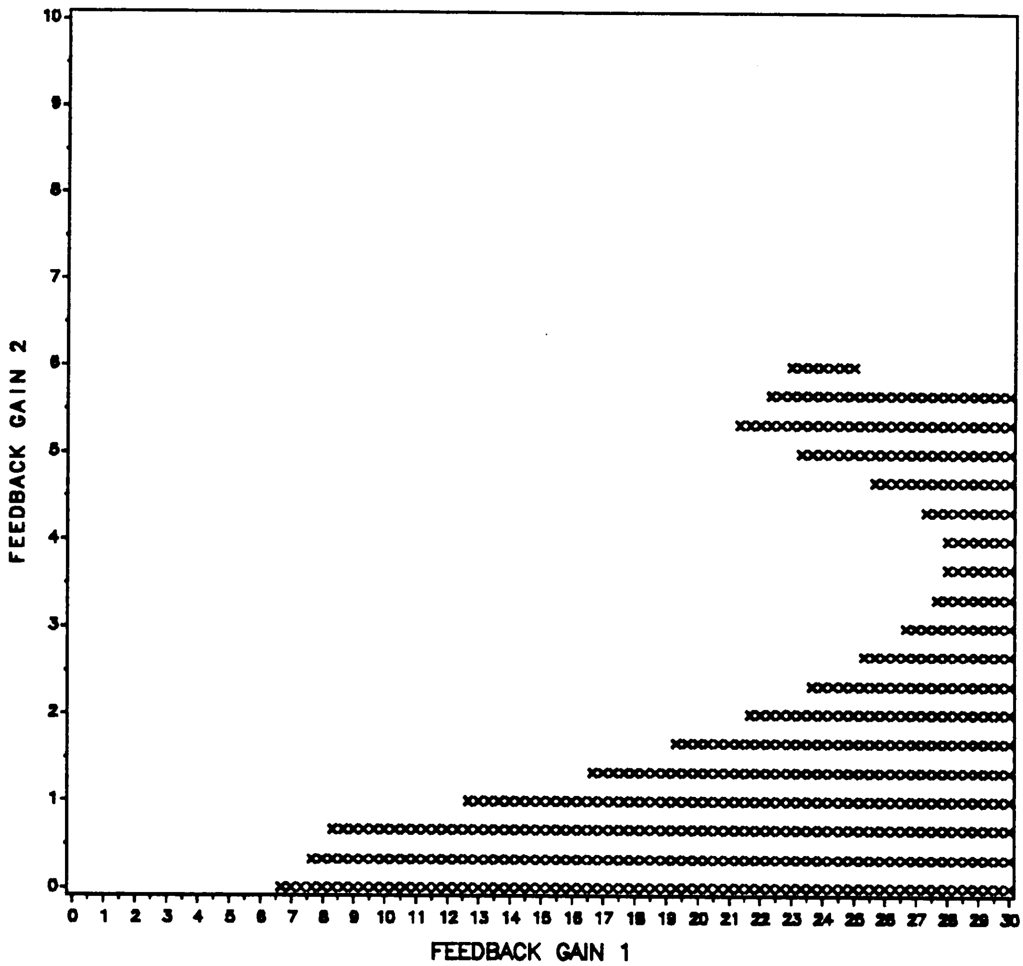

Fig. 6.2: Stability region for T = 0.5 and $\Delta T = 0$

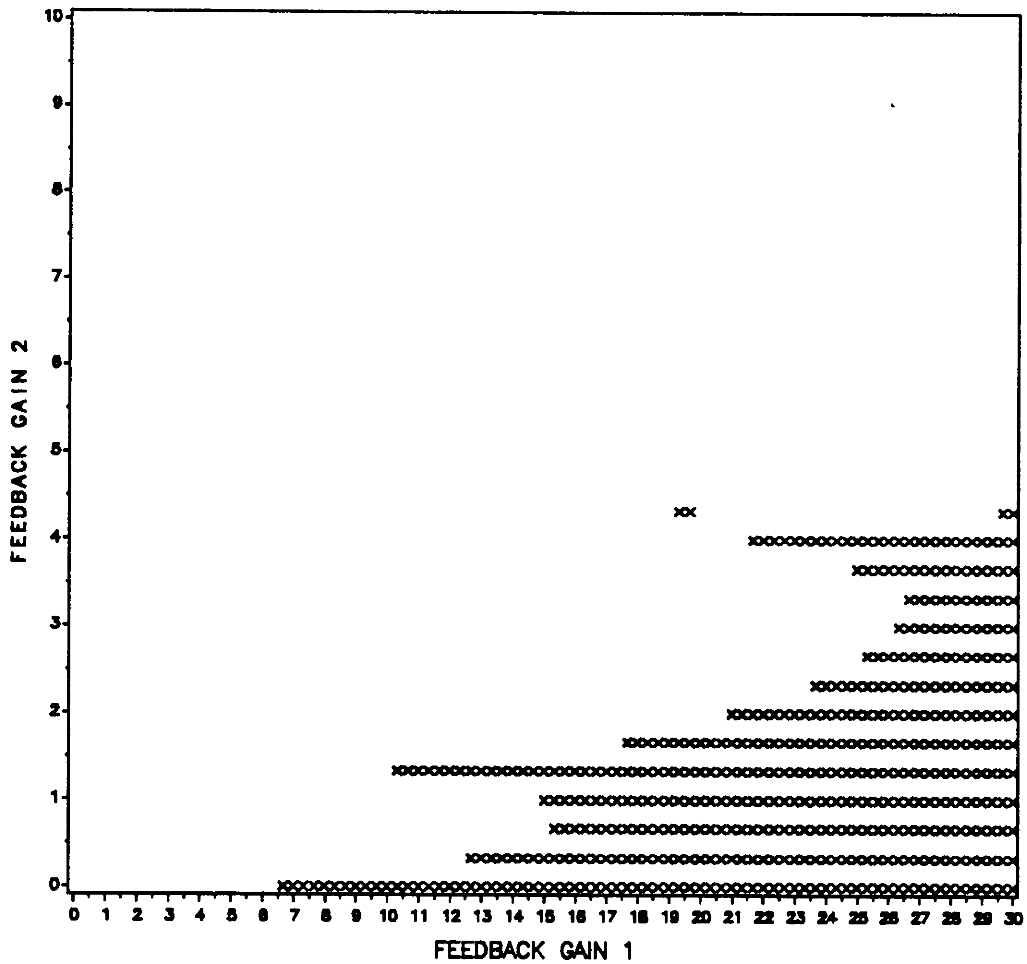

Fig. 6.3: Stability region for T = 1 and $\Delta T = 0$

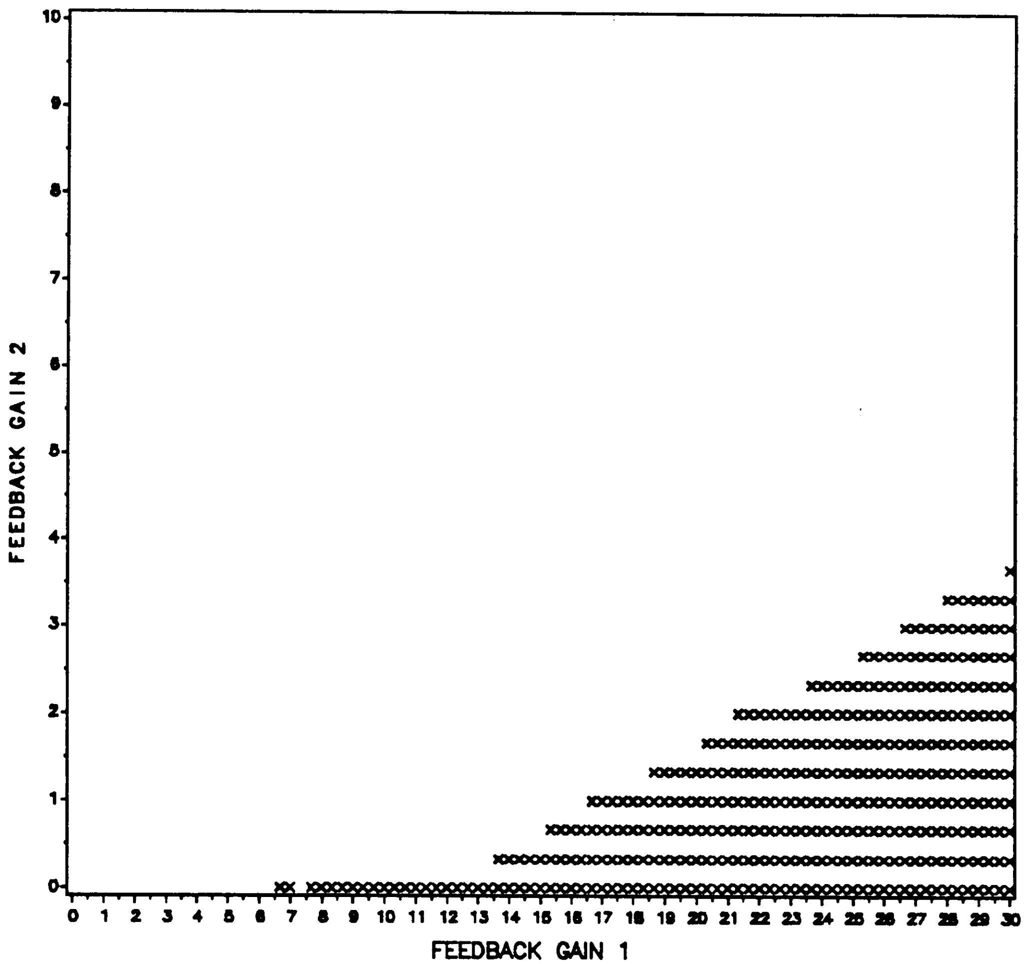

Fig. 6.4: Stability region for T = 10 and $\Delta T = 0$

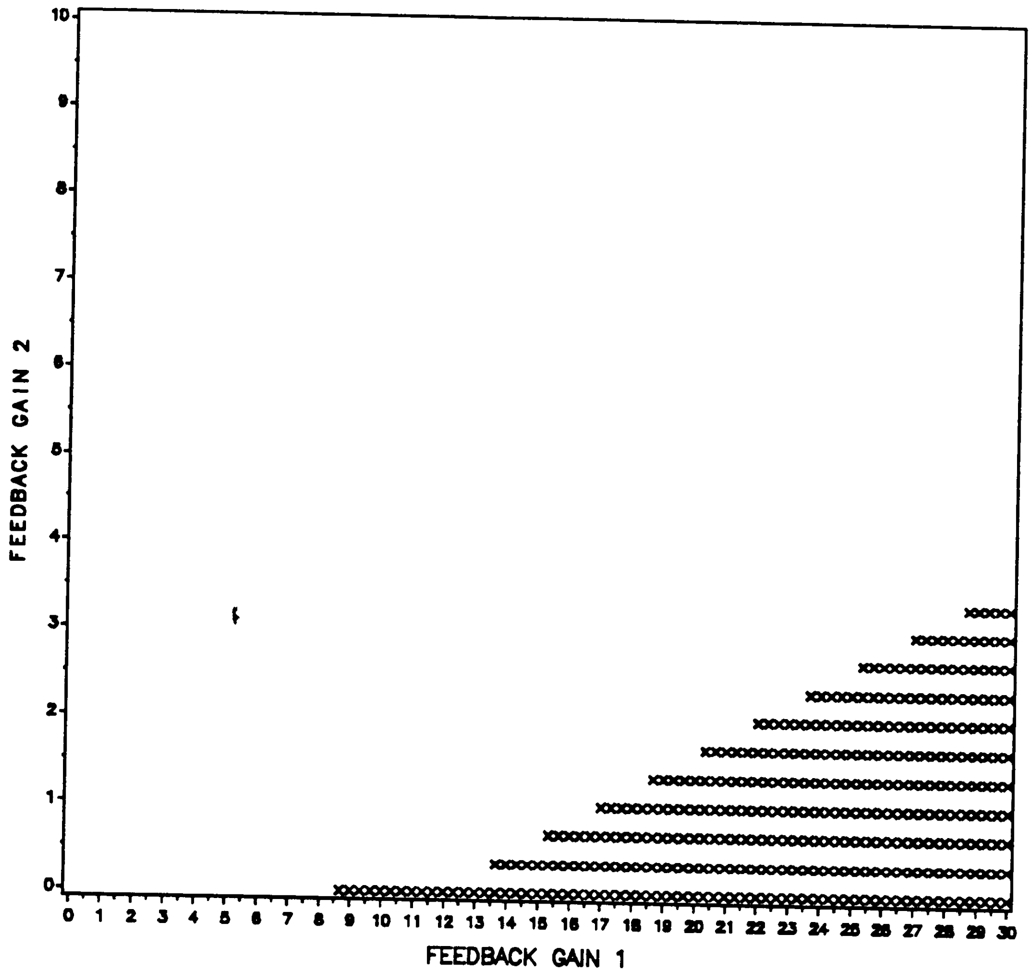

Fig. 6.5: Stability region for $T = 10^3$ and $\Delta T = 0$

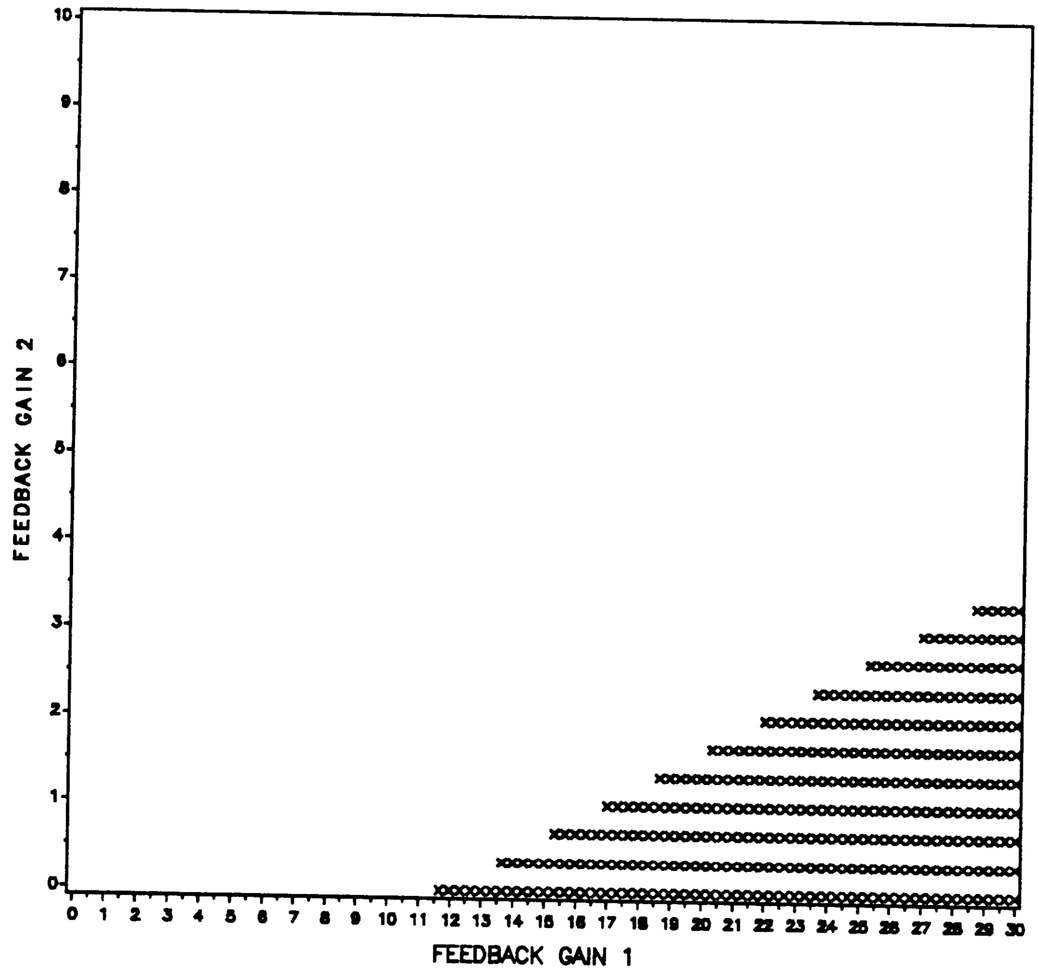

Fig. 6.6: Stability region for $T = 10^2$ and $\Delta T = 0$

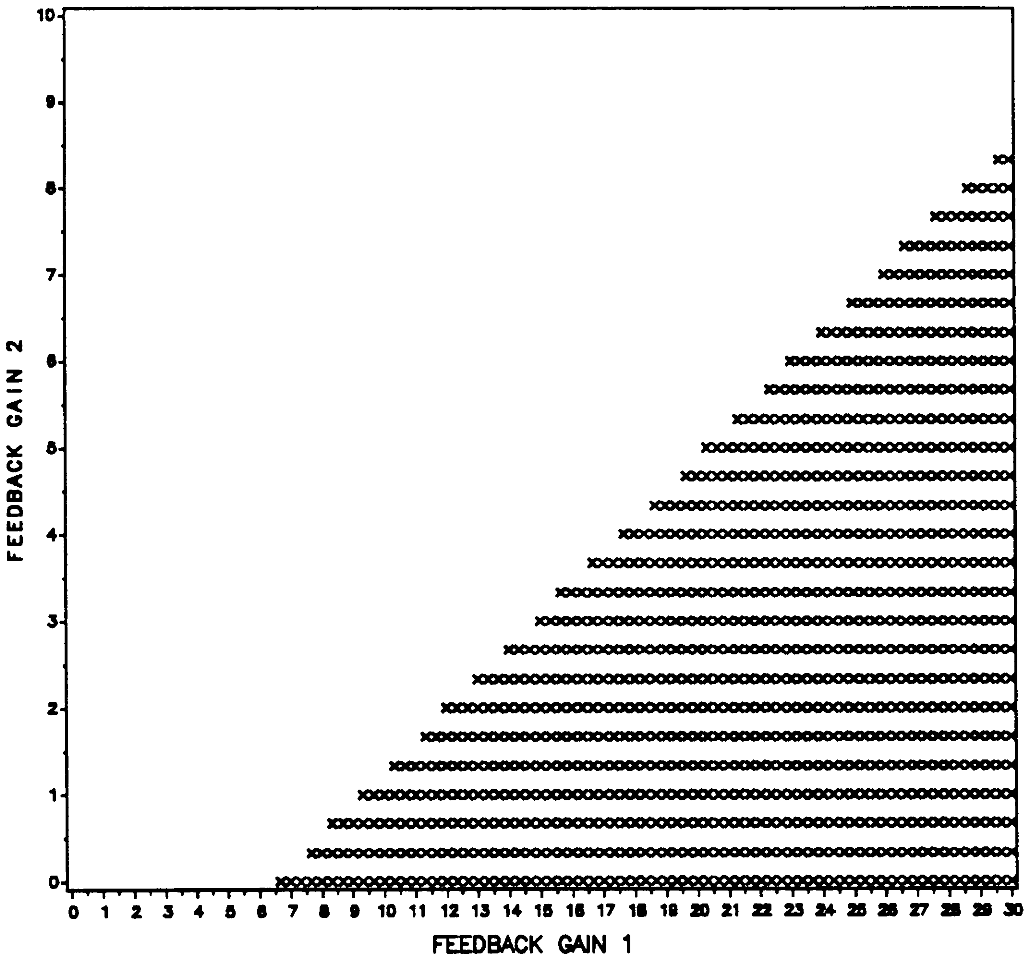

Fig. 6.7: Stability region for $T = 10^{-1}$ and $\Delta T = T/2$

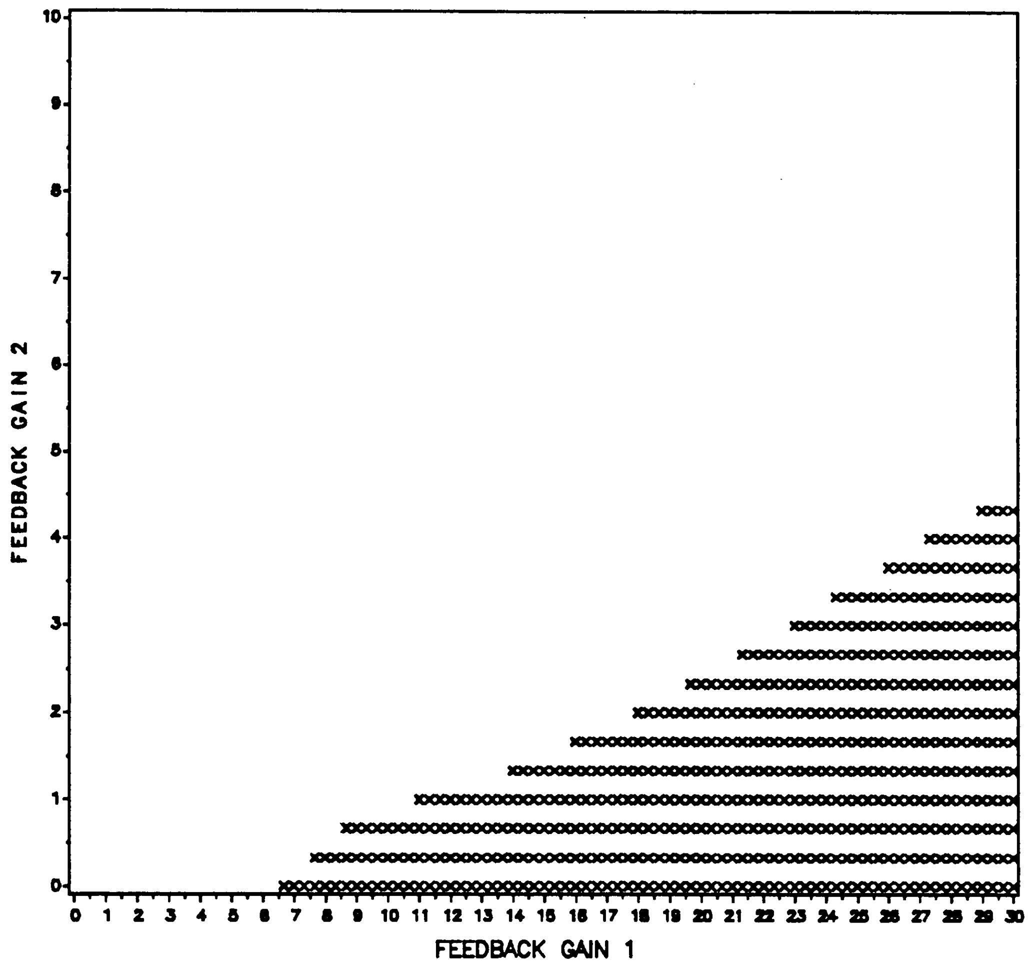

Fig. 6.8: Stability region for T = 0.5 and $\Delta T = T/2$

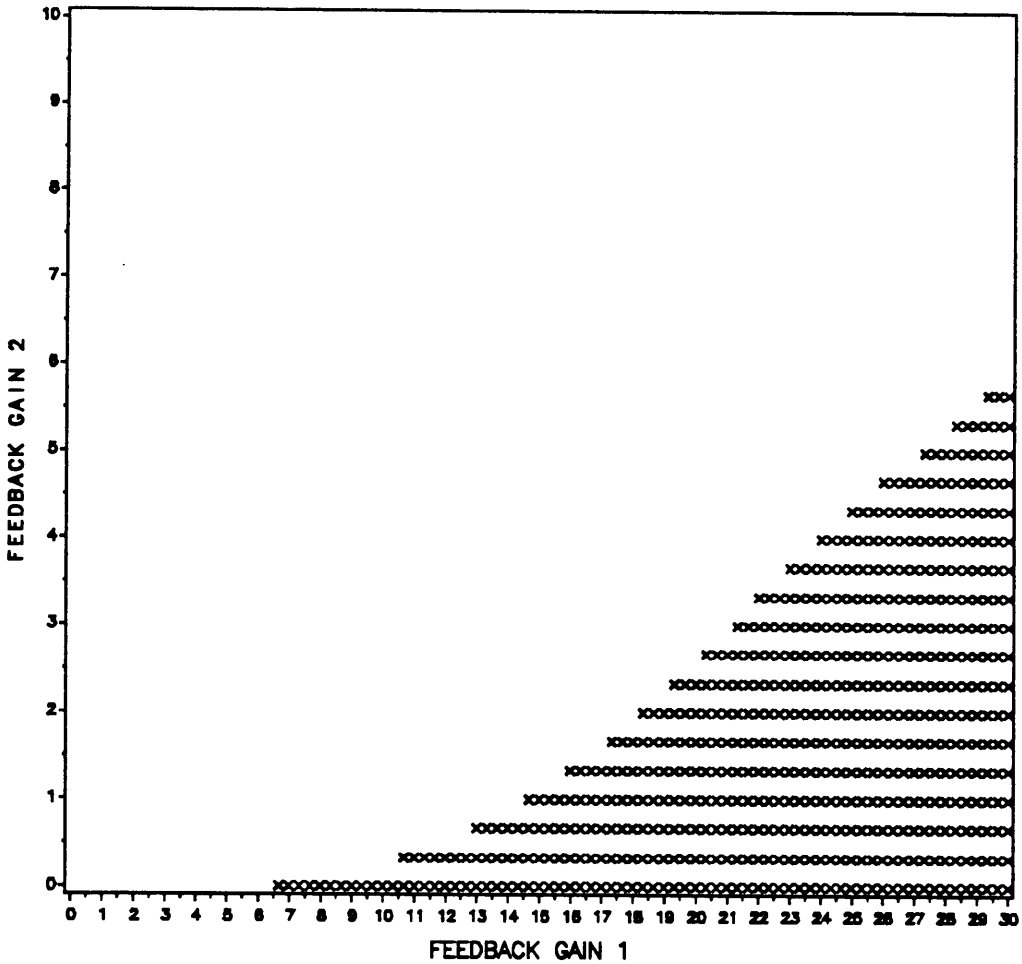

Fig. 6.9: Stability region for T = 1 and $\Delta T = T/2$

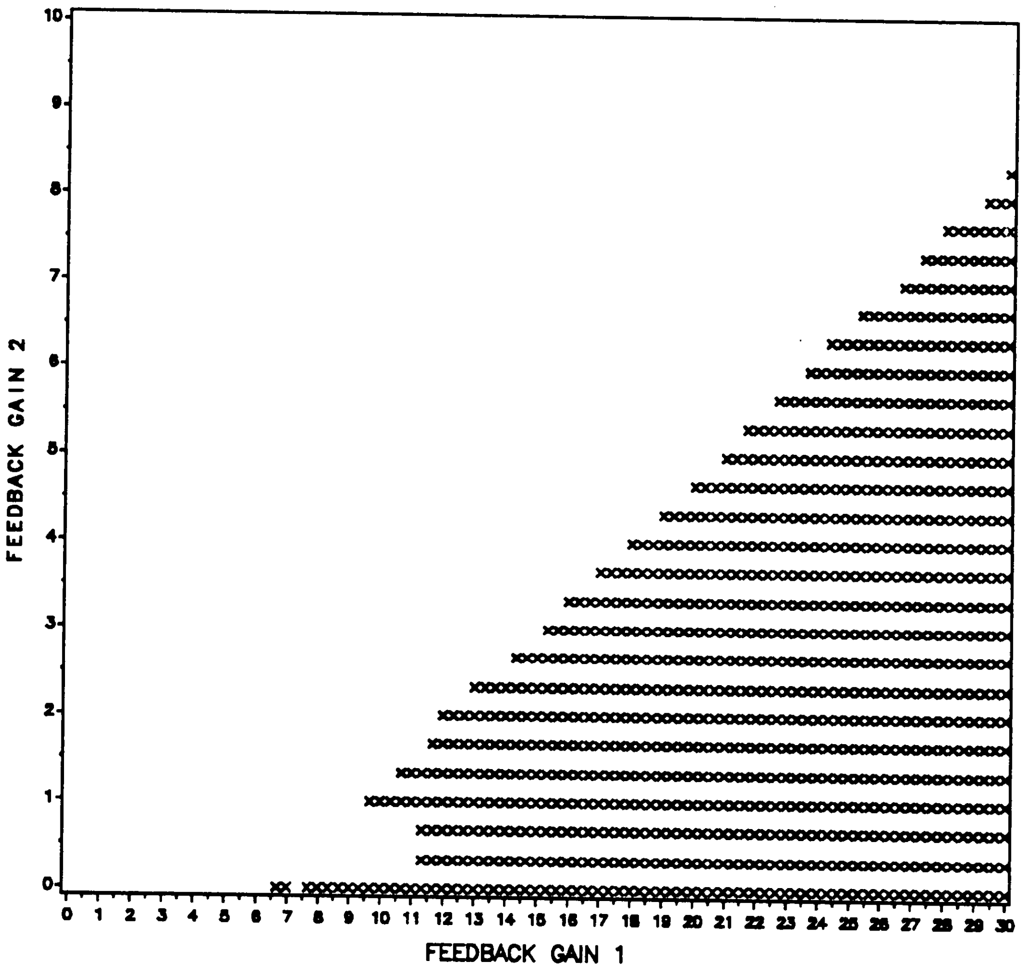

Fig. 6.10: Stability region for T = 10 and $\Delta T = T/2$

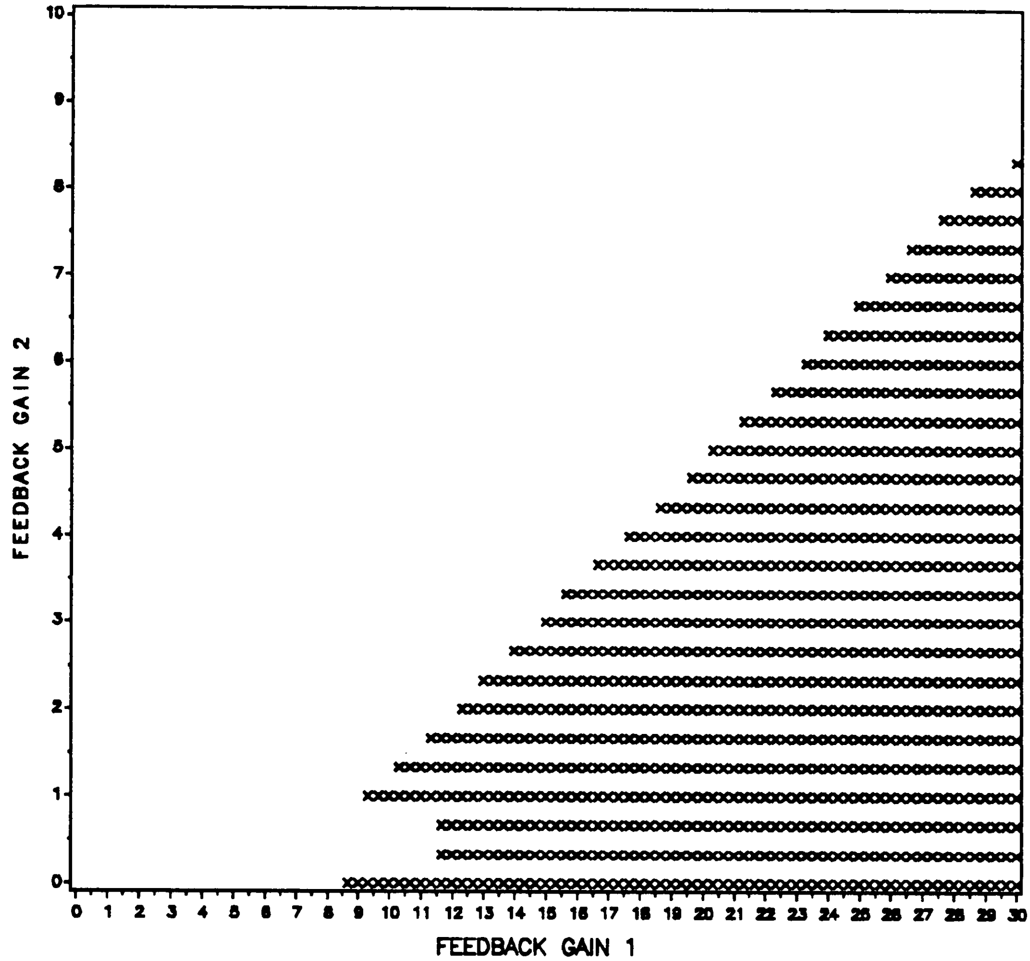

Fig. 6.11: Stability region for $T = 10^2$ and $\Delta T = T/2$

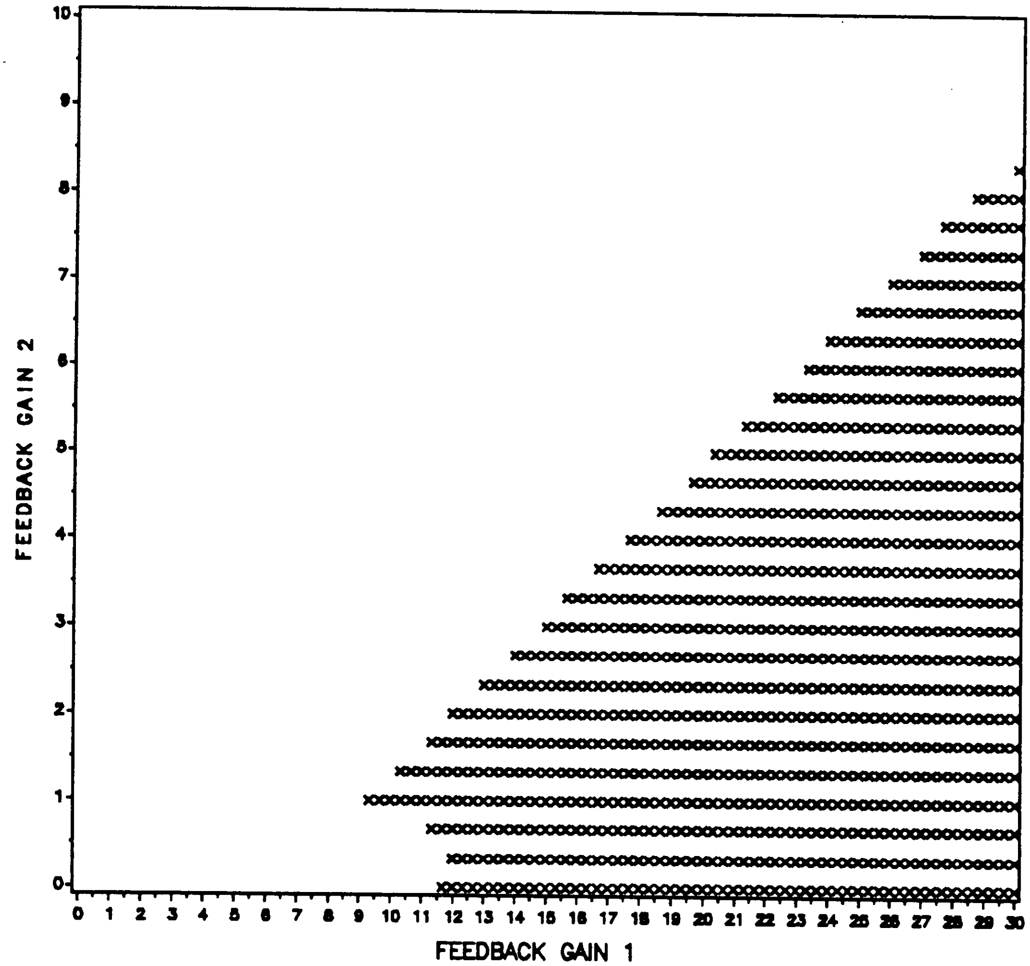

Fig. 6.12: Stability region for $T = 10^3$ and $\Delta T = T/2$

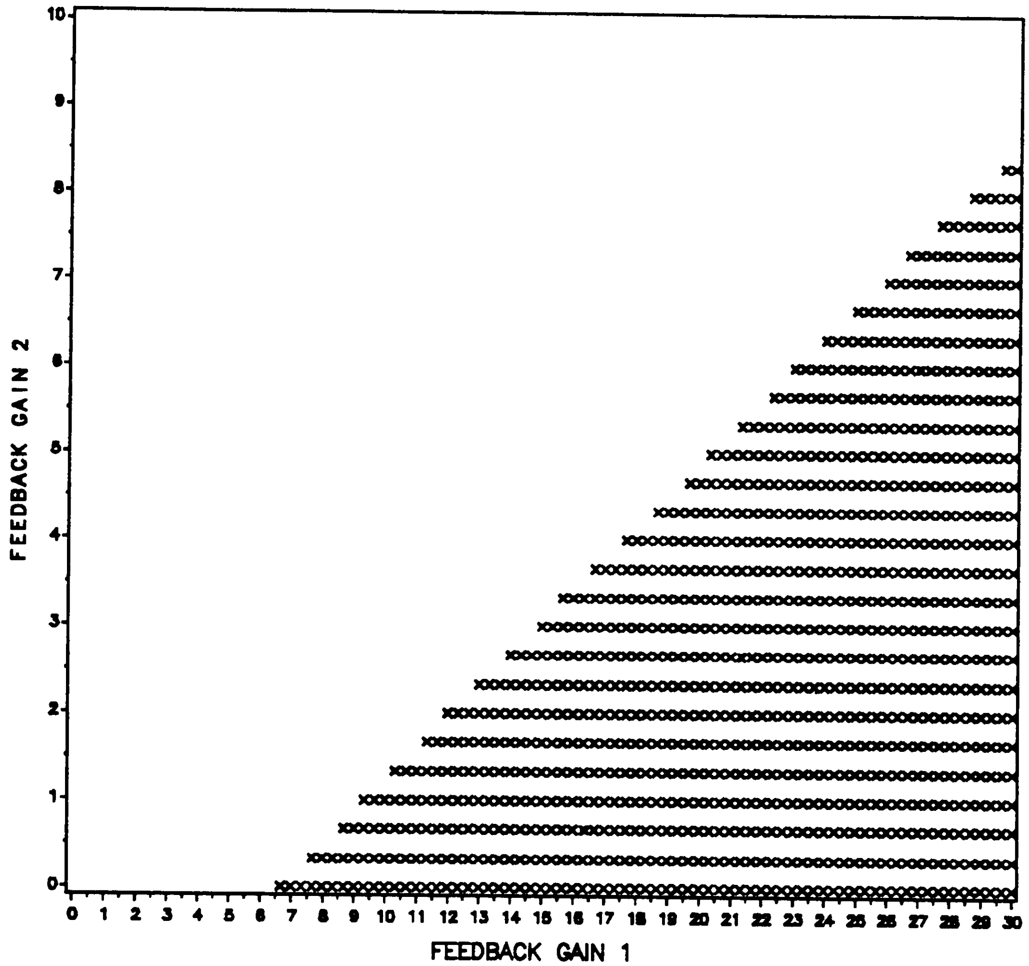

Fig. 6.13: Stability region for $T = 10^{-1}$ and $\Delta T = 0.2T$

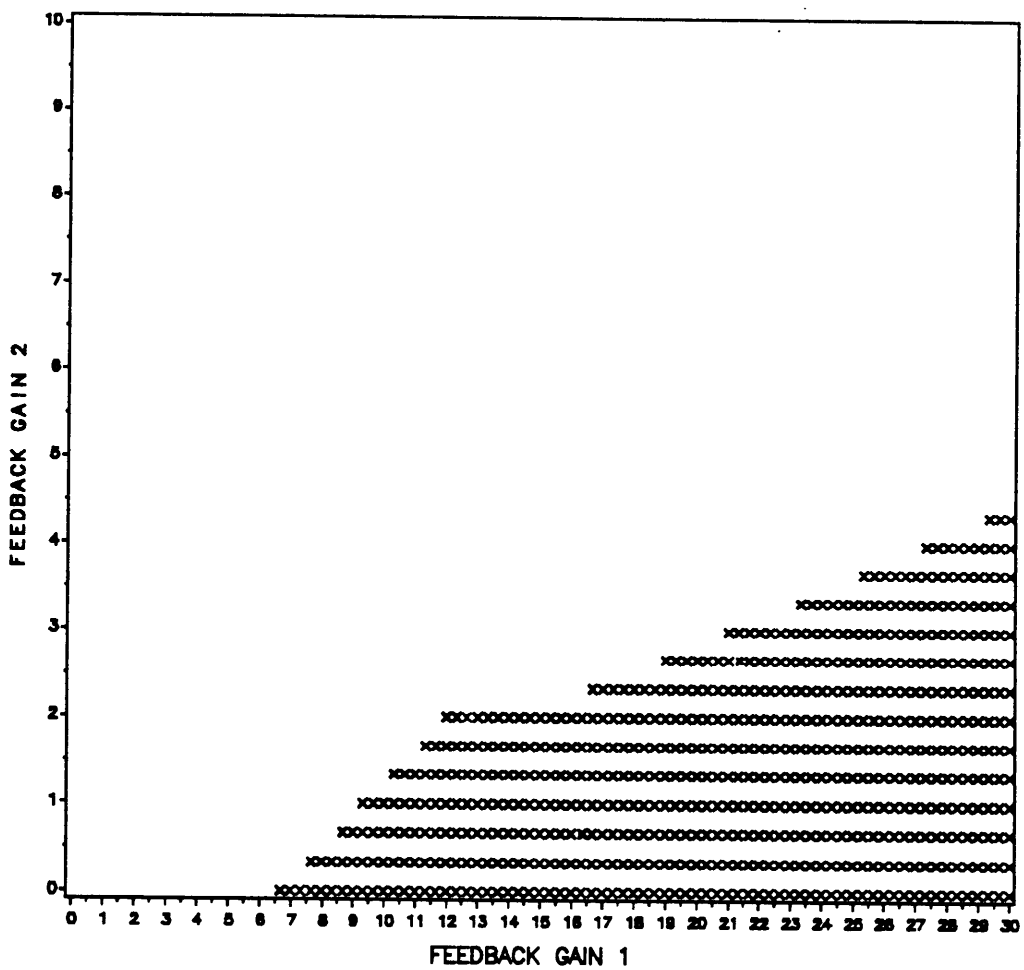

Fig. 6.14: Stability region for T = 0.5 and $\Delta T = 0.2T$

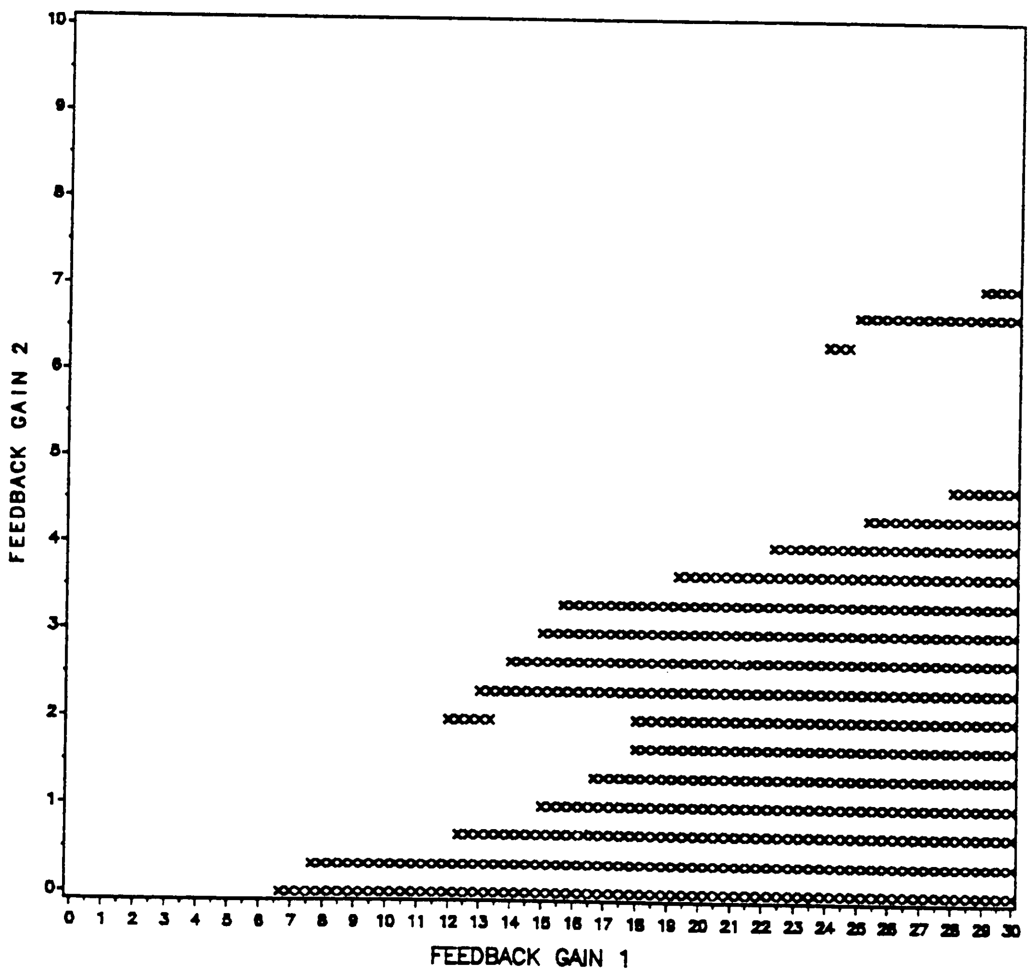

