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# Paraxial localized waves in free space

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**Abstract:** Subluminal, luminal and superluminal localized wave solutions to the paraxial pulsed beam equation in free space are determined. A clarification is also made to recent work on pulsed beams of arbitrary speed which are solutions of a narrowband temporal spectrum version of the forward pulsed beam equation.



**OCIS codes: (**260.1960) Diffraction theory (320.5540) Pulse shaping (350.5500) Propagation.

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## 1. Introduction

In recent years, there has been increasing interest in novel classes of spatio-temporally localized solutions to various hyperbolic equations governing acoustic, electromagnetic and quantum mechanical wave phenomena. The bulk of the research along these lines has been performed in connection to the basic formulation, generation, propagation, guidance, scattering and diffraction properties of electromagnetic and acoustic localized waves (LWs) in free space (see [1-6] for pertinent review literature). However, some work has been done in the area of propagation of localized waves in dispersive (see [7] and references therein) and nonlinear (see [8] and references therein) media. This interest has been sustained by advancements in ultrafast acoustical, optical and electrical devices capable of generating and shaping very short pulsed wave fields (see, e.g., [9]). Localized wave pulses exhibit distinct advantages in their performance by comparison to conventional quasi-monochromatic signals. It has been shown, in particular, that such pulses have extended ranges of localization in the near-to-far field regions. These properties render LW fields very useful in diverse physical applications, such as remote sensing, secure signaling, nondestructive testing, ultrafast microscopy, high resolution imaging, tissue characterization and photodynamic therapy.

There exist physical situations where a paraxial approximation to the scalar wave equation is pertinent. In this paper, a systematic approach to deriving paraxial spatio-temporally localized waves in free space is provided. Two distinct classes of such packet-like solutions are identified. The first class, which is based on a narrow angular spectrum assumption, is discussed in Section 2. The second one, based on both a narrow angular spectrum and a narrowband temporal spectrum approximation, is described in Section 3. Both classes incorporate subluminal, luminal and superluminal paraxial localized waves. For the second class, the subluminal and superluminal paraxial localized waves are shown in Section 4 to arise from subluminal and superluminal Lorentz boosts of two distinct types of general luminal solutions. Finally, a situation is addressed in Section 5, whereby exact localized wave solutions to the scalar wave equation are embedded into approximate paraxial solutions. Concluding remarks are made in Section 6.

## 2. Paraxial localized waves based on a narrow angular spectrum approximation of the scalar wave equation

The conventional paraxial approximation to a solution of the free-space homogeneous 3D Helmholtz equation

$$\left(\nabla_{\vec{\rho}}^2+\frac{\partial^2}{\partial z^2}+\frac{\omega^2}{c^2}\right)\hat{u}(\vec{r},\omega)=0;\;\;\vec{\rho}=(x,y),\tag{2.1}$$

viz., $\hat{u}_{\pm}(\vec{r},\omega)=\exp\left[\mp i(\omega/c)z\right]\hat{v}_{\pm}(\vec{r},\omega)$, with $\hat{v}_{\pm}(\vec{r},\omega)$ governed by the complex parabolic equations

$$i\frac{\partial}{\partial z}\hat{v}_{\pm}(\vec{r},\omega)=\pm\frac{c}{2\omega}\nabla_{\perp}^2\hat{v}_{\pm}(\vec{r},\omega),\tag{2.2}$$

is based on the assumption of a narrow angular spectrum with respect to the $z-$axis. In Eqs. (2.1) and (2.2), $c$ is the speed of light in vacuum and $\omega$ denotes an angular frequency. The


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space-time paraxial solutions $u_{\pm}(\vec{r},t)$ can be expressed in terms of the Fourier spectral representations

$$u_{\pm}(\vec{\rho},\tau_{\pm},z)=\int_{R_1} d\omega \int_{R_2} d\vec{\kappa} \exp[i\left(\omega\,\tau_{\pm}-\vec{\kappa}\cdot\vec{\rho}\right)\exp\left[\pm i(c\kappa^2 z)/(2\omega)\right]\tilde{u}_0(\vec{\kappa},\omega), \qquad (2.3)$$

where $\tau_{\pm}=t\mp z/c$, $\vec{\kappa}=\left(k_x,k_y\right)$ and $\kappa=\left|\vec{\kappa}\right|$. These representations allow one to determine the equations governing $u_{\pm}(\vec{\rho},\tau_{\pm},z)$; specifically,

$$\left(\nabla_{\vec{\rho}}^2\mp\frac{2}{c}\frac{\partial^2}{\partial\tau_{\pm}\,\partial z}\right)u_{\pm}(\vec{\rho},\tau_{\pm},z)=0. \qquad (2.4)$$

These relations are known as the forward and backward *pulsed beam equations* [10]. The equation for $u_{+}(\vec{\rho},\tau_{+},z)$ has been used extensively recently (cf., e.g., [11, 12]), especially in connection with ultra-wideband (few-cycle) signals.

*2.1 Paraxial luminal pulsed beams*

A solution to the forward pulsed beam equation is assumed as follows:

$$u_{+}(\vec{\rho},\tau_{+},z;\alpha)=\exp(-i2\alpha c\tau_{+})v_{+}(\vec{\rho},z;\alpha). \qquad (2.5)$$

Then, $v_{+}(\vec{\rho},z;\alpha)$ obeys the Schrödinger-like equation

$$i4\alpha\frac{\partial}{\partial z}v_{+}(\vec{\rho},z;\alpha)=-\nabla_{\vec{\rho}}^2\,v_{+}(\vec{\rho},z;\alpha), \qquad (2.6)$$

which has a large number of known solutions. Among them are the Hermite-Gauss, the Laguerre-Gauss and Bessel-Gauss beams. A specific class of axisymmetric Laguerre-Gauss beams is given as follows:

$$v_n(\rho,z;\alpha)=A_0\frac{a_1}{\left(a_1+iz\right)^{n+1}}\exp\left(-\alpha\frac{\rho^2}{a_1+iz}\right)L_n^{(0)}\left(\alpha\frac{\rho^2}{a_1+iz}\right);\; n=0,1,2,... \qquad (2.7)$$

Here, $\rho=\left|\vec{\rho}\right|=\sqrt{x^2+y^2}$, $a_1$ is a free positive parameter and $L_n^{(0)}(\cdot)$ denotes the *nth* order Laguerre polynomial. A general solution to the forward pulsed beam equation can be obtained by using Eq. (2.7) in conjunction with Eq. (2.5) and superimposing over the free parameter $\alpha$; specifically,

$$u_{+}(\rho,z,\tau_{+})=\int_0^{\infty} d\alpha\,\exp(-2i\alpha c\tau_{+})v_n(\rho,z;\alpha)\tilde{F}(\alpha). \qquad (2.8)$$

