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Coupling constant thresholds of perturbed periodic Hamiltonians

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We consider Schrödinger operators of the form $H_\lambda = -\Delta + V + \lambda W$ on $L^2(\mathbf{R}^\nu)$ ($\nu=1$, 2, or 3) with V periodic, W short range, and λ a real non-negative parameter. Then the continuous spectrum of H_λ has the typical band structure consisting of intervals, separated by gaps. In the gaps there may be discrete eigenvalues of H_λ that are functions of the parameter λ. Let (a,b) be a gap and $E(\lambda)\in(a,b)$ an eigenvalue of H_λ. We study the asymptotic behavior of $E(\lambda)$ as λ approaches a critical value λ_0, called a coupling constant threshold, at which the eigenvalue either emerges from or is absorbed into the continuous spectrum. A typical question is the following: Assuming $E(\lambda)\downarrow a$ as $\lambda\downarrow\lambda_0$, is $E(\lambda)-a\sim c(\lambda-\lambda_0)^\alpha$ for some $\alpha>0$ and $c\neq 0$, or is there an expansion in some other quantity? As one expects from previous work in the case $V=0$, the answer strongly depends on ν. © *1998 American Institute of Physics.* [S0022-2488(98)03007-2]

I. INTRODUCTION

We consider the family of Schrödinger operators,

$$H_\lambda = H_0 + \lambda W, \tag{1.1}$$

on $L^2(\mathbf{R}^\nu)$, $\nu=1,2,3$, where

$$H_0 = -\Delta + V, \tag{1.2}$$

V is periodic and W is a short-range perturbation. The parameter λ, called the coupling constant, is assumed to be real and non-negative. Specific conditions will be placed on V and W below; they will guarantee that the spectra of H_0 and H_λ exhibit the qualitative behavior described in this Introduction. One important restriction on V that we should mention at the outset is that in two and three dimensions we assume V to be of a form which allows us to solve the eigenvalue problem for H_0 by separation of variables. As is typical for Schrödinger operators with a periodic potential, the spectrum $\sigma(H_0)$ of H_0 consists of bands separated by gaps, where in one dimension the generic situation is that there are infinitely many gaps, while in two and three dimensions the number of gaps is finite. By a gap in $\sigma(H_0)$ we mean an interval (a,b) such that $a,b\in\sigma(H_0)$ and $(a,b)\cap\sigma(H_0)=\emptyset$. For $\lambda>0$, the spectrum of H_λ is made up of a continuous part which coincides with the spectrum of H_0 and a discrete part consisting of at most a finite number of eigenvalues (counting multiplicities) in every gap. Since there are no eigenvalues when $\lambda=0$, by general results from perturbation theory, eigenvalues of H_λ can appear in a gap only by emerging from one of its endpoints as λ is varied. Likewise, eigenvalues can disappear from a gap only by converging to an endpoint. A value λ_0 is called a coupling constant threshold (henceforth abbreviated as c.c.th.) for the family H_λ at the endpoint a, resp. b, of a gap (a,b), if there is an eigenvalue branch $E(\lambda)$ of H_λ such that $E(\lambda)\downarrow a$, resp. $E(\lambda)\uparrow b$, as $\lambda\downarrow\lambda_0$ or $\lambda\uparrow\lambda_0$. The case where $E(\lambda)\downarrow a$ as $\lambda\uparrow\lambda_0$ describes an eigenvalue that approaches the endpoint a from the right and is absorbed into the continuous spectrum at $\lambda=\lambda_0$. The case where $E(\lambda)\downarrow a$ as $\lambda\downarrow\lambda_0$ corresponds to the situation where an eigenvalue appears at a as $\lambda\uparrow\lambda_0+\epsilon$ $(\epsilon>0)$. The third possibility is that

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there is an eigenvalue converging to a both as $\lambda\uparrow\lambda_0$ and as $\lambda\downarrow\lambda_0$. In other words, an eigenvalue comes in from the right and ''turns around'' at a as λ is increased. Of course, analogous possibilities exist at the endpoint b.

The main object of this paper is to study the analytic behavior of $E(\lambda)$ when λ is near a c.c.th. λ_0. Our analysis rests on the Birman–Schwinger principle which allows us to recast the eigenvalue problem for H_λ as an eigenvalue problem for a compact integral operator. Depending on the dimension we will employ slightly different versions of the Birman–Schwinger principle. For any fixed $\lambda\geqslant 0$ we introduce the operators

$$K_{\lambda,E}=W^{1/2}(H_\lambda-E)^{-1}|W|^{1/2},\quad W^{1/2}=|W|^{1/2}\mathrm{sgn}\ W, \tag{1.3}$$

where sgn denotes the sign function, and call them Birman–Schwinger kernels, since in the applications these operators will be represented by explicit integral kernels. In one dimension the relevant kernel will be $K_{\lambda_0,E}$, where λ_0 is a c.c.th.; in two and three dimensions it will be $K_{0,E}$. The Birman–Schwinger principle says that

$$H_\lambda\psi=E\psi, \tag{1.4}$$

with $E\in(a,b)$ and $\psi\in L^2(\mathbf{R}^\nu)$ if and only if

$$(a)\ \ (\lambda-\lambda_0)K_{\lambda_0,E}f=-f,\quad \nu=1,$$

$$(b)\ \ \lambda K_{0,E}f=-f,\quad \nu=2,3,$$

where f and ψ are related by $f=W^{1/2}\psi$ and $\psi=-[\lambda-\lambda_0](H_{\lambda_0}-E)^{-1}|W|^{1/2}f$ in case (a), resp. $\psi=-\lambda(H_0-E)^{-1}|W|^{1/2}f$, in case (b). Moreover, the multiplicity of E as an eigenvalue of H_λ is the same as the (geometric) multiplicity of -1 as an eigenvalue of $(\lambda-\lambda_0)K_{\lambda_0,E}$, resp. $\lambda K_{0,E}$. Note that the Birman–Schwinger kernel is not self-adjoint unless W has constant sign, so that in general one has to distinguish between the algebraic and geometric multiplicity of an eigenvalue.

The main reason for using different kernels depending on the dimension is that in one dimension it is useful to think of $(\lambda-\lambda_0)W$ as a perturbation of H_{λ_0} because the solutions to $H_{\lambda_0}\psi=E\psi$ can be constructed using familiar techniques from the theory of integral equations and $K_{\lambda_0,E}$ can be expressed in terms of these solutions. One could also use $K_{0,E}$ in one dimension, but we were able to obtain stronger results by using $K_{\lambda_0,E}$. In two and three dimensions it is more convenient to view λW as a perturbation of H_0 because our assumptions on V will allow us to get detailed information about the resolvent $(H_0-E)^{-1}$ and, therefore, about $K_{0,E}$.

For later use we give some more details of our approach when $\nu=1$ or 2. We choose the notation appropriate for the one-dimensional case but keep it sufficiently general so that the two-dimensional case is included by setting $\lambda_0=0$. Consider the specific situation where $E(\lambda)\uparrow b$ as $\lambda\downarrow\lambda_0$. It turns out that as in the case $V=0$, the kernel $K_{\lambda_0,E}$ can be decomposed into two parts: a singular part and a regular part. The singular part is of finite rank (rank one if $\nu=1$) and diverges as $E\uparrow b$, while the regular part has a limit (in Hilbert–Schmidt norm) as $E\uparrow b$. The general form of this decomposition is

$$K_{\lambda_0,E}=Q_{\lambda_0,E}+R_{\lambda_0,E}, \tag{1.5}$$

$$Q_{\lambda_0,E}=\sum_{j=1}^{N} d_{\lambda_0;j}(E)L_{\lambda_0,E;j},\quad L_{\lambda_0,E;j}\cdot=\omega_{\lambda_0,E;j}(\tilde{\omega}_{\lambda_0 E;j},\cdot). \tag{1.6}$$

Here and in the sequel $(\ ,\ )$ denotes the inner product in $L^2(\mathbf{R}^\nu)$ such that $(f,g)=\int_{\mathbf{R}^\nu}\overline{f(x)}g(x)d^\nu x$. The sets of vectors $\{\omega_{\lambda_0,E;j}\}_{j=1}^N$ and $\{\tilde{\omega}_{\lambda_0,E;j}\}_{j=1}^N$ are separately linearly independent and $\lim_{E\uparrow b}\ \omega_{\lambda_0,E;j}$ and $\lim_{E\uparrow b}\ \omega_{\lambda_0,E;j}$ exist in the L^2-norm. The coefficients $d_{\lambda_0;j}(E)$ are real and diverge as $E\uparrow b$ in a specific manner depending on ν. So, in (1.5), $Q_{\lambda_0,E}$ is the singular part and $R_{\lambda_0,E}$ is the regular part. In one dimension $N=1$, but in two dimensions any $N>1$ is possible. Note that for $|\lambda-\lambda_0|$ small enough,

$$(\lambda-\lambda_0)K_{\lambda_0,E}f=-f \Leftrightarrow [\lambda-\lambda_0](1+[\lambda-\lambda_0]R_{\lambda_0,E})^{-1}Q_{\lambda_0,E}f=-f.$$

By using (1.6), the right-hand is seen to be equivalent to

$$(\mathbf{I}_N+\mathbf{A})\mathbf{v}=0, \tag{1.7}$$

where $\mathbf{I}_N$ denotes the identity matrix in $\mathbf{C}^N$, $\mathbf{A}$ is the $N\times N$ matrix with entries

$$A_{i,j}=(\lambda-\lambda_0)d_{\lambda_0;j}(E)(\tilde{\omega}_{\lambda_0,E;i},(1+[\lambda-\lambda_0]R_{\lambda_0,E})^{-1}\omega_{\lambda_0,E;j}),$$

and $\mathbf{v}$ is the vector with components $v_j=(\tilde{\omega}_{\lambda_0,E;j},f)$, $j=1,...,N$. If $\mathbf{v}$ is known, then f is given by

$$f=-(\lambda-\lambda_0)\sum_{j=1}^{N} d_{\lambda_0;j}(E)(1+[\lambda-\lambda_0]R_{\lambda_0,E})^{-1}\omega_{\lambda_0,E;j}v_j.$$

From (1.7) we conclude that a nontrivial solution $\mathbf{v}$ exists if and only if $\det[\mathbf{I}_N+\mathbf{A}]=0$, that is,

$$\det[\delta_{ij}+(\lambda-\lambda_0)d_{\lambda_0;j}(E)(\tilde{\omega}_{\lambda_0,E;i},(1+[\lambda-\lambda_0]R_{\lambda_0,E})^{-1}\omega_{\lambda_0,E;j})]_{i,j=1}^{N}=0, \tag{1.8}$$

where δ_{ij} is the Kronecker delta. This is the equation which we will use to investigate $E(\lambda)$.

We remark that a similar approach has been used by Simon[1] and Holden[2] when $V=0$ and $\nu=2$. An alternative method to study $E(\lambda)$ is based on eigenvalue perturbation theory. The reader is referred to Refs. 1 and 3–5, and, in the special case when V is a Kronig–Penney potential, to Ref. 6. The eigenvalue perturbation method has the drawback that in some cases, when certain integrals involving W are zero, it becomes rather cumbersome and causes complications because $K_{\lambda_0,E}$ is in general not self-adjoint. At this point we should mention that for most results in this paper we do not require W to have constant sign. It is mainly for these reasons that we have decided not to use eigenvalue perturbation theory in one and two dimensions in this paper.

In three dimensions the situation is different in that the operator $K_{0,E}$ converges to a limit, $K_{0,b}$, as $E\uparrow b$, where $K_{0,b}$ is compact. Then an eigenvalue $E(\lambda)$ is associated with an eigenvalue branch $\tau(E)$ of $K_{0,E}$ such that $\lim_{E\uparrow b}\tau(E)$ exists and is finite, and the eigenvalue $E(\lambda)$ is found by solving $\lambda\tau(E)=-1$ for $E(\lambda)$. Moreover, the c.c.th.'s are given by $-1/\tau_j(b)$, where $\tau_j(b)$ $(j=1,2,...)$ denotes the negative eigenvalues of $K_{0,b}$. Thus, in three dimensions, $\lambda_0=0$ is never a c.c.th. at b (under our assumptions on V and W). Similarly, $\lambda_0=0$ is never a c.c.th. at a and hence, if λ is sufficiently small, H_λ has no eigenvalues in the gap (a,b).

We would like to mention that part of our motivation for studying the c.c.th. behavior of perturbed periodic Hamiltonians came from seminars given by two of our colleagues, Chris Beattie and George Hagedorn, on spectral properties of operators of the type considered here. Both of these individuals have also done numerical calculations of eigenvalue branches in dimensions one and two.[7,8] Further motivation came from earlier work on the subject in the case $V=0$,[1–5] the Kronig–Penney case,[6] and from related work on c.c.th.'s[9,10] and on eigenvalues in spectral gaps.[11–15]

This paper is organized as follows. Section II is devoted to the one-dimensional case. We describe the various threshold behaviors that can occur and find the leading order asymptotics of the eigenvalues. In some cases we also find higher order corrections to the leading term. We will see that generally $E(\lambda)-b\sim-c(\lambda-\lambda_0)^{2m}$ as $\lambda\downarrow\lambda_0$ or $\lambda\uparrow\lambda_0$, with $c>0$ and m a positive integer. Similar results hold at a. However, there are various situations where we can say more. For example, if $b=\inf\sigma(H_0)$ and $\lambda_0=0$, then only $m=1$ and 2 can occur, while if $\lambda_0>0$, then only $m=1$ is possible (Theorems 2.11 and 2.12). If b (or a) is the endpoint of a finite gap, then a similar result holds provided the support of W is suitably restricted (Theorem 2.16). Without such a restriction any m seems to be possible. For an eigenvalue that ''turns around'' we always have $m\geqslant 2$ with m even [Theorem 2.12(ii) and (v)]. Because we have made an effort to keep the assumptions on W as weak as possible, there are technical details that require a careful discussion and which have caused this section to become larger than originally intended. In order to avoid extensive digressions we have moved the proofs of five technical lemmas to an Appendix.

In Section III we consider some special cases and extensions of the results of Section II, and compare them with similar results in the literature. In particular, we present an example illustrating the case $\lambda_0 = 0$, $E(\lambda) - a \sim c\lambda^8 (c>0)$, and we briefly discuss the case when $W(x)$ is of the form $\pm\delta(x-x_0)$.

In Section IV we consider the two-dimensional case. In view of the length of Section II we content ourselves with a discussion of the case $\lambda_0 = 0$. The typical leading behavior at the right endpoint of a gap in two dimensions is $E(\lambda) - b \sim -c_1 e^{-c_2/\lambda^s}$ as $\lambda\downarrow 0$, with c_1, $c_2 > 0$, and s a positive integer. If $b = \inf\sigma(H_0)$, then generally $s=1$, but $s=2$ is also possible if a certain integral involving W vanishes. At other endpoints any s seems to be possible, but $s=1$ is the generic situation. There is also the possibility that more than one distinct eigenvalue approaches an endpoint as $\lambda\downarrow 0$, or that an eigenvalue is degenerate.

