Finite Element Formulation of a Thin-Walled Beam
with Improved Response to Warping Restraint

by

Dhrubajyoti Ghose

Thesis submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of
Master of Science
in
Aerospace & Ocean Engineering

APPROVED:

[Signature]
O.F. Hughes, Chairman

[Signature]
E.R. Johnson

[Signature]
E. Nikolaidis

March 1991
Blackburg, Virginia
Finite Element Formulation of a Thin-Walled Beam

with Improved Response to Warping Restraint

by

Dhrubajyoti Ghose

O.F. Hughes, Chairman

Aerospace & Ocean Engineering

(ABSTRACT)

Linear elastic theory of torsion and flexure of thin-walled beams as developed by Vlasov and Timoshenko respectively are well known and commonly used in everyday engineering practice. However there are noticeable differences between calculations and experimental results. The difference is partly due to one of the basic assumption of classical theory, namely that the secondary shear strains due to warping are negligible.

In the present work a new three noded, with C⁰ continuity, isoparametric beam finite element is developed based on a torsion theory by Benscoter. In the classical theory warping is assumed to be proportional to the rate of twist, whereas in Benscoter's theory it is assumed to be proportional to an independent quantity called the "warping function". The exact form of this function can be evaluated from equilibrium equations. This assumption of Benscoter's allows the formulation of a C⁰ element based on the assumed displacement method. The other advantage of Benscoter's theory is that it takes into account the effects of secondary shear strains. These effects are quite significant for closed sections. The element is validated for several cases of a cantilevered beam of rectangular cross section and in each case the results are in good agreement with the exact solution.
It is also shown that the element gives a very good representation of curved beams, for which there is torsional-flexural coupling. A number of cases of a curved I-beam under various loading and boundary conditions are analysed, and in every case the results agree closely with the analytical solution.

In order to represent the torsional response the element uses seven degrees of freedom per node. This seventh degree of freedom is the "warping function" mentioned earlier. To make the element compatible with standard finite-element programs which have six degrees of freedom per node, static condensation is used.
I am indebted to Prof. Owen F. Hughes for his guidance and encouragement during my thesis work. His constant financial support helped me greatly in completing my degree. I wish to thank Dr. E. R. Johnson and Dr. E. Nikolaidis for serving on my committee.

I would also like to thank my Lemoyer office mates and all other friends for their valuable assistance and good company.

Lastly I take this opportunity to thank my parents whose love and sacrifices made all this possible.
# Table of Contents

Chapter 1: Introduction ................................................................. 1
   Background ............................................................................ 1
   Literature Review ................................................................. 3
   Objective ............................................................................... 6

Chapter 2: Theories Of Torsion ...................................................... 9
   2.1 Classical Torsion Theory .................................................. 10
      2.1.1 Uniform Torsion ....................................................... 10
      2.1.2 Non-Uniform Torsion .............................................. 14
   2.2 Besselet’s Torsion Theory .............................................. 18
      2.2.1 Displacements ......................................................... 18
      2.2.2 Stress and Strains ................................................... 19
      2.2.3 Governing Equations ............................................. 22

Chapter 3: Finite Element Formulation ....................................... 29
   3.1 Kinematics ..................................................................... 30
      3.1.1 Geometrical Description ......................................... 30
List of Illustrations

Figure 1. Load P on a solid cross section replaced by four sets of loads .......... 7
Figure 2. Components of load P on a I-section causing warping ................. 8
Figure 3. Tangential displacement along the profile of thin-walled section ...... 26
Figure 4. Stresses due to warping restraint ........................................ 27
Figure 5. Forces acting on an element of beam ...................................... 28
Figure 6. Cylindrical co-ordinate system for curved beam ....................... 49
Figure 7. Typical thin-walled beam cross-section .................................. 50
Figure 8. Finite element representation of the beam ................................. 51
Figure 9. Coordinate axis system for transformation of eccentric beam ....... 58
Figure 10. Cantilevered thin-walled box beam under distributed Torque ....... 62
Figure 11. Twist Angle, Warping function & Bimoment for the Box beam ...... 63
Figure 12. Cantilevered I-beam under tip Torque ................................... 64
Figure 13. Twist Angle, Normal Stress & Shear Stress for the I-beam .......... 65
Figure 14. Different Loading and Boundary conditions on a Curved beam ...... 66
Figure 15. Results for the Curved Beam (Case A) ................................. 67
Figure 16. Results for the Curved Beam (Case B) ................................... 68
Figure 17. Results for the Curved Beam (Case C) ................................... 69
Figure 18. Results for the Curved Beam (Case D) ................................... 70
Figure 19. Results for the Curved Beam (Case E) ................................... 71
Chapter 1: Introduction

Background

Thin-walled beams can be categorized as structures whose three dimensions all differ by an order of magnitude. That is, the length is much greater than either the depth or breadth of the section which, in turn, are much greater than the wall thickness of each web and flange element. Thus a thin-walled beam combines the essential features of both beams and shells. While the beam action is dominant in the longitudinal direction, other thin-walled effects more closely associated with plates and shells have an important bearing on the cross-sectional behavior.

There are several reasons why thin-walled structures must be given special consideration in their analysis and design. In solid and thick-walled sections, displacement of the beam’s cross-section out of its plane, referred to as “warping”, is usually sufficiently small such that if this warping is restrained the resulting secondary stresses, known as warping stresses, are negligible. A second way in which a thin-walled beam differs from a beam
of solid compact section is in the way that stresses attenuate along its length. This point has been demonstrated by Murray [13], by considering a solid, square cross-section beam which is loaded axially at one corner by a load P acting lengthwise, or parallel to the z-axis, as shown in Fig. 1a. This load can be constituted from the four sets of loads as indicated in Fig. 1b. The first three sets represent axial loading and bending moments about the x and y axes. However the last set of loads is self-equilibrating, and by virtue of the Principle of St. Venant the effect of this set of loads can be ignored. In fact this set of loads gives rise to a set of stresses which attenuate very rapidly towards zero. When we consider the thin-walled I-beam in the same way, Fig. 2a, it is found that the first three sets of loads can be treated as before. However while the fourth set of loads is also self-equilibrating as before, in this case the stresses attenuate very slowly along the length of the beam. This is because the web acts as a sort of insulator separating the loads into two subsets, one in each flange. Each subset is not self-equilibrating and causes longitudinal in-plane bending stress in each flange. The cross-section does not remain plane but warps, and the resulting stress is called warping stress. Therefore the longitudinal stress at a point is made up of axial stress, bending stresses due to simultaneous overall bending about two principal axes, and the longitudinal warping stress. The warping stress can be as large or even larger than the bending stresses and therefore it cannot be ignored.

Two basic types of torsion are uniform torsion, also known as St. Venant torsion, and non-uniform or warping torsion. When a thin-walled beam is twisted, it may involve either St. Venant type torsion only or a combination of St. Venant and non-uniform torsion. St. Venant torsion gives rise only to shear stress; there is no longitudinal normal stress. Warping stresses are a combination of normal stress and shear stress. Warping stresses can occur if either: (a) warping is restrained, or (b) the torsion is non-uniform.
In St. Venant torsion there may be warping, but since the warping is not restrained there are no warping stresses.

**Literature Review**

Vlasov [1] is generally recognized as the first to have presented a rigorous theoretical treatment of the torsion of thin-walled beams. According to him the warping, i.e. the out-of-plane displacement of an open thin-walled beam subjected to torsion, is $u = -\omega(s)\theta_s'$ where $\omega(s)$ is the sectorial coordinate which gives the variation of warping over the cross-section and $\theta_s'$ is the rate of twist.

Von Karman and Christensen [2] gave an analysis for closed sections known as the classical or approximate theory. By mathematical arguments these authors arrived at a transverse distribution of axial displacement similar to the one put forward by Vlasov. However they neglect the effect of secondary (warping) shear stress due to non-uniform torsion on the warping of the beam, and they arrive at the same expression for warping as Vlasov; i.e. $u = -\omega(s)\theta_s'$. This expression is only exact for thin-walled beams under St. Venant torsion; i.e. when warping stresses are absent.

Benscoter [3] overcame the above limitation by proposing a form of solution which accounts for the effect of warping shear stress on the warping of the beam, and which is better suited for finite-element formulations based on assumed displacement methods. In Benscoter's approach warping $u$ due to twisting is assumed to be of the form $u = -\omega(s)f(x)$, where $\omega(s)$ is the same transverse distribution as proposed in [1] and [2].

*Chapter 1: Introduction*
The function $f(x)$ is an independent function which gives the lengthwise distribution of warping and can be obtained from the conditions of equilibrium.

Based on the above theories various authors have proposed different one-dimensional models for thin-walled beams. In the world of ship structures, one developed by Kawai [4] is widely used due to its convenience and speed. However, Kawai’s method is based on the classical torsion theory in which the effect of the secondary shear stress on the warping of the beam is neglected. In order to take into account the effect of the secondary shear stress under the condition of restrained warping Kawai proposed an iterative procedure, in which the results from the shear deformation analysis at a certain step are used as additional nodal forces at the next step.

Chen and Hu [5], in order to improve the accuracy of the calculation and to avoid the iteration, proposed a torsional stiffness matrix of a thin-walled beam based on Benscoter’s theory, where the warping is of the form $u = -\omega(s)f(x)$. They consider a two-noded element with no curvature and use a cubic polynomial to interpolate the twisting angle and a quadratic polynomial for $f(x)$.

Besides the above formulations based on assumed displacement method, there are some based on a hybrid element formulation by Tralli [6] and Noor [7]. However, these formulations are complex and harder to program and involve larger number of beam cross-section degrees of freedom than the assumed displacement formulations.

In recent years there has been a tendency to develop special purpose finite element techniques for the analysis of stiffened plates and shells. The traditional procedure for analysis is the orthotropic model, where the stiffeners are not considered as discrete but their effects are averaged or smeared out. However, this orthotropic approach involves gross
simplifications, and much effort has been directed at developing a beam element that can be used in conjunction with shell elements. Beam elements compatible with the thick shell elements may be obtained either as a special form of the general three dimensional, isoparametric elements or as an application of the one dimensional isoparametric concept and an appropriate beam theory.