As a particular example, let $\tilde{F}(\alpha)=\exp\left(-2a_2\alpha c\right)$, $a_2$ being a real positive parameter. Then, from Erdelyi (cf., Ref. [13], p. 174), one obtains

$$u_+(\rho,z,\tau_+) = A_0 \frac{a_1}{(a_1+iz)^{n+1}} \frac{\Gamma(n+1)}{n!} \frac{\left[2c(a_2+i\tau_+)\right]^n}{\left[2c(a_2+i\tau_+)+\rho^2/(a_1+iz)\right]^{n+1}}. \tag{2.9}$$

As this finite-energy pulsed beam propagates in the positive $z-$direction with speed $c$, it sustains loss of amplitude as well as broadening. However, these distortions can be minimized by tweaking the free parameters $a_1$ and $a_2$. An interesting property of the solution given in Eq. (2.9) is that with the formal replacement $z \to \varsigma_-/2$; $\varsigma_- \equiv z+ct$, it becomes the *nth* order *splash mode* [1], which, in turn, belongs to the class of *focus wave mode* (FWM)-type exact solutions to the homogeneous 3D scalar wave equation. Recently, the splash mode corresponding to $n=0$ has been used as a Hertz potential in an extensive study of the spatio-temporal evolution of focused single-cycle terahertz electromagnetic pulses [14]. Special attention has been paid to the limiting case $a_1 << a_2$ corresponding to the paraxial regime. This is a particular situation whereby an exact solution to the homogeneous 3D scalar wave equation behaves as a paraxial pulsed beam under certain parametrization. An analogous result, but in a different setting, has been discussed by Saari [15] recently.

Luminal pulsed beams analogous to those in Eq. (2.8) can be found for $u_-(\rho,z,\tau_-)$; however, these wavepackets propagate in the negative $z-$direction.

*2.2 Paraxial superluminal localized waves*

It will be more convenient in the subsequent discussion of paraxial localized waves to recast equations (2.4) into new, but equivalent, forms, viz.,

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial z} \pm \frac{1}{c}\frac{\partial}{\partial t}\right) u_\pm(\vec{\rho},z,t) = \pm\frac{c}{2}\nabla^2_{\vec{\rho}}\, u_\pm(\vec{\rho},z,t). \tag{2.10}$$

The following change of variables is undertaken in the equation for $u_+(\vec{\rho},z,t)$: $\varsigma_+ = z-ct,\ \eta_+ = z-vt;\ v>c.$ One, then, obtains the transformed equation

$$-2\left(\frac{v}{c}-1\right)\frac{\partial^2}{\partial\varsigma_+\partial\eta_+}u_+(\vec{\rho},\varsigma_+,\eta_+) - 2\frac{v}{c}\left(\frac{v}{c}-1\right)\frac{\partial^2}{\partial\eta_+}u_+(\vec{\rho},\varsigma_+,\eta_+) = -\nabla^2_{\vec{\rho}}\, u_+(\vec{\rho},\varsigma_+,\eta_+). \tag{2.11}$$

An elementary solution is chosen, next, in the form

$$u_+^{(e)}(\vec{\rho},\varsigma_+,\eta_+;\alpha,\beta,\vec{\kappa}) = \exp\left(-i\vec{\kappa}\cdot\vec{\rho}\right)\exp\left(-i\alpha\varsigma_+\right)\exp\left(-i\beta\eta_+\right), \tag{2.12}$$

where $\alpha$ and $\beta$ are real positive free parameters with units of $m^{-1}$. Substitution into Eq. (2.11) leads to the dispersion relation

$$\kappa^2 = 2\frac{v}{c}\left(\frac{v}{c}-1\right)\beta^2 + 2\left(\frac{v}{c}-1\right)\alpha\beta. \tag{2.13}$$

A general solution to the forward pulsed beam equation can be written as

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$$u_+(\vec{\rho},z,t)=\int_0^\infty d\alpha\int_0^\infty d\beta\int_{R_2} d\vec{\kappa}\, u_+^{(e)}(\vec{\rho},\varsigma_+,\eta_+;\alpha,\beta,\vec{\kappa}) \\ \times\delta\left[\kappa^2-2\frac{v}{c}\left(\frac{v}{c}-1\right)\beta^2-2\left(\frac{v}{c}-1\right)\alpha\beta\right]\tilde{u}_0(\alpha,\beta,\vec{\kappa}), \tag{2.14}$$

where $\delta(\cdot)$ denotes a Dirac delta function. For an azimuthally independent spectrum, viz., $\tilde{u}_0(\alpha,\beta,\vec{\kappa})=\tilde{u}_0(\alpha,\beta,\kappa)$, one obtains, in particular, the axisymmetric solution

$$u_+(\rho,z,t)=\int_0^\infty d\alpha\int_0^\infty d\beta\exp\left[-i\left(\alpha\varsigma_++\beta\eta_+\right)\right]J_0\left[\rho\sqrt{2\frac{v}{c}\left(\frac{v}{c}-1\right)}\sqrt{\beta^2+\frac{c}{v}\alpha\beta}\right]\tilde{u}_1(\alpha,\beta), \tag{2.15}$$

where $J_0(\cdot)$ is the zero-order ordinary Bessel function. If $\tilde{u}_1(\alpha,\beta)=\exp(-a_1\beta)\tilde{F}(\alpha)$, $a_1$ being a positive parameter, the integration over $\beta$ can be carried out explicitly (cf. [13], p. 192). As a result, one has

$$u_+(\rho,z,t)=\int_0^\infty d\alpha v_+(\rho,z,t;\alpha)\tilde{F}(\alpha), \tag{2.16}$$

where

$$v_+(\rho,z,t;\alpha)=\exp\left[-i\alpha\left(1-\frac{c}{2v}\right)\left(z-v_{ph}t\right)\right]\left[2\left(\frac{v}{c}\right)\left(\frac{v}{c}-1\right)\rho^2+\left(a_1+i(z-vt)\right)^2\right]^{-1/2} \\ \times\exp\left[-\frac{c\alpha}{2v}\sqrt{2\left(\frac{v}{c}\right)\left(\frac{v}{c}-1\right)\rho^2+\left(a_1+i(z-vt)\right)^2}\right], \tag{2.17}$$

with $v_{ph}=v/\left[2(v/c)-1\right]$. It should be noted that $v_{ph}\to c/2$ as $v\to\infty$. The solution given in Eq. (2.17) is the paraxial version of the *focus X wave* (FXW) (cf. Ref. [3]). If, in Eq. (2.16), one chooses the singular spectrum $\tilde{F}(\alpha)=\delta(\alpha)$, one obtains the paraxial version of the infinite-energy zero-order X wave [16,17]; specifically,

$$u_+(\rho,z,t)=\left\{2(v/c)\left[(v/c)-1\right]\rho^2+\left(a_1+i(z-vt)\right)^2\right\}^{-1/2}. \tag{2.18}$$

