TECHNIQUES FOR DISCRETE,
TIME DOMAIN SYSTEM IDENTIFICATION

by

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(ABSTRACT)  

Effective and efficient system identification techniques for discrete, time domain, linear, MIMO, heavily damped modal systems from input/output sequences have been developed and simulated. This will facilitate a better understanding of the possible errors in the estimated model and lead to a more accurate compensator and estimator design. Three different time domain system identification algorithms have been developed in this work. The first algorithm determines the state space model in a pseudo controllable/observable canonical form. The second method is a computational simplification of the Eigensystem Realization Algorithm using pseudo observability and controllability indices. The third algorithm tested is the Pseudo Linear Identification Algorithm (PLID). The PLID algorithm is extensively tested on simulated data. This algorithm is also applied to identify a rectangular plate which gives a realistic idea of the identification capabilities of the PLID algorithm to real measured data.
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1.0 INTRODUCTION

1.1 Preview

System identification is the process by which the input to the system and its measured response or output are used in deriving an analytical model of the system. This analytical model is used to predict the system’s response to different types of inputs. System identification gives a deeper understanding of the system and facilitates better estimator and controller designs. Though not always recognized, system identification is, in practice, statistical estimation. Properties of the system derived from samples of noisy data vary depending on the particular data set. Parametric system identification involves the determination of certain parameters associated with the system which characterize either the physical or modal coordinates (i.e. mass, stiffness, and damping properties) derived from its output response to a carefully designed input. In contrast, Nonparametric system identification aims at identifying a broad empirical input-output map for the structure. Thus, the identification problem is the inverse of the analysis problem for dynamical systems. Identification problems are intrinsically nonlinear compared to analysis problems which are often linear. Applications and the
methods for system identification are numerous. There is no one identification method which is good for every problem. The choice of the method is dependent on several factors such as the size and shape of the system, type of input excitation and frequency content of the output signals. Owing to the breadth and depth of the area of system identification, this investigation focuses only on system identification techniques for discrete time, linear, multi input-multi output systems, having some heavily damped modes.

1.2 Literature Review

During the last twenty years a great deal of research has been done on the subject of system identification, and several textbooks and papers have been written on this subject. System identification literature can be broadly classified into three distinct categories:

1. Identification of Input-Output Relationships
2. Identification of Modal Characteristics
3. Identification of Model Parameters

The classifications are based on assumptions made in the techniques. Input-output relationships do not require linearity, while modal characteristic identification requires both the assumptions of linearity and the existence of specific characteristics. Assumptions of the existence of the model and the knowledge of the structure are needed for model parameter identification. The three approaches
have a significant level of dependency. Model parameters can be derived from modal characteristics or input-output relationships. Input-output relationships can be used to estimate the modal characteristics. Finally, the measured input and output data are used to identify the input-output relationships. There are a significant number of survey papers in the literature. The survey papers, [1] and [2], give an overview of the various methods in experimental modal analysis (identification of modal characteristics). The advances in the areas of data acquisition, transducers, analysis equipments, signal processing methods, excitation techniques and modal parameter estimation procedures over the period of last twenty years has been reviewed in [3]. Another significant review paper, [4], deals with the recent trends and new developments in experimental modal analysis emphasizing instrumentation, measurement methods, evaluation criteria for excitation techniques, parameter estimation methods as well as time and frequency domain methods.

A detailed literature review was conducted only on discrete, time domain identification algorithms. Different time domain methods have been developed and tested for modal parameter identification of physical structures in the area of system identification. These methods start from the transfer function matrix yielding Markov parameters which are used to construct the Hankel matrix, [5]-[7], whence a realization of the state space discrete time model is obtained. The concept of minimal realization using Markov parameters was first developed in [7]. The Ho-Kalman procedure has been modified and extended to develop the Eigensystem Realization Algorithm (ERA), [8]-[9], for identification of modal parameters from measured input and output data as well as accuracy indicators of amplitude coherence and phase colinearity for quantitative identification of the
system and noise modes.

A landmark application of canonical form realization for modal parameter identification of multi input and multi output systems using frequency response functions has been developed in reference [10]. This method is known as the Polyreference technique. The canonical form realization thus obtained can be reduced to a minimal realization, but would no longer be in a canonical form. The use of singular value decomposition for calculating the orthonormal matrices to realize companion form state matrix required in the Polyreference technique proves to yield a computationally well behaved canonical form realization, [11]. The stability of the realization is due to the closeness of the orthonormal matrices to the identity matrices.

Methods that use least squares regression for system identification have been discussed in [12]-[13]. The least squares regression has been rederived and enhanced for modal parameter identification using the free responses of the system, [14]. However, least squares regression does not minimize the order of the system as ERA and some other techniques.

The Least squares, weighted least squares and Bayesian methods, [13], for parametric identification are very common in the literature. The least squares, minimal variance and maximum likelihood estimation methods do not use statistical data of the initial parameter estimates. The Bayesian method process a priori estimates of the parameters and the associated covariance matrix, also known as confidence levels of the estimates. The least squares method processes the measured data in batch form and hence is not suitable for on-line identification.
The estimator based on the Bayesian method processes the data sequentially and hence can be used both for on-line and off-line identification, but the need for initial parameter covariance matrices is an limitation of this method.

Fast Fourier transform (FFT) techniques for system identification are very popular and efficient in determining the frequency content of the measured data. Identification of system parameters from frequency spectrum data requires intensive computing facilities. The FFT method also has difficulty in identifying closely spaced modes with small real parts in their eigenvalues. These modes are lightly damped and difficult to identify as their frequency response curves overlap.

The maximum likelihood estimator (MLE) [56] does not depend on full characterization of noise for estimation of the system parameters. This method computes the covariance of the noise and iteratively updates the covariance at each stage, but requires a large number of computations. The identification computations and covariance calculations comprise a large computational burden when implemented on a digital machine. The MLE and the least squares method are the same when the noise is zero mean Gaussian.

The auto-regressive-moving-average (ARMA) [57] model is widely used for adaptive estimation. This method is popular where it is required to have the control signals for adaptive algorithms and the exact model parameters are not necessary. The ARMA model can be implemented on fast digital computers for on line parameter identification in real time. This algorithm has a limitation of being applicable to systems with small order. Therefore, this technique is applicable to small systems (order 10 or less) and where only approximate system parameters are not necessary.
for generating the appropriate control signals.

1.3 Scope of Work

Effective and Efficient System Identification techniques for discrete-time, linear, multi-input and multi-output, heavily damped modal systems from input/output sequences have been developed and simulated. This has facilitated a better understanding of the possible model errors in the model and a more accurate compensator and estimator design.

Three different discrete, time domain identification techniques have been developed and tested in this research work.

Chapter 2 presents the development of the first system identification algorithm to determine the state-space model in a pseudo controllable/observable canonical form. The advantage of this technique is that it does not require structural identification, i.e. determination of controllability/observability indices. Instead, a MIMO system is obtained in pseudo controllable/observable form, based on a set of admissible of pseudo controllable/observable indices.

Chapter 3 covers the computational simplification of the Eigensystem Realization Algorithm. The ERA algorithm involves singular-value decomposition of the Hankel matrix formed by the Markov Parameters. This Hankel matrix grows as the system order and number of inputs and outputs increase. With the computational simplification a minimal state space representation is obtained by a
simple and appropriate selection (based on pseudo observability and controllability indices) of the columns or rows from the Hankel matrix. The identified state space model is obtained in canonical forms.

The Pseudo-Linear Identification (PLID) algorithm for simultaneous state and parameter identification is addressed in Chapter 4. The problem of joint state and parameter estimation is a non-linear problem for a linear system. The PLID algorithm involves forming an extended system. The extended state vector which comprises of the states and parameters of the system is estimated by a basic discrete time Kalman filter, which performs a linearization of the system about the estimated state, at each iteration. Various systems, with different operating conditions, have been simulated and tested with the PLID algorithm.

The main goal of developing these system identification techniques is for application to experiments with a simply supported rectangular plate as well as for identification of generalized complex modal structures. As a part of this investigation the Pseudo Linear Identification Algorithm is tested on a rectangular steel plate experiment (in cooperation with the Mechanical Engineering department). The results of this experiment are presented in Chapter 5 which are very encouraging.
2.0 DETERMINISTIC IDENTIFICATION

2.1 Introduction

Structural identification is an intrinsic part of multivariable system identification. In the early seventies a MIMO system with \( p \) outputs was usually treated as a set of \( p \) single output (MISO) systems \([13],[15],[33],[35],[36]\) and \([48]\). Another approach requires that the system structure, (i.e., the observability indices) be determined prior to the parameter estimation \([16],[17],[24],[27],[30]\) and \([31]\). The disadvantage of this approach is that the determination of observability indices is very critical, i.e. if the indices have been wrongly determined, the parameter estimates are not consistent.

The advent of pseudocanonical forms (overlapping parameterizations) removes the need of evaluating the observability indices, allowing simultaneous determination of both structure and system parameters \([21]-[23],[25],[26],[32],[34]\) and \([46]\). This is due to the fact that a given system may be represented by several pseudo canonical forms each based on an appropriate set of indices, referred to as pseudo observability indices.
Consider, a linear time-invariant discrete multi input and multi output system. It is known, [25] and [47], that based on a selected set of admissible pseudo observability indices:

\[ \eta = \{ \eta_1, \eta_2, \ldots, \eta_p \} \]

\( p \) being the number of outputs, any \( n^{th} \) order MIMO discrete system could be represented by the following pseudo-observable canonical form:

\[ x(k+1) = A_o x(k) + B_o u(k) \]  \hspace{1cm} (2.1.1)  

\[ y(k) = C_o x(k) + D_o u(k) \] \hspace{1cm} (2.1.2)  

\[ x(0) = x_o \] \hspace{1cm} (2.1.3)

where \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, \) and \( y \in \mathbb{R}^p \) are the state, input and output vectors respectively, while \( A_o, B_o, C_o \) and \( D_o \) are matrices of compatible dimensions. In [47], it has also been shown that the total number of sets of admissible pseudo observability indices is less than or equal to \( I \),

\[ I = \frac{(n-1)!}{(p-1)! (n-p)} \] \hspace{1cm} (2.1.4)

The pair \( (A_o, C_o) \) in the pseudo observable form is characterized by the following structure.
\( A_o = \begin{bmatrix}
0 & 0 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 0 \\
a_{i1} & \cdots & a_{ij} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{in} \\
0 & \cdots & 0 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 0 \\
a_{p1} & \cdots & a_{pj} & \cdots & \cdots & \cdots & \cdots & a_{pn} 
\end{bmatrix} \)  \hspace{2cm} (2.1.5)

\[ C_o = \begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 1 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 
\end{bmatrix} \]  \hspace{2cm} (2.1.6)

From (2.1.5) & (2.1.6), it can be concluded that \( A_o \) has only \( p \) rows with non-zero and non-unity elements. Locations of these rows

\[ s = \{ s_1, s_2, \ldots, s_p \} \]

are uniquely determined by the set of assumed pseudo observability indices. The remaining \((n-p)\) rows of \( A_o \) correspond to the last \((n-p)\) rows of the identity matrix \( I_n \). The first \( p \) rows of \( I_n \) correspond to the rows in \( C_o \). The matrices \( B_o \) and \( D_o \) do not have any specific structure.
\[
B_o = \begin{bmatrix}
    b_{i1} & \ldots & b_{im} \\
    \vdots & \ddots & \vdots \\
    b_{ni} & \ldots & b_{nm}
\end{bmatrix}
\]  

(2.1.7)

\[
D_o = \begin{bmatrix}
    d_{i1} & \ldots & d_{im} \\
    \vdots & \ddots & \vdots \\
    d_{pi} & \ldots & d_{pm}
\end{bmatrix}
\]  

(2.1.8)

2.2 Identification Identity

It is assumed, that the Input-output sequences:

\[
\left\{ u(k), y(k) \right\}; u(k) = u_k, y(k) = y_k; k = 0, 1, 2, \ldots , N - 1 ;
\]

corresponding to an \( n^{th} \) order system are available. The identification method presented here determines a system representation satisfying (2.1.1) - (2.1.3), where \( A_o, B_o, C_o \) and \( D_o \) are given by (2.1.5) - (2.1.8). In order to determine non-zero and non-unity parameters in \( A_o, B_o, C_o, D_o \) and \( x_o \), the following procedure is suggested.

From (2.1.1) - (2.1.3), the following equation is obtained:

\[
Y(k) = Q_o \ z(k) + H \ U(k)
\]  

(2.2.1)
where $x(k) = x_k$ is the $n \times 1$ dimensional state vector. $Y(k)$ and $U(k)$ are $(l+1)p$ and $(l+1)m$ column vectors, while $Q_o$ and $H$ are matrices of the dimensions $(l+1)p \times n$, and $(l+1)p \times (l+1)m$, respectively. $Q_o$ is the observability matrix of the pair $(A_o, C_o)$ with $l = n - p + 1$. The structure of these matrices is as follows:

$$U(k) = \begin{bmatrix} u_k \\ \vdots \\ u_{k+i} \\ \vdots \\ u_{k+l} \end{bmatrix}, \quad Y(k) = \begin{bmatrix} y_k \\ \vdots \\ y_{k+i} \\ \vdots \\ y_{k+l} \end{bmatrix}$$

$$Q_o = \begin{bmatrix} C_o \\ C_o A_o \\ \vdots \\ C_o A_o^i \\ \vdots \\ C_o A_o^l \end{bmatrix} \quad (2.2.2)$$
Elements of vectors $y_{k+1}^T x_k$ and $u_{k+1}^T$ are related to the elements in $A_o$, $B_o$, $C_o$ and $D_o$ by the set of $(l+1)p$ scalar equations given by (2.2.1). It can be verified that among the $(l+1)p$ rows of the observability matrix $Q_o$, there are $n$ rows equal to the $n$ rows of $I_n$ and $p$ rows corresponding to the non-zero, non-unity rows from $A_o$. The locations of the $n$ rows corresponding to $I_n$ and the $p$ rows corresponding to $A_o$ are uniquely determined by the assumed set of admissible pseudo-observability indices.

