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Closed first- and second-order moment equations for stochastic nonlinear problems with applications to model hydrodynamic and Vlasov-plasma turbulence

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Working along the lines of a procedure outlined by Keller, a technique is developed for deriving closed first- and second-order moment equations for a general class of stochastic nonlinear equations by performing a renormalization at the level of the second moment. The work of Weinstock, as reformulated recently by Balescu and Misguich, is extended in order to obtain two equivalent representations for the second moment using an exact, nonperturbative, statistical approach. These general results, when specialized to the weak-coupling limit, lead to a complete set of closed equations for the first two moments within the framework of an approximation corresponding to Kraichnan's direct-interaction approximation. Additional restrictions result in a self-consistent set of equations for the first two moments in the stochastic quasilinear approximation. Finally, the technique is illustrated by considering its application to two specific physical problems: (1) model hydrodynamic turbulence and (2) Vlasov-plasma turbulence in the presence of an external stochastic electric field.

1. INTRODUCTION

Several significant advances have been made in the area of stochastic nonlinear problems over the past few years. Kraichnan[1] has introduced a technique, known as the *direct-interaction approximation*, wherein the true problems of interest are replaced by stochastic dynamical models that lead, without approximation, to closed equations for covariances and averaged Green's functions. This method has been used extensively in the theories of hydrodynamic turbulence (cf. Ref. 1) and plasma turbulence (cf. Ref. 2). In an effort to understand Kraichnan's direct-interaction approximation, as well as peripheral contributions (cf. Ref. 3) related primarily to the problem of Vlasov-plasma turbulence, Weinstock[4] has presented a generalization based on an exact, nonperturbative statistical approach valid for both strong and weak turbulence. In the weak-coupling limit, Orszag's and Kraichnan's equations for the mean Green's function (cf. Ref. 2), as well as Dupree's turbulence equations (cf. Ref. 3), are recovered. Further restrictions lead to the well-known *quasilinear approximation*. Weinstock's work has been recently reformulated by Balescu and Misguich,[5] and, within the quasilinear approximation, it has been applied to the Vlasov-plasma turbulence problem, with allowance for the presence of an external, stochastic electric field. Furthermore, a modified Weinstock weak-coupling limit, referred to as the *renormalized quasilinear approximation*, has been introduced,[6] and its connection with Kraichnan's direct-interaction approximation has been discussed.

It was pointed out earlier in the introduction that in Kraichnan's direct-interaction approximation, the main results are expressed in terms of closed equations for covariances and averaged Green's functions. On the other hand, Weinstock obtained for a Vlasov plasma a general set of closed equations in terms of smoothed and fluctuating quantities. Although a connection was established in the weak-coupling limit with Orszag's and Kraichnan's equation for the averaged Green's function, no attempt was made to derive closed equations for statistical moments of relevant field quantities. Along the same vein, in Balescu's and Misguich's work on the Vlasov equation with an external stochastic electric field, an equation is established for the first moment in the quasilinear approximation (cf. Ref. 5). This equation, however, is not closed, as it contains a term proportional to the covariance. This difficulty is remedied by solving the equation for the first moment using an iterative procedure. In their most recent work, Misguich and Balescu (cf. Ref. 6) *do* close the equations for the first two moments by resorting to a renormalization at the level of the first moment. Given that $\mu(t)$ is a field quantity of interest, they derive expressions for its mean, $\mathcal{E}\{\mu(t)\}$, and fluctuating, $\delta\mu(t)$, part within the framework of the renormalized quasilinear approximation (a level related to Kraichnan's direct-interaction approximation). From the expression for $\delta\mu(t)$, a relationship is set up for the covariance $\mathcal{E}\{\delta\mu(t)\delta\mu(t')\}$. The relations for $\mathcal{E}\{\mu(t)\}$ and $\mathcal{E}\{\delta\mu(t)\delta\mu(t')\}$, together with an expression for a mean propagator (related to Kraichnan's averaged Green's function), form, then, a self-consistent set.

The procedure followed by Misguich and Balescu in order to close the equations for the first moment and the correlation function, when specialized to linear stochastic problems considered in the first-order smoothing approximation, has led in the past into serious difficulties, as pointed out by Morrison and McKenna.[7] At this stage, it is difficult to assess the degree to which these difficulties are alleviated when working with nonlinear stochastic problems at the level of Misguich and Balescu's renormalized quasilinear approximation. A clarification of this ambiguity is highly desirable; however, it will not be pursued in this paper, especially as a radically different approach to the closure problem will be followed instead.

It is our intent in this paper to present a technique for closing the equations for the first two moments of a field quantity $\mu(t)$ governed by a stochastic nonlinear equation of the form $(\partial/\partial t)\mu(t)=\Omega\mu(t)$ in the special case that the operator Ω depends linearly on $\mu(t)$. This is achieved via the Weinstock—Balescu—Misguich formalism, working, however, at the level of the second moment. We believe this approach is new and eliminates the closure difficulties mentioned earlier in connection with the work of Misguich and Balescu. Our work has been significantly motivated by a procedure outlined by Keller.[8]

In order for the discussion in this paper to be self-contained, the work of Weinstock, as reformulated by Balescu and Misguich, is briefly outlined in Sec. 2. In Sec. 3, the Weinstock—Balescu—Misguich formalism is extended in order to derive two equivalent equations for the second moment using an exact, nonperturbative, statistical approach valid for an arbitrary stochastic nonlinear operator. These general results are specialized in Sec. 4 to the weak-coupling limit, and a complete set of closed equations is obtained for the first two moments of the field $\mu(t)$ on the basis of an approximation corresponding to Kraichnan's direct-interaction approximation. Further simplifications lead to a complete self-consistent set of equations for the first two moments of $\mu(t)$ in the stochastic quasilinear approximation. Finally, the method developed in this paper is applied to two physically important areas: (1) model hydrodynamic turbulence (cf. Sec. 5), and (2) Vlasov-plasma turbulence with an external stochastic electric field (cf. Sec. 6).

2. REVIEW OF THE WEINSTOCK-BALESCU-MISGUICH FORMALISM

Consider the general nonlinear stochastic equation

$$\frac{\partial}{\partial t}\mu(t;\alpha)=\Omega(t;\alpha)\mu(t;\alpha), \quad t\geq t_0, \tag{2.1a}$$

$$\mu(t_0;\alpha)=\mu_0(\alpha). \tag{2.1b}$$

Here, $\Omega(t;\alpha)$ is a nonlinear stochastic operator depending on a parameter $\alpha\in\mathcal{A}$, $\mathcal{A}$ being a probability measure space, and $\mu(t;\alpha)$, the random field quantity, is an element of an infinitely dimensional vector space $\mathcal{H}$ and can be either a scalar or a vector quantity. The discussion in this section is general and applies independently of the precise definition of the field $\mu(t;\alpha)$ and the operator $\Omega(t;\alpha)$.[9]

The stochastic operator Ω is split into two parts as follows: $\Omega=\Omega_0+\Omega_1$. The field μ is also decomposed abstractly into two mutually independent terms, viz., $\mu=A\mu+F\mu$ by means of the formal introduction of the two operators A and F. $A\mu$ is called the *average* (or *mean*) component, and $F\mu$ is the *fluctuating* part of μ. The uniqueness of the decomposition as well as the mutual independence of the two components are ensured by prescribing the properties $A+F=I$, $A^2=A$, $F^2=F$, $AF=FA=0$, where I is the identity operator.

