OPTIMIZATION OF SLENDER SPACE TRUSSES
UTILIZING A CONTINUUM MODEL

by

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(ABSTRACT)

A method for the incorporation of continuum modeling in the optimization of large discrete structures is presented. The use of a continuum model facilitates decomposition of optimization problems and augments the scope and applicability of the multilevel decomposition method. This new concept is demonstrated by the optimization of slender, multi-bay, beam-like trusses with large numbers of members. An algorithm for the continuum model optimization of the truss is developed and tested against a traditional algorithm that might be used to solve the problem. Data are presented that reflect the advantages of the continuum model method over the traditional in the areas of computational efficiency and robustness. Additionally, design results for the beam-like truss are presented.
Acknowledgements

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1.0 Introduction

It is well known to the designers of load carrying structures that more than one structural design will suit their needs and specifications. With many adequate designs from which to choose, engineers are constantly in search of the better designs, those which are more economical, easier to manufacture, or lighter. This searching for the "optimal" design has led to the field of study known as structural optimization. If a design is to be accepted or rejected on the basis of its performance in certain structural analyses, as opposed to experimental procedures, then numerical techniques can be used to search for the optimal design. Structural optimization has become the study and use of such numerical techniques.

The standard structural optimization problem is stated as

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{such that} & \quad g_p(x) \leq 0 & p = 1, \ldots, q \\
& \quad x^l_r \leq x_r \leq x^u_r & r = 1, \ldots, s.
\end{align*}
\]

The vector \( x \) is a set of \( s \) design variables that completely defines a structural design. The design variables are restricted by upper and lower bounds which are stored in the vectors \( x^u \) and \( x^l \). Minimization is performed on the objective function \( f(x) \), which is a mathematical representation of the objective of the optimization, such as weight or cost. The minimization is restricted by the constraint functions in the vector \( g(x) \), which represent the \( q \) constraints on the design. Each of the components of \( g(x) \) should be zero or negative for a particular value of \( x \) to be within the feasible design space.
Many successful algorithms have been developed for the solution of optimization problems. Called numerical search techniques, they start from an initial design and search the design space for the design which has the minimum value of the objective function. Many structural designs have been optimized using these algorithms. The techniques have been continuously developed and refined over the years such that they can take on more complex design problems. The reason for this ongoing research is that there are still factors which can impede and often stall numerical search techniques. One of these is an excessive amount of nonlinearity in the objective and constraint functions. Probably the most obstructive factor, however, is the mere size of the problem. Almost all of the algorithms developed so far become less and less robust, or reliable, as the size of the optimization problem increases. Some problems have so many design variables or constraints that they become intractable. Other large problems can be solved but it is infeasible to do so because of the large amounts of computer storage and processor time required.

An attractive solution to the difficulties caused by large problems has been developed in the past several years. The process is called decomposition and involves dividing a large optimization problem into a group of smaller problems which are only weakly connected. Decomposition is advantageous from the tractability point of view in that generally the smaller problems are independently soluble. Benefits are also gained in the computational expense because the processor time required for an optimization problem is usually more than a linear function of the problem size (Haftka, Gürdal, and Kamat, 1990). For example, doubling the number of variables or constraints in a problem usually more than doubles the time required to solve it. Accordingly, significant time savings are realized when a large problem is broken down into smaller ones. Because it is generally not possible to completely uncouple a group of subproblems, large problems are often broken down into a group of \( w \) subproblems and a coordination problem, which involves the coupling variables and coupling constraints. The vector \( \mathbf{x}^T \) is partitioned into \( \{ \mathbf{y}, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_w \}^T \), where the design variables in \( \mathbf{y} \) are those that affect two or more of the subproblems, thereby coupling them. Design variables in the vector \( \mathbf{x}_v \) only affect the \( v \)th subproblem. The constraint vector \( \mathbf{g}^T \) is similarly partitioned into \( \{ \mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_w \}^T \), where the constraints
in $g_0$ are functions of $y$ only and those in $g_v$ are functions of $y$ and $x_v$. The objective function is similarly divided and the problem is rewritten:

minimize $f_0(y) + \sum_{v=1}^{w} f_v(y, x_v)$

such that $g_{0p}(y) \leq 0 \quad p = 1, \ldots, q_0$

$g_{vp}(y, x_v) \leq 0 \quad v = 1, \ldots, w; \quad p = 1, \ldots, q_v$

$y^l_r \leq y_r \leq y^u_r \quad r = 1, \ldots, s_0$

$x^l_{vr} \leq x_{vr} \leq x^u_{vr} \quad v = 1, \ldots, w; \quad r = 1, \ldots, s_v$

The integer $q_0$ is the number of system constraints, $q_v$ is the number of constraints in the subsystem $v$, $s_0$ is the number of system variables, and $s_v$ is the number of variables in the subsystem $v$. After decomposition, the $w$ subproblems are

minimize $f_v(y, x_v)$

such that $g_{vp}(y, x_v) \leq 0$

$x^l_{vr} \leq x_{vr} \leq x^u_{vr}$

where $v$ varies from 1 to $w$. 

Introduction
The foundation of the coordination problem is

\[
\begin{align*}
\text{minimize} & \quad f_0(y) \\
\text{such that} & \quad g_{op}(y) \leq 0 \\
& \quad y^l_r \leq y_r \leq y^u_r.
\end{align*}
\]

Extra terms concerning the subproblems need to be added to this foundation in order to effect overall coordination.

Decomposition of a structural optimization problem into a number of mutually independent subproblems linked by a coordination problem leads naturally to using a multilevel scheme for the simultaneous solution of the problems. Multilevel optimization is a method in which more than one optimization process is involved in the overall scheme. Minimization is carried out by iterating back and forth amongst the two or more processes, which are arranged in a hierarchical structure. Each process has control of a particular subset of the design variables, and while it is optimizing that subset all others are held constant. Iteration proceeds until all the processes have converged. Note that decomposition is not necessary for the use of a multilevel scheme. Multilevel optimization is an independent methodology that can be used on problems which are not divided into subproblems, but for some reason it is advantageous to optimize certain groups of variables separately.

Decomposition and the subsequent use of multilevel optimization have proven to be useful on several design problems. A classical demonstration example of decomposition was introduced by Sobieszczanski-Sobieski (1982) and has been used by other researchers in later work. The problem involves minimizing the weight of the portal frame shown in Figure 1.1. There are three I-beams whose cross-sectional dimensions are to be designed to carry the loads shown. Constraints on the design include maximum displacements of the structure and maximum
Figure 1.1 Standard portal frame example
allowable stresses in each of the three members. For the purposes of optimization, the structure is decomposed into three subsystems, each beam being a subsystem. Upper level, or system, variables are introduced which include the area and moment of inertia of each beam \( (y^T = \{ A_1, I_1, A_2, I_2, A_3, I_3 \}^T) \). These six quantities are all that are needed to perform global structural analysis of the framework. The maximum displacement constraints are, therefore, global constraints \( g_{op}(y) \) and are included on the system level. The six cross-sectional dimensions of each beam become the subsystem variables \( (x_v^T = \{ b_1, b_2, h, t_1, t_2, t_3 \}^T, v = 1, 2, 3) \). They are optimized while the upper level variables of the structure are held constant. Note that when this set of system variables is held constant the internal forces between the members are also constant. Because the forces acting on the subsystems are invariant during subsystem optimization, the maximum stress constraints can be relegated to the lower level, becoming subsystem constraints \( g_{up}(y, x_v) \). In fact, it is only because the forces are invariant that the local designs can be handled independently of one another during the lower level stage. If lower level variables in one subsystem perturb the force in other subsystems, then the two subproblems would be coupled and decomposition would not be possible. It was shown by Sobieszczanski-Sobieski and coworkers that this type of formulation leads to the same final designs as a conventional formulation (Sobieszczanski-Sobieski, James, and Dovi, 1983).

It has been shown that decomposition and multilevel schemes have beneficial effects when used in structural optimization. There has been a limit, however, on how far decomposition can be taken. The example problem just shown demonstrated that individual structural members can be subsystems. Any skeletal structure can, therefore, be decomposed into its individual members for multilevel optimization. Difficulties related to problem size can still be present (in the upper level optimization) if there are many structural members. If there are too many members the problem may still be intractable in spite of decomposition. In order to solve such a problem, further decomposition is necessary. Larger subsystems need to be implemented to create yet another level in the multilevel problem. These larger subsystems will be composed of several structural members and will actually be independent trusses or frames. Illustrated in Figure 1.2 is a frame divided into such subsystems. Under this formulation, the substructures will be
Figure 1.2  Example of subsystems with many members
designed on the top level, the structural members on the middle level, and the cross-sectional detail on the lower level. This hierarchy should allow each individual optimization problem to be small enough in size to be soluble.

The foregoing multilevel formulation is only possible if a sufficient set of system variables can be found. Recall that in the portal frame example extra variables were added so that the stress constraints of different beams would be uncoupled and decomposition could take place. More specifically, the areas and moments of inertia were added to the list of variables so that they could represent the beam in the upper level structural analysis. If they were not introduced, then beam dimensions could not have been put on the lower level, as they (the beam dimensions) would be coupled to the response and therefore the design of the other beams. Multilevel optimization was possible only because the area and moment of inertia variables were available to uncouple the subsystem problems. In the same manner, larger subsystems can be used only if uncoupling system variables can be found.

It is herein suggested that a viable solution to the lack of system variables is the use of a continuum model in the optimization scheme. A continuum model for a skeletal structure is a continuous solid which has approximately the same mass and stiffness characteristics as the structure. The usefulness of continuum models thus far has been in approximate structural analyses. A skeletal structure with many degrees of freedom can be replaced by a segmented solid structure which, when using the basic strength of materials formulae, can be analyzed with fewer degrees of freedom. Continuum models have been developed and continuously improved for a number of truss and frame configurations. Included are plate-like, beam-like, and general block truss structures. Previous researchers have found that structural analyses carried out using appropriate continuum representations lead to results that are highly accurate.

If large, multi-member subsystems of a truss are replaced by continuum models on the upper level of optimization, then system variables are provided by the models themselves. The equivalent mass and stiffnesses of a representative solid can be system variables in that they can completely represent the subsystem on the upper level. Reasonably accurate structural analysis is possible using just
the continuum properties and, therefore, the upper level constraints can be calculated. The upper level process would become, in effect, the optimization of a segmented solid structure. Lower level optimization would involve the weight minimization of a skeletal substructure restricted by local constraints (the local forces can be approximated from upper level structural analysis), and the requirement that it have specified gross stiffnesses when it is transformed into a solid.

It is the intent of this investigation to demonstrate the usefulness of continuum modeling in the design optimization of skeletal structures. To accomplish this goal, a particular type of truss is selected and used as an example. An appropriate continuum model is developed that accurately approximates the properties of the truss. Using this model as a tool for decomposition, a multilevel optimization scheme for weight minimization is created. In order to gauge the benefits that this type of formulation provides, the continuum model method is compared with a traditional algorithm that would be used to minimize the weight of the truss.

The structure used in this study is a long, slender (beam-like) truss. As illustrated in Figure 1.3, the truss is one bay wide, one bay tall, and several bays long. Because it is long and slender its deformations strongly resemble those of a structural beam. Each bay is, therefore, modeled as a one-dimensional beam segment, as depicted in Figure 1.3. Under the decomposition scheme used, each bay is a subsystem. The beam stiffnesses of each continuous beam segment are the variables used on the upper level of the multilevel scheme. The actual design variables are moved to the lower level. Constraints used on the upper level are restrictions on the global deflections. On the lower level there are local buckling constraints.
Figure 1.3  Truss to be used in this study modeled as a segmented beam
This particular truss is of importance to the U.S. space program. It has been included in candidate designs of the future orbiting space station because it is lightweight, easy to assemble, and can span large distances. The truss is used to separate station components and to position directional equipment such as solar panels and radio antennae. As with any component which is to be launched into space, light weight is critical for the truss. Equally as critical is that the truss should not fail while in use, since repairs would be very costly if not impossible. For these two reasons, the beam-like truss is ideal for the application of constrained weight minimization.

The following chapters provide a detailed account of the current research. Chapter 2 contains a review of the pertinent literature. In Chapter 3 the continuum model is developed and both methods of calculating structural response (by the continuum approximation and by traditional finite element techniques) are formulated. The actual optimization problem is outlined in Chapter 4, along with both methods that are developed to solve it. In Chapter 5 the two methods are compared in their performance in optimizing a set of sample problems. Conclusions drawn from these comparisons are given in Chapter 6.
2.0 Literature Review

To the extent of the author's knowledge, there has been no published research wherein a continuum model was used to permit decomposition of a structural optimization problem. That being the case, the literature review for the current research includes reviews of the two main topics, (1) multilevel decomposition, and (2) continuum models for beam-like trusses.

2.1 MULTILEVEL DECOMPOSITION

The first approach to decomposition appears to have been made in 1972. Kirsch, Reiss, and Shamir introduced a method where substructures are identified and optimized independently using approximate reanalysis. First an accurate global analysis is carried and then each substructure is optimized while the forces from adjoining substructures are held constant. The entire structure is then analyzed again and the forces updated. There is no global coordination in the method and therefore no comprehensive global minimization. Additionally, it is not applicable to global constraints.

Two multilevel coordination schemes for decomposed problems were later developed by Kirsch (1975). Called model and goal coordination, the schemes subject the design to global minimization but can not handle variable internal forces or global constraints. Kirsch and Moses (1979) solved this problem by using internal forces and displacements as system variables in the earlier methods and thereby were able to apply them to statically indeterminate problems with global constraints.
Schmit and Ramanathan (1978) were able to achieve fully coordinated multilevel optimization that included varying forces. Due to innovative selection of the system variables and subsystem objectives, their method can carry out optimization on the subsystem level without greatly changing the forces. The objective is on the system level and all types of global constraints can be included.

In 1982, Sobieszczanski-Sobieski outlined a "Blueprint for Development" of a coordinated, multilevel decomposition method which improves on the method of Schmit and Ramanathan by using optimum sensitivity derivatives of the subsystem optimizations to modify variables on the system level. The method was developed to serve in large multidisciplinary design problems. In keeping with the guidelines of the blueprint, Sobieszczanski-Sobieski, Barthelemy, and Riley (1982) demonstrated methods for calculating analytically the sensitivity derivatives of optimum solutions with respect to problem parameters and discussed applications of these derivatives. Sobieszczanski-Sobieski, James, and Dovi (1983) later fully developed the method and implemented it in a two-level optimization of the portal frame example with satisfactory results. In 1987, Sobieszczanski-Sobieski, James, Riley demonstrated the versatility of multilevel optimization using optimum sensitivity by extending the portal frame example to three levels.

Haftka (1984) pointed out the discontinuous nature of subsystem optimum sensitivity derivatives. To combat this problem, he developed an improved decomposition scheme which, by use of penalty functions on the lower levels, smoothed the sensitivity derivatives. To further understanding of multilevel decomposition methods, Thareja and Haftka (1986) presented a study of the effects of equality constraints when they are used on lower levels. Numerical difficulties in the sensitivity derivatives were examined in detail.

An alternative approach to those of Sobieszczanski-Sobieski and Haftka was formulated by Kirsch (1985). Structural behavior quantities such as nodal displacements are included in the list of system variables. Use of multilevel optimization resulted in greater benefits because fewer analyses were necessary.
Barthelemy and Riley (1988) also presented a detailed study which furthered the understanding of decomposition methods. They investigated the potential use of convex constraint approximations and temporary constraint deletions in decomposition schemes.