Finally, in Section V we briefly consider the three-dimensional case, restricting ourselves to c.c.th.'s at the bottom of the essential spectrum which we assume to be at zero. The main result, Theorem 5.3, shows that there are two possible behaviors: either $E(\lambda) \sim -c(\lambda-\lambda_0)^2$ or $E(\lambda) \sim -c(\lambda-\lambda_0)$ as $\lambda\downarrow\lambda_0$, with $c>0$, where the second possibility can only occur if zero is an eigenvalue of H_{λ_0}. This type of result was to be expected in light of previous work on the subject.[9]

II. ONE DIMENSION: GENERAL RESULTS

In this section we consider the operator family (1.1) on $L^2(\mathbf{R})$ under the assumptions that V and W are real-valued,

$$V \in L^1_{loc}(\mathbf{R}), \quad V(x+p) = V(x), \quad \text{for some } p>0, \tag{2.1}$$

and that W satisfies one of the following conditions:

$$\int_{-\infty}^{\infty} (1+|x|)|W(x)|dx < \infty, \tag{H1}$$

$$\int_{-\infty}^{\infty} (1+x^2)|W(x)|dx < \infty, \tag{H2}$$

$$\int_{-\infty}^{\infty} e^{c|x|}|W(x)|dx < \infty, \quad c>0. \tag{H3}$$

Condition (2.1) implies that V is $-d^2/dx^2$-form bounded with relative bound zero (see Ref. 16, p. 8). Thus $H_0 = -d^2/dx^2 + V$ can be defined as a self-adjoint, semi-bounded operator by means of the KLMN theorem (see Ref. 17, Theorem VI.2.1; Ref. 18, Theorem X.17). Alternatively, the differential operator defined by H_0 is limit-point at $\pm\infty$. Hence H_0 can be interpreted as the unique self-adjoint extension of the minimal operator $H_{0,\min}$ with domain $D(H_{0,\min}) = \{\psi \in L^2 : \psi, \psi' \in AC_{loc}$, supp ψ compact, $H_0\psi \in L^2\}$ (see Ref. 19, p. 41). Here AC_{loc} denotes the set of locally absolutely continuous functions on $\mathbf{R}$. If (H1) holds, then the multiplication operator W is a relatively form compact perturbation of H_0 (cf. Ref. 20, pp. 114 and 369) and this is equivalent to the compactness of the Birman–Schwinger kernel. It is well-known that the spectrum of H_0 is absolutely continuous and of the form

$$\sigma(H_0) = \bigcup_{n=0}^{\infty} [E_{2n}, E_{2n+1}],$$

where $-\infty < E_0 < E_1 \leqslant E_2 < E_3 \leqslant E_4 \cdots$. So the intervals $[E_{2n}, E_{2n+1}]$ are the bands and the intervals (E_{2n-1}, E_n) (with $E_{-1} = -\infty$) are the gaps. In general a gap may be empty (E_{2n-1}

$=E_n$), but in this paper we will only be concerned with nonempty gaps. The spectral parameter E will always be assumed to lie in a gap or at an endpoint of a gap. We first summarize some well-known facts about the equation

$$H_0\psi=E\psi. \tag{2.2}$$

This equation has the two solutions (also known as Floquet solutions or Bloch waves),

$$\psi_0^{(\pm)}(x,E)=\theta_0(x,E)+m_0^{(\pm)}(E)\phi_0(x,E), \tag{2.3}$$

where $\theta_0(x,E)$ and $\phi_0(x,E)$ are solutions of (2.2) satisfying for all E the initial conditions $\theta_0(0,E)=\phi_0'(0,E)=1$, $\theta_0'(0,E)=\phi_0(0,E)=0$, and $m_0^{(\pm)}(E)$ are the Titchmarsh–Weyl m-functions associated with the intervals $\mathbf{R}_+=[0,\infty)$ and $\mathbf{R}_-=(-\infty,0]$, respectively. Explicitly,

$$m_0^{(\pm)}(E)=\frac{\phi_0'(p,E)-\theta_0(p,E)}{2\phi_0(p,E)}\pm i\,\frac{\sin[q(E)]}{\phi_0(p,E)}, \tag{2.4}$$

$$q(E)=\cos^{-1}[\Delta(E)],\quad \Delta(E)=\frac{1}{2}\,[\phi_0'(p,E)+\theta_0(p,E)]. \tag{2.5}$$

Here $\Delta(E)$ is the Floquet determinant which is related to the spectrum of H_0 by $\sigma(H_0)=\{E:|\Delta(E)|\leqslant 1\}$. The branch of $\cos^{-1}$ in (2.5) is chosen such that $q(E)$ is continuous, $\mathrm{Im}[q(E)]>0$ for $E<E_0$, and $q(E_0)=0$. The solutions (2.3) are linearly independent for E inside a gap and satisfy

$$[\psi_0^{(+)}(\cdot,E);\psi_0^{(-)}(\cdot,E)]=-\frac{2i\,\sin[q(E)]}{\phi_0(p,E)}, \tag{2.6}$$

where $[f;g]$ denotes the Wronskian of f and g. Moreover, $\psi_0^{(\pm)}(x,E)$ can alternatively be written in the form

$$\psi_0^{(\pm)}(x,E)=e^{\pm ikx}\xi_0^{(\pm)}(x,E), \tag{2.7}$$

where $k=k(E)=q(E)/p$ is called the quasi-momentum, $\xi_0^{(\pm)}(x,E)$ have period p and $\xi_0^{(\pm)}(0,E)=1$ [since $\psi_0^{(\pm)}(0,E)=1$]. We further introduce the notation

$$\mathscr{G}_n=(E_{2n-1},E_{2n}),\quad n=0,1,2,...,$$

$$\mathscr{B}_n=[E_{2n-2},E_{2n-1}],\quad n=1,2,...,$$

so that $\mathscr{G}_n$ denotes the nth gap starting with the zeroth gap $(-\infty,E_0)$, and $\mathscr{B}_n$ denotes the nth band. The function $k(E)$ maps[21,22] $\mathscr{B}_n$ onto the interval $[(n-1)\pi/p,n\pi/p]$ in the k-plane, and it maps $\mathscr{G}_0$ onto the positive imaginary axis, and each $\mathscr{G}_n$ $(n=1,2,...)$ onto a segment of the form $\mathscr{S}_n=\{k:k=n\pi/p+i\mu,0<\mu\leqslant\delta_n\}$ for some $\delta_n>0$, where $\mu=\mu(E)=\mathrm{Im}[k(E)]$. Here $\delta_n>0$ is determined by the equation $k(\tilde{E}_n)=n\pi/p+i\delta_n$, where $\tilde{E}_n$ is the unique point (cf. Ref. 23, p. 295) in $\mathscr{G}_n$ such that $\Delta'(\tilde{E}_n)=0$. The two intervals $(E_{2n-1},\tilde{E}_n]$ and $[\tilde{E}_n,E_{2n})$ are both mapped onto the segment $\mathscr{S}_n$ and hence the function $\mu\mapsto E(n\pi/p+i\mu)$ has two branches, $E^{(1)}(n\pi/p+i\mu)$ and $E^{(2)}(n\pi/p+i\mu)$, such that $E^{(1)}(n\pi/p+i\mu)\to E_{2n-1}$ and $E^{(2)}(n\pi/p+i\mu)\to E_{2n}$ as $\mu\downarrow 0$. Moreover, the functions $\mu\mapsto E^{(j)}(n\pi/p+i\mu)$ $(j=1,2)$ are analytic, even and satisfy

$$E^{(j)}(n\pi/p+i\mu)-E_{2n-2+j}=\frac{(-1)^n p^2}{2\Delta'(E_{2n-2+j})}\,\mu^2+O(\mu^4),\quad \mu\downarrow 0. \tag{2.8}$$

This follows from (2.5) by expanding $\Delta(E)$ about the point E_{2n-2+j}, using $\Delta(E_{2n-2+j})=(-1)^n$ and the fact that $\phi_0(x,E)$ and $\theta_0(x,E)$ are analytic functions of E, hence even functions of μ. From now on we assume that

$$\phi_0(p,E_n)\neq 0, \tag{2.9}$$

at the endpoint under consideration. The case $\phi_0(p,E_n)=0$ will be discussed separately, since it requires special arguments. Note that for $n=0$, (2.9) is automatically satisfied because E_0 is the lowest point of $\sigma(H_0)$. In fact, $\phi_0(p,E_0)=0$ implies that $\phi_0(x,E_0)$ is periodic (with period p) and hence has infinitely many zeros, contradicting that E_0 is the lowest point of $\sigma(H_0)$. Moreover, the solution $\psi_0^{(+)}(x,E_0)$, which is well-defined because $\phi_0(p,E_0)\neq 0$, is periodic and thus has no zeros. Hence, by the Sturm separation theorem, $\phi_0(x,E_0)$ has exactly one zero (at $x=0$). There are two solutions, $F_\lambda^{(\pm)}(x,E)$, of (1.4) which are defined as solutions of the integral equations

$$F_\lambda^{(\pm)}(x,E)=\psi_0^{(\pm)}(x,E)-\lambda\int_x^{\pm\infty}A_0(x,y;E)W(y)F_\lambda^{(\pm)}(y,E)dy, \tag{2.10}$$

$$A_0(x,y;E)=-\frac{i\phi_0(p,E)}{2\sin[q(E)]}[\psi_0^{(+)}(x,E)\psi_0^{(-)}(y,E)-\psi_0^{(-)}(x,E)\psi_0^{(+)}(y,E)]. \tag{2.11}$$

The solutions $F_\lambda^{(\pm)}(x,E)$ correspond to the standard Jost solutions in the case when $V=0$ and, as usual, (2.10)–(2.11) are obtained by applying the variation of constants formula to (1.4) and using (2.6). For reasons given below, the relevant case for us will be that in which for a fixed $\lambda\geqslant 0$, $F_\lambda^{(+)}(x,E_n)$ and $F_\lambda^{(-)}(x,E_n)$ are linearly dependent. This case will also be called the exceptional case and we will call λ an exceptional value (at a given endpoint E_n) if $[F_\lambda^{(+)}(\cdot,E_n);F_\lambda^{(-)}(\cdot,E_n)]=0$. In the exceptional case we define the constant a_n by

$$F_\lambda^{(+)}(x,E_n)=a_n\,F_\lambda^{(-)}(x,E_n). \tag{2.12}$$

The case when $[F_\lambda^{(+)}(\cdot,E_n);F_\lambda^{(-)}(\cdot,E_n)]\neq 0$ will be called the generic case.

Next we introduce the transmission coefficient $T_\lambda(E)$ associated with (1.4),

$$T_\lambda(E)=-\frac{2i\,\sin[q(E)]}{[F_\lambda^{(+)}(\cdot,E);F_\lambda^{(-)}(\cdot,E)]\phi_0(p,E)}. \tag{2.13}$$

The transmission coefficient will enter naturally into our analysis and bring the advantage that we can draw on recent work on its asymptotic properties.[24] We remark that usually $T_\lambda(E)$ is first defined for $E\in\sigma(H_0)$ and then analytically continued into the gaps. The results of Ref. 24 were shown to be valid when E lies in a gap.

It will sometimes be convenient to number the gaps by using the symbol

$$\tilde{n}=\left[\frac{n+1}{2}\right], \tag{2.14}$$

where $[x]$ denotes the largest integer less than or equal to x. Then E_n is the left (right) endpoint of the gap $\mathscr{G}_{\tilde{n}}$ if n is odd (even). The statement $E\to E_n$ is understood to mean that E approaches E_n from within the gap $\mathscr{G}_{\tilde{n}}$, where E_n may be either endpoint of $\mathscr{G}_{\tilde{n}}$. In terms of μ this can be simply stated as $\mu\downarrow 0$, with the understanding that E is viewed as a function of μ, namely, $E=E^{(j)}(\tilde{n}\pi/p+i\mu)$, with the proper branch labeled by j $(j=n+2-2\tilde{n})$.

The following theorem was proved in Ref. 24 (Theorem 3.3).

Theorem 2.1: Suppose that (H1) and (2.9) hold. Fix $\lambda\geqslant 0$. Then $T_\lambda(E)$ is continuous at every E_n and there are two possible asymptotic behaviors of $T_\lambda(E)$ as $\mu\downarrow 0$ $(E\in\mathscr{G}_{\tilde{n}})$:

(i) In the generic case,

$$T_\lambda(E)=\beta_n\mu+o(\mu), \tag{2.15}$$

with β_n real and nonzero.

(ii) In the exceptional case,

$$T_\lambda(E)=T_\lambda(0)+o(1),\quad T_\lambda(0)=\frac{2a_n}{1+a_n^2}. \tag{2.16}$$

Of course, (2.15) follows immediately from (2.13), along with the fact that $F_\lambda^{(\pm)}(x,E_n)$ are real and

$$\sin[q(E)]=i(-1)^{\tilde{n}}\sinh\mu p,\quad E\in\mathscr{G}_{\tilde{n}}. \tag{2.17}$$

The proof of (2.16) is more involved. Note that if $\lambda=0$, then $F_0^{(\pm)}(x,E)=\psi_0^{(\pm)}(x,E)$, so $T_0(E)=1$ for all E, and we are in case (ii) with $a_n=1$.

Lemma 2.2: Suppose that (H1) and (2.9) hold. Then $[F_\lambda^{(+)}(\cdot,E_n);F_\lambda^{(-)}(\cdot,E_n)]=0$ if and only if

$$\int_{-\infty}^{\infty}\psi_0^{(+)}(x,E_n)W(x)F_\lambda^{(+)}(x,E_n)dx=0. \tag{2.18}$$

Proof: This result was proved in Ref. 24 as a consequence of Lemma 2.1 there. We briefly sketch the argument here, since we will use one formula from the proof later. For $E=E_n$ the integral equation (2.10) for $F_\lambda^{(+)}(x,E_n)$ becomes

$$F_\lambda^{(+)}(x,E_n)=\psi_0^{(+)}(x,E_n)-\lambda\int_x^\infty A_0(x,y;E_n)W(y)F_\lambda^{(+)}(y,E_n)dy, \tag{2.19}$$

$$A_0(x,y;E_n)=\lim_{E\to E_n}A_0(x,y;E)=\phi_0(x,E_n)\psi_0^{(+)}(y,E_n)-\psi_0^{(+)}(x,E_n)\phi_0(y,E_n). \tag{2.20}$$

Since $[F_\lambda^{(+)}(\cdot,E_n);F_\lambda^{(-)}(\cdot,E_n)]=0$ if and only if $H_\lambda\psi=E_n\psi$ has a bounded (nontrivial) solution, the assertion (2.18) follows on letting $x\to-\infty$ in (2.19) and using standard estimates (see Ref. 24, or Lemma 2.3 below). ■

For the following it is convenient to normalize the solutions $F_\lambda^{(\pm)}(x,E)$ by introducing

$$\psi_\lambda^{(\pm)}(x,E)=\frac{F_\lambda^{(\pm)}(x,E)}{F_\lambda^{(\pm)}(0,E)}, \tag{2.21}$$

so that $\psi_\lambda^{(\pm)}(0,E)=1$. Therefore, in addition to (2.9), we make the assumption

$$F_\lambda^{(\pm)}(0,E_n)\neq 0, \tag{2.22}$$

so that by continuity $F_\lambda^{(\pm)}(0,E)\neq 0$ for E near E_n. It turns out that (2.22) is not a serious restriction, since by shifting the origin if necessary, we can always make sure that it holds. This point will be discussed in more detail later when we remove condition (2.9) by the same method.