Fig. 6.15: Stability region for T = 1 and $\Delta T = 0.2T$

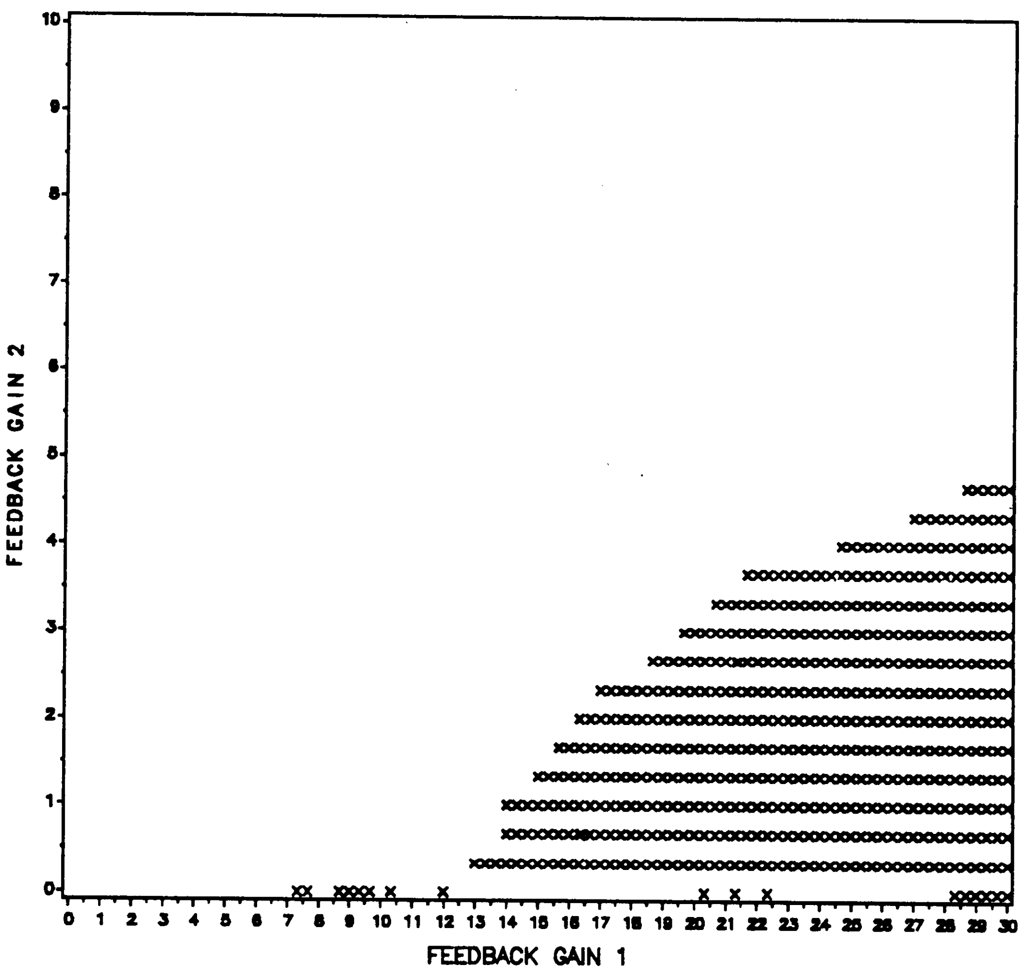

Fig. 6.16: Stability region for T = 10 and $\Delta T = 0.2T$

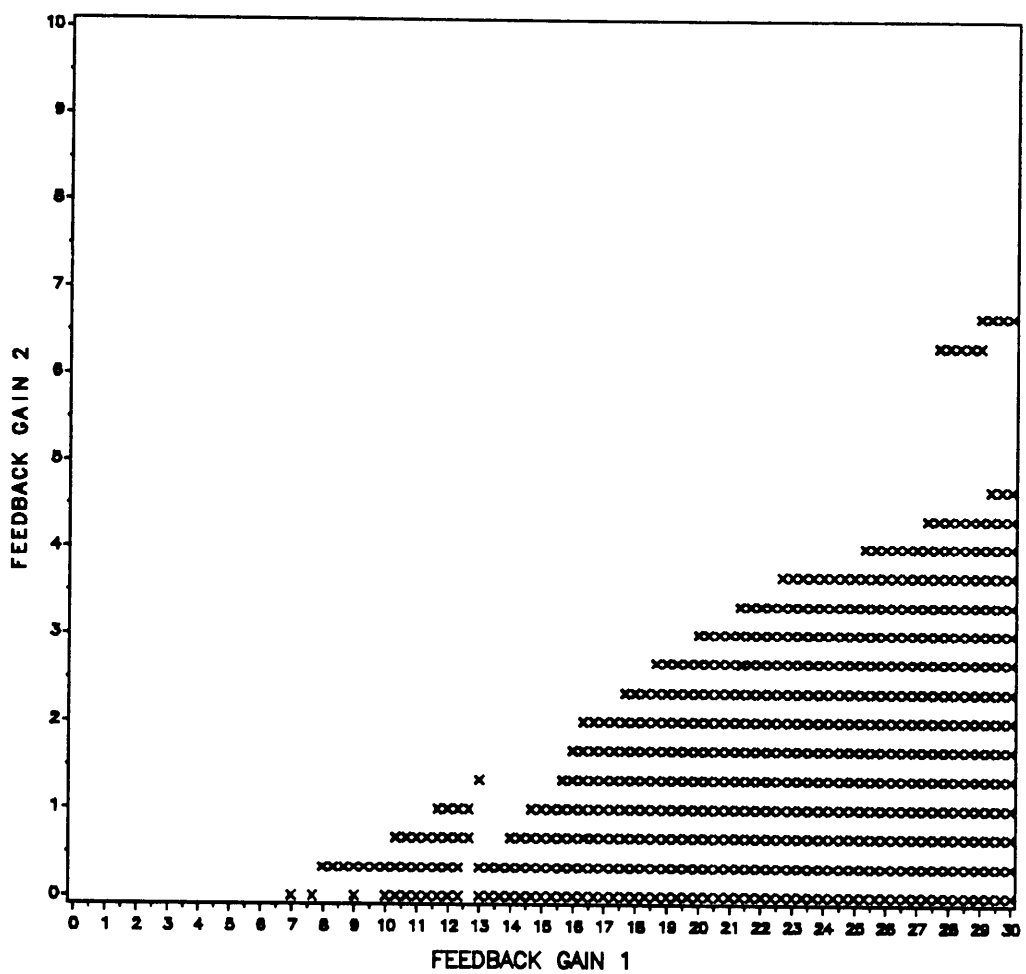

Fig. 6.17: Stability region for $T = 10^2$ and $\Delta T = 0.2T$

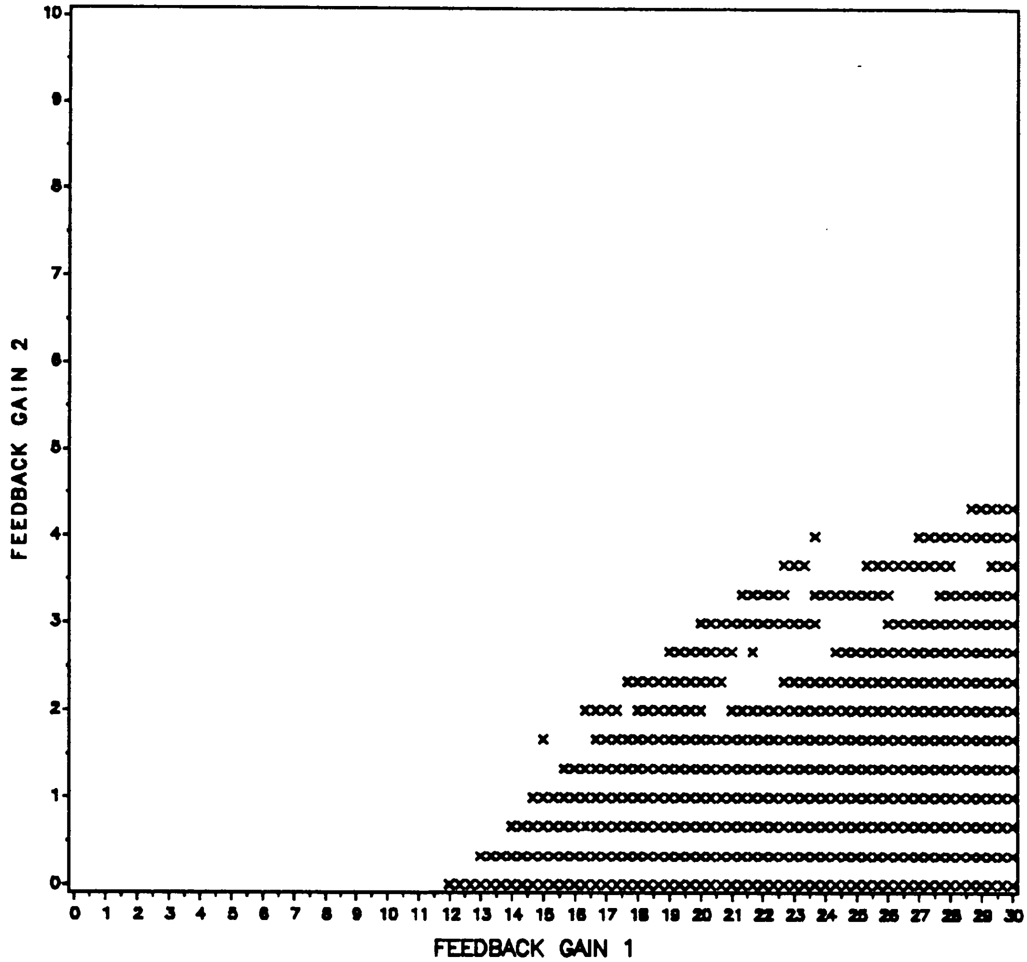

Fig. 6.18: Stability region for $T = 10^3$ and $\Delta T = 0.2T$

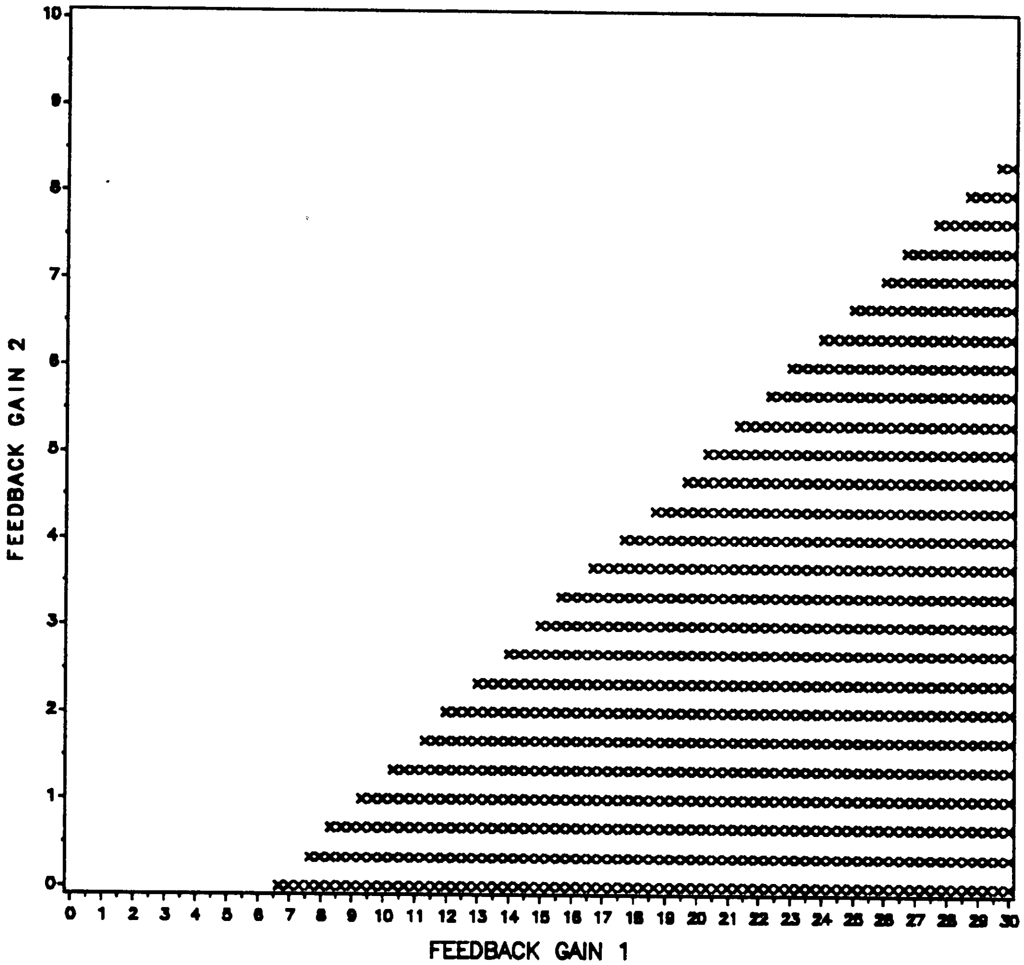

Fig. 6.19: Stability region for $T = 10^{-1}$ and $\Delta T = -0.2T$

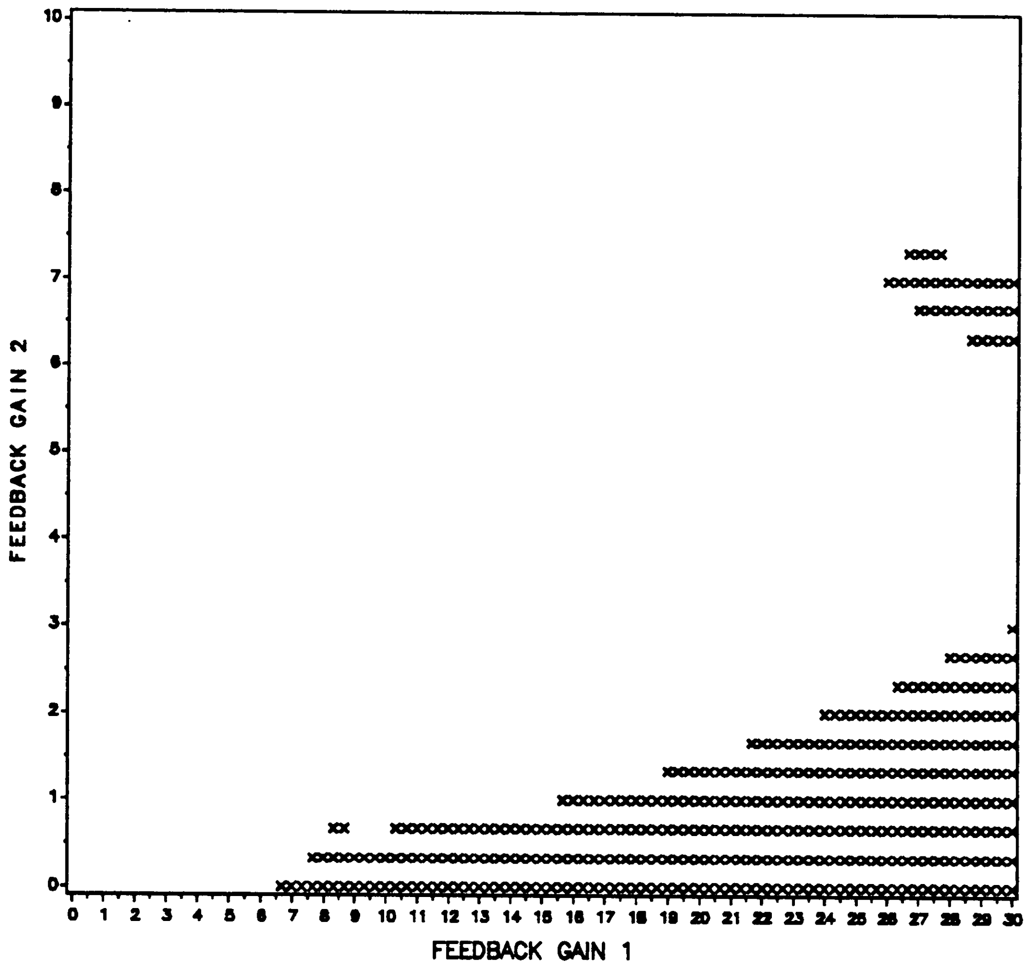

Fig. 6.20: Stability region for T = 0.5 and $\Delta T = -0.2T$

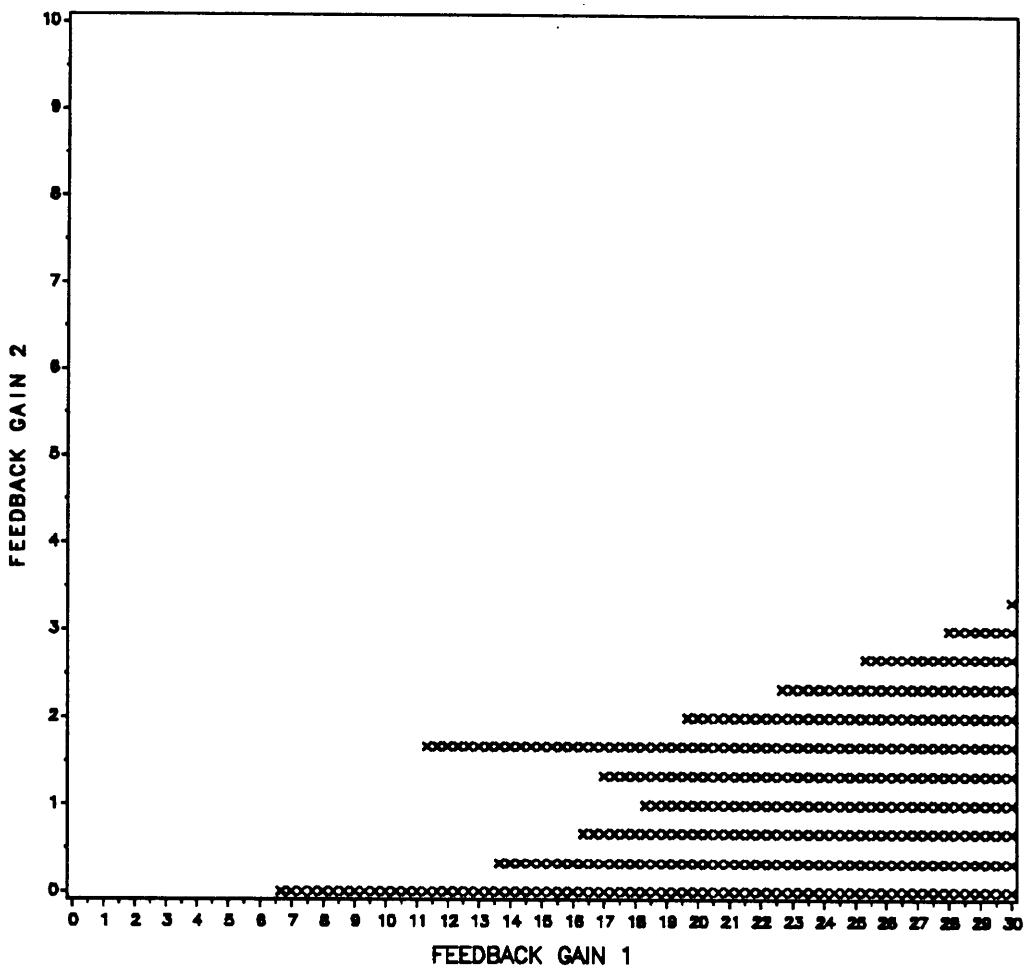

Fig. 6.21: Stability region for T = 1 and $\Delta T = -0.2T$

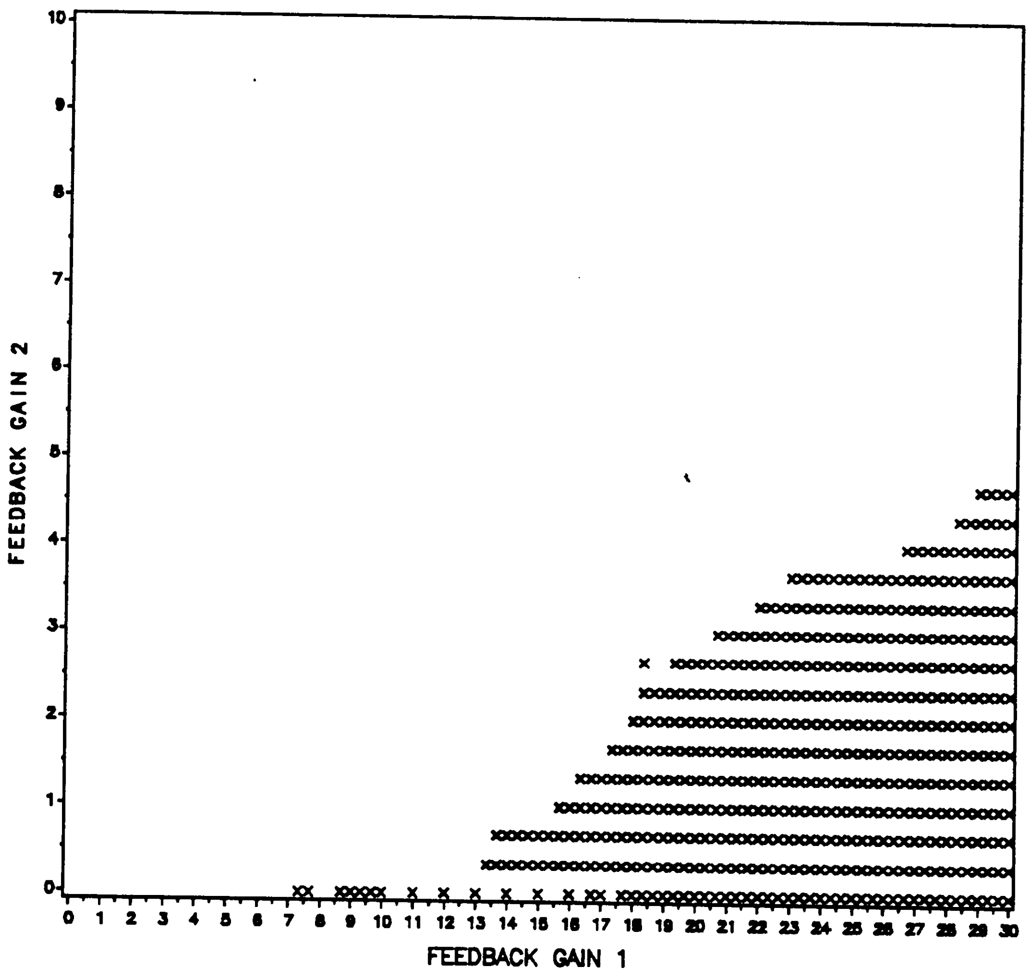

Fig. 6.22: Stability region for T = 10 and $\Delta T = -0.2T$

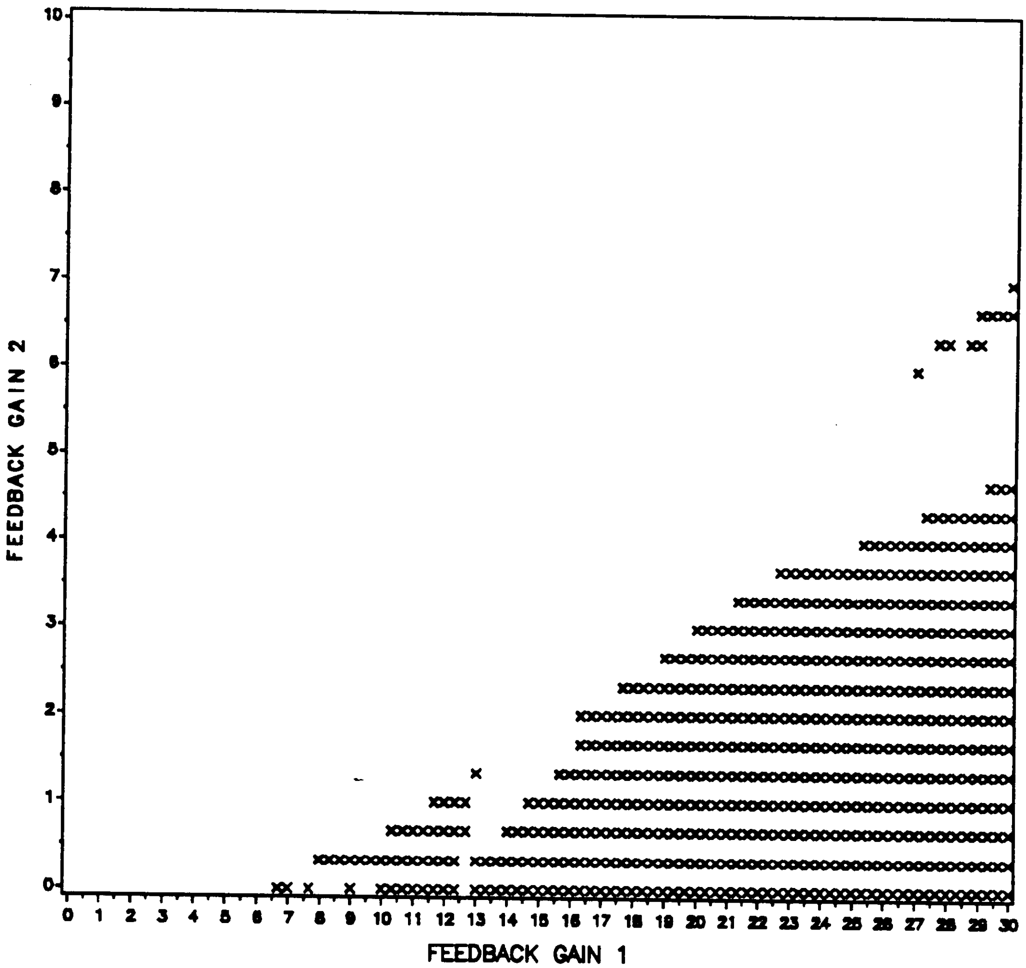

Fig. 6.23: Stability region for $T = 10^2$ and $\Delta T = -0.2T$

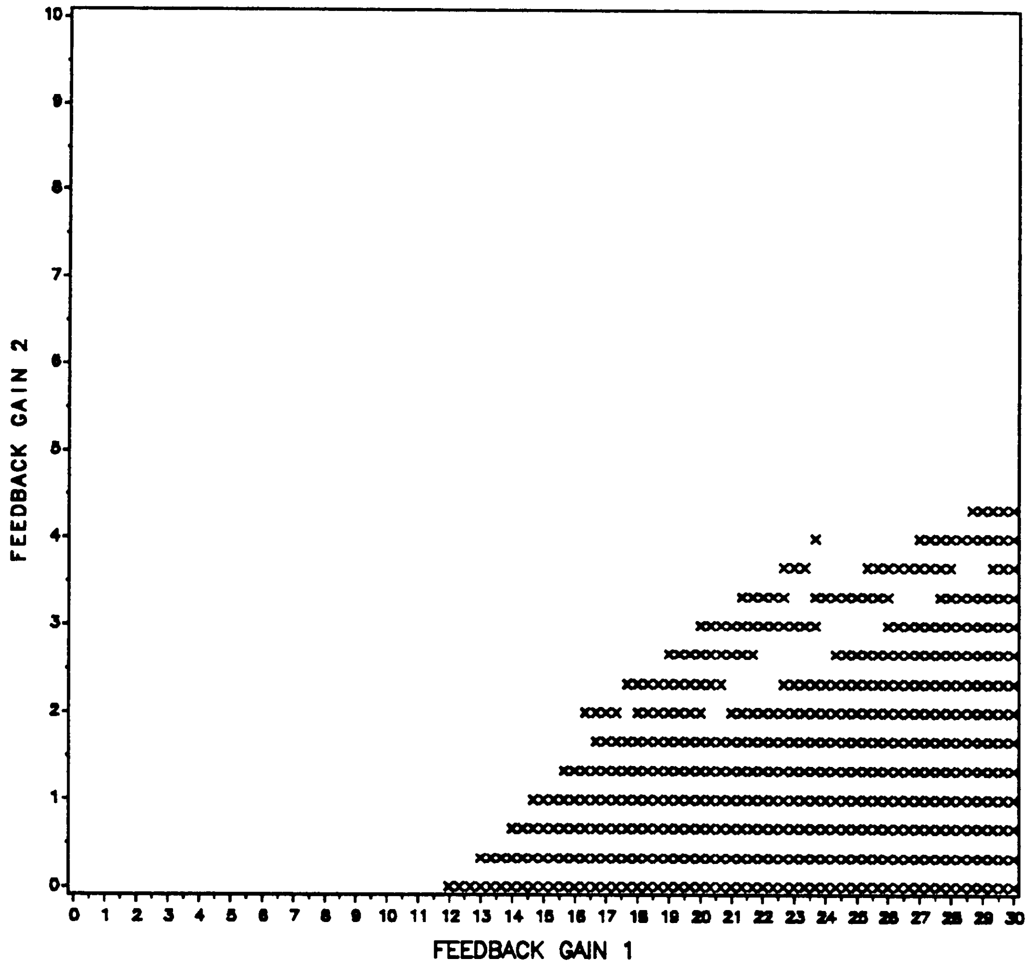

Fig. 6.24: Stability region for $T = 10^3$ and $\Delta T = -0.2T$

7. ROBUST STABILIZATION USING SMALL GAIN THEOREM

Introduction

The small gain theorem is a very general theorem which gives sufficient conditions under which a bounded input produces a bounded output. It states that for an open-loop stable system, a sufficient condition for stability is to keep the loop gain less than unity and provides the multivariable extension of the Bode stability criterion. The small gain theorem provides a conservative criterion for stability but it is useful because it can handle not only real perturbations but also complex perturbations.