The interconnection between the decompositions for the operator Ω and the field μ is contained in the commutation relations $[\Omega_0,A]_-=0$ and $[\Omega_0,F]_-=0$ which constitute a mathematical statement of the fact that the fluctuating part of μ is due only to Ω_1. Therefore, Ω_0 must commute with A, and also with $F=I-A$.

The specific realization of the "projection" operators A and F which will be used in the ensuing work is the following: $A\mu\rightarrow\mathcal{E}\{\mu\}$, $F\mu\rightarrow\delta\mu$, where $\mathcal{E}\{\mu\}$ and $\delta\mu$ are the ensemble average and fluctuating (incoherent) parts of the random field $\mu(t;\alpha)$, respectively. Within the framework of this specific realization, the aforementioned commutation relations signify that Ω_0 is a deterministic operator and Ω_1 is a generally noncentered stochastic operator.[10]

Operating on (2.1a) with the operator A yields the following equation:

$$\frac{\partial}{\partial t}\mathcal{E}\{\mu(t)\}=\Omega_0(t)\mathcal{E}\{\mu(t)\}+A\Omega_1(t)\delta\mu(t). \tag{2.2}$$

On the other hand, operating on (2.1a) with the operator $F=I-A$ results in the following two equivalent equations for the fluctuating part of the field μ:

$$\frac{\partial}{\partial t}\delta\mu(t)=(I-A)\Omega(t)\delta\mu(t)+\Omega_1(t)\mathcal{E}\{\mu(t)\}, \tag{2.3}$$

$$\frac{\partial}{\partial t}\delta\mu(t)=\Omega_0(t)\delta\mu(t)+(I-A)\Omega_1(t)\delta\mu(t)+\Omega_1(t)\mathcal{E}\{\mu(t)\}. \tag{2.4}$$

[An additional equivalent equation for $\delta\mu(t)$ given by Balescu and Misguich (cf. Ref. 5) is not being presented here since it will not play a significant role in our discussion.]

Equation (2.3) can be solved for $\delta\mu(t)$ in terms of the mean field and the initial value of the fluctuating part of μ,

$$\delta\mu(t)=U_A(t,t_0)\delta\mu(t_0)+\int_{t_0}^{t}dt'\,U_A(t,t')\Omega_1(t')\mathcal{E}\{\mu(t')\}. \tag{2.5}$$

The Weinstock propagator U_A is defined as the solution of the initial value problem

$$\frac{\partial}{\partial t}U_A(t,t_0)=(I-A)\Omega(t)U_A(t,t_0), \quad t\geq t_0, \tag{2.6a}$$

$$U_A(t_0,t_0)=I. \tag{2.6b}$$

In the case of infinite space, the solution for the propagator U_A can be written symbolically as

$$U_A(t,t_0)=X\exp\left[\int_{t_0}^{t}dt'\,(I-A)\Omega(t')\right], \tag{2.7}$$

where X denotes a time-ordering operator. [In general, the solution (2.7) must be modified to account for boundary conditions.] Inserting (2.5) into (2.2) results in the equation

$$\frac{\partial}{\partial t}\mathcal{E}\{\mu(t)\}=\Omega_0(t)\mathcal{E}\{\mu(t)\}+A\Omega_1(t)U_A(t,t_0)\delta\mu(t_0)+\int_{t_0}^{t}dt'\,\mathcal{E}\{\Omega_1(t)U_A(t,t')\Omega_1(t')\}\mathcal{E}\{\mu(t')\}. \tag{2.8}$$

In order to integrate (2.4), a propagator, $W(t, t_0)$, is introduced first by means of the equation

$$\frac{\partial}{\partial t} W(t, t_0) = \Omega_0(t) W(t, t_0), \quad t \geq t_0, \tag{2.9a}$$

$$W(t_0, t_0) = I, \tag{2.9b}$$

whose solution, for an unbounded region, can be formally written as follows:

$$W(t, t_0) = X \exp\left[\int_{t_0}^{t} dt' \, \Omega_0(t')\right]. \tag{2.10}$$

In terms of this propagator, the integral of (2.4) is given by

$$\delta\mu(t) = W(t, t_0)\delta\mu(t_0) + \int_{t_0}^{t} dt' \, W(t, t') \times [(I - A)\Omega_1(t')\delta\mu(t') + \Omega_1(t')\mathcal{E}\{\mu(t')\}]. \tag{2.11}$$

Iterating the last expression, we obtain the explicit solution

$$\delta\mu(t) = \Lambda_W(t, t_0)\delta\mu(t_0) + \int_{t_0}^{t} dt' \, \Lambda_W(t, t')\Omega_1(t')\mathcal{E}\{\mu(t')\}, \tag{2.12}$$

where

$$\Lambda_W(t, t_0) = \sum_{n=0}^{\infty} \left[\int_{t_0}^{t} dt' \, W(t, t')(I - A)\Omega_1(t')\right]^n W(t, t_0). \tag{2.13}$$

Finally, substituting (2.12) into (2.2) we arrive at the following alternative equation for the first moment:

$$\frac{\partial}{\partial t}\mathcal{E}\{\mu(t)\} = \Omega_0(t)\mathcal{E}\{\mu(t)\} + A\Omega_1(t)\Lambda_W(t, t_0)\delta\mu(t_0) + \int_{t_0}^{t} dt' \, \mathcal{E}\{\Omega_1(t)\Lambda_W(t, t')\Omega_1(t')\}\mathcal{E}\{\mu(t')\}. \tag{2.14}$$

The formal expressions (2.8) and (2.14) derived by means of a nonperturbative statistical approach are valid for both weak and strong random fluctuations. It should be pointed out, however, that neither (2.8) nor (2.14) constitutes a closed equation for $\mathcal{E}\{\mu(t)\}$. This would definitely be the case if Ω were a linear operator. Here, however, Ω depends on the field μ by assumption.

In the sequel we shall make extensive use of expressions (2.2), (2.5), (2.8), (2.11), and (2.14). For the sake of simplicity we shall neglect in these relations the parts proportional to $\delta\mu(t_0)$. It must be emphasized, however, that this is a matter of convenience only and it will not detract from the generality of the formalism which will be developed in the following sections.