A final reference for multilevel decomposition is Chapter 10 in the structural optimization textbook by Haftka, Güral, and Kamat (1990), which includes an informative monograph on the subject.
2.2 BEAM MODELS

The first continuum model for a beam-like truss appears to have been formulated by Noor, Anderson, and Greene in 1978. They used assumed temperature and displacement fields to relate a triangular truss to a continuum model and equated the kinetic and thermoelastic energies of the two. Numerical experiments using the resulting beam stiffnesses demonstrated the good accuracy of the model when applied in static and dynamic analyses of a cantilevered truss. Later, Noor and Andersen (1978) expanded the theory to rectangular trusses. The model included six Timoshenko beam strains as well as warping and shear deformation in the plane of the cross section. Again, good results were obtained when the model stiffnesses were used in solving the differential beam equations. Further developing the work, Noor and Weisstein (1981) presented a procedure for calculating the geometric stiffnesses of the continuum model. Using these stiffnesses in elastic stability analyses, they were able to accurately calculate the critical buckling loads of rectangular cantilevered trusses. Noor and Russell (1986) extended the method once more. In the last paper potential anisotropy was included in the derivations. The resulting beam model accurately represented the coupling between the various beam forces and deflections.

Sun, Kim, and Bogdanoff (1981) presented an alternate method of calculating the beam stiffnesses. They performed numerical experiments on the repeating cell of a beam-like truss to obtain the stiffnesses semi-empirically. Similar testing of the model produced highly accurate results. Necib and Sun (1989) continued the work by including coupling between different types of beam deflections. A high order Timoshenko beam finite element was formulated for use in vibration analyses.

An alternative formulation of the beam model was offered by Renton (1984). He found the characteristic deflections of the discrete truss in terms of polynomial functions. These were used to find the equivalent beam stiffnesses.
Abrate and Sun considered geometrically nonlinear trusses in 1983. It was shown that geometrically nonlinear beam models could predict the behavior of the trusses in both static and dynamic analyses. In 1986, Berry and Yang developed a continuous, nonlinear finite element to be used in multi-element continuum modeling of beam-like trusses. They demonstrated that using more than one element in a continuum model increased the accuracy of nonlinear static, vibration and stability analysis.

Continuum model technology was extended to beam-like frames by McCallen and Romstad (1988). They developed a finite element that could accurately model the nonlinear behavior of beam-like structures that have either pin joints or rigid joints.

In 1990, Lee developed a less complicated model that uses energy equivalence and transformation matrices between the continuum and discrete degrees of freedom. His model was shown to predict structural response as reliably as the former models.
3.0 Formulations

3.1 Conventions

This investigation considers the design of a particular type of flexible space truss. Pictured in Figure 3.1, it is a rectangular truss whose bays span in only one direction. This longitudinal direction will be the 1-direction. The other two coordinate directions, 2 and 3, are perpendicular to the 1 direction and are oriented parallel to the sides of the structure. The bays that compose the truss are numbered in the positive 1 direction. Joints are of the ball and socket type and the ideal truss assumption will be employed in its analysis. When cantilevered at one end the structure is stable and statically indeterminate. The two-force members that connect the joints have a cylindrical cross section, providing the largest moment of inertia per unit volume. They have a large enough slenderness ratio so that the Euler buckling for pinned-pinned columns is applicable and can be assumed to be the dominant failure mode.

For the purpose of later developments, a typical bay is separated from the rest of the structure. It is cut away at the two planes that contain the joints that define the bay. Members that reside in these two planes are divided. Part of their mass and stiffness belongs to the isolated bay and the rest remains with the two adjacent bays. The bay which is separated in this way is an independent structure with no supports. Illustrated in Figure 3.2, it has twenty-four members and eight joints. There are three bay dimensions and forty-eight cross-sectional dimensions. The bay dimensions are depicted in Figure 3.2. The variable \( L \) is the length of the bay in the 1-direction, \( W \) is the width of the bay in the 2-direction, and \( H \) is the
Figure 3.1  Coordinate directions for the beam-like space truss
Figure 3.2 Individual bay
height in the 3-direction. Expressions involving quantities associated with the diagonal members tend to be lengthy when written in terms of \( L, W, \) and \( H. \) For this reason, more variables are defined. The lengths of the three sets of diagonals are defined by \( B_1, B_2, \) and \( B_3 \) where

\[
B_1 = \sqrt{L^2 + W^2}, \quad B_2 = \sqrt{W^2 + H^2}, \quad B_3 = \sqrt{L^2 + H^2},
\]

and the corresponding direction cosines are defined by \( c_1 \) through \( c_6 \):

\[
c_1 = \frac{W}{\sqrt{L^2 + W^2}} \quad c_2 = \frac{L}{\sqrt{L^2 + W^2}}
\]

\[
c_3 = \frac{W}{\sqrt{W^2 + H^2}} \quad c_4 = \frac{H}{\sqrt{W^2 + H^2}} \quad (3.1)
\]

\[
c_5 = \frac{H}{\sqrt{L^2 + H^2}} \quad c_6 = \frac{L}{\sqrt{L^2 + H^2}}
\]

Finally a set of special relationships (whose utility will become obvious in the following Subsections) are defined by \( D_1 \) through \( D_9 \):

\[
D_1 = \frac{L^2}{(L^2 + W^2)^{3/2}} \quad D_4 = \frac{W^2}{(W^2 + H^2)^{3/2}} \quad D_7 = \frac{L^2}{(L^2 + H^2)^{3/2}}
\]

\[
D_2 = \frac{W^2}{(L^2 + W^2)^{3/2}} \quad D_5 = \frac{H^2}{(W^2 + H^2)^{3/2}} \quad D_8 = \frac{H^2}{(L^2 + H^2)^{3/2}} \quad (3.2)
\]

\[
D_3 = \frac{LW}{(L^2 + W^2)^{3/2}} \quad D_4 = \frac{WH}{(W^2 + H^2)^{3/2}} \quad D_7 = \frac{LH}{(L^2 + H^2)^{3/2}}
\]
The forty-eight cross-sectional dimensions include the nominal radii \((r)\) and the wall thicknesses \((t)\) of the twenty-four members. In each member the nominal radius will be sufficiently larger than the wall thickness so that approximate formulae for the area \((A)\) and the moment of inertia \((I)\) can be used:

\[
A \approx 2\pi rt
\]
\[
I \approx \pi r^3 t
\]

In later portions of this study, trusses are completely divided into individual bays. In addition, sometimes trusses are assembled from individual bays. Each time a transformation occurs, members at the interface between two adjacent bays have to be divided or combined. This dividing and recombining will be carried according to the requirement that the composite member have the sum of the areas and the sum of the moments of inertia of the two component members. Mathematically these requirements are

\[
A = A_1 + A_2
\]
\[
I = I_1 + I_2,
\]

where the component values are subscripted and the composite values are not. These are replaced with actual cross-sectional dimensions:

\[
2\pi rt = 2\pi r_1 t_1 + 2\pi r_2 t_2
\]
\[
\pi r^3 t = \pi r_1^3 t_1 + \pi r_2^3 t_2
\]

Simultaneous solution of Equations 3.7 and 3.8 for the composite member dimensions yields the necessary relations between the six dimensions involved:
\[ r^2 = \frac{(r_1 t_1 + r_2 t_2)}{r_1 t_1 + r_2 t_2} \]  \hspace{1cm} (3.9)

\[ t^2 = \frac{(r_1 t_1 + r_2 t_2)^3}{r_1^3 t_1 + r_2^3 t_2} \]  \hspace{1cm} (3.10)

When these requirements are met, the composite member can carry, without failure, the sum of the loads that the component members can carry without failure.

For the sake of generality, the dimensional variables and every other variable that will be used in analysis and optimization will be nondimensionalized. The only dimensional parameters to be used in the formulations will be the elastic modulus of the material, \( E^* \), the maximum allowable radius of the members, \( r^* \), and an arbitrary unit of time, which will be seconds (s\(^*\)). All other quantities will be written in terms of these variables. Using a plain variable to denote a nondimensional quantity and adding a superscript asterisk to denote its dimensional counterpart, nondimensionalization is achieved:

- a mass, \( m^* = m E^* r^* s^* \)
- a density, \( \rho^* = \frac{\rho E^* s^2}{r^*} \)
- a force, \( f^* = f E^* r^* \)
- a length, \( l^* = l r^* \)
- an area, \( a^* = a r^* \)
- a volume, \( v^* = v r^* \)

- etc.

As an example, a simply supported twenty foot member made of 6061-T6 aluminum alloy with a maximum radius of six inches and a half inch wall thickness has the following data:
<table>
<thead>
<tr>
<th></th>
<th>dimensional</th>
<th>nondimensional</th>
</tr>
</thead>
<tbody>
<tr>
<td>length</td>
<td>20 ft.</td>
<td>40</td>
</tr>
<tr>
<td>cross. area</td>
<td>18.85 in²</td>
<td>0.5236</td>
</tr>
<tr>
<td>volume</td>
<td>4524 in³</td>
<td>20.94</td>
</tr>
<tr>
<td>specific weight</td>
<td>0.098 lb/in³</td>
<td>5.88 ×10⁻⁸</td>
</tr>
<tr>
<td>weight</td>
<td>443.4 lb.</td>
<td>1.232 ×10⁻⁶</td>
</tr>
<tr>
<td>elastic modulus</td>
<td>10 ×10⁶ psi</td>
<td>1.0</td>
</tr>
<tr>
<td>moment of inertia</td>
<td>339 in⁴</td>
<td>0.2618</td>
</tr>
<tr>
<td>Euler buckling load</td>
<td>581400 lb.</td>
<td>0.001615</td>
</tr>
</tbody>
</table>

This particular nondimensionalization causes the stiffness values of the truss members to be on the order of unity and the force ranges to be three to four orders of magnitude lower. The methods and analyses developed under this system can be used for a truss of any size by appropriately scaling the input variables. More importantly, the methods and analyses can be studied without regard to any particular design problem.
3.2 CONTINUUM MODEL

3.2.1 Derivation of Stiffness Matrix

The development of the continuum model used for the current research is based on the development by Noor and Russell for an anisotropic beam model (Noor and Russell, 1986). The stiffness portion of the model is developed from a single truss bay. Given that the bay is only loaded at its joints, the twenty-four structural members can be considered to be two-force members. They can, therefore, be modeled with a bar finite element. This element has six degrees of freedom and is shown along with its stiffness matrix in Figure 3.3. The twenty-four element stiffness matrices can be assembled into the global stiffness matrix, \([ K ]\), for the representative bay. This matrix relates the force vector, \(\{ Q \}\), and the displacement vector \(\{ q \}\):

\[
\{ Q \} = \begin{bmatrix} K \end{bmatrix} \{ q \}
\]  

(3.11)

Note that the bay structure has twenty-four degrees of freedom so the stiffness matrix is 24 by 24 and the force and displacement vectors are 24 by 1.

In order to develop the stiffness model from Equation 3.11, the force and displacement vectors are first separated into components. In each case, twenty-four modal vectors are defined such that they are orthogonal to one another. A set of \(n\) orthogonal vectors serves as a basis of \(n\) dimensional space and therefore any \(n\)
\( C_1 = \Delta X_1 \)
\( C_2 = \Delta X_2 \)
\( C_3 = \Delta X_3 \)

\[
L = \sqrt{\Delta X_1^2 + \Delta X_2^2 + \Delta X_3^2}
\]

\[
K = \frac{EA}{L^3}
\begin{bmatrix}
  C_1 C_1 & C_1 C_2 & C_1 C_3 & -C_1 C_1 & -C_1 C_2 & -C_1 C_3 \\
  C_2 C_2 & C_2 C_3 & -C_2 C_2 & -C_2 C_3 & -C_2 C_3 & -C_2 C_3 \\
  C_3 C_3 & -C_3 C_3 & -C_3 C_3 & -C_3 C_3 & -C_3 C_3 & -C_3 C_3 \\
  C_1 C_1 & C_1 C_2 & C_1 C_3 & & & \\
  C_2 C_2 & C_2 C_3 & & & & \\
  C_3 C_3 & & & & &
\end{bmatrix}
\]

sym.

**Figure 3.3** Bar finite element and stiffness matrix
dimensional vector can be defined as a weighted sum of the modal vectors. When these modal vectors, or modes, are placed in the columns of a matrix, the result is an orthogonal transformation matrix. Defining the transformation matrix as \[ [ T ] \]
for the force vector and \[ [ t ] \]
for the displacement vector, the transformations are

\[
\{ Q \} = [ T ] \{ \hat{F} \}
\]

(3.12)

\[
\{ q \} = [ t ] \{ \hat{e} \},
\]

(3.13)

where \( \{ \hat{F} \} \) is the vector of magnitudes for the force modes and \( \{ \hat{e} \} \) is the vector of magnitudes for the displacement modes.

The modes used in the transformations are carefully chosen for the purposes of this development. In the case of the force modes, the first six are chosen to represent forces that act on a continuous beam. That is, the modes are created to closely model six individual beam forces. The first six force modes (group 1) are illustrated in Figures 3.4 and 3.5, along with the continuum forces they represent. Included are the forces that are normally considered in Timoshenko beam theory: the axial force in the longitudinal direction \( (N) \), the bending moments in two perpendicular transverse planes \((M_2 \text{ and } M_3)\), the shear forces in two perpendicular transverse planes \((Q_2 \text{ and } Q_3)\), and the torque about the longitudinal axis \( (T) \). Continuing with the force modes, the next twelve (group 2) represent forces that are neglected in standard beam theories. Included here are axial forces in transverse directions, bending moments and shear forces in transverse planes, and warping forces. The last six modes (group 3) are separated from the others for the reason that they are not self equilibrating. In this group are force modes whose resultants are forces in the three coordinate directions and moments about the three coordinate axes. The six modes are illustrated in Figure 3.6. Summarizing, there are three groups of modes: six Timoshenko forces, twelve higher order forces, and six unbalanced force modes.
Figure 3.4  First three Timoshenko force modes
Figure 3.5 Last three Timoshenko force modes
Figure 3.6  Nonequilibrium force modes
The 24 displacement modes are separated in a similar manner. The first six modes (group 1) represent strain quantities in Timoshenko beam theory. Illustrated in Figures 3.7 and 3.8, these quantities are axial strain in the 1 direction ($\epsilon$), two curvatures in transverse planes ($\kappa_2$ and $\kappa_3$), two shear strains in transverse planes ($\gamma_2$ and $\gamma_3$), and an angle of twist about the longitudinal axis ($\theta$). Note that the strain quantities match one by one with the Timoshenko forces in beam equations. Likewise, the middle twelve displacement modes (group 2) match the middle twelve force modes, being transverse strains, transverse curvatures and warpings. Finally, the last six displacement modes (group 3, Figure 3.9) are separated from the others because they represent rigid body motions. Included are rigid body translations in the three coordinate directions and rigid body rotations about the three coordinate axes. Here again the displacement modes match the force modes. In all cases, a force mode and its corresponding displacement mode are equivalent. The result of this equality is that the force and displacement vectors undergo the same transformation. The single matrix $[T]$ will be used to denote the transformation.