One of the early versions of the small gain theorem has been proved by Zames [42]. A detailed development of the subject was provided by Desoer and Vidyasagar [39].

Small Gain Theorem

The block diagram of a typical feedback control system is shown in Figure 7.1. The symbols u_1, u_2 denote the inputs, y_1, y_2 the outputs and e_1, e_2 the errors. H_1 and H_2 are operators which act on the errors e_1 and e_2 respectively, to produce the outputs y_1 and y_2. In simpler words H_1 and H_2 are the symbolic notations representing the differential equations relating the errors and outputs of the individual blocks.

Let's define the infinity-norm of the operators H_1 and H_2 as

$$|| H_i ||_\infty = \sup \frac{|| H_i x ||}{|| x ||} \qquad i = 1,2 \qquad (7.1)$$

where the supremum is taken over all x in the domaind of H_i and all t for which $x \neq 0$. Also $|| \cdot ||$ denotes the 2-norm which is defined as the square root of the integral of the square of x. The following version of the small gain theorem is from [38].

Theorem 7.1. If

$$|| H_1 ||_\infty \cdot || H_2 ||_\infty < 1 \qquad (7.2)$$

then the closed loop system given in Figure 7.1 is bounded-input, bounded-output stable, i.e., bounded inputs produce bounded outputs. The proof of this theorem is based on the well known Contraction Principle.

Now let's consider the special case where

$$H_1 = \Delta(t) = \mathrm{diag}(\delta_1(t),\ldots,\delta_n(t)) \qquad (7.3)$$

where

$$|\ \delta_i(t)\ | < 1\ , \qquad t \geq 0 \qquad (7.4)$$

$$H_2 = P \qquad (7.5)$$

Δ corresponds to the "structured" uncertainty of the system and P is the plant which is subject to the uncertainty Δ.

It is

$$||\ \Delta\ ||_\infty \leq \max\ \{\ |\delta_i(t)|\ /\ t \geq 0\ ,\ i = 1,\ldots,n\}\ . \qquad (7.6)$$

Therefore, according to the small gain theorem, a sufficient condition for stability of the closed loop system is

$$|\delta_i(t)| < \frac{1}{||\ P\ ||_\infty} \qquad i = 1,\ldots,n \qquad (7.7)$$

Application of the Small Gain Theorem

Consider the uncertain system (2.6) subject to the feedback control law $u = -\ [k_1\ k_2]\ x$
and let

$$\gamma_i(t) = \gamma_m + \delta_i(t) \cdot \gamma_m \qquad (7.8)$$

where

$$|\delta_i(t)| < 1 \qquad i = 1,2 \qquad (7.9)$$

where

$$|\delta_i(t)| < 1 \qquad i = 1,2 \tag{7.9}$$

and γ_m is the mean value of the uncertainty.

Using the scaling $\tau = \gamma_m \cdot t$ we can write the closed loop system as

$$\dot{x}(t) = A \cdot x(t) + \Delta(t) \cdot R \cdot x(t) \tag{7.10}$$

where

$$A = \begin{bmatrix} 1 - \dfrac{k_1}{\gamma_m} & -\dfrac{k_2}{\gamma_m} \\ k_1 & k_2 \end{bmatrix} \tag{7.11}$$