3. EXTENSION OF THE WEINSTOCK-BALESCU-MISGUICH FORMALISM TO SECOND MOMENTS

The exact, nonperturbative, statistical Weinstock–Balescu–Misguich formalism outlined in the previous section culminated in the derivations of two alternative equations for the first moment [cf. Eqs. (2.8) and (2.14)] which, as pointed out earlier, are not closed by virtue of the nonlinearity of the stochastic operator Ω.

In this section we shall extend the work of Weinstock, Balescu, and Misguich in order to derive two equivalent representations for the second moment [analogous to Eqs. (2.8) and (2.14)] using, again, an exact, nonperturbative, statistical approach.

Let us assume that μ depends on a set of variables $\mathbf{s}$ and on time, viz., $\mu = \mu(\mathbf{s}, t)$. Moreover, let $\Omega = \Omega[\mathbf{s}, \partial/\partial\mathbf{s}, t, \mu(\mathbf{s}, t)]$.[11] [If there is not danger for ambiguity, we shall use in the subsequent discussion the shorter notation $\mu = \mu(t)$ and $\Omega = \Omega(\mathbf{s}, t)$].

Next consider the quantity $R(t, t') \equiv \mu(\mathbf{s}, t)\mu(\mathbf{s}', t')$. Differentiating it with respect to t and using the original Eq. (2.1) for $\mu(\mathbf{s}, t)$ results in the following equation:

$$\frac{\partial}{\partial t} R(t, t') = \Omega(\mathbf{s}, t)R(t, t'). \tag{3.1}$$

Similarly, differentiating $R(t, t')$ with respect to t' and using the equation of evolution for $\mu(\mathbf{s}'t')$, i.e., $(\partial/\partial t')\mu(\mathbf{s}', t') = \Omega(\mathbf{s}', t')\mu(\mathbf{s}', t')$, we obtain the expression

$$\frac{\partial}{\partial t'} R(t, t') = \Omega(\mathbf{s}', t')R(t, t'). \tag{3.2}$$

It should be noted that both Eqs. (3.1) and (3.2) are of the general form (2.1); hence, the Weinstock–Balescu–Misguich formalism introduced in the previous section is applicable. However, several basic departures from their general theory have to be made in order to account for the simultaneous manipulation of Eqs. (3.1) and (3.2).

Operating on (3.1) with the operator A yields the following equation for the coherent part of $R(t, t')$:

$$\frac{\partial}{\partial t}\mathcal{E}\{R(t, t')\} = \Omega_0(\mathbf{s}, t)\mathcal{E}\{R(t, t')\} + A\Omega_1(\mathbf{s}, t)\delta R(t, t'). \tag{3.3}$$

On the other hand, corresponding to (3.1) and (3.2), respectively, and using the Weinstock formulation (2.3), we obtain the following two equations for the fluctuating part of $R(t, t')$:

$$\frac{\partial}{\partial t}\delta R(t, t') = (I - A)\Omega(\mathbf{s}, t)\delta R(t, t') + \Omega_1(\mathbf{s}, t)\mathcal{E}\{R(t, t')\}, \tag{3.4}$$

$$\frac{\partial}{\partial t'}\delta R(t, t') = (I - A)\Omega(\mathbf{s}', t')\delta R(t, t') + \Omega_1(\mathbf{s}', t') \times \mathcal{E}\{R(t, t')\}. \tag{3.5}$$

Proceeding as in (2.5), Eq. (3.4) can be formally integrated as follows:

$$\delta R(t, t') = U_A(\mathbf{s}, t, t_0)\delta R(t_0, t') + \int_{t_0}^{t} d\tau \, U_A(\mathbf{s}, t, \tau)\Omega_1(\mathbf{s}, \tau)\mathcal{E}\{R(\tau, t')\}. \tag{3.6}$$

Analogously, the integral of (3.5) (evaluated at $t = t_0$) is given by

$$\delta R(t_0, t') = U_A(\mathbf{s}', t', t_0)\delta R(t_0, t_0) + \int_{t_0}^{t'} d\tau \, U_A(\mathbf{s}', t', \tau)\Omega_1(\mathbf{s}', \tau)\mathcal{E}\{R(t_0, \tau)\}. \tag{3.7}$$

The first part of the right-hand side of (3.7), i.e., the term proportional to $\delta R(t_0, t_0)$, is neglected for convenience. [It should be stressed, however, that in contradistinction to the initial value term $\delta\mu(t_0)$ in (2.5) and (2.8), which has been shown by Weinstock to be negligible for large t, no justification has ever been made for the neglect of such terms as $\delta R(t_0, t_0)$.] The resulting expression is introduced next in Eq. (3.6),

$$\delta R(t,t') = \int_{t_0}^{t} d\tau\, U_A(\mathbf{s},t,\tau)\Omega_1(\mathbf{s},\tau)\mathcal{E}\{R(\tau,t')\} + \int_{t_0}^{t'} d\tau\, U_A(\mathbf{s},t,t_0)U_A(\mathbf{s}',t',\tau)\Omega_1(\mathbf{s}',\tau)\mathcal{E}\{R(t_0,\tau)\}. \tag{3.8}$$

This expression for $\delta R(t,t')$ is substituted next into (3.3) in order to obtain the final form of the equation for the second moment [analogous to Eq. (2.8) for the first moment],

$$\frac{\partial}{\partial t}\mathcal{E}\{R(t,t')\} = \Omega_0(\mathbf{s},t)\mathcal{E}\{R(t,t')\} + \int_{t_0}^{t} d\tau\, \mathcal{E}\{\Omega_1(\mathbf{s},t)U_A(\mathbf{s},t,\tau)\Omega_1(\mathbf{s},\tau)\} \times\mathcal{E}\{R(\tau,t')\} + \int_{t_0}^{t'} d\tau\, \mathcal{E}\{\Omega_1(\mathbf{s},t)U_A(\mathbf{s},t,t_0) \times U_A(\mathbf{s}',t',\tau)\Omega_1(\mathbf{s}',\tau)\}\mathcal{E}\{R(t_0,\tau)\}. \tag{3.9}$$

In order to derive an alternative equation for the second moment [analogous to Eq. (2.14) for the first moment], we proceed as follows: Corresponding to (3.1) and (3.2), respectively, and using (2.4), we have

$$\frac{\partial}{\partial t}\delta R(t,t') = \Omega_0(\mathbf{s},t)\delta R(t,t') + (I-A)\Omega_1(\mathbf{s},t)\delta R(t,t') + \Omega_1(\mathbf{s},t)\mathcal{E}\{R(t,t')\}, \tag{3.10}$$

$$\frac{\partial}{\partial t'}\delta R(t,t') = \Omega_0(\mathbf{s}',t')\delta R(t,t') + (I-A)\Omega_1(\mathbf{s}',t')\delta R(t,t') + \Omega_1(\mathbf{s}',t')\mathcal{E}\{R(t,t')\}. \tag{3.11}$$