The two modal transformations are substituted into the stiffness equation:

$$
\begin{bmatrix}
T
\end{bmatrix}
\begin{Bmatrix}
\hat{F}
\end{Bmatrix}
= 
\begin{bmatrix}
K
\end{bmatrix}
\begin{bmatrix}
T
\end{bmatrix}
\begin{Bmatrix}
\hat{\epsilon}
\end{Bmatrix}
$$

(3.14)

Rearranging,

$$
\begin{Bmatrix}
\hat{F}
\end{Bmatrix}
= 
\begin{bmatrix}
T
\end{bmatrix}^{-1}
\begin{bmatrix}
K
\end{bmatrix}
\begin{bmatrix}
T
\end{bmatrix}
\begin{Bmatrix}
\hat{\epsilon}
\end{Bmatrix}.
$$

(3.15)

A new stiffness matrix can now be defined that relates the magnitudes of the force modes to the magnitudes of the deflection modes:

$$
\begin{Bmatrix}
\hat{F}
\end{Bmatrix}
= 
\begin{bmatrix}
\hat{K}
\end{bmatrix}
\begin{Bmatrix}
\hat{\epsilon}
\end{Bmatrix}
$$

(3.16)
Figure 3.7  First three Timoshenko displacement modes
Figure 3.8  Last three Timoshenko displacement modes

shear strain in 12 plane

shear strain in 13 plane

angular twist about 1 axis
Figure 3.9  Rigid body displacement modes
where
\[
\begin{bmatrix}
\hat{\mathbf{K}}
\end{bmatrix} = 
\begin{bmatrix}
T
\end{bmatrix}^{-1}
\begin{bmatrix}
\mathbf{K}
\end{bmatrix}
\begin{bmatrix}
T
\end{bmatrix}.
\]

The modes can now be divided into their three groups. The six Timoshenko modes are listed first, then the twelve higher order modes, then the six rigid body modes. The stiffness matrix is also split into nine submatrices:

\[
\begin{align*}
\begin{bmatrix}
\{\hat{F}_1\} \\
\{\hat{F}_2\} \\
\{\hat{F}_3\}
\end{bmatrix} &= 
\begin{bmatrix}
[\hat{K}_{11}] & [\hat{K}_{12}] & [\hat{K}_{13}] \\
[\hat{K}_{21}] & [\hat{K}_{22}] & [\hat{K}_{23}] \\
[\hat{K}_{31}] & [\hat{K}_{32}] & [\hat{K}_{33}]
\end{bmatrix}
\begin{bmatrix}
\{\varepsilon_1\} \\
\{\varepsilon_2\} \\
\{\varepsilon_3\}
\end{bmatrix} \\
\text{(3.17)}
\end{align*}
\]

Some of these force and displacement components can be neglected. Forces on the bay must be in equilibrium so \(\{\hat{F}_3\}\) must be zero. Also rigid body motions, \(\{\varepsilon_3\}\), cause no stress in the bay members so they will be excluded from the stiffness formulation. It is assumed that the slender truss will behave in a manner similar to a beam and that the beam forces, \(\{\hat{F}_1\}\), will dominate the others. This is the continuum beam approximation. As a result, \(\{\hat{F}_2\}\) is set equal to zero, leaving

\[
\begin{align*}
\begin{bmatrix}
\{\hat{F}_1\} \\
0 \\
0
\end{bmatrix} &= 
\begin{bmatrix}
[\hat{K}_{11}] & [\hat{K}_{12}] & [\hat{K}_{13}] \\
[\hat{K}_{21}] & [\hat{K}_{22}] & [\hat{K}_{23}] \\
[\hat{K}_{31}] & [\hat{K}_{32}] & [\hat{K}_{33}]
\end{bmatrix}
\begin{bmatrix}
\{\varepsilon_1\} \\
\{\varepsilon_2\} \\
0
\end{bmatrix}. \\
\text{(3.18)}
\end{align*}
\]
The first two equations can be used to relate the beam forces to the beam strains:

\[
\begin{aligned}
\{ \hat{F}_1 \} &= \begin{bmatrix} \hat{K}_{11} \end{bmatrix} \{ \hat{\epsilon}_1 \} - \begin{bmatrix} \hat{K}_{12} \\ \hat{K}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \hat{K}_{21} \end{bmatrix} \{ \hat{\epsilon}_1 \} \\
\end{aligned}
\]

(3.19)

Or,

\[
\{ \hat{F}_1 \} = \begin{bmatrix} \hat{C} \end{bmatrix} \{ \hat{\epsilon}_1 \},
\]

where

\[
\begin{aligned}
\begin{bmatrix} \hat{C} \end{bmatrix} &= \begin{bmatrix} \hat{K}_{11} \end{bmatrix} - \begin{bmatrix} \hat{K}_{12} \\ \hat{K}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \hat{K}_{21} \end{bmatrix}.
\end{aligned}
\]

(3.20)

The entries of \( \{ \hat{F}_1 \} \) and \( \{ \hat{\epsilon}_1 \} \) are the magnitudes of the force and displacement modes. They can be related to the traditional definitions of the force and strain quantities for a continuous beam. This involves approximation of the quantities with respect to the truss bay dimensions. In each case, the beam quantity can be expressed as the magnitude of the mode multiplied by a factor. The factors are functions of the dimensions of the bay. The twelve magnitudes are related as follows:

**axial deformation:**

\[
\begin{aligned}
( \hat{F}_1 )_1 &= (1/4) N \\
( \hat{\epsilon}_1 )_1 &= (L/2) \epsilon
\end{aligned}
\]

**bending in the 12 plane:**

\[
\begin{aligned}
( \hat{F}_1 )_2 &= (1/2W) M_2 \\
( \hat{\epsilon}_1 )_2 &= (WL/4) \kappa_2
\end{aligned}
\]
bending in the 13 plane:

\[
\begin{align*}
\left( \bar{F}_1 \right)_3 &= \left( \frac{1}{2} \bar{H} \right) M_3 \\
\left( \bar{\varepsilon}_1 \right)_3 &= \left( \frac{W L}{4} \right) \kappa_3
\end{align*}
\]

shear in the 12 plane:

\[
\begin{align*}
\left( \bar{F}_1 \right)_4 &= \left( \frac{\sqrt{L^2 + W^2}}{4W} \right) Q_2 \\
\left( \bar{\varepsilon}_1 \right)_4 &= \left( \frac{\sqrt{L^2 + W^2}}{2 \left( \frac{W}{L} + L/W \right)} \right) \gamma_2
\end{align*}
\]

shear in the 13 plane:

\[
\begin{align*}
\left( \bar{F}_1 \right)_5 &= \left( \frac{\sqrt{L^2 + H^2}}{4H} \right) Q_3 \\
\left( \bar{\varepsilon}_1 \right)_5 &= \left( \frac{\sqrt{L^2 + H^2}}{2 \left( \frac{H}{L} + L/H \right)} \right) \gamma_3
\end{align*}
\]

torsion:

\[
\begin{align*}
\left( \bar{F}_1 \right)_6 &= \left( \frac{1}{2 \sqrt{W^2 + H^2}} \right) T \\
\left( \bar{\varepsilon}_1 \right)_6 &= \left( \frac{L \sqrt{W^2 + H^2}}{2} \right) \theta
\end{align*}
\]

(3.21)

Derivations of the axial, bending, shear, and torsion factors are included in Figures 3.10 through 3.13. The factors can be brought out of \{ \bar{F}_1 \} and \{ \bar{\varepsilon}_1 \} and multiplied into the stiffness matrix \( \left[ \bar{C} \right] \). The result is the final continuum stiffness matrix \( \left[ C \right] \) which relates the continuum force and strain quantities:

\[
\begin{align*}
\{ F \} &= \left[ C \right] \{ \varepsilon \}
\end{align*}
\]

(3.22)
Figure 3.10 Relation of modal magnitudes to axial quantities

Continuum Model
\[ M_2 = 2FW \]

\[
F = \left( \frac{1}{2W} \right) M_2
\]

\[
\kappa_2 = 2 \left( \frac{2\Delta}{W} \right) / L
\]

\[
\Delta = \left( \frac{WL}{4} \right) \kappa_2
\]

Figure 3.11  Relation of modal magnitudes to bending quantities
Figure 3.12 Relationship of modal magnitudes to shear quantities
\[ T = 4 \left( \frac{\sqrt{W^2 + H^2}}{2} \right)^F \]
\[ F = \left( \frac{1}{2\sqrt{W^2 + H^2}} \right) T \]
\[ \theta = \frac{2}{L} \left( \frac{\Delta}{\sqrt{W^2 + H^2}} \right) \]
\[ \Delta = \left( \frac{L \sqrt{W^2 + H^2}}{2} \right) \]

Figure 3.13  Relationship of modal magnitudes to torsional quantities
or

\[
\begin{align*}
\begin{bmatrix}
N \\ M_2 \\ M_3 \\ Q_2 \\ Q_3 \\ T
\end{bmatrix} &=
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\
C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\
C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon \\ \kappa_2 \\ \kappa_3 \\ \gamma_2 \\ \gamma_3 \\ \theta
\end{bmatrix}
\end{align*}
\]

(3.23)

Note that the main diagonal of \( [C] \) contains the effective beam stiffnesses of the continuum model:

\[
\begin{align*}
C_{11} &= EA \\
C_{22} &= (EI)_2 \\
C_{33} &= (EI)_3 \\
C_{44} &= (GA)_2 \\
C_{55} &= (GA)_3 \\
C_{66} &= GJ
\end{align*}
\]

The off-diagonal entries of the continuum stiffness matrix are coupling coefficients between the different forces and strains. By the previous procedure, the stiffnesses for any bay can be calculated from its 51 dimensions.
3.2.2 Uncoupling of Stiffness Matrix

When each of the twenty-four bar stiffnesses in a bay varies independently of the others, coupling is possible between all of the six forces and strains. For example, if the stiffnesses of the diagonal braces on one side of a bay are different than those on the other side, it is possible that the truss would bend under an axial load. The scope of this study will be restricted to beam-like trusses which have no coupling between the Timoshenko forces and strains. If the bar stiffnesses are constrained such that they are symmetrical within the bay, then coupling among the Timoshenko modes is removed. Symmetry is accomplished by dividing the bar stiffnesses of each bay into groups and specifying that the members of each group have a common bar stiffness. For this study, the twenty-four stiffnesses are divided into six groups of four. Because the nondimensionalized elastic modulus is unity and the same for all of the bars, the stiffnesses $(EA)_i$ can be defined by the areas of the six groups. Each group that shares a common area, or more exactly a common radius and wall thickness, is identified in Figure 3.14. The groups are numbered 1 through 6 as shown in the figure. Note that now the design of the uncoupled truss bay can be completely defined by fifteen variables, the three dimensions $(L, W, H)$, the six radii, and the six thicknesses. The resulting areas define the continuum stiffness and are designated $A_1$ through $A_6$.

The result of reducing the number of independent bar areas is that the calculation of the continuum model is greatly simplified. Many of the coupling terms that would normally exist between the initial twenty-four modes are removed. Population of the 18 by 18 portion of the matrix $[\mathbf{\tilde{K}}]$ that was used in the development of the continuum stiffness matrix is illustrated in Figure 3.15. Recall that the first six rows and the first six columns represent coupling of the Timoshenko forces and strains. It is seen that the axial force mode is coupled to the axial strain mode and two others. These other two deflection modes can be described as representing Poisson contraction in the two transverse directions. Each of the two bending modes is coupled to one other mode besides its corresponding strain mode. These modes also represent Poisson behavior, only
Figure 3.14 Six independent groups of bar areas
Figure 3.15 Population of stiffness matrix that relates modes
behavior which would accompany an impressed curvature (i.e. contracted at the tensile surface and expanded at the compressive surface). The two shear modes are not coupled to any other mode. The torsion mode, however, is coupled to two other modes which can be best defined as warping of the beam cross section and the derivative of transverse shear. These two deformations can be found coupled to torsion in higher order beam theories. The result of this relatively small amount of coupling between the different modes is that the continuum stiffness formulation is simplified and can be easily carried out explicitly. The resulting expressions for the main diagonal stiffnesses are

\[
[EA] = (4E) A_1 + (4LD_1E) A_4 + (4LD_7E) A_6 - \\
\frac{4EL \left\{ D_3^2 A_4^2 \left( \frac{A_3}{H} + D_5 A_5 D_8 A_6 \right) - 2D_3 D_9 D_6 A_4 A_5 A_6 + D_9^2 A_6^2 \left( \frac{A_2}{W} + D_2 A_4 + D_4 A_5 \right) \right\}}{(A_3 + D_5 A_5 + D_8 A_6)(A_2 + D_2 A_4 + D_4 A_5) - D_6^2 A_6^2},
\]

\[
[EI_2] = (W^2E) A_1 + (W^2LD_7E) A_6 - \frac{(W^2LD_9E) A_6^2}{A_3/H + D_8 A_6},
\]

\[
[EI_3] = (H^2E) A_1 + (H^2LD_1E) A_4 - \frac{(H^2LD_3E) A_4^2}{A_2/W + D_2 A_4},
\]

\[
[GA_2] = \frac{WE (4L^2D_1 + 4W^2D_2 + 8L^2D_2)}{(L/W + W/L) (L^2 + W^2)} A_4, \quad (3.24)
\]

\[
[GA_3] = \frac{HE (4L^2D_7 + 4H^2D_8 + 8L^2D_8)}{(L/H + H/L) (L^2 + H^2)} A_6,
\]
\[ \{ GJ \} = (lH^2D_2E) A_4 + (lW^2D_8E) A_6 - \]

\[
\begin{align*}
&\frac{(-WHD_2A_4 + WHD_8A_6)^2 (D_1A_4 + D_7A_6)}{EL} \\
&+ \left( W^2D_2A_4 + (W^2D_4 + H^2D_5 + 2WHD_6)A_5 + H^2D_8A_6 \right) \left( -HD_3A_4 + WD_9A_5 \right)^2 \\
&- 2 \left( WHD_2A_4 + WHD_8A_6 \right) \left( HD_3A_4 + WD_9A_6 \right) \left( WD_9A_4 + HD_9A_6 \right) \\
&\left( W^2D_2A_4 + (W^2D_4 + H^2D_5 + 2WHD_6)A_5 + H^2D_8A_6 \right) \left( D_1A_4 + D_7A_6 \right) - \left( WD_3A_4 + HD_9A_6 \right)^2,
\end{align*}
\]

where the $D$ terms are as defined in Section 3.1. As previously stated there are no nonzero coupling terms left in the continuum stiffness matrix:

\[
\begin{pmatrix}
N \\
M_2 \\
M_3 \\
Q_2 \\
Q_3 \\
T
\end{pmatrix}
= \begin{bmatrix}
EA & (EI)_2 & 0 \\
& (EI)_3 & \gamma_2 \\
& (GA)_2 & 0 \\
& (GA)_3 & GJ
\end{bmatrix}
\begin{pmatrix}
\epsilon \\
\kappa_2 \\
\kappa_3 \\
\gamma_2 \\
\gamma_3 \\
\theta
\end{pmatrix}
\]

These stiffnesses can effectively represent the original bay in all manners of structural analysis, including static, dynamic, and stability calculations.
3.3 DEFLECTION CALCULATIONS

A space truss must be designed with an adequate stiffness in order to perform its function without undergoing unacceptable deflections. For the case of the rectangular beam-like truss, there are four major deflections. These are axial, two cases of transverse, and torsional deflections. For the symmetrical truss of this investigation, the deflections are uncoupled and can be calculated independently. Correspondingly, there will be four global deflection constraints on the design of the cantilevered truss. Shown in Figure 3.16, these are constraints on the free end deflections when loaded with six end loads. The axial constraint mandates that the axial deflection $\delta_1$ due to the applied axial force $P_1$ be less than a given value $\hat{\delta}_1$. The 2 direction bending constraint requires that the deflection $\delta_2$ due to the transverse force $P_2$ and moment $M_2$ be less than $\hat{\delta}_2$. A similar constraint is set on the global bending stiffness in the 3 direction. Torsional stiffness is constrained by requiring that the angle of twist $\delta_4$ due to the torque $P_4$ be less than a given $\hat{\delta}_4$. Values of the deflections $\delta_1$-$\delta_4$ will be calculated in two ways. The standard finite element method will be used for constraint evaluation in the traditional optimization, while the continuum model will be used to calculate constraint values in the new optimization method. These two methods of calculating deflections are detailed in the following subsections.
Figure 3.16 Deflection constraints
3.3.1 Traditional Finite Element Computation

Calculation of $\delta_1$-$\delta_4$ by finite element analysis is straightforward. Each joint becomes a node with a degree of freedom in each of the three coordinate directions except for the four joints on the cantilevered end which are fixed. A truss with $n$ bays therefore has $12n$ degrees of freedom. The axial bar element shown in Figure 3.3 is used to represent the structural members. A global stiffness matrix is assembled from the local stiffness matrices of all the members and the boundary conditions are used to reduce the matrix to a nonsingular form pertaining only to the unconstrained degrees of freedom. The nonsingular stiffness matrix $[K]$ now relates the external forces at the nodes to the nodal displacements:

$$\{f\} = [K]\{d\}$$

where $\{f\}$ is the force vector and $\{d\}$ is the displacement vector. Because there are four separate load cases, the stiffness matrix is inverted rather than solving the all four systems of equations. The result is the flexibility matrix $[S]$:

$$\{d\} = [S]\{f\}$$

(3.27)

For each load case, a set of nodal forces is selected which resembles the end load or loads. The forces chosen are the same as were used in the development of the continuum model (see again Figures 3.4 and 3.5) except they are applied only to the four nodes on the end of the truss. The force vector for each loading ($\{f^1\}$ through $\{f^4\}$) is multiplied by the flexibility matrix to yield the four displacement vectors ($\{d^1\}$ through $\{d^4\}$). End deflection values are now calculated from the
resulting displacements. The axial and transverse deflections are taken as the average of the displacements of the four end nodes. The angle of twist of the end is calculated by defining a displacement mode for the four nodes that represents a rigid body rotation of the end plane and computing the magnitude of that mode in the makeup of the actual displacements. For each loading case, the deflection value $\delta_i$ is a linear function of the displacement values. In indicial notation,

$$\delta_k = a^k_j d^k_j,$$

(3.28)

where $\{a^k\}$ are vectors of constant multipliers.

