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Conservative neutron transport theory*

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A functional analytic development of the Case full-range and half-range expansions for the neutron transport equation for a conservative medium is presented. A technique suggested by Larsen is used to overcome the difficulties presented by the noninvertibility of the transport operator K^{-1} on its range. The method applied has considerable advantages over other approaches and is applicable to a class of abstract integro–differential equations.

I. INTRODUCTION

The neutron transport equation for a "conservative" half-space ($c=1$ in one-speed theory) presents special complications for essentially technical reasons. The orthodox Case approach to the one-speed situation was originally worked out by Shure and Natelson,[1] while Greenberg and Zweifel[2] used the Larsen—Habetler resolvent integration technique[3] to treat the same equation. We restrict our attention in this paper to the resolvent integration method and point out that the special difficulties encountered for $c=1$ (cf. Ref. 2 for a detailed discussion) occur because the transport operator K^{-1} is not invertible on its range for that situation. The standard technique, originally introduced by Lekkerkerker[4] is to restrict K^{-1} to a suitable subspace of its domain on which it is invertible, deal with the restricted operator of the standard Larsen—Habetler scheme, and eventually extend the result to the full domain. While this technique in fact works, it is somewhat cumbersome and introduces notational complexities, especially in the conservative multigroup case[5] which is, of course, a generalization of the one-speed situation and has been treated by the same technique. (We should point out that the solutions to the conservative transport equation are of considerable physical importance, especially in obtaining asymptotic solutions to ordinary transport equations in the boundary layer.[6,7])

Recently, we have been studying some problems in plasma oscillations and rarefied gas dynamics where the ordinary Larsen—Habetler technique is not directly applicable because the operator corresponding to $K=(K^{-1})^{-1}$ of the neutron transport equation is unbounded. (In the neutron case for $c=1$ the operator K^{-1} is unbounded but K is bounded. In the plasma and gas case both operators are unbounded.) It is in fact possible to integrate the resolvent about an unbounded spectrum, as has been done by Bareiss,[8] but the technique involves approximating the transport operator by a sequence of bounded operators and is somewhat cumbersome. Larsen suggested another approach, namely to define an operator $S=(K-zI)^{-1}$, where z is some complex number not in the spectrum of K.[9] Then S is a bounded, invertible operator, and the whole machinery of the resolvent integration technique can be applied to S. This technique has proved extremely fruitful in treating the plasma and gas problems and has, in fact been generalized to treat a class of abstract integro-differential equations.[10] In the process of writing out these cases, we suddenly realized that the same technique could be applied to the conservative neutron transport case, with considerable simplification over the treatments of Refs. 2 and 5. We present the analysis in this paper omitting many of the calculational details because they have already appeared in the above cited references. In Sec. II we treat the one-speed case and Sec. III the multigroup equations.

II. THE CASE EIGENFUNCTION EXPANSION FOR A CONSERVATIVE MEDIUM

We follow the notation of Refs. 2 and 3 to write the transport equation for $c=1$ in the form

$$\frac{\partial\psi}{\partial x}(x,\mu)+K^{-1}\psi(x,\mu)=\frac{q(x,\mu)}{\mu},\quad \mu\neq 0, \tag{1a}$$

with

$$(K^{-1}\psi)(\mu)=\frac{1}{\mu}\left[\psi(\mu)-\tfrac{1}{2}\int_{-1}^{+1}\psi(s)\,ds\right]. \tag{1b}$$

We note that K^{-1} is not invertible on its range. In fact, the vectors $e_0(\mu)=1$ and $e_1(\mu)=\mu$, $-1<\mu<1$, span the $\lambda=0$ root linear manifold of K^{-1}.[2] Furthermore, as is well known, the spectrum of K^{-1} is confined to the real line. Thus, the operator $S=(K^{-1}-iI)^{-1}$ is a bounded invertible operator. We easily compute

$$(S\psi)(\mu)=\frac{\mu}{1-i\mu}\psi(\mu)+\frac{(1-i\mu)^{-1}}{2\Lambda(i)}\int_{-1}^{+1}\frac{s\psi(s)\,ds}{1-is}. \tag{2}$$