Among others Buraghoin et al. [8] and Ferguson and Clark [9] developed curved beam elements of rectangular cross section, which have been obtained as a special form of the general three dimensional isoparametric element. However they do not take into account the torsional properties. Thus the degenerate three dimensional element placed eccentric to the shell fails to conveniently represent the torsional response of the beam.

Jirousek [10] based his approach on an appropriate beam theory, which properly accounts for the stiffer eccentricity with respect to the shell middle surface, the transverse shear and the effect of the shear center location. This approach was computationally economical since the numerical integration was performed only along the element axis. However, since he uses only six degrees of freedom per node, the element cannot model the warping torsional response, which involves $\theta_i'$ or some other function of $x$: $f(x)$.

To account for the influence of torsional warping, Voros [11] includes an additional independent seventh degree of freedom at each node, corresponding to bimoment. This seventh degree of freedom creates a problem when the beam element is to be used with shell elements which have six degrees of freedom at each node. To overcome this problem, Voros suggests augmenting the shell stiffness matrix with extra rows and columns of zeroes corresponding to the seventh degree of freedom of the beam. But the difficulty of this solution is that finite element models of typical structures already have thousands of de-
degrees of freedom, and this procedure would further increase the size of the structure stiffness matrix.

Although Jirousek and Voros consider curved beams, they neglect the effect of curvature on the strains; that is they cannot model beams of large curvature where such effects are quite significant.

Objective

The purpose of the present work is to develop a simple, isoparametric, $C^0$, three-noded curved beam finite element formulation which fully accounts for warping deformation and the effects of non-uniform torsion (such as warping stresses) for any open or closed section thin-walled beam. Three nodes are used in order that the beam element can be used in conjunction with an isoparametric plate element with midside nodes. The element accounts for warping by having seven degrees of freedom per node, but the extra degree of freedom is then eliminated by static condensation.
Figure 1. Load $P$ on a solid cross section replaced by four sets of loads
Figure 2. Components of load $P$ on a I-section causing warping
Chapter 2: Theories Of Torsion

This chapter presents relationships between the applied loads, stresses, deformations and sectional properties for a thin-walled beam under uniform and non-uniform torsion, based on the classical theory and the Benscoter theory. Detailed derivations can be found in texts by Hughes [12] and Murray [13].

In both these theories, three basic assumptions are made. Firstly, it is assumed that the shape of the cross-section is preserved in its own plane. Secondly, warping is assumed to be the product of two independent functions, one of which gives the transverse or in-plane distribution usually denoted as \( \omega(s) \), and the other which gives the axial distribution. Thirdly \( \omega(s) \) is assumed to be the same under uniform torsion and non-uniform torsion.
2.1 Classical Torsion Theory

In the classical torsion theory, by Von-Karman and Christensen, the axial displacement is assumed to be of the form \( u(x) = -\omega(s)\theta'_x \), where \( \omega(s) \) is the sectorial coordinate and is defined in slightly different ways for open and closed sections, and \( \theta'_x \) is the rate of twist. In all following expressions, a prime denotes differentiation with respect to \( x \).

2.1.1 Uniform Torsion

Under uniform torsion with no restraint of warping, there is no warping shear stress \( \tau_w \) and hence no warping shear strain \( \gamma_w \). Only St. Venant shear stress \( \tau_s \) is present.

Open Section: For open sections, the distribution of \( \tau_s \) and hence of \( \gamma_s \) is linear through the thickness, with the maximum being at the walls and a zero value at the mid-thickness. Hence at the mid-thickness,

\[
\gamma_{xs} = \gamma_x = \gamma_L = 0 = \frac{\partial u}{\partial s} + \frac{\partial \xi}{\partial x} = 0 \tag{2.1}
\]

where \( \gamma_L \) denotes the linearly varying strain through the thickness, \( s \) is the contour coordinate and \( \xi \) is the tangential displacement along the contour, Fig. 3, and is defined as

\[
\xi = h\theta_x \tag{2.2}
\]

where \( h \) is the perpendicular distance from the shear center to the line tangent to the point on the contour.
Substituting the expression for tangential displacement in Eq.(2.1) for the shear strain we get

\[ \gamma_{xs} = \frac{\partial u}{\partial s} + h \theta_x' = 0 \] (2.3)

Solving the above differential equation for \( u \), we get

\[ u = - \int_0^s h \theta_x' \, dz = - \int_0^s h \, ds \, \theta_x' \] (2.4)

Therefore the warping displacement of any point in the cross-section as a function of the contour coordinate \( s \) around the section is proportional to \( \omega(s) \), which for an open section is defined as

\[ \omega_o(s) = \int_0^s h \, ds \] (2.5)

where the subscript \( o \) denotes open section. Thus,

\[ u = - \omega_o(s) \theta_x' \] (2.6)

**Closed Section:** For a closed section, the St. Venant shear stress contains two components: the relatively large Bredt stress \( \tau_b \) which is constant through the thickness, and the linearly varying distribution \( \tau_L \) through the thickness. The latter is the same as for an open section, but is small compared to the Bredt stress and can be neglected. The associated shear strain, \( \gamma_L \), is likewise negligible compared to the Bredt strain \( \gamma_b \).

For a closed section it is convenient to define the shear flow \( q_b \)

\[ q_b = \tau_b t \] (2.7)
because, as may be shown from equilibrium of a differential element, \( q_b \) is constant around the contour. It may also be shown, by equating the twisting moment \( M_x \) to the total torsional moment of the shear flow around the contour, that for closed sections the shear flow is given by

\[
q_b = \frac{M_x}{2A}
\]  \hspace{1cm} (2.8)

where \( A \) is the area enclosed by the closed section.

The first order differential equation which governs the uniform torsion of open or closed section is

\[
M_x = GJ\theta_x
\]  \hspace{1cm} (2.9)

where \( M_x = \) twisting moment

\( G = \) Shear Modulus

\( J = \) St. Venant torsional constant

It can be shown (e.g. from strain energy) that for open sections

\[
J = J_s = \frac{bt^3}{3}
\]  \hspace{1cm} (2.10)

where \( b \) is the length of the contour. It can likewise be shown that for closed sections,

\[
J = J_B = \frac{4A^2}{\int_C \frac{1}{t} \, ds}
\]  \hspace{1cm} (2.11)

where the symbol \( C \) indicates integration around the contour. Substituting the expression for \( M_x \) from Eq.(2.9) into Eq.(2.8), we get
\[ q_b = \frac{GJ_B \theta_x'}{2A} \] (2.12)

As noted above, in a closed section the St. Venant shear strain is effectively the Bredt strain

\[ \gamma_{xs} = \frac{\partial u}{\partial s} + \frac{\partial \xi}{\partial x} = \gamma_b \] (2.13)

As done previously for open sections, substituting the expression for the tangential displacement \( \xi \) in the above expression for strain, we get

\[ \frac{\partial u}{\partial s} + h \theta_x' = \gamma_b \] (2.14)

Solving the above differential equation for \( u \) gives

\[ u = - \int_0^s (h \theta_x' - \gamma_b) \, ds \] (2.15)

Now \( \gamma_b = q_b / Gt \), and \( q_b \) is given by Eq.(2.12) Substituting the latter gives

\[ u = - \int_0^s \left( h - \frac{J_B}{2At} \right) \, ds \, \theta_x' \] (2.16)

Therefore the warping displacement \( u \) for a closed section can be written as

\[ u = - \omega(s) \theta_x' \] (2.17)

where \( \omega(s) \) gives the transverse distribution of warping as a function of contour coordinate \( s \) and is defined as
\[ \omega(s) = \int_0^s \left( h - \frac{J_B}{2At} \right) ds \] 

From Eq.(2.11), one can note that \( J_B \) is zero for open sections since \( A \) is zero for open sections. Thus the above expression for the sectorial coordinate \( \omega(s) \) is a general one which can be used for both open and closed sections, whereas Eq.(2.5) applies only to open sections.

2.1.2 Non-Uniform Torsion

Open Sections: For open sections undergoing non-uniform torsion, the warping shear strain is negligible and so it is assumed that \( \gamma_w = 0 \). This means that the in-plane warping shear stresses cannot be derived directly from the stress-strain equation \( \tau_w = G\gamma_w \), since it would also be zero. This difficulty is overcome by first expressing the warping normal stress as a function of axial displacement \( u \) through the stress-strain relationship, and then using the equilibrium equations to obtain the warping shear stress.

Consider the axial equilibrium of a differential element shown in Fig. 4

\[ (\sigma_{wt} + \frac{\partial \sigma_w}{\partial x} tdx)ds - \sigma_w tds + (\tau_{wt} + \frac{\partial \tau_w}{\partial s} tds)dx - \tau_w tdx = 0 \] 

(2.19)

Therefore

\[ \frac{\partial \tau_w}{\partial s} + \frac{t}{A} \frac{\partial \sigma_w}{\partial x} = 0 \] 

(2.20)

Chapter 2: Theories Of Torsion
Solving the above equation for the warping shear flow \( q_w = \tau_s t \) we get

\[
q_w = - \int_0^s t \frac{\partial \sigma_w}{\partial x} \, ds + (q_w)_0
\]  
\hspace{1cm} (2.21)

The constant of integration \((q_w)_0\) is zero if we start integrating from the free edge.