Now we select from the vector $Y(k)$ subvectors $Y_{1k}$ and $Y_{2k}$ corresponding to the $n$ rows in $Q_o$ containing rows equal to $I_n$ and the $p$ rows in $Q_o$ containing non-zero, non-unity rows from $A_o$. Let the corresponding rows from $H$ be $H_1$ and $H_2$. Then, from equation (2.2.1) the following equations can be formed.

$$Y_{1k} = I x_k + H_1 U_k$$  \hspace{1cm} (2.2.4)
\[ Y_{2k} = A' x_k + H_2 U_k \]  \hspace{1cm} (2.2.5)

where
\[
A' = \begin{bmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & & \vdots \\
a_{p1} & \cdots & a_{pn}
\end{bmatrix}
\hspace{1cm} (2.2.6)
\]

Therefore, eliminating the state vector \( x_k \) from (2.2.4) and (2.2.5) the following Identification Identity is obtained;

\[
Y_{2k} = \begin{bmatrix} B' & A' \end{bmatrix} \begin{bmatrix} U_{1k} \\ \vdots \\ Y_{1k} \end{bmatrix} \hspace{1cm} (2.2.7)
\]

where \( U_{1k} \) contains the first \( m(\eta' + 1) \) elements from \( U_k \) with

\[
\eta' = \max \{ \eta_i \}, \text{ for } i = 1, \ldots, p
\]

and

\[
B' = H_2 - A' H_1
\]

The matrix \( \begin{bmatrix} B' & A' \end{bmatrix} \) will be called the “parameter matrix”, since it is obvious that it depends only on the elements of \( A_o, B_o, C_o \) and \( D_o \). Using a set of \( q \) available measurements, where \( q \) satisfies the condition,

\[
n + (\eta' + 1)m \leq q
\]

we define:

\[
Y = \begin{bmatrix} Y_{2k} & Y_{2(k+1)} & \cdots & Y_{2(k+q-1)} \end{bmatrix} \hspace{1cm} (2.2.8)
\]
\[
Z = \begin{bmatrix}
U_{1k} & \cdots & \cdots & U_{1(k+q-1)}
\hline
Y_{1k} & \cdots & \cdots & Y_{1(k+q-1)}
\end{bmatrix}
\]  \hspace{1cm} (2.2.9)

The dimensions of \(Y\) and \(Z\) are \(p \times q\) and \((m \eta' + m + n) \times q\), respectively. Using the Least Squares Method, [13], from (2.2.7) - (2.2.9), the parameter matrix can be expressed as:

\[
\begin{bmatrix}
B' & A'
\end{bmatrix} = YZ^T (ZZ^T)^{-1}
\]  \hspace{1cm} (2.2.10)

where, in the case of a sufficiently rich input signal \(u(k)\) and admissible set of pseudo observability indices, \(Z\) is a full row rank matrix. Obviously the matrix \(Z\) depends on the choice of an admissible set of pseudo observability indices. Therefore, among all possible sets of admissible pseudo observability indices it is advisable to use the set which leads to the smallest condition number of the matrix \(Z\). Since from (2.2.10) it is possible to determine \(A'\), the matrix \(A_o\) is determined directly from (2.2.10) and (2.1.5), while \(C_o\) is known to have the structure as in (2.1.6).

**Determination of \(B_o\)**

It can be easily shown that \(B_o = Q_c^e B^x\)

where

\[
Q_c^e = \begin{bmatrix}
B_e & \cdots & A_oB_e & \cdots & A_o\eta'
\end{bmatrix}
\]

The columns \(b_{ej}\) of \((n \times p)\) "equivalent input matrix", \(B_e\), contain \((n - 1)\) zeros and
only one unity, whose location is determined by the integers \( s_i \) of the set \( s \) defining locations where \( A_o \) has non-zero, non-unity rows. \( B^* \) is formed by the following partitioning of \( B' \).

\[
B' = \begin{bmatrix}
B'_0 & B'_1 & \cdots & B'_{\eta'}
\end{bmatrix}
\]  \hspace{1cm} (2.2.11)

\[
B^* = \begin{bmatrix}
B'_0 \\
\vdots \\
B'_1 \\
\vdots \\
\vdots \\
B'_{\eta'}
\end{bmatrix}
\]  \hspace{1cm} (2.2.12)

Note that the dimensions of \( B' \), \( B^* \) and \( B'_i \), \( i = 0, \ldots, \eta' \) are \( p \times m(\eta'+1) \), \( p(\eta'+1) \times m \) and \( p \times m \), respectively.

**Determination of \( D_o \)**

From the definition of a transfer function matrix we have:

\[
G(z) = D^{-1}(z)N(z) = C_o(Iz - A_o)^{-1}B_o + D_o
\]  \hspace{1cm} (2.2.13)

or

\[
D_o = D^{-1}(z)D(z) - C_o(Iz - A_o)^{-1}B_o
\]  \hspace{1cm} (2.2.14)

where \( N(z) \) and \( D(z) \) are \((p \times m)\) and \((p \times p)\) co-prime polynomial matrices, [6], where

\[
N(z) = \sum_{i=0}^{\eta'} N_i z^i
\]
\[ D(z) = \sum_{i=0}^{n'} D_i z^i \]

Since \( D_o \) on the l.h.s of (2.2.14) does not depend on the Z-transform variable \( z \), it could be calculated by evaluating the r.h.s of (2.2.14) for an arbitrary value of \( z \). Since a discrete time system has no eigenvalue at \( z = 0 \), it follows that \( D(0) \) and \( A_o \) are always non-singular. From [6], it can be concluded that \( N_o = B_o' \); see equation (2.2.11) and \( D(0) \) is given by the first \( p \) columns of \( A' \), leading finally to:

\[ D_o = D^{-1}(0) N_o + C_o A_o^{-1} B_o \]

(2.2.15)

**Determination of Initial Condition \( x_o \)**

The initial condition vector \( x_o \), corresponding to (2.1.1) - (2.1.3), could be calculated directly from (2.2.4), by setting \( k = 0 \) i.e:

\[ x_o = Y_{10} - H_1 U_0 \]

(2.2.16)

**2.3 Location Vector Algorithm**

An algorithm which, using the assumed set of admissible pseudo observability indices \( \eta \), uniquely determines the location vectors \( s, s_c, \bar{h} \) and \( r \), specifying

- a) the locations \( s \) of the \( p \) non zero, non unity rows in \( A_o \),
- b) the locations \( s_c \) of the last \( n-p \) rows of \( I_n \) in \( A_o \),
c) the locations $h$ of the $n$ rows of $A_n$ in the observability matrix $Q_o$.

d) the locations $r$ of the $p$ rows of non zero, non unity rows of $A_o$ in $Q_o$.

is given below.

1. Define a set $\eta = \{\eta_1, \ldots, \eta_p\}$ of admissible pseudo observability indices where $p$ is the number of outputs of the system.

2. Set $n = \sum_{i=1}^{p} \eta_i$; $n =$ system order, $\eta_M = \max \{\eta_i\}$, $\eta_p = (\eta_M + 1)p$.

3. Set $i = 1$.

4. Set $k = i$.

5. For $j = 1$ through $\eta_p$, set $V(k) = \eta_i + 1 - j$, $k = k + p$.

6. Set $i = i + 1$.

7. If $i > p$, go to 8, else go to 4.

8. Set $i_1 = 0$, $i_2 = 0$, $k = 1$.

9. If $V(k) \leq 0$, go to 11, else go to 10.

10. Set $i_1 = i_1 + 1$, $h(i_1) = k$.

11. If $V(k) \neq 0$, go to 13, else go to 12.

12. Set $i_2 = i_2 + 1$, $r(i_2) = k$.

13. Set $k = k + 1$.

14. If $k \leq \eta_p$, go to 9, else go to 15.

15. Set $k = p + 1$, $i_a = 0$, $i_\alpha = 0$, $i_\gamma = 0$.

16. Set $i_p = i_p + 1$.

17. If $V(k) < 0$, set $i_p = i_p - 1$ and go to 21, else go to 18.

18. If $V(k) \neq 0$, go to 19, else go to 20.

19. Set $i_1 = i_1 + 1$, $s_c(i_1) = i_p$, go to 21.

20. Set $i_a = i_a + 1$, $s_c(i_a) = i_p$.

21. Set $k = k + 1$. 
22. If $k \leq \eta_p$, go to 16, else, STOP.

A more intuitive explanation of the above algorithm is given by [20]. Consider, for example, the case where $p = 3$ and $\eta = \{1, 4, 2\}$. Then, the following location vectors are obtained:

\[
\begin{align*}
\mathbf{s} &= \{1, 5, 7\} \\
\mathbf{s}_c &= \{2, 3, 4, 6\} \\
\mathbf{h} &= \{1, 2, 3, 4, 5, 6, 8, 11\} \\
\mathbf{r} &= \{4, 9, 14\}
\end{align*}
\]

In case of another set of admissible pseudo observability indices, $\eta = \{2, 3, 2\}$, the location vectors obtained are as follows:

\[
\begin{align*}
\mathbf{s} &= \{4, 6, 7\} \\
\mathbf{s}_c &= \{1, 2, 3, 5\} \\
\mathbf{h} &= \{1, 2, 3, 4, 5, 6, 8\} \\
\mathbf{r} &= \{7, 9, 11\}
\end{align*}
\]

Note that the location set $\mathbf{s}_c$ is the "complement" of the set $\mathbf{s}$ in the sense that among $n$ integers in the set $I = 1, 2, \ldots, n$, some $p$ integers are in $\mathbf{s}$ while the remaining $n-p$ integers are in $\mathbf{s}_c$.

### 2.4 Algorithm for Identification

According to the previous discussion the following algorithm may be formulated:
1. Given Input-Output sequences \( \{ u(k), y(k) \} \), \( k = 0, 1, \ldots, N - 1 \), corresponding to an \( n^{th} \) order linear MIMO discrete system,

\[
u(k) \in \mathbb{R}^m, \ y(k) \in \mathbb{R}^p
\]  

(2.4.1)

where \( 1 \leq m \leq n \) and \( 1 \leq p \leq n \).

2. Assume a set \( \eta = \{ \eta_1, \eta_2, \ldots, \eta_p \} \) of admissible pseudo observability indices which satisfy the condition

\[
n = \sum_{j=1}^{p} \eta_j \ ; \ 1 \leq \eta_j \leq n - p + 1
\]  

(2.4.2)

Let \( \eta' = \max \{ \eta_1, \eta_2, \ldots, \eta_p \} \)

Build the \( m(\eta' + 1) \times N_1 \) matrix \( U \), where \( N_1 \leq N - (\eta' + 1) \), defined by:

\[
U = \begin{bmatrix}
U(0) & \ldots & U(N_1 - 1) \\
U(1) & \ldots & U(N_1) \\
\vdots & \vdots & \vdots \\
U(\eta') & \ldots & U(N_1 + \eta' - 1)
\end{bmatrix}
\]  

(2.4.3)

If the matrix \( U \) is of the full row rank, i.e

\[
\rho | U | = (\eta' + 1)m
\]  

(2.4.4)

then the input sequence is "sufficiently rich" and is capable of exciting all \( n \) modes of the system to be identified.

If \( \rho | U | < (\eta' + 1)m \), then either:

(a) Select another set of pseudo observability indices having a smaller value of \( \eta' \) or
(b) Select a different, more rich input sequence which would satisfy the richness condition (2.4.4).

3. Using the location vector algorithm determine the following:

(a) A set of integers, \( s = \{s_1, \ldots, s_p\} \) \hspace{1cm} (2.4.5)
corresponding to the locations of non-zero, non-unity rows

\[
a_i = |a_{i1}, \ldots, a_{in}|
\]

(2.4.6)
of the matrix \( A_o \) of the pseudo observable canonical form.

(b) A set of \( (n-p) \) integers \( s_c \) (complement to the set \( s \)) corresponding to the locations of the last \( (n-p) \) rows from the identity matrix \( I_n \) in the matrix \( A_o \).

(c) A set of \( n \) integers: \( h = \{h_1, \ldots, h_n\} \) \hspace{1cm} (2.4.7)
corresponding to the location of the \( n \) rows of \( I_n \) in the observability matrix:

\[
Q_o = \begin{bmatrix}
C_o \\
\vdots \\
C_o A_o \\
\vdots \\
C_o A_o^{n-p+1}
\end{bmatrix}
\]

(2.4.8)

(d) A set of \( p \) integers: \( r = \{r_1, \ldots, r_p\} \) \hspace{1cm} (2.4.9)
corresponding to the locations of the rows \( a_j \) in the observability matrix \( Q_o \).
4. Build the $p(\eta' + 1) \times N_1$ matrix $Y$, defined by,

$$
Y = \begin{bmatrix}
    Y(0) & \cdots & Y(N_1 - 1) \\
    Y(1) & \cdots & Y(N_1) \\
    \vdots & \ddots & \vdots \\
    Y(\eta') & \cdots & Y(N_1 + \eta' - 1)
\end{bmatrix}
$$

(2.4.10)

5. From the matrix $Y$ select the $(n \times N_1)$ matrix $Y_1$ with $n$ rows corresponding to the elements in $\mathcal{h}$. This selection could be represented by the following premultiplication;

$$
Y_1 = S_\mathcal{h} \times Y
$$

(2.4.11)

where $S_\mathcal{h}$ is the "selector matrix" of dimension $(n \times p(\eta'+1))$. The $k^{th}$ row of $S_\mathcal{h}$ has $(p(\eta'+1) - 1)$ zeros and a unity at the location specified by the integer $h_k$ of the set $\mathcal{h}$.

6. From the matrix $Y$ select the $(p \times N_1)$ matrix $Y_2$ with $p$ rows corresponding to the elements in $\mathcal{r}$. This selection could be represented by the following premultiplication;

$$
Y_2 = S_\mathcal{r} \times Y
$$

(2.4.12)

where $S_\mathcal{r}$ is the "selector matrix" of dimension $(p \times p(\eta'+1))$. The $k^{th}$ row of $S_\mathcal{r}$ has $(p(\eta'+1) - 1)$ zeros and a unity at the location specified by the integer $r_k$ of the set $\mathcal{r}$.

7. Build the matrix $Z$;
\[ Z = \begin{bmatrix} U \\ \vdots \\ Y_1 \end{bmatrix} \] (2.4.13)

If the matrix \( Z \) is of full row rank, then the selected set of pseudo observability indices is admissible.