As in Ref. 2, we work in the space $X_p=\{f\mid \mu f\in L_p(-1,1)\}$ but restrict S to the space $H_p=\{f\in X_p\mid f \text{ is of class } H^*\}$; the final results can then be extended to X_p by continuity. Here $\Lambda(z)$ is the usual dispersion function for $c=1$:

$$\Lambda(z)=1-\frac{z}{2}\int_{-1}^{+1}\frac{ds}{z-s}. \tag{3}$$

We now proceed to deal with S by the technique of Ref. 3, i.e., we compute the resolvent and by contour integration of the resolvent about the spectrum of S we obtain the desired Case eigenfunction expansion. The resolvent is seen to be

$$(zI-S)^{-1}\psi(\mu)=\frac{(1+iz)^{-1}}{t^{-1}(z)-\mu}\left\{(1-i\mu)\psi(\mu)+\frac{(1+iz)^{-1}}{2\Lambda(t^{-1}(z))}\int_{-1}^{+1}\frac{s\psi(s)\,ds}{t^{-1}(z)-s}\right\}, \tag{4a}$$

with

$$t(z) = z/(1 - iz) \tag{4b}$$

and

$$t(t^{-1}(z)) = z, \tag{4c}$$

so that

$$t^{-1}(z) = z/(1 + iz) \tag{4d}$$

The spectrum of S can be computed by studying the analytic structure of the resolvent or one can use the spectral mapping theorem to transform the spectrum of K^{-1}. In either case one finds

$$\sigma(S) = P\sigma(S) \cup C\sigma(S), \tag{5a}$$

with

$$P\sigma(S) = \{i\} \tag{5b}$$

and

$$C\sigma(S) = \{z \,|\, z = \tfrac{1}{2}(i + e^{i\theta}), \quad -\pi \leq \theta \leq 0\}. \tag{5c}$$

$C\sigma(S)$ is, of course, a semicircle. Furthermore, the point i is an eigenvalue of multiplicity 2.

We now utilize the identity

$$\frac{1}{2\pi i} \int_C (zI - S)^{-1} dz = I, \tag{5d}$$

where the contour C surrounds the spectrum of S. As usual C is "squeezed" down into a contour Γ surrounding $C\sigma(S)$ and a contour Γ_i surrounding the eigenvalue i. We compute the two contributions to (5d) separately. First consider

$$\frac{1}{2\pi i} \int_\Gamma (zI - S)^{-1}\psi(\mu)\, dz$$
$$= \frac{1}{2\pi i} \int_{\Gamma'} \frac{1}{z' - \mu} \left(\psi(\mu) + \frac{\Lambda^{-1}(z')}{2} \int_{-1}^{+1} \frac{s\psi(s)\, ds}{z' - s}\right) dz'. \tag{6}$$

Here Γ' is any contour surrounding the cut $[-1, 1]$. Equation (6) was obtained simply by integrating (4a) around the semicircle $C\sigma(S)$ and introducing the change of variable $z' = t^{-1}(z) = z/(1 + iz)$. This is precisely the result of Ref. 2 for the branch cut integration. Thus we are led directly to the standard formula[2,3]

$$\frac{1}{2\pi i} \int_\Gamma (zI - S)^{-1}\psi(\mu)\, dz = \int_{-1}^{+1} A(\nu)\, \phi_\nu(\mu)\, d\nu, \tag{7a}$$

with

$$A(\nu) = \frac{1}{N(\nu)} \int_{-1}^{+1} \mu\psi(\mu)\, \phi_\nu(\mu)\, d\mu, \tag{7b}$$

$$\phi_\nu(\mu) = \frac{\nu}{2} P \frac{1}{z - \mu} + \tfrac{1}{2}[\Lambda^+(\nu) + \Lambda^-(\nu)], \tag{7c}$$

and

$$N(\nu) = \nu\Lambda^+(\nu)\Lambda^-(\nu). \tag{7d}$$

As usual we denote

$$\Lambda^\pm(\nu) = \lim_{\epsilon \to 0} \Lambda(\nu \pm i\epsilon), \quad -1 < \nu < 1.$$

