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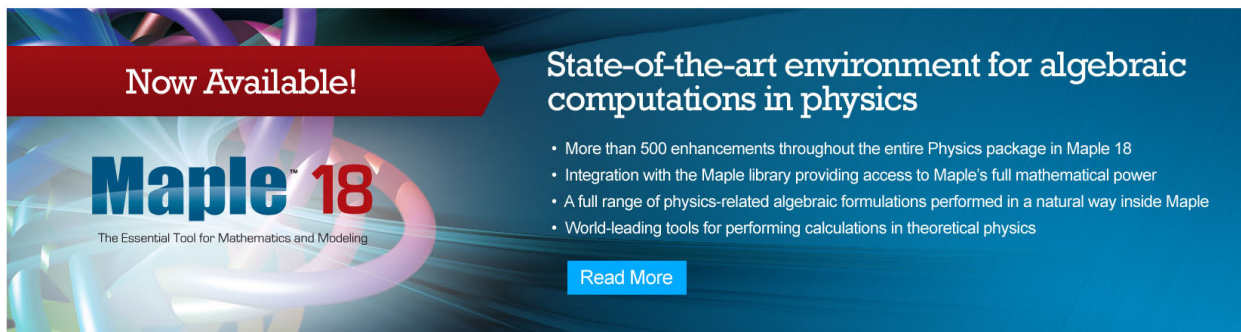
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Cauchy problem for the linearized version of the Generalized Polynomial KdV equation

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In the present paper results about the "Generalized Polynomial Korteweg-de Vries equation" (GPKdV) are obtained, extending the ones by Sachs [SIAM J. Math. Anal. 14, 674 (1983)] for the Korteweg-de Vries (KdV) equation. Namely, the evolution of the so-called "prolonged squared" eigenfunctions of the associated spectral problem according to the linearized GPKdV is proven, the Lax pairs associated with the "prolonged" eigenfunctions as well as "prolonged squared" eigenfunctions are derived, and on the basis of some expansion formulas the Cauchy problem for the linearized GPKdV with a decreasing at infinity initial condition is solved.

I. INTRODUCTION

Gardner, Green, Kruskal, and Miura observed in their classic paper¹ that the squares of the eigenfunctions satisfy the formal adjoint of the linearized KdV equation, and the derivatives of these squares satisfy the linearized KdV equation itself. In Ref. 2, Sachs used this result and solved the Cauchy problem for the linearized KdV by applying an expansion formula for the squares of the eigenfunctions of the Schrödinger equation.

The present paper is a generalization of these results for a wider class of equations (GPKdV reduces to KdV when $n = 1, N = 1$). It makes use of the expansion formulas obtained in Ref. 3.

Analogous questions have been considered in Ref. 4 for the Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy. The problem of deriving orthogonality relations and expansion formulas for the Schrödinger equation^{2,5,6} and other similar equations^{3,7} has been treated by many authors.

Let us consider the spectral problem

$$l\psi \equiv -\psi''(x, \lambda) + \left[\sum_{r=0}^{N-1} \lambda^r v_r(x) \right] \psi(x, \lambda) = \lambda^N \psi(x, \lambda), \quad (1)$$

$$\lambda \in \mathbb{C}, \quad v_r \in \mathcal{G}, \quad ' = \partial/\partial x,$$

where \mathcal{G} is the Schwartz space of functions $v: \mathbb{R} \rightarrow \mathbb{C}$ and N is a natural number. As usual, the functions v_r, ψ are assumed to depend also on a "time variable" t in connection with the evolution equations associated with (1).

Equation (1) is a generalization of the Schrödinger equation ($N = 1$) and we will call it a polynomial Schrödinger equation. It is known to be the spectral problem associated with a certain class of nonlinear evolution equations (here referred to as GPKdV) solvable⁸ by the inverse scattering method. Strictly speaking, the associ-

ated linear problem is not (1) but, according to the Lax pair for GPKdV (Theorem 1), the equivalent matrix equation

$$Lf = \lambda f \quad (2)$$

with L independent of λ , namely,

$$L = \left(\begin{array}{c|cccc} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ -\partial_{xx} + v_0 & v_1 & v_2 & \dots & v_{N-1} \end{array} \right), \quad f = \begin{pmatrix} \psi \\ \lambda\psi \\ \vdots \\ \lambda^{N-1}\psi \end{pmatrix}.$$

Here $f = \psi(x, \lambda)\sigma(\lambda)$, $\sigma(\lambda) = (1, \lambda, \dots, \lambda^{N-1})^T$, $T =$ transposition, is called a prolonged solution, ψ being a solution of (1). Also, if ϕ, ψ are solutions of (1), then the product $\phi\psi$ is a solution of

$$\left[-\frac{1}{4}\partial_{xxx} + \sum_{r=0}^{N-1} \lambda^r j(v_r) \right] (\phi\psi) = \lambda^N \partial_x(\phi\psi), \quad (3)$$

$$j(v_r) \equiv v_r \partial_x + \frac{1}{2} v_{r,x}, \quad (\cdot)_x \equiv \partial_x(\cdot),$$

or, equivalently, $MF = \lambda JF$ where

$$J = \begin{pmatrix} -j(v_1) & \dots & -j(v_{N-1}) & \partial_x \\ -j(v_2) & \ddots & \partial_x & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \partial_x & 0 & \dots & 0 \end{pmatrix},$$

$$M = \left(\begin{array}{c|cccc} -\frac{1}{4}\partial_{xxx} + j(v_0) & 0 & 0 & \dots & 0 \\ 0 & -j(v_2) & -j(v_3) & \dots & \partial_x \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & -j(v_{N-1}) & \partial_x & \ddots & \vdots \\ 0 & \partial_x & 0 & \dots & 0 \end{array} \right),$$

and $F(x,\lambda) = \phi(x,\lambda)\psi(x,\lambda)\sigma(\lambda)$ is the so-called prolonged squared solution. For convenience we define a product “ $f \circ g$ ” of $f = \psi\sigma$ and $g = \phi\sigma$ as $f \circ g = \phi\psi\sigma$, so that $F = f \circ g$.