Now using the stress-strain law

\[
\sigma_w = E\varepsilon_x = E \frac{\partial u}{\partial x} = E \frac{\partial [\omega(s)\theta_x']}{\partial x} 
\]  
\hspace{1cm} (2.22)

or \( \sigma_w = E\omega(s)\theta_x'' \)  
\hspace{1cm} (2.23)

Substituting the above expression for \( \sigma_w \) into Eq.(2.21) gives

\[
q_w = - \int_0^s E\theta_x'''' \omega t \, ds = -E\theta_x'''' \int_0^s \omega t \, ds
\]  
\hspace{1cm} (2.24)

The warping shear flow \( q_w \) sets up a secondary twisting moment \( M_w \) which may be evaluated as follows,

\[
M_w = \int_0^b q_w h \, ds
\]  
\hspace{1cm} (2.25)

Substituting the expression for \( q_w \) from Eq.(2.24) in the above equation

\[
M_w = -E\theta_x'''' \int_0^b \left[ \int_0^s \omega t \, ds \right] h \, ds = -EI\omega \theta_x''''
\]  
\hspace{1cm} (2.26)
where $I_w$ is called the warping moment of inertia and is defined as

$$I_w = \int_0^b \left[ \int_0^s \omega t \, ds \right] h \, ds$$

which on simplification yields (see Hughes [12] for details)

$$I_w = \int_0^b \left[ \omega(s) \right]^2 t \, ds \tag{2.27}$$

The total twisting moment due to the non-uniform torsion of an open section is

$$M = M_v + M_w \tag{2.28}$$

where $M_v$ is the twisting moment set up due to the St. Venant shear stress $\tau$, and $M_w$ is the twisting moment due to warping shear stress $\tau_w$. Therefore the governing equation is

$$M = GJ\theta_x' - EI_w \theta_x'''	ag{2.29}$$

**Closed Sections**: For a closed section, the derivation of warping strain exactly parallels the derivation given above for an open section, with the sectorial coordinate $\omega(s)$ now given by Eq.(2.18). The only difference now is that the constant of integration ($q_w)_0$ in Eq.(2.21) can no longer be automatically set to zero because there is now no free edge at which $q_w$ is zero. Thus Eq.(2.21) now becomes,

$$q_w = -E\theta_x''' \int_0^s \omega t \, ds + (q_w)_0 \tag{2.30}$$
In order to evaluate \((q_w)\), we make use of the fact that since the section is closed, \(q_w(s)\) must be such that the net total warping displacement which it causes over one full cycle of the section is zero. Therefore we make a cyclic evaluation of the warping caused by \(q_w\). To do so, we divide \(q_w\) into two parts: one part, say \(q_w^*\), which corresponds to the first term of Eq. (2.30) and for which the value at the starting point of the integration is taken to be zero; and a second part which is the true, but unknown, starting value \((q_w)_0\). The latter is constant and hence the foregoing condition is,

\[
\frac{1}{G} \left[ \int \frac{q_w^*}{t} \, ds + \left( (q_w)_0 \right) \int \frac{1}{t} \, ds \right] = 0
\]  

(2.31)

from which we get the constant of integration \((q_w)_0\),

\[
(q_w)_0 = -\frac{\int \frac{q_w^*}{t} \, ds}{\int \frac{1}{t} \, ds}
\]  

(2.32)

After solving for \((q_w)_0\), the additional warping moment \(M_w\) which results from \(q_w\) is found in the same way as for an open section.
2.2 Benscoter's Torsion Theory

The main drawback of Classical theory is that although it does calculate a value for warping shear stress $\tau_\omega$, it still begins with the assumption that the warping $\omega$ is directly proportional to the rate of twist $\theta'_s$, whereas in reality the existence of $\tau_\omega$ changes the lengthwise distribution of the warping. Some closed sections undergo large warping shear strain $\gamma_\omega$ and its effect cannot be ignored. Obviously if $\gamma_\omega$ is large, the warping shear stress $\tau_\omega$ will also be large and the classical theory gives unsatisfactory results. In such cases the more accurate theory due to Benscoter should be used. Benscoter's theory overcomes the difficulty in finding $\tau_\omega$ by starting with an assumed form for the axial component of deformation. In so doing, the strains can be found by simple differentiation and then the stresses $\sigma_\omega$ and $\tau_\omega$ are obtained by using the stress-strain relationship. The final solution bears a close resemblance to that of the classical torsion theory and a relationship between the warping function $f(x)$ and the rate of twist $\theta'_s$, can be derived.

2.2.1 Displacements

As shown in Fig. 3 the displacement $\xi(x,s)$ is tangential to the contour of the profile and it is assumed to be of the same form as in the Classical torsion theory; i.e.

$$\xi(x,s) = h(s)\theta'_s(x)$$  \hspace{1cm} (2.33)

where $h(s)$ is the length of the perpendicular from the shear center $S$ to the tangent to the middle line of the profile and $\theta'_s$ is the angle of rotation about $S$. The out-of-plane displacement of the cross-section $u(x,s)$ is assumed to be of the following form:
\[ u(x,s) = -\omega(s)f(x) \]  \hspace{1cm} (2.34)

where \( \omega(s) \) is the sectorial coordinate and \( f(x) \) is the warping function. As shown earlier, in Eq.(2.18), the sectorial coordinate \( \omega(s) \) is

\[ \omega(s) = \int_{\rho}^{s} \left( h - \frac{J_B}{2At} \right) ds \]  \hspace{1cm} (2.35)

where \( J_B \) is the Bredt torsional constant given by Eq.(2.11).

### 2.2.2 Stress and Strains

The longitudinal and shear strains are

\[ \varepsilon_x = \frac{\partial u}{\partial x} = -\omega f'(x) \]  \hspace{1cm} (2.36)

\[ \gamma_{xs} = \frac{\partial u}{\partial s} + \frac{\partial \xi}{\partial x} = h \theta_x' - \omega f' \]  \hspace{1cm} (2.37)

where a dot over \( \omega \) denotes differentiation with respect to \( s \). Therefore,

\[ \dot{\omega} = \frac{\partial \omega}{\partial s} = h - \frac{J_B}{2At} \]  \hspace{1cm} (2.38)

The stress resultants can be determined in terms of the displacements \( \theta_x \) and \( f \) from the stress-strain relationship, thus

\[ \sigma_x = E \frac{\partial u}{\partial x} = -Ef' \omega \]  \hspace{1cm} (2.39)
\[ \tau_{xs} = G(h\theta_x' - \dot{\omega}f) \]  

(2.40)

The normal force \( N_x \) acting in the axial direction and bending moments \( M_z \) and \( M_y \) about the principal axes are therefore

\[ N_x = \int_C \sigma_x t \, ds = -Ef' \int_C \omega t \, ds = 0 \]  

(2.41)

\[ M_y = \int_C \sigma_{xzt} \, ds = -Ef' \int_C \omega z t \, ds = 0 \]  

(2.42)

\[ M_z = \int_C \sigma_{xyt} \, ds = -Ef' \int_C \omega y t \, ds = 0 \]  

(2.43)

We assume that the shear centre is the origin and the reference axes \( y \) and \( z \) coincide with the principal axes of the cross-section. Under pure torsion, transverse shear forces must be zero to satisfy equilibrium; i.e. \( V_y = V_z = 0 \).

The torque \( M_x \) is found by integrating around the profile

\[ M_x = \int_C \tau_{xn} h t \, ds = G\theta_x' \int_C h^2 t \, ds - Gf \int_C h \frac{\partial \omega}{\partial s} t \, ds \]  

(2.44)

and the bimoment, \( B \) is

\[ B = \int_C \sigma_x \omega t \, ds = -Ef' \int_C \omega^2 t \, ds \]  

(2.45)

From Eq. (2.38), \( \frac{\partial \omega}{\partial s} = h - \frac{J_b}{2At} \). Therefore Eq. (2.44) becomes

Chapter 2 : Theories Of Torsion
\[ M_x = G \theta_x' \int_C h'^2 t \, ds - G \int_C h(h - \frac{J_B}{2A})t \, ds \]  

(2.46)

or

\[ M_x = G \theta_x' \int_C h'^2 t \, ds - G f \left[ \int_C h'^2 t \, ds - \int_C h \frac{J_B}{2A} \, ds \right] \]  

(2.47)

Now introducing certain section properties the above equation can be simplified. For a closed section \( \int_C h \, ds = 2A \) [12] and hence the last term is simply \( J_B \). Another section property is \( I_p \) or the Polar Constant. It is defined as

\[ I_p = \int_C h'^2 t \, ds \]  

(2.48)

and as shown earlier, Eq.(2.27), \( I_o = \int_C \omega^2 t \, ds \) is the Sectorial Moment of Inertia.

Substituting the above expressions in Eq.(2.47) and (2.45) gives

\[ M_x = GI_p \theta_x' - G(I_p - J_B)f \]  

(2.49)

and

\[ B = -EI_\omega f'' \]  

(2.50)
2.2.3 Governing Equations

The displacement functions $\theta_s(x)$ and $f(x)$ are determined by setting up the governing equations. Consider equilibrium of the element illustrated in Fig. 5. Taking moments about an axis through the shear centre

$$\int C \frac{\partial \tau_{xx}}{\partial x} h_t \, ds + \int C (p_s h + p_n h_n) \, ds = 0 \tag{2.51}$$

where

$p_s = \text{load per unit length acting in tangential direction},$  
$p_n = \text{load per unit length acting in normal direction},$  
$h = \text{distance from shear centre S to tangent of the surface at the point being considered},$  
$h_n = \text{distance from shear centre S to the normal of the surface at the point being considered}.$

The second term in Eq.(2.57) is simply the external torque, $m(x)$ applied to the surface of the element per unit length of the beam.