Derivatives of the four deflections with respect to the cross-sectional dimensions $r$ and $t$ of every member are needed in order to carry out optimization of the truss with deflection constraints. They can be calculated by finite difference approximations but this is very costly due to the necessary reanalysis during the perturbation of every individual variable. For a truss with ten bays there would be 120 reanalyses. Instead, analytical derivatives are sought. According to Haftka, Gürdal, and Kamat (1990) if a constraint $g$ is a function of a variable $x$ and the entries of a nodal displacement vector $\{ u \}$, then the derivative of $g$ with respect to $x$ is

$$\frac{dg}{dz} = \frac{\partial g}{\partial x} + \{ \lambda \}^T \left( \frac{df}{dz} \right) - \frac{d}{dz} \left( K \right) \{ u \},$$

(3.29)

where $\{ u \}$ is the solution of the finite element problem

$$\{ f \} = \left[ K \right] \{ u \},$$

(3.30)

and $\{ \lambda \}$ is the solution of
\[
\{ z \} = \left[ \begin{array}{c} K \end{array} \right] \{ \lambda \},
\]

(3.31)

where \( \{ z \} \) is the vector of derivatives of \( g \) with respect to the nodal displacements \( u_j \):

\[
z_j = \frac{\partial g}{\partial u_j}
\]

(3.32)

For the cases of \( \delta_1, \delta_4 \), \( z_i \) are the constants \( a_i \) and the expression for the derivatives is

\[
\frac{d\delta_i}{dx} = -\left\{ \lambda^i \right\}^T \frac{d}{dx} \left[ \begin{array}{c} K \end{array} \right] \left\{ d^i \right\},
\]

(3.33)

where

\[
\left\{ \lambda^i \right\} = \left[ \begin{array}{c} S \end{array} \right] \{ a^i \}.
\]

(3.34)

This approach to calculating the derivatives is called the adjoint method and is very efficient when the finite element problem is solved by inverting the stiffness matrix rather than solving the system of equations. Note that the only new quantity needed for a derivative calculation is the derivative of the stiffness matrix with respect to the design variable. Because the member areas appear in the stiffness matrix and the radii and wall thicknesses do not, the derivatives with respect to those variables are calculated from the those taken with respect to the areas:

Deflection Calculations
\[ \frac{\partial \delta_r}{\partial r} = \frac{\delta \delta}{\partial A} \frac{\partial A}{\partial r} = \frac{\delta \delta}{\partial A} (2\pi r) \] (3.35)

\[ \frac{\partial \delta}{\partial t} = \frac{\delta \delta}{\partial A} \frac{\partial A}{\partial t} = \frac{\delta \delta}{\partial A} (2\pi r) \] (3.36)

### 3.3.2 Continuum Model Computation

For this approximate analysis the truss is replaced with the solid continuum model. Finite elements are used where each bay becomes a single element. The elements will correspond exactly to the beam segments in the continuum model that in turn correspond to the bays of the truss. Each finite element therefore has the same six beam stiffnesses as the beam segment it represents. The finite element to be used is the twelve degree of freedom beam element that includes shear deflection. This element and its degrees of freedom are illustrated in Figure 3.17. The twelve degrees of freedom can be separated due to lack of coupling. For axial deflections the stiffness equation is

\[
\begin{bmatrix}
N_i \\
N_j
\end{bmatrix} = \frac{EA}{L} \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix} \begin{bmatrix}
u_i \\
u_j
\end{bmatrix}
\] (3.37)

Where \( N \) is the axial force and \( u \) is the displacement at nodes \( i \) and \( j \). For pure transverse deflections the stiffness equation is
Figure 3.17  Beam element with twelve degrees of freedom
\[
\begin{align*}
\begin{cases}
Q_i \\
M_i \\
Q_j \\
M_j
\end{cases} & = \frac{EI}{L^3 + \frac{EI}{GA}} \begin{bmatrix}
1 & \frac{L}{2} & -1 & \frac{L}{2} \\
\frac{L^2}{4} & -\frac{L}{2} & \frac{L^2}{4} & \\
sym & & \frac{L^2}{4} & \\
sym & & 1 & \\
\end{bmatrix} \begin{cases}
y_i \\
\psi_i \\
y_j \\
\psi_j
\end{cases} \\
+ \frac{EI}{L} \begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & -1 & \\
0 & 0 & \\
sym & & 1 & \\
\end{bmatrix} \begin{cases}
y_i \\
\psi_i \\
y_j \\
\psi_j
\end{cases},
\end{align*}
\] (3.38)

where \( M \) is the moment, \( Q \) is the shear force, \( y \) is the transverse displacement, and \( \psi \) is the angular rotation at the nodes \( i \) and \( j \). Finally, the stiffness equation for torsional deflection is

\[
\begin{align*}
\begin{cases}
T_i \\
T_j
\end{cases} & = \frac{GJ}{L} \begin{bmatrix}
1 & -1 \\
-1 & 1 \\
\end{bmatrix} \begin{cases}
\theta_i \\
\theta_j
\end{cases},
\end{align*}
\] (3.39)

where \( T \) is the torque and \( \psi \) is the angle of twist at the nodes \( i \) and \( j \).

The implementation of the finite element method in each of these cases
results in a narrowly banded global stiffness matrix. In each constraint used in this study, a closed form solution for the deflection can be obtained relatively easily. Because the problems are statically determinant, the forces on each element are known beforehand and are always the same for a particular loading. The deflections of each element can be written in terms of the forces and the element stiffnesses.

Considering the axial deflection constraint, it is obvious that each element carries the same load, \( P_1 \). From Equation 3.37 it is seen that the deflection of the \( j \)th node of each element can be written in terms of the \( i \)th node:

\[
\Delta_j = \Delta_i + \frac{P_1 L}{EA} \tag{3.40}
\]

Moving from left to right, the deflection at each node is the sum of the deflections of the elements before the node. Numbering the unconstrained nodes and the elements from left to right, the deflection of a given node is

\[
\Delta_j = \sum_{i=1}^{j} \frac{P_1 L_i}{(EA)_i} \tag{3.41}
\]

The deflection of the node furthest to the right is of course the global axial deflection of the beam. Therefore, for a beam with \( n \) segments, the global axial deflection under load \( P_1 \) is

\[
\delta_1 = \sum_{i=1}^{n} \frac{P_1 L_i}{(EA)_i} \tag{3.42}
\]

The derivative of \( \delta_1 \) with respect to the axial stiffness of the \( i \)th beam
segment follows:

\[
\frac{\partial \delta_4}{\partial (EA)_i} = - \frac{P_i L_i}{(EA)_i^2}
\]  
(3.43)

Derivations for the angle of twist constraint and its derivatives are exactly the same. They are

\[
\delta_4 = \sum_{i=1}^{n} \frac{P_i L_i}{(GJ)_i}
\]  
(3.44)

\[
\frac{\partial \delta_4}{\partial (GJ)_i} = - \frac{P_i L_i}{(GJ)_i^2}.
\]  
(3.45)

Deriving a closed form expression for the bending deflection is more involved. The shear acting on each beam segment is the same, but the moment varies. If the nodes and elements are numbered as before then the moment at node \( j \) of a beam loaded with the shear \( P_2 \) and moment \( M_2 \) is

\[
M_j = -M_2 - (n - j) P_2 L.
\]  
(3.46)

Using this equation along with the bending/shear stiffness matrix, the transverse deflection at a node can be written in terms of the deflection and rotation of the previous node:
\[ y_j = y_i + \psi_i L + \left[ (n - j + 1)/2 - 1/6 \right] \frac{P_2 L^3}{(EI)_j} + \frac{P_2 L}{(GA)_j} + \frac{M_2 L^2}{2(EI)_j} \]  \hspace{1cm} (3.47)

Once again noting the deflection and rotation at the support are zero and moving from left to right, the deflection at a given node \( j \) can be written:

\[ y_j = \sum_{i=1}^{j} \left[ (j - i)(n - i + 1/2) + (n - i + 2/3)/2 \right] \frac{P_2 L^3}{(EI)_i} + \frac{P_2 L}{(GA)_i} + \left[ j - i + 1/2 \right] \frac{M_2 L^2}{(EI)_i} \]  \hspace{1cm} (3.48)

The deflection at node \( n \) is the transverse deflection constraint in the 2 direction (\( \delta_2 \)):

\[ \delta_2 = \sum_{i=1}^{n} \left[ (n - i)^2 + (n - i) + 1/3 \right] \frac{P_2 L^3}{(EI)_i} + \frac{P_2 L}{(GA)_i} + \left[ n - i + 1/2 \right] \frac{M_2 L^2}{(EI)_i} \]  \hspace{1cm} (3.49)

And the derivatives of the deflection with respect to the bending and shear stiffnesses are

\[ \frac{\partial \delta_2}{\partial (EI)_i} = - \left[ (n - i)^2 + (n - i) + 1/3 \right] \frac{P_2 L^3}{(EI)_i^2} - \left[ n - i + 1/2 \right] \frac{M_2 L^2}{(EI)_i^2} \]  \hspace{1cm} (3.50)

\[ \frac{\partial \delta_2}{\partial (GA)_i} = - \frac{P_2 L}{(GA)_i^2} \]  \hspace{1cm} (3.51)

3.3.3 Verification of Continuum Model Computation
To verify the accuracy of the continuum model in calculating beam
deflections, 1000 deflection values were computed using both the traditional
method (Subsection 3.3.1) and the continuum model method (Subsection 3.3.2).
The first 500 values are the four end deflections of 125 randomly chosen trusses. In
the random truss generation all variables were random: number of bays,
dimensions, member sizes, and end loads. The upper graph in Figure 3.18 contains
the comparison of the results. For each deflection a point is plotted that represents
the continuum value divided by the traditional value. It is seen that deflections
calculated by the continuum model are consistently larger than those calculated by
the axial element model. For entirely random trusses, it appears that the error can
be as large as nineteen percent. The lower graph is a similar plot for the second
500 values. These 125 trusses were also randomly chosen, but the member sizes are
constrained relative to one another such that the designs are similar to those which
occur when the optimization methods of the next chapter are implemented. The
constraints are that the group 1 members are the largest in each bay, groups 2, 3,
and 5 are the smallest, and groups 4 and 6 are not very different from one another.
As can be seen, the continuum model calculations are more accurate for these types
of trusses. Deflection values are still always larger, but they are now within an
accuracy of five percent.
Figure 3.18  Accuracy of the continuum model in calculating end deflections
3.4 LOCAL STRESS CALCULATIONS

An important consideration in the design of any truss is that of local failure. Every truss member must be able to withstand its maximum design load without failing. It is assumed in this investigation that Euler buckling will be the dominant failure mode. The reasoning for this assumption is that for many materials, the critical buckling stress for a slender pinned-pinned column is less in absolute magnitude than either the yield stress in compression, yield stress in tension, or the critical shell buckling stress. Because the loading on the beam-like truss will be considered fully reversible, buckling of a member will occur before yielding. Therefore every member of every bay in the truss should be designed against buckling. These constraints will be formulated in terms of buckling factors. For a pinned-pinned Euler column the critical buckling load is

\[ P_{cr} = \frac{\pi^2 EI}{l^2}. \]  \hspace{1cm} (3.52)

where \( l \) is the length of the column. The Euler critical stress is

\[ \sigma_{cr} = \frac{\pi^2 EI}{At^2}. \]  \hspace{1cm} (3.53)

Substituting for the cross-sectional variables gives

\[ \sigma_{cr} = \frac{\pi^2 E(\pi r^3 t)}{(2\pi r^2 t)^2} = \frac{\pi^2 Er^2}{2l^2}. \]  \hspace{1cm} (3.54)
The buckling factor is defined as

\[ \lambda = \frac{\chi \sigma}{\sigma_{cr}}. \]  \hspace{1cm} (3.55)

where \( \sigma \) is the actual stress in the member and \( \chi \) is a factor of safety. When the buckling factor of a member is less than one, the member is considered safe from buckling under the given loading. When the factor is greater than one the member is not considered safe. Substituting for the critical stress results in

\[ \lambda = \frac{2\chi \sigma}{\pi^2 E \frac{I^2}{r^2}}. \]  \hspace{1cm} (3.56)

What is needed, therefore, from structural analysis is the maximum compressive stress in each member due to the four reversible end loads and the derivative of that stress with respect to the areas of all the members. Detailed in the next two subsections is the analysis carried out both in the traditional way and with the aid of the continuum model.

3.4.1 Traditional Finite Element Computation

The stresses in the bar members are easily found from the results of the static analyses described in section 3.2.1. Therein the four loadings (\( P_1, P_2 \) and \( M_2, P_3 \) and \( M_3, P_4 \)) were applied to the end of the truss and the four displacement vectors \( \{ \delta_i \} \) were calculated. Values of stress in each member can be found from the linear superposition of the four sets of displacement data. Stress in an axial member can be related to the axial displacements of its endpoints: 
\[
\sigma = E\epsilon = E \frac{\Delta l}{l} = E \frac{(u_2 - u_1)}{l}
\] (3.57)

where \(d_1\) and \(d_2\) are the displacements of the two endpoints. Stress can be related to the coordinate displacements of the endpoints by the direction cosines of the member. Accordingly, each stress in the truss can be written as a linear sum of the displacements:

\[
\sigma = \frac{E}{l} (a_j d_j)
\] (3.58)

Note that the \(a_j\) are either zero or a direction cosine of the member in question. Because the four loadings are considered reversible, either \(\{d^k\}\) or \(-\{d^k\}\) can be used in the equation for stress. The total stress in a particular member from the superposed loadings must be the worst case. The algebraic signs of the four displacement vectors should cause the highest possible stress in the member. To ensure the four terms add to the highest possible sum, absolute values are used:

\[
\sigma = \frac{E}{l} \left\{ |a_j d_1^3| + |a_j d_2^2| + |a_j d_3^3| + |a_j d_4^4| \right\}
\] (3.59)

Derivatives of the stresses with respect to the cross-sectional dimensions are calculated exactly as in Section 3.3.1 Attention must be paid, however, to the fact that the derivatives of \(\sigma\) with respect to the displacements are not always \(a_j\). Occasionally the derivative with respect to \(d^k\) is \(-a_j\). The sign of the derivative is dependent on the sign of the term inside the absolute value brackets:

Local stress calculations
\[ z^k_m = \frac{\partial \sigma}{\partial d^k_m} = \frac{E}{I} \frac{a_j^{d^k_j}}{a_j^{d^k_j}} a_m \] (3.60)

This formulation must be used because a positive \( a_m \) only indicates a positive derivative when the sum itself is positive. It should be noted that the derivatives are undefined, and discontinuous, when the stress is zero. For this investigation, the derivative is set to zero when the stress vanishes. This is of no consequence since a zero stress in a member is not of great concern in an optimization problem.

### 3.4.2 Continuum Model Computation

In this formulation, the stresses in each member are calculated from the internal forces computed in the continuum model deflection analysis (Subsection 3.3.2). The internal loading on each beam segment in the model (6 Timoshenko beam forces) is transferred to the corresponding truss bay. Each force is converted into its corresponding force mode (Figures 3.4, 3.5) which is applied to the bay. Therefore, the loading on each bay is approximated as the sum of six force modes. Because linear elastic theory is being used, the stress in each member can be considered the sum of the stresses due to each of the six loadings being applied independently. The derivation of stresses due to each Timoshenko force mode will, therefore, be carried out independently.