Also, $M = \Lambda J$ with

$$\Lambda = \left(\begin{array}{ccc|c} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{array} \right) \left(\begin{array}{c} (-\frac{1}{4}\partial_{xxx} + j(v_0))\partial_x^{-1} \\ j(v_1)\partial_x^{-1} \\ j(v_2)\partial_x^{-1} \\ \vdots \\ j(v_{N-1})\partial_x^{-1} \end{array} \right), \partial_x^{-1} = \int_{-\infty}^x \cdot$$

$$\Lambda^* = \left(\begin{array}{ccc|ccc} 0 & & & 1 & 0 & \dots & 0 \\ 0 & & & 0 & 1 & \ddots & \vdots \\ \vdots & & & \vdots & \ddots & \ddots & 0 \\ 0 & & & 0 & \dots & 0 & 1 \\ \hline \partial_x^{-1}(-\frac{1}{4}\partial_{xxx} + j(v_0)) & & & \partial_x^{-1}j(v_1) & \partial_x^{-1}j(v_2) & \dots & \partial_x^{-1}j(v_{N-1}) \end{array} \right)$$

Using the operator Λ we can⁸ write the GPKdV equations in the form

$$v_t = \Omega(\Lambda)v_x, \quad v(x,t) = (v_0, v_1, \dots, v_{N-1})^T,$$

$$\Omega(\mu) = a_0\mu^n + a_1\mu^{n-1} + \dots + a_n. \tag{6}$$

Here Ω is used for generality. Often only the case $\Omega(\mu) = \mu^n$, $n = 0, 1, 2, \dots$, is considered, referred to as a hierarchy of evolution equations. For $N = 1$ this is the KdV hierarchy and for $N = 2$ it is the Jaulent–Miodek hierarchy.

Despite the presence of ∂_x^{-1} in Λ , (6) represents a system of differential equations. Indeed, as a consequence of Appendix B and formulas (3.12) and (3.18) in Ref. 8, we conclude that the functions

$$p_0 = 2, \quad p_m = \int_{-\infty}^x c_{m-1, N-1}(s) ds, \quad m \geq 1,$$

$$\text{where } \mathbf{c}_m \equiv \begin{pmatrix} c_{m,0} \\ \vdots \\ c_{m, N-1} \end{pmatrix} = \Lambda^m v_x,$$

depend polynomially on v_r , $r = \overline{1, N-1}$, and their derivatives only.

Therefore, $MF = \lambda JF$ can be represented as

$$\Lambda(JF) = \lambda(JF), \tag{4}$$

where $F = \phi\psi\sigma$ is a prolonged squared solution, $F \xrightarrow{x \rightarrow \pm \infty} 0$.

Moreover, the following is true:

$$\Lambda^*F = \lambda F \tag{5}$$

with

II. LAX PAIRS AND TIME EVOLUTION OF EIGENFUNCTIONS

Let us introduce the matrix

$$C_n = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ c_{n,0} & \dots & c_{n, N-1} \end{pmatrix}. \tag{7}$$

For simplicity the following theorem will consider only the case $a_0 = 1$, $a_1 = \dots = a_n = 0$, the general result being written as a corollary.

Theorem 1: The GPKdV equations (6) for $a_0 = 1$, $a_1 = \dots = a_n = 0$ have the Lax representation

$$L_t = A_n L - L A_n, \tag{8}$$

where

$$A_n = \sum_{i=0}^n \left(-\frac{1}{4} p_{i,x} + \frac{1}{2} p_i \partial_x \right) L^{n-i}.$$

Proof: Using the approach in Ref. 9 (p. 216) we will show first that

$$\begin{aligned} & \left(-\frac{1}{4} p_{i,x} + \frac{1}{2} p_i \partial_x \right) L - L \left(-\frac{1}{4} p_{i,x} + \frac{1}{2} p_i \partial_x \right) \\ &= C_i - C_{i-1} L, \quad i \geq 0, \quad C_{-1} = 0. \end{aligned} \tag{9}$$

Indeed, at $i = 0$ we obtain

$$\partial_x L - L\partial_x = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ v_{0,x} & \dots & v_{N-1,x} \end{pmatrix} = C_0.$$

If $i > 0$, then

$$\begin{aligned} & \left(-\frac{1}{4}p_{i,x} + \frac{1}{2}p_i\partial_x \right) L - L \left(-\frac{1}{4}p_{i,x} + \frac{1}{2}p_i\partial_x \right) \\ &= \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ b_0 & \dots & b_{N-1} \end{pmatrix}, \end{aligned}$$

where, for $r = \overline{1, N-1}$,

$$b_r = \left(-\frac{1}{4}p_{i,x} + \frac{1}{2}p_i\partial_x \right) v_r - v_r \left(-\frac{1}{4}p_{i,x} + \frac{1}{2}p_i\partial_x \right) = \frac{1}{2}p_i v_{r,x}$$

and (using the above at $r = 0$)

$$\begin{aligned} b_0 &= \left(-\frac{1}{4}p_{i,x} + \frac{1}{2}p_i\partial_x \right) (-\partial_{xx} + v_0) \\ &\quad - (-\partial_{xx} + v_0) \left(-\frac{1}{4}p_{i,x} + \frac{1}{2}p_i\partial_x \right) \\ &= \frac{1}{2}p_i v_{0,x} + \left[\frac{1}{4}p_{i,x}\partial_{xx} - \frac{1}{2}p_i\partial_{xxx} - \frac{1}{4}\partial_{xx}p_{i,x} + \frac{1}{2}\partial_{xx}p_i\partial_x \right] \\ &= \frac{1}{2}p_i v_{0,x} + (-\frac{1}{4}p_{i,xxx} + p_{i,x}\partial_{xx}). \end{aligned}$$

On the other hand,

$$C_i - C_{i-1}L = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ d_0 & \dots & d_{N-1} \end{pmatrix}$$

where, due to $c_i = \Lambda c_{i-1}$, we have

$$\begin{aligned} d_r &= [c_{i-1,r-1} + j(v_r)\partial_x^{-1}c_{i-1,N-1}] \\ &\quad - [c_{i-1,r-1} + c_{i-1,N-1}v_r] \\ &= \frac{1}{2}v_{r,x}\partial_x^{-1}c_{i-1,N-1} = b_r, \quad r = \overline{1, N-1}, \\ d_0 &= \left(-\frac{1}{4}\partial_{xxx} + j(v_0) \right) \partial_x^{-1}c_{i-1,N-1} - c_{i-1,N-1} \\ &\quad \times (-\partial_{xx} + v_0) \\ &= \frac{1}{2}v_{0,x}(\partial_x^{-1}c_{i-1,N-1}) - \frac{1}{4}c_{i-1,N-1,xx} \\ &\quad + c_{i-1,N-1}\partial_{xx} = b_0. \end{aligned}$$