The integration around Γ_i of (4a) involves the evaluation of a residue at a second order pole [since $\Lambda(t^{-1}(i)) = \Lambda'(t^{-1}(i)) = 0$]. Using the standard residue formula

$$\frac{1}{2\pi i} \oint \frac{p(z)}{q(z)}\, dz$$
$$= \frac{2}{3[q''(z_0)]^2} [3p'(z_0)\, q''(z_0) - p(z_0)\, q'''(z_0)], \tag{8a}$$

if $q(z_0) = q'(z_0) = 0$, and identifying

$$p(z) = \tfrac{1}{2}[z(1 - i\mu) - \mu]^{-1} \int_{-1}^{+1} \frac{s\psi(s)\, ds}{z(1 - is) - s} \tag{8b}$$

and

$$q(z) = \Lambda(t^{-1}(z)), \tag{8c}$$

one easily finds

$$\frac{1}{2\pi i} \int_{\Gamma_i} (zI - S)^{-1}\, \psi(\mu)\, dz$$
$$= \frac{3}{2} \left[\mu \int_{-1}^{+1} s\psi(s)\, ds + \int_{-1}^{+1} s^2\psi(s)\, ds\right]. \tag{9}$$

If one now combines Eqs. (6) and (9), one obtains Eq. (10) of Ref. 2, i.e., the Case full-range expansion formula for $c = 1$,

$$\psi(\mu) = \tfrac{1}{2}a_0 - \tfrac{1}{2}a_1\mu + \int_{-1}^{+1} A(\nu)\, \phi_\nu(\mu)\, d\nu, \tag{10}$$

where the expansion coefficients a_i are defined by

$$a_i = 3 \int_{-1}^{+1} (-\mu)^{2-i}\, \psi(\mu)\, d\mu. \tag{11}$$

We now sketch the procedure that can be used to obtain the Case half-range expansion. As usual, we define an operator $E: X_p' \to X_p$, where X_p' is the space of functions $f: [0, 1] \to \mathbb{C}$ with

$$\|f\|_{p'} = [\int_0^1 |uf(u)|^p\, du]^{1/p} < \infty,$$

and we require

(i) $(E\psi)(\mu) = \psi(\mu), \quad \mu > 0,$

(ii) $(zI - S)^{-1}E\psi$ is analytic for $\mathrm{Re}z < 0$,

(iii) $(zI - S)^{-1}E\psi$ has at worst a simple pole at $z = i$.

(12)

Condition (ii) will guarantee that in the integral of $(zI - S)^{-1}E\psi$ around a contour containing the spectrum of S there will be no contribution from the portion of $C\sigma(S)$ with $\mathrm{Re}z < 0$. Because the transformation $S \to K^{-1}$ maps $C\sigma(S)$, $\mathrm{Re}z < 0$ into $[-1, 0)$, this assures that no negative Case continuum modes will occur in the full range expansion of $E\psi$, i.e., the half-range expansion of $\psi \in X_p'$. Condition (iii) guarantees that the discrete coefficient a_1 does not enter into the half-range expansion of ψ. These conditions could be used to derive the operator E, but the result would be the same as that used in Ref. 2. Therefore, we shall only verify that the operator E as given in Ref. 2,

$$Ef(\mu) = \begin{cases} \dfrac{1}{X(\mu)} \dfrac{3}{2} \displaystyle\int_0^1 \frac{sf(s)\, ds}{X(-s)(s - \mu)}, & \mu < 0, \\ f(\mu), \quad \mu > 0, \end{cases} \tag{13}$$

has the correct properties. Here $X(z)$ provides the

Wiener–Hopf factorization of $\Lambda(z)^2$:

$$X(z)X(-z)=3\Lambda(z),$$

where $X(z)$ is analytic in $\mathbb{C}\setminus[0,1]$ and vanishes as $1/z$ as $|z|\to\infty$.