$$R = \begin{bmatrix} 1 & 0 \\ k_1 & k_2 \end{bmatrix} \tag{7.12}$$

$$\Delta(t) = \begin{bmatrix} \delta_1(t) & 0 \\ 0 & \delta_2(t) \end{bmatrix} \tag{7.13}$$

and the derivative in (7.10) is with respect to τ.

To obtain the largest bounds of the perturbations, a constant transformation need to be performed, such that the full structure of the uncertainty is revealed.

Therefore, let's consider the scaling matrix

$$D = \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} \tag{7.14}$$

where d is a scaling constant. Then it is

$$\Delta(t) = D \cdot \Delta(t) \cdot D^{-1} \tag{7.15}$$

and the transfer function of the plant has the form

$$P(s) = D^{-1} \cdot R(sI - A)^{-1} D \tag{7.16}$$

After some calculations we can find that

$$P(s) = \frac{1}{s^2 + a_1 \cdot s + a_2} \cdot \begin{bmatrix} s - k_2 & -\dfrac{k_2 d}{\gamma_m} \\ \dfrac{k_1 \cdot s}{d} & k_2 s - k_2 \end{bmatrix} \tag{7.17}$$

where

$$a_1 = \frac{k_1}{\gamma_m} - k_2 - 1 \tag{7.18}$$

and

$$a_2 = k_2 \tag{7.19}$$

Let's define

$$\bar{s} = \frac{s}{k_2} \tag{7.20}$$

$$\bar{d} = \frac{d}{\gamma_m} \tag{7.21}$$

$$\bar{k}_1 = \frac{k_1}{\gamma_m} \tag{7.22}$$

and

$$\rho = \frac{\bar{k}_1}{k_2} \tag{7.23}$$

Then

$$P(\bar{s}) = \frac{1}{\bar{s}^2 + \bar{a}_1 \cdot \bar{s} + \bar{a}_2} \cdot \begin{bmatrix} \bar{s} - 1 & - \bar{d} \\ \frac{\rho}{\bar{d}} k_2 \bar{s} & k_2 \bar{s} - 1 \end{bmatrix} \qquad (7.24)$$