Proceeding as in (2.12), Eq. (3.10) can be integrated formally as follows:

$$\delta R(t,t') = \Lambda_W(\mathbf{s},t,t_0)\delta R(t_0,t') + \int_{t_0}^{t} d\tau\, \Lambda_W(\mathbf{s},t,\tau)\Omega_1(\mathbf{s},\tau)\mathcal{E}\{R(\tau,t')\}. \tag{3.12}$$

Similarly, the integral of (3.11) (evaluated at $t=t_0$) is given by

$$\delta R(t_0,t') = \Lambda_W(\mathbf{s}',t',t_0)\delta R(t_0,t_0) + \int_{t_0}^{t'} d\tau\, \Lambda_W(\mathbf{s}',t',\tau)\Omega_1(\mathbf{s}',\tau)\mathcal{E}\{R(t_0,\tau)\}. \tag{3.13}$$

Neglecting the first term on the right-hand side of (3.13) and using the resulting expression for $\delta R(t_0,t')$ in conjunction with (3.12), we find that

$$\delta R(t,t') = \int_{t_0}^{t} d\tau\, \Lambda_W(\mathbf{s},t,\tau)\Omega_1(\mathbf{s},\tau)\mathcal{E}\{R(\tau,t')\} + \int_{t_0}^{t'} d\tau\, \Lambda_W(\mathbf{s},t,t_0)\Lambda_W(\mathbf{s}',t',\tau)\Omega_1(\mathbf{s}',\tau)\mathcal{E}\{R(t_0,\tau)\}. \tag{3.14}$$

Finally, inserting (3.14) into (3.3) we obtain the desired alternative equation for the second moment,

$$\frac{\partial}{\partial t}\mathcal{E}\{R(t,t')\} = \Omega_0(\mathbf{s},t)\mathcal{E}\{R(t,t')\} + \int_{t_0}^{t} d\tau\, \mathcal{E}\{\Omega_1(\mathbf{s},t)\Lambda_W(\mathbf{s},t,\tau)\Omega_1(\mathbf{s},\tau)\} \times\mathcal{E}\{R(\tau,t')\} + \int_{t_0}^{t'} d\tau\, \mathcal{E}\{\Omega_1(\mathbf{s},t)\Lambda_W(\mathbf{s},t,t_0) \times\Lambda_W(\mathbf{s}',t',\tau)\Omega_1(\mathbf{s}',\tau)\}\mathcal{E}\{R(t_0,\tau)\}. \tag{3.15}$$

We close this section with two important remarks: (1) As in the case of Eqs. (2.8) and (2.14) for the first moment, neither of the equivalent equations (3.9) and (3.15) for the second moment is closed, again because of the nonlinearity of the stochastic operator Ω; (2) the procedure outlined in this section can obviously be extended in order to derive equations for higher moments.

4. CLOSED FIRST- AND SECOND-ORDER MOMENT EQUATIONS

The exact nonperturbative results contained in the previous two sections are valid for an arbitrary nonlinear stochastic operator Ω. In the sequel we shall restrict the discussion to the special case that Ω depends linearly on the field μ. The class of nonlinear stochastic equations (2.1) spanned by Ω under this assumption includes two physically important problems: (1) model hydrodynamic turbulence, and (2) plasma turbulence.

A. Direct-interaction approximation

In the first part of this section we shall present a procedure for obtaining a complete set of closed equations for the first two moments of the field μ in the framework of an approximation corresponding to Kraichnan's direct-interaction approximation.

We introduce the propagator

$$U(t,t_0) = X\exp\left[\int_{t_0}^{t} dt'\, \Omega(t')\right] \tag{4.1}$$

as the solution of the initial value problem

$$\frac{\partial}{\partial t}U(t,t_0) = \Omega(t)U(t,t_0), \quad t \geq t_0, \tag{4.2a}$$

$$U(t_0,t_0) = I \tag{4.2b}$$

in an unbounded domain. This, of course, is the fundamental problem associated with (2.1), viz.,

$$\mu(t) = U(t,t_0)\mu(t_0), \tag{4.3}$$

whence

$$\mathcal{E}\{\mu(t)\} = AU(t,t_0)\mu(t_0), \tag{4.4a}$$

$$\delta\mu(t) = (I-A)U(t,t_0)\mu(t_0), \tag{4.4b}$$

and, consequently, since $\mu(t_0)$ is specified, $\mathcal{E}\{U(t,t_0)\}$ may be chosen as the basic quantity in the place of $\mathcal{E}\{\mu(t)\}$.

Equation (2.8) for the first moment can be expressed in terms of $U(t,t_0)$, instead of $\mu(t)$, by substituting (4.4) as follows:

$$\frac{\partial}{\partial t}AU(t,t_0) = \Omega_0(t)AU(t,t_0) + A\Omega_1(t)U_A(t,t_0)(I-A) + \int_{t_0}^{t} dt'\, \mathcal{E}\{\Omega_1(t)U_A(t,t')\Omega_1(t')\}AU(t',t_0). \tag{4.5}$$

Weinstock has established that in the weak-coupling approximation,

$$U_A(t, t_0)(I - A) \approx (I - A)\mathcal{E}\{U(t, t_0)\}, \tag{4.6}$$

$$U(t, t_0) \approx \mathcal{E}\{U(t, t_0)\}. \tag{4.7}$$

Assuming for simplicity that $\mu(t_0)$ is deterministic, (4.5) simplifies in this case to

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{E}\{U(t, t_0)\} &= \Omega_0(t)\mathcal{E}\{U(t, t_0)\} \\ &\quad + \int_{t_0}^{t} dt' \,\mathcal{E}\{\Omega_1(t)\mathcal{E}\{U(t, t')\}\Omega_1(t')\}\mathcal{E}\{U(t', t_0)\}\end{aligned} \tag{4.8}$$

and (3.9) assumes the form

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{E}\{R(t, t')\} &= \Omega_0(\mathbf{s}, t)\mathcal{E}\{R(t, t')\} \\ &\quad + \int_{t_0}^{t} d\tau\, \mathcal{E}\{\Omega_1(\mathbf{s}, t)\mathcal{E}\{U(\mathbf{s}, t, \tau)\}\Omega_1(\mathbf{s}, \tau)\}\mathcal{E}\{R(\tau, t')\} \\ &\quad + \int_{t_0}^{t'} d\tau\, \mathcal{E}\{\Omega_1(\mathbf{s}, t)\mathcal{E}\{U(\mathbf{s}', t', \tau)\}\Omega_1(\mathbf{s}', \tau)\}\mathcal{E}\{R(t, \tau)\}.\end{aligned} \tag{4.9}$$

In deriving this equation we have made use of the fact that the propagators $U_A(\mathbf{s}, t, t_0)$ and $U_A(\mathbf{s}', t', \tau)$ in (3.9) commute. Furthermore, we have used the relations

$$\begin{aligned}U_A(\mathbf{s}, t, t_0)(I - A)\mathcal{E}\{R(t_0, \tau)\} &\approx (I - A)\mathcal{E}\{U(\mathbf{s}, t, t_0)\}\mathcal{E}\{R(t_0, \tau)\} \\ &\approx (I - A)\mathcal{E}\{U(\mathbf{s}, t, t_0)R(t_0, \tau)\} \\ &= (I - A)\mathcal{E}\{R(t, \tau)\},\end{aligned} \tag{4.10}$$

the last equality following from the semigroup property of the propagator U.

