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The conditional entropy in the microcanonical ensemble

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The existence of the configurational microcanonical conditional entropy in classical statistical mechanics is proved in the thermodynamic limit for a class of long-range multiparticle observables. This result generalizes a theorem of Lanford for finite range observables.

Although a considerable amount of research has been directed in recent years toward proving the existence of the thermodynamic limit for the classical ensembles, the microcanonical ensemble for long-range interactions has presented certain difficulties. In this article we provide a proof of the existence of the configurational conditional entropy for a class of systems including long-range interactions falling off in ν dimensions faster than $1/r^\nu$.

Griffiths, using arguments of Fisher,[1,2] has outlined proofs of the existence of the microcanonical entropy for variously tempered two-body interactions. There the microcanonical energy is studied as a function of the entropy, and the entropy is recovered implicitly after the infinite volume limit has been taken. Similar results for two-body interactions were obtained by Minlos and Povzner.[3]

Lanford[4] has pointed out that methods of Ruelle[5] can be employed to obtain the entropy directly, and has used this approach to prove the existence of the configurational conditional entropy for strictly finite-range observables.

We use the Lanford approach to extend the existence theorem to observables with long-range behavior.

1. LIMIT ALONG A SPECIAL SEQUENCE OF CUBES

Let T^ν designate either the ν-dimensional lattice Z^ν or ν-dimensional real Euclidean space $\mathbb{R}^\nu$ with counting or Lebesgue measure μ, and denote the corresponding phase space by $\mathcal{G}$, $\mathcal{G}=\cup_{n=1}^{\infty}(T^\nu)^n$. The extension of μ to $(T^\nu)^n$ and $\mathcal{G}$ will also be written μ. If $Q_i\in\mathcal{G}\cap(T^\nu)^{n_i}$, $Q_i=(q_{i1},\ldots,q_{in_i})$, $i=1,2$, write $N(Q_i)=n_j$, $q_{ij}\in Q_i$, and $d(Q_1,Q_2)=\inf\{\tilde{d}(q_1,q_2)\mid q_i\in Q_i\}$, where $\tilde{d}:T^\nu\times T^\nu\to\mathbb{R}$ is the Euclidean metric. Let $\mathcal{S}$ be the set of bounded, measurable subsets of T^ν, and $\mathcal{C}_t$, the set of bounded open convex subsets of $\mathbb{R}^t$, $t\in Z_+$. If $J\in\mathcal{C}_t$ and $\epsilon>0$, then $J^\epsilon=\{x\in J\mid \|x-y\|>\epsilon,\ \forall\, y\in\mathbb{R}^t/J\}$ is the ϵ-contraction of J, and $J^{-\epsilon}=\{x\in\mathbb{R}^t\mid\|x-y\|<\epsilon$ for some $y\in J\}$.

Definition 1.1: The real linear space $\mathcal{A}_t^\lambda$ of t-valued observables, $t\in Z_+$ and $\lambda\in\mathbb{R}$, is the set of μ-measurable functions f: $\mathcal{G}\to\mathbb{R}^t$ satisfying the following:

(i) $f(Q+q)=f(Q)$, $Q\in\mathcal{G}$, $q\in T^\nu$, and $Q+q=\{p\in T^\nu\mid p-q\in Q\}$;

(ii) $f(Q)=f(Q')$ if Q' is a permutation of Q;

(iii) there exist $A>0$ and $R_0>0$ such that, for all $m\in Z_+$ and $Q_1,\ldots,Q_m\in\mathcal{G}$, $d(Q_i,Q_j)\geq r\geq R_0$ for all $i\neq j$ implies

$$\left\| f(Q_1,\ldots,Q_m)-\sum_{i=1}^m f(Q_i)\right\| \leq \frac{A}{r^\lambda}\left(\sum_{i=1}^m N(Q_i)\right)^2.$$

For $\Lambda\in\mathcal{S}$, $J\in\mathcal{C}_t$, $n\in Z_+$, and $f\in\mathcal{A}_t^\lambda$, the conditional phase space volume $\mathcal{V}_f$ is

$$\mathcal{V}_f(\Lambda,n,J)=(1/n!)\,\mu\{Q\in\Lambda^n\mid(1/n)f(Q)\in J\}.$$

The vector-valued observable f is to be viewed as a set of t translation-invariant symmetric scalar-valued observables, with a decrease condition (tempering) for each at large distances. For example, tempering would require a pair potential interaction generating a Hamiltonian to fall off at least as fast as $r^{-\lambda}$. Since the observable f will be fixed, the subscript on $\mathcal{V}$ will be dropped. Also, throughout it will be necessary to assume that $\lambda>\nu$.

Proposition 1.2: (a) If $J\in\mathcal{C}_t$ and $\Lambda\subset\Lambda'$, Λ, $\Lambda'\in\mathcal{S}$, then $\mathcal{V}(\Lambda',n,J)\geq\mathcal{V}(\Lambda,n,J)$.

(b) If $\{\Lambda_i\}_{i=1}^m\subset\mathcal{S}$, $d(\Lambda_i,\Lambda_j)\geq r\geq R_0$ for $i\neq j$, $n=\sum_{i=1}^m n_i$, $n_i\in Z_+$, and $\{J_i\}_{i=1}^m\subset\mathcal{C}_t$, then

$$\mathcal{V}\left(\bigcup_{i=1}^m\Lambda_i,\ \sum_{i=1}^m n_i,\ \sum_{i=1}^m\left(\frac{n_i}{n}\right)J_i\right)\geq\prod_{i=1}^m\mathcal{V}(\Lambda_i,n_i,J_i^{Amr^{-\lambda}}).$$