where $\bar{a}_1$ and $\bar{a}_2$ depends on d, k_2 and ρ. So P is independent of the mean value of the uncertainty γ_m. Therefore sufficient condition for the uncertain system (2.6) to be stable is

$$|\delta_i(t)| < \frac{1}{|| P(s) ||_\infty} \qquad (7.25)$$

for i=1,2. So our objective is to find the minimum of $||P(s)||_\infty$ over all possible values of ρ, k_2 and d.

For the computation of $||P||_\infty$ we will use the following formula (see [38])

$$|| P ||_\infty = \sup_w \left\{ \lambda_{max}[P(jw) \; P^T (-jw)] \right\}^{1/2} \qquad (7.26)$$

where λ_{max} $(\cdot)$ denotes the maximum eigenvalue of a matrix. An analytic expression for λ_{max} $[P(jw)P^T (-jw)]$ and the derivative of this with respect to w can be found, so the maximization problem (7.26) can be solved computing numerically the roots of

$$\frac{d}{dw} \left\{ \lambda_{max} [P (jw) \cdot P^T (-jw)] \right\} \qquad (7.27)$$

and evaluating $\{\lambda_{max} [P(jw) \cdot P^T (-jw)]\}^{1/2}$ on those roots. In an attempt to find the global minimum of $||P||_\infty$ we varied d, k_2, and ρ doing a step by step increment on these parameters.

Results

Using this numerical procedure we found that $\| P \|_\infty$ satisfies

$$\min \| P \|_\infty \leq 1.4144 \tag{7.28}$$

Therefore, a sufficient condition for the uncertain system to be stable is