The integral kernel of the resolvent $(H_\lambda-E)^{-1}$, i.e., the Green's function of H_λ, will be denoted by $G_{\lambda,E}(x,y)$. It is given by

$$G_{\lambda,E}(x,y)=\frac{1}{[\psi_\lambda^{(+)}(\cdot,E);\psi_\lambda^{(-)}(\cdot,E)]}\begin{cases}\psi_\lambda^{(+)}(x,E)\psi_\lambda^{(-)}(y,E), & y<x,\\ \psi_\lambda^{(-)}(x,E)\psi_\lambda^{(+)}(y,E), & y>x.\end{cases} \tag{2.23}$$

For fixed λ and E inside a gap, the Wronskian in the denominator of (2.23) vanishes exactly when E is an eigenvalue of H_λ, and so it is nonzero for E near but different from an endpoint E_n. The Birman–Schwinger kernel (1.3) is given by

$$K_{\lambda,E}(x,y)=W(x)^{1/2}G_{\lambda,E}(x,y)|W(y)|^{1/2},\quad W(x)^{1/2}=|W(x)|^{1/2}\operatorname{sgn}W(x).$$

In order to split $K_{\lambda,E}$ into a singular and a regular part we write

$$G_{\lambda,E}(x,y)=\frac{1}{G_{\lambda,E}(0,0)}G_{\lambda,E}(x,0)G_{\lambda,E}(0,y)+G_{\lambda,E}^D(x,y). \tag{2.24}$$

where $G^D_{\lambda,E}(x,y)$ is the Green's function of H_λ with an additional Dirichlet condition at the origin. The decomposition (2.24) was used in Ref. 15 to study the number of eigenvalues in $\mathcal{G}_n$ for n large. Then the corresponding decomposition of $K_{\lambda,E}$ is

$$K_{\lambda,E}=d_\lambda(E)L_{\lambda,E}+R_{\lambda,E}, \tag{2.25}$$

where $L_{\lambda,E}$ and $R_{\lambda,E}$ are integral operators with kernels $L_{\lambda,E}(x,y)$ and $R_{\lambda,E}(x,y)$, respectively, given by

$$\begin{aligned} L_{\lambda,E}(x,y)&=\omega_\lambda(x,E)\tilde{\omega}_\lambda(y,E),\\ R_{\lambda,E}(x,y)&=W(x)^{1/2}G^D_{\lambda,E}(x,y)|W(y)|^{1/2}, \end{aligned} \tag{2.26}$$

$$\begin{aligned} \omega_\lambda(x,E)&=\begin{cases} W(x)^{1/2}\psi^{(+)}_\lambda(x,E), & x>0,\\ W(x)^{1/2}\psi^{(-)}_\lambda(x,E), & x<0,\end{cases}\\ \tilde{\omega}_\lambda(y,E)&=\omega_\lambda(y,E)\operatorname{sgn} W(y), \end{aligned} \tag{2.27}$$

$$G^D_{\lambda,E}(x,y)=\begin{cases} \phi_\lambda(x,E)\psi^{(+)}_\lambda(y,E), & 0<x<y,\\ \psi^{(+)}_\lambda(x,E)\phi_\lambda(y,E), & 0<y<x,\\ -\phi_\lambda(x,E)\psi^{(-)}_\lambda(y,E), & y<x<0,\\ -\psi^{(-)}_\lambda(x,E)\phi_\lambda(y,E), & x<y<0,\\ 0, & \text{otherwise.}\end{cases} \tag{2.28}$$

Moreover,

$$d_\lambda(E)=G_{\lambda,E}(0,0)=\frac{1}{[\psi^{(+)}_\lambda(\cdot,E);\psi^{(-)}_\lambda(\cdot,E)]}, \tag{2.29}$$

so that, by (2.6) and (2.17),

$$d_0(E)=\frac{(-1)^{\tilde{n}}\phi_0(p,E)}{2\sinh \mu p},\quad E\in\mathcal{G}_{\tilde{n}}. \tag{2.30}$$

Another useful expression for $d_\lambda(E)$ follows from (2.13), (2.17), (2.21), and (2.29), namely,

$$d_\lambda(E)=\frac{(-1)^{\tilde{n}}\phi_0(p,E)T_\lambda(E)F^{(+)}_\lambda(0,E)F^{(-)}_\lambda(0,E)}{2\sinh \mu p},\quad E\in\mathcal{G}_{\tilde{n}}. \tag{2.31}$$

The implicit equation (1.8) for $E(\lambda)$ can be written as

$$[\lambda-\lambda_0](\tilde{\omega}_{\lambda_0,E},(1+[\lambda-\lambda_0]R_{\lambda_0,E})^{-1}\omega_{\lambda_0,E})=-\frac{1}{d_{\lambda_0}(E)}, \tag{2.32}$$

where λ_0 is a c.c.th. and $\omega_{\lambda,E}$ (resp., $\tilde{\omega}_{\lambda,E}$) denotes the function $\omega_\lambda(x,E)$ [resp., $\tilde{\omega}_\lambda(x,E)$] defined in (2.27). Notice that the decomposition (2.25) is of the form (1.5), (1.6) with $N=1$ if we set $\lambda=\lambda_0$, $d_{\lambda_0;1}(E)=d_{\lambda_0}(E)$, $L_{\lambda_0,E;1}=L_{\lambda_0,E}$, $\omega_{\lambda_0,E;1}=\omega_{\lambda_0,E}$, etc.

Our goal now is to find solutions $E=E(\lambda)$ of (2.32) such that $E(\lambda)\to E_n$ as $\lambda\to\lambda_0$. We do this by first finding the solutions in the form $\mu(\lambda)$ and then using (2.8) to obtain $E(\lambda)$. Expanding the left-hand side of (2.32) in powers of $\lambda-\lambda_0$ yields

$$\sum_{j=0}^{\infty}(-1)^j[\lambda-\lambda_0]^{j+1}(\tilde{\omega}_{\lambda_0,E},R^j_{\lambda_0,E}\omega_{\lambda_0,E})=-\frac{1}{d_{\lambda_0}(E)}. \tag{2.33}$$

Now we would like to expand the coefficients $(\tilde{\omega}_{\lambda_0,E}, R^j_{\lambda_0,E}\omega_{\lambda_0,E})$ and the right-hand side of (2.33) in powers of μ and then obtain an expansion of $\mu(\lambda)$ in powers of $\lambda-\lambda_0$. It is not hard to see that a full expansion will generally not be possible under assumptions (H1) or (H2); one needs stronger assumptions like (H3) to do this. Under (H1) or (H2) one can only expect to find a few terms in an expansion of $\mu(\lambda)$. In particular, since the right-hand side of (2.33) vanishes linearly in μ, it will be easy to obtain the leading order term of $\mu(\lambda)$ provided the coefficient with $j=0$, $(\tilde{\omega}_{\lambda_0,E_n},\omega_{\lambda_0,E_n})$, is nonzero. However, we are also interested in the case when $(\tilde{\omega}_{\lambda_0,E_n},\omega_{\lambda_0,E_n})=0$, and, more generally, in the case when $(\tilde{\omega}_{\lambda_0,E_n},R^j_{\lambda_0,E_n}\omega_{\lambda_0,E_n})=0$ for $j=0,1,...,M-1$ and $(\tilde{\omega}_{\lambda,E_n},R^M_{\lambda_0,E_n}\omega_{\lambda_0,E_n})\neq 0$ for some M. Or, what if $(\tilde{\omega}_{\lambda_0,E_n},R^j_{\lambda_0,E_n}\omega_{\lambda_0,E_n})=0$ for all j, if this is possible? In order to answer these questions we need some technical lemmas. Since the proofs of these lemmas are somewhat lengthy but mostly standard and do not contribute much to the understanding of our main results, they have all been consigned to the Appendix. In the following we will always think of E as a function of μ [as explained below (2.14)]. The phrase ''for E near E_n'' shall mean that there is a $\mu_0>0$ sufficiently small, so that the given statement holds for $0\leq\mu\leq\mu_0$. The letter C will be used to denote various constants that may depend on μ_0 (but not on μ) and are not necessarily the same at each appearance. Differentiation with respect to μ will be denoted by a dot.

Lemma 2.3: Assume (H1), (2.9), (2.22), and fix $\lambda\geq 0$. Then for E near E_n we have the estimates

(i) $|\psi^{(\pm)}_\lambda(x,E)|\leq Ce^{\mp\mu x}$, $x\in\mathbf{R}_\pm$,

(ii) $|\phi_\lambda(x,E)|\leq C(1+|x|)e^{\mu|x|}$,

(iii) $|\dot{\phi}_\lambda(x,E)|\leq C[\mu(1+|x|)^3/(1+\mu|x|)^2]e^{\mu|x|}$.

Furthermore, if $\lambda=0$ and (H1) holds, or if $\lambda>0$ and (H2) holds, then

(iv) $|\dot{\psi}^{(\pm)}_\lambda(x,E)|\leq C(1+|x|)e^{\mp\mu x}$, $x\in\mathbf{R}_\pm$.

Proof: See the Appendix. ■

It is true that for E near but different from E_n, the function $\psi^{(\pm)}_\lambda(x,E)$ is differentiable with respect to μ under condition (H1) alone, for any $\lambda\geq 0$. However, if $\lambda>0$ and (H2) is not satisfied, then $\dot{\psi}_\lambda(x,E)$ will in general not remain bounded as $\mu\downarrow 0$. In other words, estimate (iv) no longer holds. The evidence for this conclusion comes from a result in Ref. 25 (Theorem 3.1) where, in the case $V=0$, it was proved that for a certain class of potentials that satisfy (H1) but violate (H2), the Jost solutions are in general not differentiable with respect to k $(k=\sqrt{-E})$ at $k=0$.

Lemma 2.4: Assume (H1), (2.9), (2.22) and fix $\lambda\geq 0$. Then for E near E_n we have

(i) $G^D_{\lambda,E}(x,y)\leq C[1+\min\{|x|,|y|\}]$,

(ii) $R_{\lambda,E}\to R_{\lambda,E_n}$ in Hilbert–Schmidt norm as $E\to E_n$,

(iii) Assume $\lambda=0$ and (H1), or $\lambda>0$ and (H2). Then

$$|\dot{G}^D_{\lambda,E}(x,y)|\leq C(1+|x|)(1+|y|). \tag{2.34}$$

Proof: See the Appendix. ■

We note that (i) implies

$$|G^D_{\lambda,E}(x,y)|\leq C(1+|x|)^{1/2}(1+|y|)^{1/2}.$$

When $\lambda=0$ a bound of this type was used in Ref. 15. Under condition (H1) all we can say in general is that $\|R_{\lambda,E}-R_{\lambda,E_n}\|_{H.S.}=o(1)$ as $E\to E_n$. However, if (H2) holds, then (2.34) implies that $\dot{R}_{\lambda,E}(x,y)$ is Hilbert–Schmidt and hence $\|R_{\lambda,E}-R_{\lambda,E_n}\|_{H.S.}=O(\mu)$.

Lemma 2.5: In addition to (2.9) and (2.22) assume $\lambda=0$ and (H1), or $\lambda>0$ and (H2). Then, for E near E_n, the functions $\mu\to(\tilde{\omega}_{\lambda,E},R^s_{\lambda,E}\omega_{\lambda,E})$, $s=0,1,2,...$, are continuously differentiable and obey

(i) $|(\tilde{\omega}_{\lambda,E},R^s_{\lambda,E}\omega_{\lambda,E})|\leq\tilde{C}r^s$,

(ii) $|d(\tilde{\omega}_{\lambda,E},R^s_{\lambda,E}\omega_{\lambda,E})/d\mu|\leq\tilde{C}r^s$,

for some $\tilde{C}>0$ and $r>0$.

Proof: See the Appendix. ■

The next two lemmas are essential for our study of (2.33). They are variants of the inverse function theorem tailored to our needs.

Lemma 2.6: Suppose $a_j(\mu)$, $j=1,2,...$, and $c(\mu)$ are continuous real-valued functions defined for $0\leqslant\mu\leqslant\mu_0$, satisfying

(a) $|a_j(\mu)|\leqslant Dr^j$, $j=1,2,3,...$, for some $D>0$, $r>0$, and $0\leqslant\mu\leqslant\mu_0$,

(b) $a_j(\mu)=O(\mu)$ as $\mu\to 0$, for $j=1,2,...N-1$, and $a_N(0)\neq 0$ for some $N\geqslant 1$,

(c) $c(\mu)=\dot{c}(0)\mu+o(\mu)$, with $\dot{c}(0)>0$, as $\mu\to 0$,

and consider the equation

$$\sum_{j=1}^{\infty} a_j(\mu)z^j=c(\mu). \tag{2.35}$$

Then the following occur:

(i) If N is odd and $a_N(0)>0$, then there is a unique, real, continuous solution $z(\mu)$ of (2.35) such that $z(\mu)=[\dot{c}(0)/a_N(0)]^{1/N}\mu^{1/N}+o(\mu^{1/N})$ as $\mu\downarrow 0$.

(ii) If N is odd and $a_N(0)<0$, then there is a unique, real, continuous solution $z(\mu)$ such that $z(\mu)=-[-\dot{c}(0)/a_N(0)]^{1/N}\mu^{1/N}+o(\mu^{1/N})$ as $\mu\downarrow 0$.

(iii) If N is even and $a_N(0)>0$, then there are exactly two real, continuous solutions $z_\pm(\mu)$ such that $z_\pm(\mu)=\pm[\dot{c}(0)/a_N(0)]^{1/N}\mu^{1/N}+o(\mu^{1/N})$ as $\mu\downarrow 0$.

(iv) If N is even and $a_N(0)<0$, then (2.35) has no real solution converging to zero as $\mu\downarrow 0$.

(v) If $a_j(0)=0$ for all j and $|a_j(\mu)|\leqslant C_1 r_1^j\mu$ for some $C_1>0$ and $r_1>0$, then there is no real solution converging to zero as $\mu\downarrow 0$.

Proof: See the Appendix. ■

Lemma 2.7: Suppose that, in addition to (a)–(c) of Lemma 2.6, $a_j(\mu)$ $(j=1,2,...)$ and $c(\mu)$ are continuously differentiable, and $|\dot{a}_j(\mu)|\leqslant C_2 r_2^j$, for some $C_2>0$ and $r_2>0$. Then we have the following:

(i) If N is even and $a_N(0)>0$, then (2.35) has a unique solution $\mu(z)$ such that

$$\mu(z)=\frac{a_N(0)}{\dot{c}(0)}\, z^N+o(z^N),\quad z\to 0. \tag{2.36}$$

(ii) If N is odd and $a_N(0)>0$, resp. $a_N(0)<0$, then (2.35) has a solution obeying (2.36) as $z\downarrow 0$, resp. $z\uparrow 0$.

(iii) Suppose that $c(\mu)=\dot{c}(0)\mu+c_2\mu^2+o(\mu^2)$ with $\dot{c}(0)>0$. Then, under the same assumptions on $a_N(0)$ as in (i) and (ii) and with z approaching zero in the same manner as there, we have that

$$\mu(z)=\frac{a_N(0)}{\dot{c}(0)}\, z^N+\left[\frac{a_{N+1}(0)}{\dot{c}(0)}+\frac{a_N(0)\dot{a}_1(0)}{\dot{c}(0)^2}-\frac{c_2 a_1(0)^2}{\dot{c}(0)^3}\,\delta_{N,1}\right]z^{N+1}+o(z^{N+1}). \tag{2.37}$$

In (2.37) the Kronecker symbol $\delta_{N,1}$ has been added only for clarity. It follows from assumption (b) of Lemma 2.6 that $a_1(0)=0$ when $N>1$; thus the coefficient c_2 has no effect on (2.37) when $N>1$. Note that if $a_N(0)=0$, then (2.37) just reduces to (2.36) but with $N+1$ in place of N.