Proof: The first part is obvious. If $Q_i\in\Lambda_i^{n_i}$, then for $J=\sum_{i=1}^m(n_i/n)J_i$, $(1/n_i)f(Q_i)\in J_i^{Amr^{-\lambda}}$ implies $(1/n)f(Q_1,\ldots,Q_m)\in J$ by 1.1 (iii), since

$$\frac{1}{n}\sum_{i=1}^m f(Q_i)\in\sum_{i=1}^m\left(\frac{n_i}{n}\right)J_i^{Amr^{-\lambda}}=J^{Amr^{-\lambda}}.$$

Hence

$$\left\{(Q_1,\ldots,Q_m)\,\middle|\,Q_i\in\Lambda_i^{n_i}\text{ and }\frac{1}{n_i}f(Q_i)\in J_i^{Amr^{-\lambda}}\right\}$$

$$\subset\left\{Q\in\left(\bigcup_{i=1}^m\Lambda_i\right)^n\,\middle|\,\frac{1}{n}f(Q)\in\sum_{i=1}^m\left(\frac{n_i}{n}\right)J_i\right\}.$$

Define the density $\rho=n/\mu(\Lambda)$, the specific volume $v=1/\rho$, and $l_v=v^{1/\nu}$. For $\epsilon=\lambda-\nu>0$ as in 1.1 (ii), $\kappa\in(0,1)$, and $m\in Z_+$, let $\theta_\kappa=2^{(\nu+\kappa\epsilon)/\lambda}$, $\varphi_\kappa=[1-(2^\nu/\theta_\kappa^\lambda)]^{-1}$, $R_\kappa=R_0(2-\theta_\kappa)^{-1}$, $R_{\kappa,m}=\theta_\kappa^m R_0$, and $\Delta_{\kappa,m}=A\varphi_\kappa 2^{(m+1)\nu}/R_{\kappa,m}^\lambda$. Denote the cube $\Lambda_{\kappa,m}(v)=\{q=(q_1,\ldots,q_\nu)\in T^\nu\mid 0<q_k<2^m l_v-\theta_\kappa^m R_\kappa,\ k=1,\ldots,\nu\}$.

The reasons for choosing $\Lambda_{\kappa,m}(v)$ and $R_{\kappa,m}$ in this manner will be apparent from Proposition 1.3. In particular, φ_κ is chosen to make valid the last equality in the proof of that proposition. Note that as $m\to\infty$ $\Delta_{\kappa,m}$ goes to zero as $2^{-\kappa\epsilon m}$, and $J^{\Delta_{\kappa,m}}\to J$. The parameter κ is specified, since later some control will be needed over the rate of convergence of $\Delta_{\kappa,m}$ to zero.

Proposition 1.3: If $J\in\mathcal{C}_t$, then

$$\mathcal{V}(\Lambda_{\kappa,m+1}(v),\ 2^{(m+1)\nu}, J^{\Delta_{\kappa,m+1}}) \geq \{\mathcal{V}(\Lambda_{\kappa,m}(v), 2^{m\nu}, J^{\Delta_{\kappa,m}})\}^{2^\nu}.$$

Proof: Since $R_{\kappa,m}=(2^{m+1}l_v-\theta_\kappa^{m+1}R_\kappa)-2(2^m l_v-\theta_\kappa^m R_\kappa)$, 2^ν disjoint translates of $\Lambda_{\kappa,m}(v)$ with mutual separation equal to or greater than $R_{\kappa,m}$ can be placed inside $\Lambda_{\kappa,m+1}(v)$. Further,

$$(J^{\Delta_{\kappa,m+1}})^{A2^{(m+1)\nu}/R^\lambda_{\kappa,m}} = J^{\Delta_{\kappa,m}}.$$

The proof is completed using 1.2.

Corollary 1.4:

$$\frac{1}{2^{(m+1)\nu}}\log\mathcal{V}(\Lambda_{\kappa,m+1}(v),\ 2^{(m+1)\nu},\ J^{\Delta_{\kappa,m+1}})$$

$$\geq \frac{1}{2^{m\nu}}\log\mathcal{V}(\Lambda_{\kappa,m}(v), 2^{m\nu}, J^{\Delta_{\kappa,m}})$$

and the limit as $m\to\infty$ is equal to the supremum over $m\in Z_+$.

Definition 1.5: For $J\in\mathcal{C}_t$ and $x\in\mathbb{R}^t$, let

$$S_\kappa(v,J)=\lim_{m\to\infty}\frac{1}{2^{m\nu}}\log\mathcal{V}(\Lambda_{\kappa,m}(v), 2^{m\nu}, J^{\Delta_{\kappa,m}})$$

and

$$s_\kappa(v,x)=\inf_{x\in J\in\mathcal{C}_t}\{S_\kappa(v,J)\}.$$

Proposition 1.6: (i) If $J\subset J'$, $J,J'\in\mathcal{C}_t$, then $S_\kappa(v,J)\leq S_\kappa(v,J')\leq 1+\log v$.

(ii) If $\{J_i\}_{i=1}^k\subset\mathcal{C}_t$, $J_0=\cup_{i=1}^k J_i\in\mathcal{C}_t$, and for $\Delta>0$ sufficiently small, $\cup_{i=1}^k J_i^\Delta=J_0^\Delta$, then $S_\kappa(v,\cup_{i=1}^k J_i)=\sup_{1\leq i\leq k}S_\kappa(v,J_i)$.

(iii) If $J\in\mathcal{C}_t$, then $S_\kappa(v,J)=\sup\{S_\kappa(v,\hat J)\,|\,\hat J\in\mathcal{C}_t, \hat J^-\subset J\}$.

(iv) $S_\kappa(v,J)=\sup_{x\in J}\ S_\kappa(v,x)$.

Proof: Routine, using 1.2 and properties of μ.

Corollary 1.7: (i) $x\to s_\kappa(v,x)$ is upper semicontinuous and concave on $\mathbb{R}^t$.

(ii) $v\to s_\kappa(v,x)$ is nondecreasing and concave on $\mathbb{R}_+$, and continuous on (v_x,∞), where $v_x=\inf\{v\,|\,s_\kappa(v,x)>-\infty\}$.