Finally, due to (9) we find

$$\begin{aligned} A_n L - L A_n &= \sum_{i=0}^n \left[\left(-\frac{1}{4}p_{i,x} + \frac{1}{2}p_i\partial_x \right) \right. \\ &\quad \times L - L \left(-\frac{1}{4}p_{i,x} + \frac{1}{2}p_i\partial_x \right) \Big] L^{n-i} \\ &= \sum_{i=0}^n (C_i - C_{i-1}L) L^{n-i} = C_n. \end{aligned}$$

Corollary: The nonlinear evolution equations (6) can be represented in the form

$$L_t = AL - LA, \quad A = \sum_{i=0}^n a_i A_{n-i} \tag{10}$$

Now we will find the time evolution of the solutions of (1), the prolonged and the prolonged squared solutions provided the potential $v(x,t)$ evolves according to (6).

Lemma 1: If $f(x,\lambda)$ and $v(x)$ satisfy (2) and (6), respectively, then $f_t - Af$ is another solution of (2).

Proof: Differentiating $Lf = \lambda f$ with respect to t yields $L_t f + L f_t = \lambda f_t$. Using (10) and (2) we obtain $L(f_t - Af) = \lambda(f_t - Af)$.

We will be especially interested in the partial case $f_t - Af = 0$.

Lemma 2: If $f(x,\lambda) = \psi\sigma$ is a solution of (2) and $f_t = Af$, then ψ evolves according to

$$\psi_t = \sum_{i=0}^n \left(-\frac{1}{4}p_{i,x} + \frac{1}{2}p_i\partial_x \right) (a_0\lambda^{n-i} + \dots + a_{n-i})\psi. \tag{11}$$

(This result was obtained independently of Ref. 10.)

Proof: Using change in order of summation we obtain

$$\begin{aligned} f_t = Af &= \left[a_0 \sum_{i=0}^n \left(-\frac{1}{4}p_{i,x} + \frac{1}{2}p_i\partial_x \right) L^{n-i} \right. \\ &\quad + a_1 \sum_{i=0}^{n-1} \left(-\frac{1}{4}p_{i,x} + \frac{1}{2}p_i\partial_x \right) L^{n-1-i} \\ &\quad + \dots + a_n \sum_{i=0}^0 \left(-\frac{1}{4}p_{i,x} + \frac{1}{2}p_i\partial_x \right) L^{0-i} \Big] f \\ &= \sum_{i=0}^n \left(-\frac{1}{4}p_{i,x} + \frac{1}{2}p_i\partial_x \right) \\ &\quad \times [a_0 L^{n-i} + a_1 L^{n-1-i} + \dots + a_{n-i}] f \\ &= \sum_{i=0}^n \left(-\frac{1}{4}p_{i,x} + \frac{1}{2}p_i\partial_x \right) \\ &\quad \times (a_0\lambda^{n-i} + \dots + a_{n-i}) f. \end{aligned}$$

Theorem 2: If the solutions $f = \psi\sigma$, $g = \phi\sigma$ of (2) satisfy the evolution equation $y_t = Ay$, then

(a) the prolonged squared solution $F = f \circ g$ evolves according to

$$F_t = \sum_{i=0}^n \left(-\frac{1}{2} p_{i,x} + \frac{1}{2} p_i \partial_x \right) \times (a_0 \lambda^{n-i} + a_1 \lambda^{n-i-1} + \dots + a_{n-i}) F; \quad (12)$$

(b) if, moreover, (6) holds, then JF evolves according to

$$(JF)_t = BJF, \quad B = \sum_{i=0}^n a_i B_{n-i} \quad (13)$$

where

$$B_n = \sum_{i=0}^n T_i \Lambda^{n-i}, \quad T_0 = \partial_x$$

$$T_m = (p_{m,x} + \frac{1}{2} p_m \partial_x) + P_m$$

$$P_m = - \begin{pmatrix} 0 & \dots & 0 & j(c_{m-1,0}) \partial_x^{-1} \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & j(c_{m-1,N-1}) \partial_x^{-1} \end{pmatrix}, \quad m > 0.$$

Proof: (a) As a consequence of Lemma 2,

$$F_t = (\psi\phi_t + \phi\psi_t)\sigma$$

$$= \sum_{i=0}^n \left[\psi \left(-\frac{1}{4} p_{i,x} + \frac{1}{2} p_i \partial_x \right) (a_0 \lambda^{n-i} + \dots + a_{n-i}) \phi + \phi \left(-\frac{1}{4} p_{i,x} + \frac{1}{2} p_i \partial_x \right) \times (a_0 \lambda^{n-i} + \dots + a_{n-i}) \psi \right] \sigma$$

$$= \sum_{i=0}^n \left[\left(-\frac{1}{2} p_{i,x} + \frac{1}{2} p_i \partial_x \right) (a_0 \lambda^{n-i} + \dots + a_{n-i}) \phi \psi \right] \sigma.$$

(b) We will prove the statement at $a_0 = 1$, $a_1 = \dots = a_n = 0$. In the general case it follows from the linearity of the right-hand sides of the equalities used.