8. Using the Least Squares algorithm determine the \( p \times (m\eta' + m + n) \) "parameter matrix" \( Q \), where \( Q = \begin{bmatrix} B' & A' \end{bmatrix} \), satisfying

\[ Y_2 = Q \times Z = \begin{bmatrix} B' & A' \end{bmatrix} \begin{bmatrix} U \\ \vdots \\ Y_1 \end{bmatrix} \] (2.4.14)

9. \( B' \) and \( A' \) are \( p \times (\eta' + 1)m \) and \( (p \times n) \) matrices, respectively. The \( p \) rows in the matrix \( A' \) correspond to the non-zero, non-unity rows, (2.4.6), of the matrix \( A_o \). Thus, having determined the parameter matrix \( Q \), i.e the matrix \( A' \),

\[ A' = \begin{bmatrix} a_1 \\ \vdots \\ \vdots \\ a_p \end{bmatrix} \] (2.4.15)

the matrix \( A_o \) in the pseudo observable canonical form can be easily determined using the sets \( S \) and \( S_c \).

10. As it has been mentioned earlier, the matrix \( C_o \) in the pseudo observable canonical form is always of the structure, \( C_o = [I_p : 0] \), i.e it contains the first \( p \) rows from the identity matrix \( I_n \).

The input matrix \( B_o \) is to be determined by the following procedure.
11. The \( p \times (\eta' + 1)m \) matrix \( B' \) is partitioned into \((\eta' + 1), (p \times m)\) submatrices \( B'_0, \ldots, B'_{\eta'} \), i.e:

\[
B' = [B'_0 : \ldots : B'_{\eta'}]
\]  

(2.4.16)

12. Let the columns \( b_{e_j} \) of the \((n \times p) B_e\), "equivalent input matrix", contain \((n - 1)\) zeros and only one unity, whose location is determined by the integer \( s_k \) of the set \( s \).

13. Build the \((n \times (\eta' + 1)p)\) controllability matrix \( Q_c \) of the pair \( \{A_o, B_{e_j}\} \):

\[
Q_c = [B_e : A_o B_e : \ldots : A_o B_{e_j}]
\]  

(2.4.17)

14. The input matrix \( B_o \) in the pseudo observable canonical form can now be easily determined from

\[
B_o = Q_c B^*
\]  

(2.4.18)

\[
B^* = \begin{bmatrix}
B'_0 \\
\vdots \\
B'_1 \\
\vdots \\
B'_n
\end{bmatrix}
\]  

(2.4.19)

The "direct path" feed through matrix \( D_o \) in the pseudo observable canonical form is determined in the following way.
15. We know that,

\[ G(z) = C_o(Iz - A_o)^{-1}B_o + D_o = D^{-1}(z)N(z) \] (2.4.20)

or

\[ D_o = D^{-1}(z)N(z) - C_o(Iz - A_o)^{-1}B_o \]

Since the matrix \( D_o \) does not depend on \( z \), \( D_o \) can be determined by evaluating the above equation at \( z = 0 \), i.e:

\[ D_o = D^{-1}(0)N(0) + C_oA_o^{-1}B_o \] (2.4.21)

In equation (2.2.15) it has been shown that \( N(0) = N_o = B'_o \) and \( D(0) = \tilde{A}_1 \), where \( \tilde{A}_1 \) contains the first \( p \) columns from \( A' \), i.e:

\[ A_r = [\tilde{A}_1 : \tilde{A}_2] \] (2.4.22)

16. The initial condition vector \( x(0) = x_o \) can be determined by the following equation:

\[ x_o = Y_{10} - H_1U_o \] (2.4.23)

where \( Y_{10} \) is the first column of the matrix \( Y_1 \) and \( U_o \) is the first column of the matrix \( U \), while \( H_1 \) is formed from \( H \) using \( S_h \), (2.4.11), as follows:

\[ H_1 = S_h \times H \] (2.4.24)
2.5 Illustrative Example

In order to generate the input-output sequence \( \{ u(k), y(k) \} \) which will be used in the identification algorithm the following example has been considered:

\[
\dot{x} = Ax + Bu
\]

\[(2.5.1)\]

\[y = Cx + Du; \quad x(0) = x_0\]

where

\[
A = \begin{bmatrix}
0.1 & 0 & 0 & 0 & 0 \\
0 & 0.2 & 0 & 0 & 0 \\
0 & 0 & 0.3 & 0 & 0 \\
0 & 0 & 0 & 0.4 & 0 \\
0 & 0 & 0 & 0 & 0.5
\end{bmatrix},
B = \begin{bmatrix}
1 & 1 \\
0.01 & 1 \\
0.02 & 1 \\
0 & 1 \\
0 & 1
\end{bmatrix},
C = \begin{bmatrix}
1 & 0.01 & 0 & 0 & 0 \\
0 & 0 & 1 & 0.01 & 0 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

\[(2.5.2)\]

\[
D = \begin{bmatrix}
0 & 0 \\
1 & 0 \\
0 & 0
\end{bmatrix},
x_0 = \begin{bmatrix}
1 \\
2 \\
1 \\
2
\end{bmatrix}
\]

i.e \( n = 5, m = 2 \) and \( p = 3 \). Random \( m \times N \), \( N = 17 \) matrix is used for the input sequence \( u(k) \). The output sequence \( y(k) \) is obtained by calculating the response of (2.5.1) to the selected random input sequence \( u(k) \). According to [47], it follows that in the case of \( n = 5 \) and \( p = 3 \), the total number of possible pseudo observable indices are \( I = 6 \). These sets of indices are:

\[
\{ 1, 1, 3 \}, \{ 1, 2, 2 \}, \{ 2, 1, 2 \}, \{ 2, 2, 1 \}, \{ 1, 3, 1 \} \text{ and } \{ 3, 1, 1 \}.
\]
In case of (2.5.2), it can be concluded that the sets \( \{ 3, 1, 1 \} \) and \( \{ 1, 3, 1 \} \) are non-admissible and that the set \( \{ 2, 2, 1 \} \) corresponds to the unique set of pseudo observability indices \([6], [28], [47]\). As was shown in [47], the non-admissibility of the sets \( \{ 3, 1, 1 \} \) and \( \{ 1, 3, 1 \} \) follows from the fact that individually the first and the second component \( y_1(k) \) and \( y_2(k) \) of the output vector \( y(k) \) "see" only two system modes each. Thus, pseudo observability indices \( n_1 \) and \( n_2 \) corresponding to \( y_1(k) \) and \( y_2(k) \) cannot be greater than two. Using the available set of input-output sequences and applying the above mentioned algorithm for all admissible sets of pseudo observable indices, we get:

**Case I:**

Using \( \eta^1 = \{ 1, 1, 3 \} \), the location vector algorithm gives:

\[
\begin{align*}
\hat{s}^1 &= \{ 1, 2, 5 \}, \quad \hat{s}_c^1 = \{ 3, 4 \},
\hat{b}^1 &= \{ 1, 2, 3, 6, 9 \},
\hat{r}^1 &= \{ 4, 5, 12 \}
\end{align*}
\]

After building the identification identity (2.2.7) and (2.2.10), the following parameter matrix \( Q \) is obtained as in (2.4.14), where \( B' \) and \( A' \) are as given below,

\[
A' = \begin{bmatrix}
0.098 & 0.000 & 0.003 & -0.015 & 0.017 \\
0.002 & 0.299 & -0.005 & 0.036 & -0.051 \\
-0.012 & 0.002 & 0.041 & -0.383 & 1.103
\end{bmatrix}
\]

\[
B' = \begin{bmatrix}
1.014 & 1.061 & -0.017 & -0.084 & 0.000 & 0.000 \\
-0.310 & 0.908 & 1.052 & 0.254 & 0.000 & 0.000 \\
0.285 & 0.808 & -1.028 & -4.015 & 1.030 & 5.000
\end{bmatrix}
\]
Thus, the system matrix $A_c$ in the pseudo observable canonical form is,

$$A_c = \begin{bmatrix}
0.098 & 0.000 & 0.003 & -0.015 & 0.017 \\
0.002 & 0.299 & -0.005 & 0.036 & -0.051 \\
0.000 & 0.000 & 0.000 & 1.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 1.000 \\
-0.012 & 0.002 & 0.041 & -0.383 & 1.103
\end{bmatrix}$$

The equivalent input matrix $B_c$ used in calculating $B_o$ is,

$$B_c = \begin{bmatrix}
1.0 & 0.0 & 0.0 \\
0.0 & 1.0 & 0.0 \\
0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 1.0
\end{bmatrix}$$

leading to $B_o =$

$$B_o = \begin{bmatrix}
1.000 & 1.010 \\
0.020 & 1.010 \\
1.030 & 5.000 \\
0.108 & 1.500 \\
0.012 & 0.550
\end{bmatrix}$$

The initial condition vector calculated using step (16) of the algorithm and the condition number of the matrix $Z$, $N_c$ are;

$$x_o^T = \begin{bmatrix}
1.020 \\
1.020 \\
7.000 \\
2.100 \\
0.750
\end{bmatrix}$$

and $N_c = 2.282 \times 10^5$

**Case II**: $r^2 = \{1, 2, 2\}$:

$$\mathbf{s}^2 = \{1, 4, 5\}, \mathbf{s}_c^2 = \{2, 3\}, \mathbf{h}_c^2 = \{1, 2, 3, 5, 6\}, \mathbf{r}_c^2 = \{4, 8, 9\}$$

$$A' = \begin{bmatrix}
0.099 & 0.099 & 0.002 & -0.331 & -0.003 \\
0.000 & -0.120 & 0.000 & 0.700 & 0.000 \\
0.041 & 5.881 & -0.101 & -19.700 & 0.701
\end{bmatrix}$$
\[
B' = \begin{bmatrix}
0.912 & 1.361 & 0.331 & 0.000 & 0.000 & 0.000 \\
0.112 & -0.493 & -0.680 & 1.010 & 1.000 & 0.000 \\
-6.102 & 17.857 & 20.698 & 5.000 & 0.000 & 0.000 \\
\end{bmatrix}
\]

\[
A_o = \begin{bmatrix}
0.099 & 0.099 & 0.002 & -0.331 & -0.003 \\
0.000 & 0.000 & 0.000 & 1.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 1.000 \\
0.000 & -0.120 & 0.000 & 0.700 & 0.000 \\
0.041 & 5.881 & -0.101 & -19.700 & 0.701 \\
\end{bmatrix}
\]

\[
B_c = \begin{bmatrix}
1.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 \\
0.0 & 1.0 & 0.0 \\
0.0 & 0.0 & 1.0 \\
\end{bmatrix}
\]

\[
B_o = \begin{bmatrix}
1.000 & 1.010 \\
0.020 & 1.610 \\
1.030 & 5.000 \\
0.006 & 0.304 \\
0.108 & 1.500 \\
\end{bmatrix}
\]

\[
\tau_o^T = [1.020 \ 1.020 \ 7.000 \ 0.308 \ 2.100 ] \text{ and } N_c = 1.058 \times 10^6
\]

**Case III:** \( \eta^3 = \{2, 1, 2\} : \)

\( s^3 = \{2, 4, 5\}, \ s_c^3 = \{1, 3\}, \ h^3 = \{1, 2, 3, 4, 6\}, \ r^3 = \{5, 7, 9\} \)

\[
A' = \begin{bmatrix}
0.298 & 0.298 & 0.005 & -3.020 & -0.010 \\
-0.020 & 0.000 & 0.000 & 0.300 & 0.000 \\
-5.820 & 0.020 & -0.201 & -59.400 & 0.902 \\
\end{bmatrix}
\]
\[
B' = \begin{bmatrix}
2.753 & 4.112 & 1.000 & 0.000 & 0.000 & 0.000 \\
-0.200 & -0.201 & 1.000 & 1.010 & 1.000 & 0.000 \\
-60.300 & 63.000 & 1.030 & 5.010 & 0.000 & 0.000 \\
\end{bmatrix}
\]

\[
A_o = \begin{bmatrix}
0.000 & 0.000 & 0.000 & 1.000 & 0.000 \\
0.298 & 0.298 & 0.005 & -3.020 & -0.010 \\
0.000 & 0.000 & 0.000 & 0.000 & 1.000 \\
-0.020 & 0.000 & 0.000 & 0.300 & 0.000 \\
-5.820 & 0.020 & -0.201 & -59.400 & 0.902 \\
\end{bmatrix}
\]

\[
B_e = \begin{bmatrix}
0.0 & 0.0 & 0.0 \\
1.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 \\
0.0 & 1.0 & 0.0 \\
0.0 & 0.0 & 1.0 \\
\end{bmatrix}
\]

\[
B_o = \begin{bmatrix}
1.000 & 1.010 \\
0.020 & 1.010 \\
1.030 & 5.000 \\
0.100 & 0.102 \\
0.108 & 1.500 \\
\end{bmatrix}
\]

\[
z_o^T = [ 1.020 \ 1.020 \ 7.000 \ 0.104 \ 2.100 ] \text{ and } N_c = 3.121 \times 10^6
\]

**Case IV: \( \eta^4 = \{ 2, 2, 1 \} \):**

\[
\beta^4 = \{ 3, 4, 5 \}, \ q_c^4 = \{ 1, 2 \}, \ t^4_c = \{ 1, 2, 3, 4, 5 \}, \ x^4 = \{ 6, 7, 8 \}
\]

\[
A' = \begin{bmatrix}
29.200 & 29.200 & 0.500 & -296.00 & -98.000 \\
-0.020 & 0.000 & 0.000 & 0.300 & 0.000 \\
0.000 & -0.120 & 0.000 & 0.000 & 0.700 \\
\end{bmatrix}
\]
\[ B' = \begin{bmatrix} 269.8 & 402.9 & 97.900 & 0.000 & 0.000 & 0.000 \\ -0.200 & -0.201 & 1.000 & 1.010 & 0.000 & 0.000 \\ 0.112 & -0.403 & -0.686 & 1.010 & 1.000 & 0.000 \end{bmatrix} \]

\[ A_o = \begin{bmatrix} 0.000 & 0.000 & 0.000 & 1.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 1.000 \\ 29.200 & 29.200 & 0.500 & -296.00 & -98.000 \\ -0.020 & 0.000 & 0.000 & 0.300 & 0.000 \\ 0.000 & -0.120 & 0.000 & 0.000 & 0.700 \end{bmatrix} \]

\[ B_e = \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.9 & 1.0 \end{bmatrix} \quad B_o = \begin{bmatrix} 1.000 & 1.010 \\ 0.020 & 1.010 \\ 1.029 & 5.000 \\ 0.100 & 0.102 \\ 0.006 & 0.304 \end{bmatrix} \]

\[ x_o^T = [1.020 \ 1.020 \ 7.000 \ 0.104 \ 0.304 ] \text{ and } N_c = 7.740 \times 10^5 \]

Since the matrix \( Z \) in the case I has the smallest condition number, the set \{1,1,3\} is the most convenient set of pseudo observability indices to be used in calculating a state space representation. This can also be concluded by comparing \( A_o \) in the above treated cases. The small condition number of the matrix \( Z \) results in the matrix \( A_o \) having elements with relatively small absolute values. Note that the set \{1, 1, 3\} does not correspond to the unique observability indices \{2, 2, 1\}.
3.0 SIMPLIFICATION IN ERA

3.1 Introduction

Consider the discrete-time representation of a dynamical system of order \( n \), the number of inputs and outputs being \( m \) and \( p \), respectively.

\[
x(k + 1) = A\, x(k) + B\, u(k)
\]

\[
y(k) = C\, x(k) ; \quad k = 0, 1, 2, \ldots \ldots \tag{3.1.1}
\]

Since experimental data are discrete in nature, the above set of equations constitute the formulation for system identification of linear, time-invariant dynamical systems. The impulse response characteristics for each input element are combined to obtain the impulse response function matrix \( Y \).

\[
Y(0) = CB, \ Y(1) = CAB, \ Y(2) = CA^2B, \ldots, \ Y(k) = CA^kB, \ldots \tag{3.1.2}
\]

These \( ( p \times m ) \) constant matrices \( Y_i \) are known as Markov parameters which are obtained from the experimental data through impulse responses or transfer
functions and can be used for deriving mathematical models of dynamical systems. The set of matrices \(\{A, B, C\}\) representing the system characteristics can be used to determine the system responses and this representation is called a realization of the system. A given system can have an infinite number of realizations giving the same response for any particular input.