When the tempering condition in the definition of $\mathcal{A}_t^\lambda$ is replaced by a finite range condition (additivity: $A=0$), then the interior of Γ_κ is nonempty if the components of f are linearly independent. More generally, a sort of asymptotic openness is required.

Proposition 1.8: Let $\Omega_\kappa(v)$ be the convex set

$$\Omega_\kappa(v)=\{x\in\mathbb{R}^t\,|\,s_\kappa(v,x)>-\infty\},$$

and, for $m\in Z_+$, let $E_m(v)=$ ess range $((1/2^{m\nu})f_{\kappa,m})$, where $f_{\kappa,m}$ is the restriction of f to $[\Lambda_{\kappa,m}(v)]^{2^{m\nu}}$. Write

$$\lim_{n\to\infty}\text{s. } E_m(v)$$
$$=\{x\in\mathbb{R}^t\,|\,0\cap E_m(v)\neq\emptyset \text{ for infinitely many } m, \text{ for each open } 0\subset\mathbb{R}^t,\ x\in 0\}.$$

Then

$$\Omega_\kappa(v)^- = \lim_{m\to\infty}\text{s. } E_m(v).$$

Proof: If $x_0\notin\Omega_\kappa(v)^-$, then there exists $J\in\mathcal{C}_t$ such that $x_0\in J$ and $S_\kappa(v,J)=-\infty$, hence $J^{\Delta_{\kappa,m}}\cap E_m(v)=\emptyset$ and $x_0\in J^{\Delta_{\kappa,m}}$ for m sufficiently large. Therefore, $x_0\notin \lim\text{s.}_{m\to\infty}E_m(v)$.

On the other hand, $x_0\notin\lim\text{s.}_{m\to\infty}E_m(v)$ implies $J\cap E_m(v)=\emptyset$ for some $J\in\mathcal{C}_t$, $x_0\in J$, and all sufficiently large m. Passing to the contraction $J^{\Delta_{\kappa,m}}$ gives the desired result.

Definition 1.9: We say that f is asymptotically open if there exists $v>0$ and $\kappa=\mathcal{K}$ such that $(\lim\text{s.}_{m\to\infty}E_m(v))^0\neq\emptyset$. We denote by $v_0(\kappa)$ the infimum over all such v for any $\kappa\leq\mathcal{K}$. In the remainder we will always assume that $\kappa\leq\mathcal{K}$.

If f is asymptotically open and $v>v_0(\kappa)$, then $\Omega_\kappa(v)^0\neq\emptyset$. Let $\Gamma_\kappa=\Gamma_\kappa(f)$ be the set $\{(v,x)\in\mathbb{R}_+\times\mathbb{R}^t\,|\,s_\kappa(v,x)>-\infty\}$.

Corollary 1.10: (i) Γ_κ^0 is convex.

(ii) If f is asymptotically open, then Γ_κ^0 is nonempty and dense in Γ_κ.

(iii) $(v,x)\to s_\kappa(v,x)$ is continuous and concave on Γ_κ^0.

Corollary 1.11: If f is asymptotically open and $0<v<v_0(\kappa)$, then $\Omega_\kappa(v)=\emptyset$. Hence $v_0(\kappa)=\inf_x v_x$.

2. INDEPENDENCE OF THE PARAMETER

We shall show in this section that the contraction parameter κ can be removed, and that the result coincides with the conditional entropy defined without contractions, at least for the limit taken along a special sequence of cubes.

Lemma 2.1: Let $f:\mathbb{R}^t\to\mathbb{R}\cup\{-\infty,\infty\}$ be upper semicontinuous and concave, $J_1,J_2\in\mathcal{C}_t$ with $J_1\cap J_2\neq\emptyset$, $f(J_1\cap J_2)\cap\mathbb{R}\neq\emptyset$, and $\{d_i\}_{i=1}^\infty$ a sequence of positive real numbers with $d_i\to 0$. Then

$$\inf_i\ \sup_{x\in J_1^{-d_i}\cap J_2} f(x)=\sup_{x\in J_1\cap J_2} f(x).$$

Proof: Since $J^{-d_i}\supset J$,

$$\inf_i\ \sup_{x\in J_1^{-d_i}\cap J_2} f(x)\geq\sup_{x\in J_1\cap J_2} f(x).$$

So assume $(J_1^{-d_i}/J_1)\cap J_2\neq\emptyset$ for all i, $\sup_{x\in J_1\cap J_2}f(x)<\infty$, and suppose

$$\inf_i\ \sup_{x\in J_1^{-d_i}\cap J_2} f(x)>\sup_{x\in J_1\cap J_2} f(x)+\tfrac{3}{2}\epsilon,\quad \epsilon>0.$$

Then, for each i,

$$\sup_{x\in(J_1^{-d_i}/J_1)\cap J_2} f(x)>\sup_{x\in J_1\cap J_2} f(x)+\tfrac{3}{2}\epsilon.$$

Let $y_i'\in(J_1^{-d_i}/J_1)\cap J_2$ be such that

$$f(y_i') > \sup_{x \in (J_1^{-d_i}/J_1) \cap J_2} f(x) - \epsilon/2$$

and $\{y_i\}_{i=1}^{\infty}$ a subsequence convergent to $y \in \partial(J_1 \cap J_2)$. If $z_i \to y$, by upper semicontinuity,

$$f(y) \geq \limsup_{i\to\infty} f(z_i) \geq \limsup_{i\to\infty} f(y_i) \geq \sup_{x \in J_1 \cap J_2} f(x) + \epsilon$$

and by concavity

$$\sup_{x \in (J_1 \cap J_2)^-} f(x) = \sup_{x \in J_1 \cap J_2} f(x),$$

which yields a contradiction.