Equations (4.8) and (4.9), together with (2.2), form a complete set of closed equations for the smoothed quantities $\mathcal{E}\{\mu(t)\}$, $\mathcal{E}\{R(t, t')\}$, and $\mathcal{E}\{U(\mathbf{s}, t, t_0)\}$. It should be noted that since $\Omega_1 \sim \delta\mu$ by assumption, terms proportional to the covariance $\mathcal{E}\{\delta\mu(t)\delta\mu(t')\}$ appear in Eqs. (4.8), (4.9), and (2.2). However, making use of the formula $\mathcal{E}\{\mu(t)\mu(t')\} = (\mathcal{E}\{R(t, t')\}) = \mathcal{E}\{\mu(t)\}\mathcal{E}\{\mu(t')\} + \mathcal{E}\{\delta\mu(t)\delta\mu(t')\}$, the covariance function can be expressed in terms of first and second moments.

The resolution of the closure problem in the weak-coupling limit presented here has been achieved at a level of approximation corresponding to Kraichnan's direct-interaction approximation; hence the title of this subsection. It should be emphasized that in contradistinction to Kraichnan's direct-interaction approximation which is based on a stochastic modeling scheme, our technique has been developed along the lines of a modified Weinstock—Balescu—Misguich formalism. Also, whereas in Kraichnan's work the main results are expressed in terms of closed equations for the mean field, the covariance, and an averaged Green's function, our results are given in terms of closed equations for the first two moments and the mean propagator $\mathcal{E}\{U\}$.[12]

We wish to close this subsection with a few remarks concerning the difference between our technique for closing the equations for the first two moments and that reported recently by Misguich and Balescu (cf. Ref. 6). Their method is directly specialized to the problem of Vlasov-plasma turbulence with an external stochastic electric field. It is presented here in a more general setting so that comparisons with our work can be made more easily.

Starting from Eq. (2.5) for the fluctuating part of μ, viz.,

$$\delta\mu(t) = \int_{t_0}^{t} dt'\, U_A(t, t')\Omega_1(t')\mathcal{E}\{\mu(t')\}, \tag{4.11}$$

where the part proportional to $\delta\mu(t_0)$ is neglected for simplicity, they obtain in the weak-coupling limit

$$\delta\mu(t) = \int_{t_0}^{t} dt'\, \mathcal{E}\{U(t, t')\}\Omega_1(t')\mathcal{E}\{\mu(t')\}, \tag{4.12}$$

and, analogously,

$$\delta\mu(\tau) = \int_{t_0}^{\tau} dt''\, \mathcal{E}\{U(\tau, t'')\}\Omega_1(t'')\mathcal{E}\{\mu(t'')\}. \tag{4.13}$$

From the last two expressions, a relationship is set up for the covariance,

$$\begin{aligned}\mathcal{E}\{\delta\mu(t)\delta\mu(\tau)\} = \int_{t_0}^{t} dt' \int_{t_0}^{\tau} dt''\, &\mathcal{E}\{U(t, t')\}\mathcal{E}\{\Omega_1(t')\mathcal{E}\{\mu(t')\} \\ &\times \mathcal{E}\{U(\tau, t'')\}\Omega_1(t'')\}\mathcal{E}\{\mu(t'')\}.\end{aligned} \tag{4.14}$$

This equation contains only covariances and mean fields; therefore, together with the equation for the mean propagator $\mathcal{E}\{U(t, t')\}$ [cf. Eq. (4.8)] and the equation for the mean field (in what they call the renormalized quasilinear approximation), viz.,

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{E}\{\mu(t)\} &= \Omega_0(t)\mathcal{E}\{\mu(t)\} \\ &\quad + \int_{t_0}^{t} dt'\, \mathcal{E}\{\Omega_1(t)\mathcal{E}\{U(t, t')\}\Omega_1(t')\}\mathcal{E}\{\mu(t')\},\end{aligned} \tag{4.15}$$

constitutes a self-consistent set.

This type of closure, when specialized to linear stochastic problems considered in the first-order smoothing approximation ($U_A \to W$; also cf. the next subsection), has been criticized by Morrison and McKenna (cf. Ref. 7). We feel that our approach to the closure problem, based on a renormalization at the level of the second moment, is fundamentally different from that of Misguich and Balescu and is devoid of the aforementioned difficulties.

B. Quasilinear approximation

In this subsection we shall outline a procedure for closing the equations for the first two moments within the confines of the quasilinear approximation. The latter is essentially a perturbational method at a level lower than the direct-interaction approximation discussed earlier. It is applicable for small correlations and corresponds to retaining only the zero-order term in the series expansion (2.13), viz., $\Lambda_W \to W$. If this approximate expression for Λ_W is substituted into (3.15), one has[13]

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{E}\{R(t, t')\} &= \Omega_0(\mathbf{s}, t)\mathcal{E}\{R(t, t')\}\end{aligned}$$

$$+\int_{t_0}^{t} d\tau\, \mathcal{E}\{\Omega_1(\mathbf{s},t)W(\mathbf{s},t,\tau)\Omega_1(\mathbf{s},\tau)\}\mathcal{E}\{R(\tau,t')\}$$
$$+\int_{t_0}^{t'} d\tau\, \mathcal{E}\{\Omega_1(\mathbf{s},t)W(\mathbf{s}',t',\tau)\Omega_1(\mathbf{s}',\tau)\}\mathcal{E}\{R(t,\tau)\}. \tag{4.16}$$

In deriving this equation we have made use of the commutation of the operators $W(\mathbf{s}',t',\tau)$ and $W(\mathbf{s},t,t_0)$, as well as the semigroup property $W(\mathbf{s},t,t_0)\mathcal{E}\{R(t_0,\tau)\} = \mathcal{E}\{W(\mathbf{s},t,t_0)R(t_0,\tau)\} \approx \mathcal{E}\{R(t,\tau)\}$. (The last equality is valid only in the quasilinear approximation.)

Equations (4.16) and (2.9), together with (2.2), form a complete set of closed equations for the averaged quantities $\mathcal{E}\{\mu(t)\}$, $\mathcal{E}\{R(t,t')\}$, and $W(\mathbf{s},t,t_0)$.