Let us denote by $J'[w]$ the matrix operator $\delta J|_{\delta v = w}$ (Here δ is the variational derivative, $\delta\Phi(v) = (d/d\epsilon) \times \Phi(v + \epsilon(\delta v))|_{\epsilon=0}$, where Φ is a functional of v or an operator depending on v .) Then for $n = 0$ we have

$$(JF)_t = J'[v_x]F + JF_t = J'[v_x]F + J\partial_x F = \partial_x JF.$$

Suppose it is true for some non-negative n . Then

$$B_n JF = J'[v_t]F + JF_t$$

$$= J'[c_n]F + J \left(\sum_{i=0}^n \left(-\frac{1}{2} p_{i,x} + \frac{1}{2} p_i \partial_x \right) \lambda^{n-i} \right) F$$

and therefore for $n + 1$ we obtain [using (7), (4), (5)]

$$(JF)_t = J'[c_{n+1}]F + J \left(\sum_{i=0}^{n+1} \left(-\frac{1}{2} p_{i,x} + \frac{1}{2} p_i \partial_x \right) \lambda^{n+1-i} \right) F$$

$$= J'[c_{n+1}]F + J \left(-\frac{1}{2} p_{n+1,x} + \frac{1}{2} p_{n+1} \partial_x \right) F + \lambda \{ B_n JF - J'[\Lambda^n v_x]F \}$$

$$= J'[\Lambda c_n]F + J \left(-\frac{1}{2} p_{n+1,x} + \frac{1}{2} p_{n+1} \partial_x \right) F + B_n \Lambda JF - J'[c_n] \Lambda^* F.$$

In order to calculate this we need to use the linearity of j , namely $j(c_{n,r-1} + j(v_r)p_{n+1}) = j(c_{n,r-1}) + j(j(v_r)p_{n+1})$, as well as

$$j(j(v_r)p_{n+1}) + j(v_r) \left(-\frac{1}{2} p_{n+1,x} + \frac{1}{2} p_{n+1} \partial_x \right) = (p_{n+1,x} + \frac{1}{2} p_{n+1} \partial_x) j(v_r). \quad (14)$$

Now we find

$$(JF)_t = B_n \Lambda JF + (p_{n+1,x} + \frac{1}{2} p_{n+1} \partial_x) JF$$

$$= (B_n \Lambda + T_{n+1}) JF = B_{n+1} JF$$

$$= \begin{pmatrix} j(c_{n,0}) & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ j(c_{n,N-1}) & 0 & \dots & 0 \end{pmatrix} F$$

Lemma 3: The following identity holds:

$$\Lambda T_m - T_m \Lambda = P_{m+1} - P_m \Lambda, \quad m \geq 0. \quad (15)$$

Proof: We express the left-hand side as

$$\Lambda T_m - T_m \Lambda = -P_m \Lambda + [\Lambda T_m - (p_{m,x} + \frac{1}{2} p_m \partial_x) \Lambda]$$

$$= -P_m \Lambda + \begin{pmatrix} 0 & \dots & 0 & s_0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & s_{N-1} \end{pmatrix},$$

where for $m > 0, r = \overline{1, N-1}$, we have, due to (14),

$$\begin{aligned}
 s_r &= j(v_r)\partial_x^{-1}(p_{m,x} + \frac{1}{2}p_m\partial_x) - (p_{m,x} + \frac{1}{2}p_m\partial_x)j(v_r)\partial_x^{-1} - j(c_{m-1,r-1})\partial_x^{-1} - j(v_r)\partial_x^{-1}j(c_{m-1,N-1})\partial_x^{-1} \\
 &= j(v_r)\partial_x^{-1}(p_{m,x} + \frac{1}{2}p_m\partial_x) - [j(j(v_r)p_m) + j(v_r)(-\frac{1}{2}p_{m,x} + \frac{1}{2}p_m\partial_x)]\partial_x^{-1} - j(c_{m-1,r-1})\partial_x^{-1} - j(v_r)\partial_x^{-1}j(p_{m,x})\partial_x^{-1} \\
 &= -[j(j(v_r)p_m) + j(c_{m-1,r-1})]\partial_x^{-1} + j(v_r)\partial_x^{-1}[\frac{1}{2}p_m\partial_{xx} + p_{m,x}\partial_x + \frac{1}{2}\partial_x p_{m,x} - \frac{1}{2}\partial_x p_m\partial_x - p_{m,x}\partial_x - \frac{1}{2}p_{m,xx}]\partial_x^{-1} \\
 &= -j(j(v_r)p_m + c_{m-1,r-1})\partial_x^{-1} = -j(c_{m,r})\partial_x^{-1}.
 \end{aligned}
 \tag{16}$$

Now we use (16) for $r = 0$ to obtain

$$\begin{aligned}
 s_0 &= [-\frac{1}{4}\partial_{xxx} + j(v_0)]\partial_x^{-1}(p_{m,x} + \frac{1}{2}p_m\partial_x) \\
 &\quad - (p_{m,x} + \frac{1}{2}p_m\partial_x)[- \frac{1}{4}\partial_{xxx} + j(v_0)]\partial_x^{-1} \\
 &\quad - [-\frac{1}{4}\partial_{xxx} + j(v_0)]\partial_x^{-1}j(c_{m-1,N-1})\partial_x^{-1} \\
 &= -j(j(v_0)p_m)\partial_x^{-1} - \frac{1}{4}[\partial_{xx}(p_{m,x} + \frac{1}{2}p_m\partial_x) \\
 &\quad - (p_{m,x} + \frac{1}{2}p_m\partial_x)\partial_{xx} - \partial_{xx}(\frac{1}{2}p_{m,xx} + p_{m,x}\partial_x)\partial_x^{-1}] \\
 &= -j(j(v_0)p_m - \frac{1}{4}p_{m,xxx})\partial_x^{-1} - \frac{1}{4}[j(p_{m,xxx})\partial_x^{-1} \\
 &\quad + \frac{1}{2}p_{m,xx}\partial_x - \frac{1}{2}\partial_{xx}p_{m,xx}\partial_x^{-1}] \\
 &= -j(c_{m,0})\partial_x^{-1}.
 \end{aligned}$$

At $m = 0$ we have

$$\begin{aligned}
 T_0 &= \partial_x, \quad P_0 = 0, \\
 s_r &= j(v_r)\partial_x^{-1}\partial_x - \partial_x j(v_r)\partial_x^{-1} \\
 &= (j(v_r)\partial_x - \partial_x j(v_r))\partial_x^{-1} \\
 &= -j(v_{r,x})\partial_x^{-1} = -j(c_{0,r})\partial_x^{-1}, \quad r = \overline{1, N-1},
 \end{aligned}$$

$$\begin{aligned}
 s_0 &= [-\frac{1}{4}\partial_{xxx} + j(v_0)]\partial_x^{-1}\partial_x - \partial_x[-\frac{1}{4}\partial_{xxx} \\
 &\quad + j(v_0)]\partial_x^{-1} = -j(v_{0,x})\partial_x^{-1} = -j(c_{0,0})\partial_x^{-1}.
 \end{aligned}$$