Proof: See the Appendix. ■

Some technical comments on Lemmas 2.6 and 2.7 may be in order. In assumption (c) of Lemma 2.6 we do not intend to imply that $c(\mu)$ is continuously differentiable on $[0,\mu_0]$, only that besides being continuous, it is also differentiable at $\mu=0$. In the application of Lemma 2.6 we will actually need the inverse function $\mu(z)$ of $z(\mu)$ [in case (iii) we need the inverse of each of the two functions $z_\pm(\mu)$]. Lemma 2.6 does not guarantee the existence of these inverse functions. For example, the solution $z(\mu)$ of the equation $[1+2\mu\,\sin(1/\mu)]z=\mu(0<\mu\leqslant\mu_0)$ does not have an inverse on $(0,\mu_0]$, no matter how small μ_0. However, in the applications we will have additional information on $\mu(z)$ which will allow us to argue that the inverse function exists. It may be also useful to give a short illustration of what can happen if $a_j(\mu)$ and $c(\mu)$ violate some of the conditions placed on them in Lemmas 2.6 and 2.7. Consider, for example, the equation $z+z^2=\mu+\mu^{1+\epsilon}$ with $1/2<\epsilon<1$, i.e., $a_1(\mu)=a_2(\mu)=1$, $c(\mu)=\mu+\mu^{1+\epsilon}$, and so $\dot{c}(0)=1$. Thus $c(\mu)$ does not fulfill condition (iii) of Lemma 2.7. Then $\mu(z)=z-z^{1+\epsilon}+z^2+o(z^2)$ and we see that a term, $-z^{1+\epsilon}$, has slipped between the linear and quadratic terms so that (2.37) no longer holds. Now replace $a_1(\mu)$ by μ, so that the equation becomes $\mu z+z^2=\mu+\mu^{1+\epsilon}$. Then $\mu(z)$

$=z^2-z^{2(1+\epsilon)}+O(z^3)$ which shows that the ϵ has no effect on the leading term. This is in agreement with Lemma 2.6. Furthermore, if $a_1(\mu)$ does not vanish linearly, uniqueness or existence may break down. For example, the equation $5\sqrt{\mu}z-4z^2=\mu$ has two solutions $\mu(z)=z^2$ and $\mu(z)=16z^2$, and the equation $\mu^{2/3}z-z^2=\mu$ has no real non-negative solution $\mu(z)$ that converges to zero as $z\to 0$.

Theorem 2.8: Assume (2.9) and (2.22). If $\lambda_0>0$ is a c.c.th. at E_n, then λ_0 is an exceptional value. The exceptional values and the c.c.th.'s each form a discrete set with no finite accumulation points.

Proof: Suppose that $[F_{\lambda_0}^{(+)}(\cdot,E_n);F_{\lambda_0}^{(-)}(\cdot,E_n)]\neq 0$. Then, by (2.15) and (2.31), the right-hand side of (2.33) approaches a finite nonzero limit as $E\to E_n$, whereas, by Lemma 2.5, the left-hand side goes to zero as $\lambda\to\lambda_0$, uniformly in E for E near E_n. Therefore, a solution $\mu(\lambda)$ of (2.33) satisfying $\mu(\lambda)\to 0$ as $\lambda\to\lambda_0$ does not exist and hence λ_0 cannot be a c.c.th. This proves the first part. For the second part we allow λ to be complex. Solving (2.19) by iteration, we see that for each x, the functions $\lambda\mapsto F_\lambda^{(\pm)}(x,E_n)$ and $\lambda\mapsto F_\lambda^{(\pm)\prime}(x,E_n)$ are entire functions of λ. Hence $\lambda\mapsto[F_\lambda^{(+)}(\cdot,E_n);F_\lambda^{(-)}(\cdot,E_n)]$ is entire and the conclusion follows. ■

In connection with Theorem 2.8 we remind the reader that $\lambda_0=0$ is always an exceptional value, but, as we will see below, it need not always be a c.c.th. [Theorem 2.11(i)]. Moreover, the restrictions (2.9) and (2.22) can be removed and the first assertion of Theorem 2.8 has a converse if $n=0$ and, under further assumptions, also when $n\geqslant 1$. This will be shown in Theorems 2.14(iii) and 2.16(i).

From now on we will always assume that we are in the exceptional case. In order to determine the threshold behavior of the eigenvalues we have to identify (2.33) with (2.35). Since $\dot{c}(0)$ in Lemma 2.6 must be positive, we need to know the sign of the derivative of $-1/d_{\lambda_0}(E)$ with respect to μ. To this end, we first note that by (2.12), (2.16), and (2.31),

$$-\frac{1}{d_{\lambda_0}(E)}=\frac{(-1)^{\tilde{n}+1}(1+a_n^2)p}{\phi_0(p,E_n)F_{\lambda_0}^{(+)}(0,E_n)^2}\,\mu+o(\mu). \tag{2.38}$$

Now it is known[14] that if (2.9) holds, then

$$\phi_0(p,E_n)>0,\quad n=4j\quad\text{and}\quad n=4j+1,$$

$$\phi_0(p,E_n)<0,\quad n=4j+2\quad\text{and}\quad n=4j+3,\ \ j=0,1,\ldots\ .$$

Therefore,

$$\operatorname{sgn}[(-1)^{\tilde{n}+1}\phi_0(p,E_n)]=(-1)^{n+1},$$

and if we set

$$c_{\lambda_0}(\mu)=\frac{(-1)^n}{d_{\lambda_0}(E)}, \tag{2.39}$$

then

$$\dot{c}_{\lambda_0}(0)=\frac{(-1)^{\tilde{n}+n}(1+a_n^2)p}{\phi_0(p,E_n)F_{\lambda_0}^{(+)}(0,E_n)^2}>0. \tag{2.40}$$

In order to achieve some computational simplications later, we introduce the kernel

$$M_{\lambda,E}(x,y)=d_\lambda(E)W(x)^{1/2}|W(y)|^{1/2}\cdot\begin{cases}B_\lambda(x,y;E), & y<x,\\ B_\lambda(y,x;E), & y>x,\end{cases}$$
$$B_\lambda(x,y;E)=\psi_\lambda^{(+)}(x,E)\psi_\lambda^{(-)}(y,E)-\psi_\lambda^{(+)}(x,E_n)\psi_\lambda^{(+)}(y,E_n), \tag{2.41}$$

and denote the associated integral operator by $M_{\lambda,E}$. We remark that the integral operator $M_{\lambda,E}$ satisfies

$$K_{\lambda,E}=d_\lambda(E)L_{\lambda,E_n}+M_{\lambda,E}, \tag{2.42}$$

and thus is associated with a decomposition of $K_{\lambda,E}$ slightly different from that in (2.25). When $V=0$ this decomposition corresponds to that used in Refs. 1 and 5. If we assume (H2), then Lemma 2.4(ii) also holds for $M_{\lambda,E}$, namely $M_{\lambda,E}\to M_{\lambda,E_n}$ in the Hilbert–Schmidt norm. Then we could use the implicit equation

$$[\lambda-\lambda_0](\tilde{\omega}_{\lambda_0,E_n},(1+[\lambda-\lambda_0]M_{\lambda_0,E})^{-1}\omega_{\lambda_0,E_n})=-\frac{1}{d_{\lambda_0}(E)},$$

instead of (2.32). In this paper we need (2.41) only when $E=E_0$ and $\lambda=0$. Then we have

$$M_{0,E_0}(x,y)=\frac{-W(x)^{1/2}|W(y)|^{1/2}}{2}\cdot\begin{cases}A_0(x,y;E_0), & y<x,\\ A_0(y,x;E_0), & y>x,\end{cases} \tag{2.43}$$

where $A_0(x,y;E_0)$ is defined in (2.20). If only (H1) is assumed, then the kernel (2.43) will in general not represent a bounded operator. However, here we will only need certain matrix elements like $(\tilde{\omega}_{0,E_0},M_{0,E_0}\omega_{0,E_0})$. This matrix element exists as an integral over $\mathbf{R}^2$ and is equal to $\lim_{E\to E_0}(\tilde{\omega}_{0,E_0},M_{0,E}\omega_{0,E_0})$, where this limit exists by the estimates in Lemma 2.3.

In preparation of our next theorem we first prove two lemmas. Let λ_0 be an exceptional value and define

$$I_{\lambda_0,j}(E)=(\tilde{\omega}_{\lambda_0,E},R^j_{\lambda_0,E}\omega_{\lambda_0,E}), \quad j=0,1,2,...; \tag{2.44}$$

in particular,

$$I_{\lambda_0,0}(E_n)=(\tilde{\omega}_{\lambda_0,E_n},\omega_{\lambda_0,E_n})=\int_{-\infty}^{\infty}W(x)\psi^{(+)}_{\lambda_0}(x,E_n)^2dx. \tag{2.45}$$

Lemma 2.9: Assume (H1), (2.9), $n=0$, and $\lambda_0=0$. Then

$$(\tilde{\omega}_{0,E_0},M_{0,E_0}\omega_{0,E_0})=(\tilde{\omega}_{0,E_0},R_{0,E_0}\omega_{0,E_0})+\frac{1}{\dot{c}_0(0)}I_{0,0}(E_0)\dot{I}_{0,0}(E_0). \tag{2.46}$$

Proof: By (2.25), (2.39), (2.42), and (2.44), we have

$$\begin{aligned}(\tilde{\omega}_{0,E_0},M_{0,E_0}\omega_{0,E_0})-(\tilde{\omega}_{0,E_0},R_{0,E_0}\omega_{0,E_0})&=\lim_{E\to E_0}d_0(E)(\tilde{\omega}_{0,E_0},[L_{0,E}-L_{0,E_0}]\omega_{0,E_0})\\&=2\frac{1}{\dot{c}_0(0)}(\tilde{\omega}_{0,E_0},\omega_{0,E_0})(\tilde{\omega}_{0,E_0},\dot{\omega}_{0,E_0})\\&=\frac{1}{\dot{c}_0(0)}I_{0,0}(E_0)\dot{I}_{0,0}(E_0),\end{aligned}$$

proving (2.46). In the last equation we have used $\dot{I}_{0,0}(E)=2(\tilde{\omega}_{0,E},\dot{\omega}_{0,E})$. Note that, by Lemma 2.3(iv), $\dot{I}_{0,0}(E_0)$ exists under condition (H1). ■

Lemma 2.10: Assume (H1), $W\neq0$, and $n=0$. Then the following are true:

(i) If $\lambda_0>0$, $[F^{(+)}_{\lambda_0}(\cdot,E_0);F^{(-)}_{\lambda_0}(\cdot,E_0)]=0$, and (2.22) holds, then $I_{\lambda_0,0}(E_0)<0$.

(ii) If $\lambda_0=0$ and $I_{0,0}(E_0)=0$, then $I_{0,1}(E_0)=(\tilde{\omega}_{0,E_0},M_{0,E_0}\omega_{0,E_0})>0$.

As we shall see later, in (i) it is not really necessary to assume (2.22). The only purpose of this assumption is to ensure that $\psi^{(+)}_{\lambda_0}(x,E_0)$ is defined by (2.21).

Proof: (i) Since $F_{\lambda_0}^{(\pm)}(x,E_0)$ are linearly dependent, (2.18) holds with $n=0$ and $\lambda=\lambda_0$. Multiplying (2.19) for $n=0$ by $W(x)F_{\lambda_0}^{(+)}(x,E_0)$, integrating over $\mathbf{R}$ using an integration by parts and Lemma 2.2, and using (2.21) and (2.45) yields

$$F_{\lambda_0}^{(+)}(0,E_0)^2 I_{\lambda_0,0}(E_0)=\int_{-\infty}^{\infty}W(x)F_{\lambda_0}^{(+)}(x,E_0)^2dx$$
$$=-2\lambda_0\int_{-\infty}^{\infty}W(x)F_{\lambda_0}^{(+)}(x,E_0)\phi_0(x,E_0)\times\left(\int_x^{\infty}W(y)\psi_0^{(+)}(y,E_0)F_{\lambda_0}^{(+)}(y,E_0)dy\right)dx. \tag{2.47}$$

Now the relation $[\phi_0(x,E_0)/\psi_0^{(+)}(x,E_0)]'=1/\psi_0^{(+)}(x,E_0)^2$ [recall that $\psi_0^{(+)}(x,E_0)$ has no zeros] together with an integration by parts yields

$$F_{\lambda_0}^{(+)}(0,E_0)^2 I_{\lambda_0,0}(E_0)=-\lambda_0\int_{-\infty}^{\infty}\frac{1}{\psi_0^{(+)}(x,E_0)^2}\times\left(\int_x^{\infty}W(y)\psi_0^{(+)}(y,E_0)F_{\lambda_0}^{(+)}(y,E_0)dy\right)^2dx<0. \tag{2.48}$$

The boundary terms from the integration by parts vanish on account of Lemma 2.3 and (H1). Hence $I_{\lambda_0,0}(E_0)<0$, proving (i).

(ii) By (2.45) and (2.46), since $I_{0,0}(E_0)=0$, we have that $I_{0,1}(E_0)=(\tilde{\omega}_{0,E_0},M_{0,E_0}\omega_{0,E_0})$. Then, from (2.43), by an integration by parts and using that $\psi_0^{(+)}(x,E_0)=\psi_0^{(-)}(x,E_0)$, we obtain

$$I_{0,1}(E_0)=2\int_{-\infty}^{\infty}W(x)\psi_0^{(+)}(x,E_0)\phi_0(x,E_0)\left(\int_x^{\infty}W(y)\psi_0^{(+)}(y,E_0)^2dy\right)dx. \tag{2.49}$$

Proceeding with (2.49) as with (2.47), we obtain

$$I_{0,1}(E_0)=\int_{-\infty}^{\infty}\frac{1}{\psi_0^{(+)}(x,E_0)^2}\left(\int_x^{\infty}W(y)\psi_0^{(+)}(y,E_0)^2dy\right)^2dx>0,$$

and (ii) is proved. ■

In the following the constant

$$\nu_n=\frac{|\phi_0(p,E_n)|F_{\lambda_0}^{(+)}(0,E_n)^2}{(1+a_n^2)\sqrt{2|\Delta'(E_n)|}} \tag{2.50}$$

will frequently appear. We do not explicitly indicate the dependence on λ_0 of this and similar constants.

Theorem 2.11: Assume (H1), $W\neq 0$, and $n=0$. Then we have the following:

(i) $\lambda_0=0$ is a c.c.th. at E_0 if and only if $I_{0,0}(E_0)\leqslant 0$. If this is the case, then there exists a unique eigenvalue $E(\lambda)$ such that

$$\sqrt{E_0-E(\lambda)}=-\nu_0 I_{0,0}(E_0)\lambda+\nu_0(\tilde{\omega}_{0,E_0},M_{0,E_0}\omega_{0,E_0})\lambda^2+o(\lambda^2),\quad \lambda\downarrow 0. \tag{2.51}$$

(ii) Suppose that $\lambda_0>0$, $[F_{\lambda_0}^{(+)}(\cdot,E_0);F_{\lambda_0}^{(-)}(\cdot,E_0)]=0$, and that (2.22) holds. Then λ_0 is a c.c.th., $I_{\lambda_0,0}(E_0)<0$, and there is a unique eigenvalue given by

$$\sqrt{E_0-E(\lambda)}=-\nu_0 I_{\lambda_0,0}(E_0)(\lambda-\lambda_0)+o(\lambda-\lambda_0),\quad \lambda\downarrow\lambda_0. \tag{2.52}$$

Proof: Suppose λ_0 is a c.c.th. at E_0. Then we identify (2.33) with (2.35) by means of the substitutions $1/d_{\lambda_0}(E_0)=c_{\lambda_0}(\mu)\to c(\mu)$ [cf. (2.39) with $n=0$], $\lambda-\lambda_0\to z$, and $(-1)^{j+1}(\tilde{\omega}_{\lambda_0,E},R^j_{\lambda_0,E}\omega_{\lambda_0,E})=(-1)^{j+1}I_{\lambda_0,j}(E)\to a_{j+1}(\mu)$ $(j=0,1,...)$. To prove (i), first note that, since $\psi_0^{(+)}(x,E_0)=\psi_0^{(-)}(x,E_0)$, $\lambda_0=0$ is an exceptional value at each E_0. By Lemma 2.6(ii), if $a_1(0)<0$, then (2.35) does not have an acceptable (i.e. positive) solution $z(\mu)$. Hence $a_1(0)=-I_{0,0}(E_0)\geqslant 0$ is necessary for $\lambda_0=0$ to be a c.c.th. To show the sufficiency of the condition $I_{0,0}(E_0)\leqslant 0$ we note that by Lemma 2.10(ii), if $I_{0,0}(E_0)=0$, then $I_{0,1}(E_0)=a_2(0)>0$. So either $a_1(0)>0$ or $a_1(0)=0$ and $a_2(0)>0$. Moreover, Lemma 2.5(ii) guarantees that the coefficients $a_k(\mu)$ are continuously differentiable, and from (2.29) and (2.39) it follows that $c(\mu)$ is continuously differentiable, $\dot{c}(0)>0$ [cf. (2.40)] and $c_2=0$ [because $\phi_0(p,E)$ is an even function of μ]. Hence Lemma 2.7(iii) applies and $\mu(z)$ is given by (2.37). This shows that $\lambda_0=0$ is a c.c.th. It remains to convert (2.37) to (2.51). Considering the bracketed term in (2.37), using (2.46), we have

$$\frac{a_2(0)}{\dot{c}(0)}+\frac{a_1(0)\dot{a}_1(0)}{\dot{c}(0)^2}=\frac{I_{0,1}(E_0)}{\dot{c}(0)}+\frac{I_{0,0}(E_0)\dot{I}_{0,0}(E_0)}{\dot{c}(0)^2}=\frac{1}{\dot{c}(0)}\,(\tilde{\omega}_{0,E_0},M_{0,E_0}\omega_{0,E_0}). \tag{2.53}$$

Now (2.51) follows from (2.37), (2.53), and using (2.8) to relate μ to $\sqrt{E_0-E(\lambda)}$. This proves (i).