5. HYDRODYNAMIC TURBULENCE

Let the motion of a fluid be described by the Navier—Stokes equations

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2 + \mathbf{v}(\mathbf{x},t)\cdot\frac{\partial}{\partial \mathbf{x}}\right)v_i(\mathbf{x},t) = -\frac{\partial}{\partial x_i}p(\mathbf{x},t) + f_i(\mathbf{x},t), \tag{5.1a}$$

$i=1,2,3$, and the incompressibility condition

$$\frac{\partial}{\partial x_i}v_i(\mathbf{x},t)=0. \tag{5.1b}$$

Here, $\mathbf{v}(\mathbf{x},t)$ is the fluid velocity vector field, ν is the kinematic viscosity, $\mathbf{f}(\mathbf{x},t)$ is an externally supplied vector forcing function, and $p(\mathbf{x},t)$ is the pressure divided by the density.

It is well known that at high Reynolds numbers the character of the fluid motion changes from laminar to turbulent. It is believed that the "chaotic" or turbulent flow is described adequately by the Navier—Stokes equations, the solutions of which (at high Reynolds numbers) are extremely unstable. Predictions concerning the turbulent flow on the basis of the Navier—Stokes equations would require the specification of initial conditions with unrealistic accuracy. Because of this, it is of interest to determine equations for smooth, mean quantities, such as the first and second moments of the velocity field. (The notion "mean" is used here synonymously with the ensemble average over various realizations of the same flow with different initial conditions.)

For simplicity we are going to deal with a "model" hydrodynamic turbulence, assuming that the pressure is uniform throughout the fluid volume. We shall also neglect the external force in (5.1a). The simplified Navier—Stokes equations are then of the general form (2.1), viz.,

$$\frac{\partial}{\partial t}v_i(\mathbf{x},t)=\Omega v_i(\mathbf{x},t) \tag{5.2a}$$

with

$$\Omega=\nu\nabla^2-\mathbf{v}(\mathbf{x},t)\cdot\frac{\partial}{\partial\mathbf{x}}. \tag{5.2b}$$

The mean and fluctuating parts of the operator Ω can be readily written down as follows:

$$\Omega_0=\nu\nabla^2-\mathcal{E}\{\mathbf{v}(\mathbf{x},t)\}\cdot\frac{\partial}{\partial\mathbf{x}}, \tag{5.3a}$$

$$\Omega_1=-\delta\mathbf{v}(\mathbf{x},t)\cdot\frac{\partial}{\partial\mathbf{x}}. \tag{5.3b}$$

A. Direct-interaction approximation

We are now in a position to write down explicitly a complete set of closed equations for the first two moments of the velocity field in the direct-interaction approximation (cf. Sec. 4).

The equation for the mean velocity field [cf. Eq. (2.2)] assumes the form

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2+\mathcal{E}\{\mathbf{v}(\mathbf{x},t)\}\cdot\frac{\partial}{\partial\mathbf{x}}\right)\mathcal{E}\{v_i(\mathbf{x},t)\} = -\mathcal{E}\{\delta\mathbf{v}(\mathbf{x},t)\cdot(\partial/\partial\mathbf{x})\delta v_i(\mathbf{x},t)\}. \tag{5.4}$$

Corresponding to Eq. (4.9) for the second moment, we have in this case

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\nu\nabla^2+\mathcal{E}\{\mathbf{v}(\mathbf{x},t)\}\cdot\frac{\partial}{\partial\mathbf{x}}\right)&\mathcal{E}\{R_{ij}(\mathbf{x},\mathbf{x}',t,t')\}\\
=\frac{\partial}{\partial x_k}\int_{t_0}^{t}d\tau\,&\mathcal{E}\{U(t,\tau)\mathcal{E}\{\delta v_k(\mathbf{x},t)\delta v_l(\mathbf{x},\tau)\}\\
&\times\frac{\partial}{\partial x_l}\mathcal{E}\{R_{ij}(\mathbf{x},\mathbf{x}',\tau,t')\}+\frac{\partial}{\partial x_k}\int_{t_0}^{t'}d\tau\,\mathcal{E}\{U(t',\tau)\}\\
&\times\mathcal{E}\{\delta v_k(\mathbf{x},t)\delta v_l(\mathbf{x}',\tau)\}\frac{\partial}{\partial x_l'}\mathcal{E}\{R_{ij}(\mathbf{x},\mathbf{x}',t,\tau)\},\end{aligned} \tag{5.5}$$

where $\mathcal{E}\{R_{ij}(\mathbf{x},\mathbf{x}',t,t')\}=\mathcal{E}\{v_i(\mathbf{x},t)v_j(\mathbf{x}',t')\}$. Finally, the equation for the mean propagator $\mathcal{E}\{U(t,t_0)\}$ [cf. Eq. (4.8)] becomes in this case

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\nu\nabla^2+\mathcal{E}\{\mathbf{v}(\mathbf{x},t)\}\cdot\frac{\partial}{\partial\mathbf{x}}\right)&\mathcal{E}\{U(t,t_0)\}\\
=\frac{\partial}{\partial x_i}\int_{t_0}^{t}d\tau\,&\mathcal{E}\{U(t,\tau)\}\mathcal{E}\{\delta v_i(\mathbf{x},t)\delta v_j(\mathbf{x},\tau)\}\\
&\times\frac{\partial}{\partial x_j}\mathcal{E}\{U(\tau,t_0)\}.\end{aligned} \tag{5.6}$$

In the derivation of (5.5) and (5.6) we made use of the incompressibility condition (5.1b).[14] The closure of Eqs. (5.4)—(5.6) is more clearly evident on recalling the formula $\mathcal{E}\{\delta v_i(\mathbf{x},t)\delta v_j(\mathbf{x}',t')\}=\mathcal{E}\{R_{ij}(\mathbf{x},\mathbf{x}',t,t')\} - \mathcal{E}\{v_i(\mathbf{x},t)\}\mathcal{E}\{v_j(\mathbf{x}',t)\}$.

B. Quasilinear approximation

In order to write a closed set of equations for the first two moments of the velocity field within the region of applicability of the quasilinear approximation (cf. Sec. 4), Eq. (5.4) for the mean velocity field is retained unaltered; however, in Eq. (5.5) for the second moment, $\mathcal{E}\{U(t,t')\}$ must be replaced by the propagator $W(t,t')$ which, in turn, satisfies in the case of hydrodynamic turbulence the following equation [also cf. Eq. (2.9)]:

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2+\mathcal{E}\{\mathbf{v}(\mathbf{x},t)\}\cdot\frac{\partial}{\partial\mathbf{x}}\right)W(t,t')=0,\quad t\geq t_0, \tag{5.7a}$$

$$W(t_0,t_0)=I. \tag{5.7b}$$

The solution of (5.7) for an unbounded region can be formally written as follows:

$$W(t,t_0)=X\exp\left[\int_{t_0}^{t}dt'\left(\nu\nabla^2-\mathcal{E}\{\mathbf{v}(\mathbf{x},t')\}\cdot\frac{\partial}{\partial\mathbf{x}}\right)\right]. \tag{5.8}$$

The basic closed equations for the first two moments derived on the basis of the quasilinear approximation by means of the procedure outlined above are valid for the general case of nonstationary, inhomogeneous, and anisotropic turbulence. These equations simplify considerably by introducing additional constraints.

