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Anomaly meltdown

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It is shown that at nonzero temperature it is possible that anomalies in representations of symmetry groups and gauge groups, present at zero temperature, disappear. Several examples are given. Thus the idea that anomalies in baryon currents might have caused the baryon imbalance in the early hot universe needs reconsideration.

I. INTRODUCTION

In quantum mechanics in the Heisenberg picture, a symmetry group G acts by *-automorphisms on the algebra $\mathscr{A}$ of observables. Thus we have a homomorphism τ: $G \to \text{Aut}\, \mathscr{A}$ in which $\tau_g(A)$ is the observable A transformed by the group element g. In the Schrödinger picture we have, for each $g \in G$, a unitary operator U_g implementing the symmetry; thus $\mathscr{A}$ is concretely represented by a representation π on a Hilbert space H, and U_g acts on H and obeys

$$U_g \pi(A) U_g^{-1} = \pi(\tau_g(A)), \quad \forall A \in \mathscr{A}$$

(unless the symmetry is spontaneously broken in which case we restrict to the subgroup of unbroken symmetries).

However, the map $g \to U_g$ is in general only a projective, not a true representation. A spectacular example of this is seen in Galilean-invariant quantum mechanics. There, the six-parameter Abelian group of space translations and Galilean (velocity) transformations is represented by the quantum-mechanical operators $U(\mathbf{a}) = e^{i\mathbf{p}\mathbf{a}}$ and $V(\mathbf{v}) = e^{mi\mathbf{x}\mathbf{v}}$.

If we include in $\mathscr{A}$ unobservable fields like fermions, then gauge groups get represented along with $\mathscr{A}$. In that case, the observables are represented reducibly, and the multipliers between some gauge groups and, e.g., time evolution are called anomalies. The gauge currents j^μ_ν, usually required to be conserved, acquire an anomalous divergence $\partial_\mu j^\mu$ which induces a violation of the law of conservation of the corresponding charge. This idea has been proposed as an explanation of the imbalance of baryons and antibaryons in the universe.

A simple example of an anomaly between energy and momentum is given in Ref. 1. There, a linear spin system in a constant magnetic field h is represented in the state with a domain boundary at some point, say $x = 0$. It is found that the time-evolution operators $U(t)$ exist, as do the (discrete) space translations $V(n)$; but these do not commute, even though the corresponding automorphisms of the algebra do commute. So we have an anomaly. In fact, $U(t)V(n) = e^{ihnt}V(n)U(t)$.

In Sec. II we study the same model at nonzero temperature, and, to our surprise, we find that there is no anomaly: $U(t)$ and $V(n)$ commute in this representation.

Henle[2] showed that in a representation with a properly infinite commutant, we can always represent a group (acting continuously) with no multiplier. We generalize this result in Sec. III to groups acting ultraweakly continuously. Thus there is no multiplier if the commutant is of type II_∞ or III. The latter is the case at nonzero temperature. Thus anomalies melt at positive temperatures, and the model of Sec. II is just one example of this. At infinite temperatures the algebra is of type II_1, and the equilibrium state is the trace. This is invariant under all automorphism, and so by the GNS construction, we get no multipliers. Thus at nonzero temperatures multipliers disappear and symmetry is restored.

In Sec. IV we collect the various results and examples.

II. AN EXAMPLE

We consider the example of Sec. III of Ref. 1, but at nonzero temperature. Thus the algebra, the spin algebra over $\mathbb{Z}$, is $\otimes_{j\in \mathbb{Z}} M_2^{(j)}$ and the time evolution is given by $\tau_t(A) = \otimes_{j\in\mathbb{Z}} e^{i\sigma^3_{(j)}t} A_j e^{-i\sigma^3_{(j)}t}$, where $A = \otimes_{j\in\mathbb{Z}} A_j$, $A_j \in M_2$. The space evolution is given by $(\sigma_n(A))_j = A_{j-n}$.