For each force mode the forces applied externally to the eight joints of the bay are specified. Each joint provides three equations of equilibrium. Therefore, twenty-four equations of equilibrium can be written in terms of the twenty-four internal stresses. However, in each of the six loading cases there are one or more planes of symmetry within the bay. The result of symmetry is that a number of the equilibrium equations are redundant. Consequently, there are more unknowns than equations. In order to provide the additional equations needed the method of
virtual work is employed. For each new equation a different internal virtual force system is applied to the bay. At most, four virtual force systems are needed. In these systems, the twenty-four internal forces are broken up such that each of the six previously defined member groups have a single force assigned to it. If these forces are labeled $p_1^*$ through $p_6^*$, then the four virtual systems to be used are

\[ \begin{align*}
    p_1^* &= 2.0 \\
    p_2^* &= 0.0 \\
    p_3^* &= 0.0 \\
    p_4^* &= -B_1/L \\
    p_5^* &= B_2/L \\
    p_6^* &= -B_3/L \\
    p_1^* &= 0.0 \\
    p_2^* &= 2.0 \\
    p_3^* &= 0.0 \\
    p_4^* &= -B_1/W \\
    p_5^* &= -B_2/W \\
    p_6^* &= B_3/W
\end{align*} \]

Close examination of these internal forces will show that each system is in equilibrium, which is the only requirement for a virtual force system. Furthermore, there are no external virtual forces, making the external virtual work zero.

Equating internal virtual work to external virtual work (zero) provides an independent equation. In the case of pinned end truss members with constant cross section the expression for internal virtual work ($W_I$) is

Local stress calculations
\[ W_I = 4 \sum_{i=1}^{6} \frac{P_i}{E A_i} p_i^* , \]  

(3.61)

where \( P_i \) is the real force in the \( i \)th member, \( l_i \) the length, \( A_i \) the area, and \( p_i^* \) the internal virtual force. The expression can easily be rewritten in terms of the twenty-four member stresses \( (\sigma_i) \) and set equal to zero:

\[ W_I = 4 \sum_{i=1}^{6} \frac{E}{l_i} \sigma_i p_i^* = 0 \]  

(3.62)

Due to the symmetry of loading and of member stiffnesses, however, there are never twenty-four independent stresses in the solution. Exactly how many variables are present is dependent on the loading case.

**Axial force mode**

The loading caused by the axial force mode (Figure 3.4) has two planes of symmetry. It can be proven from symmetry arguments that six independent stresses are present. Each stress is present in the four members of a particular member group. The six stresses will be designated \( \sigma_1 \) through \( \sigma_6 \), the numbers corresponding to the areas \( A_1 \) through \( A_6 \). With the number of variables in the problem reduced to these six, it can be seen from the figure that only three equations of equilibrium can be written. All eight joints provide the same three equations:

\[ \sum F_1 = (\sigma_1 A_1) + c_2 (\sigma_4 A_4) + c_6 (\sigma_6 A_6) = N/4 \]

\[ \sum F_2 = (\sigma_2 A_2) + c_1 (\sigma_4 A_4) + c_3 (\sigma_5 A_5) = 0 \]  

(3.63)

\[ \sum F_3 = (\sigma_3 A_3) + c_4 (\sigma_5 A_5) + c_5 (\sigma_6 A_6) = 0 \]
Three more equations are needed to solve for the six stresses. If the first three virtual systems are used in Equation 3.62 the resulting equations are

\[
2L^2\sigma_1 - B_1^2 \sigma_4 + B_2^2 \sigma_5 - B_3^2 \sigma_6 = 0
\]

\[
2W^2\sigma_2 - B_1^2 \sigma_4 - B_2^2 \sigma_5 + B_3^2 \sigma_6 = 0
\]

\[
2H^2\sigma_3 + B_1^2 \sigma_4 - B_2^2 \sigma_5 - B_3^2 \sigma_6 = 0.
\]

(3.64)

The six stresses are calculated from the simultaneous solution of the Equations 3.63 and 3.64. Closed-form solutions for \(\sigma_1\) through \(\sigma_6\) can be obtained through symbolic manipulation. These solutions are very lengthy and are included (along with the stresses resulting from the following modes) in the appendix.

**Bending moment modes**

The two bending modes have only one plane of symmetry. Examination of the loading reveals that there are nine independent stresses. This is due to the fact that members on the tension side of the bending will have different stresses than their counterparts on the compression side. The result is that three of the six member groups have subgroups; one subgroup of two members on the tension side and the other subgroup of two members on the compression side. The three new subgroups represent three additional unknowns. Here will be outlined the problem of finding the stresses due to \(M_2\), the moment acting in the the 1-2 plane. Stresses due to \(M_3\) are found using the same procedure.

When \(M_2\) is applied, the stresses in the 1, 3, and 6 member groups are the ones that are divided into subgroups. The stresses on the compression side will be called \(\sigma_1\), \(\sigma_3\), and \(\sigma_6\). Stresses on the tension side will be called \(\tau_1\), \(\tau_3\), and \(\tau_6\). Stresses in the members in between will still be \(\sigma_2\), \(\sigma_4\), and \(\sigma_5\). With these nine independent variables, five independent equilibrium equations can be written.
\[ \sum F_1 = (\sigma_1 A_1) + c_2 (\sigma_4 A_4) + c_6 (\sigma_6 A_6) = -M_2/2W \]
\[ \sum F_1 = (\tau_1 A_1) + c_2 (\sigma_4 A_4) + c_6 (\sigma_6 A_6) = M_2/2W \]
\[ \sum F_2 = (\sigma_2 A_2) + c_1 (\sigma_4 A_4) + c_3 (\sigma_5 A_5) = 0 \]  \hspace{1cm} (3.65)
\[ \sum F_3 = (\sigma_3 A_3) + c_4 (\sigma_4 A_4) + c_5 (\sigma_6 A_6) = 0 \]
\[ \sum F_3 = (\tau_3 A_3) + c_4 (\sigma_4 A_4) + c_5 (\tau_6 A_6) = 0 \]

Two of these come from joints on the tension side, two come from joints on the compression side, and one is common to both. With five equations in hand, the four virtual systems are used along with Equation 3.62 to yield

\[ L^2 \sigma_1 + L^2 \tau_1 - B_1^2 \sigma_4 + B_2^2 \sigma_5 - .5B_3^2 \sigma_6 - .5B_3^2 \tau_6 = 0 \]
\[ 2W^2 \sigma_1 - B_1^2 \sigma_4 - B_2^2 \sigma_5 + .5B_3^2 \sigma_6 + .5B_3^2 \tau_6 = 0 \]  \hspace{1cm} (3.66)
\[ H^2 \sigma_1 + H^2 \tau_1 + B_1^2 \sigma_4 - B_2^2 \sigma_5 - .5B_3^2 \sigma_6 - .5B_3^2 \tau_6 = 0 \]
\[ L^2 \sigma_1 + L^2 \tau_1 + H^2 \sigma_3 + H^2 \tau_3 - B_3^2 \sigma_6 - B_3^2 \tau_6 = 0. \]

Symbolic manipulation of this set of nine equations reveal that three of the stresses (\(\sigma_2, \sigma_4,\) and \(\sigma_5\)) are always zero. These are the members that connect the tension side to the compression side. Furthermore, the members on the tension side carry equal and opposite stresses to those on the compression side.

Local stress calculations
Shear mode

It was indicated in the previous section that the shear displacement mode in a particular transverse plane was coupled only to the shear force mode in that plane. The useful result of this fact is that deflections due to application of the shear force mode are known. For this reason, having a set of equations is not necessary to solve for the internal stresses caused by the shear force mode. The stresses are found by applying the appropriate deflections of the shear displacement mode to each member. When this is done, it is seen that for each shear mode, only one group of members carries stress. For a shear force in the 1-2 plane, the diagonal members in group 4 are loaded, the diagonals in group 5 are loaded by a shear force in the 1-3 plane.

Torsion mode

Unlike the shear mode, the torsion mode is coupled and a system of equations is needed to solve for the stresses. For this loading case there are no planes of symmetry, but three planes of anti-symmetry do exist. Consideration of the anti-symmetry leads to the identification of nine independent stresses. The first, second, and third member groups each carry a different stress. The twelve members of the other three groups are all diagonal braces and they have six more stresses between them. Each face of the bay has two diagonals whose stresses are different in magnitude, if not different in both magnitude and sign. The result is that the 4th, 5th, and 6th member groups are each divided into two subgroups. When these nine stresses are independent there are six equations available from equilibrium at the joints. Calling stresses in the diagonals $\sigma_4$, $\tau_4$, $\sigma_5$, $\tau_5$, $\sigma_6$, and $\tau_6$, the equations are
\[ \sum F_1 = (\sigma_1 A_1) + c_2 (\sigma_4 A_4) + c_6 (\sigma_6 A_6) = 0 \]
\[ \sum F_1 = (\sigma_1 A_1) + c_2 (\tau_4 A_4) + c_6 (\tau_6 A_6) = 0 \]
\[ \sum F_2 = (\sigma_2 A_2) + c_1 (\sigma_4 A_4) + c_3 (\sigma_5 A_5) = TH/(2B_2^2) \]
\[ \sum F_2 = (\sigma_2 A_2) + c_1 (\tau_4 A_4) + c_3 (\tau_5 A_5) = -TH/(2B_2^2) \] \hspace{1cm} (3.67)
\[ \sum F_3 = (\sigma_3 A_3) + c_4 (\sigma_5 A_5) + c_5 (\sigma_6 A_6) = -TW/(2B_2^2) \]
\[ \sum F_3 = (\sigma_3 A_3) + c_4 (\tau_5 A_5) + c_5 (\tau_6 A_6) = TW/(2B_2^2). \]

Utilizing the first three virtual systems and Equation 3.62 yields the three additional equations needed:

\[ 4L^2\sigma_1 - B_1^2 \sigma_4 - B_1^2 \tau_4 + B_2^2 \sigma_5 + B_2^2 \tau_5 - B_3^2 \sigma_6 - B_3^2 \tau_6 = 0 \]
\[ 4W^2\sigma_2 - B_1^2 \sigma_4 - B_1^2 \tau_4 - B_2^2 \sigma_5 - B_2^2 \tau_5 + B_3^2 \sigma_6 + B_3^2 \tau_6 = 0 \] \hspace{1cm} (3.68)
\[ 4H^2\sigma_3 + B_1^2 \sigma_4 + B_1^2 \tau_4 - B_2^2 \sigma_5 - B_2^2 \tau_5 - B_3^2 \sigma_6 + B_3^2 \tau_6 = 0 \]

The stresses in the members of a bay due to the application of the torsion force mode are found from the simultaneous solution of the nine equations.

As stated before, the stresses in a bay due to a particular loading combination are calculated as the linear sum of the stresses caused by each of the six components of the loading. Derivatives of the six stresses with respect to the six areas are easily found analytically.
3.4.3 Verification of Continuum Model Computation

The method of calculating local stresses from the continuum model structural analysis was also verified through comparison with the traditional approach. Buckling factors were calculated by both methods for the same random and partially random trusses as used in Subsection 3.3.3. In the cases of members on the boundary between two trusses (groups 2, 3, and 5), the stress was found in each of the two parts of the divided member by application of the continuum model computation to the two separate bays. The stresses were added together and the two parts of the member were joined as outlined in Section 3.1. The composite stress is applied to the composite member to generate the buckling factor. As before, the data given are the buckling factors calculated by the continuum model method divided by the factors calculated from finite elements. The upper graph in Figure 3.19 has the data from the totally random trusses. A wide scatter in accuracy is seen. The approximate analysis can be as much as 98 percent in error. The lower graph shows that when more realistic trusses are used, the accuracy of the buckling factors by continuum model analysis improves greatly. Careful examination of the data reveals that at least ten percent accuracy can be expected. This is considered sufficient for the current optimization technique.
Figure 3.19  Accuracy of the continuum model in calculating buckling factors
4.0 Optimization Methods

4.1 OPTIMIZATION PROBLEM

The optimization problem is defined to be the weight minimization of the beam-like space truss, designed for use as a cantilever. Such is the use of the truss in the candidate designs for the space station. One end is anchored on a relatively large mass, such as the space station body, while the other end holds a smaller piece of equipment, such as a dish antenna. Loading of the truss is usually due to the inertia of the smaller mass during attitude maneuvers. Because one of the masses is generally much larger than the other, the truss can be modeled as a cantilever. Alternatively, the cantilever can serve as a half model of a free-free truss under symmetry conditions. The final design must be able to withstand the simultaneous action of a set of static end loads without excessive end deflections or failure by local buckling.

Design variables to be used in this problem are the cross-sectional dimensions of the truss members. Not every member will be designed independently, however. As described in the previous chapter, the members of each bay are divided into six groups which are identified with each other in design. These six groups are illustrated in Figure 4.1. Group 1 are the four longerons in the 1-direction. Group 2 and 3 are the two battens in the 2 and 3-directions respectively. Groups 4, 5, and 6 are the four, two, and four diagonals in the 1-2, 2-3, and 1-3 planes, respectively. The members of each group have identical cross-sectional dimensions, \( r \) and \( t \). There are, therefore, \( 12n \) design variables for a truss with \( n \) bays. These will be labeled \( r_{ij} \) and \( t_{ij} \), where \( i \) designates the bay number.
Figure 4.1 Separation of bay members into groups
(from 1 to n) and \( j \) designates the group number (1-6). In this design problem there are upper and lower bounds on the design variables. The nondimensionalization of length in this study is based on the maximum allowable radius (section 3.1) so its value is 1.0. Minimum radius is chosen to be 0.5. Bounds on the nondimensionalized wall thickness are chosen to be 0.04 minimum and 0.2 maximum. The extreme designs under these bounds are depicted in correct scale relative to each other in Figure 4.2. Values of the minimum and maximum dimensions are arbitrary and were chosen such that a wide selection of members was available in the design process. Cross-sectional areas of the members \( (A_{i,j}) \) vary over exactly one order of magnitude. The smallest possible area is .125 and the largest is 1.25.

Mathematical formulation of the design constraints is as follows. Six design loads are chosen. Applied to the free end, these are as defined in Chapter 3: an axial force \( P_1 \), transverse forces in the 2 and 3 directions \( (P_2 \text{ and } P_3) \), transverse moments in the 2 and 3 directions \( (M_2 \text{ and } M_3) \), and a torque about the longitudinal axis \( (P_4) \). For these loads, four uncoupled end deflections \( (\delta_1-\delta_4) \) can be calculated as outlined in Chapter 3. The deflection constraints will require that \( \delta_k \leq \delta_k^* \), where \( (\delta_1-\delta_4) \) are specified maximum allowable values. In addition, local buckling constraints must be satisfied. Symmetry due to the choice of design variable identification causes there to be only six distinct buckling factors per bay. That is, members in the same group have identical buckling factors. The \( 6n \) local buckling constraints can be written as \( \lambda_{ij} < 1.0 \) where \( \lambda_{ij} \) is the buckling factor of the \( j \)th group in the \( i \)th bay.

Given the preceding variables and constraints, a design problem in this study is completely specified by fifteen values: the number of bays \( (n) \), the dimensions of the bays \( (L, W, H) \), the six end loads \( (P_1, P_2, M_2, P_3, M_3, P_4) \), the maximum allowable deflections \( (\delta_1-\delta_4) \), and the factor of safety \( (\chi) \) used for the local buckling constraints. This optimization problem in standard form is

Optimization Problem

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Figure 4.2  Relative sizes of the maximum and minimum member designs
minimize $\WT(r_{ij}, t_{ij})$

such that

\[ \delta_k(r_{ij}, t_{ij}) \leq \delta_k \quad k = 1, \ldots, 4 \]

\[ \lambda_{ij}(r_{ij}, t_{ij}) < 1.0 \quad i = 1, \ldots, n, \quad j = 1, \ldots, 6 \]

\[ 0.5 \leq r_{ij} \leq 1.0 \quad i = 1, \ldots, n, \quad j = 1, \ldots, 6 \]

\[ 0.04 \leq t_{ij} \leq 0.2 \quad i = 1, \ldots, n, \quad j = 1, \ldots, 6, \quad (4.1) \]

where $\WT$ is the weight function.