As an illustration, we consider here the case of stationary turbulence. In this special case we need only be concerned with the equation of evolution of the correlation tensor of the velocity field, $\Gamma_{ij}(\mathbf{x},\mathbf{x}',\tau)\equiv\mathcal{E}\{v_i(\mathbf{x},t)\times v_j(\mathbf{x}',t-\tau)\}$. [The quantity $\mathcal{E}\{v_i(\mathbf{x},t)\}=\mathcal{E}\{v_i(\mathbf{x},t_0)\}$ is assumed to be given.] We put, also, $\nu=0$ in the expression (5.8) for the propagator $W(t,t_0)$. This means that we neglect the effect of viscosity on turbulence, but not on dissipation, an assumption that seems reasonable for well-developed turbulence. Under these restrictions, Eq. (5.8) reduces to

$$W(t,t_0)=\exp\left[\left(-\mathcal{E}\{\mathbf{v}(\mathbf{x},t_0)\}\cdot\frac{\partial}{\partial\mathbf{x}}\right)(t-t_0)\right] \tag{5.9}$$

(the time-ordering operator is the identity operator in this case), and the correlation tensor evolves in time as follows:

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2+\mathcal{E}\{\mathbf{v}(\mathbf{x},t_0)\}\cdot\frac{\partial}{\partial\mathbf{x}}\right)\Gamma_{ij}(\mathbf{x},\mathbf{x}',\tau)$$
$$=\int_0^\tau dt\,\mathcal{E}\left\{\delta v_k(\mathbf{x},\tau)\frac{\partial}{\partial x_k}\exp\left[\left(-\mathcal{E}\{\mathbf{v}(\mathbf{x},t_0)\}\cdot\frac{\partial}{\partial\mathbf{x}}\right)\times(\tau-t)\right]\delta v_l(\mathbf{x},t)\right\}\frac{\partial}{\partial x_l}\Gamma_{ij}(\mathbf{x},\mathbf{x}',t). \tag{5.10}$$

In order to evaluate explicitly the effect of the operator $\exp[-(\mathcal{E}\{\mathbf{v}\}\cdot(\partial/\partial\mathbf{x}))(\tau-t)]$ in (5.10) we will make additional assumptions, concerning the mean flow. We recall that $\exp[a(\partial/\partial x)]f(x)=f(x+a)$, and $\exp(A+B)=\exp A\exp B$ provided that $[A,B]_-=0$. We assume that the exponential operator in (5.10) is factorizable, i.e., the commutators are close to zero $[(\partial/\partial x_j)\mathcal{E}\{v_i\}\approx 0,\ i\neq j]$ or, for example, that we have a parallel mean flow, e.g., $\mathcal{E}\{\mathbf{v}\}=(0,0,\mathcal{E}\{v_3(\mathbf{x},t_0)\})$. In this case (5.10) reduces to the simpler equation

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2+\mathcal{E}\{\mathbf{v}(\mathbf{x},t_0)\}\cdot\frac{\partial}{\partial\mathbf{x}}\right)\Gamma_{ij}(\mathbf{x},\mathbf{x}',\tau)$$
$$=+\int_0^\tau dt\frac{\partial}{\partial x_k}\Gamma_{kl}[\mathbf{x},\mathbf{x}-\mathcal{E}\{\mathbf{v}\}(\tau-t),\tau-t]$$
$$\times\frac{\partial}{\partial x_l}\Gamma_{ij}[\mathbf{x}-\mathcal{E}\{\mathbf{v}\}(\tau-t),\mathbf{x}',t]-\int_0^\tau dt\,\mathcal{E}\{v_k(\mathbf{x},t_0)\}$$
$$\times\frac{\partial}{\partial x_k}\mathcal{E}\{v_l[\mathbf{x}-\mathcal{E}\{\mathbf{v}\}(\tau-t),t_0]\}$$
$$\times\frac{\partial}{\partial x_l}\Gamma_{ij}[\mathbf{x}-\mathcal{E}\{\mathbf{v}\}(\tau-t),\mathbf{x}',t]. \tag{5.11}$$

This equation for the correlation tensor is rendered closed on specifying the initial condition $\Gamma_{ij}(\mathbf{x},\mathbf{x}',0)$.

6. VLASOV-PLASMA TURBULENCE WITH AN EXTERNAL STOCHASTIC ELECTRIC FIELD

A collisionless plasma in the presence of an external stochastic electric field is governed by the one-species, self-consistent Vlasov—Poisson equation

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}}+\frac{e}{m}[\mathbf{E}^s(\mathbf{x},t)+\mathbf{E}^e(\mathbf{x},t)]\cdot\frac{\partial}{\partial\mathbf{v}}\right)f(\mathbf{x},\mathbf{v},t)=0, \tag{6.1a}$$

$$\nabla\cdot\mathbf{E}^s(\mathbf{x},t)=4\pi e\int_{R^3}d\mathbf{v}\,[f(\mathbf{x},\mathbf{v},t)-n_0\delta(\mathbf{v})]. \tag{6.1b}$$

Here, $f(\mathbf{x},\mathbf{v},t)$ is the particle distribution function, e and m are the charge and mass of the particle, respectively, n_0 is the density of a uniform background of neutralizing charge, $\mathbf{E}^s(\mathbf{x},t)$ is the self-electric field, and $\mathbf{E}^e(\mathbf{x},t)$ denotes the external stochastic electric field.

Equations (6.1) can be brought into the general form (2.1), viz.,

$$\frac{\partial}{\partial t}f(\mathbf{x},\mathbf{v},t)=\Omega f(\mathbf{x},\mathbf{v},t), \tag{6.2a}$$

with

$$\Omega=-\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}}-\frac{e}{m}[\mathbf{E}^s(\mathbf{x},t)+\mathbf{E}^e(\mathbf{x},t)]\cdot\frac{\partial}{\partial\mathbf{v}}, \tag{6.2b}$$

on introducing the relationship

$$\mathbf{E}^s(\mathbf{x},t)=L(\mathbf{x})[f(\mathbf{x},\mathbf{v},t)-n_0\delta(\mathbf{v})], \tag{6.3a}$$

where the operator $L(\mathbf{x})$ is defined by

$$L(\mathbf{x})f(\mathbf{x},\mathbf{v},t)=4\pi e\frac{\partial}{\partial\mathbf{x}}\int_{R^3}d\mathbf{x}'\int_{R^3}d\mathbf{v}'\,Q(\mathbf{x},\mathbf{x}')f(\mathbf{x}',\mathbf{v}',t). \tag{6.3b}$$