These automorphisms commute. Now consider this algebra in the representation given by $\otimes_{j\in\mathbb{Z}} \rho_\beta^{(j)} = \omega_\beta$ and $\rho_\beta^{(j)}(A) = \text{Tr}(e^{-h\sigma^3\beta}A)/\text{Tr}(e^{-h\sigma^3\beta})$. This describes the spin chain in a constant magnetic field h in the third direction at the temperature $T = \beta^{-1}$. In this section we put $h = 1$. The ground state is $\lim_{\beta\to\infty} \omega_\beta = \omega_\infty$. The GNS Hilbert space of $\rho_\beta^{(j)}$ is four dimensional. The state ω_β is a space-time invariant state which obeys the KMS condition. The Hilbert space H_ω of the latter coming from GNS construction can be expressed as a von Neumann incomplete infinite tensor product $\otimes_{j\in\mathbb{Z}} \mathbb{C}^4$. Let $(\pi_\beta, H_\omega, \Omega)$ be the GNS representation of this state. Let γ: $\mathscr{A} \to \mathscr{A}$ be the automorphism which reduces to I for operators in $\otimes_{-\infty}^{-1} M_2^{(j)}$ and to $\otimes(\sigma^1 \cdots \sigma^1 \cdots)$ for operators in $\otimes_{j=0}^{\infty} M_2^{(j)}$. Then $\pi_\beta \circ \gamma$ is the representation given by the state

$$\omega_\beta \circ \gamma = \cdots \otimes \rho_\beta^{(-2)} \otimes \rho_\beta^{(-1)} \otimes \sigma^1 \rho_\beta^{(0)} \sigma^1 \otimes \sigma^1 \rho_\beta^{(1)} \sigma^1 \otimes \cdots$$

giving the state

$$\cdots \otimes \uparrow \otimes \uparrow \cdots \underset{-10}{\uparrow\downarrow} \otimes \downarrow \cdots$$

warmed up to temperature $1/\beta$. Let Ω_0 be the cyclic vector of the state $\omega_\beta \circ \gamma$. At zero temperature, the implementing operators of time and space transformations do not commute. But in $\pi_\omega \circ \gamma$ we can construct commuting implemen-

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ters, i.e., the anomaly has gone. They are: $U(t)$ given by the GNS construction from the invariant state; $V(n)$: restricted to Ω_0 is given by the operator (n positive)

$$\cdots \otimes I \otimes \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}} \otimes I \cdots .$$

The first V, given below, is at the position zero, and is unitarily equivalent to the representation of σ^1 on $\mathbb{C}^4$. Thus $V(n)$ shifts the domain wall in Ω_0 n steps to the right.

Similarly one argues for n negative.

$U(t)$ restricted to Ω_0 is given by $\cdots \otimes U_t \otimes U_t \otimes U_t \cdots$ (simply by tensoring up U_t). V is given as a matrix

$$V = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and U_t is given by

$$U_t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e^{2it} & 0 & 0 \\ 0 & 0 & e^{-2it} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The construction of U_t involves the modular antiunitary in representation of $M_2(\mathbb{C})$ by the state ρ_β.[3] $U_t = e^{iHt}$, where

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

H is defined by ρ_3 as

$$H = \pi(\sigma^3) - \pi'(\sigma^3),$$

where $\pi(A)|\rho_\beta^{1/2}\rangle = |A\rho_\beta^{1/2}\rangle$, and $\pi'(A)|\rho_\beta^{1/2}\rangle = |\rho_\beta^{1/2} A^*\rangle = J\pi(A)J$, for A real, $(A \in M_2(\mathbb{C}))$, where

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In $\otimes_j \mathbb{C}_j^4$, Ω_0 is given from the GNS construction as $\otimes_{-\infty}^{j=0} \Omega_1 \otimes_{j=1}^{\infty} \Omega_2$, where $\Omega_1 = (1/\alpha, 0, 0, 1/\beta)$, and $\Omega_2 = V\Omega_1$; $1/\alpha = (s_1)^{1/2}$, $1/\beta = (s_2)^{1/2}$. The parameters s_1 and s_2 come from

$$\rho_\beta = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix};$$

s_1 and s_2 are positive and add up to 1. One sees, that $U(t)$ and $V(n)$ commute on Ω_0, so by the properties of the GNS construction and the fact that the automorphisms commute, they commute.