(ii) When $\lambda_0>0$, then $a_1(0)=-I_{\lambda_0,0}(E_0)>0$ by Lemma 2.10(i), and hence $z(\mu)=[\dot{c}(0)/a_1(0)]\mu+o(\mu)$ by Lemma 2.6(ii). Note that Lemma 2.7 is not applicable, since $a_k(\mu)$ and $c(\mu)$ are not necessarily continuously differentiable at $\mu=0$. Now we show that the function $z(\mu)$ is invertible for small enough μ. We argue by contradiction. Suppose that $z(\mu_1)=z(\mu_2)$ for $0<\mu_1<\mu_2$. Then $z(\mu)$ assumes an extremum at an interior point of this interval, say at μ^*. If $z(\mu^*)$ is a strict extremum, then there are two sequences $\{\mu_{n,\pm}\}$ such that $\mu_{n,+}\downarrow\mu^*$, $\mu_{n,-}\uparrow\mu^*$, $z(\mu_{n,+})=z(\mu_{n,-})$, and $z(\mu_{n,\pm})\to z(\mu^*)$ as $\mu_{n,\pm}\to\mu^*$. This means that, as $\lambda\to\lambda_0+z(\mu^*)$, two eigenvalue sequences $\{E(i\mu_{n,\pm})\}$ belonging to H_{λ_n} with $\lambda_n=\lambda_0+z(\mu_{n,+})=\lambda_0+z(\mu_{n,-})$ converge to the eigenvalue $E(i\mu^*)$ of $H_{\lambda_0+z(\mu^*)}$. Consequently, by standard results from perturbation theory, $H_{\lambda_0+z(\mu^*)}$ has a discrete eigenvalue of multiplicity two. But this is impossible; hence $z(\mu)$ cannot have a strict minimum at μ^*. If the extremum at μ^* is not strict, then there exists a sequence $\{\mu_n\}$ such that $\mu_n\to\mu^*$ and $z(\mu_n)=z(\mu^*)$. This implies that the values $E(i\mu^*)$ and $E(i\mu_n)$ are all eigenvalues of $H_{\lambda_0+z(\mu^*)}$ and that $E(i\mu^*)$ is an accumulation point of these eigenvalues. Again, this is impossible. Hence $z(\mu)$ is a monotone function. Inverting it we get $\mu(z)=[a_1(0)/\dot{c}(0)]z+o(z)$ and (2.52) follows from (2.8). ∎

In the statement of the next theorem, in order to reduce the number of case distinctions, we temporarily lift the restriction $\lambda\geqslant 0$ to the extent that we allow λ to approach 0 from below if the c.c.th. is $\lambda_0=0$.

Theorem 2.12: Suppose that $\lambda_0=0$ or $\lambda_0>0$ and $[F^{(+)}_{\lambda_0}(\cdot,E_n);F^{(-)}_{\lambda_0}(\cdot,E_n)]=0$, $n\geqslant 1$, and that (2.9) and (2.22) hold. Also, suppose that for some $M\geqslant 0$,

$$I_{\lambda_0,j}(E_n)=0,\quad j=0,...,M-1,\quad I_{\lambda_0,M}(E_n)\neq 0. \tag{2.54}$$

Further, assume one of the following:

(a) $\lambda_0=0$ and (H1),

(b) $\lambda_0>0$, $M=0$, and (H1),

(c) $\lambda_0>0$, $M\geqslant 1$, and (H2).

Then the following holds:

(i) If n is odd, M is even, and $I_{\lambda_0,M}(E_n)>0$ [resp., $I_{\lambda_0,M}(E_n)<0$], then there is a unique eigenvalue $E(\lambda)$ of H_λ obeying

$$\sqrt{E(\lambda)-E_n}=\nu_n I_{\lambda_0,M}(E_n)(\lambda-\lambda_0)^{M+1}[1+o(1)],\quad \lambda\downarrow\lambda_0\quad (\text{resp. }\lambda\uparrow\lambda_0). \tag{2.55}$$

(ii) If n is odd, M is odd, and $I_{\lambda_0,M}(E_n)<0$, then there is an eigenvalue obeying

$$\sqrt{E(\lambda)-E_n}=-\nu_n I_{\lambda_0,M}(E_n)(\lambda-\lambda_0)^{M+1}[1+o(1)],\quad \lambda\downarrow\lambda_0\quad\text{and}\quad\lambda\uparrow\lambda_0. \tag{2.56}$$

(iii) If n is odd, M is odd, and $I_{\lambda_0,M}(E_n)>0$, then there is no eigenvalue converging to E_n as either $\lambda\downarrow\lambda_0$ or $\lambda\uparrow\lambda_0$.

(iv) If n is even, M is even, and $I_{\lambda_0,M}(E_n)<0$ [resp. $I_{\lambda_0,M}(E_n)>0$], then there is a unique eigenvalue $E(\lambda)$ of H_λ obeying

$$\sqrt{E_n-E(\lambda)}=-\nu_n I_{\lambda_0,M}(E_n)(\lambda-\lambda_0)^{M+1}[1+o(1)],\quad \lambda\downarrow\lambda_0 \quad (\text{resp. } \lambda\uparrow\lambda_0). \tag{2.57}$$

(v) If n is even, M is odd and $I_{\lambda_0,M}(E_n)>0$, then there is an eigenvalue obeying

$$\sqrt{E_n-E(\lambda)}=\nu_n I_{\lambda_0,M}(E_n)(\lambda-\lambda_0)^{M+1}[1+o(1)],\quad \lambda\downarrow\lambda_0 \quad \text{and} \quad \lambda\uparrow\lambda_0. \tag{2.58}$$

(vi) If n is even, M is odd, and $I_{\lambda_0,M}(E_n)<0$, then there is no eigenvalue converging to E_n as either $\lambda\downarrow\lambda_0$ or $\lambda\uparrow\lambda_0$.

Proof: Apply Lemma 2.7 if (a) or (c) are assumed and use Lemma 2.6, together with the argument about inverting $z(\mu)$ used in the proof of Theorem 2.11, when (b) is assumed. In identifying (2.33) with (2.35) use (2.39) and the correspondence $(-1)^{j+n+1}I_{\lambda_0,j}(E)\to a_{j+1}(\mu)$; further details are omitted. ■

Some comments about Theorems 2.11 and 2.12 may be in order. According to (2.51) and (2.52) eigenvalues can approach E_0 only as $\lambda\downarrow\lambda_0$. In other words, eigenvalues that appear at E_0 can only move to the left as λ increases. This is well-known and not only true for λ near λ_0 but for all $\lambda>\lambda_0$ (Ref. 20, p. 79; Ref. 26, footnote on p. 89). For an endpoint E_n with n odd, i.e., the left endpoint of a gap, part (i) of Theorem 2.12 describes the two situations where an eigenvalue either appears at E_n as $\lambda\uparrow\lambda_0+\epsilon$ [if $I_{\lambda_0,M}(E_n)>0$] or where an eigenvalue comes in from the right and gets absorbed into the continuous spectrum at E_n [if $I_{\lambda_0,M}(E_n)<0$]. Case (ii) describes the situation where an eigenvalue ''turns around'' at E_n. Note that $M+1$ is even and hence $M+1\geqslant 2$, so that $E(\lambda)-E_n=O(\lambda^4)$ at least. In case (iii), λ_0 is an exceptional value, but there is no eigenvalue converging to E_n as $\lambda\downarrow\lambda_0$ or $\lambda\uparrow\lambda_0$. In other words, λ_0 is an exceptional value but not a c.c.th. For n even, the possibilities are similar as can be seen from (iv)–(vi).

Theorem 2.13: Suppose that W satisfies (H3). Then $\sqrt{E(\lambda)-E_n}$ is given by a convergent power series in $\lambda-\lambda_0$.

Proof: Both sides of (2.32) can be analytically continued, as functions of μ, to a complex neighborhood of $\mu=0$. This follows by generalizing the estimates in Lemmas 2.3 and 2.4 to the case when μ (resp., E) is complex. We omit a detailed discussion and only sketch a few steps. The estimates $|\phi_{\lambda_0}(x,E)|\leqslant C(1+|x|)e^{|\mathrm{Re}\,\mu||x|}$ and $|\psi^{(\pm)}_{\lambda_0}(x,E)|\leqslant Ce^{\mp(\mathrm{Re}\,\mu)x}$ for $x\in\mathbf{R}_\pm$ imply that $|G^D_{\lambda_0,E}(x,y)|\leqslant C[1+\min\{|x|,|y|\}]e^{|\mathrm{Re}\,\mu|(|x|+|y|)}$. The exponential factor appearing here will be absorbed by the decay of W when used in the expressions for $R_{\lambda_0,E}$ and $\omega_{\lambda_0,E}$, provided $2|\mathrm{Re}\,\mu|<c$, where c is the constant in (H3). As a result, the function $h(\mu,z)$ in (A30) (see the Appendix) is jointly analytic in μ and z near $\mu=z=0$, $h(0,0)=0$, and $h_\mu(0,0)\neq 0$. Therefore, by the implicit function theorem for analytic functions, the equation $h(\mu,z)=0$ has, for small z, a unique analytic solution $\mu(z)$. ■

Next we discuss how the above results have to be modified when (2.9) or (2.22) are not satisfied, that is, when $\phi_0(p,E_n)=0$, or $\phi_0(p,E_n)\neq 0$ but $F^{(\pm)}_{\lambda_0}(0,E_n)=0$. These special situations may easily occur for reasons of symmetry. For example, if V and W are even, then the subspaces of odd and even parity, respectively, are reducing subspaces for H_λ. Hence on the subspace of odd parity we have $F^{(\pm)}_{\lambda_0}(0,E_n)=0$ at the threshold. Moreover, either E_1 or E_2 is the first Dirichlet eigenvalue for the eigenvalue problem (2.2) on the interval $[0,p]$ and thus $\phi_0(p,E_j)=0$ for $j=1$ or 2. That our approach breaks down when $\phi_0(p,E_n)=0$ is evident from (2.4) which shows that the Titchmarsh–Weyl coefficient $m_0^{(+)}(E)$ [or $m_0^{(-)}(E)$] becomes singular as $E\to E_n$; the imaginary part of $m_0^{(+)}(E)$ blows up, while the real part approaches a finite limit. Note that $\phi_0(p,E)$ vanishes quadratically in μ as $\mu\downarrow 0$, while $\sin[q(E)]$ only vanishes linearly. This implies that the decomposition (2.25) is no longer useful. One way to deal with this situation would be to choose a different decomposition of the Green's function, and this is also necessary if one wants to give a complete treatment of the case $\phi_0(p,E_n)=0$. However, here we are interested only in obtaining the leading terms of some of the eigenvalue expansions, and so we choose a different approach. We perform a shift of the origin so that for the shifted problem conditions (2.9) and (2.22) are satisfied. Then we relate the expressions for the shifted problem back to the original problem. We will often refer to Ref. 24, where some details of this method have already been worked out.

Let $b \in \mathbf{R}$, and replace $V(x)$ and $W(x)$ in (1.1) and (1.2) by $V(x+b)$ and $W(x+b)$. Clearly, the spectrum of H_λ is invariant under the shift. We denote the various solutions associated with the shifted problem by means of an additional argument or subscript b, e.g., $\psi_\lambda^{(+)}(x,E;b)$, $\phi_\lambda(x,E;b)$, $\nu_{n;b}$, etc. From (2.45) and (2.50) we have

$$\nu_{n;b} I_{\lambda_0,0}(E_n;b) = \frac{|\phi_0(p,E_n;b)|}{(1+a_n^2)\sqrt{2|\Delta'(E_n)|}} \int_{-\infty}^{\infty} W(x+b) F_{\lambda_0}^{(+)}(x,E_n;b)^2 dx, \tag{2.59}$$

where we used (2.21) for the shifted problem and the fact that the constant a_n [defined in (2.12)] and the discriminant $\Delta(E)$ are invariant under the shift. We now distinguish between two cases.