Corollary 1: For each $n \geq 0$ we have

$$\Lambda B_n - B_n \Lambda = P_{n+1}. \tag{17}$$

Proof: By changing the order of summation we obtain

$$\begin{aligned}
 \Lambda B_n - B_n \Lambda &= \Lambda \left(\sum_{i=0}^n T_i \Lambda^{n-i} \right) - \left(\sum_{i=0}^n T_i \Lambda^{n-i} \right) \Lambda \\
 &= \sum_{i=0}^n (\Lambda T_i - T_i \Lambda) \Lambda^{n-i} \\
 &= \sum_{i=0}^n (P_{i+1} - P_i \Lambda) \Lambda^{n-i}.
 \end{aligned}$$

Corollary 2: If (6) is satisfied, then the following Lax representation holds:

$$\Lambda_t = B \Lambda - \Lambda B. \tag{18}$$

Proof: Now we use Corollary 1 to obtain

$$\begin{aligned}
 B \Lambda - \Lambda B &= \sum_{i=0}^n a_i (B_{n-i} \Lambda - \Lambda B_{n-i}) \\
 &= - \sum_{i=0}^n a_i P_{n-i+1} \\
 &= \sum_{i=0}^n a_i (\delta \Lambda) |_{\delta v = \Lambda^{n-i} v_x} \\
 &= (\delta \Lambda) |_{\delta v = \Omega(\Lambda) v_x} = \Lambda_r.
 \end{aligned}$$

Corollary 3: The operator B_n can also be represented as

$$B_n = \sum_{i=0}^n \Lambda^{n-i} (T_i - P_i).$$

Proof: For $n = 0$ it is trivial. Let it be true for some n . Then

$$\begin{aligned}
 B_{n+1} &= B_n \Lambda + T_{n+1} \\
 &= \left\{ \sum_{i=0}^n \Lambda^{n-i} (T_i - P_i) \Lambda \right\} + T_{n+1} \\
 &= \left\{ \sum_{i=0}^n \Lambda^{n-i} [\Lambda (T_i - P_i) + (\Lambda P_i - P_{i+1})] \right\} \\
 &\quad + T_{n+1} = \left\{ \sum_{i=0}^n \Lambda^{n-i} [\Lambda (T_i - P_i)] \right\} \\
 &\quad + (T_{n+1} - P_{n+1}).
 \end{aligned}$$

Theorem 3: Let the conditions in Theorem 2 hold [including the one in (b)]. Then JF satisfies the linearized GPKdV equation

$$h_t = \delta(\Omega(\Lambda)v_x)|_{\delta v = h} \tag{19}$$

Proof: Again we will prove the statement for $a_0 = 1, a_1 = \dots = a_n = 0$ only. We have to show that

$$\delta(\Lambda^n v_x) = B_n(\delta v). \tag{20}$$

First we will prove (20) in the case when $\delta v \in \mathfrak{G}$. Then it will follow for the general case as well because $\Lambda^n v_x$ depends on v, v_x, \dots , polynomially and therefore does not contain ∂_x^{-1} . As a byproduct we get that B_n is also a differential operator, not containing ∂_x^{-1} .

For $n = 0$ we have

$$\delta(\Lambda^0 v_x) = \partial_x(\delta v) = T_0(\delta v) = B_0(\delta v).$$

If it is true for some $n > 0$, then

$$\begin{aligned}
 \delta(\Lambda^{n+1} v_x) &= (\delta \Lambda) \Lambda^n v_x + \Lambda \delta(\Lambda^n v_x) \\
 &= \begin{pmatrix} 0 & \dots & 0 & j(\delta v_0) \partial_x^{-1} \\ \vdots & & \vdots & \\ 0 & \dots & 0 & j(\delta v_{N-1}) \partial_x^{-1} \end{pmatrix} \begin{pmatrix} c_{n,0} \\ \vdots \\ c_{n,N-2} \\ p_{n+1,x} \end{pmatrix} + \Lambda B_n(\delta v) \\
 &= (p_{n+1,x} + \frac{1}{2} p_{n+1} \partial_x)(\delta v) + (B_n \Lambda + P_{n+1})(\delta v) \\
 &= (B_n \Lambda + T_{n+1})(\delta v) = B_{n+1}(\delta v).
 \end{aligned}$$

III. CAUCHY PROBLEM FOR THE LINEARIZED GPKdV EQUATIONS

Now we will find specific solutions of (1)–(4) for which the statements above apply. Equation (1) has Jost solutions

$$\psi_{\pm}(x, \lambda) \sim e^{\pm ikx} (x \rightarrow \pm \infty), \quad k = \lambda^{N/2}, \quad \text{Im } k \geq 0, \tag{21}$$

analytic^{3,11} in the sectors $\Omega_s = \{\lambda: -(s-1)2\pi/N < \arg \lambda < s2\pi/N\}$, $s = 1, N$ and continuous up to the rays $l_s = \{\lambda: \arg \lambda = s2\pi/N\}$, $s = 0, N$ ($l_0 \equiv l_N$). The same is true for the function

$$\begin{aligned}
 a(\lambda) &= W(\psi_-(x, \lambda), \psi_+(x, \lambda)) \\
 &\equiv (2ik)^{-1} [\psi_-(x, \lambda) \psi'_+(x, \lambda) \\
 &\quad - \psi'_-(x, \lambda) \psi_+(x, \lambda)].
 \end{aligned}$$

In addition, on the rays l_s we have

$$\begin{aligned}
 \psi_+(x, \lambda_{\pm}) &= b_{\pm}(\lambda) \psi_-(x, \lambda_{\pm}) \\
 &\quad + a_{\pm}(\lambda) \psi_-(x, \lambda_{\mp}), \quad \lambda \in l_s, \tag{22}
 \end{aligned}$$

where for any function ϕ we define $\phi(\lambda_{\pm}) = \lim_{\epsilon \rightarrow 0^+} \phi(\lambda \pm i\epsilon), \lambda \in l_s$. Notice that $k_{\pm} = \lambda_{\pm}^{N/2} \gtrless 0$ and $a_{\pm}(\lambda) = a(\lambda_{\pm})$. Also

$$a_+(\lambda) a_-(\lambda) - b_+(\lambda) b_-(\lambda) = 1 \tag{23}$$

which, together with (21), implies

$$\begin{aligned}
 \psi_-(x, \lambda_{\pm}) &= -b_{\mp}(\lambda) \psi_+(x, \lambda_{\pm}) \\
 &\quad + a_{\pm}(\lambda) \psi_+(x, \lambda_{\mp}), \quad \lambda \in l_s, \tag{24}
 \end{aligned}$$

We suppose that $a(\lambda)$ has a finite number of simple zeros $\lambda_l, l = 1, M$. Then

$$\psi_+(x, \lambda_l) = b_l \psi_-(x, \lambda_l). \tag{25}$$

Define $f_{\pm}(x, \lambda) = \psi_{\pm}(x, \lambda) \sigma(\lambda)$. Then $f_{\pm}(x, \lambda)$ also satisfy (22), (24), and (25) as well as $f_{\pm}(x, \lambda) \sim e^{\pm ikx} \sigma(\lambda)$ for $x \rightarrow \pm \infty$.