The simplest way to obtain the second equilibrium equation, in the longitudinal direction, is to introduce a virtual displacement $f(x) = -1$, in which case the longitudinal displacement is, from Eq.(2.34)

$$u = \omega \tag{2.52}$$

The corresponding shear strain is

$$\gamma_{xs} = \frac{\partial u}{\partial s} = \frac{\partial \omega}{\partial s} = h - \frac{J_B}{2At} \tag{2.53}$$
The sum of the internal energy and the external work done during the virtual displacement is zero. Therefore,

$$\int_C \frac{\partial \sigma_x}{\partial x} \omega t \ ds - \int_C \tau_{xS} \frac{\partial \omega}{\partial s} \ t \ ds + \int_C p_x \omega \ ds = 0$$  \hspace{1cm} (2.54)$$

The last term is the bimoment $b(x)$ applied to the element per unit length. On substituting the expression for $\sigma_x$ and $\tau_{xS}$ from Equ. (2.39) and (2.40) into Equ. (2.51) and (2.54), we get

$$G \theta_x'' \int_C h^2 t \ ds - Gf' \int_C h \frac{\partial \omega}{\partial s} \ t \ ds + m(x) = 0$$  \hspace{1cm} (2.55)$$

$$-Ef'' \int_C \omega^2 t \ ds - G \theta_x' \int_C h \frac{\partial \omega}{\partial s} \ t \ ds + Gf \int_C \left( \frac{\partial \omega}{\partial s} \right)^2 \ t \ ds + b(x) = 0$$  \hspace{1cm} (2.56)$$

From Equ.(2.38), $\frac{\partial \omega}{\partial s} = h - \frac{J_B}{2At}$. Therefore

$$\left( \frac{\partial \omega}{\partial s} \right)^2 = \left( h - \frac{J_B}{2At} \right)^2 = h^2 - h \frac{J_B}{At} + \left( \frac{J_B}{2At} \right)^2$$

Therefore

$$\int_C \left( \frac{\partial \omega}{\partial s} \right)^2 \ t \ ds = \int_C h^2 t \ ds - \int_C h \frac{J_B}{A} \ ds + \int_C \left( \frac{J_B}{2At} \right)^2 \ t \ ds$$  \hspace{1cm} (2.57)$$

Both $J_B$ and $A$ are constants and can be brought outside of the integral. In the middle term the result is $-J_B \int_C h \ ds$ which is equal to $-2J_B$ since $\int_C h \ ds = 2A$. In the third term the result is $J_B^2 \left[ \frac{1}{4A^2} \int_C \frac{1}{t} \ ds \right]$ and from Equ.(2.11) the term in the square brackets is
simply \( \frac{1}{J_B} \). And finally we note that the first term is simply \( I_P \). Making these three substitutions converts Eq.(2.57) into

\[
\int_C \left( \frac{\partial \omega}{\partial s} \right)^2 t \, ds = I_P - J_B
\]

Therefore Eq.(2.55) and Eq.(2.56) in terms of \( I_w, J_B \) and \( I_P \) can be written as

\[
GI_P \theta_x'' - G(I_P - J_B)f' + m(x) = 0 \tag{2.58}
\]

\[
-EI_\omega f'' - G(I_P - J_B)(\theta_x' - f) + b(x) = 0 \tag{2.59}
\]

From Eq.(2.58) we get

\[
f' = \frac{I_P}{I_P - J_B} \theta_x'' + \frac{m(x)}{G(I_P - J_B)} \tag{2.60}
\]

If we define \( \eta^2 = 1 - \frac{J_B}{I_P} \)

\[
f' = \frac{1}{\eta^2} \left[ \theta_x'' + \frac{m(x)}{GI_P} \right] \tag{2.61}
\]

Eq.(2.61) is the relation between the warping function \( f \) and angle of twist \( \theta_x \). Manipulating the governing Eq.(2.58) and (2.59) using (2.61) we get Benscoter's governing equation for the twisting of a beam

\[
E \frac{I_w}{\eta^2} \theta_x'''' - GJ_B \theta_x'' = m(x) + b'(x) - \frac{EI_\omega}{\eta^2 GI_P} m''(x) \tag{2.62}
\]

or

Chapter 2 : Theories Of Torsion
\[ EI_0^0 \theta_x''' - GJ_B \theta_x'' = m(x) + b'(x) - \frac{EI_0^0}{\eta^2 GI_P} m''(x) \] (2.63)

where

\[ I_0^0 = \frac{I_0}{\eta^2} = \left( \frac{I_p}{I_p - J_B} \right) I_0 \] (2.64)

The above governing equation is similar in form to the one obtained by classical torsion theory excepting for the factor \( \eta^2 \). For open sections \( J_B = 0 \), thus \( \eta^2 = 1 \). Therefore the governing equations for open sections are same whether obtained from Benscoter's theory or Classical theory but for closed sections, \( I_0 \) is replaced by \( I_0^9 \) in Benscoter's theory.
Figure 3. Tangential displacement along the profile of thin-walled section
Figure 4. Stresses due to warping restraint
Figure 5. Forces acting on an element of beam
Chapter 3: Finite Element Formulation

In this chapter the element stiffness matrix for a curved beam element of open or closed section, which includes the effects of both uniform (St. Venant's) torsion as well as non-uniform torsion, is presented. To include the torsional effects, a seventh independent degree of freedom at each node, in addition to the usual six degrees of freedom is used. Besides being able to give an improved representation of the warping stresses that accompany warping restraint torsion, the formulation also considers transverse shear stresses which are quite significant in deep beams. In essence this beam formulation is a result of a combination of Timoshenko beam theory and Benscoter's theory for thin-walled beams. Also, the effect of curvature on the various strain terms has been taken into account; this gives rise to coupling among the flexural and torsional moments along a curved beam. To eliminate shear-locking due to the presence of shear terms, a selective reduced numerical integration scheme has been adopted to compute the element stiffness matrix.

In the present work, the following assumptions have been made for thin-walled curved beams:
i) if \( t \) denotes the thickness of the cross-section and \( h \) the maximum dimension of the cross-section, then \( t/h < 0.1 \)

(ii) the cross-section does not distort in its own plane during deformation

(iii) the largest cross-sectional dimension is less than one-tenth the radius of curvature, \( R \) of the beam.

(iv) the Kirchoff-Love assumption holds, i.e. shear strain through the thickness is zero.

### 3.1 Kinematics

#### 3.1.1 Geometrical Description

A curved beam segment is defined with respect to cylindrical coordinates \((r, \phi, y)\) as shown in Fig. 6. The centroidal axis passing through the centroids of each cross-section normal to the circumferential direction is part of a circular arc of radius \( R \) measured from the origin \( O \). The tangential axis to the centroidal arc is denoted by \( x \) and is measured as \( x = R\phi \). At a particular cross-section, the cartesian coordinates \( x, y \) and \( z \) are defined by the right-hand screw rule. The cross-section is assumed to be symmetric about the \( r - \phi \) (or \( x - z \)) plane, and therefore the \( y \) and \( z \) axes are the principal centroidal axes of the cross-section. A typical thin-walled beam cross-section is depicted in Fig. 7. The centroid is denoted by \( C \) \((0,0)\). The shear center for the section lies along the \( z \)-axis at \( z = z_s \), because of assumed symmetry. An arbitrary point \( P \) on the cross-section Fig. 7, can be defined either by the cartesian coordinates \( y_s \) and \( z_s \) or by a curvilinear contour coordinate \( \xi \). The contour coordinate with its origin at \( D \), is defined along the contour profile on a
line that is at the middle of the wall thickness. The perpendicular distance between the
tangent at the point P and the shear center S is denoted by h and α is the angle between
the y-axis and the tangential direction at point P.

The finite element discretization of the beam is shown in Fig. 8. As shown in the figure,
the element is a 3-noded curvilinear line element with the nodes lying on the centroidal
axis. A dimensionless, curvilinear coordinate ξ that varies between -1 at node 1 to +1
at node 3 is used to define the centroidal axis of the beam. The local x, y and z axes at a
particular cross-section are fixed in a way such that the x-axis is always tangent to the
ξ-axis and the y-z axis are the principal axes. The coordinate of any point on the beam
in terms of the nodal coordinates and ξ can be expressed as

\[
\begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix} = \sum_{i=1}^{3} N_i(\xi) \begin{bmatrix}
    X_i \\
    Y_i \\
    Z_i
\end{bmatrix}
\]

(3.1)

where \( N_i(\xi), i = 1,3 \) are the quadratic Lagrangian shape funtions.

3.1.2 Displacement

The displacement components in the \( \phi, y \) and \( z \) directions are designated \( u, v \) and \( w \) re-
spectively. The longitudinal displacement \( u \) can be expressed as the sum of axial de-
formation \( u_{x1} \) due to bending in the \( x - y \) and \( x - z \) plane, and \( u_{x2} \) due to out-of-plane warping
deformation due to torsion. Therefore \( u = u_{x1} + u_{x2} \). The axial deformation due to bend-
ing, \( u_{x1} \), obeys Timoshenko’s beam theory which is
\[ u_{x1} = U(\phi) - y\theta_z(\phi) + z\theta_y(\phi) \]

where \( U(\phi) \) is the circumferential displacement of the centroid, and \( \theta_y \) and \( \theta_z \) are rotations about the \( y \) and \( z \) axes respectively. Axial deformation due to warping follows Benscoter's theory whereby,

\[ u_{x2} = -\omega(s)\hat{f}(\phi) \]

Therefore the axial displacement is,

\[ u(\phi, y, z) = U(\phi) - y\theta_z(\phi) + z\theta_y(\phi) - \omega(s)\hat{f}(\phi) \tag{3.2} \]

Assuming that the cross-section does not distort in its own plane, the transverse displacements \( v \) and \( w \) are expressed as

\[ v(\phi, y, z) = V(\phi) - (z - z_s)\theta_x(\phi) \tag{3.3} \]

\[ w(\phi, y, z) = W(\phi) - y\theta_x(\phi) \tag{3.4} \]

where \( V(\phi) \) and \( W(\phi) \) are the \( y \) and \( z \) direction displacements respectively of the shear center and \( \theta_x(\phi) \) is the rotation of the cross-section about the shear center. Since we are considering bending shear deformation, \( \theta_x(\phi) \) and \( \theta_s(\phi) \) are not equal to the derivatives of the corresponding translations; i.e. \( \theta_s \neq \frac{-dW}{dx} \) and \( \theta_s \neq \frac{dV}{dx} \). Here we can draw an analogy between bending shear deformation and warping shear deformation due to non-uniform torsion. Therefore \( f \) is not equal to the rate of twist \( \theta_s' \). Thus the displacements can be expressed in terms of the seven nodal degrees of freedom \( u, v, w, \theta_x, \theta_y, \theta_z \) and \( f \). These seven degrees of freedom constitute the nodal displacement vector \( \{\delta_i\} \). As this is an isoparametric formulation, the same shape functions are used to interpolate the nodal degrees of freedom as the ones used to interpolate the coordinates.
3.2 Strain - Displacement Relations