On the other hand, a smooth spectrum, e.g.,

$$\tilde{F}(\alpha)=\begin{cases}0, & b>\alpha>0,\\ \dfrac{1}{\Gamma(q)}(\alpha-b)^{q-1}\exp\left[-a_2(\alpha-b)\right], & \alpha\ge b;\ b,q\ge 0,\end{cases} \tag{2.19}$$

results in the paraxial version of the finite-energy modified focus X wave (MFXW) pulse [3]

$$u_+(\rho,z,t)=\frac{\left[2(v/c)\left[(v/c)-1\right]\rho^2+\left(a_1+i(z-vt)\right)^2\right]^{-1/2}\exp(ib\lambda)}{\left[c/(2v)\sqrt{2(v/c)\left[(v/c)-1\right]\rho^2+\left(a_1+i(z-vt)\right)^2}+(a_2-i\lambda)\right]^q} \tag{2.20}$$
$$\times\exp\left\{-(bc)/(2v)\sqrt{2(v/c)\left[(v/c)-1\right]\rho^2+\left(a_1+i(z-vt)\right)^2}\right\},$$

where $\lambda\equiv\left[1-c/(2v)\right]\left(z-v_{ph}t\right)$. It should be noted that although $v_+(\rho,z,t;\alpha)$ in Eq. (2.17) is unidirectional, this is not necessarily true of $u_+(\rho,z,t)$ in Eq. (2.20); the latter may contain both forward and backward propagating components. As in the case of the exact MFXW solution to the scalar wave equation, superluminality in the paraxial version given in Eq. (2.20) does not contradict relativity theory. If the parameters are chosen so that $u_+(\rho,z,t)$ contains mostly forward propagating components, the pulse moves superluminally with almost no distortion up to a certain distance $z_d$, and then slows down to a luminal speed $c$, with significant accompanying distortion. Although the peak of the pulse does move superluminally up to $z_d$, it is not causally related at two distinct ranges $z_1,z_2\in\left[0,z_d\right)$. Thus, no information can be transferred superluminally from $z_1$ to $z_2$. The physical significance of $u_+(\rho,z,t)$ is due to its spatio-temporal localization.

*2.3 Paraxial subluminal localized waves*

For $v<c$, the dispersion relation in Eq. (2.13) can be recast into the form

$$\kappa^2=2\frac{v}{c}\left(1-\frac{v}{c}\right)\left[\left(\frac{\alpha c}{2v}\right)^2-\bar{\beta}^2\right];\ \bar{\beta}\equiv\beta+\frac{\alpha c}{2v}. \tag{2.21}$$

Then, a general axisymmetric solution to Eq. (2.10) can be written as follows:

$$u_+(\rho,\varsigma_+,\eta_+)=\int_0^\infty d\alpha\int_{(\alpha c)/(2v)}^\infty d\bar{\beta}\,J_0\left\{\rho\left[2\frac{v}{c}\left(1-\frac{v}{c}\right)\right]^{1/2}\left[\left(\frac{\alpha c}{2v}\right)^2-\bar{\beta}^2\right]^{1/2}\right\} \tag{2.22}$$
$$\times\exp\left(-i\alpha\varsigma_+\right)\exp\left(-i\bar{\beta}\eta_+\right)\exp\left[i\eta_+(\alpha c)/(2v)\right]\tilde{u}_2(\alpha,\bar{\beta}).$$

A particular solution is given by

$$u_+(\rho,z,t)=\int_0^\infty d\alpha\frac{\sin\left[(\alpha c)/(2v)\sqrt{2(v/c)\left[1-(v/c)\right]\rho^2+\left(z-vt\right)^2}\right]}{\sqrt{2(v/c)\left[1-(v/c)\right]\rho^2+\left(z-vt\right)^2}} \tag{2.23}$$
$$\times\exp\left\{-i\alpha\left[1-c/(2v)\right](z-v_{ph}t)\right\}\tilde{F}(\alpha),$$

where $v_{ph}=v/\left[2\left(c/v\right)-1\right]$. The following should be noted: $v_{ph}\to[0,-\infty)$ as $v\to[0,c/2)$ and $v_{ph}\to(\infty,c]$ as $v\to(c/2,c]$.The solution in Eq. (2.23) consists of a superposition of paraxial MacKinnon-type wavepackets (cf. Ref. [3]). Finite-energy solutions can be obtained by choosing smooth spectra $\tilde{F}(\alpha)$.

*2.4 Localized waves for* $u_{-}(\vec{\rho},z,t)$

The following change of variables is undertaken in the equation for $u_{-}(\vec{\rho},z,t)$ in Eq. (2.10): $\varsigma_{-}=z+ct,\ \eta_{+}=z-vt$ . Then, one obtains

$$2\left(1+\frac{v}{c}\right)\frac{\partial^2}{\partial\varsigma_{-}\partial\eta_{+}}u_{-}(\vec{\rho},\varsigma_{-},\eta_{+})-2\frac{v}{c}\left(1+\frac{v}{c}\right)\frac{\partial^2}{\partial\eta_{+}^2}u_{-}(\vec{\rho},\varsigma_{-},\eta_{+})=-\nabla_{\vec{\rho}}^2\,u_{-}(\vec{\rho},\varsigma_{-},\eta_{+}). \quad (2.24)$$

An elementary solution is chosen, next, of the form

$$u_{-}^{(e)}(\vec{\rho},\varsigma_{-},\eta_{+};\alpha,\beta,\vec{\kappa})=\exp\left(-i\vec{\kappa}\cdot\vec{\rho}\right)\exp\left(i\alpha\varsigma_{-}\right)\exp\left(-i\beta\eta_{+}\right), \quad (2.25)$$

where $\alpha$ and $\beta$ are real positive free parameters. Substitution into Eq. (2.24) leads to the dispersion relation

$$\kappa^2=2\frac{v}{c}\left(1+\frac{v}{c}\right)\beta^2+2\left(1+\frac{v}{c}\right)\alpha\beta, \quad (2.26)$$

A general solution can be written as

$$\begin{aligned}u_{-}(\vec{\rho},z,t)=&\int_0^\infty d\alpha\int_0^\infty d\beta\int_{R_2}d\vec{\kappa}\,u_{-}^{(e)}(\vec{\rho},\varsigma_{-},\eta_{+};\alpha,\beta,\vec{\kappa})\\&\times\delta\left[\kappa^2-2\frac{v}{c}\left(\frac{v}{c}-1\right)\beta^2-2\left(\frac{v}{c}-1\right)\alpha\beta\right]\tilde{u}_0(\alpha,\beta,\vec{\kappa}),\end{aligned} \quad (2.27)$$

For an azimuthally independent spectrum, viz., $\tilde{u}_0(\alpha,\beta,\vec{\kappa})=\tilde{u}_0(\alpha,\beta,\kappa)$, one obtains, in particular, the axisymmetric solution

$$u_{-}(\rho,z,t)=\int_0^\infty d\alpha\int_0^\infty d\beta\exp\left[i\left(\alpha\varsigma_{-}-\beta\eta_{+}\right)\right]J_0\left[\rho\sqrt{2\frac{v}{c}\left(1+\frac{v}{c}\right)}\sqrt{\beta^2+\frac{c}{v}\alpha\beta}\right]\tilde{u}_1(\alpha,\beta). \quad (2.28)$$