Lemma 4: If the solutions $f = \psi \sigma, g = \phi \sigma$ of (2) satisfy $y_t = Ay$, then $[W(\psi, \phi)]_t = 0$.

Proof: According to Lemma 2,

$$\begin{aligned}
 [W(\psi, \phi)]_t &= W(\psi, \phi) + W(\psi, \phi_t) \\
 &= \sum_{i=0}^n (a_0 \lambda^{n-i} + \dots + a_{n-i}) \\
 &\quad \times \{W((-\frac{1}{4} p_{i,x} + \frac{1}{2} p_i \partial_x) \psi, \phi) \\
 &\quad + W(\psi, (-\frac{1}{4} p_{i,x} + \frac{1}{2} p_i \partial_x) \phi)\} = 0.
 \end{aligned}$$

Lemma 5: If (6) holds, then

- (a) $f_{\pm,t}(x,\lambda) = Af_{\pm}(x,\lambda) \mp ik\Omega(\lambda)f_{\pm}(x,\lambda)$;
- (b) $e^{\pm ik\Omega(\lambda)t}f_{\pm}(x,\lambda)$ satisfies $y_t = Ay$;
- (c) $a_i(\lambda) = 0$;
- (d) $b_{\pm,t}(\lambda) = -2ik_{\pm}\Omega(\lambda)b_{\pm}(\lambda)$, $\lambda \in I_s$;
- (e) $b_{l,t} = -2ik_{\pm}\Omega(\lambda)b_{\pm}$, where $k_l = \lambda_l^{N/2}$, $\text{Im } k_l \geq 0$;
- (f) the vector function $g_l(x) = f_+(x,\lambda_l) - b_l f_-(x,\lambda_l)$, where $\cdot = \partial_{\lambda}$, is a solution of (2) and $g_{l,t}(x) = Ag_l(x) - ik_{\pm}\Omega(\lambda_l)g_l(x) - (2ik_{\pm}\Omega(\lambda_l)) \cdot f_+(x,\lambda_l) - (2ik_{\pm}\Omega(\lambda_l)) \cdot [g_l(x) + (2ik_{\pm}\Omega(\lambda_l)) \cdot f_+(x,\lambda_l)]$ satisfies $y_t = Ay$.

Proof: Statement (a) follows from Lemma 1, the asymptotics (21) and the fact that $f_{\pm,t}(x,\lambda) \sim o(e^{\pm ikx})\sigma(\lambda)$ as $x \rightarrow \pm \infty$, as well as $p_i(x) \in \mathcal{O}$ for $i > 0$ and $p_0(x) = 2$.

Statement (b) is a corollary of (a), and (c) and (d) follow from (b), Lemma 4 and the formulas

$$a(\lambda) = (2ik)^{-1}W(\psi_-(x,\lambda)e^{-ik\Omega(\lambda)t}, \psi_+(x,\lambda)e^{ik\Omega(\lambda)t})$$

$$b_{\pm}(\lambda)e^{2ik_{\pm}\Omega(\lambda)t} = (2ik_{\pm})^{-1}W(\psi_-(x,\lambda_{\mp})e^{-ik_{\mp}\Omega(\lambda)t}, \psi_+(x,\lambda_{\pm})e^{ik_{\pm}\Omega(\lambda)t})$$

Next, (e) can be obtained by differentiating (25) with respect to t and using (a). Notice that $\lambda_{l,t} = 0$ due to (c).

For (f), in order to show that $g_l(x)$ solves (2) we differentiate (2) with respect to λ and use (25) again.

This fact is not surprising since $g_l(x) = [\dot{\psi}_+(x,\lambda_l) - b_l \dot{\psi}_-(x,\lambda_l)]\sigma(\lambda_l)$. Concerning the second part of (f),

$$g_{l,t}(x) = (\dot{f}_+ - b_l \dot{f}_-)|_{\lambda=\lambda_l}$$

$$= \{\dot{f}_{+,t} - b_l \dot{f}_{-,t}\} - b_l \dot{f}_-|_{\lambda=\lambda_l}$$

$$= \{A(\dot{f}_+ - b_l \dot{f}_-) - [ik\Omega(\lambda)] \cdot (f_+ + b_l f_-) - ik\Omega(\lambda)(\dot{f}_+ + b_l \dot{f}_-)\}|_{\lambda=\lambda_l}$$

$$+ 2ik_{\pm}\Omega(\lambda_l)b_l \dot{f}_-(x,\lambda_l).$$

Now we use (25).

Finally, (g) is a direct consequence of (f) and (a).

Remark 1: According to (g) of Lemma 5 the formula for $\tilde{g}_j(x,t)$ in Ref. 2 is incorrect.