Definition 2.2: For $v > 0$ and $\kappa \in (0, \mathcal{K})$, let $\widetilde{\mathcal{C}}_t(v, \kappa) = \{J \in \mathcal{C}_t \,|\, (\{v\} \times J) \cap \Gamma_\kappa \neq \emptyset$ or $d(\{v\} \times J, \Gamma_\kappa) > 0\}$. For $J \in \widetilde{\mathcal{C}}_t(v, \kappa)$ and $x \in \mathbb{R}^t$, define

$$S_{\kappa,0}(v, J) = \lim_{m\to\infty} \frac{1}{2^{m\nu}} \log \mathcal{V}(\Lambda_{\kappa,m}(v), 2^{m\nu}, J),$$

$$s_{\kappa,0}(v, x) = \inf_{x \in J \in \widetilde{\mathcal{C}}_t} S_{\kappa,0}(v, J),$$

and for $\kappa_1, \kappa_2 \in (0, \mathcal{K})$, $\alpha > 0$,

$$S_{\kappa_1,\kappa_2,\alpha}(v, J) = \lim_{m\to\infty} \frac{1}{2^{m\nu}} \log \mathcal{V}(\Lambda_{\kappa_1,m}(v), 2^{m\nu}, J^{\alpha\Delta_{\kappa_2,m}}$$

Theorem 2.3: If $0 < v \neq v_0(\kappa_i)$, $\alpha > 0$, $\kappa_1, \kappa_2 \in (0, \mathcal{K})$, and $J \in \widetilde{\mathcal{C}}_t(v, \kappa_1) \cap \widetilde{\mathcal{C}}_t(v, \kappa_2)$, then

(i) $S_{\kappa_1,0}(v, J) = S_{\kappa_1}(v, J)$,

(ii) $S_{\kappa_1}(v, J) = S_{\kappa_2}(v, J)$,

(iii) $S_{\kappa_1,\kappa_2,\alpha}(v, J) = S_{\kappa_1}(v, J)$.

Proof: For each $d > 0$ there exists a positive integer $m(d)$ satisfying

$$(1/2^{m\nu}) \log \mathcal{V}(\Lambda_{\kappa,m}(v), 2^{m\nu}, J) \leq (1/2^{m\nu}) \log \mathcal{V}(\Lambda_{\kappa,m}(v), \times 2^{m\nu}, J^{-d+\Delta_{\kappa,m}})$$

for $m > m(d)$. Hence, if $\{d_i\}_{i=1}^{\infty}$ is a sequence of positive numbers with $d_i \to 0$,

$$\limsup_{m\to\infty} (1/2^{m\nu}) \log \mathcal{V}(\Lambda_{\kappa,m}(v), 2^{m\nu}, J) \leq \inf_i S_\kappa(v, J^{-d_i}).$$

Now, by Proposition 1.6(iv) and the concavity of $x \to s_\kappa(v, x)$,

$$S_\kappa(v, J^{-d_i}) = \sup\{s_\kappa(v, x) \,|\, x \in J^{-d_i} \cap \Omega_\kappa(v)^0\}.$$

Therefore, for $(\{v\} \times J) \cap \Gamma_\kappa^0 \neq \emptyset$, by 1.7 and 2.1, $\inf_i S_\kappa(v, J^{-d_i}) = S_\kappa(v, J)$, and so

$$\limsup_{m\to\infty} (1/2^{m\nu}) \log \mathcal{V}(\Lambda_{\kappa,m}(v), 2^{m\nu}, J) \leq S_\kappa(v, J).$$

In the case $d(\{v\} \times J, \Gamma_\kappa) > 0$, if δ satisfies $(\{v\} \times J^{-\delta}) \cap \Gamma_\kappa = \emptyset$, then $S_\kappa(v, J^{-\delta}) = -\infty$, so that $S_\kappa(v, J) = -\infty$.

Assuming $\kappa_2 > \kappa_1$ and noting $\Lambda_{\kappa_2,m}(v) \subset \Lambda_{\kappa_1,m}(v)$, obtain $S_{\kappa_2,0}(v, J) \leq S_{\kappa_1,0}(v, J)$. On the other hand, letting $0 < v' < v$ if $v < v_0(\kappa_1)$, or $v_0(\kappa_1) < v' < v$ otherwise, compute

$$\lim_{m\to\infty} \frac{2^{m+1} l_v - \theta_{\kappa_2}^{m+1} R_{\kappa_2}}{2(2^m l_{v'})} = \frac{l_v}{l_{v'}} > 1,$$

so that 2^ν disjoint translates of the cube with sides $(0, 2^m l_{v'})$ can be placed inside $\Delta_{\kappa_2,m+1}(v)$ for all sufficiently large m, and hence 2^ν disjoint translates of $\Lambda_{\kappa,m}(v')$ with mutual separations equal to or greater than $R_{\kappa_1,m}$. Since

$$J^{\Delta_{\kappa_1,m+1}} \subset J^{\Delta_{\kappa_2,m+1}}$$

for sufficiently large m, therefore

$$\mathcal{V}(\Lambda_{\kappa_2,m+1}(v), 2^{(m+1)\nu}, J^{\Delta_{\kappa_2,m+1}}) \geq [\mathcal{V}(\Lambda_{\kappa_1,m}(v'), 2^{m\nu}, J^{\Delta_{\kappa_1,m}})]^{2^\nu}$$

and so

$$S_{\kappa_2}(v, J) \geq \sup_{v' < v} S_{\kappa_1}(v', J) = S_{\kappa_1}(v, J).$$

Thus (ii) follows from (i).