When we substitute Eq. (13) into Eq. (4a), we find after simplification that $(zI-S)^{-1}E\psi$ satisfies

$$[(zI-S)^{-1}E\psi](\mu)$$
$$=[z(1-i\mu)-\mu]^{-1}\quad(1-i\mu)\psi(\mu)+\frac{3}{2}\frac{1}{X(t^{-1}(z))}$$
$$\times\int_0^1\frac{s\psi(s)\,ds}{X(-s)[z(1-is)-s]}\Bigg\},\quad\mu>0 \qquad (14a)$$

$$[(zI-S)^{-1}E\psi](\mu)$$
$$=\frac{3}{2[z(1-i\mu)-\mu]}\Bigg\{\int_0^1\frac{s\psi(s)}{X(-s)}\Bigg[\frac{1-i\mu}{X(\mu)(s-\mu)}$$
$$+\frac{1}{X(t^{-1}(z))[z(1-is)-s]}\quad ds\Bigg\},\quad\mu<0. \qquad (14b)$$

Equation (14) can be used to quickly verify that $(zI-S)^{-1}E\psi$ satisfies properties (ii) and (iii). To see this, note that t^{-1} maps the left half complex plane into itself and is analytic except for a simple pole at $z=i$. Thus $X(t^{-1}(z))$ is analytic for $\operatorname{Re}z<0$. Moreover, for $\mu>0$ and $\operatorname{Re}z<0$, $z(1-i\mu)-\mu$ does not vanish. Therefore, from Eq. (14a) we have that $(zI-S)^{-1}E\psi(\mu)$ is analytic in z for $\operatorname{Re}z<0$ and $\mu>0$. To see that $(zI-S)^{-1}E\psi$ is analytic for $\operatorname{Re}z<0$ when $\mu<0$, we need only check that $z=\mu/(1-i\mu)$ is not a singularity of $(zI-S)^{-1}E\psi$. This is done by recalling from Eq. (4d) that $t^{-1}(\mu/(1-i\mu))=\mu$. Thus $(zI-S)^{-1}E\psi$ is analytic for $\operatorname{Re}z<0$. At $z=i$, we note from Eq. (14) that $(zI-S)^{-1}E\psi$ has a simple pole induced by the zero of $X(t^{-1}(z))$.

Integrating $(zI-S)^{-1}E\psi(\mu)$ on z along a contour containing the point i and the semicircle $\{z\,|\,z=\frac{1}{2}(i+e^{i\theta}),\ -\pi/2<\theta<0\}$ yields the Case half-range eigenfunction expansion.

III. CONSERVATIVE MULTIGROUP TRANSPORT

We now derive the result of Ref. 5 in the same simple manner used in Sec. II. We define

$$(K^{-1}\psi)(x,\mu)$$
$$=(1/\mu)[\Sigma\psi(x,\mu)-C\int_{-1}^{+1}\psi(x,s)\,ds],\quad\mu\neq0. \qquad (15)$$

Here ψ is an N-component vector where the ith component represents the neutron angular densities in the ith group, Σ is the diagonal cross-section matrix, and C the group–group transfer matrix. The appropriate space to seek a solution is, as in Ref. 5, the space

$$X_p^N=\bigotimes_{j=1}^{N}X_p.$$

As in the one-speed case, the computations are done in a dense subspace of Hölder continuous functions, and can be extended to X_p^n by continuity.[11]