The integrand is a *paraxial monochromatic Bessel beam.* Its difference from an exact monochromatic Bessel beam solution to the homogeneous 3D scalar wave equation is discussed in Appendix A. If the spectrum $\tilde{u}_1(\alpha,\beta)$ in Eq. (2.28) equals $\exp(-a_1\beta)\tilde{F}(\alpha)$, $a_1$ being a positive parameter, the integration over $\beta$ can be carried out explicitly [13]. As a result, one has

$$u_{-}(\rho,z,t)=\int_0^\infty d\alpha v_{-}(\rho,z,t;\alpha)\tilde{F}(\alpha), \quad (2.29)$$

where

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$$v_-(\rho,z,t;\alpha)=\exp\left[i\alpha\left(1+\frac{c}{2v}\right)(z+v_{ph}t)\right]\left[2\frac{v}{c}\left(1+\frac{v}{c}\right)\rho^2+\left(a_1+i(z-vt)\right)^2\right]^{-1/2}$$
$$\times\exp\left\{-(c\alpha)/(2v)\sqrt{2(v/c)\left[(1+v/c)\right]\rho^2+\left(a_1+i(z-vt)\right)^2}\right\}, \quad (2.30)$$

with $v_{ph}=c/\left(2+c/v\right)$.

An interesting property of the localized solution in Eq. (2.29) is that it is valid for $0<v<\infty$. If $v=c$, for example, with $\tilde{F}(\alpha)=\delta(\alpha-\alpha_0)$, one obtains the "luminal FXW"

$$u_-(\rho,z,t)=\exp\left\{i(3/2)\alpha_0\left[z+\left(c/3\right)t\right]\right\}\left[4\rho^2+\left(a_1+i(z-ct)\right)^2\right]^{-1/2}$$
$$\times\exp\left\{-\alpha_0/(2c)\sqrt{4\rho^2+\left(a_1+i(z-ct)\right)^2}\right\}, \quad (2.31)$$

and for $\tilde{F}(\alpha)=\delta(\alpha)$, the "luminal zero-order X wave," viz.,

$$u_-(\rho,z,t)=\left[4\rho^2+\left(a_1+i(z-ct)\right)^2\right]^{-1/2} \quad (2.32)$$

Finite-energy paraxial localized waves can be determined by using smooth spectra $\tilde{F}(\alpha)$ in Eq. (2.29).

## 3. Paraxial localized waves based on a narrowband temporal spectrum approximation of the forward and backward pulsed beam equations

The function $\tilde{u}_0(\vec{\kappa},\omega)\equiv\tilde{u}_1(\vec{\kappa},\omega-\omega_0)$ in Eq. (2.3) is assumed to be narrowband around the frequency $\omega=\omega_0$. Furthermore, the phase term $\beta(\kappa,\omega)\equiv\left(c\kappa^2\right)/(2\omega)$ is expanded in a Taylor series around $\omega=\omega_0$ and only the first term in the expansion is retained; i.e.,

$$\beta(\kappa,\omega)\approx\beta(\kappa,\omega_0)=\left(c\kappa^2\right)/(2\omega_0). \quad (3.1)$$

Within the framework of this additional approximation the expressions in Eq. (2.3) assume the forms

$$\psi_\pm(\vec{\rho},z,t)\equiv\exp\left(i\omega_0\tau_\pm\right)\int_{R_1}d\Omega\int_{R_2}d\vec{\kappa}\exp\left(i\Omega\tau_\pm\right)\exp\left(-i\vec{\kappa}\cdot\vec{\rho}\right)$$
$$\times\exp\left[\pm i(c\kappa^2 z)/(2\omega_0)\right]\tilde{u}_1(\vec{\kappa},\Omega), \quad (3.2)$$

where $\Omega=\omega-\omega_0$, or

$$\psi_\pm(\vec{\rho},z,t)=\exp\left(i\omega_0\tau_\pm\right)\phi_\pm(\vec{\rho},z,t), \quad (3.3)$$

with $\phi_\pm(\vec{\rho},z,t)$ governed by the equations


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$$i\left(\frac{\partial}{\partial z}\pm\frac{1}{c}\frac{\partial}{\partial t}\right)\phi_{\pm}(\vec{\rho},z,t)=\pm\frac{1}{2k_0}\nabla_{\vec{\rho}}^2\,\phi_{\pm}(\vec{\rho},z,t);\; k_0\equiv\omega_0/c\,. \tag{3.4}$$

*3.1 Subluminal and superluminal pulsed beams*

A solution to Eq. (3.4) for $\phi_+(\vec{\rho},z,t)$ is assumed of the form

$$\phi_+(\vec{\rho},z,t)=f(\tau_+)\Phi(\vec{\rho},\eta_+), \tag{3.5}$$

where $\eta_+=z-vt,\ v\neq c$, and $f(\tau_+)$ is an arbitrary function (at least differentiable). It follows, then, that the wave function $\Phi(\vec{\rho},\eta_+)$ obeys the equation

$$i\left(1-\frac{v}{c}\right)\frac{\partial}{\partial\eta_+}\Phi(\vec{\rho},\eta_+)=\frac{1}{2k_0}\nabla_{\vec{\rho}}^2\,\Phi(\vec{\rho},\eta_+). \tag{3.6}$$

Thus, a narrow angular spectrum and a narrowband temporal spectrum result in the following approximate nonluminal solution to the homogeneous 3D scalar wave equation:

$$\psi_+(\vec{\rho},\tau_+,\eta_+)=\exp\left(i\omega_0\tau_+\right)f(\tau_+)\Phi(\vec{\rho},\eta_+), \tag{3.7}$$

This general solution was originally reported by Wunsche [18] and, independently, by Besieris *et al*. [19]. The special case with $f(\tau_+)=\text{constant}$ was rediscovered by Longhi [20] recently. Longhi mistakenly attributed his solution to a "generalized paraxial approximation," instead of to a narrowband approximation of the forward pulsed beam equation.

It will be convenient for the discussion in the sequel to introduce new variables as follows: $\sigma_{\pm}=\pm 2\eta_+/\left(k_0\left|(v/c)-1\right|\right)$. The plus sign is associated with the superluminal case $v>c$ and the minus sign to the subluminal case $v<c$. In terms of the new variables, Eqs. (3.6) and (3.7) assume the simpler forms

$$i4\frac{\partial}{\partial\sigma_{\pm}}\Phi(\vec{\rho},\sigma_{\pm})=-\nabla_{\vec{\rho}}^2\,\Phi(\vec{\rho},\sigma_{\pm}), \tag{3.8}$$

$$\psi_+(\vec{\rho},\tau_+,\sigma_{\pm})=\exp\left(i\omega_0\tau_+\right)f(\tau_+)\Phi(\vec{\rho},\sigma_{\pm})\,, \tag{3.9}$$

respectively. Cited below are specific examples of superluminal/subluminal pulsed beams based on three distinct classes of solutions of Eq. (3.8).