A model with the smallest state space dimension among all the possible realizations having the same input-output relations is called the "Minimal realization", \(R = \{A, B, C\}\), of the system. The use of a generalized Hankel matrix, which is composed of Markov parameters, \(Y_i\), is a beginning step in the time domain system identification of structures.

In their seminal paper [7], Ho and Kalman introduced the principles of minimum realization theory. They suggested a procedure for calculating a minimal state space representation \(R = \{A, B, C\}\) of a linear MIMO system from noise free data, using the generalized Hankel matrix formed by the Markov Parameters \(Y_i\), defined by

\[
Y_i = CA^iB \quad ; \quad i = 0, 1, \ldots, N
\]  

Since publication of that paper a number of papers and textbooks, [6], [7], [13], [37]-[41], have referred to this minimal realization procedure and have suggested slightly modified versions. Recently, particularly in the area of large flexible structures, this algorithm has been modified and substantially extended to develop the Eigensystem Realization Algorithm (ERA), [8], [42], [43] and [45], to identify the modal parameters from noisy measurement data.
The basic steps in the Eigensystem Realization Algorithm are as given below.

1: Define Markov Parameters, i.e. \( p \in m \) matrices \( Y_i \), of an \( n^{th} \) order MIMO system with \( m \) inputs and \( p \) outputs.

2: Build the \( (r \times r m) \) Hankel matrix \( H_r \) given by,

\[
H_r = \begin{bmatrix}
Y_o & \cdots & Y_{r-1} \\
\vdots & & \vdots \\
Y_{r-1} & \cdots & Y_{2r-2}
\end{bmatrix}
\]  

for an arbitrary integer \( r \).

3: Increase the integer \( r \) sequentially, until the rank of \( H_r \) does not increase, i.e. when

\[
\rho \mid H_r \mid = \rho \mid H_{r+1} \mid
\]  

then

\[
n = \rho \mid H_r \mid
\]

represents the order of a minimal state space representation \( \{ A, B, C \} \), satisfying (3.1.3).

4: Perform the singular value decomposition (SVD) of \( H_r \), i.e. calculate nonsingular matrices \( U \) and \( V \) and the singular values \( \sigma_i \) of \( H_r \) along the diagonal matrix \( \Sigma_n \) satisfying
\[ H_r = U \begin{bmatrix} \Sigma_n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} V^T \text{ where } \Sigma_n = \text{diag} \{ \sigma_1, \ldots, \sigma_n \}, \sigma_i > 0 \quad (3.1.7) \]

5: The matrices \( A, B, C \) in a minimal realization are given by

\[
A = (\Sigma_n)^{-1/2} U_1^T \tilde{H}_r V_1 (\Sigma_n)^{-1/2}
\]

\[
B = (\Sigma_n)^{1/2} V_{11}
\]

\[
C = U_{11} (\Sigma_n)^{1/2}
\]

where

\[
\tilde{H}_r = \begin{bmatrix}
Y_1 & \cdots & Y_r \\
\vdots & \ddots & \vdots \\
Y_r & \cdots & Y_{2r-1}
\end{bmatrix}
\]

while \((r_p \times n)\), \((r_m \times n)\), \((n \times m)\) and \((p \times n)\) matrices \( U_1, V_1, U_{11} \) and \( V_{11} \), respectively, are defined by

\[
U = [U_1 : U_2]
\]

where \( U_1 = \begin{bmatrix} U_{11} \\ \vdots \\ U_{21} \end{bmatrix} \)

\[
V = [V_1 : V_2]
\]

where \( V_1 = \begin{bmatrix} V_{11} \\ \vdots \\ V_{21} \end{bmatrix} \)

In other words \( U_1 \) contains the first \( n \) columns from \( U \) and \( V_1 \) contains the first \( n \)
columns from $V$. Similarly, $U_{11}$ contains the first $p$ rows from $U_1$ and $V_{11}$ contains the first $m$ columns from $(V_1)^T$. The important property of the above algorithm, is in the fact that it makes use of orthogonal matrices $U$ and $V$.

\section*{3.2 Simplified ERA.}

The computational simplification in Eigensystem Realization Algorithm suggested in this chapter is based on the use of pseudo controllability indices (p.c.i) and pseudo observability indices (p.o.i), introduced in Chapter 2. Consequently, as explained in [25], [44], [46] and [47], there are more equivalent state space representations in pseudo controllable/observable forms corresponding to the given Markov parameters, $Y_i$. As it is explained in later sections, after calculating the rank of $H_r$, i.e order $n$ of a minimal realization (3.1.6), (steps 1, 2, 3 of the original ERA algorithm), the only calculations that should be done in the case of representing the system in a pseudo controllable form are:

i. Select $n$ appropriate columns from $H_r$ with locations $\nu_1^c, \ldots, \nu_n^c; \nu_i^c < \nu_{i+1}^c$, into the $(rp \times n)$ matrix $H_1$.

ii. Select $n$ columns from $H_r$ with locations $m + \nu_1^c, \ldots, m + \nu_n^c$, into the $(rp \times n)$ matrix $H_2$.

iii. The system matrix $A_c$ in a pseudo controllable form is given by the least square solution of

$$H_1 A_c = H_2$$
or

\[ A_c = (H_1^T H_1)^{-1} H_1^T H_2 \]  

(3.2.1)

The input matrix \( B_c \) is fixed and is always given by

\[
B_c = \begin{bmatrix}
I_m \\
\ldots \\
O
\end{bmatrix}
\]  

(3.2.2)

while the output matrix \( C_c \) contains the first \( p \) rows from \( H_1 \).

Similarly, in order to obtain a state space representation in a pseudo observable form, the procedure is dual to the procedure for \( \{A_c, B_c, C_c\} \), i.e.

i. From the Hankel matrix \( H_r \) select \( n \) rows with locations \( \nu_1^o, \ldots, \nu_n^o; \nu_i^o < \nu_{i+1}^o \) into the (\( n \times rm \)) matrix \( H_1 \).

ii. From \( H_r \) select \( n \) rows with locations \( p + \nu_1^o, \ldots, p + \nu_n^o \) into the (\( n \times rm \)) matrix \( H_2 \).

iii. The system matrix \( A_o \) in a pseudo observable form is given by the least square solution of

\[
A_o H_1 = H_2
\]

or

\[
A_o = H_2 H_1^T (H_1^T H_1)^{-1}
\]  

(3.2.3)
$B_o$ contains the first $m$ columns from $H_1$, while $C_o$ is fixed and is always given by

\[
C_o = \begin{bmatrix} I_p & 0 \end{bmatrix} \tag{3.2.4}
\]

Obviously, the calculation required by (3.2.1) or (3.2.3), is much simpler than those required in the step 5 of the original ERA algorithm, equations (3.1.7)-(3.1.9).

It can be easily shown that columns of the matrix $H_2$ lie in the range space of $H_1$, i.e.

$H_2 \in \mathbb{R}(H_1)$

So, the system (3.2.1) has the unique solution for the matrix $A_c$. Moreover, instead of calculating $A_c$ by solving the least squares problem (3.2.1), the matrix $A_c$ could also be calculated by:

\[
A_c = H_{11}^{-1} H_{21}
\]

where $H_{11}$ contains any $n$ linearly independent rows from $H_1$, let the locations of these rows be:

$\eta_1, \ldots, \eta_n$

while $H_{21}$ contains the $n$ rows from $H_2$ having the same locations $\eta_1, \ldots, \eta_n$.

The least squares solution of the system matrix $A_c$ can also be obtained from the Hankel matrices $H_1$ and $H_2$. The singular value decomposition of the matrix $H_1$,

\[
H_1 = U \Sigma_n V^T \tag{3.2.5}
\]
where \( U \) and \( V \) are \( (pr \times n) \) and \( (n \times n) \) matrices, and \( \Sigma_n = \text{diag} \{ \sigma_1, \ldots, \sigma_n \} \).

respectively.

From (3.2.1)

\[
A_c = (H_1^T H_1)^{-1} H_1^T H_2
\]

Substituting for \( H_1 \) from (3.2.5)

\[
A_c = \left( V \Sigma_n^{-1} U^T U \Sigma_n V^T \right)^{-1} V \Sigma_n^{-1} U^T H_2
\]

From the definition of singular value decomposition of a matrix

\[
U^T U = V V^T = \Sigma_n^{-1} \Sigma_n = I
\]

We get

\[
A_c = V \Sigma_n^{-1} U^T H_2
\] (3.2.6)

The expression (3.2.6) is given merely to indicate that the least squares solution for \( A_c \), could also be obtained using orthogonal matrices. The same argument is valid in the dual sense for the calculation the matrix \( A_o \), equation (3.2.3).

The \( n \) location numbers \( \nu^c_1, \ldots, \nu^c_n \) or \( \nu_1^o, \ldots, \nu_n^o \), referred to in [25] as “nice indices”, defining the columns or rows of \( H_r \) to be selected in matrices \( H_1 \) and \( H_2 \) are uniquely related to the assumed set of admissible pseudo controllability or pseudo observability indices, respectively, [47].
3.3 Review of Pseudo Controllable/Observable Forms.

As it was explained in [47], for an $n^{th}$ order MIMO system represented by \{ $A$, $B$, $C$ \}, with $m$ inputs and $p$ outputs, the total numbers of admissible sets of pseudo controllable and pseudo observable indices are less than or equal to:

$$I_c = \frac{(n-1)!}{(m-1)! \cdot (n-m)!} \quad \text{for p.c.i} \quad (3.3.1)$$

$$I_o = \frac{(n-1)!}{(p-1)! \cdot (n-p)!} \quad \text{for p.o.i} \quad (3.3.2)$$

An admissible set of p.c.i is defined by $m$ positive integers $n_1^c, \ldots, n_m^c$ satisfying the following conditions:

a) $\sum_{i=1}^{m} n_i^c = n, \ n_i^c > 0$.

b) The $n$ columns

$$[ b_1 \ldots A^{n_1^c-1} b_1 : b_2 \ldots \ldots A^{n_2^c-1} b_2 : \ldots : b_m \ldots \ldots A^{n_m^c-1} b_m ] \quad (3.3.3)$$

should form a set of linearly independent columns.

The location numbers $\nu_1^c, \ldots, \nu_n^c$ used in selecting columns from $H_r$ into matrices $H_1$ and $H_2$, equation (3.2.1), are defined by locations of $n$ columns in (3.3.3), in the controllability matrix;
\[ Q_c = [ \quad B \quad : \quad A B \quad : \quad \ldots \quad : \quad A^{n-m} B \quad ] \]  

(3.3.4)

For instance, it is easy to verify that in the case of p.c.i, given by:

\[ \{ n_1^c, n_2^c, n_3^c \} = \{ 2, 1, 4 \} \]

corresponding to \( m = 3 \) and \( n = 7 \), the location numbers \( \{ \nu_1^c, \ldots, \nu_7^c \} \) are \( \{ 1, 2, 3, 4, 6, 9, 12 \} \) which could be verified from:

<table>
<thead>
<tr>
<th>Column#s</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q_c )</td>
<td>( b_1 )</td>
<td>( b_2 )</td>
<td>( b_3 )</td>
<td>( A b_1 )</td>
<td>( A b_2 )</td>
<td>( A b_3 )</td>
<td>( A^2 b_1 )</td>
<td>( A^2 b_2 )</td>
<td>( A^2 b_3 )</td>
<td>( A^3 b_1 )</td>
<td>( A^3 b_2 )</td>
<td>( A^3 b_3 )</td>
</tr>
<tr>
<td>Location#s</td>
<td>( \nu_1 )</td>
<td>( \nu_2 )</td>
<td>( \nu_3 )</td>
<td>( \nu_4 )</td>
<td>( \nu_5 )</td>
<td>( \nu_6 )</td>
<td>( \nu_7 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(3.3.5)

If, for a selected set of indices \( n_i^c \), \( i = 1, \ldots, m \), the \( n \) columns in (3.3.3) are linearly dependent, then this set is considered non-admissible and cannot be used in generating the location numbers \( \nu_i^c \) and a pseudo controllable form \( \{ A_c, B_c, C_c \} \).