An alternative problem where the radius and thickness variables are replaced by a single cross-sectional area variable is also studied. Every member is assumed to have the maximum allowable radius (1.0). Recall from Chapter 3 that all constraints can be calculated from just the cross-sectional areas. The reduced problem is therefore

minimize $\WT(A_{ij})$

such that

\[ \delta_k(A_{ij}) \leq \delta_k \quad k = 1, \ldots, 4 \]

\[ \lambda_{ij}(A_{ij}) < 1.0 \quad i = 1, \ldots, n, \quad j = 1, \ldots, 6 \]

\[ 0.125 \leq A_{ij} \leq 1.25 \quad i = 1, \ldots, n, \quad j = 1, \ldots, 6. \quad (4.2) \]

Two schemes are developed to solve the foregoing design problem. One will be a single level scheme using traditional methods. It is detailed in Section 4.2. The other will be a multilevel scheme using the proposed continuum model method and is detailed in Section 4.3.
4.2 TRADITIONAL SCHEME

This scheme uses standard numerical optimization techniques to solve the problems described in the last section. Every design variable is considered in a single level numerical search. Each constraint and its derivatives are calculated using the standard finite element techniques outlined in Subsections 3.3.1 and 3.4.1. These analyses are computationally intensive and largely dictate the time needed by any search algorithm. For an $n$ bay truss, the computation of the deflection and buckling constraints involves, along with other calculations, the inversion of a $12n$ by $12n$ stiffness matrix. Because the constraint calculations are costly, an optimization strategy that approximates the constraints is utilized. Known as sequential linear programming, the strategy involves representing the constraint functions by a first-order Taylor series expansion about the current design point. Use of the approximation allows the optimization to proceed with fewer actual constraint evaluations, therefore saving a substantial amount of computer time.

Sequential linear programming (SLP) works by replacing the given nonlinear optimization problem with a series of linear ones. Using the standard notation of optimization literature, the objective function $f(x)$ and $q$ constraint functions $g_j(x)$ are expanded about the current design point $x_o$:

$$f(x) = f(x_o) + \sum_{i=1}^{n} (x_i - x_{oi}) \frac{\partial f(x_0)}{\partial x_i}$$  \hspace{1cm} (4.3)

$$g_j(x) = g_j(x_o) + \sum_{i=1}^{n} (x_i - x_{oi}) \frac{\partial g_j(x_0)}{\partial x_i}$$  \hspace{1cm} (4.4)
When the functions values are represented in this manner, the problem is a linear programming problem and can be solved readily and efficiently by the Simplex method (Haftka, Gürdal, and Kamat, 1990). When SLP is started, the objective and constraint functions and their gradients are approximated about the initial guess \( z_0 \). The linear programming problem that is thereby formed is then solved, resulting in a new point \( z_1 \) that is closer to the optimum than \( z_0 \). The process is then repeated from \( z_1 \) and so on until the objective function converges. There must be limits on the variation of the design variables (\( x_i - x_{oi} \) or \( \Delta x_i \)) so that the approximations are not used beyond their regions of accuracy. These limits, called move limits, are usually a small percentage of the current value of the variable. The initial move limit used in this study is five percent, or 0.05\( x_i \), with the move limit occasionally decreased by half as the objective function converges. This causes large moves at the beginning of the process, and smaller, refining moves as the design approaches the optimum.

The truss problem (Equations 4.1 or 4.2) is well suited to be solved by sequential linear programming in that exact derivatives are available for both the objective and constraint functions. For this investigation, the SLP option of the optimization program ADS is used to solve the truss problem (Vanderplaats, 1985). The exact algorithm in terms of the truss variables is as follows:

1. Determine initial guess \( r_{ij} \), \( t_{ij} \) and initial move limit \( a \)
2. Evaluate \( WT \), \( \delta_k \), \( \lambda_{ij} \), and their derivatives w.r.t \( r_{ij} \), \( t_{ij} \)
3. Assemble linear problem from derivatives and the move limit.
4. Solve linear problem for \( \Delta r_{ij} \), \( \Delta t_{ij} \).
5. Perturb \( r_{ij} \), \( t_{ij} \) by \( \Delta r_{ij} \), \( \Delta t_{ij} \) and calculate new weight \( WT \).
6. If \( WT \) has converged (\( \Delta WT < \epsilon \)) then stop.
7. Go to 2.
4.3 CONTINUUM MODEL SCHEME

The continuum model scheme for optimization of the truss problem involves multilevel decomposition. Before the scheme is described in detail, a discussion of decomposition is warranted.

4.3.1 Multilevel Decomposition

The method of multilevel decomposition is an attractive solution to the difficulties associated with very large optimization problems. It is attractive because it replaces a large, costly, possibly intractable problem with a set of smaller problems which are more easily dealt with. The manner and method of decomposition is repeated here in detail. Once again, the standard structural optimization problem is

\[
\text{minimize} \quad f(x) \\
\text{such that} \quad g_p(x) \leq 0 \quad p = 1, \ldots, q \\\n\quad x^l_r \leq x_r \leq x^u_r \quad r = 1, \ldots, s. \quad (4.5)
\]
In general, not every constraint $g$ is a function of every design variable $x_i$. The constraints and design variables can each be divided into well defined groups that reflect this lack of universal coupling. Usually, subsets of variables can be found that affect only corresponding subsets of constraints. These subsets will almost definitely correspond to particular substructures of the original structure. This weak coupling can be taken advantage of by decomposing the large problem into a set of smaller problems (subsystems) and a coupling problem (system). If $(x_1, \ldots, x_w)$ are the subsets of the design variables that affect only corresponding subsets of the constraints, then $x$ can be partitioned as

$$ x^T = \{y^T, x_1^T, \ldots, x_w^T\}, \quad (4.6) $$

where $y$ is the vector of remaining variables which each affect more than one subsystem. The weakly connected groups of constraints can likewise be separated:

$$ g^T = \{g_0^T(y), g_1^T(y, x_1), \ldots, g_w^T(y, x_w)\} \quad (4.7) $$

Additionally, the objective function is usually separable as well:

$$ f = f_0(y) + f_1(y, x_1) + \ldots + f_w(y, x_w) \quad (4.8) $$

Notice that with the variables, constraints, and objectives divided in this manner that there are $w$ smaller design problems which are connected only by the design variables $y$. These coupled problems are
minimize \[ f_v(y, x_v) \]
such that \[ g_{vp}(y, x_v) \leq 0 \]
\[ x^l_{vr} \leq x_{vr} \leq x^u_{vr} \] (4.9)

where \( v \) varies from 1 to \( w \). If the design variables in \( y \) are held constant then the \( w \) problems are entirely independent of one another. These coupling variables must be designed as well, however, because the remaining constraints and part of the objective function may depend on them. Elements of the original problem not accounted for in the subsystems are grouped in the system problem:

minimize \[ f_0(y) \]
such that \[ g_{0p}(y) \leq 0 \]
\[ y^l_r \leq y_r \leq y^u_r \] (4.10)

Optimizing the decomposed problem involves the simultaneous solution of Equations 4.9 and 4.10. In the optimization process the coupling must be preserved so that all necessary interaction and tradeoff between the system and subsystems can take place. Multilevel optimization is the process that has been used to accomplish this interaction. In multilevel optimization a group of different optimization processes are executed semi-concurrently. Overall optimization is effected by moving control back and forth among the different processes. Each has its own optimizer and its own separate design space. While one optimizer is modifying its design variables, all other variables are held constant.
When multilevel optimization is applied to a decomposed problem, the system problem is optimized on the upper level and the subsystem problems are optimized in individual lower level processes. The overall process starts with the system optimizer, which operates as usual except that at every new design point the subsystem optimizer is called. Each of the subproblems is optimized one at a time or in parallel if multiprocessing technology is available. These mutually independent optimizations are carried out using the current value of $y$ as problem parameters. To enforce coupling and compromise between the two levels, the system problem (4.10) must be rewritten such that it is responsive to the relative success of the subsystem problems. Two schemes for accomplishing this coordination between the levels are given in the next paragraph. The result is that the system optimizer modifies its design variables to produce the minimum results from the secondary optimizations. Optimization is complete when all the variables have converged.

Sobiesczanski-Sobieski and coworkers developed a method of coordination in which the total violation of the subsystem constraints is minimized on the lower level (Sobiesczanski-Sobieski, James, and Dovi, 1983). In the example problems solved, the objective function could be calculated entirely on the upper level. The lower level objective became the minimization of the sum of the constraints. The minimized sums were incorporated on the upper level as constraints. In standard form their upper level coordination problem was

\[
\begin{align*}
\text{minimize} & \quad f_0(y) \\
\text{such that} & \quad g_{op}(y) \leq 0 \\
& \quad \left[ \sum g_{or}(y, x_r) \right]_{\text{min}} \leq 0 \\
& \quad y'_r \leq y_r \leq y''_r, \quad (4.11)
\end{align*}
\]
and their lower level problems were

\[
\text{minimize} \quad \sum g_{vp}(y, x_v) \\
\quad x_{vr}^l \leq x_{vr} \leq x_{vr}^u. \quad (4.12)
\]

In order to provide derivative information about the minimized subsystem objective to the upper level, Sobieszczanski-Sobieski et. al. calculated optimum sensitivity derivatives by the analytical methods of an earlier publication (Sobieszczanski-Sobieski and Barthelemy, 1983). There are difficulties, however, related to the discontinuity of optimum sensitivity derivatives. Haftka developed a method which alleviates the problems of discontinuity (Haftka 1984). He proposed using the penalty function approach on both levels and demonstrated that this smooths the derivatives of the lower level optimums. His method could handle calculation of all or part of the objective function on the lower level. The upper level coordination problem was

\[
\text{minimize} \quad f_0(y) + P[\beta, g_{0v}(y)] + \sum(\phi_v)_{\text{min}} \\
y_{l_r}^l \leq y_r \leq y_{r}^u \quad (4.13)
\]

where \(P\) is a penalty function, \(\beta\) is a penalty function multiplier, and \(\phi_v\) are the augmented objectives of the subsystems. Accordingly, the lower level problems were

\[
\text{minimize} \quad \phi_v = f_v(y, x_v) + P[\beta, g_{vp}(y, x_v)] \\
\quad x_{vr}^l \leq x_{vr} \leq x_{vr}^u \quad (4.14)
\]
Derivatives passed to the upper level were calculated analytically from the equation for $\phi_v$. This multilevel method is the one utilized in the current optimization scheme. It is applicable because, as shown in the next subsection, the objective function is not available on the upper level.

### 4.3.2 Decomposition of Truss Problem

As stated in the introduction, use of a continuum model makes decomposition of the beam-like truss problem possible. If each bay is made to be a subsystem and replaced with its equivalent beam segment, the stiffnesses of the solid segment can serve as upper level or system variables. The overall stiffness of the bay subsystem is represented by the segment and, therefore, the internal forces in the subsystem will remain constant during subsystem optimization. The beam stiffnesses of the continuum model are added to the problem in order to decompose it. They will be labeled $K_{ij}$:

\[
\begin{align*}
K_{i1} &= (EA)_i \\
K_{i2} &= (EI_2)_i \\
K_{i3} &= (EI_3)_i \\
K_{i4} &= (GA_2)_i \\
K_{i5} &= (GA_3)_i \\
K_{i6} &= (GJ)_i
\end{align*}
\]  

(4.15)

The natural variables of the problem ($r_{ij}$, $t_{ij}$) are moved to the subsystem level to be designed independently. Constraints in the problem are similarly divided between the two levels. Global deflections are functions of the beam stiffnesses of each bay so the deflection constraints remain on the upper level. Local buckling factors are dependent on the cross sectional dimensions of the bay members and are moved to the lower level. An additional constraint must be added to the lower level in order for the levels to be compatible. The upper level will be specifying
values of the beam stiffnesses \(K_{ij}\) and the lower level variables must be designed to match them. However, not all combinations of the six beam stiffnesses that can be specified at the system level can be achieved at the lower level. An equality constraint is added to each subsystem to ensure compatibility. To avoid confusion between the required and actual stiffnesses, the system level variables will be renamed \(KR_{ij}\). The decomposition of the truss problem is as follows.

The system level problem is

\[
\begin{align*}
\text{minimize} & \quad WT(KR_{ij}) = \sum_{i=1}^{n} (wt_i)_{\text{min}} & i = 1, \ldots, n \ , \ j = 1, \ldots, 6 \\
\text{such that} & \quad \delta_k(KR_{ij}) \leq \delta_k & k = 1, \ldots, 4 \\
& \quad KR_{ij}^l \leq KR_{ij} \leq KR_{ij}^u, \quad (4.16)
\end{align*}
\]

where \(WT\) is the total weight of the truss and \(wt_i\) is the weight of the \(i\)th bay. The upper and lower bounds on \(KR_{ij}\) are calculated from applying the continuum model to the heaviest truss (maximum \(r\) and \(t\) in for all members) and the lightest truss (minimum dimensions). The calculation of the deflection values \(\delta_k\) is as demonstrated in Section 3.3.2 using \(KR_{ij}\) as the beam stiffnesses of the segmented beam. The subsystem level problem for the \(m\)th bay is:

\[
\begin{align*}
\text{minimize} & \quad wt(r_{mj}, t_{mj}) \\
\text{such that} & \quad \lambda_{mj}(r_{mj}, t_{mj}) < 1.0 & j = 1, \ldots, 6 \\
& \quad K_{mj}(r_{mj}, t_{mj}) = KR_{mj} & j = 1, \ldots, 6 \\
& \quad 0.5 \leq r_{mj} \leq 1.0 & j = 1, \ldots, 6 \\
& \quad 0.04 \leq t_{mj} \leq 0.2 & j = 1, \ldots, 6 \quad (4.17)
\end{align*}
\]
The calculation of the local buckling factors ($\lambda_{ij}$) and their derivatives is as was demonstrated in Subsection 3.4.2. Note that internal forces acting on the subsystem ($Q_{mj}$) can be calculated during upper level structural analysis and held constant during lower level optimization. The actual beam stiffnesses ($K_{mj}$) and their derivatives are calculated according to Subsection 3.2.2

As mentioned in the last subsection, the foregoing multilevel optimization problem will be solved using the method proposed by Haftka. Lower level optimization is therefore carried out using the penalty function approach. The augmented subsystem weights used are

$$
\phi_i = wt_i + \alpha \sum_{j=1}^{6} \left(1 - \frac{K_{ij}}{K_{R_{ij}}} \right)^2 + \beta \sum_{j=1}^{6} \frac{1}{1 - \lambda_{ij}}.
$$

(4.18)

where $\alpha$ and $\beta$ are penalty multipliers. The first penalty term is designed to grow large whenever any of the six beam stiffnesses deviates appreciably from the values required by the upper level. The second term grows large when a buckling factor approaches unity. This type of penalty is called a barrier penalty and suffers from the drawback that it requires a feasible initial design. In order to start from a potentially infeasible design point, a different equation is used close to unity and in the infeasible region. Called the extended interior penalty function (Haftka, Gürdal, and Kamat, 1990), it involves defining the overall penalty as follows:

$$
g = 1 - \lambda
$$

$$
P(g) = \begin{cases} 
1/g & \text{for } g \geq g_0 \\
1/g_0 [3 - 3(g/g_0) + (g/g_0)^2] & \text{for } g < g_0 
\end{cases}
$$

(4.19)

The term $g_0$ is a small number that specifies the transition point where the polynomial form of the penalty takes over. It is moved closer and closer to zero as optimization proceeds. As this happens, the lenient barrier provided by the polynomial becomes stricter and stricter.
System level optimization is now

minimize \[ \Phi(KR_{ij}) = \sum_{i=1}^{n} (\phi_i)_{\text{min}} \]

such that \[ \delta_k(KR_{ij}) \leq \delta_k \] \[ k = 1, \ldots, 4 \]

\[ KR^L_{ij} \leq KR_{ij} \leq KR^U_{ij} \] \[ i = 1, \ldots, n \] \[ j = 1, \ldots, 6, \ (4.20) \]

while the subsystem optimization for bay \( m \) are

minimize \[ \phi_m(r_{mj}, t_{mj}) \]

such that \[ 0.5 \leq r_{mj} \leq 1.0 \] \[ j = 1, \ldots, 6 \]

\[ 0.04 \leq t_{mj} \leq 0.2 \] \[ j = 1, \ldots, 6. \ (4.21) \]

4.3.3 Optimization Process

The continuum model scheme is executed in typical multilevel fashion. There are two separate optimization routines, one for the system level and one for the subsystem level. The upper level optimizer runs as normal, calling the lower level optimizer at every new design point. Further complexity is introduced here, however, due to the use of the penalty functions. Normally when using the penalty approach, the penalty multiplier starts at an appropriate initial value and then is modified for more strict constraints between every successive unconstrained minimization. Multipliers like \( \alpha \) which gauge the enforcement of equality constraints are increased so that the optimizer is compelled to further minimize the constraint violation. Multipliers like \( \beta \) which gauge inequality constraints are
decreased so that the design may approach the constraint boundary and become active. It has been recommended in the literature that subsystem minimization not be carried to completion on every call to the lower level optimizer (Haftka, 1984). It was recommended that the penalty multipliers be held constant during subsystem optimization and altered on the upper level between calls to the lower level. The method used in this investigation takes this concept one level higher. Here, the penalty multipliers are held constant during complete multilevel optimizations. That is, each of a series of system level optimizations is run to completion using fixed values of the two penalty multipliers. This change was made because the earlier method of modifying the multipliers did not work well in the present scheme with sequential linear programming being used on the system level (the earlier work implemented penalty functions on both levels). With SLP, the upper level tended to converge to erroneous local minima which were created by the too rapid increase of $\alpha$. The modification of $\alpha$ could be fine tuned, of course, but the tuning was problem dependent and not reliable. The present method is implemented because SLP is needed for a similar reason as it was in the traditional scheme. There it was needed because of the large computational effort expended in the calculation of the constraints. Here the approximation is needed because of the effort expended in the calculation of the objective function, namely $n$ separate optimizations.