In cylindrical coordinates, the linear strain-displacement relations as given in Johnson [14] are

\[
\begin{align*}
\varepsilon_{\phi\phi} &= \frac{1}{(R - z)} (u_{\phi} - w) \\
\varepsilon_{yy} &= v_{,y} \\
\varepsilon_{zz} &= w_{,z} \\
\gamma_{\phi y} &= u_{,y} + \frac{v_{,\phi}}{(R - z)} \\
\gamma_{\phi z} &= u_{,z} + \frac{(w_{,\phi} + u)}{(R - z)} \\
\gamma_{yz} &= v_{,z} + w_{,y}
\end{align*}
\] (3.5)

where \(\cdot\) denotes partial differentiation with respect to the coordinate following it. Now if we substitute the expressions for displacements and their derivatives into the above expressions for strain, we get the following expressions for strain in terms of the degrees of freedom. Consider first the axial normal strain,

\[
(R - z)\varepsilon_{\phi\phi} = U_{,\phi} - y\theta_{z,\phi} + z\theta_{y,\phi} - \omega f_{,\phi} - W - y\theta_{x}
\]

or

\[
(R - z)\varepsilon_{\phi\phi} = U_{,\phi} - W - y(\theta_{z,\phi} + \theta_{x}) + z\theta_{y,\phi} - \omega f_{,\phi}
\]

For thin curved girders \((R - z) \approx R\), therefore

\[
\varepsilon_{\phi\phi} = \frac{U_{,\phi}}{R} - \frac{W}{R} - y\left(\frac{\theta_{z,\phi}}{R} + \frac{\theta_{x}}{R}\right) + z\frac{\theta_{y,\phi}}{R} - \omega \frac{f_{,\phi}}{R}
\] (3.6)

The other normal strain turn out to be zero. That is
\( \varepsilon_{yy} = 0 \) \hspace{1cm} (3.7)

\( \varepsilon_{zz} = 0 \) \hspace{1cm} (3.8)

The shear strain component \( \gamma_{\phi y} \) becomes,

\[
(R - z)\gamma_{\phi y} = (R - z)u_{,y} + v_{,\phi} \\
= -(R - z)\theta_{x} - (R - z)\omega_{,y}f + V_{,\phi} - (z - z_s)\theta_{x,\phi} \\
= -(R - z)\theta_{x} + (z - z_s)(\theta_{x,\phi} + (R - z)\omega_{,y}f)
\]

Substituting \( \gamma_{\phi y} = V_{,\phi} - (R - z_s)\theta_{x} \), and adding and subtracting \( \frac{V_{,\phi}}{R - z_s} \) in the third term of the above expression we get,

\[
(R - z)\gamma_{\phi y} = \gamma_{\phi y}^\phi - (z - z_s)(\theta_{x,\phi} - \frac{V_{,\phi}}{R - z_s} + \frac{V_{,\phi}}{R - z_s} - \theta_{x}) - (R - z)\omega_{,y}f
\]

In the above expression, if we denote \( \theta_{x,\phi} - \frac{V_{,\phi}}{R - z_s} \) by \( \beta_{x,\phi} \), we get

\[
(R - z)\gamma_{\phi y} = \gamma_{\phi y}^\phi - (z - z_s)(\beta_{x,\phi} + \frac{\gamma_{\phi y}^\phi}{R - z_s}) - (R - z)\omega_{,y}f
\]

or \( (R - z)\gamma_{\phi y} = \gamma_{\phi y}^\phi(1 - \frac{z - z_s}{R - z_s}) - (z - z_s)\beta_{x,\phi} - (R - z)\omega_{,y}f \)

For thin curved girders \( 1 - \frac{z - z_s}{R - z_s} \approx 1 \),

\[
\text{or} \quad (R - z)\gamma_{\phi y} = \gamma_{\phi y}^\phi - (z - z_s)\beta_{x,\phi} - (R - z)\omega_{,y}f \quad (3.9)
\]

The other shear strain term, \( \gamma_{zz} \), can be written as
\[ (R - z)\gamma_{\phi z} = (R - z)u_{z} + (w_{,\phi} + u) \]
\[ = (R - z)\theta_{y} - (R - z)\omega_{z, z}f + W_{,\phi} + y\theta_{x, \phi} + U - y\theta_{z} + z\theta_{y} - \omega f \]
\[ = W_{,\phi} + U + R\theta_{y} + y(\theta_{x, \phi} - \theta_{z}) - [(R - z)\omega_{z, z} + \omega]f \]

If we substitute \(\gamma_{\phi z} = W_{,\phi} + U + R\theta_{y}\) and as we did previously, add and subtract \(\frac{V_{,\phi}}{R - z, z}\) in the fourth term of the above expression we get

\[ (R - z)\gamma_{\phi z} = \gamma^{\phi}_{\phi z} + y(\theta_{x, \phi} - \frac{V_{,\phi}}{R - z, z} + \frac{V_{,\phi}}{R - z, z} - \theta_{z}) - [(R - z)\omega_{z, z} + \omega]f \]
\[ = \gamma^{\phi}_{\phi z} + y(\beta^{\phi, \phi} + \frac{V_{,\phi}}{R - z, z}) - [(R - z)\omega_{z, z} + \omega]f \]
\[ = \gamma^{\phi}_{\phi z} + y\beta^{\phi, \phi} + \frac{y}{R - z, z} \gamma^{\phi}_{\phi y} - [(R - z)\omega_{z, z} + \omega]f \]

For thin curved girders \(\frac{y}{R - z, z}\) is negligible. Therefore

\[ (R - z)\gamma_{\phi z} = \gamma^{\phi}_{\phi z} + y\beta^{\phi, \phi} - [(R - z)\omega_{z, z} + \omega]f \]  
(3.10)

Now \(\gamma_{x} = \gamma_{x y} \cos \alpha + \gamma_{x z} \sin \alpha\)

or \( (R - z)\gamma_{x} = (R - z)\gamma_{x y} \cos \alpha + (R - z)\gamma_{x z} \sin \alpha \)  
(3.11)

Therefore substituting for \((R - z)\gamma_{x y}\) and \((R - z)\gamma_{x z}\) from Eq.(3.9) and Eq.(3.10) respectively in the above expression, we get

\[ (R - z)\gamma_{\phi z} = \gamma^{\phi}_{\phi y} \cos \alpha - (z - z_{0})\beta^{\phi, \phi} \cos \alpha - (R - z)\omega_{, y}f \cos \alpha + \]
\[ \gamma^{\phi}_{\phi z} \sin \alpha + y\beta^{\phi, \phi} \sin \alpha - [(R - z)\omega_{, z} + \omega]f \sin \alpha \]
\[ = \gamma^{\phi}_{\phi y} \cos \alpha + \gamma^{\phi}_{\phi z} \sin \alpha + \[y \sin \alpha - (z - z_{0}) \cos \alpha\] \beta^{\phi, \phi} \]
\[ - [(R - z)\omega_{, y} \cos \alpha + (R - z)\omega_{, z} \sin \alpha]f - \omega f \sin \alpha \]  
(3.12)
Now \( h \), which is the perpendicular distance from the shear centre \( S \) to the point \( P \) on the contour, can be written as

\[
h = y \sin \alpha - (z - z_c) \cos \alpha
\]

Using the chain rule of differentiation, partial differentiation with respect to \( s \) can be expressed as

\[
\frac{\partial(\ )}{\partial s} = \frac{\partial(\ )}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial(\ )}{\partial z} \frac{\partial z}{\partial s}
\]

And since \( \cos \alpha = \frac{\partial y}{\partial s} \) and \( \sin \alpha = \frac{\partial z}{\partial s} \), we can write

\[
\omega_\alpha = \omega_y \cos \alpha + \omega_z \sin \alpha
\]

Making the above substitutions in Eq.(3.12), we get

\[
(R - z)\phi_x = \phi_{yy} \cos \alpha + \phi_{y2} \sin \alpha + h\beta_{\phi, \phi} - [(R - z)\omega_\alpha + \omega \sin \alpha] f
\]

\[
= \phi_{yy} \cos \alpha + \phi_{y2} \sin \alpha + h\beta_{\phi, \phi} - (R - z)^2 \frac{\partial(\omega)}{\partial s} f
\]

For thin curved girders \((R - z) \approx R\). Therefore

\[
R\phi_x = \phi_{yy} \cos \alpha + \phi_{y2} \sin \alpha + h\beta_{\phi, \phi} - R^2 \frac{\partial(\omega)}{\partial s} f
\]

or

\[
\phi_x = \frac{\phi_{yy}}{R} \cos \alpha + \frac{\phi_{y2}}{R} \sin \alpha + h\frac{\beta_{\phi, \phi}}{R} - \frac{\partial \omega}{\partial s} f
\]

(3.13)

It can be shown that the other shear strain term is zero. That is

\[
\gamma_{yz} = 0
\]

(3.14)
Now since the \( x \)-axis is assumed to be tangential to the curved centroidal axis, we can say \( x = R \theta \). Therefore

\[
\frac{\partial ( \theta )}{\partial x} = \frac{\partial ( \phi )}{\partial \phi} \frac{\partial \phi}{\partial x} = \frac{1}{R} \frac{\partial ( \theta )}{\partial \phi}
\]

Using the above relationship, the axial normal strain \( \varepsilon_{xx} \) and the shear strain \( \gamma_{xx} \), can be written as

\[
\varepsilon_{xx} = U' - \frac{W'}{R} - y(\theta_z' + \frac{\theta_z}{R}) + z\theta_y' - \omega'f''
\]

\[
= e - y\kappa_z + z\kappa_y - \omega'f''
\]  