$$|\delta_i(t)| < \frac{1}{1.4144} = 0.707 \tag{7.29}$$

Note that, as we pointed out at the introduction of this chapter, the Small Gain Theorem admits complex perturbations, therefore the bound (7.29) is valid also when δ_i is a complex perturbation.

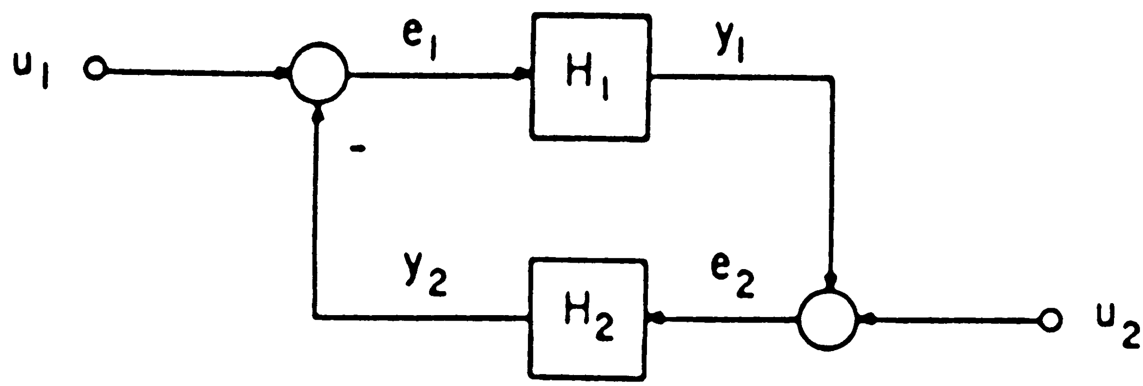

Fig. 7.1: Feedback control system

8. NECESSARY AND SUFFICIENT CONDITIONS FOR QUADRATIC STABILIZABILITY

Introduction

One popular approach to solve the problem of robust stabilization of uncertain systems is the so-called quadratic stabilization approach. Basically this method amount to construct (if possible) a state feedback controller such that the closed loop system admits a fixed (parameter independent) quadratic Lyapunov function. Hollot and Barmish [5] derived necessary and sufficient conditions for quadratic stabilizability of an uncertain linear system where the uncertainty enters only into the system matrix. Barmish [7] showed that the quadratic stabilizability of a linear system which contains uncertainty in both the state and input matrices is equivalent to the quadratic stabilizability of an augmented system which contains uncertainty only into the system matrix. Finally Barmish [18] gave necessary and sufficient conditions for quadratic stabilizability for general linear uncertain systems and proved that when these conditions are satisfied, a nonlinear stabilizing state feedback exists.