Due to the necessity of repeating the multilevel scheme with different values of $\alpha$ and $\beta$, an external algorithm is required to initialize and control them. Called the controller, it determines the appropriate values of the multipliers and then calls the system optimizer. It continually monitors the progress of the system optimizations and terminates the process when they converge. The controller uses three additional quantities besides the weight and design variables to initialize and modify the multipliers. They are

$$\sigma = \sqrt{\frac{1}{6n} \sum_{i=1}^{n} \sum_{j=1}^{6} \left(1 - \frac{K_{ij}}{KR_{ij}}\right)^2}$$  \hspace{1cm} (4.22)
\[ \gamma = \frac{\phi}{\bar{W}} \]  
\[ \lambda_{\text{max}} = \max(\lambda_{ij}). \]  

The quantity \( \sigma \) measures the disagreement between the the required beam stiffnesses \( KR_{ij} \) and the actual stiffnesses \( K_{ij} \) of a final system design. It is used to ensure that the final bay designs actually have the required stiffnesses upon which the constraint calculations are based. The second quantity, \( \gamma \), is the ratio of the augmented weight to the actual weight. It is used to ensure that \( \beta \) has decreased to the point that the buckling constraints can become active. Lastly, \( \lambda_{\text{max}} \) is useful in forcing the initial design to honor the buckling constraints sufficiently (no members buckling).

The three nested algorithms are as follows:

**Controller**

1. Determine initial guess \( KR_{ij} \)
2. Initialize \( \alpha \) and increase it until \( \sigma < 0.05 \)
3. Initialize \( \beta \) and increase it until \( \lambda_{\text{max}} < 1.0 \)
4. Call system optimizer, receive back \( \phi, \sigma, \) and \( \gamma \)
5. If convergence criteria are met \( (\sigma \leq 0.015, \gamma \leq 1.01) \) then stop
6. Increase \( \alpha \) and/or decrease \( \beta \)
7. go to 4
System Optimizer

1. Receive $\alpha$, $\beta$, and initial design $KR_{ij}$ from controller

2. Evaluate $\delta_k$, and their derivatives w.r.t $KR_{ij}$

3. Evaluate internal forces $Q_{ij}$

4. Call subsystem optimizer $n$ times to evaluate $\Phi = \sum(\phi_{i})_{\text{min}}$ and its derivatives

5. Assemble linear problem from derivatives and the move limit

6. Solve linear problem for $\Delta KR_{ij}$

7. Perturb $KR_{ij}$ by $\Delta KR_{ij}$

8. If $\Phi$ has converged ($\Delta \Phi < \epsilon$) then go to 10

9. Go to 2

10. Return final $\Phi$, $\sigma$, and $\gamma$ to controller

Subsystem Optimizer

1. Receive from system level $KR_{mj}$, $Q_{mj}$, $\alpha$, and $\beta$

2. Perform unconstrained minimization on $\phi_m(r_{mj}, t_{mj})$

3. Calculate sensitivity derivatives $\partial(\phi_{m})_{\text{min}}/\partial KR_{mj}$

4. Return $(\phi_m)_{\text{min}}$, its derivatives, $wt$, and $K_{ij}$ to system optimizer

Details of the three algorithms are contained in the following paragraphs.
Controller

Given the initial design $KR_{ij}$, the first thing the controller does is find appropriate starting values of $\alpha$ and $\beta$. The stiffnesses in the initial design are those calculated from an initial design of the natural variables $r_{ij}$ and $t_{ij}$ and are a realistic set of required upper level variables. This fact is utilized in the choosing of the starting value of $\alpha$. It is required that the first $\alpha$ be large enough in magnitude that when subsystem optimizations are called, the resulting actual stiffnesses match the required ones to a certain degree. Obviously if $\alpha$ is large enough the requirements will be met exactly. It is not advisable to start a penalty function algorithm with too large a penalty, however, because the optimizer will focus on reducing the penalty instead of the objective function. The compliance with this requirement will be measured by $\sigma$. The subsystem optimizations are performed with $\alpha = 1, 2, 4, \ldots$ until $\sigma$ is less than 0.05. Once the correct value of $\alpha$ is found, the controller turns its attention to $\beta$. Subsystem optimizations are run again with increasing $\beta$'s until $\lambda_{max}$ is less than 1.0. This procedure simply insures that $\beta$ is large enough to start the optimization in the nonbuckling part of the design space. With the multipliers defined, the controller calls the system optimizer which runs to completion. The process is not over, however, if one or both of two conditions are not met. The first is that the actual beam stiffnesses represented by the natural variables are close enough to those being used as system variables in global structural analysis. Once again, $\sigma$ is used to measure this. If $\sigma$ at the end of the optimization is not less than 0.015, $\alpha$ is doubled and another system optimization occurs. The second condition is that the local buckling constraints are being allowed to approach unity. If they are not, this fact will be demonstrated by the size of the penalty added to the weight. If $\gamma$ is greater than 1.01 (the penalty is more than one percent of the weight), then $\beta$ is divided by ten and another optimization is effected. The result of this initialization and modification of the penalty parameters is that the subsystem constraints are lax in the beginning and become tighter as optimization progresses. This leads to large weight cutting steps at first and then small refining steps towards the end. Usually there are four to seven optimization cycles with different penalty parameters.
System Optimizer

As mentioned previously, sequential linear programming is used on the system level. The actual optimizer is the ADS subroutine developed for NASA (Vanderplaats, 1985). Move limits on the system level are kept relatively small. They start at five percent and then decrease by half as the problem converges. It was found in studying this multilevel algorithm as a whole that convergence is actually faster when the move limits were started at smaller percentages. One possible reason for this is that moves are kept in the more accurate region of the Taylor series approximation and the number of misguided moves is reduced. The system level is converged when the decrease in the objective due to a move is less than 0.1 percent.

Subsystem Optimizer

Subsystem optimization is also carried out using the ADS subroutine. The function $\phi$ for a given set of upper level variables, upper level forces, and penalty multipliers is found using the Fletcher-Reeves algorithm for unconstrained minimization (Vanderplaats, 1985). It was discovered through study that a convergence limit of 0.5 percent works well for the unconstrained minimization. After the optimum design is reached the sensitivity derivatives are calculated:

$$\frac{\partial(\phi_i)_{\min}}{\partial KR_{ij}} = 2a \sum_{j=1}^{6} \left(1 - \frac{K_{ij}}{KR_{ij}}\right) \left(\frac{K_{ij}}{KR_{ij}^2}\right)$$

(4.25)

This equation is direct partial differentiation of the objective with respect to parameter. It was proposed by Haftka and works rather well in this scheme (Haftka, 1984).
5.0 Results

The two optimization methods developed in the last chapter were applied to a series of truss design problems to study the performance of the proposed approach. Selected results from these studies are presented in the following three sections. First are the final truss designs that resulted from the two methods. Next is a study of the computational expense required. Last is a comparison of the reliability or robustness of the methods.

5.1 FINAL DESIGNS

The design problems considered in this study are involved with finding the least weight truss that satisfies constraints on the global deflections and the local buckling factors. Presented in this section are the final designs that were produced by the continuum model and traditional optimization schemes. Recall that the design variables for the problem are the radius and wall thickness of six groups of members in each bay. Each radius is constrained within the interval (0.5, 1.0) and each thickness is constrained within the interval (0.04, 0.2).

Presented first are the minimum weight designs of a ten bay truss that are obtained when only one deflection constraint is used. In these cases the buckling constraints are present but not active. This will demonstrate how material is optimally distributed to provide a particular type of stiffness. If the only constraint is on the axial stiffness, then the resulting material distribution is qualitatively like that shown in Table 5.1. Observe that all members are minimized except those in group one, which are the longerons. Apparently it is most weight effective to control the axial stiffness only with the longerons. They are fine tuned to meet the axial deflection constraint exactly. Also worth noting is
that the stiffness is equally distributed among the bays. This is not the case when
the only constraint is on the global bending stiffness. Table 5.2 tabulates the
results from design optimization where there were equivalent constraints on the
two perpendicular bending stiffnesses. Once again, the longerons (group one) were
used to tune the stiffness while the rest of the truss was minimized. In the case of
bending the longerons are made thicker and larger in diameter near the wall
support and gradually become thinner and smaller towards the free end. As
mentioned, these bending results were generated using equivalent global stiffness
constraints in the two transverse directions. If differing stiffness requirements are
used the resulting truss is qualitatively the same. The optimization uses the
longerons to satisfy the higher of the two stiffness requirements and the other is
over satisfied. Moving on to torsion, the results for a single constraint on the
torsional stiffness are presented in Table 5.3. Therein it is demonstrated that the
most weight effective way to design for torsion is to minimize all members except
those in groups four and six. These are the two sets of diagonals that are directed
partially in the longitudinal direction. Material is equally distributed between
those eight members in every bay in the correct amount to tune the torsional
stiffness.

The previous three cases were combined to demonstrate the final designs
that occur when there are constraints on all of the stiffnesses. Results from the
optimization are in Table 5.4. Notice that the design of groups four and six is
about the same as it was in Table 5.3. The longerons are tapered down from the
wall as in the case of bending constraints only, but they are thinner. This is due to
the fact that groups four and six are providing some bending stiffness. The
remaining groups (two, three, and five) are still for the most part minimized.
These are the groups whose members are perpendicular to longitudinal axis. As
might be expected, they are not important to the design of a beam-like truss. The
optimization algorithms select only the outside longerons and diagonals to provide
the stiffnesses of the truss. Groups four and six provide the torsional and shear
stiffnesses and group one provides the axial and bending stiffnesses. This design
can be seen qualitatively as a box beam. The internal cross members are only
present to provide support and stability to the main members. There is a usual set
of tradeoffs between the main members. Normally groups four and six are tuned to
provide exactly the torsional stiffness. Group one is used to add the right amount of stiffness to satisfy the more stringent of the two other constraints, bending and axial. If the axial constraint is more demanding then the longerons will be uniform. If the reverse is true, then the longerons will be tapered. There are many situations where the constraints demand approximately the same amount of material and a tapered design is the result.

If the loads used in the last case are increased, the local buckling constraints will become active and start controlling the design. Table 5.5 contains the results of an optimization with the same stiffness constraints as before but with higher end loads. The effects of active buckling constraints can be seen in the increased radii of groups two, three, and five. These members have been made larger in order to shunt off some of the load and prevent the buckling of groups four, five, and six. Groups four and six are always the members to become critical first. After that, group five becomes critical. The remaining members never become critical because four and six will buckle before that can happen. It is seen, therefore, that material is added to cross member groups two, three, and five only to forestall failure by local buckling.

One last important fact that can be drawn from the five tables is that the continuum model scheme converges to designs that are equivalent to or better than (Table 5.5) the designs arrived at by the traditional scheme. The difference in the two final weights in Table 5.5 is due to the traditional method converging to a local, rather than global, minimum.
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**Continuum Model Method**

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**Weight** 889.27

**Traditional Method**

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**Weight** 904.32
Table 5.2  Optimization Results for Bending Constraints Only

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Final Designs
### Table 5.3  Optimization Results for Torsional Constraint Only

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Final Designs
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Traditional Method

| Radii |    |    |    |    |    |    |    |    |    |    |
| 1     | 0.914 | 0.938 | 0.927 | 0.940 | 0.938 | 0.905 | 0.871 | 0.836 | 0.822 | 0.799 |
| 2     | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 |
| 3     | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 |
| 4     | 0.918 | 0.957 | 0.965 | 0.981 | 0.970 | 0.933 | 0.943 | 0.946 | 0.944 | 0.956 |
| 5     | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 |
| 6     | 0.918 | 0.957 | 0.965 | 0.981 | 0.970 | 0.933 | 0.943 | 0.946 | 0.944 | 0.956 |
| Thicknesses |    |    |    |    |    |    |    |    |    |    |
| 1     | 0.133 | 0.126 | 0.114 | 0.103 | 0.093 | 0.087 | 0.081 | 0.076 | 0.072 | 0.066 |
| 2     | 0.040 | 0.040 | 0.040 | 0.040 | 0.040 | 0.040 | 0.040 | 0.040 | 0.040 | 0.040 |
| 3     | 0.040 | 0.040 | 0.040 | 0.040 | 0.040 | 0.040 | 0.040 | 0.040 | 0.040 | 0.040 |
| 4     | 0.061 | 0.057 | 0.057 | 0.057 | 0.057 | 0.060 | 0.059 | 0.059 | 0.060 | 0.060 |
| 5     | 0.040 | 0.040 | 0.040 | 0.040 | 0.040 | 0.040 | 0.040 | 0.040 | 0.040 | 0.040 |
| 6     | 0.061 | 0.057 | 0.057 | 0.057 | 0.060 | 0.059 | 0.059 | 0.059 | 0.060 | 0.060 |
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### Table 5.5  Optimization Results for Active Buckling Constraints

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<td>Weight</td>
<td>1470.43</td>
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### Traditional Method

| Radii |      |      |      |      |      |      |      |      |      |      |
| 1    | 0.981 | 0.963 | 0.950 | 0.926 | 0.922 | 0.905 | 0.912 | 0.901 | 0.913 | 0.901 |
| 2    | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.510 | 0.500 |
| 3    | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.510 | 0.500 |
| 4    | 0.999 | 0.999 | 0.999 | 0.999 | 0.999 | 0.999 | 0.999 | 0.999 | 0.999 | 0.999 |
| 5    | 0.594 | 0.597 | 0.611 | 0.618 | 0.623 | 0.624 | 0.626 | 0.626 | 0.626 | 0.539 |
| 6    | 0.999 | 0.999 | 0.999 | 0.999 | 0.999 | 0.999 | 0.999 | 0.999 | 0.999 | 0.999 |
| Thicknesses |      |      |      |      |      |      |      |      |      |      |
| 1    | 0.115 | 0.109 | 0.103 | 0.100 | 0.097 | 0.095 | 0.094 | 0.093 | 0.092 | 0.085 |
| 2    | 0.076 | 0.072 | 0.073 | 0.074 | 0.075 | 0.076 | 0.077 | 0.078 | 0.082 | 0.067 |
| 3    | 0.076 | 0.072 | 0.073 | 0.074 | 0.075 | 0.076 | 0.077 | 0.078 | 0.082 | 0.067 |
| 4    | 0.057 | 0.053 | 0.055 | 0.056 | 0.056 | 0.057 | 0.056 | 0.057 | 0.056 | 0.059 |
| 5    | 0.073 | 0.067 | 0.069 | 0.071 | 0.073 | 0.074 | 0.075 | 0.076 | 0.083 | 0.063 |
| 6    | 0.057 | 0.053 | 0.055 | 0.056 | 0.056 | 0.057 | 0.056 | 0.057 | 0.056 | 0.059 |
| Weight | 1603.73 |      |      |      |      |      |      |      |      |      |
5.2 COMPUTATIONAL EXPENSE

As stated in the introduction, savings in the amount of computational effort is the main motivation behind this investigation. In order to study the relative behavior of the continuum model algorithm, both algorithms were used to optimize a series of test cases. Each case consisted of a group of seven related design problems, each problem with a different number of bays \((n = 6, 10, 14, 18, 22, 26, 30)\). The final minimum weights and the CPU times required for each method were recorded as data.