We have the dispersion function

$$\Lambda(z)=(\Sigma-2C)C^{-1}\Sigma-\int_{-1}^{+1}\mu D(z,\mu)\,d\mu, \qquad (16a)$$

where

$$D(z,\mu)=(zI-\mu\Sigma^{-1})^{-1}. \qquad (16b)$$

As in Ref. 5, we consider the conservative case for which $\det(\Sigma-2c)=0$. In this case K^{-1} given by Eq. (14) is not invertible on its range. Thus defining S as before, i.e., $S=(K^{-1}-iI)^{-1}$, we find

$$S\eta(\mu)=B(\mu)\{\mu\eta(\mu)+\Sigma[C^{-1}-\int_{-1}^{+1}B(s)\,ds]^{-1}\int_{-1}^{+1}sB(s)\eta(s)\,ds\}, \qquad (17a)$$

where

$$B(\mu)=(\Sigma-i\mu I)^{-1}. \qquad (17b)$$

We have assumed $z=i$ is in the resolvent set of K^{-1}. If not, any other point could be chosen assuming the spectrum of K does not consist of the entire complex plane. Furthermore, we have assumed that $\det\Lambda(z)$ vanishes as $1/z^2$ as $|z|\to\infty$.

It is convenient to define

$$F(z,\mu)=(zI-\mu B(\mu))^{-1}. \qquad (18)$$

Then a direct computation gives

$$(zI-S)^{-1}\psi(\mu)=F(z,\mu)\{\psi(\mu)+B(\mu)R^{-1}(z)$$
$$\times[C^{-1}-\int_{-1}^{+1}B(s)\,ds]^{-1}\int_{-1}^{+1}tB(t)F(z,t)\psi(t)\,dt\}. \qquad (19a)$$

Here we have defined

$$R(z)=I-[C^{-1}-\int_{-1}^{+1}B(s)\,ds]^{-1}\int_{-1}^{+1}tB^2(t)F(z,t)\,dt. \qquad (19b)$$

R is actually related to the dispersion matrix $\Lambda(z)$, Eq. (16a), by

$$R(z)=[C^{-1}-\int_{-1}^{+1}B(s)\,ds]^{-1}\Sigma^{-1}\Lambda(t^{-1}(z))\Sigma^{-1}. \qquad (20)$$

Since $\det\Lambda(z)$ has a double zero at infinity, it follows that $\det R(z)$ will have a double zero at $t(\infty)=i$. The continuous spectrum of K transforms into the semicircle given by Eq. (5c) and the additional eigenvalues of K [zeros of $\Lambda(z)$] transform by $\nu_i\to t(\nu_i)$.

The eigenfunction expansion is again obtained by integrating the resolvent around the spectrum. The integration around the continuous spectrum can be transformed into the identical form found in Ref. 5 (or see the result for the subcritical situation which is also identical)[12] by the change of variable $z'=t^{-1}(z)$. Similarly the integrals about the isolated point eigenvalues ν_i can, by the same change of variables, be transformed into the expansions met in Refs. 13 and 5. Only the contribution from the double pole at $+i$ remains to be evaluated. Again the appropriate residue for a second order pole must must be used.

We proceed to evaluate $(1/2\pi i)\int_{\Gamma_i}(zI-S)^{-1}\psi(\mu)\,dz=I_1$. We have from Eqs. (19a) and (20)

$$I_1=\frac{1}{2\pi i}\int_{\Gamma_i}\Bigg(F(z,\mu)B(\mu)\frac{\Lambda_c(t^{-1}(z))}{\det\Lambda(t^{-1}(z))}$$
$$\times\Sigma\int_{-1}^{+1}sB(s)F(z,s)\psi(s)\,ds\Bigg)\,dz. \qquad (21)$$