In order to solve the linearized GPKdV equation we need an expansion formula. We introduce the bilinear form

$$\langle f_1, f_2 \rangle = \int_{-\infty}^{\infty} \sum_{r=0}^{N-1} f_1^{(r)}(x) f_2^{(r)}(x) dx,$$

$$f_i = (f_i^{(0)}(x), f_i^{(1)}(x), \dots, f_i^{(N-1)}(x))^T,$$

$$i = 1, 2.$$

Then^{3,11} for any vector function $h(x) \in L_1^N(-\infty, \infty) \equiv L_1(-\infty, \infty) \times \dots \times L_1(-\infty, \infty)$ we have

$$h(x) = \mp (2\pi i)^{-1} \sum_{s=0}^{N-1} \int_{I_s} \left[\frac{JF_{\mp}(x,\lambda_+)}{2\lambda^N a_+^2(\lambda)} \langle F_{\pm}(\lambda_+), h \rangle - \frac{JF_{\mp}(x,\lambda_-)}{2\lambda^N a_-^2(\lambda)} \langle F_{\pm}(\lambda_-), h \rangle \right] d\lambda \mp \sum_{l=1}^M (2\lambda_l^N \dot{a}^2(\lambda_l))^{-1}$$

$$\times \left[\left(\frac{N}{\lambda_l} + \frac{\ddot{a}(\lambda_l)}{\dot{a}(\lambda_l)} \right) JF_{\mp}(x,\lambda_l) - J\dot{F}_{\mp}(x,\lambda_l) \right] \langle F_{\pm}(\lambda_l), h \rangle - JF_{\mp}(x,\lambda_l) \langle \dot{F}_{\pm}(\lambda_l), h \rangle \quad (26)$$

with $F_{\pm}(x,\lambda) = f_{\pm}(x,\lambda) \circ f_{\pm}(x,\lambda)$, provided $\lambda \in I_s$, $l = 1, M$, $s = 1, N$.

Remark 2: According to Refs. 3 and 11 for $N > 2$ the expansions above are formal when some $v_r \neq 0$, $r = [N/2 + 1], N - 1$ due to lack of results on the asymptotics of $\psi_{\pm}(x,\lambda)$ as $|\lambda| \rightarrow \infty$ for λ close to the rays I_s . Let us not forget also about the assumption that $a(\lambda)$ has a finite number of simple zeros.

By adding the two formulas in (26) we will obtain a new expansion formula containing prolonged squared solutions only (and not their λ -derivatives as well), namely

$$P(x,\lambda) = f_+(x,\lambda_+) \circ f_-(x,\lambda_-),$$

$$Q(x,\lambda) = f_+(x,\lambda_-) \circ f_-(x,\lambda_+), \quad \lambda \in I_s,$$

$$P_l(x) = F_+(x,\lambda_l), \quad Q_l(x) = f_+(x,\lambda_l) \circ g_l(x),$$

$$l = \overline{1, M}. \quad (27)$$

In order to do that we need the following.

Lemma 6: The functions in (27) satisfy the identities

$$\begin{aligned}
 & \text{(a) } [P(x,\lambda)Q^\top(y,\lambda) \\
 & \quad - Q(x,\lambda)P^\top(y,\lambda)]2/a_+(\lambda)a_-(\lambda) \\
 & = \frac{F_-(x,\lambda_-)F_+^\top(y,\lambda_-)}{a_-^2(\lambda)} - \frac{F_-(x,\lambda_+)F_+^\top(y,\lambda_+)}{a_+^2(\lambda)} \\
 & \quad - \frac{F_+(x,\lambda_-)F_-^\top(y,\lambda_-)}{a_-^2(\lambda)} \\
 & \quad + \frac{F_+(x,\lambda_+)F_-^\top(y,\lambda_+)}{a_+^2(\lambda)}; \tag{28}
 \end{aligned}$$

$$\begin{aligned}
 & \text{(b) } [P_l(x)Q_l^\top(y) - Q_l(x)P_l^\top(y)]2/b_l^2 \\
 & = \left[\left(\frac{N}{\lambda_l} + \frac{\ddot{a}(\lambda_l)}{\dot{a}(\lambda_l)} \right) F_+(x,\lambda_l) - \dot{F}_+(x,\lambda_l) \right] \\
 & \quad \times F_-^\top(y,\lambda_l) - F_+(x,\lambda_l)\dot{F}_-^\top(y,\lambda_l) \\
 & \quad - \left[\left(\frac{N}{\lambda_l} + \frac{\ddot{a}(\lambda_l)}{\dot{a}(\lambda_l)} \right) F_-(x,\lambda_l) - \dot{F}_-(x,\lambda_l) \right] \\
 & \quad \times F_+^\top(y,\lambda_l) + F_-(x,\lambda_l)\dot{F}_+^\top(y,\lambda_l). \tag{29}
 \end{aligned}$$

Proof: (a) Let $p(x,\lambda) = \psi_+(x,\lambda_+)\psi_-(x,\lambda_-)$, $q(x,\lambda) = \psi_+(x,\lambda_-)\psi_-(x,\lambda_+)$. Using (22) and (24) several times we obtain

$$\begin{aligned}
 & P(x,\lambda)Q^\top(y,\lambda) - Q(x,\lambda)P^\top(y,\lambda) \\
 & = \sigma(\lambda)\sigma^\top(\lambda)\{\psi_-(x,\lambda_-)[b_+\psi_-(x,\lambda_+) + a_+\psi_-(x,\lambda_-)]\psi_+(y,\lambda_-)[-b_-\psi_+(y,\lambda_+) + a_+\psi_+(y,\lambda_-)] \\
 & \quad - \psi_-(x,\lambda_+)[b_-\psi_-(x,\lambda_-) + a_-\psi_-(x,\lambda_+)]\psi_+(y,\lambda_+)[-b_+\psi_+(y,\lambda_-) + a_-\psi_+(y,\lambda_+)]\} \\
 & = \sigma\sigma^\top\{a_+^2\psi_-^2(x,\lambda_-)\psi_+^2(y,\lambda_-) - a_-^2\psi_-^2(x,\lambda_+)\psi_+^2(y,\lambda_+) + \psi_-(x,\lambda_-)\psi_-(x,\lambda_+)[b_+a_+\psi_+^2(y,\lambda_-) \\
 & \quad - b_+a_-\psi_+^2(y,\lambda_+)] + [-b_+a_+\psi_-^2(x,\lambda_-) + b_+a_-\psi_-^2(x,\lambda_+)]\psi_+(y,\lambda_-)\psi_+(y,\lambda_+)\}.
 \end{aligned}$$