Hermite-Gauss pulsed beams:

$$\begin{aligned}\psi_+^{(mn)}(x,y,\tau_+,\sigma_{\pm})&=\exp\left(i\omega_0\tau_+\right)f(\tau_+)\frac{\exp\left[-x^2/\left(\gamma_1+i\sigma_{\pm}\right)\right]\exp\left[-y^2/\left(\gamma_2+i\sigma_{\pm}\right)\right]}{\left(\gamma_1+i\sigma_{\pm}\right)^{(m+1)/2}\left(\gamma_2+i\sigma_{\pm}\right)^{(n+1)/2}}\\&\times H_m\left(x/\sqrt{\gamma_1+i\sigma_{\pm}}\right)H_n\left(y/\sqrt{\gamma_2+i\sigma_{\pm}}\right).\end{aligned} \tag{3.10}$$

Here, $\gamma_{1,2}$ are free positive parameters and $H_m(\cdot)$ denotes the *mth* order Hermite polynomial.

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Axisymmetric Laguerre-Gauss pulsed beams:

$$\psi_+^{(n)}(\vec{\rho},\tau_+,\sigma_\pm)=\exp\left(i\omega_0\tau_+\right)f(\tau_+)\frac{\gamma_0}{\left(\gamma_0+i\sigma_\pm\right)^{n+1}}\exp\left(-\frac{\rho^2}{\gamma_0+i\sigma_\pm}\right)L_n^{(0)}\left[\frac{\rho^2}{\left(\gamma_0+i\sigma_\pm\right)}\right]. \tag{3.11}$$

Here, $\gamma_0$ is a free positive parameter and $L_n^{(0)}(\cdot)$ denotes the *nth* order Laguerre polynomial.

For $n=0,$ the Laguerre-Gauss solution in Eq. (3.11) becomes the axisymmetric "modified" fundamental Gaussian pulsed beam

$$\psi_+^{(0)}(\rho,\tau_+,\sigma_\pm)=\exp\left(i\omega_0\tau_+\right)f(\tau_+)\frac{\gamma_0}{\gamma_0+i\sigma_\pm}\exp\left(-\frac{\rho^2}{\gamma_0+i\sigma_\pm}\right). \tag{3.12}$$

With $\gamma_0=2a/\left[k_0\left|(v/c)-1\right|\right],$ $a$ being a real positive parameter, this solution can be rewritten as

$$\psi_+^{(0)}(\rho,\tau_+,\eta_+)=\exp\left(i\omega_0\tau_+\right)f(\tau_+)\frac{a}{a\pm i\eta_+}\exp\left(-\frac{\omega_0}{2c}\left|\frac{v}{c}-1\right|\frac{\rho^2}{a\pm i\eta_+}\right). \tag{3.13}$$

It should be noted that the factor in $\psi_+^{(0)}(\rho,\tau_+,\eta_+)$ multiplying $\exp(i\omega_0\tau_+)f(\tau_+)$ is an infinite energy invariant wavepacket propagating along the positive $z-$direction with fixed speed $v$, either superluminal or subluminal. The arbitrary time-limiting function $f(\tau_+)$ in Eq. (3.13) can be chosen so that the entire wavepacket $\psi_+^{(0)}(\rho,\tau_+,\eta_+)$ has finite energy and propagates to a large distance $z$ with almost no distortion, except for local deformations. For example, the function

$$f(\tau_+)=\exp\left(-\frac{\tau_+^2}{4T^2}\right)=\exp\left\{-\frac{1}{4T^2}\left[\left(t-\frac{z}{v}\right)-\left(\frac{v-c}{vc}\right)z\right]^2\right\} \tag{3.14}$$

can be used to achieve this goal for values of the speed $v$ close to $c$ and a large values of $\omega_0$ so that $\omega_0\left|(v/c)-1\right|/c=O(1)$.

Axisymmetric Bessel-Gauss pulsed beam:

$$\begin{aligned}\psi_+(\rho,\tau_+,\sigma_\pm)=\exp\left(i\omega_0\tau_+\right)f(\tau_+)\frac{\gamma_0}{\gamma_0+i\sigma_\pm}J_0\left(\frac{\gamma_0k_0\rho\sin\theta}{\gamma_0+i\sigma_\pm}\right)\exp\left(-\frac{\rho^2}{\gamma_0+i\sigma_\pm}\right)\\ \times\exp\left[-i\frac{\sigma_\pm}{4}\left(k_0^2\gamma_0\sin^2\theta\right)/\left(\gamma_0+i\sigma_\pm\right)\right].\end{aligned} \tag{3.15}$$

Here, $J_0(\cdot)$ denotes the zero-order ordinary Bessel function and $\theta$ is an arbitrary real angle. It should be noted that for $\theta=0,$ this solution reduces to the pulsed Gaussian beam in Eq. (3.12).

*3.2 Luminal pulsed beams*

A solution to Eq. (3.4) for $\phi_+(\vec{\rho},z,t)$ is assumed of the form

$$\phi_+(\vec{\rho},\tau_+,\sigma_z^-) = f(\tau_+)\Phi(\vec{\rho},\sigma_z^-), \tag{3.16}$$

where $\sigma_z^- = -(2z)/k_0$. It follows, then, that the wave function $\Phi(\vec{\rho},\sigma_z^-)$ obeys the Schrödinger equation

$$i4\frac{\partial}{\partial\sigma_z^-}\Phi(\vec{\rho},\sigma_z^-) = -\nabla_{\vec{\rho}}^2\,\Phi(\vec{\rho},\sigma_z^-). \tag{3.17}$$

Thus, a narrow angular spectrum and a narrowband frequency spectrum result in the following approximate luminal solution to the homogeneous 3D scalar wave equation:

$$\psi_+(\vec{\rho},\tau_+,\sigma_z^-) = \exp\left(i\omega_0\tau_+\right) f(\tau_+)\Phi(\vec{\rho},\sigma_z^-). \tag{3.18}$$

The Hermite-Gauss solutions in Eq. (3.10), the Laguerre-Gauss solutions in Eq. (3.11) and the Bessel-Gauss solution in Eq. (3.15) are still applicable; however, $\sigma_\pm$ must be replaced by $\sigma_z^- = -(2z)/k_0$ for the luminal case under consideration.