Finally, for $\kappa_2 > \kappa_1$, $d > 0$, and m sufficiently large,

$$\mathcal{V}(\Lambda_{\kappa_2,m}(v), 2^{m\nu}, J^{\Delta_{\kappa_2,m}}) \geq \mathcal{V}(\Lambda_{\kappa_2,m}(v), 2^{m\nu}, J^{\alpha\Delta_{\kappa_1,m}})$$

and

$$\mathcal{V}(\Lambda_{\kappa_2,m}(v), 2^{m\nu}, J^{\alpha\Delta_{\kappa_1,m}}) \geq \mathcal{V}(\Lambda_{\kappa_2,m}(v), 2^{m\nu}, J^{d+\Delta_{\kappa_2,m}})$$

so that

$$S_{\kappa_2}(v, J) \geq \limsup_{m\to\infty} (1/2^{m\nu}) \log \mathcal{V}(\Lambda_{\kappa_2,m}(v), 2^{m\nu}, J^{\alpha\Delta_{\kappa_1,m}})$$

and

$$\liminf_{m\to\infty} (1/2^{m\nu}) \log \mathcal{V}(\Lambda_{\kappa_2,m}(v), 2^{m\nu}, J^{\alpha\Delta_{\kappa_1,m}}) \geq \sup_{d>0} S_{\kappa_2}(v, J^d)$$

$$\geq \sup\{S_{\kappa_2}(v, \hat{J}) \,|\, \hat{J} \in \mathcal{C}_t \text{ and } \hat{J}^- \subset J\} = S_{\kappa_2}(v, J).$$

If $\kappa_2 < \kappa_1$, then, for m large,

$$\mathcal{V}(\Lambda_{\kappa_2,m}(v), 2^{m\nu}, J^{\alpha\Delta_{\kappa_1,m}}) \geq \mathcal{V}(\Lambda_{\kappa_2,m}(v), 2^{m\nu}, J^{\Delta_{\kappa_2,m}})$$

so that

$$\liminf_{m\to\infty} (1/2^{m\nu}) \log \mathcal{V}(\Lambda_{\kappa_2,m}(v), 2^{m\nu}, J^{\alpha\Delta_{\kappa_1,m}}) \geq S_{\kappa_2}(v, J).$$

On the other hand,

$$S_{\kappa_2}(v, J) = S_{\kappa_2,0}(v, J) \geq \limsup_{m\to\infty} (1/2^{m\nu}) \log \mathcal{V}(\Lambda_{\kappa_2,m}(v), 2^{m\nu}, J^{\alpha\Delta_{\kappa_1,m}}).$$

Corollary 2.4: For all $v > 0$, $v \neq v_0(\kappa_1)$, $\kappa_1, \kappa_2, \kappa_3 \in (0, \mathcal{K})$, and $x \in \mathbb{R}^t$,

$$S_{\kappa_1}(v, x) = S_{\kappa_2}(v, x) = S_{\kappa_3,0}(v, x).$$

Corollary 2.5: $\Gamma(f) = \Gamma_\kappa(f)$, $\widetilde{C}_t(v) = \widetilde{C}_t(v, \kappa)$, and $v_0 = v_0(\kappa)$ are independent of κ.

3. LIMIT ALONG GENERAL SEQUENCES

We wish to extend the results of the previous sections, derived for limits taken along the *standard* sequences of cubes $\{\Lambda_{\kappa,m}\}_{m=1}^{\infty}$, to limits along a more general sequence of domains.

For $\Lambda \in \mathcal{S}$ with boundary $\partial\Lambda$, let $V_\partial(r; \Lambda)$

$=\mu\{x\in T^\nu | d(x,\partial\Lambda)\le r\}$. A sequence $\{\Lambda_i\}_{i=1}^\infty\subset\mathcal{S}$ is said to tend to infinity in the sense of Fisher, $\Lambda_i\to\infty$ (Fisher) if[6] $V(\Lambda_i)\to\infty$ and there exists an $i_0\in Z_+$ such that

$$\lim_{\alpha\to 0}\sup_{i\ge i_0} V_\partial(\alpha V(\Lambda_i)^{1/\nu};\Lambda_i)/V(\Lambda_i)=0.$$

Lemma 3.1[1]: If $\Lambda\in\mathcal{S}$ is filled with cubes of edge-length d lying entirely within Λ, the volume remaining after maximal filling is less than $V_\partial(\nu^{1/\nu}d;\Lambda)$.

Suppose $\{\Lambda_i\}_{i=1}^\infty\subset\mathcal{S}$, $\Lambda_i\to\infty$ (Fisher), and $\{n_i\}_{i=1}^\infty\subset Z_+$ satisfy $\lim_{i\to\infty}V(\Lambda_i)/n_i=v\in(0,\infty)$. Then we will say that $\{(\Lambda_i,n_i)\}_{i=1}^\infty$ is a Fisher system tending to density $1/v$. In this case, let $\kappa,\kappa',\kappa''\in(0,\mathcal{K})$ be such that $\kappa''>\kappa'>\kappa$ and $\kappa''-\kappa'>\kappa$, for $n\in Z_+$; let $m(n)=\mathcal{N}((\log_2 n)[\nu+\kappa'(\lambda-\nu)]^{-1})$, where $\mathcal{N}(x)$ is the greatest integer in x, and define $n_i=m_i2^{m(n_i)\nu}+r_i$ with $0\le r_i<2^{m(n_i)\nu}$ and

$$r_i=\sum_{j=0}^{m(n_i)-1}C_{ij}2^{j\nu}$$

For $0<v'<v$, let $\zeta>1$ satisfy $v'<\zeta^\nu v'<v$, and write $\Lambda_m(\zeta^\nu v')$, $m\in Z_+$, for the cube in T^ν with edges $(0,2^m\zeta l_{v'})$. Finally, let $\mathcal{J}_i$ be the number of cubes in a maximal filling of Λ_i with translates of cubes $\Lambda_{m(n_i)}(\zeta^\nu v')$.

Lemma 3.2: For i sufficiently large, $m_i+\sum_{j=0}^{m(n_i)-1}C_{ij}$ disjoint translates of $\Lambda_{m(n_i)}(\zeta^\nu v')$ may be placed inside Λ_i.