The above discussion about the unique relationship between pseudo controllable indices \( \{ n_i^c \} \), and location numbers \( \{ \nu_i^c \} \), is applicable in the dual sense to the relationship between pseudo observable indices \( \{ n_i^o \} \) and location numbers \( \{ \nu_i^o \} \).
A pseudo controllable form \( \{ A_c, B_c, C_c \} \), of an arbitrary controllable representation \( \{ A, B, C \} \), corresponding to the assumed set of pseudo controllability indices \( n^c_i \), is defined by the similarity transformation;

\[
A_c = T^{-1} A T \\
B_c = T^{-1} B \\
C_c = C T
\]

(3.3.6)

where the similarity transformation matrix \( T \) contains \( n \) columns from the controllability matrix \( Q_c \) with location numbers given by \( \{ \nu^c_1, \ldots, \nu^c_n \} \). Since the matrix \( T \) is non-singular and contains the matrix \( B \) in the first \( m \) columns it, it is obvious that

\[
B_c = \begin{bmatrix}
I_m \\
\vdots \\
O
\end{bmatrix}
\]

Similarly, considering the equation \( A_c = T^{-1} A T \) or \( T A_c = A T \), it is easy to check that among the \( n \) columns of \( A_c \), there are \( (n - m) \) columns equal to the last \( (n - m) \) columns of the identity matrix \( I_n \). The locations of these columns in \( A_c \) are uniquely determined by the set of admissible pseudo controllability indices used. In the above illustrated example for \( \{ n^c_1, n^c_2, n^c_3 \} = \{ 2, 1, 4 \} \) the structure of \( A_c \) is:

\[
A_c = \begin{bmatrix}
0 & x & 0 & 0 & x \\
0 & x & 0 & \vdots & \vdots \\
0 & \vdots & \vdots & \vdots & \vdots \\
1 & 0 \\
0 & 1 & 0 \\
\vdots & 0 & 1 & 0 & \vdots \\
0 & x & 0 & x & 0 & 1 & x
\end{bmatrix}
\]

(3.3.7)
The columns with $x$ indicate non-zero elements. In the controllability matrix of the pair $\{A_c, B_c\}$ there are $n$ columns equal to the $n$ columns of the identity matrix $I_n$ whose locations are exactly equal to the locations given by $\nu_j^c$, $j = 1, \ldots, n$.

### 3.4 Justification of the Simplified ERA.

Assume that, given the Markov Parameters $Y$, it is desired to obtain a state space representation $\{A_c, B_c, C_c\}$ in a pseudo controllable form based on a selected set of admissible pseudo controllability indices $\{n_i^c\}$. From the definition of the Hankel matrix $H_r$ and the matrix $\tilde{H}_r$, equations (3.1.4) and (3.1.8), it is evident that

$$H_r = Q_o A_c Q_c$$ (3.4.1)

$$\tilde{H}_r = Q_o A_c Q_c$$

where $Q_o$ and $Q_c$ are observability and controllability matrices of the desired representation in pseudo controllable form, $\{A_c, B_c, C_c\}$, i.e,
\[ Q_c = [B_c : A_c B_c : \ldots : \tilde{A}_c^{-1} B_c] \]

\[ Q_o = \begin{bmatrix}
C_c \\
\vdots \\
C_c A_c \\
\vdots \\
\vdots \\
C_c \tilde{A}_c^{-1}
\end{bmatrix} \]

Note that the columns at locations \( \nu_1^c, \ldots, \nu_n^c \) in \( Q_c \) correspond to the \( n \) columns of the identity matrix \( I_n \). Postmultiplying \( H_r \) and \( \tilde{H}_r \) by a "selector matrix" \( S_c \) gives

\[ H_r S_c = Q_o Q_c S_c \]

\[ \tilde{H}_r S_c = Q_o A_c Q_c S_c \]

If the selector matrix \( S_c \) is so chosen as to select from \( Q_c \) the \( n \) columns equal to the columns of identity matrix then,

\[ Q_c S_c = I_n \]

which leads to

\[ H_r S_c = Q_o \]

\[ \tilde{H}_r S_c = Q_o A_c \]

thus;

\[ H_1 A_c = H_2 \]
where \( H_1 = H_r S_c \) and \( H_2 = \tilde{H}_r S_c \) \hspace{1cm} (3.4.8)

which verifies the result in (3.2.1). Comparing (3.4.2), (3.4.5), (3.4.6) and (3.4.8), it can be concluded that the output matrix \( C_c \) corresponds to the first \( p \) rows of \( H_1 \), see (3.2.2). The version of the simplified algorithm given by (3.2.3) - (3.2.4), corresponding to a pseudo observable form could be derived using the principle of duality.

### 3.5 Simulation Example

As an illustration of the proposed algorithm, consider the example given in [5], pp 491-492. Given the Markov Parameters \( Y_i \), \( i = 0, \ldots, 5 \):

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
[0 0] & [0 0] & [-1 1] & [0 1] & [1 1] & [1 2] \\
\end{array}
\]

The order of the minimal realization and the number of inputs and outputs are \( n = 4 \), \( m = 2 \) and \( p = 2 \).

**Calculation of the Pseudo-Controllable Form**

According to (3.3.1) and (3.3.3), the possible sets of pseudo controllable indices are \{1, 3\}, \{2, 2\} and \{3, 1\}. The Hankel matrix \( H \) becomes:

\[
H = \begin{bmatrix}
Y_0 & Y_1 & Y_2 & Y_3 \\
Y_1 & Y_2 & Y_3 & Y_4 \\
Y_2 & Y_3 & Y_4 & Y_5
\end{bmatrix} = \begin{bmatrix}
-1 & 1 & 0 & 1 & 1 & 1 & 2 \\
0 & 0 & 0 & -1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 2 & 1 & 4 \\
0 & 0 & -1 & 1 & 0 & 1 & 1 \\
1 & 1 & 2 & 1 & 4 & 2 & 7 \\
-1 & 1 & 0 & 1 & 1 & 1 & 2
\end{bmatrix}
\]
The locations \( \{ \nu \} \) corresponding to the sets of pseudo controllability indices are:

\[
\text{p.c.i : } \quad \{ 1, 3 \} \quad \{ 2, 2 \} \quad \{ 3, 1 \}
\]

column

locations : \quad \{ 1, 2, 4, 6 \} \quad \{ 1, 2, 3, 4 \} \quad \{ 1, 2, 3, 5 \}

The matrices \( H_1 \) and \( H_2 \) as defined in section 3.2, corresponding to all sets of pseudo controllability indices are:

1: p.c.i = \{1, 3\} \quad H_1 = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 4 \\ -1 & 1 & 1 & 1 \end{bmatrix} \quad H_2 = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ -1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 7 \\ 0 & 1 & 1 & 2 \end{bmatrix}

2: p.c.i = \{2, 2\} \quad H_1 = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ -1 & 1 & 0 & 1 \end{bmatrix} \quad H_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ -1 & 1 & 0 & 1 \\ 1 & 2 & 4 & 7 \\ 0 & 1 & 1 & 1 \end{bmatrix}

3: p.c.i = \{3, 1\} \quad H_1 = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 1 \end{bmatrix} \quad H_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}

It may be concluded that the matrix \( H_1 \) in the case of p.c.i \{2, 2\} is not of full rank, therefore, this set of p.c.i is not admissible.
Pseudo Controllable Forms.

According to (3.2.1) and (3.2.2), the pseudo controllable forms \( \{A_c, B_c, C_c\} \) corresponding to pseudo controllability indices \( \{1, 3\} \) and \( \{3, 1\} \) are:

\[
\text{p.c.i \{1, 3\}: \quad A_c = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 2 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C_c = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

\[
\text{p.c.i \{3, 1\}: \quad A_c = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C_c = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}
\]

Calculation of the Pseudo-Observable Form.

According to (3.3.2), the possible sets of pseudo observability indices are \( \{1, 3\} \), \( \{2, 2\} \) & \( \{3, 1\} \).

The Hankel matrix \( H \) becomes:

\[
H = \begin{bmatrix}
Y_0 & Y_1 & Y_2 \\
Y_1 & Y_2 & Y_3 \\
Y_2 & Y_3 & Y_4 \\
Y_3 & Y_4 & Y_5 \\
\end{bmatrix} = \begin{bmatrix}
-1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & -1 & 1 \\
0 & 1 & 1 & 1 & 1 & 2 \\
0 & 0 & -1 & 1 & 0 & 1 \\
1 & 1 & 1 & 2 & 1 & 4 \\
-1 & 1 & 0 & 1 & 1 & 1 \\
1 & 2 & 1 & 4 & 2 & 7 \\
0 & 1 & 1 & 1 & 1 & 2 \\
\end{bmatrix}
\]

The locations \( \{ \nu_i^o \} \) corresponding to the sets of pseudo observability indices are:
p.o.i: $\{1, 3\}$ $\{2, 2\}$ $\{3, 1\}$

row locations: $\{1, 2, 4, 6\}$ $\{1, 2, 3, 4\}$ $\{1, 2, 3, 5\}$

The matrices $H_1$ and $H_2$ as defined in section 3.2, equation (3.2.3) corresponding to all sets of pseudo observability indices are:

1: p.o.i $= \{1, 3\}$
$$H_1 = \begin{bmatrix} -1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$
$$H_2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & -1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

2: p.o.i $= \{2, 2\}$
$$H_1 = \begin{bmatrix} -1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & -1 & 1 & 0 & 1 \end{bmatrix}$$
$$H_2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & -1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 & 1 & 4 \\ -1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

3: p.o.i $= \{3, 1\}$
$$H_1 = \begin{bmatrix} -1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 1 & 4 \end{bmatrix}$$
$$H_2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & -1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 & 1 & 4 \\ 1 & 2 & 1 & 4 & 2 & 7 \end{bmatrix}$$

It may be concluded that the matrix $H_1$ in the case of p.o.i $\{1, 3\}$ is not of full rank, therefore this set of p.o.i is not admissible.

**Pseudo Observable Forms.**

According to (3.2.3) and (3.2.4), the pseudo observable forms $\{A_o, B_o, C_o\}$ corresponding to p.o.i $\{2, 2\}$ and $\{3, 1\}$ are:

p.o.i $\{2, 2\}$: $A_o = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ $B_o = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $C_o = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$
p.o.i \{3, 1\}: 

\[ A_o = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & -1 & 2 \end{bmatrix} \]

\[ B_o = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \]

\[ C_o = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \]
4.0 PSEUDO LINEAR IDENTIFICATION

4.1 Background

Simultaneous identification of the states and the parameters of a system has been of much interest to researchers for a long time. Among the various algorithms for joint estimation of states and parameters of a system, one standard approach uses the extended Kalman filter to estimate an extended state vector, which consists of both the states and the parameters of the system. This is a non-linear problem, but is well approximated by a linearization about the current state. The major limitation with the extended Kalman filter is that it is not globally convergent, hence, a good initial guess is required for better estimates. The requirement of a good or in fact any initial guess of the system parameters is not a desirable feature for any identification algorithm.

The other popular method is an iterative algorithm, where the states are estimated first and the estimated states are then used to identify the system parameters. These parameters are in turn used to estimate the states and so on. This iterative process continues until convergence is obtained. The limitation of this iterative
algorithm is in the difficulty in using the error covariance data from state to parameter estimates and vice versa.

In [49] a new approach is presented which uses the structure of observable canonical forms in obtaining an extended system. The state vector of the extended system consists of the unknown states and the unknown parameters of the system. In the system matrix the states are replaced by measured output. This makes the non-linearities implicit and then the estimator is linear in the estimates. The limitation of this method is in the fact that the autocovariance of the state noise, system inputs and outputs, should be known exactly. This prior knowledge of noise statistics is difficult to obtain in some physical systems. The work in [49] has been extended for unknown autocovariance [50], but the method results in a sub-optimal estimator.

4.2 SISO Pseudo Linear Identification

The Pseudo Linear Identification (PLID), an algorithm for simultaneous state and parameter estimation, developed extensively in [51]-[52] starts with the concept of extended state representation. The PLID algorithm requires the system to be completely observable and controllable. The observability condition is required to represent the original system in an extended state representation form. The system has to be completely controllable else the input will not excite all the states of the system. This requirement in system identification literature is known by the term "persistently exciting". This is a necessary condition for all the modes of the
system to be excited for a sufficient amount of time for an accurate and unique identification.

Consider a completely observable and controllable single input, single output (SISO) discrete time system.

\[ x_{k+1} = Ax_k + B(u_k + v_k) + \Sigma_x(k) \]  \hspace{1cm} (4.2.1)

\[ y_k = Cx_k + w_k \]  \hspace{1cm} (4.2.2)

where

\[ A = \begin{bmatrix}
0 & 0 & \ldots & 0 & 0 & a_0 \\
1 & 0 & \ldots & 0 & 0 & a_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & 1 & 0 & a_{n-2} \\
0 & 0 & \ldots & 0 & 1 & a_{n-1}
\end{bmatrix} \]

\[ B = \begin{bmatrix}
b_0 \\
b_1 \\
\vdots \\
b_{b-2} \\
b_{b-1}
\end{bmatrix} \]

\[ C = \begin{bmatrix}
0 & 0 & \ldots & 0 & 0 & 1
\end{bmatrix} \]  \hspace{1cm} (4.2.3)

and \( v_k \) is the unknown Gaussian noise with variance \( q \) on the system input, \( w_k \) is the unknown Gaussian noise with variance \( r \) affecting the output measurement and \( \Sigma_x \) being the zero mean white Gaussian noise adding directly to the states of the system.

The same linear \( n^{th} \) order controllable and observable system can be represented
by a strictly proper transfer function with $a_i$ and $b_i$, $i = 0, 1, \ldots, n - 1$, being the system parameters.

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b_{n-1}z^{n-1} + \ldots + b_1z + b_0}{z^n - a_{n-1}z^{n-1} - \ldots - a_1z - a_0} \quad (4.2.4)$$

Cross multiplying (4.2.4), we get

$$\{z^n - a_{n-1}z^{n-1} - \ldots - a_1z - a_0\}Y(z) = \{b_{n-1}z^{n-1} + \ldots + b_1z + b_0\}U(z) \quad (4.2.5)$$

Premultiplying both sides by $z^{-n}$,

$$\{1 - a_{n-1}z^{-1} - \ldots - a_0z^{-n}\}Y(z) = \{b_{n-1}z^{-1} + \ldots + b_0z^{-n}\}U(z) \quad (4.2.6)$$

Taking the inverse $z$-transform of (4.2.5)

$$y_k = a_{n-1}y_{k-1} + \ldots + a_0y_{k-n} + b_{n-1}u_{k-1} + \ldots + b_0u_{k-n} \quad (4.2.7)$$

From the state space representation (4.2.1)-(4.2.3), the general form of the state equations can be written as

$$x_i(k+1) = x_{i-1}(k) + a_{i-1}x_n(k) + b_{i-1}[u_k + v_k] \quad (4.2.8)$$

$$y_k = x_n(k) + w_k \quad (4.2.9)$$

for $i = 1, \ldots, n$. 