From the diagonal expansion of the $\det\Lambda(z)$,[14] we find

$$\det\Lambda(z)=\det(\Sigma C^{-1}\Sigma-2\Sigma)+\frac{2}{3z^2}\operatorname{Tr}\Sigma^{-1}\Lambda_c(z)+O(1/z^4), \tag{22}$$

where $\Lambda_c(z)$ is the cofactor matrix of $\Lambda(z)$. Note by definition of the critical multigroup problem, the first term of the rhs of Eq. (22) vanishes. The second term gives the residue which we need. The result is

$$I_1=\tfrac{3}{2}[\operatorname{Tr}\Sigma^{-1}\Lambda_c(\infty)]^{-1}\{\mu\Sigma^{-1}\Lambda_c(\infty)\int_{-1}^{+1}s\psi(s)\,ds + \Lambda_c(\infty)\Sigma^{-1}\int_{-1}^{+1}s^2\psi(s)\,ds\}. \tag{23}$$

For use in solving transport problems, it is convenient, if not essential, to recast this result as expansion coefficients multiplying eigenvectors of K (or K^{-1}). This, in fact, is the form in which the result was expressed in Ref. 5. This is accomplished by representing $\Lambda_c(\infty)$ as (details found in Ref. 15)

$$\Lambda_c(\infty)=\tfrac{2}{3}\operatorname{Tr}[\Sigma^{-1}\Lambda_c(\infty)]\ \Sigma\xi\otimes\xi, \tag{24a}$$

where $\hat{\xi}$ and ξ are certain null vectors introduced by Ref. 5,

$$\Lambda(\infty)\xi=0 \tag{24b}$$

and

$$\Lambda^T(\infty)\Sigma\hat{\xi}=0. \tag{24c}$$

The normalization $\hat{\xi}^T\xi=\frac{3}{2}$ has been imposed. Using this representation, we obtain finally the eigenfunction expansion of Ref. 5, which is

$$\psi(\mu)=\sum_{i=1}^{2n}\psi_{\nu_i}+\psi_\Gamma+\int_{-1}^{+1}d\mu\,\mu^2[\psi(\mu),\hat{\xi}]\,\xi + \left(\int_{-1}^{+1}d\mu\,\mu[\psi(\mu),\Sigma\hat{\xi}]\right)\mu\Sigma^{-1}\xi. \tag{25}$$

The first term on the rhs is surely the contribution from the finite eigenvalues of K. This, along with ψ_I, is identical with the subcritical result obtained in Ref. 12. Only the contribution from the eigenvalue at infinity is essentially different in the critical case.

For the half-space expansion, again an "albedo operator" E must be introduced. This operator has precisely the same properties as in the one-speed case, Sec. II. The appropriate E is

$$(E\psi)_i(\sigma_i\mu)=\begin{cases}-[X^{-1}(\mu)\int_0^1 s(\mu-s)^{-1}Y^{-1}(-s)\Sigma^2\psi_\Sigma(s)\,ds]_i, & -1\le\sigma_i\mu\le 0,\\ \psi_i(\sigma_i\mu), & \mu>0,\end{cases} \tag{26}$$

where X and Y provide the Wiener–Hopf factorization of Λ, as in Ref. 5:

$$Y(-z)X(z)=\Lambda(z). \tag{27}$$

We now compute

$$(1/2\pi i)\int(zI-S)^{-1}E\psi(\mu)\,dz \tag{28}$$

about the spectrum of S and thereby obtain the half-range expansion formula,

$$\psi(\mu)=\sum_{i=1}^{n}\psi_{\Gamma_i}+\psi_\Gamma+\left(\int_{-1}^{+1}s^2\,ds[E\psi(s),\hat{\xi}]\right)\xi, \tag{29}$$

where ψ_{Γ_i} and ψ_Γ are defined in Ref. 12.

IV. CONCLUSION

We feel that the results of Refs. 2 and 5 have been obtained obtained in the present paper in a much simpler and thereby more elegant fashion. In particular we have avoided the introduction of subspaces Y_p and restrictions of operators, etc. However, we point out that the method described here is quite general and will permit us to study large classes of unbounded and/or noninvertible operators. The problems posed by critical neutron transport is that the point spectrum extends to infinity. The "Larsen transform" utilized here reduces both classes of problems to tractable form.

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