The first test case is considered the benchmark or base case. The end loads are contrived in conjunction with the maximum allowable deflections so that all four deflection constraints are active at the optimum design. Defining a maximum deflection constraint along with a prescribed end load is equivalent to requiring the truss to have a certain overall stiffness in that deflection mode. End loads are scaled for each problem such that the overall beam stiffnesses required by the constraints are the same, regardless of the number of bays in the problem. The nondimensionalized global beam stiffnesses are:

\[
\begin{align*}
EA &= 0.125 \\
EI_2 &= 0.0333 \\
EI_3 &= 0.0333 \\
GJ &= 5.0
\end{align*}
\]

The maximum deflections \((\delta_1, \delta_4)\) are set equal to unity, and the loads \((P_1, P_4)\) are calculated from standard strength of materials formulae:
\[ P_1 = \frac{0.125(1.0)}{l} \]
\[ P_2 = \frac{0.0333(1.0)}{l^3} \]
\[ P_3 = \frac{0.0333(1.0)}{l^3} \]
\[ P_4 = \frac{(5.0)(1.0)}{l} \]  

(5.1)

When the stiffness requirements are held constant in this way, the design problems at different numbers of bays are very similar to one another. Local buckling constraints are included also in the benchmark case, but very few, if any, become active and they do not control the design. Both algorithms were started from the same initial designs for the seven problems. The initial guesses used were good designs that were developed by using the continuum model to forecast the bay stiffnesses that would be required. Data from the fourteen optimizations is presented in Figure 5.1. The top graph contains the CPU time required for each optimization method to reach a final design on an IBM 3090 mainframe computer. Observe the substantial savings in computational expense that the continuum model method allows. The bottom graph is a display of the ratio of the continuum final design weights to the traditional final weights. It demonstrates that the multilevel method converged to a comparable design or a better one with the time savings illustrated in the upper graph. This improvement over the traditional optimization method is attributed to two different factors. The major part of the savings is due to the decomposition used in the new method. Solving the coordination problem simultaneously with the \( n \) smaller subproblems is simply faster than solving the large design problem by itself. This is due to the well known nonlinear dependence of computational expense on problem size. Notice that the ratio of the time required for the two methods grows larger with the size of the problem. That is a key result; the new method becomes more and more useful as the numbers of design variables and constraints grow. The second factor contributing to this savings is that the approximate global structural analysis in the continuum model method is much less time consuming than the full finite element analysis used in the traditional method. This case has demonstrated the
Figure 5.1  Case 1 - benchmark case
savings in computational expense that are possible with the proposed new method. The next few cases involve perturbations of the benchmark case so that the causes of the savings can be further studied.

Case two involves reducing the number of design variables in the benchmark case by fifty percent. It is the alternative design problem which was mentioned briefly in the last chapter. Only the cross-sectional areas of the members are used as design variables. All of the constraints, however, remain unchanged. Results from the same fourteen problems with this new formulation are shown in Figure 5.2. Notice from the time scale on the left that the amount of time taken by the traditional algorithm has been reduced by a factor of five. Given that the multilevel data did not change significantly, it can be concluded that the time savings are much less when half the number of design variables is used.

In case three the local buckling constraints are removed, leaving only the four maximum allowable deflection constraints. These results are in Figure 5.3. Again a decrease in the time savings is evident, due mainly to the decrease in time taken by the traditional method. The savings were greater in case three than in case two though. From this fact it can be concluded that implementation of the multilevel method is more beneficial when there is a large number of design variables than when there is a large number of constraints.

All variables and constraints are restored to the benchmark case for case four. The difference lies in that the end loads are increased so that local buckling constraints partially control the design. Allowable deflections are also increased so the effective stiffness constraints remain the same. It is seen from the scale in Figure 5.4 that the difference between the times required for the two methods is increased over the benchmark case. Increasing the number of active constraints causes the optimization problem to be more complex numerically. It can be concluded that the multilevel method is not affected as much by this complexity and is actually more beneficial when it is present.
Figure 5.2  Case 2 - design variables halved
Figure 5.3 Case 3 - local buckling constraints removed
Figure 5.4  Case 4 - controlling constraints increased
Lastly, case five is the benchmark case started from a slightly worse initial design. Plotted in Figure 5.5 are the data from this case. It is plainly seen that once again the new condition has a worse effect on the traditional method than on the continuum model method.

It can be concluded from these five test cases that the multilevel method has a definite advantage in computational expense. The extent of that advantage is highly dependent on the size and complexity of the problem.
Figure 5.5  Case 5 - poor initial design
5.3 \textit{ROBUSTNESS}

The final set of results deal with the robustness, or the ability to overcome numerical difficulty, of the two methods. The benchmark case from the last section is optimized with with poorer initial designs. Final designs from these optimizations were compared to the final designs that resulted when the benchmark case was optimized with the good starting values. What is being measured is the ability of each method to find the global optimum starting from a point far away from it. The top graph in Figure 5.6 is a plot of the results from the use of an "average" initial design. To come up with the initial designs, each member in the first bay is maximized and each member in the last bay is minimized. Linear interpolation is used to size the initial values of members in bays in between. Points in the plot represent the ratio of the new final weight to the original final weight. The proximity of the points to unity reflects the robustness of the algorithms. It is seen in this graph that the traditional method on average is better at returning to close to its previous optimum. The lower graph in Figure 5.6 contains data from the use of a worse initial guess. This time the two methods are started from an initial design with every member maximized. Data in this graph show that for the worst initial design the continuum model is more robust. The squares high above the line at 1.0 indicate that the traditional method converged to local minima. It can be concluded that the continuum model method is relatively less robust when started close to the optimum but becomes relatively more robust once the initial design is removed from that region.
Figure 5.6 Ability of methods to reach optimums from poor initial designs
6.0 Conclusions

In this investigation an alternative method of optimization for a truss design problem was developed and employed. Using a continuum model in the context of a multilevel decomposition scheme, the method demonstrated superiority over a traditional scheme in terms of computational efficiency. In addition, the method exhibited greater robustness in converging to optimal designs than did the traditional method. The success of the continuum model method is a testament to the substantial ability of multilevel decomposition in making large scale optimization problems less expensive and more tractable. The decomposition used in the continuum model method was not possible with standard design variables. Because the continuum model was used, subsystems with many members could be employed. Considering that decomposition into structural elements was already available, the continuum model extended the potential decomposition to one more upper level. It has been demonstrated, through example, that continuum modeling can increase the utility of multilevel decomposition.

It can be argued that with the current state of super computing technology, taking the time and effort to develop a continuum model based multilevel algorithm is unnecessary and more costly than simply using a traditional method. There will soon be design problems, however, with so many variables and constraints that decomposition will be of significant benefit, if not indispensable to their solution. An excellent example of these are the new topology and shape optimization problems which currently have thousands of design variables. Another point of view is that decomposition and continuum modeling can extend the capabilities of even the fastest computers by bringing larger and larger problems into their range.
The example problem used for this investigation is not the only structural optimization problem to which continuum modeling can be applied. As mentioned previously, continuum models have been developed for beam-like, plate-like, shell-like and general block truss and frame structures. In addition, continuum models could potentially be developed for highly specific structural subsystems with many members. A good example of this would be the repeating cells of an aircraft wing.

The accuracy of the continuum model used in this study was relatively high. Assumptions used in its development turned out to be well founded. It is doubtful that the continuum model method would have the same success in reaching global optimums if this were not the case. Continuum model methods with less accuracy may still be useful, however. They might be used to generate high quality initial designs to be used in single level optimizations.

Suggestions for future work include three possible avenues of research. First, the method can be applied using different continuum models and different skeletal structures. These might include continuum models that have already been developed or new models that could be developed for specific structures. Research of this type would add more general knowledge about continuum model optimization. Second, the method could include a wider range of global constraints. The model presented can be expanded to include quantities like masses, geometric stiffnesses, coupling coefficients, and thermal coefficients. With these extra characteristics available more comprehensive design problems can be handled by the method. Lastly, the mechanics of the continuum model method might be improved. The multilevel scheme used in the current research could very well be fine tuned or completely altered for better results.
References


References


Appendix

Local Stresses Calculated from Continuum Forces

Stresses due to axial force $N$:

$$\sigma_j = \frac{N}{4} \left| \frac{NUM_j}{DEN} \right|$$

$$DEN = A_1 \left[ \frac{A_3 B_1^2 B_3^2 (A_3 W^3 + A_2 H^3)}{B_2} + \frac{A_4 A_6 B_2^2 H^3 W^3}{B_3 B_1} + A_2 A_3 B_1 B_2^2 B_3^2 \right]$$

$$+ \frac{A_4 L^3}{B_1} \left[ \frac{A_5 B_3^2 (A_3 W^3 + A_2 H^3)}{B_2} + \frac{A_6 H^3 (2 A_5 W^3 / B_2 + A_2 B_2^2)}{B_3} + A_2 A_3 B_2^2 B_3^2 \right]$$

$$+ \frac{A_1}{B_2} \left[ \frac{A_4 A_5 B_3^2 H^3 W^3}{B_1} + \frac{A_4 A_3 B_3^2 B_2^2 W^3}{B_3} + \frac{A_5 A_6 B_1^2 H^3 W^3}{B_3} + A_2 A_6 B_1 B_2^3 H^3 \right]$$

$$+ \frac{A_6 L^3}{B_3} \left[ \frac{A_5 B_1^2 (A_3 W^3 + A_2 H^3)}{B_2} + \frac{A_4 W^3 (2 A_5 H^3 / B_2 + A_3 B_2^2)}{B_1} + A_2 A_3 B_2^2 B_3^2 \right]$$

$$NUM_1 = \frac{W^3 H^3 (A_4 A_5 B_3^3 + A_4 A_6 B_2^3 + A_5 A_6 B_1^3) + B_3^3 W^3 (A_3 A_4 B_2^3 + A_3 A_5 B_1^3)}{B_1 B_2 B_3}$$

$$+ B_1^3 H^3 (A_2 A_5 B_3^3 + A_2 A_6 B_2^3) + A_2 A_3 B_1^3 B_2^3 B_3^3$$

$$NUM_2 = \frac{L^2 W}{B_1 B_2 B_3} \left[ H^3 (A_5 A_6 B_1^3 - A_4 A_5 B_3^3 - A_4 A_6 B_2^3) - A_3 A_4 B_2^3 B_3^3 \right]$$
\[ NUM_3 = \frac{L^2 H}{B_1 B_2 B_3} \left[ W^3 (A_4 A_5 B_3^3 - A_5 A_6 B_1^3 - A_4 A_6 B_2^3) - A_2 A_6 B_1^3 B_2^3 \right] \]

\[ NUM_4 = \frac{L^2}{B_2 B_3} \left[ W^3 (2A_5 A_6 H^3 + A_3 A_5 B_3^3) + H^3 (A_2 A_5 B_3^3 + A_2 A_6 B_2^3) + A_2 A_3 B_2^3 B_3^3 \right] \]

\[ NUM_5 = -\frac{L^2}{B_1 B_3} \left[ 2A_4 A_6 W^3 H^3 + A_3 A_4 B_3^3 W^3 + A_2 A_6 B_1^3 H^3 \right] \]

\[ NUM_6 = \frac{L^2}{B_1 B_2} \left[ H^3 (2A_4 A_5 W^3 + A_2 A_5 B_1^3) + W^3 (A_3 A_4 B_2^3 + A_3 A_5 B_1^3) + A_2 A_3 B_1^3 B_2^3 \right] \]

**Stresses due to moment \( M_2 \):**

\[ \sigma_1 = \frac{M_2}{2W} \left| \frac{A_6 H^3 + A_3 B_3^3}{A_1 A_6 H^3 + A_3 A_6 L^3 + A_1 A_3 B_3^3} \right| \]

\[ \sigma_2 = 0 \]

\[ \sigma_3 = \frac{M_2}{2W} \left| \frac{-A_6 H L^2}{A_1 A_6 H^3 + A_3 A_6 L^3 + A_1 A_3 B_3^3} \right| \]

\[ \sigma_4 = 0 \]

\[ \sigma_5 = 0 \]

\[ \sigma_6 = \frac{M_2}{2W} \left| \frac{A_3 L^2 B_3}{A_1 A_6 H^3 + A_3 A_6 L^3 + A_1 A_3 B_3^3} \right| \]
Stresses due to the moment $M_3$:

$$\sigma_1 = \frac{M_3}{2H} \frac{A_4 W^3 + A_2 B_1^3}{A_1 A_4 W^3 + A_1 A_2 B_1^3 + A_2 A_4 L^3}$$

$$\sigma_2 = \frac{M_3}{2H} \frac{-A_4 L^2 W}{A_1 A_4 W^3 + A_1 A_2 B_1^3 + A_2 A_4 L^3}$$

$$\sigma_3 = 0$$

$$\sigma_4 = \frac{M_3}{2H} \frac{A_2 L^2 B_1}{A_1 A_4 W^3 + A_1 A_2 B_1^3 + A_2 A_4 L^3}$$

$$\sigma_5 = 0$$

$$\sigma_6 = 0$$

Stresses due to the shear $Q_2$:

$$\sigma_1 = 0$$

$$\sigma_2 = 0$$

$$\sigma_3 = 0$$

$$\sigma_4 = \frac{Q_2 B_1}{4 W A_4}$$

$$\sigma_5 = 0$$

$$\sigma_6 = 0$$
Stresses due to the shear $Q_3$:

\[ \sigma_1 = 0 \]
\[ \sigma_2 = 0 \]
\[ \sigma_3 = 0 \]
\[ \sigma_4 = 0 \]
\[ \sigma_5 = 0 \]
\[ \sigma_6 = \frac{Q_3 B_3}{4 H A_6} \]

Stresses due to the torque $T$:

\[ \sigma_4 = \frac{TB_1}{2B_2} \left| \begin{array}{c} H \\ 2A_4 W \end{array} \right| - \frac{W}{2A_4 H} \]
\[ \sigma_5 = \frac{TB_2}{2B_2} \left| \begin{array}{c} H \\ 2A_5 W \end{array} \right| - \frac{W}{2A_5 H} \]
\[ \sigma_6 = \frac{TB_3}{2B_2} \left| \begin{array}{c} H \\ 2A_6 W \end{array} \right| - \left( \frac{H}{2A_6 W} + \frac{W}{2A_6 H} \right) \]
Vita

Keith William Yates was born in High Point, North Carolina on November 22, 1967. He later moved to Bassett, Virginia where he attended J. D. Bassett High School. In the Fall of 1986, after graduation, he entered Virginia Polytechnic Institute & State University as an engineering freshman. At the end of that year he chose Engineering Science and Mechanics for his major and structural and stress analysis as his area of concentration. After graduating Magna Cum Laude in Spring 1990, he returned to Virginia Tech to pursue a Master of Science degree in Engineering Mechanics.

Keith Yates