PSEUDO LINEAR IDENTIFICATION
Rearranging (4.2.8),

\[
x_{i+1} = x_{i-1} + x_n a_{i-1} + [u_k + v_k] b_{i-1}
\]  \hspace{1cm} (4.2.10)

Substituting for \(x_n\) from (4.2.9) in the above equation, we get

\[
x_{i+1} = x_{i-1} + [y_k - w_k] a_{i-1} + [u_k + v_k] b_{i-1}
\]  \hspace{1cm} (4.2.11)

The above equation can be rewritten as

\[
x_{k+1} = \begin{bmatrix}
0 & 0 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & 1 \\
\end{bmatrix} x_k + \begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_{n-2} \\
a_{n-1} \\
\end{bmatrix} [y_k - w_k] + \begin{bmatrix}
b_0 \\
b_1 \\
\vdots \\
b_{b-2} \\
b_{b-1} \\
\end{bmatrix} [u_k + v_k]
\]  \hspace{1cm} (4.2.12)

Defining the extended state vector as

\[
S_k = \begin{bmatrix}
x_k \\
\theta_A \\
\theta_B
\end{bmatrix}
\]  \hspace{1cm} (4.2.13)

where

\[
\theta_A = \begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_{n-2} \\
a_{n-1}
\end{bmatrix} \quad \text{and} \quad \theta_B = \begin{bmatrix}
b_0 \\
b_1 \\
\vdots \\
b_{b-2} \\
b_{b-1}
\end{bmatrix}
\]  \hspace{1cm} (4.2.14)
In other words $\theta_A$ is the $n^{th}$ column of the system matrix $A$ in observable canonical form and $\theta_B$ being the input column matrix $B$. Using (4.2.13)-(4.2.15), the extended state model can be derived as follows:

\[
S_{k+1} = F_k(y_k, u_k) S_k + N_k
\]  \hspace{1cm} (4.2.15)

\[
y_{k+1} = H_{k+1} S_{k+1} + v_k
\]  \hspace{1cm} (4.2.16)

where $F_k(y_k, u_k)$ clearly shows that the system matrix of the extended system model is a function of the input and measured output of the system. The structure of the system matrix $F_k$ is as follows

\[
F_k = \begin{bmatrix}
J_n & y_k I_n & u_k I_n \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
O & \cdots & I_{2n}
\end{bmatrix}
\]  \hspace{1cm} (4.2.17)

$I$ being an identity matrix of dimension indicated by the subscript and $O$ being a matrix whose elements are all zeros of appropriate dimensions to make $F_k$ of dimensions $(3n \times 3n)$. $J_n$ denotes the $(n \times n)$ lower Jordan block of zero eigen values.

The output matrix $H_{k+1}$ has the structure measuring only the last state of the system,

\[
H_{k+1} = \begin{bmatrix}
0 & \cdots & 1 & 0 & \cdots & 0
\end{bmatrix}
\]  \hspace{1cm} (4.2.18)
The above development clearly shows that the order of the system increases when a system is represented in extended state model form. A system of order \( n \) gives an extended system of order \( 3n \).

In case the system has a feed through term, i.e., in state space terminology a \( D \) matrix, then as developed in [51], the extended state vector is augmented by the feed through term resulting in a system of order \( 3n + 1 \). The extended state vector \( S_k \) becomes

\[
S_k = \begin{bmatrix}
x_k \\
\theta^A_k \\
\theta^B_k \\
b_n
\end{bmatrix}
\] (4.2.19)

and the system matrix \( F_k \) and the output matrix \( H_{k+1} \) have the following structure

\[
F_k' = \begin{bmatrix}
J_n & y_k I_n & u_k J_n & O \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
O & \vdots & I_{n+1}
\end{bmatrix}
\] (4.2.20)

\( I \) being an identity matrix of dimension indicated by the subscript and \( O \) being a matrix whose elements are all zeros of appropriate dimensions to make \( F_k \) of dimensions \( [(3n + 1) \times (3n + 1)] \).

The output matrix \( H_{k+1} \) has the structure measuring only the last state of the system,

\[
H_{k+1} = \begin{bmatrix}
0 & \cdots & 1 & \vdots & 0 & \cdots & 0 & \vdots & u_{k+1}
\end{bmatrix}
\] (4.2.21)
4.3 MIMO Pseudo Linear Identification

Consider a \( n^{th} \) order, linear, time-invariant, completely controllable and observable multi input, multi output, discrete time system \( \{A, B, C\} \). Let the system have \( m \) inputs, \( u_1(k), \ldots, u_m(k) \) and \( p \) outputs, \( y_1(k), \ldots, y_p(k) \). Since the pair \( \{A, C\} \) is completely observable, let the observability indices be \( n_1, \ldots, n_p \). To form the extended state vector for the MIMO system, \( A \) and \( B \) are transformed into a observable canonical form. For multi input-multi output systems as explained in [6], the observability matrix \( Q_o \) of the pair \( \{A, C\} \), of the following structure is formed.

\[
Q_o^T = \begin{bmatrix}
    c_1^T & (c_1A)^T & \ldots & (c_1A^{n_1-1})^T & \ldots & c_p^T & (c_pA)^T & \ldots & (c_pA^{n_p-1})^T \\
\end{bmatrix}
\]

(4.3.1)

c_1, \ldots, c_p \text{ are the rows of the output matrix } C. \text{ Let } M \text{ denote the inverse of the observability matrix } Q_o. M \text{ is of the form}

\[
M = \begin{bmatrix}
    e_{11} & e_{12} & \ldots & e_{1n_1} & \vdots & \vdots & e_{p1} & e_{p2} & \ldots & e_{pn_p} \\
\end{bmatrix}
\]

(4.3.2)

Using columns of \( M \) and \( A \), we form the transformation matrix \( T \)

\[
T = \begin{bmatrix}
    e_{1n_1} & e_{1n_1}A & \ldots & e_{1n_1}A^{n_1-1} & \vdots & \vdots & e_{pn_p} & e_{pn_p}A & \ldots & e_{pn_p}A^{n_p-1} \\
\end{bmatrix}
\]

(4.3.3)
The pair \( \{A, B\} \) is transformed into observable canonical forms, \([28]\).

\[
\bar{A} = T^{-1}AT \\
\bar{B} = T^{-1}B \\
\bar{C} = C T
\]  \hspace{1cm} (4.3.4)

The matrices \( \bar{A} \) and \( \bar{B} \) are of the form

\[
\begin{pmatrix}
\theta_A^1 & \theta_A^2 & \cdots & \theta_A^p \\
\downarrow & \downarrow & \cdots & \downarrow
\end{pmatrix}
\]

\[
\bar{A} =
\begin{bmatrix}
0 & 0 & \cdots & x & x & x \\
1 & 0 & \cdots & x & x & x \\
0 & 1 & \cdots & x & x & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\theta & 1 & \cdots & x & x & \vdots \\
0 & 0 & \cdots & x & x & \vdots \\
1 & 0 & \cdots & x & x & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\theta & 0 & \cdots & 1 & x & \vdots \\
\end{bmatrix}
\]  \hspace{1cm} (4.3.5)

\[
\begin{pmatrix}
\theta_B^1 & \theta_B^2 & \cdots & \theta_B^m \\
\downarrow & \downarrow & \cdots & \downarrow
\end{pmatrix}
\]

\[
\bar{B} =
\begin{bmatrix}
x & x & \cdots & x \\
x & x & \cdots & x \\
\vdots & \vdots & \ddots & \vdots \\
x & x & \cdots & x \\
\end{bmatrix}
\]  \hspace{1cm} (4.3.6)

The columns with \( x \) denote non-zero, non-unity elements. The \( x \) in the matrix \( \bar{A} \)
and \( \bar{B} \) represent the denominator and numerator coefficients of the individual transfer functions in the transfer function matrix, respectively. In other words the elements \( x \) in \( \bar{A} \) are the parameters associated with output feedback and those in \( \bar{B} \) are associated with the input. The structure of the system matrix \( \bar{A} \) is known as the Luenberger multivariable observable canonical form \([28]\). The columns vectors \( \theta^i_A \) and \( \theta^i_B \), \( i = 1, \ldots, p \), constitute the extended state vector \( S_k \) of the MIMO system.

\[
S_k = \begin{bmatrix}
x_k \\
\theta^1_A \\
\vdots \\
\theta^p_A \\
\theta^1_B \\
\vdots \\
\theta^m_B
\end{bmatrix}
\]  
\[(4.3.7)\]

The state space equations of the system can be written as

\[
x_{k+1} = \bar{A} \ x_k + \bar{B} \ u_k
\]
\[(4.3.8)\]

\[
y_k = \bar{C} \ x_k
\]

where the matrix \( \bar{C} \) and the columns vectors \( x_k \), \( u_k \) and \( y_k \) are of the form

\[
\bar{C} = \begin{bmatrix}
0 & 0 & 1 & \ldots & \ldots & 0 & 0 & 0 \\
\vdots
\end{bmatrix}
\]
\[(4.3.9)\]
From the discussion of the SISO Pseudo Linear Identification development, we know how to write the individual state equations of the MIMO system, using from (4.3.5)-(4.3.10).

\[
x_{i,j}(k+1) = x_{i,j-1}(k) + a_{1,i}^{-1} x_{1,n_1}(k) + \ldots + a_{p,i}^{-1} x_{p,n_p}(k) + b_{1,i}^{-1} u_{1}(k) + \ldots + b_{m,i}^{-1} u_{m}(k)
\]

(4.3.11)

where \(i = 1, \ldots, p\) and \(j = 1, \ldots, n_i\).

\[
y_i(k) = x_{i,n_i}(k)
\]

(4.3.12)

where \(i = 1, \ldots, p\).

By substituting (4.3.12) into (4.3.11)

\[
x_{i,j}(k+1) = x_{i,j-1}(k) + a_{1,i}^{-1} y_1(k) + \ldots + a_{p,i}^{-1} y_p(k) + b_{1,i}^{-1} u_1(k) + \ldots + b_{m,i}^{-1} u_m(k)
\]

(4.3.13)

Using the extended state vector \(S_k\) is defined by (4.3.7), the system (4.3.8) is
modified and the extended state model for the MIMO system is written as

\[ S_{k+1} = F_k S_k \]

(4.3.14)

\[ y_k = H_k S_k \]

(4.3.15)

where the system matrix \( F_k \) and output matrix \( H_k \) are of the following structure

\[
F_k = \begin{bmatrix}
J_{n_1} & \cdots & 0 & y_1(k)I_n & \cdots & y_p(k)I_n & u_1(k)I_n & \cdots & u_m(k)I_n \\
0 & \cdots & J_{n_p} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \cdots \\
O_{(m+p)n \times n} & \vdots & I_{(m+n)n} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\]

(4.3.16)

where \( J_{n_i} \) is \( n_i \times n_i \) lower Jordan block of zero eigenvalues.

\[
H_k = \begin{bmatrix}
0 & \cdots & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 \\
\end{bmatrix}
\]

(4.3.17)

As can be seen from (4.3.16) and (4.3.17), the size of system with \( n \) states, \( m \) inputs and \( p \) outputs becomes \((m + p + 1)n\).
4.4 Recursive Estimator

The extended state model for both the SISO, (4.2.15)-(4.2.18) and MIMO, (4.3.14)-(4.3.17), systems have been developed in the previous two sections and a more detailed development can be found in the original work, [51]-[52]. From (4.2.17)-(4.2.18) and (4.3.16)-(4.3.17) it can be concluded that the system matrix $F_k$ and the output matrix $H_k$ are dependent on the measurements at the current time instant and not on the past or the future measurements. Once again for a clear understanding of the recursive estimator, the state equations (4.2.15)-(4.2.16) are written below

$$S_{k+1} = F_k S_k + N_k$$

(4.4.1)

$$y_{k+1} = H_{k+1} S_{k+1} + v_{k+1}$$

(4.4.2)

where the vector $N_k$ models the plant noise $\Sigma_x(k)$, (4.2.1) where $\Sigma_x(k)$ is zero mean, white Gaussian noise affecting the states directly and $v_k$ is assumed to be zero mean, white Gaussian noise with variance $R_k$. The vector $N_k$ is of the form

$$N_k = \begin{bmatrix} \Sigma_x(k) \\ O_{(2n+1) \times m} \end{bmatrix}$$

(4.4.3)

It can reasonably be assumed that $\Sigma_x(k)$ and $v_k$ are uncorrelated and that $\Sigma_x(k)$ has a covariance matrix

$$E \left[ \Sigma_x(k) \Sigma_x^T(k) \right] = Q_1(k)$$

(4.4.4)
Based on the assumptions and (4.4.3)-(4.4.4) it can be concluded that the discrete
time Kalman filter is the optimal (minimum mean square) recursive estimator for
$S_k$. For completing the PLID algorithm we need to formulate the Kalman filter
equations. Let us assume an initial estimate $\hat{S}_0$, with a covariance matrix $P_0$. $P_0$
represents the uncertainty of the initial estimate. The detailed derivations of the
basic discrete time Kalman filter equations can be found in [53] or any text on
estimation theory.

The time update equation starting with $P_0$ is

$$M_{k+1} = [ F_k P_k F_k^T + Q_k ] \tag{4.4.5}$$

The optimal gain matrix $K_{k+1}$ is given by

$$K_{k+1} = M_{k+1} H_{k+1}^T [ H_{k+1} M_{k+1} H_{k+1}^T + R_k ]^{-1} \tag{4.4.6}$$

The measurement update equation $P_{k+1}$ is

$$P_{k+1} = [ I - K_{k+1} H_{k+1} ] M_{k+1} \tag{4.4.7}$$

And finally the filter or the estimate equation is

$$\tilde{S}_{k+1} = F_k \tilde{S}_k + K_{k+1} [ y_{k+1} - H_{k+1} F_k \tilde{S}_k ] \tag{4.4.8}$$

where

$$Q_k = \begin{bmatrix} Q_{1.(k)} & 0 \\ 0 & 0 \end{bmatrix}$$
Equations (4.4.1) to (4.4.8) along with the initial observability modeling constitute the Pseudo Linear Identification (PLID) Algorithm. The same set of equations are valid for both the SISO and MIMO systems with only appropriate dimensional changes.

4.5 Illustrative Examples

The Pseudo Linear Identification (PLID) Algorithm has been extensively tested on simulated examples and realistic experimental data. The identification algorithm has been programmed in the MATLAB environment and various runs with different noise statistics, step size and size of systems have been carried out as a part of this research effort. The PLID algorithm has proven to be highly accurate in a noise free environment, and the performance under realistic noise conditions has been very encouraging.

The algorithm takes a minimum of $3n + 1$ steps of recursive estimation to converge to accurate system parameters. The initial estimates on all the examples and experimental data have been assumed to be zeros, i.e. no initial guess, which is a strong advantage for system identification purposes of physical structures. The equations (4.4.1) and (4.4.2) have been used to generate the output sequences from various input sequences for different systems. The input and the output sequences thus obtained are used in forming the system matrix and output matrix (if a feed through term exists) and the basic Kalman filter equations are recursively computed till convergence is obtained. The algorithms identifies the system
parameters, i.e. the numerator and denominator coefficients of the system transfer function in discrete time. The PLID algorithm has been used in an experimental set up which is explained in detail in chapter 5.0.