Case 1: $F_{\lambda_0}^{(\pm)}(0,E_n)=0$ and $\phi_0(p,E_n)\neq 0$: Since

$$\psi_0^{(\pm)}(x,E;b) = \frac{\psi_0^{(\pm)}(x+b,E)}{\psi_0^{(\pm)}(b,E)},$$

$$F_{\lambda_0}^{(\pm)}(x,E;b) = \psi_0^{(\pm)}(x,E;b) + o(1), \quad x \to \pm\infty,$$

it follows that

$$F_{\lambda_0}^{(\pm)}(x,E;b) = \frac{F_{\lambda_0}^{(\pm)}(x+b,E)}{\psi_0^{(\pm)}(b,E)}. \tag{2.60}$$

So we can choose b such that $\psi_0^{(\pm)}(b,E_n)\neq 0$ and $F_{\lambda_0}^{(\pm)}(b,E_n)\neq 0$, and hence

$$F_{\lambda_0}^{(\pm)}(0,\, E_n;b) = \frac{F_{\lambda_0}^{(\pm)}(b,E_n)}{\psi_0^{(\pm)}(b,E_n)} \neq 0.$$

Furthermore, we have [cf. Ref. 24, Eqs. (3.37)–(3.38)]

$$\phi_0(p,E_n;b) = \psi_0^{(+)}(b,E_n)^2 \phi_0(p,E_n) \neq 0, \tag{2.61}$$

so that (2.9) and (2.22) are both satisfied for the shifted problem. Since $F_{\lambda_0}^{(\pm)}(0,\, E_n)=0$, we have

$$F_{\lambda_0}^{(\pm)}(x,E_n) = F_{\lambda_0}^{(\pm)\prime}(0,\, E_n)\,\phi_{\lambda_0}(x,E_n), \tag{2.62}$$

and so (2.59) can be written as

$$\nu_{n;b} I_{\lambda_0,0}(E_n;b) = \tilde{\nu}_n \tilde{I}_{\lambda_0,0}(E_n), \tag{2.63}$$

$$\tilde{\nu}_n = \frac{|\phi_0(p,E_n)| F_{\lambda_0}^{(+)\prime}(0,E_n)^2}{(1+a_n^2)\sqrt{2|\Delta'(E_n)|}}, \quad \tilde{I}_{\lambda_0,0}(E_n) = \int_{-\infty}^{\infty} W(x)\,\phi_{\lambda_0}(x,E_n)^2 dx. \tag{2.64}$$

Finally, we find an expression for $I_{\lambda_0,1}(E_n;b) = (\tilde{\omega}_{\lambda_0,E_n;b}, R_{\lambda_0,E_n;b}\omega_{\lambda_0,E_n;b})$ under the assumption that $\tilde{I}_{\lambda_0,0}(E_n)=0$, so that we can also determine the leading order term in the eigenvalue expansion when $\tilde{I}_{\lambda_0,0}(E_n)=0$. By using (2.26) and $\tilde{I}_{\lambda_0,0}(E_n)=0=I_{\lambda_0,0}(E_n;b)$, we obtain

$$I_{\lambda_0,1}(E_n;b)=2\int_{-\infty}^{\infty}W(x+b)\,\phi_{\lambda_0}(x,E_n;b)\,\psi_{\lambda_0}^{(+)}(x,E_n;b)\left(\int_x^{\infty}W(y+b)\,\psi_{\lambda_0}^{(+)}(y,E_n;b)^2dy\right)dx. \tag{2.65}$$

Now using

$$\phi_{\lambda_0}(x,E_n;b)=\theta_{\lambda_0}(b,E_n)\,\phi_{\lambda_0}(x+b,E_n)-\phi_{\lambda_0}(b,E_n)\,\theta_{\lambda_0}(x+b,E_n), \tag{2.66}$$

along with (2.60), (2.61), (2.62) for $x=b$, and $\tilde{I}_{\lambda_0,0}(E_n)=0$, we get

$$\begin{gathered}\nu_{n;b}I_{\lambda_0,1}(E_n;b)=\tilde{\nu}_n\tilde{I}_{\lambda_0,1}(E),\\ \tilde{I}_{\lambda_0,1}(E_n)=-2\int_{-\infty}^{\infty}W(x)\,\phi_{\lambda_0}(x,E_n)\,\theta_{\lambda_0}(x,E_n)\left(\int_x^{\infty}W(y)\,\phi_{\lambda_0}(y,E_n)^2dy\right)dx.\end{gathered} \tag{2.67}$$

Case 2: $\phi_0(p,E_n)=0(n\geqslant 1)$: In place of (2.61) we now have [cf. Ref. 24, Eq. (3.37)]

$$\phi_0(p,E_n;b)=-\phi_0(b,E_n)^2\,\theta_0'(p,E_n), \tag{2.68}$$

and we see that by choosing b such that $\phi_0(b,E_n)\neq 0$ we can make sure that $\phi_0(p,E_n;b)\neq 0$. The exceptional case is now characterized by the property that $H_{\lambda_0}\psi=E_n\psi$ has a unique (up to constant factors) nontrivial solution. This is the same as saying that the shifted problem is exceptional in the sense defined below Theorem 2.1. Assuming we are in the exceptional case, following Ref. 24 we introduce the solutions

$$\eta_{\lambda_0}^{(\pm)}(x,E)=\frac{F_{\lambda_0}^{(\pm)}(x,E)}{m_0^{(\pm)}(E)}, \tag{2.69}$$

of $H_{\lambda_0}\psi=E\psi$, and note that

$$\eta_{\lambda_0}^{(\pm)}(x,E_n)=\lim_{E\to E_n}\eta_{\lambda_0}^{(\pm)}(x,E) \tag{2.70}$$

exist, and that $\eta_{\lambda_0}^{(\pm)}(x,E_n)$ represent bounded nontrivial solutions of $H_{\lambda_0}\psi=E_n\psi$. Moreover, $\eta_{\lambda_0}^{(+)}(x,E_n)=b_n\eta_{\lambda_0}^{(-)}(x,E_n)$ for some constant b_n, analogous to (2.12), and $\eta_0^{(\pm)}(x,E_n)=\phi_0(x,E_n)$ if $\lambda_0=0$ and $\eta_{\lambda_0}^{(\pm)}(x,E_n)=\phi_0(x,E_n)+o(1)$ as $x\to\pm\infty$ if $\lambda_0>0$. These asymptotics follow from the integral equation that $\eta_{\lambda_0}^{(\pm)}(x,E_n)$ satisfy and which can be obtained from (2.69) and (2.10):

$$\eta_{\lambda_0}^{(\pm)}(x,E_n)=\phi_0(x,E_n)-\lambda_0\int_x^{\pm\infty}\tilde{A}_0(x,y;E_n)W(y)\,\eta_{\lambda_0}^{(\pm)}(y,E_n)dy, \tag{2.71}$$

where $\tilde{A}_0(x,y;E_n)=\phi_0(x,E_n)\,\theta_0(y,E_n)-\theta_0(x,E_n)\,\phi_0(y,E_n)$. Note that $\tilde{A}_0(x,y;E_n)=\lim_{E\to E_n}A_0(x,y;E)$, where $A_0(x,y;E)$ is defined in (2.11), as can be seen by inserting (2.3) and (2.4) in (2.11). $\tilde{A}_0(x,y;E_n)$ replaces the kernel in (2.20) which becomes formally undefined because $\phi_0(p,E_n)=0$. Since $\psi_0^{(\pm)}(b,E)/m_0^{(\pm)}(E)\to\phi_0(b,E_n)$ as $E\to E_n$, by (2.60), (2.69), and (2.70) we obtain

$$F_{\lambda_0}^{(\pm)}(x,E_n;b)=\frac{\eta_{\lambda_0}^{(\pm)}(x+b,E_n)}{\phi_0(b,E_n)},\quad \psi_{\lambda_0}^{(+)}(x,E_n;b)=\frac{\eta_{\lambda_0}^{(+)}(x+b,E_n)}{\eta_{\lambda_0}^{(+)}(b,E_n)}, \tag{2.72}$$

and hence

$$F_{\lambda_0}^{(+)}(x,E_n;b)=b_nF_{\lambda_0}^{(-)}(x,E_n;b). \tag{2.73}$$

Upon inserting (2.68), (2.72), and (2.73) in (2.59), we find that

$$\nu_{n;b} I_{\lambda_0,0}(E_n;b) = \kappa_n J_{\lambda_0,0}(E_n), \tag{2.74}$$

$$\kappa_n = \frac{|\theta_0'(p,E_n)|}{(1+b_n^2)\sqrt{2|\Delta'(E_n)|}}, \quad J_{\lambda_0,0}(E_n) = \int_{-\infty}^{\infty} W(x)\,\eta_{\lambda_0}^{(+)}(x,E_n)^2 dx. \tag{2.75}$$

As in Case 1, by assuming $J_{\lambda_0,0}(E_n)=0$, we can obtain an expression for the product $\nu_{n;b} I_{\lambda_0,1}(E_n;b)$ which we state without a detailed derivation:

$$\nu_{n;b} I_{\lambda_0,1}(E_n;b) = \kappa_n J_{\lambda_0,1}(E_n), \tag{2.76}$$

$$J_{\lambda_0,1}(E_n) = 2\int_{-\infty}^{\infty} W(x) h_{\lambda_0}(x,E_n)\,\eta_{\lambda_0}^{(+)}(x,E_n)\left(\int_x^{\infty} W(y)\,\eta_{\lambda_0}^{(+)}(y,E_n)^2 dy\right) dx. \tag{2.77}$$

Here $h_{\lambda_0}(x,E_n)$ is any solution of $H_{\lambda_0}\psi = E_n\psi$ such that $[\eta_{\lambda_0}(\cdot,E_n);h_{\lambda_0}(\cdot,E_n)]=1$. This concludes our discussion of Case 2.

The expressions on the right-hand sides of (2.63) and (2.74) can be written in a common form that involves an arbitrary bounded, nontrivial solution $\tilde{\psi}_{\lambda_0}(x,E_n)$ of $H_{\lambda_0}\psi = E_n\psi$. If $\phi_0(p,E_n) \neq 0$, we have that $\tilde{\psi}_{\lambda_0}(x,E_n) = \tilde{\psi}_{\pm}\psi_0^{(\pm)}(x,E_n) + o(1)$ as $x\to\pm\infty$ for some nonzero constants $\tilde{\psi}_{\pm}$. Similarly, if $\phi_0(p,E_n)=0$, since $\tilde{\psi}_{\lambda_0}(x,E_n)$ is a multiple of $\eta_{\lambda_0}^{(+)}(x,E_n)$, there are constants $\tilde{\psi}_{\pm}$ such that $\tilde{\psi}_{\lambda_0}(x,E_n) = \tilde{\psi}_{\pm}\phi_0(x,E_n) + o(1)$ as $x\to\pm\infty$. If $\phi_0(p,E_n)\neq 0$, it follows that

$$\nu_n I_{\lambda_0,0}(E_n), \text{ resp. } \tilde{\nu}_n \tilde{I}_{\lambda_0,0}(E_n) = \frac{|\phi_0(p,E_n)|}{\sqrt{2|\Delta'(E_n)|}}\left[\frac{\int_{-\infty}^{\infty} W(x)\,\tilde{\psi}_{\lambda_0}(x,E_n)^2 dx}{\tilde{\psi}_+^2 + \tilde{\psi}_-^2}\right], \tag{2.78}$$

and, if $\phi_0(p,E_n)=0$, then

$$\kappa_n J_{\lambda_0,0}(E_n) = \frac{|\theta_0'(p,E_n)|}{\sqrt{2|\Delta'(E_n)|}}\left[\frac{\int_{-\infty}^{\infty} W(x)\,\tilde{\psi}_{\lambda_0}(x,E_n)^2 dx}{\tilde{\psi}_+^2 + \tilde{\psi}_-^2}\right]. \tag{2.79}$$

When $\lambda_0 = 0$, the right-hand sides of (2.78) and (2.79) can be expressed in yet another form, one that involves the effective mass. Recall [see Ref. 27 and (2.8)] that at the endpoint E_n the effective mass m_n^* is given by

$$m_n^* = \frac{|\Delta'(E_n)|}{p^2}. \tag{2.80}$$

Furthermore, as a consequence of the variation of constants formula [see (2.3.7) and (2.3.9) in Ref. 28] we have

$$\Delta'(E_n) = \begin{cases} -\dfrac{1}{2}\,\phi_0(p,E_n)\displaystyle\int_0^p \psi_0^{(+)}(x,E_n)^2 dx, & \phi_0(p,E_n)\neq 0, \\ \dfrac{1}{2}\,\theta_0'(p,E_n)\displaystyle\int_0^p \phi_0(x,E_n)^2 dx, & \phi_0(p,E_n)=0 \end{cases}. \tag{2.81}$$

If $\lambda_0 = 0$, then $\tilde{\psi}_0(x,E_n) = \tilde{\psi}_+\psi_0^{(+)}(x,E_n)$ [resp., $\tilde{\psi}_0(x,E_n) = \tilde{\psi}_+\phi_0^{(+)}(x,E_n)$] and $\tilde{\psi}_+ = \tilde{\psi}_-$. Therefore, from (2.78)–(2.81) it follows that

$$\nu_n I_{0,0}(E_n), \ \tilde{\nu}_n \tilde{I}_{0,0}(E_n), \text{ resp. } \kappa_n J_{0,0}(E_n) = \frac{\sqrt{m_n^*}\,p}{\sqrt{2}}\left[\frac{\int_{-\infty}^{\infty} W(x)\,\tilde{\psi}_0(x,E_n)^2 dx}{\int_0^p \tilde{\psi}_0(x,E_n)^2 dx}\right].$$

In the next theorem we specialize some of the results contained in Lemma 2.10, Theorem 2.11, and Theorem 2.12 to the situations described in Cases 1 and 2. We confine ourselves to the cases $M=0$ and $M=1$ [cf. (2.54)], where, in the present context, the case $M=0$ corresponds to either $\tilde{I}_{\lambda_0,0}(E_n)\neq 0$ or $J_{\lambda_0,0}(E_n)\neq 0$, and the case $M=1$ corresponds to either $\tilde{I}_{\lambda_0,0}(E_n)=0$ and $\tilde{I}_{\lambda_0,1}(E_n)\neq 0$, or $J_{\lambda_0,0}(E_n)=0$ and $J_{\lambda_0,1}(E_n)\neq 0$.

Theorem 2.14: Assume (H1), $W\neq 0$, and $\lambda\geqslant 0$.

(i) If $F_{\lambda_0}^{(\pm)}(0,E_0)=0$, then $\lambda_0>0$, $\tilde{I}_{\lambda_0,0}(E_0)<0$, and there exists a unique eigenvalue obeying

$$\sqrt{E_0-E(\lambda)}=-\tilde{\nu}_0\tilde{I}_{\lambda_0,0}(E_0)(\lambda-\lambda_0)+o(\lambda-\lambda_0),\quad \lambda\downarrow\lambda_0. \tag{2.82}$$

(ii) Suppose that either $\phi_0(p,E_n)\neq 0$, $n\geqslant 1$, $\lambda_0>0$, and $F_{\lambda_0}^{(\pm)}(0,E_n)=0$, or $\phi_0(p,E_n)=0$ and λ_0 is an exceptional value. Also assume (H2) if $M=1$. If $M=0$ or 1, then Theorem 2.12 applies with $I_{\lambda_0,M}(E_n)$ replaced by $\tilde{I}_{\lambda_0,M}(E_n)$ [resp., $J_{\lambda_0,M}(E_n)$], and the constants ν_n replaced by $\tilde{\nu}_n$ (resp., κ_n).

(iii) $\lambda_0>0$ is a c.c.th. at E_0 if and only if λ_0 is an exceptional value. If $\lambda_0>0$ is a c.c.th. at E_n with $n\geqslant 1$, then λ_0 is an exceptional value.

Proof: (i) Since $F_0^{(\pm)}(0,E_0)=1$, it follows that $\lambda_0>0$ and that λ_0 is an exceptional value. The rest follows from Lemma 2.10(i) and Theorem 2.11(ii) by applying a shift and using (2.63). (ii) follows from Theorem 2.12, along with (2.63), (2.67), (2.74), and (2.76). The assertion in (iii) concerning E_0 is a direct consequence of Theorems 2.8 and 2.11, and, in Case 1, of (i) and (ii). The statement concerning E_n with $n\geqslant 1$ has been proved in Theorem 2.8 when $\phi_0(p,E_n)\neq 0$ and $F_{\lambda_0}^{(\pm)}(0,E_n)\neq 0$. If one of these restrictions is dropped, then the result follows by using a shift as above and following the proof of Theorem 2.8. Note that the Wronskian $[F_{\lambda_0}^{(+)}(\cdot,E_n;b);\ F_{\lambda_0}^{(-)}(\cdot,E_n;b)]$ is independent of b by (2.60) and (2.72). ■

We remark that there are connections between some of our results concerning eigenvalue absorption at E_0 and earlier work by Gesztesy and Zhao.[29] In Ref. 29 the concepts of criticality, subcriticality, and supercriticality were defined for general second order linear differential operators. Without going into details, we note that in our context H_λ is critical if and only if $\lambda=0$ or $\lambda=\lambda_{0,\min}$, where $\lambda_{0,\min}=\min\{\lambda_0:\lambda_0$ is a c.c.th. at $E_0\}$. Moreover, assuming $\lambda_{0,\min}>0$, H_λ is subcritical if $0<\lambda<\lambda_{0,\min}$ and supercritical if $\lambda>\lambda_{0,\min}$. This follows (after some arguments) from Lemma 2.10 and Theorems 2.11 and 2.14 here, and Definition 3.2 and Theorem 3.6 in Ref. 29. Also, Lemma 3.18 in Ref. 29 and our Lemma 2.10(i) correspond to each other (ignoring the differences in the assumptions on V).