Proof: Observe

$$\lim_{i\to\infty}\frac{\mathcal{J}_i2^{m(n_i)\nu}}{n_i}=\frac{v}{\zeta^\nu v'}\lim_{i\to\infty}\frac{\mathcal{J}_iV(\Lambda_{m(n_i)}(\zeta^\nu v'))}{V(\Lambda_i)}.$$

Therefore, if $\lim_{i\to\infty}\mathcal{J}_iV(\Lambda_{m(n_i)}(\zeta^\nu v'))/V(\Lambda_i)=1$, then $\lim_{i\to\infty}\mathcal{J}_i/m_i\ge v/\zeta^\nu v'$, and so there exists $\xi\in\mathbb{R}$, $\zeta^\nu v'/v<\xi<1$ such that $\mathcal{J}_i>m_i+(\xi v/\zeta^\nu v'-1)m_i$ for all large enough i. Hence $m_i+\mathcal{N}(m_i\xi v/\zeta^\nu v'-m_i)$ disjoint translates of $\Lambda_{m(n_i)}(\zeta^\nu v')$ may be placed inside Λ_i, and the lemma follows from obvious estimates.

Using 3.1 for $i\ge i_0$ and

$$H(\alpha)=\sup_{i\ge i_0}V_\partial(\alpha V(\Lambda_i)^{1/\nu},\Lambda_i)/V(\Lambda_i),$$

$$1-\frac{\mathcal{J}_iV(\Lambda_{m(n_i)}(\zeta^\nu v'))}{V(\Lambda_i)}\le\frac{V_\partial(\nu^{1/\nu}2^{m(n_i)}\zeta l_{v'};\Lambda_i)}{V(\Lambda_i)}$$

$$\le H\left(\frac{\nu^{1/\nu}2^{m(n_i)}\zeta l_{v'}}{V(\Lambda_i)^{1/\nu}}\right).$$

Choose $w\in\mathbb{R}$, $\zeta l_{v'}/l_v<w<1$, so that, for i large, $V(\Lambda_i)^{1/\nu}>wl_vn_i^{1/\nu}$. Then $\zeta l_{v'}/V(\Lambda_i)^{1/\nu}<n_i^{-1/\nu}$, and

$$H(\nu^{1/\nu}2^{m(n_i)}\zeta l_{v'}/V(\Lambda_i)^{1/\nu})\le H(\nu^{1/\nu}n_i^{1/[\nu+\kappa'(\lambda-\nu)]-1/\nu}).$$

But

$$\lim_{i\to\infty}n_i^{1/[\nu+\kappa'(\lambda-\nu)]-1/\nu}=0.$$

Theorem 3.3: Suppose $\{(\Lambda_i,n_i)\}_{i=1}^\infty$ is a Fisher system tending to density $1/v$, $v\ne v_0$. Let $\{d_i\}_{i=1}^\infty$ be a sequence of nonnegative real numbers, $d_i\to 0$, $\kappa\in(0,\mathcal{K})$, and $J\in\tilde{\mathcal{C}}_t(v)$. Then

$$\liminf_{i\to\infty}\frac{1}{n_i}\log\mathcal{V}(\Lambda_i,n_i,J^{d_i})\ge S_{\kappa,0}(v,J).$$

Proof: From 1.6 and 2.3, it is sufficient to show that there exists $\kappa''\in(0,\mathcal{K})$ such that for all $v_0<v'<v$ and $\hat{J}\in\mathcal{C}_t$ with $\hat{J}^-\subset J$,

$$\liminf_{i\to\infty}(1/n_i)\log\mathcal{V}(\Lambda_i,n_i,J^{d_i})\ge S_{\kappa''}(v',\hat{J}).$$

Suppose $\kappa,\kappa',\kappa''\in(0,\mathcal{K})$ as before, Lemma 3.2. By that lemma, for sufficiently large i, $m_i+\sum_{j=0}^{m(n_i)-1}C_{ij}$ disjoint translates of $\Lambda_{\kappa'',m(n_i)}(\zeta^\nu v')$ with mutual separation at least $\theta_{\kappa''}^{m(n_i)}R_0$ may be placed inside Λ_i. Therefore, for any $\tilde{J}\in\mathcal{C}_t$ and i sufficiently large,

$$\mathcal{V}_i\equiv\mathcal{V}\left(\Lambda_i,n_i,\frac{m_i2^{m(n_i)\nu}}{n_i}\hat{J}+\sum_{j=0}^{m(n_i)-1}\left(\frac{C_{ij}2^{j\nu}}{n_i}\right)\tilde{J}\right)$$

$$\ge[\mathcal{V}(\Lambda_{\kappa'',m(n_i)}(\zeta^\nu v'),2^{m(n_i)\nu},\hat{J}^{\Delta_i})]^{m_i}$$

$$\times\prod_{j=0}^{m(n_i)-1}[\mathcal{V}(\Lambda_{\kappa'',m(n_i)}(\zeta^\nu v'),2^{j\nu},\tilde{J}^{\Delta_i})]^{C_{ij}},$$

where $\Delta_i=An_i/\theta_{\kappa''}^{\lambda m(n_i)}R_0^\lambda$.

A short computation gives $\Delta_i<C\Delta_{\kappa''-\kappa',m(n_i)}$ for $C=2^{\kappa'(\lambda-\nu)}\varphi_{\kappa''-\kappa'}^{-1}$, and $\Delta_{\kappa,j}\ge A\varphi_\kappa2^{\nu+\kappa(\lambda-\nu)}/R_0^\lambda2^{\kappa(\lambda-\nu)m(n_i)}$ for $0\le j<m(n_i)$. Hence, for large enough i, $\Delta_i<\Delta_{\kappa,j}$, so that $\tilde{J}^{\Delta_{\kappa,j}}\subset\tilde{J}^{\Delta_i}$. Similarly, it is seen that $\Lambda_{\kappa,j}(v')\subset\Lambda_{\kappa'',m(n_i)}(\zeta^\nu v')$ and $\Lambda_{\kappa'',m(n_i)}(v')\subset\Lambda_{\kappa'',m(n_i)}(\zeta^\nu v')$. Writing $N_1=\inf\{m\in Z_+|\Lambda_{\kappa,m}(v')\ne\emptyset\}\in\{0,1,\ldots,m(n_i)-1\}$, we obtain

$$\mathcal{V}_i\ge[\mathcal{V}(\Lambda_{\kappa'',m(n_i)}(v'),2^{m(n_i)\nu},\hat{J}^{C\Delta_{\kappa''-\kappa',m(n_i)}})]^{m_i}$$

$$\times\prod_{j=0}^{N_1-1}[\mathcal{V}(\Lambda_{\kappa,N_1}(v'),2^{j\nu},J^{\Delta_{\kappa,j}})]^{C_{ij}}$$

$$\times\prod_{j=N_1}^{m(n_i)-1}[\mathcal{V}(\Lambda_{\kappa,j}(v'),2^{j\nu},\tilde{J}^{\Delta_{\kappa,j}})]^{C_{ij}}.$$