The various examples tested on PLID are as follows.

\( \{A\} \). 4\(^{th}\) order SISO (single input-single output) noise free system.

\[
G(z) = \frac{b_3 z^3 + b_2 z^2 + b_1 z + b_0}{z^4 + a_3 z^3 + a_2 z^2 + a_1 + a_0}
\]

Table 4.4.

4th order SISO system with negligible noise

<table>
<thead>
<tr>
<th>Coefficients</th>
<th>Actual Parameters</th>
<th>Identified Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_0 )</td>
<td>0.2187</td>
<td>0.2187</td>
</tr>
<tr>
<td>( a_1 )</td>
<td>0.3675</td>
<td>0.3675</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>1.0720</td>
<td>1.0720</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>0.7756</td>
<td>0.7756</td>
</tr>
<tr>
<td>( b_0 )</td>
<td>0.1582</td>
<td>0.1582</td>
</tr>
<tr>
<td>( b_1 )</td>
<td>0.0510</td>
<td>0.0510</td>
</tr>
<tr>
<td>( b_2 )</td>
<td>0.2237</td>
<td>0.2237</td>
</tr>
<tr>
<td>( b_3 )</td>
<td>0.3309</td>
<td>0.3309</td>
</tr>
</tbody>
</table>

The plots and the identified parameters show the PLID has identified the system accurately.
"-" Identified System & "-" Actual System

Fig. 4.A.1 Frequency response of the Identified and the Actual system.
Fig. 4.A.2 Plot shows the poles of the Identified and the Actual system.
" + " Identified Zeros & " o " Actual Zeros

Fig. 4.A.3 Plot shows the zeros of the Identified and the Actual system.
\{B\} \ 4^{th} \ \text{order SISO (single input-single output)} \ \text{in} \ \{A\} \ \text{with a Signal to Noise ratio of 19.09 (12.8 dB)}

\[ G(z) = \frac{b_3 z^3 + b_2 z^2 + b_1 z + b_0}{z^4 + a_3 z^3 + a_2 z^2 + a_1 + a_0} \]

Table 4.B.

4th order SISO system with \( \text{SNR} = 19.09 \)

<table>
<thead>
<tr>
<th>Coefficients</th>
<th>Actual Parameters</th>
<th>Identified Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_0 )</td>
<td>0.2187</td>
<td>0.2081</td>
</tr>
<tr>
<td>( a_1 )</td>
<td>0.3675</td>
<td>0.3312</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>1.0720</td>
<td>1.0391</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>0.7756</td>
<td>0.7477</td>
</tr>
<tr>
<td>( b_0 )</td>
<td>0.1582</td>
<td>0.1482</td>
</tr>
<tr>
<td>( b_1 )</td>
<td>0.0510</td>
<td>0.0273</td>
</tr>
<tr>
<td>( b_2 )</td>
<td>0.2237</td>
<td>0.2239</td>
</tr>
<tr>
<td>( b_3 )</td>
<td>0.3309</td>
<td>0.3255</td>
</tr>
</tbody>
</table>

The frequency responses are almost the same, the poles of the identified system are very close to the actual poles and the zeros have been identified accurately.
"-" Identified System & "--" Actual System

Fig. 4.B.1 Frequency response of the Identified and the Actual system.
"+" Identified Poles & "x" Actual Poles

Fig. 4.B.2 Plot shows the poles of the Identified and the Actual system.
" + " Identified Zeros & " o " Actual Zeros

Fig. 4.3 Plot shows the zeros of the Identified and the Actual system.
8th order SISO (single input-single output) system with 2% damping and negligible noise on the measurements of the system.

\[ G(z) = \frac{b_7z^7 + b_6z^6 + b_5z^5 + b_4z^4 + b_3z^3 + b_2z^2 + b_1z + b_0}{z^8 + a_7z^7 + a_6z^6 + a_5z^5 + a_4z^4 + a_3z^3 + a_2z^2 + a_1 + a_0} \]

Table 4.C.

8th order SISO system with 2% damping and negligible noise

<table>
<thead>
<tr>
<th>Coefficients</th>
<th>Actual Parameters</th>
<th>Identified Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_0 )</td>
<td>0.8367</td>
<td>0.8365</td>
</tr>
<tr>
<td>( a_1 )</td>
<td>-2.7673</td>
<td>-2.7664</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>5.6262</td>
<td>5.6246</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>-7.7974</td>
<td>-7.7954</td>
</tr>
<tr>
<td>( a_4 )</td>
<td>8.9315</td>
<td>8.9294</td>
</tr>
<tr>
<td>( a_5 )</td>
<td>-8.2964</td>
<td>-8.2946</td>
</tr>
<tr>
<td>( a_6 )</td>
<td>6.2958</td>
<td>6.2947</td>
</tr>
<tr>
<td>( a_7 )</td>
<td>-3.2257</td>
<td>-3.2253</td>
</tr>
<tr>
<td>( b_0 )</td>
<td>0.0448</td>
<td>0.0448</td>
</tr>
<tr>
<td>( b_1 )</td>
<td>-0.1103</td>
<td>-0.1103</td>
</tr>
<tr>
<td>( b_2 )</td>
<td>0.1338</td>
<td>0.1338</td>
</tr>
<tr>
<td>( b_3 )</td>
<td>-0.0645</td>
<td>-0.0645</td>
</tr>
<tr>
<td>( b_4 )</td>
<td>-0.0699</td>
<td>-0.0699</td>
</tr>
<tr>
<td>( b_5 )</td>
<td>0.1452</td>
<td>0.1452</td>
</tr>
<tr>
<td>( b_6 )</td>
<td>-0.1240</td>
<td>-0.1240</td>
</tr>
<tr>
<td>( b_7 )</td>
<td>0.0512</td>
<td>0.0512</td>
</tr>
</tbody>
</table>
Fig. 4.C.1 Frequency response of the Identified and the Actual system.
Fig. 4.C.2 Plot shows the poles of the Identified and the Actual system.
Fig. 4.C.3 Plot shows the zeros of the Identified and the Actual system.
The plots show that PLID has identified the system accurately though the system parameters are not exactly the same.
\{D\}. 8^{th} order SISO (single input-single output) system with 20% damping and negligible noise on the measurements of the system.

\[ G(z) = \frac{b_7 z^7 + b_6 z^6 + b_5 z^5 + b_4 z^4 + b_3 z^3 + b_2 z^2 + b_1 z + b_0}{z^8 + a_7 z^7 + a_6 z^6 + a_5 z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 + a_0} \]

Table 4.D.

8th order SISO system with 20% damping and negligible noise

<table>
<thead>
<tr>
<th>Coefficients</th>
<th>Actual Parameters</th>
<th>Identified Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_0)</td>
<td>0.1639</td>
<td>0.1649</td>
</tr>
<tr>
<td>(a_1)</td>
<td>-0.6016</td>
<td>-0.6058</td>
</tr>
<tr>
<td>(a_2)</td>
<td>1.4974</td>
<td>1.5064</td>
</tr>
<tr>
<td>(a_3)</td>
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<tr>
<td>(a_4)</td>
<td>4.0937</td>
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<tr>
<td>(a_5)</td>
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<td>(a_6)</td>
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<td>-2.8778</td>
<td>-2.8858</td>
</tr>
<tr>
<td>(b_0)</td>
<td>0.0109</td>
<td>0.0110</td>
</tr>
<tr>
<td>(b_1)</td>
<td>-0.0299</td>
<td>-0.0301</td>
</tr>
<tr>
<td>(b_2)</td>
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<td>0.0420</td>
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<tr>
<td>(b_3)</td>
<td>-0.0192</td>
<td>-0.0192</td>
</tr>
<tr>
<td>(b_4)</td>
<td>-0.0441</td>
<td>-0.0444</td>
</tr>
<tr>
<td>(b_5)</td>
<td>0.0959</td>
<td>0.0965</td>
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<td>(b_6)</td>
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<td>-0.0963</td>
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<tr>
<td>(b_7)</td>
<td>0.0431</td>
<td>0.0413</td>
</tr>
</tbody>
</table>
Fig. 4. D.1 Frequency response of the Identified and the Actual system.
"+" Identified Poles & "x" Actual Poles

Fig. 4.D.2 Plot shows the poles of the Identified and the Actual system.
"+" Identified Zeros & "o" Actual Zeros

Fig. 4.D.3 Plot shows the zeros of the Identified and the Actual system.
(E). 8th order SISO (single input-single output) system with 20% damping and Signal to Noise ratio of 21.800 (13.38 dB).

\[ G(z) = \frac{b_7 z^7 + b_6 z^6 + b_5 z^5 + b_4 z^4 + b_3 z^3 + b_2 z^2 + b_1 z + b_0}{z^8 + a_7 z^7 + a_6 z^6 + a_5 z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 + a_0} \]

Table 4.E.

8th order SISO system with 20% damping and SNR=21.800

<table>
<thead>
<tr>
<th>Coefficients</th>
<th>Actual Parameters</th>
<th>Identified Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_0)</td>
<td>0.1639</td>
<td>0.1509</td>
</tr>
<tr>
<td>(a_1)</td>
<td>-0.6016</td>
<td>-0.6047</td>
</tr>
<tr>
<td>(a_2)</td>
<td>1.4974</td>
<td>1.4960</td>
</tr>
<tr>
<td>(a_3)</td>
<td>-2.6644</td>
<td>-2.6706</td>
</tr>
<tr>
<td>(a_4)</td>
<td>4.0937</td>
<td>4.1038</td>
</tr>
<tr>
<td>(a_5)</td>
<td>-4.9927</td>
<td>-5.0488</td>
</tr>
<tr>
<td>(a_6)</td>
<td>4.7033</td>
<td>4.7629</td>
</tr>
<tr>
<td>(a_7)</td>
<td>-2.8778</td>
<td>-2.9099</td>
</tr>
<tr>
<td>(b_0)</td>
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</tr>
<tr>
<td>(b_1)</td>
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<td>-0.0301</td>
</tr>
<tr>
<td>(b_2)</td>
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<td>(b_3)</td>
<td>-0.0192</td>
<td>-0.0204</td>
</tr>
<tr>
<td>(b_4)</td>
<td>-0.0441</td>
<td>-0.0442</td>
</tr>
<tr>
<td>(b_5)</td>
<td>0.0959</td>
<td>0.0980</td>
</tr>
<tr>
<td>(b_6)</td>
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<td>-0.0982</td>
</tr>
<tr>
<td>(b_7)</td>
<td>0.0431</td>
<td>0.0434</td>
</tr>
</tbody>
</table>

PSSUDO LINEAR IDENTIFICATION
Fig. 4.E.1 Frequency response of the Identified and the Actual system.
Fig. 4.E.2 Plot shows the poles of the Identified and the Actual system.
" + " Identified Zeros & " o " Actual Zeros

Fig. 4.E.3 Plot shows the zeros of the Identified and the Actual system.
The plots show that PLID has identified the system almost completely with a negligible differences. If closely noticed, the numerator coefficients of the identified system are much closer to the numerator coefficients of the actual system than the denominator coefficients of both the systems. From the structure of the system matrix $F_k$ and the extended state vector $S_k$, the numerator coefficients are multiplied by the outputs and the denominator coefficients are multiplied by the inputs. The reason for the difference in denominator coefficients stated above seems to be that the measurements of the outputs of the system are corrupted by noise, whereas the inputs were not corrupted by noise. This was achieved during simulation by increasing the measurement noise covariance and keeping the process noise covariance negligible.
\{F\}. 8^{th} order SISO (single input-single output) system with 2\% damping and Signal to Noise ratio of 27.8667 (14.45 dB).

\[ G(z) = \frac{b_1 z^7 + b_6 z^6 + b_5 z^5 + b_4 z^4 + b_3 z^3 + b_2 z^2 + b_1 z + b_0}{z^8 + a_7 z^7 + a_6 z^6 + a_5 z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 + a_0} \]

Table 4. F.
8th order SISO system with 2\% damping and SNR=27.8667

<table>
<thead>
<tr>
<th>Coefficients</th>
<th>Actual Parameters</th>
<th>Identified Parameters</th>
</tr>
</thead>
<tbody>
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<tr>
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<td>-2.7673</td>
<td>-2.8110</td>
</tr>
<tr>
<td>(a_2)</td>
<td>5.6262</td>
<td>5.7043</td>
</tr>
<tr>
<td>(a_3)</td>
<td>-7.7974</td>
<td>-7.9013</td>
</tr>
<tr>
<td>(a_4)</td>
<td>8.9315</td>
<td>9.0341</td>
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<td>(a_5)</td>
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<td>-8.3717</td>
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<td>(a_6)</td>
<td>6.2958</td>
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<td>(a_7)</td>
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<td>0.0448</td>
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<tr>
<td>(b_1)</td>
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<td>-0.1135</td>
</tr>
<tr>
<td>(b_2)</td>
<td>0.1338</td>
<td>0.1350</td>
</tr>
<tr>
<td>(b_3)</td>
<td>-0.0645</td>
<td>-0.0626</td>
</tr>
<tr>
<td>(b_4)</td>
<td>-0.0699</td>
<td>-0.0722</td>
</tr>
<tr>
<td>(b_5)</td>
<td>0.1452</td>
<td>0.1478</td>
</tr>
<tr>
<td>(b_6)</td>
<td>-0.1240</td>
<td>-0.1269</td>
</tr>
<tr>
<td>(b_7)</td>
<td>0.0512</td>
<td>0.0514</td>
</tr>
</tbody>
</table>
Fig. 4. F.1 Frequency response of the Identified and the Actual system.
Fig. 4.2 Plot shows the poles of the Identified and the Actual system.
Fig. 4.3 Plot shows the zeros of the Identified and the Actual system.
The plots show that PLID has identified the system accurately. The explanation for the previous example \{E\} holds good in this example also.

As a part of the current investigation a comparison of various system identification algorithms has been conducted. The main focus of this investigation has been to compare the various time-domain system identification algorithms developed and tested in this research work to the frequency domain system identification algorithms developed in the ME department at VPI as well as also those algorithms commercially available through SDRC IDEAS software package. Primary emphasis has been on the ability to identify heavily damped eigenvalues. The system order is known exactly and damping values are less than 15\% of critical damping.