$Q(\mathbf{x},\mathbf{x}')$ is the Green's function for the Poisson equation. In the case of an infinite plasma,

$$Q(\mathbf{x},\mathbf{x}')=-\frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}'|}. \tag{6.3c}$$

The coherent and fluctuating parts of the operator Ω [cf. Eq. (6.2b)] are given as follows:

$$\Omega_0=-\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}}-\frac{e}{m}[\mathcal{E}\{\mathbf{E}^s(\mathbf{x},t)\}+\mathcal{E}\{\mathbf{E}^e(\mathbf{x},t)\}]\cdot\frac{\partial}{\partial\mathbf{v}}, \tag{6.4a}$$

$$\Omega_1=-\frac{e}{m}[\delta\mathbf{E}^s(\mathbf{x},t)+\delta\mathbf{E}^e(\mathbf{x},t)]\cdot\frac{\partial}{\partial\mathbf{v}}. \tag{6.4b}$$

A. Direct-interaction approximation

We next present a complete set of closed equations for the first two moments of the particle distribution function in the direct-interaction approximation.

The equation for the mean distribution function is given by

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}}+\frac{e}{m}\mathcal{E}\{\mathbf{E}^t(\mathbf{x},t)\}\cdot\frac{\partial}{\partial\mathbf{v}}\right)\mathcal{E}\{f(\mathbf{x},\mathbf{v},t)\}$$
$$=-\frac{e}{m}\mathcal{E}\{\delta\mathbf{E}^t(\mathbf{x},t)\cdot(\partial/\partial\mathbf{v})\delta f(\mathbf{x},\mathbf{v},t)\}, \tag{6.5}$$

where, for simplicity, we have used the shorter notation $\mathbf{E}^t(\mathbf{x},t)=\mathbf{E}^s(\mathbf{x},t)+\mathbf{E}^e(\mathbf{x},t)$. The equation for the second moment assumes in this case the following form:

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}}+\frac{e}{m}\mathcal{E}\{\mathbf{E}^t(\mathbf{x},t)\}\cdot\frac{\partial}{\partial\mathbf{v}}\right)\mathcal{E}\{R(\mathbf{s},\mathbf{s}',t,t')\}$$
$$=\left(\frac{e}{m}\right)^2\frac{\partial}{\partial v_i}\int_{t_0}^t d\tau\,\mathcal{E}\{U(\mathbf{s},t,\tau)\}\mathcal{E}\{\delta E_i^t(\mathbf{x},t)\delta E_j^t(\mathbf{x},\tau)\}$$
$$\times\frac{\partial}{\partial v_j}\mathcal{E}\{R(\mathbf{s},\mathbf{s}',\tau,t')\}+\left(\frac{e}{m}\right)^2\frac{\partial}{\partial v_i}\int_{t_0}^{t'}d\tau\,\mathcal{E}\{U(\mathbf{s}',t',\tau)\}$$
$$\times\mathcal{E}\{\delta E_i^t(\mathbf{x},t)\delta E_j^t(\mathbf{x}',\tau)\}\frac{\partial}{\partial v_j'}\mathcal{E}\{R(\mathbf{s},\mathbf{s}',t,\tau)\}, \tag{6.6}$$

where $\mathbf{s}=(\mathbf{x},\mathbf{v})$ and $\mathcal{E}\{R(\mathbf{s},\mathbf{s}',t,t')\}=\mathcal{E}\{f(\mathbf{s},t)f(\mathbf{s}',t')\}$. Finally, the equation for the mean propagator $\mathcal{E}\{U(\mathbf{s},t,t_0)\}$ becomes in this case

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}}+\frac{e}{m}\mathcal{E}\{\mathbf{E}^t(\mathbf{x},t)\}\cdot\frac{\partial}{\partial\mathbf{v}}\right)\mathcal{E}\{U(\mathbf{s},t,t_0)\}$$
$$=\left(\frac{e}{m}\right)^2\frac{\partial}{\partial v_i}\int_{t_0}^{t}d\tau\,\mathcal{E}\{U(\mathbf{s},t,\tau)\}\mathcal{E}\{\delta E_i^t(\mathbf{x},t)\delta E_j^t(\mathbf{x},\tau)\}$$
$$\times\frac{\partial}{\partial v_j}\mathcal{E}\{U(\mathbf{s},\tau,t_0)\}. \qquad (6.7)$$

Equations (6.5)–(6.7) constitute a closed self-consistent set (1) in the absence of an external stochastic electric field, and (2) in the case that $\mathcal{E}\{\delta\mathbf{E}^e(\mathbf{x},t)\delta f(\mathbf{x},\mathbf{v},t)\}=0$. Both of these restrictions can be lifted without too much difficulty. However, we shall not pursue this issue further in this paper.

B. Quasilinear approximation

Within the domain of validity of the quasilinear approximation, Eq. (6.5) for the mean particle distribution function is retained as it stands; however, in Eq. (6.6) for the second moment, $\mathcal{E}\{U(\mathbf{s},t,t')\}$ must be replaced by the propagator $W(\mathbf{s},t,t')$ which, for the problem under consideration here, satisfies the question

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}}+\frac{e}{m}\mathcal{E}\{\mathbf{E}^t(\mathbf{x},t)\}\cdot\frac{\partial}{\partial\mathbf{v}}\right)W(\mathbf{s},t,t_0)=0,\quad t\geq t_0, \qquad (6.8a)$$

$$W(\mathbf{s},t_0,t_0)=I. \qquad (6.8b)$$

The solution of (6.8) for an unbounded Vlasov plasma can be formally written down as follows:

$$W(\mathbf{s},t,t_0)=X\exp\left[\int_{t_0}^{t}dt'\left(-\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}}-\frac{e}{m}\mathcal{E}\{\mathbf{E}^s(\mathbf{x},t')\}\cdot\frac{\partial}{\partial\mathbf{v}}\right)\right]. \qquad (6.9)$$

The remarks at the end of the previous subsection concerning the closure of the resulting equations for the first two moments apply here as well.

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[9]The parameter α will be usually suppressed for convenience.
[10]For the sake of simplicity, we shall assume in the following discussion that Ω_1 has zero mean. This condition is stated mathematically as $A\Omega_1 A=0$.
[11]The explicit appearance of the arguments will permit the free interchange of the order of noncommuting quantities without committing an error.
[12]Strictly speaking, $\mathcal{E}\{U(\mathbf{s},t,t_0)\}=G^*$, where G is Kraichnan's averaged Green's function, and "*" is the operator of convolution with respect to the variables **s**.
[13]Equation (4.16) can also be obtained directly from (4.9) since Weinstock has shown that $\mathcal{E}\{U\}\to W$ for weak correlations.
[14]On the right-hand side of the Eqs. (5.5) and (5.6), the propagator $\mathcal{E}\{U(t,\tau)\}$ operates only on those functions that have τ as the time argument.