Next $\tilde{J}\in\mathcal{C}_t$ will be fixed to guarantee that each of the factors above will be nonzero. Choose $x_0\in\mathbb{R}^t$ such that $s_\kappa(v',x_0)>-\infty$ and let $J_{-1}\in\mathcal{C}_t$ be a rectangular solid with edges $\{(a_{-1k},b_{-1k})\}_{k=1}^t$ containing x_0. Then, for all $j\ge N_2$, $\mathcal{V}(\Lambda_{\kappa,j}(v'),2^{j\nu},J_{-1}^{\Delta_{\kappa,j}})>0$, where $N_2=\inf\{j\in Z_+|\mathcal{V}(\Lambda_{\kappa,j}(v'),2^{j\nu},J_{-1}^{\Delta_{\kappa,j}})>0\}$, and, for i large, $0\le N_2<m(n_i)$. Let $J_j\in\mathcal{C}_t$ be a rectangular solid with edges $\{(a_{jk},b_{jk})\}_{k=1}^t$ such that $\mathcal{V}(\Lambda_{\kappa,N_1}(v'),2^{j\nu},J_j^{\Delta_{\kappa,j}})>0$ for $0\le j<N_1$, and $\mathcal{V}(\Lambda_{\kappa,j}(v'),2^{j\nu},J_j^{\Delta_{\kappa,j}})>0$ for $N_1\le j<N_2$. Then define J to be a rectangle with sides $\{(a_k,b_k)\}_{k=1}^t$, where

$$a_k=(\inf_{-1\le j<N_3}a_{jk})-\Delta_{\kappa,0},\quad b_k=(\inf_{-1\le j<N_3}b_{jk})+\Delta_{\kappa,0},$$

$$N_3=\sup\{N_1,N_2\}.$$

Since

$$\lim_{i\to\infty}\sum_{j=0}^{m(n_i)-1}\frac{C_{ij}2^{j\nu}}{n_i}=0\quad\text{and}\quad\lim_{i\to\infty}\frac{m_i2^{m(n_i)\nu}}{n_i}=1,$$

for i large enough,

$$\frac{m_i 2^{m(n_i)\nu}}{n_i}\hat{J}+\sum_{j=0}^{m(n_i)-1}\left(\frac{C_{ij}2^{j\nu}}{n_i}\right)\tilde{J}\subset J^{d_i}.$$

Using this and 2.3(iii),

$$\liminf_{i\to\infty}\frac{1}{n_i}\log \mathcal{V}(\Lambda_i,n_i,J^{d_i})\geq S_{\kappa'}(v',\hat{J})+\liminf_{i\to\infty}\sigma_i$$

for

$$|\sigma_i|=\frac{1}{n_i}\left|\sum_{j=0}^{N_1-1}C_{ij}2^{j\nu}\frac{1}{2^{j\nu}}\log \mathcal{V}(\Lambda_{\kappa,N_1}(v'),2^{j\nu},\tilde{J}^{\Delta_{\kappa,j}})\right.$$
$$\left.+\sum_{j=N_1}^{m(n_i)-1}C_{ij}2^{j\nu}\frac{1}{2^{j\nu}}\log \mathcal{V}(\Lambda_{\kappa,j}(v'),2^{j\nu},\tilde{J}^{\Delta_{\kappa,j}})\right|$$

$$\leq\frac{m(n_i)2^{m(n_i)\nu}}{n_i}\sup\left\{\sup_{0\leq j<N_1}\left|\frac{1}{2^{j\nu}}\log \mathcal{V}(\Lambda_{\kappa,N_1}(v'),2^{j\nu},\tilde{J}^{\Delta_{\kappa,j}})\right|,\right.$$
$$\left.\sup_{N_1\leq j<m(n_i)}\left|\frac{1}{2^{j\nu}}\log \mathcal{V}(\Lambda_{\kappa,j}(v'),2^{j\nu},\tilde{J}^{\Delta_{\kappa,j}})\right|\right\}.$$

But the first term in the supremum is finite by construction of $\tilde{J}$ and nondependence of N_1 on i, and the second term is equal to

$$\sup\left\{\frac{1}{2^{N_1\nu}}\left|\log \mathcal{V}(\Lambda_{\kappa,N_1}(v'),2^{N_1\nu},\tilde{J}^{\Delta_{\kappa,N_1}})\right|,\right.$$
$$\left.\frac{1}{2^{(m(n_i)-1)\nu}}\left|\log \mathcal{V}(\Lambda_{\kappa,m(n_i)-1}(v'),2^{(m(n_i)-1)\nu},\tilde{J}^{\Delta_{\kappa,m(n_i)-1}})\right|\right\}.$$

Here again the first term is finite by choice of $\tilde{J}$, and the second term is bounded by $2+\log v'$, hence $\lim_{i\to\infty}|\sigma_i|=0$.

Lemma 3.4[1]: If $\{\Lambda_i\}_{i=1}^{\infty}\subset\mathcal{S}$, each Λ_i is connected, and $\Lambda_i\to\infty$ (Fisher), then $\inf_i V(\Lambda_i)/V(\tilde{\Lambda}_i)>0$, where $\tilde{\Lambda}_i$ is any minimal cube containing Λ_i.