One may ask whether Lemma 2.10, Theorem 2.11, and Theorem 2.14 [(i) and (iii)] can be extended to endpoints other than E_0. It turns out that for compactly supported W, under some further assumptions on the location of the support of W relative to the zeros of the bounded (nontrivial) solution of $H_0\psi=E_n\psi$, this is possible. The fact that zeros of solutions play a role in threshold problems has been noted earlier (see Ref. 10, Theorem 3.2) and in a related context in Ref. 30 (Theorem 8.2).

Note that when $n\geqslant 1$, then any nontrivial solution of $H_0\psi=E_n\psi$ has infinitely many zeros. If $\phi_0(p,E_n)\neq 0$, let ζ_k, $k\in\mathbf{Z}$, denote the zeros of $\psi_0^{(+)}(x,E_n)$ arranged in ascending order, i.e., $\cdots<\zeta_{-1}<\zeta_0<\zeta_1<\cdots$. Similarly, if $\phi_0(p,E_n)=0$, let η_k denote the zeros of $\phi_0(x,E_n)$ arranged in ascending order.

Lemma 2.15: Assume (H1), $W\neq 0$, and $n\geqslant 1$. Let $\lambda_0\geqslant 0$ be an exceptional value at the endpoint E_n.

(i) Suppose $\phi_0(p,E_n)\neq 0$, $\lambda_0>0$, and supp $W\subset[\zeta_k,\zeta_{k+1}]$ for some fixed $k\in\mathbf{Z}$. Then $I_{\lambda_0,0}(E_n)<0$ [resp., $\tilde{I}_{\lambda_0,0}(E_n)<0$ if $F_{\lambda_0}^{(\pm)}(0,E_n)=0$]. If $\phi_0(p,E_n)=0$ and supp $W\subset[\eta_k,\eta_{k+1}]$ for some fixed $k\in\mathbf{Z}$, then $J_{\lambda_0,0}(E_n)<0$.

(ii) Suppose that $\lambda_0=0$. If $\phi_0(p,E_n)\neq 0$ and

$$\int_{\zeta_k}^{\zeta_{k+1}} W(x)\psi_0^{(+)}(x,E_n)^2\,dx=0, \tag{2.83}$$

for all $k \in \mathbf{Z}$, then $I_{0,0}(E_n)=0$ and $I_{0,1}(E_n)>0$.

If $\phi_0(p,E_n)=0$ and

$$\int_{\eta_k}^{\eta_{k+1}} W(x)\phi_0(x,E_n)^2 dx=0,$$

for all $k \in \mathbf{Z}$, then $J_{0,0}(E_n)=0$ and $J_{0,1}(E_n)>0$.

Proof: (i) If $\phi_0(p,E_n)\neq 0$, we can mimic the proof of Lemma 2.10(i) except that the relevant integrals now only involve $x\in[\zeta_k,\zeta_{k+1}]$. As in (2.47) and (2.49) we now get

$$\int_{\zeta_k}^{\zeta_{k+1}} W(x)F_{\lambda_0}^{(+)}(x,E_n)^2 dx<0,$$

so that $I_{\lambda_0,0}(E_n)<0$, resp., $\tilde{I}_{\lambda_0,0}(E_n)<0$ [see (2.62) and (2.64)]. When $\phi_0(p,E_n)=0$, we use (2.71), the relation $[\theta_0(x,E_n)/\phi_0(x,E_n)]'=-1/\phi_0(x,E_n)^2$, and the analog of Lemma 2.2, namely, the fact that

$$\int_{\eta_k}^{\eta_{k+1}} \phi_0(x,E_n)W(x)\eta_{\lambda_0}^{(+)}(x,E_n)dx=0,$$

to derive

$$J_{\lambda_0,0}(E_n)=-\lambda_0\int_{\eta_k}^{\eta_{k+1}} \frac{1}{\phi_0(x,E_n)^2}\left(\int_x^{\eta_{k+1}}\phi_0(y,E_n)W(y)\eta_{\lambda_0}^{(+)}(y,E_n)dy\right)^2 dx<0.$$

This proves (i). Considering (ii), it is obvious by summing over all intervals $[\zeta_k,\zeta_{k+1}]$ that $I_{0,0}(E_n)=0$ [resp., $J_{0,0}(E_n)=0$]. If $\phi_0(p,E_n)\neq 0$ and (2.83) holds, then (2.49) becomes

$$I_{0,1}(E_n)=2\sum_{k\in\mathbf{Z}}\int_{\zeta_k}^{\zeta_{k+1}} W(x)\psi_0^{(+)}(x,E_n)\phi_0(x,E_n)\left(\int_x^{\zeta_{k+1}} W(y)\psi_0^{(+)}(y,E_n)^2 dy\right)dx.$$

Using $[\phi_0(x,E_n)/\psi_0^{(+)}(x,E_n)]'=1/\psi_0^{(+)}(x,E_n)^2$ on each interval $[\zeta_k,\zeta_{k+1}]$, along with an integration by parts, we obtain

$$I_{0,1}(E_n)=\sum_{k\in\mathbf{Z}}\int_{\zeta_k}^{\zeta_{k+1}}\frac{1}{\psi_0^{(+)}(x,E_n)^2}\left(\int_x^{\zeta_{k+1}} W(y)\psi_0^{(+)}(y,E_n)^2 dy\right)^2 dx>0. \qquad \blacksquare$$

If $\phi_0(p,E_n)=0$, then we argue similarly by using (2.77) with $\lambda_0=0$, $\eta_0^{(+)}(x,E_n)=\phi_0(x,E_n)$, and $[h_0(x,E_n)/\phi_0(x,E_n)]'=1/\phi_0(x,E_n)^2$; thus $J_{0,1}(E_n)>0$. ■

Theorem 2.16: Suppose supp $W\subset[\zeta_k,\zeta_{k+1}]$, resp., supp $W\subset[\eta_k,\eta_{k+1}]$ if $\phi_0(p,E_n)=0$, for some $k\in\mathbf{Z}$ and $W\neq 0$. Then the following holds:

(i) $\lambda_0>0$ is a c.c.th. at E_n if and only if λ_0 is an exceptional value. Then $I_{\lambda_0,0}(E_n)<0$ [resp., $\tilde{I}_{\lambda_0,0}(E_n)<0$, resp., $J_{\lambda_0,0}(E_n)<0$] and the corresponding eigenvalue obeys (2.55) as $\lambda\uparrow\lambda_0$, with $M=0$, if n is odd, and it obeys (2.57) as $\lambda\downarrow\lambda_0$, with $M=0$, if n is even.

(ii) If n is even, then $\lambda_0=0$ is a c.c.th. at E_n if and only if $I_{0,0}(E_n)\leqslant 0$ [resp., $J_{0,0}(E_n)\leqslant 0$]. The corresponding eigenvalue obeys (2.57) as $\lambda\downarrow\lambda_0$, with $M=0$, if $I_{0,0}(E_n)<0$, and it obeys (2.58) as $\lambda\downarrow\lambda_0$ and $\lambda\uparrow\lambda_0$, with $M=1$, if $I_{0,0}(E_n)=0$.

(iii) If n is odd, then $\lambda_0=0$ is a c.c.th. at E_n if and only if $I_{0,0}(E_n)>0$ [resp., $J_{0,0}(E_n)>0$]. The corresponding eigenvalue obeys (2.55) as $\lambda\downarrow\lambda_0$, with $M=0$.

Proof: (i) If $n=0$, then the result has already been established in Theorem 2.14(iii) together with Theorem 2.11(ii) and Theorem 2.14(i). If $n\geqslant 1$ and $\lambda_0>0$ is a c.c.th., then Theorem 2.14(iii) implies that λ_0 is an exceptional value. The converse follows from Lemma 2.15(i), Theorem 2.14(ii), and parts (i) and (iv) of Theorem 2.12. Note that Lemma 2.15(i) implies $I_{\lambda_0,0}(E_n)<0$ [resp., $\tilde{I}_{\lambda_0,0}(E_n)<0$, resp., $J_{\lambda_0,0}(E_n)<0$], and so $M=0$. This proves (i). To prove (ii), suppose that $\lambda_0=0$ is a c.c.th. and that $I_{0,0}(E_n)>0$ [resp., $J_{0,0}(E_n)>0$]. Then, by Theorem 2.12(iv), an eigenvalue converging to E_n as $\lambda\downarrow 0$ cannot exist. Hence $I_{0,0}(E_n)\leq 0$ [resp., $J_{0,0}(E_n)\leq 0$]. The converse follows from Theorem 2.12(iv) if $I_{0,0}(E_n)<0$ [resp., $J_{0,0}(E_n)<0$], and from Lemma 2.15(ii) combined with Theorem 2.12(v) if $I_{0,0}(E_n)=0$ [resp., $J_{0,0}(E_n)=0$]. In (iii), if $I_{0,0}(E_n)>0$ [resp., $J_{0,0}(E_n)>0$], then $\lambda_0=0$ is a c.c.th. by Theorem 2.12(i). Conversely, if $\lambda_0=0$ is a c.c.th., then necessarily $I_{0,0}(E_n)\geq 0$ [resp., $J_{0,0}(E_n)\geq 0$]. However, if $I_{0,0}(E_n)=0$ [resp., $J_{0,0}(E_n)=0$], then Lemma 2.15(ii) implies that we are in case (iii) of Theorem 2.12 with $M=1$. Hence $\lambda_0=0$ is not a c.c.th. This is a contradiction and thus $I_{0,0}(E_n)>0$ [resp., $J_{0,0}(E_n)>0$]. ■

Theorem 2.16(i) implies that, under the stated restrictions on the support of W, at the right endpoint of a gap an eigenvalue can only appear as $\lambda\uparrow\lambda_0+\epsilon$; it cannot get absorbed as $\lambda\uparrow\lambda_0$. Similarly, at the left endpoint an eigenvalue can only get absorbed as $\lambda\uparrow\lambda_0$; there is no eigenvalue near the endpoint for λ slightly larger than λ_0.

Note that the points ζ_k (resp.,η_k) are the zeros of the unique (up to constant multiples) bounded, nontrivial solution of $H_0\psi=E_n\psi$. Let $\tilde{\psi}_0(x,E_n)$ denote any such solution. It turns out that if we assume W to be non-negative, then the results of Theorem 2.16 can be strengthened as follows.

Theorem 2.17: Suppose that the support of W lies between two consecutive zeros of $\tilde{\psi}_0(x,E_n)$ and that $W\geq 0$ $(W\neq 0)$. If n is even, then there are no c.c.th.'s at E_n. If n is odd, then $\lambda_0=0$ is the only c.c.th. at E_n.

Proof: For any c.c.th. $\lambda_0\geq 0$ we have $I_{\lambda_0,0}(E_n)>0$ [resp., $\tilde{I}_{\lambda_0,0}(E_n)>0$, resp., $J_{\lambda_0,0}(E_n)>0$], by (2.45), (2.64), (2.75), and the assumption on W. Now the assertions follow from Theorem 2.16. ■

The final theorem of this section tells us what happens if W satisfies the support condition of Theorem 2.17 at both endpoints of a gap.

Theorem 2.18: Suppose that at each endpoint of the gap (E_n,E_{n+1}) (n odd) the support of W satisfies the condition of Theorem 2.17, and that $W\geq 0$ $(W\neq 0)$. Then there is a unique eigenvalue $E(\lambda)$ such that $\lim_{\lambda\downarrow 0}E(\lambda)=E_n$, $E(\lambda)$ is strictly increasing for $\lambda>0$, $\lim_{\lambda\uparrow\infty}E(\lambda)=E_\infty$ exists, and $E_\infty\in(E_n,E_{n+1}]$.

Proof: By Theorem 2.17, $\lambda_0=0$ is the only c.c.th. at E_n. Moreover, by perturbation theory, since W is non-negative (and nontrivial), $E'(\lambda)>0$ for $\lambda>0$. Furthermore, the eigenvalue cannot get absorbed at E_{n+1} by Theorem 2.17. Thus $E(\lambda)$ has to converge to a limit as $\lambda\to+\infty$, and this limit must lie in $(E_n,E_{n+1}]$ (it may be the point E_{n+1}). ■

Theorem 2.18 describes the situation where an eigenvalue $E(\lambda)$ is "trapped." The trapping phenomenon has been discussed in detail, and in greater generality, in Ref. 13. There, an interpretation of E_∞ as a Dirichlet eigenvalue of H_0 on $\mathbf{R}\backslash\mathrm{supp}\,W$ was given. Theorems 2.17 and 2.18 also fit together with more general results obtained in Ref. 11 (see, e.g., Corollary 3.2) which, in the present context, imply that the number of c.c.th.'s at a left endpoint of a gap is always finite, provided W has compact support and is non-negative. Some further results along these lines will appear elsewhere.

III. ONE DIMENSION: FURTHER RESULTS AND SPECIAL CASES

In this section we discuss some special aspects of the results of Section II and we establish the connection with related results that have appeared in the literature.

In Lemma 2.10(ii) and Lemma 2.15(ii), we were able to conclude that $I_{\lambda_0,1}(E_n)>0$ provided that $I_{\lambda_0,0}(E_n)=0$. So it is natural to ask whether it is possible that $I_{\lambda_0,0}(E_n)=I_{\lambda_0,1}(E_n)=0$ ($W\neq 0$, of course). As the following example shows the answer is in the affirmative, at least when $\lambda_0=0$. We give the construction for the case when $\phi_0(p,E_n)\neq 0$, but it will be obvious that similar examples can be constructed if $\phi_0(p,E_n)=0$, showing that $J_{0,0}(E_n)=J_{0,1}(E_n)=0$ is possible. The construction of an example with $\lambda_0>0$ will not be attempted here. Suppose that V is even and W is odd. Then $\psi_0^{(+)}(x,E_n)$ is even, in fact $\psi_0^{(+)}(x,E_n)=\theta_0(x,E_n)$, $\phi_0(x,E_n)$ is odd, and hence $I_{0,0}(E_n)=0$ by symmetry and (2.45). Furthermore, symmetry implies that $I_{0,j}(E_n)=0$ for j even. Consider now a specific W of the form

$$W(x)=\begin{cases}1, & x\in[c,c+\epsilon],\\ -1, & x\in[-c-\epsilon,-c],\\ 0, & \text{otherwise},\end{cases}$$

where $c>0$ and $\epsilon>0$ are parameters. The function $h(x)=\int_x^\infty W(y)\psi_0^{(+)}(y,E_n)^2dy$ is even and non-negative. Hence, by symmetry, we can write (2.49) as

$$I_{0,1}(E_n)=4\int_c^{c+\epsilon}W(x)\psi_0^{(+)}(x,E_n)\phi_0(x,E_n)\left(\int_x^{c+\epsilon}W(y)\psi_0^{(+)}(y,E_n)^2dy\right)dx. \tag{3.1}$$