The performance of the PLID algorithm in identifying the lower order systems (typically 4\(^{th}\) and 8\(^{th}\) order) with damping values ranging from 2\% to 20\% has been very successful. These cases have not been illustrated in this report, as the performance of PLID on lower order systems can be seen in \{A\} to \{F\}. However, to illustrate the performance of the PLID algorithm on a higher order system, the following example have been included.
A $14^{th}$ order SISO system was generated using natural frequencies, damping ratios and residues.

Table 4.G.

14th order SISO system

<table>
<thead>
<tr>
<th>Natural Frequency (Hz)</th>
<th>Damping (% critical)</th>
<th>Residue</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.0</td>
<td>0.04</td>
<td>$-j^2$</td>
</tr>
<tr>
<td>12.5</td>
<td>0.04</td>
<td>$j^2$</td>
</tr>
<tr>
<td>18.0</td>
<td>0.15</td>
<td>$j^{10}$</td>
</tr>
<tr>
<td>26.5</td>
<td>0.09</td>
<td>$j^1$</td>
</tr>
<tr>
<td>35.0</td>
<td>0.05</td>
<td>$j^5$</td>
</tr>
<tr>
<td>50.0</td>
<td>0.08</td>
<td>$j^4$</td>
</tr>
<tr>
<td>65.0</td>
<td>0.15</td>
<td>$j^6$</td>
</tr>
</tbody>
</table>

From the frequency response and the pole-zeros plots it can be concluded that the PLID algorithm has identified the system accurately.
Fig. 4.G.1 Frequency response of the Identified and the Actual system.
Fig. 4.G.2 Plot shows the poles of the Identified and the Actual system.
"+" Identified Zeros & "o" Actual Zeros

Fig. 4.G.3 Plot shows the zeros of the Identified and the Actual system.
A 4\textsuperscript{th} order multi input-multi output system. Number of inputs \(= 2\), number of outputs \(= 2\). The given system was first transformed into observable canonical form, as explained in section 4.3.

The given system: \(n = 4\), \(m = 2\), \(p = 2\)

\[
A = \begin{bmatrix}
0.3619 & 0.0015 & 0.0000 & 0.0000 \\
-558.7859 & 0.3575 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.8865 & 0.0019 \\
0.0000 & 0.0000 & -110.4479 & 0.8468 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 & 0 \\
0.0091 & -0.055 \\
-0.0001 & 0 \\
-0.0024 & -0.0152 \\
\end{bmatrix}
\]

\[
C = 1.0 \times 10^4 \begin{bmatrix}
-0.5671 & 0.0018 & 0.3327 & -0.0007 \\
-1.1773 & 0.0006 & -0.0990 & -0.0015 \\
\end{bmatrix}
\]

This system is transformed into multi-variable observable canonical form:

\[
A = \begin{bmatrix}
0.0000 & -0.3131 & 0.0000 & -0.7648 \\
1.0000 & 0.8451 & 0.0000 & 0.4573 \\
0.0000 & 0.7556 & 0.0000 & -1.1986 \\
0.0000 & -0.7208 & 1.0000 & 1.6077 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
-0.3614 & -0.1544 \\
-0.1831 & -0.1793 \\
-0.2270 & 0.2080 \\
-0.1793 & -0.1692 \\
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
Hence, the extended vector becomes

<table>
<thead>
<tr>
<th>Coefficients</th>
<th>Actual Parameters</th>
<th>Identified Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta^1_A$</td>
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<td>-0.3132</td>
</tr>
<tr>
<td></td>
<td>0.8451</td>
<td>0.8452</td>
</tr>
<tr>
<td></td>
<td>0.7556</td>
<td>0.7556</td>
</tr>
<tr>
<td>$\theta^2_A$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.7208</td>
<td>-0.7208</td>
</tr>
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<td></td>
<td>-0.7648</td>
<td>-0.7647</td>
</tr>
<tr>
<td></td>
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<td>$\theta^1_B$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.1831</td>
<td>-0.1831</td>
</tr>
<tr>
<td></td>
<td>-0.2270</td>
<td>-0.2270</td>
</tr>
<tr>
<td>$\theta^2_B$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.1793</td>
<td>-0.1793</td>
</tr>
<tr>
<td></td>
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<td>-0.1544</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.2050</td>
<td>0.2080</td>
</tr>
<tr>
<td></td>
<td>-0.1692</td>
<td>-0.1692</td>
</tr>
</tbody>
</table>

Table 4.H.
4th order MIMO system
Fig. 4.H.1 Frequency response of the Identified and the Actual system for the 1st input to 1st output.
Fig. 4.2 Frequency response of the Identified and the Actual system for the 2\textsuperscript{nd} input to 2\textsuperscript{nd} output.
Fig. 4.H.3 Frequency response of the Identified and the Actual system for the $2^{nd}$ input to $1^{st}$ output.
Fig. 4. H.4 Frequency response of the Identified and the Actual system for the 2\textsuperscript{nd} input to 2\textsuperscript{nd} output.
5.0 RECTANGULAR PLATE EXPERIMENT

5.1 Preview

The best practical approach to test the performance of any system identification algorithm is to apply it to a physical experiment. There are several issues to be addressed during the design process of experiments for identification purposes. The important aspects to be considered are the signals to be measured (outputs), the manipulated variables (inputs) and the locations of sensors and actuators for measuring these signals. A very important variable in the design of the identification process is sampling interval. In most cases, the signals are sampled using a constant sampling interval $T$, and this variable has to be specified. The choice of the sampling interval $T$ is dependent on the available data acquisition facilities. Many physical structures have a very large number of modes or natural frequencies. A decision has to be made regarding the range of frequencies that are of interest to a particular problem. The choice of the input signals substantially influence the measured (output) data. As stated in earlier chapters, the input must be persistent enough to excite all the modes of the system that are to be identified.
5.2 Modal Testing

Modal testing, the process of testing physical structures to obtain a mathematical description of their vibrations or dynamic behavior, is the primary focus of system identification experiments. These experiments are conducted to obtain the natural frequencies, damping factors and mode shapes of the structure. Experimental study of structural vibrations provides a major contribution in understanding and controlling the vibration phenomena of structures. The main objectives of the experiment are the determination of the nature and extent of system responses and verification of analytical models and algorithms. Modal testing can be broadly classified into two different types of tests. In the first, the vibration responses are measured during natural operation of the structure, obtaining the free responses of the system. In the second type of test the structure is vibrated with a known excitation and the responses are measured. The second type yields more accurate and detailed information of the plant due to the controlled nature of the vibrations. In the experiments carried out in this work known inputs are used to excite the system. The results obtained from the experimental study are compared with those of the mathematical models.

The experimental verification of system identification algorithms, primarily the Pseudo Linear Identification (PLID) algorithm, is carried out on a simply supported rectangular steel plate. The experimental setup was constructed in the M. E. department under DARPA funding. The test structure is excited with two types of input signals. The first is a random excitation and the second being impact or burst random excitation. The two types of excitations together enable
better identification of the natural frequencies, damping ratios and mode shapes. The rectangular plate is 0.6 m in length, 0.5 m in breadth and 1 inch thick. It is supported at one end on a platform.

5.3 Mathematical Model for a Rectangular Plate

Any physical system can be modeled by nonhomogeneous partial differential equations describing the dynamics of motion in space and time coordinates. The solution of these partial differential equations consists of two parts. The time solution gives the natural frequencies and damping ratios of the structure, while the space solution gives the modes and mode shapes of the structure. The differential equation describing the displacement of an undamped vibrating plate in vacuum is developed by [54] and is briefly reviewed in [55].

For a simply supported plate in free vibration and lying in the x-y plane with sides \( x = (0, a) \) and \( y = (0, b) \), the natural (modal) frequencies are given by

\[
\omega_{kl} = \pi^2 \left[ \left( \frac{k}{a} \right)^2 + \left( \frac{l}{b} \right)^2 \right] \sqrt{\frac{D_E}{\rho}} \quad k, l = 1, 2, \ldots \tag{5.3.1}
\]

where

\[
D_E = \frac{E h^3}{12(1 - \nu^2)} \tag{5.3.2}
\]

and

\[D_E = \text{Plate flexural rigidity}\]
\[ \rho = \text{mass per unit area} \]

\[ E = \text{Young’s modulus} \]

\[ h = \text{Plate thickness} \]

\[ \nu = \text{Poisson’s ratio} \]

and the orthogonal mode shapes are given by

\[ \phi_{kl}(x,y) = \frac{2}{\sqrt{ab}} \sin \left( \frac{k \pi}{a} x \right) \sin \left( \frac{l \pi}{b} y \right) k, l = 1, 2, \ldots \]  \hspace{1cm} (5.3.3) 

The natural frequencies of the first four modes of the plate in the experimental setup are given in following table:

**Table 5.1**

First Four modes of the rectangular plate

<table>
<thead>
<tr>
<th>Mode</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>i k l</td>
<td>Hz</td>
</tr>
<tr>
<td>1 1 1</td>
<td>48.0127</td>
</tr>
<tr>
<td>2 2 1</td>
<td>107.0446</td>
</tr>
<tr>
<td>3 1 2</td>
<td>133.0187</td>
</tr>
<tr>
<td>4 2 2</td>
<td>192.0507</td>
</tr>
</tbody>
</table>
5.4 Experimental Procedure

The procedure of the experiment is as follows. A random number sequence is generated in the computer. The random number sequence is converted to analog voltage using a data translation board and D/A converter. The data translation board, a 12-bit board, converts the random number sequence into 4096 quantization levels giving voltage in the range of ±10 volts. This analog voltage is applied to an amplifier. The amplifier outputs current which follows the voltage command. This input current is applied to the shaker, a piezoelectric transducer, thereby applying force to the plate. This force input is also applied to a load cell. The load cell gives current which is fed to a charge amplifier. The voltage output of the charge amplifier is passed through a low pass filter. The low pass filter is used since here we are only interested in the range of 0-200 Hz, i.e. the first four modes of vibration of the plate. The signal from the low pass filter is converted to an analog signal using an A/D converter. This is the input sequence for the system identification algorithm.

There are 12 accelerometers mounted on the plate to measure the acceleration of the plate to the force applied. The accelerometers measure acceleration in terms of voltage. The voltages are, in turn fed to a bank of charge amplifiers. The output of these amplifiers are passed through a low pass filter to filter out frequencies above 200 Hz. The signals from the low pass filters are converted to analog signals using A/D converters. This is the output sequence for the system identification algorithm.

A block diagram of the experimental set up is shown in Fig. 5.4.1.
The entire set up is monitored by a computer running a software specially developed for modal testing of the rectangular plate. For our system identification algorithm 3 sets of input and output sequences of 1024 data points are measured for each input and output pair, i.e. the shaker and each accelerometer pair.

The Pseudo Linear Identification (PLID) algorithm, covered in the previous chapter and in [51]-[52], was used for identifying the mechanical plate in the frequency range of interest. For each accelerometer and shaker pair 3 different runs with 100, 200 and 300 iterations were computed to understand better the rate of convergence. Three sets of system parameters were identified for each input/output pair and averaged for a more accurate set of system parameters. These identified parameters were compared with the theoretical frequencies calculated from the mathematical model of the plate. The plots in Fig. 5.4.2-5.4.5 show the first four modes of the plate identified at each accelerometer location as compared to the theoretical frequencies. The identified natural frequencies are much closer to the experimentally measured frequencies of the first four modes (50.4 Hz, 110.5 Hz, 133.0 Hz, 190.5 Hz) than to the theoretical (analytical) natural frequencies derived using a assumed mathematical model.
Figure 5.4.1. Block Diagram of the Experimental Setup
Figure 5.4.2. First Mode of the rectangular plate
Second Mode: "***" 100; "x" 200; "+" 300 iterations

Theoretical Second Mode 107.0446 Hz

Rectangular Plate Experiment

Figure 5.4.3. Second Mode of the rectangular plate.
Figure 5.4.4. Third Mode of the rectangular plate
Figure 5.4.5. Fourth Mode of the rectangular plate
6.0 CONCLUSIONS

Three different algorithms for system identification for discrete-time, linear, multi-input and multi-output, heavily damped modal systems from input/output sequences have been developed and simulated.

The first system identification algorithm determines the state-space model in a pseudo controllable/observable canonical forms. The advantage of this technique was that it does not require structural identification, i.e. determination of controllability/observability indices. Instead, a MIMO system is obtained in pseudo controllable/observable form, based on one of the possible sets of admissible pseudo controllable/observable indices, not necessarily the classical set. This algorithm leads to the most convenient representation involving manipulation of well conditioned matrices.

The second algorithm presented was the computational simplification of the Eigensystem Realization Algorithm. In practice the system order $n$ is usually assumed in advance in any experiment. Hence, it is not necessary to perform singular value decomposition of the Hankel matrix formed from the Markov parameters for calculating the unitary matrices $U$ and $V$. The computational
burden of singular value decomposition of the Hankel matrix increases as the system order and the number of inputs and outputs increase. Instead, as suggested in the simplification developed in this work, it is sufficient to select $n$ appropriate columns or rows from the Hankel matrix and by solving the least squares problem having the unique solution involving matrices with only $n$ columns or rows, determines a minimal state space representation of a considered system. For the systems tested on this algorithm a computational improvement of about a factor of 10 was achieved, which is considerable when identifying large systems.

The Pseudo-Linear Identification (PLID) algorithm for simultaneous state and parameter identification was the third algorithm. This algorithm was simulated and tested on a rectangular steel plate. This method involved forming an augmented system that is time-varying due to the multiplication of the state by measurements. The augmented state vector is estimated by a standard Kalman filter. Various systems with differing operating conditions have been simulated and tested with the PLID algorithm. The PLID algorithm has been proved to have deadbeat convergence in the noise free case. It has been noticed that with a considerable amount of noise it takes a larger number of iterations to converge. For MIMO systems the order of the extended system increases considerably, and thereby requires more computational time. This algorithm can be used both for on-line and off-line system identification. The iterative algorithm is the same for both the SISO and MIMO systems, with appropriate matrix dimensions. The identification problem is reduced to estimating the parameters of the transfer function or transfer function matrix, respectively. The PLID algorithm performed very well when applied to identify the modal frequencies of the experimental rectangular plate. The PLID algorithm needed only one good set of data for each
sensor and actuator pair for identifying all the modes of interest, which is in contrast to other methods used for identification in modal analysis which typically require many identical measurement sets for averaging. The PLID algorithm performs very well in identifying heavily damped modes which is of great importance in system identification of structures.
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