It is important to discuss the basic differences between the luminal solutions to the pulsed beam equation [cf. Sec. 1], which are based on the narrow angular spectrum approximation, and those given in Eq. (3.18). A particularly simple example of the former is the monochromatic Gaussian beam

$$u_+(\rho,z;\alpha) = \exp\left(-2i\alpha c\tau_+\right)\frac{a_1}{(a_1+iz)}\exp\left(-\alpha\frac{\rho^2}{a_1+iz}\right), \tag{3.19}$$

where $a_1$ is a real positive parameter and $\alpha$ is an arbitrary real positive quantity. A superposition over the latter, e.g.,

$$u_+(\rho,z,t) = \frac{1}{\pi}\int_0^\infty d\alpha\tilde{F}(\alpha)\exp\left(-2i\alpha c\tau_+\right)\frac{a_1}{(a_1+iz)}\exp\left(-\alpha\frac{\rho^2}{a_1+iz}\right), \tag{3.20}$$

yields the forward pulsed beam solution

$$u_+(\rho,z,t) = \frac{a_1}{(a_1+iz)}\hat{f}\left(t-z/c-i\frac{1}{2c}\frac{\rho^2}{a_1+iz}\right), \tag{3.21}$$

where $\hat{f}(t)$ denotes the complex analytic signal associated with the spectrum $\tilde{F}(\omega)$.

A particularly simple example of a luminal pulsed beam based on a narrow angular spectrum and a narrowband temporal spectrum is the following:

$$\psi_+(\rho,z,t) = \exp\left[i\omega_0(t-z/c)\right] f(t-z/c)\frac{\gamma_0}{\gamma_0-i2z/k_0}\exp\left(-\frac{\rho^2}{\gamma_0-i2z/k_0}\right). \tag{3.22}$$


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It consists of a product of two factors; a plane wave modulated by a longitudinal envelope function traveling along the $z-$direction at the speed of light *in vacuo* and a "standing" fundamental Gaussian mode. In Eq. (3.22), $\omega_0 = ck_0$ is fixed and $\gamma_0$ is an arbitrary positive parameter. Thus, the pulsed beams given in Eqs. (3.21) and (3.22) differ substantially.

*3.3 Localized waves for $\psi_-(\vec{\rho},z,t)$*

A solution to Eq. (3.4) for $\phi_-(\vec{\rho},z,t)$ is assumed of the form

$$\phi_-(\vec{\rho},z,t) = f(\tau_-)\Phi(\vec{\rho},\eta_+), \tag{3.23}$$

where $f(\tau_-)$ is an arbitrary function (at least differentiable). It follows, then, that the wave function $\Phi(\vec{\rho},\eta_+)$ obeys the equation

$$i\left(1+\frac{v}{c}\right)\frac{\partial}{\partial\eta_+}\Phi(\vec{\rho},\eta_+) = -\frac{1}{2k_0}\nabla^2_{\vec{\rho}}\,\Phi(\vec{\rho},\eta_+). \tag{3.24}$$

It is convenient to introduce a new variable as follows: $\bar{\sigma}_+ = 2\eta_+/\left[k_0(1+v/c)\right]$. Then Eq. (3.24) changes to

$$i4\frac{\partial}{\partial\bar{\sigma}_+}\Phi(\vec{\rho},\bar{\sigma}_+) = -\nabla^2_{\vec{\rho}}\,\Phi(\vec{\rho},\bar{\sigma}_+). \tag{3.25}$$

Thus, a narrow angular spectrum and a narrowband temporal spectrum approximation result in the solution

$$\psi_-(\vec{\rho},\tau_-,\bar{\sigma}_+) = \exp\left(i\omega_0\tau_-\right) f(\tau_-)\Phi(\vec{\rho},\bar{\sigma}_+)\,. \tag{3.26}$$

By construction, this solution is valid for $0 \le v < \infty$. Since $\Phi(\vec{\rho},\bar{\sigma}_+)$ obeys the Schrödinger equation (3.25), one can have in a single setting Hermite-Gauss, Laguerre-Gauss and Bessel-Gauss subluminal, luminal and superluminal solutions. It must be pointed out, however, that whereas the "envelope" function $\Phi(\vec{\rho},\bar{\sigma}_+)$ moves in the $+z-$direction with an arbitrary speed $v \in [0,\infty)$, the factor $\exp(i\omega_0\tau_-)f(\tau_-)$ in Eq. (3.26) travels in the opposite direction. For $v = 0$, one obtains

$$\psi_-(\vec{\rho},\tau_-,\sigma_z^+) = \exp\left(i\omega_0\tau_-\right) f(\tau_-)\Phi(\vec{\rho},\sigma_z^+);\; \sigma_z^+ \equiv -\sigma_z^- = 2z/k_0, \tag{3.27}$$

an expression dual to that for $\psi_+(\vec{\rho},\tau_+,\sigma_z^-)$ in Eq. (3.18).

## 4. Derivation of paraxial subluminal and superluminal localized waves by means of Lorentz relativistic boosts

*4.1 Subluminal boosts*

It should be noted that Eq. (3.4) is Lorentz invariant. Specifically, under the subluminal Lorentz transformations $x = x', y = y', z = \bar{\gamma}\left(z'+vt'\right), t = \bar{\gamma}\left[t'+\left(v/c^2\right)z'\right]$, where $v < c$ and $\bar{\gamma} = 1/\sqrt{1-(v/c)^2}$, the solution $\psi_\pm(\vec{\rho},z,t) = \exp\left(i\omega_0\tau_\pm\right)\phi_\pm(\vec{\rho},z,t)$ [cf. Eq. (3.3)] transforms to $\psi_\pm(\vec{\rho},z',t') = \exp\left[i\omega_0\bar{\gamma}(1\mp v/c)\tau'_\pm\right]\phi_\pm(\vec{\rho},z',t')$, where $\tau'_\pm = t' \mp z'/c$ and


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$$i\left(\frac{\partial}{\partial z'}\pm\frac{1}{c}\frac{\partial}{\partial t'}\right)\phi_\pm(\vec{\rho},z',t')=\pm\frac{1}{2k_\pm}\nabla^2_{\vec{\rho}}\,\phi_\pm(\vec{\rho},z',t');\ k_\pm\equiv\overline{\gamma}\left(1\mp v/c\right)k_0. \tag{4.1}$$

Consider, next, the following general luminal solutions:

$$\begin{gathered}\psi_\pm(\vec{\rho},\tau'_\pm,\overline{\sigma}^\mp_{z'})=\exp\left[i\omega_0\overline{\gamma}\left(1\mp\frac{v}{c}\right)\tau'_\pm\right]f\left[\overline{\gamma}\left(1\mp\frac{v}{c}\right)\tau'_\pm\right]\Phi(\vec{\rho},\overline{\sigma}^\mp_{z'});\\ \tau'_\pm\equiv t'\mp\frac{z'}{c};\ \overline{\sigma}^\mp_{z'}\equiv\mp\frac{2z'}{k_\pm},\end{gathered} \tag{4.2}$$

where $\Phi(\vec{\rho},\overline{\sigma}^-_{z'})$ satisfies the parabolic equation (3.17) with the interchange $z\to z'$, and $\Phi(\vec{\rho},\overline{\sigma}^+_{z'})$ is governed by an analogous equation. Application of the inverse boosting $x'=x,\ y'=y,\ z'=\overline{\gamma}(z-vt),\ ct'=-\overline{\gamma}(v/c)\left[z-\left(c^2/v\right)t\right]$ to Eq. (4.2) yields the general paraxial subluminal solutions [cf. Eqs. (3.9) and (3.26)]

$$\psi_+(\vec{\rho},\tau_+,\sigma_-)=\exp\left(i\omega_0\tau_+\right)f\left(\tau_+\right)\Phi\left(\vec{\rho},\sigma_-\right) \tag{4.3}$$

and

$$\psi_-(\vec{\rho},\tau_-,\overline{\sigma}_+)=\exp\left(i\omega_0\tau_-\right)f\left(\tau_-\right)\Phi\left(\vec{\rho},\overline{\sigma}_+\right). \tag{4.4}$$

This observation has also been made by Longhi [19] for $\psi_+(\vec{\rho},\tau'_+,\sigma^-_{z'})$, except for the additional function $f\left[\overline{\gamma}\left(1-v/c\right)\tau'_+\right]$ appearing in Eq. (4.2).