As another example, consider a Boson field algebra $\mathscr{A}$ at temperature $1/\beta$. To any canonical pair of coordinates p, q of the field, the group $\mathbb{R}^2$ acts on A by $(p,q) \to (p+x, q+y)$, $(x, y \in \mathbb{R}^2)$. In the Fock representation this automorphism is spatial and is implemented by the Weyl operators of the mode in question: $W(x,y) = e^{i(py - qx)}$. This provides a projective representation of $\mathbb{R}^2$. But at non-zero temperature, the same automorphism can be implemented by

$$W(x,y) = \pi_\beta(W(x,y)) \otimes J(\pi_\beta(W(x,y)))J^{-1}.$$

By the general KMS theory, the latter factor lies in $\pi_\beta(A)'$ and so does not interfere with the implementation, which is done by the first factor. The presence of J however, ensures that $J(\pi_\beta(W(x,y)))J^{-1}$ represents the Heisenberg commutation rules with a change of sign of i, and so the generators of $W(x,y)$ commute; if we regard canonical variables as generators of automorphisms, then they can be represented by classical (commuting) variables at $1/\beta > 0$. If, however, these variables are observable in their own right, we use π, not $\pi \otimes J\pi J^{-1}$ to represent them. Currents, typically, are defined only as generators of transformations, and so we have a mechanism for the cancellation of anomalies.

III. HENLE'S THEOREM

In 1970, J. Henle proved a theorem that can be applied directly to the example of Sec. II; it implies that in $\pi_\beta \circ \gamma$ there exists a true representation of the automorphism group of space-time.

Theorem (Henle[2]): Let $\mathscr{A}$ be a W^*-algebra $\subseteq B(H)$, G be a locally compact group, H a separable Hilbert space. Moreover, let $\mathscr{A}$ have a properly infinite commutant, and let τ: $G \to \text{Aut}\, \mathscr{A}$ be a norm-continuous homomorphism of G into Aut $\mathscr{A}$, i.e., $\|(\tau_g - I)A\| \to 0$ as $g \to e_G$. Then there exists a strongly continuous representation of G, U_g, on H, implementing τ_g.

Since in the example of Sec. II, τ_t and $\sigma(n)$ are norm continuous, the existence of commuting $U(t)$, $V(n)$ is ensured. We have found just one choice.

The norm continuity of Henle's theorem is only rarely true in models. On the other hand, Henle does not assume (but proves) that τ_g is spatial. In our approach, as we are only interested in g such that τ_g is spatial, there is room to weaken the continuity assumption to a representation-dependent one. Thus we prove the following.

Modified Henle Theorem: Let $\mathscr{A}$ be a W^* algebra with a properly infinite commutant, acting on a separable Hilbert space. Let G be a locally compact group of automorphisms of $\mathscr{A}$ implemented by $U_g \in \text{Aut}\, H$. Suppose U_g is weakly measurable. Then there exist implementing operators U_g' without multiplier.

Proof: Let $K = L^2(G,H)$ and define for each $A \in \mathscr{A}$ the operators $\tau(A)$ on $L^2(G,H)$ by

$$(\tau(A))(\psi)(g) := \tau_g(A)\psi(g) = U_g A U_g^{-1} \psi(g)$$

(see M. Henle[2]). The right-hand side of the latter is measurable in g by assumption. The operator $\tau(A)$ is bounded:

$$\|\tau(A)\| = \operatorname*{esssup}_g \|\tau_g(A)\| = \|A\|.$$

The map $A \to \tau_g(A)$ is normal: If E_n is an increasing net of projections with upper bound E, then $\tau_g(E_n) = U_g E_n U_g^{-1}$ is an increasing net with upper bound $U_g E U_g^{-1} = (\tau(E))(g)$. Hence, the map $A \to \tau(A)$ is normal. It implies that $\tau(\mathscr{A})$ is a W^*-algebra. Then on $L^2(G,H)$ define

$$(V_g\psi)(h) := \psi(g^{-1}h),$$

a unitary continuous representation. The map τ is spatial: $\exists\, R: H \to L^2(G,H)$ such, that $\tau(A) = RAR^{-1}$ for, as Henle notes, any isomorphism between W^*-algebras, both with infinite commutants, is spatial. Then $R^{-1}V_gR$ is a continuous true representation of G on H and implements τ_g, Q.E.D.