(3.15)

\[
\gamma_{xx} = \gamma^o_{xy} \cos \alpha + \gamma^o_{xz} \sin \alpha + h\beta' - \omega_{,s}f
\]

(3.16)

where primes denote differentiation with respect to \( x \). The symbol \( e, \kappa_y, \kappa_z, \gamma^o_{xy}, \gamma^o_{xz} \), and \( \beta' \) are the generalized strains which are defined as

\[
e = U' - \frac{W'}{R}
\]

\[
\kappa_y = \theta_y'
\]

\[
\kappa_z = \theta_z' + \frac{\theta_z}{R}
\]

\[
\gamma^o_{xy} = V' - \theta_z
\]

\[
\gamma^o_{xz} = W' + \frac{U'}{R} + \theta_y
\]

\[
\beta' = \theta_x' - \frac{V'}{R}
\]

(3.17)

Now as shown in chapter 2, Eq.(2.38),

\[
\omega_{,s} = \frac{\partial \omega}{\partial s} = (h - \frac{J_B}{2At})
\]

Therefore substituting this expression for \( \omega_{,s} \) in Eq.(3.16), we get

Chapter 3 : Finite Element Formulation
\[ \gamma_{xx} = \gamma_{xy} \cos \alpha + \gamma_{xz} \sin \alpha + h \beta' - \left( h - \frac{J_B}{2At} \right) f \]

or

\[ \gamma_{xx} = \gamma_{xy} \cos \alpha + \gamma_{xz} \sin \alpha + \frac{J_B}{2At} \beta' + \left( h - \frac{J_B}{2At} \right) (\beta' - f) \quad (3.18) \]

In matrix form, the strain components, \( \varepsilon_{xx} \) and \( \gamma_{xx} \) can be written as

\[
\{ \varepsilon \} = \begin{bmatrix} \varepsilon_{xx} \\ \gamma_{xx} \end{bmatrix} = \begin{bmatrix} e - y \kappa_{z} + z \kappa_{y} - \omega f' \\ \gamma_{xy} \cos \alpha + \gamma_{xz} \sin \alpha + \frac{J_B}{2At} \beta' + \left( h - \frac{J_B}{2At} \right) (\beta' - f) \end{bmatrix} \quad (3.19)
\]

### 3.2.1 Generalized Strains

For later work it will be useful to define a quantity \( \{ \bar{\varepsilon} \} \) which denotes the vector of generalized strain components. That is \( \{ \bar{\varepsilon} \} \) represents,

\[
\{ \bar{\varepsilon} \} = \begin{bmatrix} e & \kappa_{y} & \kappa_{z} & \gamma_{xy} & \gamma_{xz} & \beta' & f' & (\beta' - f) \end{bmatrix}^{T}
\]

The component \( e \) represents the circumferential stretching strain of the centroidal axis, \( \kappa_{z} \) is the change in curvature out of the plane of the curved beam (the \( x - z \) plane) \( \kappa_{y} \) is the change in curvature in the plane of the curved beam, \( \gamma_{xy} \) and \( \gamma_{xz} \) are the transverse shearing strains due to transverse bending, \( \beta' \) is the shear strain due to St. Venant's torsion, \( (\beta' - f) \) is the warping shear strain due to non-uniform torsion and \( f' \) is the warping normal strain. Therefore the actual strain \( \{ \varepsilon \} \) can be expressed in terms of generalised strain \( \{ \bar{\varepsilon} \} \) and a transfer matrix \([G]\) such that
\[
\{ \epsilon \} = [ G ] \{ \bar{\epsilon} \}
\]  \hspace{1cm} (3.20)

where the transfer matrix is defined as

\[
[ G ] = \begin{bmatrix}
1 & z & -y & 0 & 0 & 0 & \omega & 0 \\
0 & 0 & 0 & \cos \alpha & \sin \alpha & \frac{J_B}{2At} & 0 & \left( h - \frac{J_B}{2At} \right)
\end{bmatrix}
\]  \hspace{1cm} (3.21)

The generalized strains can be expressed in terms of nodal displacements as

\[
\{ \bar{\epsilon} \} = \sum_{i=1}^{N} [B_i] \{ \delta_i \}
\]  \hspace{1cm} (3.22)

where

\[ N \] is the number of nodes per element, in this case three.

\{ \delta_i \} is the nodal displacement vector consisting of \( u, \ v, \ w, \ \theta_x, \ \theta_y, \ \theta_z \) and \( f \).

\[ [B_i] \] is the strain-displacement matrix.

The strain-displacement matrix \([B_i]\) can be expressed as

\[
[B_i] = \begin{bmatrix}
N_i & 0 & -N_i & 0 & 0 & 0 \\
0 & 0 & 0 & N_i & 0 & 0 \\
0 & 0 & 0 & \frac{N_i}{R} & 0 & N_i \\
0 & N_i & 0 & 0 & 0 & -N_i \\
\frac{N_i}{R} & 0 & N_i & 0 & N_i & 0 \\
0 & -\frac{N_i}{R} & 0 & N_i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & N_i \\
0 & -\frac{N_i}{R} & 0 & N_i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -N_i
\end{bmatrix}
\]  \hspace{1cm} (3.23)
where the prime stands for differentiation with respect to \( x \); i.e. \( N'_y = \frac{dN_y}{dx} \). The differentiation with respect to \( x \) is performed by setting

\[
\frac{d()}{dx} = \frac{1}{t} \frac{d()}{d\xi}
\]

where \( t \) stands for the length of the tangent vector

\[
\hat{t}(\xi) = \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix} = \sum_{i=1}^{3} dN_i(\xi) \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix}
\]

therefore

\[
t = \sqrt{(t_x^2 + t_y^2 + t_z^2)}
\]

### 3.3 Stress - Strain Relations

The two stress components acting on a thin-walled beam are the normal axial stress \( \sigma_{xx} \) and the in-plane shearing stress \( \tau_{xy} \). The corresponding strain components are \( \varepsilon_{xx} \) and \( \gamma_{xy} \). The constitutive equations expressed in matrix form can be written as,

\[
\{\sigma\} = [D] \{\varepsilon\}
\]

(3.24)

where the stress vector \( \{\sigma\} = [\sigma_{xx} \tau_{xy}]^T \) and the strain vector \( \{\varepsilon\} = [\varepsilon_{xx} \gamma_{xy}]^T \).

\([D]\) is known as the Rigidity Matrix, and in this case it is,
\[ [D] = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix} \] (3.25)

### 3.3.1 Generalized Rigidity Matrix

Corresponding to the generalized strains \( \bar{\epsilon} \) defined in section 3.2.1, let there be generalized stresses represented by \( \bar{\sigma} \) which are related to the generalized strains by the generalized rigidity matrix \( \bar{D} \). Thus in matrix form the relation can be expressed as,

\[ \{\bar{\sigma}\} = [\bar{D}]{\bar{\epsilon}} \] (3.26)

The generalized rigidity matrix can be derived using the Principle of Virtual Work. According to this principle, the work done by the generalized stress and strain - for a virtual displacement - should be equal to that due to the actual stress and strain. Hence

\[ \int_{Vol} \bar{\epsilon}^T \bar{\sigma} \, dVol = \int_{Vol} \epsilon^T \sigma \, dVol \] (3.27)

Now from Eq(3.23) and Eq(3.20) we get

\[ \{\sigma\} = [D]{\epsilon} = [D][G]{\bar{\epsilon}} \]

\[ \{\epsilon\}^T = \{\bar{\epsilon}\}^T [G]^T \]

Therefore substituting the above expressions in Eq(3.26) we get
\[
\int_{Vol} \{\vec{e}\}^T [\overline{D}] \{\vec{e}\} \, dVol = \int_{Vol} \{\vec{e}\}^T [G]^T [D][G] \{\vec{e}\} \, dVol
\]

\[
= \int_{0}^{l} \{\vec{e}\}^T \left\{ \int_{F} [G]^T [D][G] \, dF \right\} \{\vec{e}\} \, dl
\]

where the integral within the curly brackets is done over the whole cross-sectional area \(F\), of the beam. Thus the generalized rigidity matrix expressed in terms of the actual rigidity matrix and the transfer matrix is

\[
[\overline{D}] = \int_{F} [G]^T [D][G] \, dF
\]

Now

\[
[D][G] = \begin{bmatrix}
E & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & G & \cos \alpha & \sin \alpha & \frac{J_B}{2At} \\
0 & 0 & 0 & 0 & \cos \alpha & \sin \alpha & \left( h - \frac{J_B}{2At} \right) \\
Ez & -Ey & 0 & 0 & 0 & -E\omega & 0 \\
0 & 0 & 0 & G \cos \alpha & G \sin \alpha & \frac{J_B}{2At} & 0 \\
0 & 0 & 0 & 0 & G \cos \alpha & G \sin \alpha & \left( h - \frac{J_B}{2At} \right)
\end{bmatrix}
\]

and therefore
\[
\begin{bmatrix}
1 & 0 \\
\gamma & 0 \\
-\omega & 0 \\
0 & \cos \alpha \\
0 & \sin \alpha \\
0 & \frac{J_B}{2At} \\
-\omega & 0 \\
0 & (h - \frac{J_B}{2At})
\end{bmatrix}
\begin{bmatrix}
E & Ez & -Ey & 0 & 0 & 0 & -E\omega & 0 \\
Ez^2 & -Ez\gamma & 0 & 0 & 0 & -Ez\omega & 0 \\
Ey^2 & 0 & 0 & 0 & 0 & E\gamma \omega & 0 \\
G \cos^2 \alpha & G \cos \alpha \sin \alpha & \frac{GJ_B}{2At} \cos \alpha & 0 & 0 & G \cos \alpha \left(h - \frac{J_B}{2At}\right) & 0 \\
G \sin^2 \alpha & G \sin \alpha \sin \alpha & \frac{GJ_B}{2At} \sin \alpha & 0 & 0 & G \sin \alpha \left(h - \frac{J_B}{2At}\right) & 0 \\
G \left(\frac{J_B}{2At}\right)^2 & 0 & 0 & 0 & E\omega^2 & G \left(h - \frac{J_B}{2At}\right)^2 & 0 \\
G \left(h - \frac{J_B}{2At}\right)^2 & 0 & 0 & 0 & 0 & 0 & E\omega^2
\end{bmatrix}
\]