*4.2 Superluminal boosts*

An interesting question is the following: Are there general luminal solutions to Eq. (3.4) which become the general paraxial superluminal solutions given in Eqs. (3.9) and (3.26) after a Lorentz transformation? In order to answer this question, we seek solutions to Eq. (3.4) of the form

$$\phi_\pm(\vec{\rho},\varsigma_\pm,t)=g(\varsigma_\pm)\Psi(\vec{\rho},t), \tag{4.5}$$

where, as defined earlier, $\varsigma_\pm=z\mp ct$. It follows, then, that the wave function $\Psi(\vec{\rho},t)$ obeys the Schrödinger equation

$$i4\frac{\partial}{\partial\sigma^\mp_t}\Psi(\vec{\rho},\sigma^\mp_t)=-\nabla^2_{\vec{\rho}}\,\Psi(\vec{\rho},\sigma^\mp_t), \tag{4.6}$$

where $\sigma^\mp_t=\mp(2ct)/k_0$. Thus, a narrow angular spectrum and a narrowband frequency spectrum result in the following approximate luminal solution to the homogeneous 3D scalar wave equation:

$$\psi_\pm(\vec{\rho},\varsigma,\sigma^\mp_t)=\exp\left(-ik_0\varsigma_\pm\right)g(\varsigma_\pm)\Psi(\vec{\rho},\sigma^\mp_t). \tag{4.7}$$

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Simple examples of such solutions are the following:

$$\psi_\pm(\rho,z,t)=\exp\left[-ik_0(z\mp ct)\right]g(z\mp ct)\frac{\gamma_0}{\gamma_0\mp i(2ct)/k_0}\exp\left(-\frac{\rho^2}{\gamma_0\mp i(2ct)/k_0}\right). \qquad (4.8)$$

Equation (3.4) is Lorentz invariant. More specifically, under the generalized (superluminal) Lorentz transformation $x=x', y=y', z=\gamma(v/c)\left[z'+\left(c^2/v\right)t'\right], t=\gamma\left(z'+vt'\right)/c,$ where $\gamma=1/\sqrt{(v/c)^2-1}$ and $v>c,$ the solution $\psi_\pm(\vec{\rho},z,t)=\exp\left(i\omega_0\tau_\pm\right)\phi_\pm(\vec{\rho},z,t)$ [cf. Eq. (3.3)] transforms to $\psi_\pm(\vec{\rho},z',t')=\exp\left\{-ik_0\gamma\left[\left(v/c\right)\mp 1\right]\varsigma'_\pm\right\}\phi_\pm(\vec{\rho},z',t'),$ where $\varsigma'_\pm=z'\mp ct'$ and $\phi_\pm(\vec{\rho},z',t')$ is given in Eq. (4.1) with $k_\pm\to\bar{k}_\pm\equiv\gamma\left[\left(v/c\right)\mp 1\right]k_0$. Consider, next, the general luminal solutions

$$\begin{gathered}\psi_\pm(\vec{\rho},\varsigma'_\pm,\sigma^\mp_{t'})=\exp\left[-ik_0\gamma\left(\frac{v}{c}\mp 1\right)\varsigma'_\pm\right]g\left[\gamma\left(\frac{v}{c}\mp 1\right)\varsigma'_\pm\right]\Psi(\vec{\rho},\sigma^\mp_{t'});\\ \varsigma'_\pm\equiv z'\mp ct';\ \sigma^\mp_{t'}\equiv\mp(2ct')/\bar{k}_\pm,\end{gathered} \qquad (4.9)$$

where $\Psi(\vec{\rho},\sigma^\mp_{t'})$ satisfies Eq. (4.6) with $t\to t'$. Application of the inverse boosting $x'=x,\ y'=y,\ ct'=-\gamma(z-vt),\ z'=\gamma(v/c)\left[z-\left(c^2/v\right)t\right]$ to Eq. (4.9) yields the general paraxial superluminal solutions [cf. Eqs. (3.9) and (3.26)]

$$\psi_+(\vec{\rho},\varsigma_+,\sigma_+)=\exp\left(-ik_0\varsigma_+\right)g\left(\varsigma_+\right)\Psi\left(\vec{\rho},\sigma_+\right) \qquad (4.10)$$

and

$$\psi_-(\vec{\rho},\varsigma_-,\bar{\sigma}_+)=\exp\left(-ik_0\varsigma_-\right)g\left(\varsigma_-\right)\Psi\left[\vec{\rho},-\bar{\sigma}_+\right]. \qquad (4.11)$$

## 5. Embedding of exact localized wave solutions of the scalar wave equation into approximate paraxial ones

The change of variables $\varsigma_+=z-ct,\ z_-=z+ct$ is introduced in Eq. (3.4). As a consequence one obtains the Schrödinger equations

$$i4\frac{\partial}{\partial\chi_\mp}\phi_\pm(\vec{\rho},\chi_\mp)=-\nabla^2_{\vec{\rho}}\,\phi_\pm(\vec{\rho},\chi_\mp), \qquad (5.1)$$

with the definition $\chi_-=-\varsigma_-/k_0$ and $\chi_+=\varsigma_+/k_0$. Thus, under the assumption of a narrow angular spectrum and a narrowband frequency spectrum one obtains the general solutions

$$\psi_+(\vec{\rho},z,t)=\exp\left(-ik_0\varsigma_+\right)\phi_+(\vec{\rho},\chi_-)=\exp\left[-ik_0(z-ct)\right]\phi_+\left[\vec{\rho},-(z+ct)/k_0\right]. \qquad (5.2)$$

and

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$$\psi_-(\vec{\rho},z,t)=\exp\left(-ik_0\varsigma_-\right)\phi_-(\vec{\rho},\chi_+)=\exp\left[-ik_0(z+ct)\right]\phi_-\left[\vec{\rho},(z-ct)/k_0\right]. \tag{5.3}$$

The simplest such solutions are the following:

$$\psi_+(\rho,z,t)=\frac{k_0}{k_0-i(z+ct)}\exp\left[-ik_0(z-ct)\right]\exp\left[-k_0\frac{\rho^2}{k_0-i(z+ct)}\right], \tag{5.4}$$

$$\psi_-(\rho,z,t)=\frac{k_0}{k_0+i(z-ct)}\exp\left[-ik_0(z+ct)\right]\exp\left[-k_0\frac{\rho^2}{k_0+i(z-ct)}\right]. \tag{5.5}$$

But $\psi_-(\rho,z,t)$ in Eq. (5.5) and $\psi_+(\rho,z,t)$ in Eq. (5.4) are, respectively, the fundamental focus wave mode (FWM) and a variant of it. Both are *exact* solutions to the homogeneous 3D scalar wave equation for an arbitrary wavenumber $k_0$! More generally, the solutions in Eqs. (5.2) and (5.3) embody Hermite-Gauss, Laguerre-Gauss and Bessel-Gauss FWM-type solutions that are also exact. This "peculiarity", whereby exact solutions of the scalar wave equation are *embedded* into approximations to this equation, has been mentioned by Wunsche [18] previously.