Because of the identities

$$\begin{aligned}
 & b_+a_+\psi_+^2(y,\lambda_-) - b_-a_-\psi_+^2(y,\lambda_+) \\
 & = b_+\psi_+(y,\lambda_-)\psi_-(y,\lambda_+) - b_-\psi_+(y,\lambda_+)\psi_-(y,\lambda_-), \\
 & -b_-a_+\psi_-^2(x,\lambda_-) + b_+a_-\psi_-^2(x,\lambda_+) \\
 & = b_+\psi_-(x,\lambda_+)\psi_+(x,\lambda_-) - b_-\psi_-(x,\lambda_-)\psi_+(x,\lambda_+), \\
 & \psi_+(y,\lambda_-)\psi_+(y,\lambda_+) \\
 & = [b_-\psi_-(y,\lambda_-) + a_-\psi_-(y,\lambda_+)]\psi_+(y,\lambda_+) \\
 & = b_-\psi_-(y,\lambda_-)\psi_+(y,\lambda_+) + \psi_-(y,\lambda_+)[\psi_-(y,\lambda_-) + b_+\psi_+(y,\lambda_-)],
 \end{aligned}$$

we obtain

$$\begin{aligned}
 & \sigma\sigma^\top[p(x,\lambda)q(y,\lambda) - q(x,\lambda)p(y,\lambda)] = a_+^2F_-(x,\lambda_-)F_+^\top(y,\lambda_-) - a_-^2F_-(x,\lambda_+)F_+^\top(y,\lambda_+) + \sigma\sigma^\top\psi_-(x,\lambda_-) \\
 & \quad \times \psi_-(x,\lambda_+)[b_+q(y,\lambda) - b_-p(y,\lambda)] + \sigma\sigma^\top[b_+q(x,\lambda) - b_-p(x,\lambda)] \\
 & \quad \times [\psi_-(y,\lambda_+)\psi_-(y,\lambda_-) + b_-p(y,\lambda) + b_+q(y,\lambda)]. \tag{30}
 \end{aligned}$$

After interchanging $x \leftrightarrow y$ in (30) and subtracting it out of (30) we obtain after cancellation

$$2[P(x,\lambda)Q^T(y,\lambda) - Q(x,\lambda)P^T(y,\lambda)] = a_+^2 F_-(x,\lambda_-)F_+^T(y,\lambda_-) - a_-^2 F_-(x,\lambda_+)F_+^T(y,\lambda_+) \\ - a_+^2 F_+(x,\lambda_-)F_-^T(y,\lambda_-) + a_-^2 F_+(x,\lambda_+)F_-^T(y,\lambda_+) \\ - \sigma(\lambda)\sigma^T(\lambda)2b_-b_+ [p(x,\lambda)q(y,\lambda) - q(x,\lambda)p(y,\lambda)].$$

Now we use (23) and obtain (28).

(b) Let $A_l(x,y)$ be the right-hand side of (29). Then due to (25) we have

$$A_l(x,y) = -\dot{F}_+(x,\lambda_l)F_-^T(y,\lambda_l) - F_+(x,\lambda_l)\dot{F}_-^T(y,\lambda_l) \\ + \dot{F}_-(x,\lambda_l)F_+^T(y,\lambda_l) + F_-(x,\lambda_l)\dot{F}_+^T(y,\lambda_l).$$

On the other hand

$$2Q_l(x) = 2\sigma(\lambda)f_+(x,\lambda_l)[\dot{f}_+(x,\lambda_l) - b_l\dot{f}_-(x,\lambda_l)] \\ = \sigma(\lambda)[f_+^2(x,\lambda_l) - b_l^2f_-^2(x,\lambda_l)] \\ = \dot{F}_+(x,\lambda_l) - b_l^2\dot{F}_-(x,\lambda_l).$$

Therefore

$$A_l(x,y) = -2Q_l(x)F_-^T(y,\lambda_l) + F_-(x,\lambda_l)2Q_l^T(y) \\ = [-Q_l(x)P_l^T(y) + P_l(x)Q_l^T(y)]2/b_l^2.$$

Using Lemma 6 we find, after adding the formulas in (26) and dividing by 2:

$$h(x,t) = (2\pi i)^{-1} \sum_{s=0}^{N-1} \int_{I_s} [e^{2ik+\Omega(\lambda)t}JP(x,\lambda)\langle Q(\lambda,t=0),h_0 \rangle - e^{-2ik+\Omega(\lambda)t}JQ(x,\lambda)\langle P(\lambda,t=0),h_0 \rangle] \\ \times \frac{d\lambda}{2\lambda^N a_+(\lambda)a_-(\lambda)} + \sum_{l=1}^M [2\lambda_l^N \dot{a}^2(\lambda_l)b_l^2(t=0)]^{-1} e^{2ik\Omega(\lambda_l)t} \{JP_l(x)\langle Q_l(t=0),h_0 \rangle \\ - [JQ_l(x) + (2ik_l\Omega(\lambda_l))\cdot JP_l(x)t]\langle P_l(t=0),h_0 \rangle\}. \tag{32}$$

[Here, P, Q, P_b, Q_l defined in (27) depend on t implicitly via $v(x,t)$.]

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$$h(x) = (2\pi i)^{-1} \sum_{s=0}^{N-1} \int_{I_s} [JP(x,\lambda)\langle Q(\lambda),h \rangle \\ - JQ(x,\lambda)\langle P(\lambda),h \rangle] \frac{d\lambda}{2\lambda^N a_+(\lambda)a_-(\lambda)} \\ + \sum_{l=1}^M [2\lambda_l^N \dot{a}^2(\lambda_l)b_l^2]^{-1} [JP_l(x)\langle Q_l,h \rangle \\ - JQ_l(x)\langle P_l,h \rangle]. \tag{31}$$

As a corollary of Theorem 3 and Lemma 5 [(b) and (g)] we find that

$$e^{2ik+\Omega(\lambda)t}JP(x,\lambda), \quad e^{-2ik+\Omega(\lambda)t}JQ(x,\lambda), \quad \lambda \in I_s, \\ e^{2ik\Omega(\lambda_l)t}JP_l(x), \\ e^{2ik\Omega(\lambda_l)t}[JQ_l(x) + (2ik_l\Omega(\lambda_l))\cdot JP_l(x)t], \quad l = \overline{1,M}$$

are solutions of (19). Therefore, applying the expansion formula (31) we obtain the following.

Theorem 4: The Cauchy problem for the linearized GPKdV equation (19) (subject to the restrictions in Remark 2) where $v(x,t)$ evolves according to (6), with initial condition $h(x,t=0) = h_0(x) \in L_1^N(-\infty, \infty)$, has a solution

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