Theorem 3.5: Suppose $\{(\Lambda_i,n_i)\}_{i=1}^{\infty}$ is a Fisher system tending to density $1/v$, each Λ_i is connected, $\kappa\in(0,\mathcal{K})$, $J\in\tilde{\mathcal{C}}_t(v)$, and $S_\kappa(v,J)>-\infty$. Then there exists a decreasing sequence $\{d_i\}_{i=1}^{\infty}$ of strictly positive real numbers converging to zero, such that

$$S_{\kappa,0}(v,J)\geq\limsup_{i\to\infty}\frac{1}{n_i}\log \mathcal{V}(\Lambda_i,n_i,J^{d_i}).$$

Proof: For $i\in Z_+$, let $p(i)=1+\inf\{m\in Z_+\,|\,\Lambda_{\kappa,m}(v)$ contains a translate Λ_i' of $\Lambda_i\}$, so that $\Lambda_{\kappa,p(i)-1}(v)$ is the minimal *standard* cube containing Λ_i. Writing $d_i=A\varphi_\kappa 2^{p(i)\nu}/\theta_\kappa^{\lambda p(i)}R_\kappa$, if $\tilde{\Lambda}_i$ is any minimal cube containing Λ_i, then

$$2^\nu\leq V(\Lambda_{\kappa,p(i)}(v))/V(\tilde{\Lambda}_i)\leq(2^\nu+1)^2$$

for i sufficiently large, so

$$\frac{1}{2^\nu}\geq\frac{V(\Lambda_i)}{V(\Lambda_{\kappa,p(i)}(v))}>\frac{1}{(2^\nu+1)^2}\inf_i\frac{V(\Lambda_i)}{V(\tilde{\Lambda}_i)}>0.$$

Now pass to a subsequence $\{\Lambda_{i_j}\}_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty}\frac{V(\Lambda_{i_j})}{V(\Lambda_{\kappa,p(i_j)}(v))}=\beta$$

and

$$\lim_{j\to\infty}\frac{1}{n_{i_j}}\log \mathcal{V}(\Lambda_{i_j},n_{i_j},J^{d_{i_j}})=\lim_{i\to\infty}\sup\frac{1}{n_i}\log \mathcal{V}(\Lambda_i,n_i,J^{d_i}).$$

Let $\Lambda_i''=\{q\in\Lambda_{\kappa,p(i)}(v)/\Lambda_i'\,|\,d(q,\Lambda_i')>\theta_\kappa^{p(i)}R_\kappa\}$ and $n_{i_j}''=2^{p(i_j)\nu}-n_{i_j}$. It can be shown, as in Ref. 1, that $\Lambda_i''\to\infty$ (Fisher) and it is easy to see that $\lim_{j\to\infty}V(\Lambda_{i_j}'')/V(\Lambda_{\kappa,p(i_j)}(v))=1-\beta$. By computing $\lim_{j\to\infty}n_{i_j}''/2^{p(i_j)\nu}=1-\beta$, therefore $\lim_{j\to\infty}V(\Lambda_{i_j}'')/n_{i_j}''=v$. Since Λ_{i_j}' and Λ_{i_j}'' may be translated inside $\Lambda_{\kappa,p(i_j)}(v)$ with mutual separation $\theta_\kappa^{p(i_j)}R_\kappa\geq R_0$, for large enough j,

$$\frac{1}{2^{p(i_j)\nu}}\log \mathcal{V}(\Lambda_{\kappa,p(i_j)}(v),2^{p(i_j)\nu},J)$$
$$\geq\frac{n_{i_j}}{2^{p(i_j)\nu}}\frac{1}{n_{i_j}}\log \mathcal{V}(\Lambda_{i_j},n_{i_j},J^{d_{i_j}})$$
$$+\frac{n_{i_j}''}{2^{p(i_j)\nu}}\frac{1}{n_{i_j}''}\log \mathcal{V}(\Lambda_{i_j}'',n_{i_j}'',J^{d_{i_j}}).$$

The theorem follows from 3.3.

Corollary 3.6: Suppose $\{(\Lambda_i,n_i)\}_{i=1}^{\infty}$ is a Fisher system tending to density $1/v$, $v\neq v_0$, with each Λ_i connected, and f is an asymptotically open t-valued observable with $\lambda>\nu$. Then, if $J\in\tilde{\mathcal{C}}_t(v)$,

$$\lim_{i\to\infty}\frac{1}{n_i}\log \mathcal{V}(\Lambda_i,n_i,J)=S(v,J)$$

exists, and

$$s(v,x)=\inf_{x\in J\in\tilde{\mathcal{C}}_t}S(v,J)$$

is given by

$$s(v,x)=s_{\kappa,0}(v,x),\quad\kappa\in(0,\mathcal{K}).$$

Therefore, s has the continuity and concavity properties of Corollaries 1.7 and 1.10.

The existence of the limit follows from 3.3 and 3.5 by removal of the contractions from J^{d_i} as in 2.3, and from an argument similar to 3.5.

We note that when $t=1$ and f is the potential energy U of a tempered interaction, $s(v,\epsilon)$ is the usual microcanonical configurational entropy per particle for a system at density $1/v$ and interaction energy per particle ϵ.

[1]M.E. Fisher, Arch. Ratl. Mech. Anal. 17, 377 (1964).
[2]R. Griffiths, J. Math. Phys. 6, 1447 (1965).
[3]R.A. Minlos and A.Ya. Povzner, Trud. Mosk. Obshch. 17, 243 (1967).
[4]O. Lanford, in *Lecture Notes in Physics*, *Vol.* 20 (Springer, New York, 1973).
[5]D. Ruelle, J. Math. Phys. 6, 201 (1965).
[6]D. Ruelle, *Statistical Mechanics* (Benjamin, New York, 1969).