It is possible to provide a more physical explanation for the "peculiarity" described above. Consider, for example, the solution given in Eq. (4.4), viz.,

$$\psi_-(\vec{\rho},\tau_-,\bar{\sigma}_+)=\exp\left(i\omega_0\tau_-\right)f\left(\tau_-\right)\Phi\left(\vec{\rho},\bar{\sigma}_+\right);\ \tau_-=t+\frac{z}{c}z,\ \bar{\sigma}_+=\frac{2(z-vt)}{k_0\left(1+v/c\right)}. \tag{5.6}$$

With $f(\tau_-)=\text{constant}$ and $v=c$, this expression simplifies to

$$\psi_-(\vec{\rho},z,t)=\exp\left[-ik_0(z+ct)\right]\Phi\left(\vec{\rho},\frac{z-ct}{k_0}\right). \tag{5.7}$$

Since $\Phi\left(\vec{\rho},z\right)$ obeys the complex parabolic equation (3.17), a simple solution in the place of the general one in Eq. (5.7) is given as follows:

$$\psi_-(\vec{\rho},z,t)=\frac{k_0}{a+i(z-ct)}\exp\left[-ik_0(z+ct)\right]\exp\left[-k_0\frac{\rho^2}{a+i(z-ct)}\right],\ a>0. \tag{5.8}$$

*Modulo* the constant multiplier $k_0$ and with $a=k_0$, one recovers the exact FWM solution given in Eq. (5.5). Retracing the steps, it follows that the FWM arises from a subluminal Lorentz transformation of the monochromatic luminal beam [cf. restriction of Eq. (4.2)]

$$\begin{gathered}\psi_-(\vec{\rho},\tau'_-,\bar{\sigma}^+_{z'})=\frac{1}{a/k_0+i\bar{\sigma}^+_{z'}}\exp\left[i\omega_0\bar{\gamma}\left(1\mp\frac{v}{c}\right)\tau'_-\right]\exp\left[-\frac{\rho^2}{a/k_0+i\bar{\sigma}^+_{z'}}\right]\\ \tau'_-=t'+\frac{z'}{c};\ \bar{\sigma}^+_{z'}=\frac{2z'}{k_-},k_-=\bar{\gamma}(1+v/c)k_0.\end{gathered} \tag{5.9}$$

The procedure is analogous to the one followed by Be′langer [21] who showed that certain Gaussian packet-like solutions to the homogeneous scalar wave equation could be explained as monochromatic Gaussian beams observed in a another inertial frame.

## 6. Concluding remarks

A systematic approach to deriving paraxial spatio-temporally localized waves has been introduced. Two distinct classes of such pulsed waves have been studied in detail. The first category deals with paraxial localized waves based on a narrow angular spectrum assumption. The second class is more restricted because it is based on both a narrow angular spectrum and a narrowband temporal spectrum approximation. Both classes allow subluminal, luminal and superluminal paraxial localized waves. For the second class, however, the subluminal and superluminal paraxial localized waves have been shown to arise from subluminal and superluminal Lorentz boosts of two types of general luminal solutions. Finally, the situation has been addressed, whereby exact localized wave solutions to the scalar wave equation are embedded into approximate paraxial solutions.

## Appendix A

The integrand in Eq. (2.28) may be rewritten as

$$B_p(\rho,z,t) = \exp\left[-i(\beta-\alpha)\left(z - v_{ph}t\right)\right] J_0\left[\rho\sqrt{2\frac{v}{c}\left(1+\frac{v}{c}\right)}\sqrt{\beta^2 + \frac{c}{v}\alpha\beta}\right], \qquad \text{(A-1)}$$

where $v_{ph} = c\left[\alpha + (v/c)\beta\right]/(\beta-\alpha)$. As mentioned earlier, this is an axisymmetric paraxial monochromatic Bessel beam solution to the homogeneous 3D scalar wave equation. It differs significantly from the exact monochromatic Bessel beam solution

$$B_e(\rho,z,t) = \exp\left[-ik_z\left(z - \frac{\omega}{k_z}t\right)\right] J_0\left[\rho\sqrt{(\omega/c)^2 - k_z^2}\right]. \qquad \text{(A-2)}$$

In order for the argument of the Bessel function in the latter to be real, one must have the inequality $|\omega|/|k_z| > c$. Assuming $\omega$ and $k_z$ to be positive, this means that the exact Bessel beam propagates along the $+z-$direction with the superluminal speed $v = \omega/k_z$. The situation is much different in Eq. (A-1) Three distinct cases will be considered in detail:

*Case (i):* $\alpha = 0, \beta > 0$

In this case, the paraxial Bessel beam simplifies as follows:

$$B_p(\rho,z,t) = \exp\left[-i\beta\left(z - vt\right)\right] J_0\left[\rho\beta\sqrt{2\frac{v}{c}\left(1+\frac{v}{c}\right)}\right] \qquad \text{(A-3)}$$

It propagates in the positive $z-$direction at any speed $v \in (0,\infty)$ .

*Case (ii):* $\alpha > 0, \beta > 0$

Let $\beta = \mu(c/v)\alpha;\ \mu > 0.$ Then, $v_{ph} = c(1+\mu)/\left[(\mu/\delta)-1\right];\ \delta \equiv v/c > 0.$ For $v_{ph} > 0,$ the inequality $\delta < \mu$ must hold. With these restrictions, one finds that $v_{ph}$ in Eq. (A-1) is subluminal, luminal or superluminal if $\delta <,=,> \mu/(2+\mu)$ , respectively.

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*Case (iii):* $\alpha < 0, \beta > (c/v)|\alpha|$

Let $\beta = \mu(c/v)|\alpha|$; $\mu > 1$. Then, $v_{ph} = c(-1+\mu)/[(\mu/\delta)+1]$. In this case, $v_{ph}$ in Eq. (A-1) is subluminal, luminal or superluminal if $\delta <,=,> \mu/(\mu-2)$; $\mu > 2$, respectively.

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