Now if we integrate each element of the above matrix over the cross-sectional area, we obtain the elements of the generalized rigidity matrix. Hence
\[
[D] = \begin{bmatrix}
 EF & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & EI_y & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & EI_z & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & GF_y & 0 & S1 & 0 & S3 \\
 0 & 0 & 0 & GF_z & S2 & 0 & S4 & 0 \\
 0 & 0 & 0 & S1 & S2 & GJ_B & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & EI_\omega & 0 \\
 0 & 0 & 0 & S3 & S4 & 0 & 0 & G(I_p - J_B)
\end{bmatrix}
\] (3.30)

In the above matrix the off-diagonal terms corresponding to \( \int x \, dF \), \( \int y \, dF \) and \( \int xy \, dF \) are all zero because we are considering principal coordinate axes passing through the centroid and the terms corresponding to \( \int \omega \, dF \), \( \int y \omega \, dF \) and \( \int x \omega \, dF \) are zero because of the condition that net warping must be zero (otherwise there would be extension of the member) and the net first moments of the warping, both horizontal and vertical, about the center of twist must be zero (since otherwise it would require a horizontal or vertical bending moment to act on the member, see [12] for details.). The other off-diagonal terms \( S1, S2, S3, \) and \( S4 \) are zero for cross-sections made up of rectangular parts e.g. I-beams. The diagonal terms correspond to the various other sectional properties of the beam which are: \( F \), the total cross-sectional area; \( F_y \) and \( F_z \), the projected cross-sectional area along the \( y \) and \( z \) axis respectively; \( I_y \) and \( I_z \), the second moment of inertia about the \( y \) and \( z \) axis respectively; \( J_B \), the Bredt Torsional constant; \( I_\omega \), the sectorial moment of inertia and \( I_p \), the polar constant.

In order to allow for St. Venant's shearing stress which varies linearly through the thickness and which is predominant for open sections, we replace \( J_B \) by \( J \) which is the sum of
St. Venant torsional constant and Bredt torsional constant; i.e. \( J = J_b + J_r \). Therefore the generalized rigidity matrix for open and closed section is

\[
[D] = \begin{bmatrix}
EF & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & EI_y & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & EI_z & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & GF_y & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & GF_z & GJ & 0 \\
0 & 0 & 0 & 0 & 0 & EI_w & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & G(I_p - J_B)
\end{bmatrix}
\]

(3.30)

\[
(3.31)
\]

3.3.2 Generalized Stresses

Generalized stresses which where introduced in the previous section and which where said to correspond to the generalized strains are nothing but the components of the stress resultant obtained by integrating the actual stresses over the cross-sectional area. The vector of generalized stress components is denoted by \( \{\vec{\sigma}\} \) which is made up of

\[
\{\vec{\sigma}\} = \begin{bmatrix} N & M_y & M_z & Q_y & Q_z & T_r & B & T_w \end{bmatrix}^T
\]

The component \( N \) represents the axial stretching force applied along the centroidal axis, \( M_z \) is the out of plane bending moment, \( M_r \) is the in-plane bending moment, \( Q_y \) and \( Q_z \) are the shear forces due to transverse bending, \( T_r \) is the torsion moment developed due to St.
Venant’s shear stress, $T_w$ is the warping torsion moment developed due to non-uniform torsion and $B$ is the Bimoment. Therefore the generalized stress components, expressed in terms of the generalized strain components and the rigidity coefficients are:

$$N = EFe$$  \hspace{1cm} (3.32)

$$M_y = EI_y \kappa_y$$  \hspace{1cm} (3.33)

$$M_z = EI_z \kappa_z$$  \hspace{1cm} (3.34)

$$Q_y = GF_y y^x \left. \right|_{xy}$$  \hspace{1cm} (3.35)

$$Q_z = GF_z y^x \left. \right|_{xz}$$  \hspace{1cm} (3.36)

$$T_s = GJ_B \beta'$$  \hspace{1cm} (3.37)

$$B = EI_\omega \phi'$$  \hspace{1cm} (3.38)

$$T_w = G(I_p - J_B)(\beta' - f)$$  \hspace{1cm} (3.39)

The applied torque is the sum of St. Venant's torsion moment $T$, and warping torsion moment $T_w$; that is $M_{app} = T_s + T_w$

### 3.4 Stiffness Matrix

The element stiffness matrix is obtained by the application of the Principle of Virtual Work. The element is given a virtual nodal displacement $\{\delta'\}$ and the external work, in-
volving actual nodal forces \( \{p\} \), is equated to the internal work involving the generalized virtual strain \( \{\varepsilon^*\} \) (expressed in terms of \( \{\delta^*\} \)) and the actual generalized stress \( \{\sigma\} \) (expressed in terms of actual displacements \( \{\delta\} \)). The virtual nodal displacements are then set to unity and the resulting matrix relating \( \{p\} \) to \( \{\delta\} \) is the required element stiffness matrix.

The external virtual work is

\[
W_{\text{ext}} = [\delta^*]^T \{p\}
\]

and the internal virtual work is

\[
W_{\text{int}} = \int_L [\varepsilon^*]^T \{\sigma\} \, dx
\]

Now since \( [\varepsilon] = [B] \{\delta\} \), therefore

\[
[\varepsilon^*]^T = [\delta^*]^T [B]^T
\]

Expressing the generalized stress in terms of the nodal displacement, we get

\[
\{\sigma\} = [\bar{D}] [\bar{\varepsilon}] = [\bar{D}][B] \{\delta\}
\]

Substituting the above expressions for \( [\varepsilon^*]^T \) and \( \{\sigma\} \) in the expression for \( W_{\text{int}} \), we get

\[
W_{\text{int}} = \int_L [\delta^*]^T [B]^T [\bar{D}][B] \{\delta\} \, dx
\]

Setting \( \{\delta^*\} \) to unity and equating the internal and external work gives,
\{p\} = \left[ \int_L [B]^T \overline{[D]} [B] \, dx \right] \{\delta\} = [K]_e \{\delta\}

therefore the element stiffness matrix \([K]_e\) is defined as

\[ [K]_e = \int_L [B]^T \overline{[D]} [B] \, dx \{\delta\} = [K]_e \{\delta\} \quad (3.40) \]

3.4.1 Computation of Element Stiffness Matrix

The terms of the element stiffness matrix are evaluated numerically using a selective-reduced integration scheme. This scheme employs a technique whereby the terms corresponding to axial, bending and warping stiffness are evaluated exactly according to the Gauss quadrature rule whereas those corresponding to transverse and warping shear are under integrated. It is necessary to use such a scheme of numerical integration because of the inclusion of shear deformation. In \(C_0\) beam elements, which includes shear deformation, a commonly occurring anomaly when analyzing shallow beams is a phenomenon called "shear locking". When shear locking occurs, the element is much too stiff in bending when the height \(h\) is small relative to the length \(l\). As shown by Melosh [15] and Utku [16], this drawback can be traced to the transverse shear energy which is of the order \((l/h)^2\) higher than the remaining terms, where \(l/h\) denotes the element length to thickness ratio. Thus as the element thickness approaches zero the computed shear stiffness completely dominates and no effect of bending stiffness remains with a finite computer word length. As a result the element produces an excessively stiff solution which does not reflect the correct bending behavior.
Figure 6. Cylindrical co-ordinate system for curved beam.

Chapter 3: Finite Element Formulation
Figure 7. Typical thin-walled beam cross-section.
Figure 8. Finite element representation of the beam.
Chapter 4: Adaptation for General Finite Element Programs

In this chapter some of the techniques which have been used to adapt the one dimensional, three noded, seven degree of freedom (d.o.f.) beam element to general finite element programs like MAESTRO are discussed.

4.1 Eccentricity

In plated structures the beams are attached to one or the other side of the plating. In other words the centroidal axis of the beam and the axis passing through the center of twist are offset from the plate/shell reference surface, which is usually the plate middle surface. In most cases this reference surface also lies along the global coordinate axis system. Hence pure forces which act along these global axes give rise to moments on the beam. In general for mono-symmetric beams, the shear center may not coincide with the
centroid of the beam cross-section. Thus we have to consider three axis systems. A global axis system which lies at the point of attachment of the stiffener to the plate and two local axis systems, one which passes through the centroid of the beam's cross section and another which passes through the shear center. As mentioned in the last chapter the nodal d.o.f. \( u, \theta, \) and \( \theta_s \) are defined with respect to the centroidal axis; \( v, w, \theta_s \) and \( f \) are defined with respect to the axis of twist.