As the last example, consider a spin chain on $\mathbb{Z}$ in a magnetic field h in the z direction, with an additional term in the Hamiltonian due to spin interaction of nearest neighbors:

$$H_J = -J \sum_{i=1}^{\infty} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_{i+1}.$$

One can see that there is an anomaly at $\beta = \infty$ (same as if $J = 0$), but that it goes away at $0 < \beta < \infty$. The last point follows from Henle's theorem, since the automorphisms generated from $H = H_0 + H_J$ are strongly continuous[4] ($H_0 = h \sum_{i=1}^{\infty} \sigma_i^{(3)}$), and commute with space translations. We now consider the representation at zero temperature. Clearly, H_0 and H_J commute and give spatial automorphisms in the representation (π, Ω, H_Ω) given by the state $\omega_\infty \circ \gamma$, where γ is the automorphism given in Sec. II. Let $U_0(t)$, $U_J(t)$ be the one-parameter groups implementing these automorphisms. It was shown in Ref. 1 that $U_0(t)V(n) = e^{inht}V(n)U_0(t)$.

Define $U(t) := U_0(t)U_J(t)$. This will have the anomaly with $V(n)$ provided $V(n)$ and $U_J(t)$ commute. The implementation of $-J\sum_{i=1}^{\infty} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_{i+1}$ is by $H_J = 3J \sum_{i=1}^{\infty} P_{i,i+1}$, where $P_{i,i+1}$ is the projection of $\mathbb{C}^2 \otimes \mathbb{C}^2$ onto the singlet state $2^{-1/2}(\uparrow \otimes \downarrow - \downarrow \otimes \uparrow)$. We see that $\mathscr{H}_J$ commutes with $V(n)$; to be precise $V(1)\mathscr{H}_J\Omega = \mathscr{H}_J V(1)\Omega$. So $V(n)\mathscr{H}_J\Psi = \mathscr{H}_J V(n)\Psi$, where $\Psi \in \mathscr{N} = \text{span}\,(A\Omega; A \in \textit{local algebra})$. $\mathscr{N}$ is dense in $\mathscr{H}_\Omega$, invariant under $V(n)$ and $\mathscr{H}_J$. Moreover, each $\Psi \in \mathscr{N}$ is an analytic vector for $\mathscr{H}_J$, so $\mathscr{H}_J$ generates a unique group of unitaries $U_J(t)$ that commutes with $V(n)$. But that means $U(t)V(n) = V(n)U(t)e^{inht}$.

IV. CONCLUSION

Suppose we have an algebra of fields $\mathscr{F}$ on which acts a locally compact gauge group $\mathscr{G}$. The $\mathscr{G}$-invariant elements of $\mathscr{F}$ comprise the observable algebra $\mathscr{A}$. Suppose G, a symmetry group, acts on $\mathscr{A}$, and each $g \in G$ is not spontaneously broken, so is spatial and given by U_g. Obviously G commutes with $\mathscr{G}$ in its action on $\mathscr{A}$, as $\mathscr{G}$ acts trivially. But even if each gauge transformation $\in \mathscr{G}$ is spatial, say $W(u)$, $u \in \mathscr{G}$, it is not always true that $W(u)$ commute with $U(g)$, $g \in G$. In particular, an anomaly arises if $W(u)$ does not commute with time evolution. But at positive temperature, we have shown that, by modifying U_g we can eliminate any multipliers of G. We interpret this as predicting that anomalies melt away if $T > 0$; we can regard $\mathscr{G} \times G$ as the group acting on $\mathscr{A}$, and so find a true representation. In particular, Galilean quantum mechanics becomes classical.

Thus, as mentioned in the abstract, the idea of Kuzmin, Rubakov, and Shaposhnikov that anomalies in baryon currents might have caused the baryon imbalance in the early hot universe needs reconsideration.[5]

The only way out of this meltdown is if the gauge group is spontaneously broken.

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