One might expect that given the linearity of the system, if there exist a nonlinear quadratically stabilizing state feedback then also exist a linear quadratically stabilizing state feedback law. However, Peterson [24] has shown that, in general, the above conjecture is false. On the other hand, Hollot and Barmish [5] showed that if the input matrix is fixed and known then quadratic stabilizability implies quadratic stabilizability via linear control.

Here we will combine these results to find the necessary and sufficient conditions for linear quadratic stabilizability of the uncertain system (2.6).

Theorems for Quadratic Stabilizability

Consider the linear uncertain system

$$\dot{x}(t) = [A + \Delta A(r(t)]x(t) + B\, u(t) \ , \quad t \geq 0 \tag{8.1}$$

where $x(t) \in R^n$, $u(t) \in R^m$, $r(t) \in \Omega$, ΔA is a continuous matrix function of r and (A,B) is stabilizable.

Let θ be any n x (n-m) left unitary matrix whose columns form a set of basis vectors for the linear space

$$N(B^T) = \{x \in R^n / \; B^T \cdot x = 0\}$$

i.e., it is

$$\theta^T \theta = I \quad \text{and} \quad B^T \theta = 0 \tag{8.2}$$

The following result is from [31].

Theorem 8.1. The system (8.1) is quadratically stabilizable if and only if there exist a positive definite symmetric matrix S such that

$$\theta^T \cdot [(A + \Delta A(r))S + S(A + \Delta A(r))^T] \theta < 0 \tag{8.3}$$

for all $r(t) \in \Omega$, where θ satisfies (8.2). Furthermore, if the above inequality holds then $V(x) = x^T S^{-1} x$ is a quadratic Lyapunov function for the closed loop system and the control law can be taken as

$$K = -\lambda B^T \cdot S^{-1} \tag{8.4}$$

where $\lambda > 0$ is some sufficiently large scalar.

The next theorem allow us to transform the general uncertain linear system (2.1) which includes uncertainty into both the state and system matrices, into form (8.1) where only the system matrix is a function of the uncertain parameter.

Consider the linear uncertain system.

$$\dot{x} = A(r(t))\ x(t) + B(s(t))\ u(t) \tag{8.5}$$

where $x(t) \in R^n$, $u(t) \in R^m$, $r(t) \in \Omega$ and $s(t) \in \Psi$. Also consider the augmented system

$$\dot{x}^+(t) = A^+\ (r(t), s(t)) \cdot x^+(t) + B^+ \cdot u^+(t) \tag{8.6}$$

where

$$A^+(r,s) \hat{=} \left[\begin{array}{c|c} A(r) & B(s) \\ \hline 0 & 0 \end{array}\right] \quad B^+ = \left[\begin{array}{c} 0 \\ \hline I \end{array}\right] \tag{8.7}$$

and $x^+(t) \in R^{n+m}$ and $u^+(t) \in R^m$.

Theorem 8.2. The linear uncertain system (8.5) is quadratically stabilizable via linear control if and only if the linear uncertain system (8.6) is quadratically stabilizable via linear control.

The proof of this theorem can be found in [7].

Application of Theorems for Quadratic Stabilizability

The following development is due to Chen [44].

Consider the linear second-order uncertain system (2.6) where the uncertainties γ_1 and γ_2 satisfy the constraints (2.7) and (2.8). The augmented state and input matrices (8.7) of the corresponding augmented system (8.6) are

$$A^+(\gamma_1,\gamma_2) = \begin{bmatrix} \gamma_1 & 0 & 1 \\ 0 & 0 & -\gamma_2 \\ 0 & 0 & 0 \end{bmatrix}, \quad B^+ = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{8.8}$$

Therefore according to Theorem (8.2) the uncertain system (2.6) and the uncertain system (8.6) where A^+ and B^+ are given by (8.8) are equivalent with respect to the quadratic stabilizability property. A matrix Θ which satisfy the conditions (8.2) is the following

$$\Theta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{8.9}$$

So using Theorem (8.1) we can easily find that a necessary and sufficient condition for quadratic stabilizability is the existence of a symmetric matrix $S = [s_{ij}] > 0$, $i,j = 1,2,3$ such that

$$\begin{bmatrix} 2 \cdot (\gamma_1 \cdot s_{11} + s_{13}) & \gamma_1 s_{12} - \gamma_2 \cdot s_{13} + s_{23} \\ \gamma_1 \cdot s_{12} - \gamma_2 \cdot s_{13} + s_{23} & -2\gamma_2 \cdot s_{23} \end{bmatrix} < 0 \tag{8.10}$$

Since s_{22} and s_{33} does not appear in (8.10) and therefore can be made arbitrarily large, the only requirement for S to be positive definite is $s_{11} > 0$. Also since s_{11} only enters into (1,1) element of the left hand side of (8.10), with coefficient $2\gamma_1 > 0$ this means that if the unequality holds for $s_{11} > 0$ it holds also for $s_{11} = 0$. Therefore, we need only to consider $s_{11} = 0$. So (8.10) takes the form

$$\begin{bmatrix} 2\, s_{13} & \gamma_1\, s_{12} - \gamma_2 \cdot s_{13} + s_{23} \\ \gamma_1 s_{12} - \gamma_2 s_{13} + s_{23} & - 2\gamma_2\, s_{23} \end{bmatrix} < 0 \qquad (8.11)$$

Therefore, must be

$$s_{13} < 0 \qquad (8.12)$$

and

$$- 4 \cdot s_{13} \cdot s_{23} \cdot \gamma_2 - (\gamma_1 \cdot s_{12} - \gamma_2\, s_{13} + s_{23})^2 < 0 \qquad (8.13)$$

This is a homogeneous form and without loss of generality we can assume that $s_{13} = -1$ and after some algebra we have that the necessary and sufficient conditions for quadratic stabilizability is the existence of s_{12} and s_{23} such that

$$s_{12} < 0\,, \quad s_{23} > 0 \qquad (8.14)$$

$$\sqrt{\gamma_2} + \sqrt{s_{23}} > \sqrt{-\, s_{12}\, \gamma_1} \qquad (8.15)$$

and

$$- \sqrt{-s_{12}\gamma_1} < \sqrt{\gamma_2} - \sqrt{s_{23}} < \sqrt{-s_{12}\, \gamma_1} \qquad (8.16)$$

for all γ_1 and γ_2 such that

$$0 < \gamma_1^- \leq \gamma_1(t) \leq \gamma_1^+ \qquad (8.17)$$

and

$$0 < \gamma_2^- \leq \gamma_2(t) \leq \gamma_2^+ \tag{8.18}$$

Therefore (8.15) and (8.16) are equivalent to

$$\sqrt{\gamma_1^-} + \sqrt{s_{23}} > \sqrt{s \cdot \gamma_1^+} \tag{8.19}$$

$$\sqrt{\gamma_1^-} - \sqrt{s_{23}} > - \sqrt{s\ \gamma_1^-} \tag{8.20}$$

and

$$\sqrt{\gamma_2^+} - \sqrt{s_{23}} < \sqrt{s \cdot \gamma_1^+} \tag{8.21}$$

where $s = - s_{12} > 0$. After some calculations we can see that equivalent expressions are

$$2 \sqrt{s\gamma_1^-} + \sqrt{\gamma_2^-} - \sqrt{\gamma_2^+} > 0 \tag{8.22}$$

and

$$2 \sqrt{\gamma_2^-} - \sqrt{s} \left(\sqrt{\gamma_1^+} - \sqrt{\gamma_1^-} \right) > 0 \tag{8.23}$$

Finally, using these two inequalities, a lengthy calculation reveals that a selection of s is possible if and only if

$$4\sqrt{\gamma_1^-}\;\sqrt{\gamma_2^-} > \left(\sqrt{\gamma_1^+} - \sqrt{\gamma_1^-}\right)\left(\sqrt{\gamma_2^+} - \sqrt{\gamma_2^-}\right). \qquad (8.24)$$

Therefore (8.24) is the necessary and sufficient condition for quadratic stabilizability via linear control, of the uncertain system (2.6).

Results

Let's consider the case where

$$\gamma_1^+ = \gamma_2^+ = \gamma^+ \qquad (8.25)$$

and

$$\gamma_1^- = \gamma_2^- = \gamma^- \qquad (8.26)$$

and let's define as previously

$$\gamma_m = \frac{\gamma^+ + \gamma^-}{2} \qquad (8.27)$$

and

$$\bar{\Delta}\gamma = \frac{\gamma^+ - \gamma^-}{2} \qquad (8.28)$$

Then using (8.24) we can easily find that a necessary and sufficient condition for quadratic stabilizability is

$$\bar{\Delta}\gamma < 0.8\ \gamma_m \qquad (8.29)$$

Therefore in this case the quadratic stabilization bound is linear with respect to the mean value of the uncertainty and has the impressive value of 80% of the mean value of the uncertainty.

9. CONCLUSIONS

Summary

In this study we examined the robust stabilization problem of a specific second-order linear uncertain system which does not satisfy the matching conditions. Our objective was to develop a linear constant feedback control law such that the closed loop system is stable for all admissible uncertainties. Different methods were discussed and the maximum bounds provided by each method were given. The results are summarized in Table 9.1.

The Elemental Perturbation Bound method given in Chapter 3 and the Mismatched Decomposition method given in Chapter 5, have as a result uncertain bounds which are of the same order of magnitude (10^{-2}) but extremely small. Therefore, both methods, which provide sufficient conditions for quadratic stabilizability, are very conservative and there is a large number of uncertain systems which are stabilizable but fail to satisfy the conditions of these methods.

The stabilizing condition for slowly time-varying uncertainties, given in Chapter 5 is very interesting because theoretically, using this method any uncertain system can be stabilized, but the resulted bounds on the derivatives of the uncertainties are extremely small (10^{-4} to 10^{-7}) and exclude any practical interest.

The case of piecewise constant periodic uncertainties was treated using Floquet Theory and reveals a very interesting behaviour of the system at different frequencies of the uncertain parameters. We found that the smallest region of stability corresponds to the case where both uncertain functions have the same period T between 0.5 and 1.0, and a shift ΔT equal to -0.2T.

The Small Gain Theorem given in Chapter 7 provides bounds of magnitude of 70% of the mean value of the uncertainty and the necessary and sufficient conditions for Quadratic Stabilizability developed in Chapter 8 reveal a bound of 80% of the mean value of the uncertainty. Therefore the uncertainty bounds given by these two methods are more than twenty times larger than the bounds of the previous methods and this fact gives a measure of the conservativeness of the methods. The bounds given by the Small Gain Theorem have the advantage that they are valid also for the case of complex perturbations.

In the methods of Chapters 3, 4 and 5 the constant feedback gains k_1 and k_2 have the same behaviour, i.e., k_1 increases almost linearly as γ_m increases, and k_2 tends to a constant value between 0.9 and 2.0. This is not surprising because according to the stability conditions for the nominal system (3.15) and (3.16), k_1 has to increase at least linearly with respect to γ_m and k_2. On the other hand, k_1 and k_2 have to remain "small" such that the uncertain terms $\gamma_2 \cdot k_1$ and $\gamma_2 \cdot k_2$, which appear in the closed loop system matrix, remain also "small". The exact asymptotic behaviour of k_1 and k_2 can be found from the corresponding analytic expressions letting γ_m go to infinity.

Motivation for Further Studies

The problem of robust stabilization of linear, time-varying, uncertain system is still an open problem. This study answers some questions for the particular uncertain system used by Petersen but more work needs to be done to determine necessary and sufficient conditions for robust stabilizability, and to answer the question of the existence of a linear stabilizing control, if a nonlinear stabilizing control exists. To be able to do that someone has to go beyond the class of quadratically stabilizing controllers which embody a restricted class of quadratic Lyapunov functions.

METHOD	$\overline{\Delta\gamma}$	$\overline{\Delta\gamma}/\gamma_m$
Elemental Perturbation Bound	0.032	0.210
Mismatched Decomposition	0.015	0.024
Small Gain Theorem	$0.707 \cdot \gamma_m$	0.707
Quadratic Stabilizability	$0.8 \cdot \gamma_m$	0.8
Slowly Varying Uncertainties	No Bound	No Bound
Piecewise Constant Uncertainties	No Bound	No Bound

Table 9.1: Perturbation bounds provided by the different methods

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