The transformation of nodal displacements from local coordinate system to global system is described now. Displacements and rotations with a bar on top indicate global displacements. From Fig. 9 it is seen that the transformation of the axial displacement \( u \) is affected by the rotation angle \( \theta_s \), so that

\[
\bar{u} = u + y_0 \theta_s
\]  

(4.1)

where \( y_0 \) is the y coordinate of the origin of the centroidal axis as measured from the global coordinate system. In a similar fashion the transformation of horizontal displacement \( w \) is affected by the twisting angle, \( \theta_s \). Therefore

\[
\bar{w} = w + e \theta_s
\]  

(4.2)

where \( e \) is the y coordinate of the origin of the axis system passing through the center of twist as measured from the global coordinate system. The vertical displacement \( v \) and rotation \( \theta_s \) remain unchanged. Warping of the element will cause an additional rotation of the cross section about the y-axis at the point \( O \); this should be considered in the transformation of rotation angle \( \theta_y \). Therefore

\[
\bar{\theta}_y = \theta_y + \left. \frac{\partial u}{\partial z} \right|_0
\]  

(4.3)
Now \( u = -\omega f \), therefore

\[
\frac{\partial u}{\partial z} = -\frac{\partial \omega}{\partial z} \bigg|_o f
\]

\[
= -\frac{\partial \omega}{\partial s} \frac{\partial s}{\partial z} \bigg|_o f
\]

\[
\frac{\partial s}{\partial z} = \frac{1}{\sin \alpha} = 1
\]

From Equ(2.38),

\[
\frac{\partial \omega}{\partial s} \bigg|_o = (e - \frac{J_b}{2At})
\]

Therefore,

\[
\frac{\partial u}{\partial z} = f(e - \frac{J_b}{2At}) \bigg|_o = -fe^*
\]

(4.4)

where \( e^* = e - \frac{J_b}{2At} \). Therefore

\[
\bar{\theta}_y = \theta_y - e^* f
\]

(4.5)

The remaining two displacements \( \theta_x \) and \( f \) don't change under transformation. Thus the transformation matrix for transforming the global displacements and rotations to the local ones is.
$$
\begin{align*}
\{\theta_x\} &= \begin{bmatrix}
1 & 0 & 0 & 0 & -y_0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & e & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
\bar{u} \\
\bar{v} \\
\bar{w} \\
\bar{\theta}_x \\
\bar{\theta}_y \\
\bar{\theta}_z \\
\bar{f} \\
\end{bmatrix} \\
\{\delta\} &= [T]\{\bar{\delta}\} \quad (4.7)
\end{align*}
$$

where \{\delta\} is the vector of nodal displacements in local coordinates,

\{\bar{\delta}\} is the vector of nodal displacements in global coordinates

and \[[T]\] is the transformation matrix between the global and local systems.

Following the usual procedure for transforming a stiffness matrix, the stiffness matrix

\[[K]\] in global coordinates incorporating eccentricity is :

$$
[K] = [T]^T[K][T] \quad (4.8)
$$

where \[[K]\] is the stiffness matrix in the local beam coordinate system.

### 4.2 Static Condensation

To implement the present beam element, which has seven d.o.f. per node, in general finite element programs which use elements with six d.o.f. per node, one has to resort to static condensation.
Static condensation (Wilson [17], Cook [18]) is the process of reducing the number of d.o.f. by a partial reduction (solution) of a system of equations; for example, by applying a Gauss elimination process but stopping before the stiffness matrix has been fully reduced. Usually the d.o.f. to be eliminated are done at the substructure level. The substructure might be a macroelement like a longitudinal girder or transverse frame built of subelements. The result obtained would be the same if they were done later; at the structural level. However the advantage of eliminating them at the element level and before assembly is that the order of the structure stiffness matrix is reduced because the d.o.f. to be eliminated are not carried into the global set of equations.

By using the concept of matrix partitioning the steps involved in static condensation can be explained in the following way. Let the equations \([K]{d} = {r}\) represent a substructure as defined above. Let the vector of d.o.f., \(d\) be partitioned so that \(d = [d_r, d_c]^T\), where \(d_r\) are the d.o.f. to be retained and \(d_c\) are the ones to be condensed out. Therefore \([K]{d} = {r}\) becomes

\[
\begin{bmatrix}
  K_{rr} & K_{rc} \\
  K_{cr} & K_{cc}
\end{bmatrix}
\begin{bmatrix}
  d_r \\
  d_c
\end{bmatrix}
= \begin{bmatrix}
  r_r \\
  r_c
\end{bmatrix}
\]

(4.9)

The lower partition is solved for \(d_c\) :

\[
\{d_c\} = [K_{cc}]^{-1}(\{r_c\} - [K_{cr}]{d_r})
\]

(4.10)

Next \(d_c\) is substituted into the upper partition of Eq.(4.9). Thus

\[
([K_{rr}] - [K_{rc}][K_{cc}]^{-1}[K_{cr}]){d_r} = \{r_r\} - [K_{rc}][K_{cc}]^{-1}\{r_c\}
\]

(4.11)

or \([K^*]{d} = \{r^*\}\)

(4.12)
where \([K']\) is the condensed stiffness matrix and \(\{r'\}\) is the condensed load matrix. The condensed stiffness and load matrices are assembled into the structure stiffness matrix \([K]\). The structural d.o.f. \(\{D\}\) are obtained in the usual way by solving the equation \([K]\{D\} = \{R\}\). Thus \(\{d'\}\) becomes known and using Eq.(4.10) \(\{d_i\}\) can be calculated.

Usually static condensation is applied to internal d.o.f. and is an exact process involving no approximation. But in the present case, condensation is employed on d.o.f. (warping function, \(f\)) which exist at the boundary. Since the external forces corresponding to these condensed d.o.f. in this case the bimoment, is zero, this introduces an approximation.

The matrix partitioning scheme used above is best for illustrating the concept of the technique; however, the matrix operations of multiplications and inversion are not efficient, and within a computer program it is more efficient to use a Gauss elimination algorithm.
Figure 9. Coordinate axis system for transformation of eccentric beam.
Chapter 5: Validation of the Element

In this chapter some test cases are presented to validate the accuracy of the proposed element. A computer program was developed which performs most of the computations usually associated with any general finite element program: data input, forming the element stiffness matrix, transforming them to global coordinates, assembling the global stiffness matrix, applying boundary conditions, solving the linear set of equations and calculating stresses.

5.1 Rectangular Box-Beam

The first test is a cantilevered, thin-walled beam of rectangular cross section subjected to a uniformly distributed external torque of 1000.0 kN cm/cm, as shown in Fig. 10. The results of the finite element solution using the element formulated in this work are shown in Fig. 11. Kawai proposed an iterative solution for such a problem to account for the secondary shear stresses. The exact solution of the differential equation obtained by
Benscoter and the result using Kawai’s [5] stiffness matrix without iterating are also plotted. Comparison shows that the present results agree well with the exact solution. The main difference between Kawai’s solution and the exact solution occurs near the clamped end of the beam. This is because warping is totally restrained at the clamped end and hence the effects of secondary shear stresses are pronounced at this end. This effect dies out at a distance from the clamped end. Hence the stiffness matrix developed here which accounts for the secondary shear stresses gives good results even at the clamped end while at a distance both results match well since the effects of secondary shear stresses are negligible.

5.2 Straight I - Beam

A second example, which validates the proposed element for an open section, is the fully restrained cantilevered I-beam under a tip torque of 25000.0 kN mm. as shown in Fig. 12, which was studied previously by Tralli [6]. The angle of twist, axial normal stress and shear stress are compared to those obtained by Tralli and are shown in Fig. 13. Tralli’s analysis is based on a hybrid element formulation whereas the present work is based on assumed displacement method. Therefore the tip rotation in Tralli’s case requires a higher number of elements (about 8) to converge to the exact solution, whereas using the present formulation convergence is obtained using only two elements. The exact distribution of the shear stress across the flange of the I-beam is parabolic. In the present work we obtain a uniform distribution across the flange, the value of which corresponds to two thirds the maximum value of the exact analysis.
5.3 Curved I - Beam

To test the validity of the element for curved beams, a curved I-Beam of length 180 in., radius 240 in. and of a standard cross section 10WF49, is analysed under five different load cases and boundary conditions, Fig. 14. Besides being loaded by torsional moments (concentrated or distributed) the curved I - Beam is loaded by vertical forces (concentrated or distributed) perpendicular to its plane of curvature. The results are compared to the analytical solution by Heins [19] in Fig. 15 through 19. In each case comparison is made between vertical deflection, St. Venant’s torsion moment and warping moment. In all cases there is good agreement.
Figure 10. Cantilevered thin-walled box beam under distributed Torque.
Figure 11. Twist Angle, Warping function & Bimoment for the Box beam.

Chapter 5: Validation of the Element
Figure 12. Cantilevered I-beam under tip Torque.
Figure 13. Twist Angle, Normal Stress & Shear Stress for the I-beam.
\[ T = 20 \text{ K in} \quad P = 2 \text{ K} \]
\[ t = 0.222 \text{ Kin/ in} \quad p = 0.022 \text{ K/ in} \]

Figure 14. Different Loading and Boundary conditions on a Curved beam.

Chapter 5: Validation of the Element
Figure 15. Results for the Curved Beam (Case A)
Figure 16. Results for the Curved Beam (Case B)

Chapter 5: Validation of the Element
Figure 17. Results for the Curved Beam (Case C)
Figure 18. Results for the Curved Beam (Case D)

Chapter 5 : Validation of the Element
Figure 19. Results for the Curved Beam (Case E)

Chapter 5: Validation of the Element
Chapter 6 : Concluding Remarks

A simple three nodded, $C^1$, isoparametric beam finite element has been developed and validated. The formulation is based on a combination of Timoshenko beam theory and Benscoter's torsion theory. While Timoshenko's theory allows for transverse shear stresses, Benscoter's torsion theory is an improvement over the classical torsion theory which neglects the secondary warping effects of non-uniform torsion. For closed thin-walled sections these effects of non-uniform torsion are quite significant, and their inclusion allows the proposed element to model thin-walled beams of both open and closed sections.

Torsional-flexural coupling which exists for curved beams has been fully incorporated by including the effects of curvature in the strain terms. Five cases of curved beams under different boundary and loading conditions have been evaluated using this model and the results are found to be in good agreement with the analytical solution.
A technique of static condensation has been employed to eliminate the seventh degree of freedom at each node. This is necessary to make this element compatible with standard finite element programs, which use six degrees of freedom per node.
List of References


Vita

The author was born on 21st September 1966, in Calcutta, India. On graduating from the Indian Institute of Technology, Kharagpur, in May 1988, with a B.Tech. (H) degree in Naval Architecture, he joined Virginia Polytechnic Institute & State University for graduate studies in Aerospace & Ocean Engineering.

Shubhajyoti Ghose