Now let ζ_1 denote the smallest positive zero of $\psi_0^{(+)}(x,E_n)$ and let η_1 denote the smallest positive zero of $\phi_0(x,E_n)$. Then $\zeta_1<\eta_1$ by the interlacing property of zeros. On $(0,\zeta_1)$ we have $\psi_0^{(+)}(x,E_n)>0$ and $\phi_0(x,E_n)>0$, and on (ζ_1,η_1) we have $\psi_0^{(+)}(x,E_n)<0$ and $\phi_0(x,E_n)>0$. Let $\epsilon<\min\{\zeta_1/2,(\eta_1-\zeta_1)/2\}$ and choose $c=\zeta_1/2$. Then $I_{0,1}(E_n)>0$. On the other hand, choosing $c=(\zeta_1+\eta_1)/2$ we get $I_{0,1}(E_n)<0$. So if we vary c from $\zeta_1/2$ to $(\zeta_1+\eta_1)/2$, then there exists a value c for which $I_{0,1}(E_n)=0$. Hence $I_{0,0}(E_n)=I_{0,1}(E_n)=0$ for this value of c. From the above argument we also see that for every sufficiently small $\epsilon>0$ there is a $c=c(\epsilon)$ such that $I_{0,1}(E_n)=0$ and that $c(\epsilon)\to\zeta_1$ as $\epsilon\to0$. The asymptotic behavior of the function $c(\epsilon)$ as $\epsilon\to0$ can be obtained by expanding the right-hand side of (3.1) in powers of ϵ and $c-\zeta_1$ using the approximations $\psi_0^{(+)}(x,E_n)=\psi_0^{(+)'}(\zeta_1,E_n)(x-\zeta_1)+o(x-\zeta_1)$ and $\phi_0(x,E_n)=\phi_0(\zeta_1,E_n)+o(1)$, and setting $I_{0,1}(E_n)=0$. Introducing variables r and φ such that $c-\zeta_1=r\cos\varphi$ and $\epsilon=r\sin\varphi$, it follows that

$$I_{0,1}(E_n)=\frac{[\psi_0^{(+)'}(\zeta_1,E_n)]^3\phi_0(\zeta_1,E_n)}{15}\cdot r^5f(\varphi)\sin^2\varphi+o(r^5),$$

$$f(\varphi)=30\cos\varphi+17\sin\varphi+11\sin3\varphi.$$

The relevant zero of $f(\varphi)$ is $\varphi_0=1.96784...$ which gives $c(\epsilon)=\zeta_1+\epsilon\cot\varphi_0+o(\epsilon)$, where $\cot\varphi_0=-0.41932...$. So $c(\epsilon)<\zeta_1$ and $c(\epsilon)+\epsilon>\zeta_1$ as it should be, since the support of W must contain the point ζ_1 in its interior. We have already mentioned that $I_{0,2}(E_n)=0$ by symmetry. We want to show that $I_{0,3}(E_n)<0$ for ϵ small and $c=c(\epsilon)$. This follows by writing

$$I_{0,3}(E_n)=4\int_c^{c+\epsilon}\phi_0(x,E_n)u(x,E_n)\left(\int_x^{c+\epsilon}\psi_0^{(+)}(y,E_n)u(y,E_n)dy\right)dx,$$

where $u(x,E_n)=[G_0^DW\psi_0^{(+)}](x,E_n)$, and expanding the right-hand side as in the case of $I_{0,1}(E_n)$. The result is

$$I_{0,3}(E_n)=\frac{[\psi_0^{(+)'}(\zeta_1,E_n)]^5\phi_0(\zeta_1,E_n)^3}{11340}\cdot r^9g(\varphi)\sin^4\varphi+o(r^9),$$

$$g(\varphi)=26838\cos\varphi-1386\cos3\varphi-2772\cos5\varphi+16556\sin\varphi+15383\sin3\varphi+1067\sin5\varphi.$$

Since $g(\varphi_0)=9.31172...>0$, $\psi_0^{(+)'}(\zeta_1,E_n)<0$, and $\phi_0(\zeta_1,E_n)>0$, we conclude that $I_{0,3}(E_n)<0$ for small enough r, resp., small enough $\epsilon(r,\epsilon>0)$. By Theorem 2.12, if n is odd, then there is an eigenvalue obeying (2.56) with $\lambda_0=0$ and $M=3$, i.e., $\sqrt{E(\lambda)-E_n}=-\nu_nI_{0,3}(E_n)\lambda^4+o(\lambda^4)$ as $\lambda\downarrow0$. On the other hand, if n is even, then case (vi) of Theorem 2.12 occurs and there is no eigenvalue approaching zero as $\lambda_0\downarrow0$ (or as $\lambda_0\uparrow0$).

Next we specialize Theorems 2.11 and 2.12 and compare them with some known results in the literature. Suppose that $V\neq0$, so that $E_0=0$ and $[0,\infty)$ is the only band. If (H1) holds and $\int_{-\infty}^\infty W(x)dx\leqslant0$, then $\lambda_0=0$ is a c.c.th. and (2.51) reduces to

$$\sqrt{-E(\lambda)}=-\frac{\lambda}{2}\int_{-\infty}^\infty W(x)dx-\frac{\lambda^2}{4}\int_{-\infty}^\infty W(x)|x-y|W(y)dxdy+o(\lambda^2),\quad \lambda\downarrow0, \tag{3.2}$$

which is a well-known result.[1,4,5] If $\lambda_0>0$, we can compare our results with a result in Ref. 10 (Theorem 3.2), although the operators considered there were of a somewhat different form, namely, $-d^2/dx^2+V+\lambda W$ with $V,W\in C_0^\infty(\mathbf{R})$, and the method used there was also different. For simplicity we set $V=0$ and assume that $\lambda_0>0$ is a c.c.th. for $H_\lambda=-d^2/dx^2+\lambda W$ at $E_0=0$. In accordance with the notation established above (2.78), we let $\tilde{\psi}_{\lambda_0}(x,0)$ be any bounded nontrivial solution of $H_{\lambda_0}\psi=0$. Then the limits $\lim_{x\to\pm\infty}\tilde{\psi}_{\lambda_0}(x,0)=\tilde{\psi}_\pm$ exist and are nonzero. Using (2.78) and the appropriate expansion for $E(\lambda)$ [(2.52), (2.82), or Theorem 2.14(ii)] gives

$$E(\lambda)=-\left[\frac{\int_{-\infty}^{\infty}W(x)\tilde{\psi}_{\lambda_0}(x,0)^2dx}{\tilde{\psi}_+^2+\tilde{\psi}_-^2}\right]^2(\lambda-\lambda_0)^2+o([\lambda-\lambda_0]^2),\quad \lambda\downarrow\lambda_0. \tag{3.3}$$

This result agrees with (3.8) in Ref. 10. Furthermore, (3.3) is easily extended to the case $V\neq 0$ with V and W obeying (2.1) and (H1), respectively. The problem where $V(x)=l_\pm/x^2$ for large $\pm x$, with constants $l_\pm$, was studied recently by Engländer and Pinsky.[31]

Our results extend to the case when V and W contain delta functions. To be specific, suppose that $W(x)=\pm\delta(x-x_0)$ and that V still obeys (2.1). We show how this example fits into the context of the present paper. We give details only for the case $W(x)=+\delta(x-x_0)$. Suppose that E is an eigenvalue of H_λ in a certain gap with eigenfunction $\tilde{\psi}_\lambda(x,E)$. Then we can assume that $\tilde{\psi}_\lambda(x,E)=\psi_0^{(+)}(x,E)$ for $x>x_0$ and $\tilde{\psi}_\lambda(x,E)=c\,\psi_0^{(-)}(x,E_n)$ for $x<x_0$ with some constant c. At x_0, the following matching conditions are satisfied:

$$\psi_0^{(+)}(x_0,E)=c\,\psi_0^{(-)}(x_0,E),\quad \psi_0^{(+)\prime}(x_0,E)-c\,\psi_0^{(-)\prime}(x_0,E)=\lambda\,\psi_0^{(+)}(x_0,E). \tag{3.4}$$

It is immediately clear that E can be an eigenvalue only when $\psi_0^{(\pm)}(x_0,E)\neq 0$. The equation determining the eigenvalues follows from (3.4) and (2.29):

$$\lambda\,\psi_0^{(+)}(x_0,E)\,\psi_0^{(-)}(x_0,E)=-\frac{1}{d_0(E)}. \tag{3.5}$$

This equation is equivalent to (2.33) with $\lambda_0=0$ as can be seen by inserting W in (2.44) which gives

$$I_{0,j}(E)=\begin{cases}\phi_0(x_0,E)^j\,\psi_0^{(+)}(x_0,E)^{j+2}, & x_0\geqslant 0,\\ (-1)^j\phi_0(x_0,E)^j\,\psi_0^{(-)}(x_0,E)^{j+2}, & x_0<0.\end{cases} \tag{3.6}$$

Summing the geometric series in (2.33) yields

$$\frac{\lambda\,\psi_0^{(+)}(x_0,E)^2}{1+\lambda\,\phi_0(x_0,E)\,\psi_0^{(+)}(x_0,E)}=-\frac{1}{d_0(E)},\quad x_0\geqslant 0, \tag{3.7}$$

$$\frac{\lambda\,\psi_0^{(-)}(x_0,E)^2}{1-\lambda\,\phi_0(x_0,E)\,\psi_0^{(-)}(x_0,E)}=-\frac{1}{d_0(E)},\quad x_0<0. \tag{3.8}$$

With the help of (A1) (see the Appendix) it is easy to see that (3.7) and (3.8) are equivalent to (3.5). If $\phi_0(p,E_n)\neq 0$ and λ_0 is a c.c.th., then from (2.3), (2.38), and (3.5) we conclude that $\lambda_0\psi_0^{(+)}(x_0,E_n)^2=0$, so $\lambda_0=0$ or $\psi_0^{(+)}(x_0,E_n)=0$. In the latter case we have $\psi_0^{(\pm)}(x_0,E)=O(\mu)$ as $E\to E_n$. So, in view of (2.38) this means that in the limit $\mu\to 0$ the two sides of (3.5) are incompatible regardless of what c.c.th. λ is converging to. Thus, if $\psi_0^{(+)}(x_0,E_n)=0$, then (3.5) has no solution $E(\lambda)$ such that $E(\lambda)\to E_n$ and so there are no c.c.th.'s in this case. This is in agreement with case (v) of Lemma 2.6, since then, by (3.6), $I_{0,j}(E_n)=0$ for all j. If $\psi_0^{(+)}(x_0,E_n)\neq 0$, then $\lambda_0=0$ is the only c.c.th. Further analysis then shows that n has to be odd, which is in agreement with the fact that W is non-negative, and that (2.55) applies with $M=0$ and $I_{0,0}(E_n)=\psi_0^{(+)}(x_0,E_n)^2$. The case $\phi_0(p,E_n)=0$ and $\psi_0^{(+)}(x_0,E_n)=0$, can be handled similarly, or by a shift as in Section II. The result is that $\lambda_0=0$ is the sole c.c.th. provided $\phi_0(x_0,E_n)\neq 0$ and n is odd. Then Theorem 2.14(ii) applies. In sum we see that at E_n with n odd an eigenvalue enters the gap as $\lambda\uparrow\lambda+\epsilon$ if and only if x_0 does not coincide with a zero of the bounded solution of $H_0\psi=E_n\psi$. An analogous result holds when $W(x)=-\delta(x-x_0)$ and n is even. Without going into detail we mention that the above results can be further specialized by taking H_0 to be a Kronig-Penney Hamiltonian of the form $V(x)=\Sigma_{m=-\infty}^{\infty}\delta(x-[2m-1]a)$, with $a>0$, so that $p=2a$. The reader is referred to Ref. 32 for detailed information about such Hamiltonians. Theorems 2.6.4–2.6.6 (Section III.2.6) of Ref. 32 deal with the existence of eigenvalues when W represents a substitutional or interstitial delta-type impurity and those results are direct consequences of the results presented above.

IV. TWO DIMENSIONS

In two dimensions we assume that V is given by

$$V(x)=V^{(1)}(x_1)+V^{(2)}(x_2),\quad x=(x_1,x_2), \tag{4.1}$$

with $V^{(j)}$ real-valued, $V^{(j)}\in L^1_{loc}(\mathbf{R})$, and

$$V^{(j)}(x_j+p_j)=V^{(j)}(x_j),\quad p_j>0,\quad j=1,2. \tag{4.2}$$

Moreover, W is real-valued and satisfies

$$\int_{\mathbf{R}^2}|W(x)|^{1+\gamma}d^2x<\infty,\quad \int_{\mathbf{R}^2}|W(x)|(1+|x|)^{\gamma}d^2x<\infty, \tag{4.3}$$

for some $\gamma>0$. Here and below the symbol $|\ |$ denotes the Euclidean norm in $\mathbf{R}^{\nu}$. The assumptions (4.1)–(4.2) guarantee that V is relatively $-\Delta$-form bounded with relative bound zero so that $H_0=-\Delta+V$ can be defined by the method of forms. This follows from the corresponding one-dimensional result. Alternatively, H_0 is the unique self-adjoint extension of the minimal operator $H_{0,\min}=H_{0,\min}^{(1)}\otimes\mathbf{I}+\mathbf{I}\otimes H_{0,\min}^{(2)}$, where $H_{0,\min}^{(j)}$ is the minimal operator associated with the one-dimensional operator,

$$H_0^{(j)}=-\frac{d^2}{dx_j^2}+V^{(j)}(x_j),$$

on $L^2(\mathbf{R})$. Since each $H_{0,\min}^{(j)}$ is essentially self-adjoint, $H_{0,\min}$ is also essentially self-adjoint (Ref. 33, Theorem 8.33), and so $H_0=\overline{H_{0,\min}}$. The conditions (4.3) imply[1] that $|W|^{1/2}(-\Delta+1)^{-1}|W|^{1/2}$ is Hilbert–Schmidt. Hence $|W|^{1/2}(-\Delta+1)^{-1/2}\in\mathscr{T}_4$, where $\mathscr{T}_p$ $(1\leqslant p\leqslant\infty)$ denotes the usual trace ideals.[34] Let $E<\inf\sigma(H_0)$. Since $(-\Delta+1)^{1/2}(H_0-E)^{-1/2}$ is bounded, we conclude that $|W|^{1/2}(H_0-E)^{-1/2}=[|W|^{1/2}(-\Delta+1)^{-1/2}][(-\Delta+1)^{1/2}(H_0-E)^{-1/2}]\in\mathscr{T}_4$. Thus $|W|^{1/2}(H_0-E)^{-1}|W|^{1/2}$ is Hilbert–Schmidt, and so W is a relatively form compact perturbation of H_0.

The spectrum of H_0 coincides with the set

$$\bigcup_{n,m=0}^{\infty}\{E_n^{(1)}(\theta_1)+E_m^{(2)}(\theta_2):\theta_1\in[0,2\pi],\theta_2\in[0,2\pi]\}, \tag{4.4}$$

where $E_n^{(j)}(\theta_j)$, $n=0,1,2,...$, are the eigenvalues, arranged in increasing order, of the spectral problem

$$\begin{gathered}H_0^{(j)}(\theta_j)\phi=\left[-\frac{d^2}{dx_j^2}+V^{(j)}(x_j)\right]\phi=E\phi,\quad j=1,2,\\ \phi(p_j)=e^{i\theta_j}\phi(0),\quad \phi'(p_j)=e^{i\theta_j}\phi'(0),\quad \theta_j\in[0,2\pi].\end{gathered} \tag{4.5}$$

The spectrum of $H_0^{(j)}$ consists of the bands $[E_0^{(j)}(0),E_0^{(j)}(\pi)]$, $[E_1^{(j)}(\pi),E_1^{(j)}(0)]$, etc. The sums $E_n^{(1)}(\theta_1)+E_m^{(2)}(\theta_2)$ $(n,m=0,1,...)$ are the eigenvalues of the closure of the operator $H_0^{(1)}(\theta_1)\otimes\mathbf{I}+\mathbf{I}\otimes H_0^{(2)}(\theta_2)$, and the corresponding normalized eigenfunctions are products of the form

$\psi_n^{(1)}(x_1,\theta_1)\psi_m^{(2)}(x_2,\theta_2)$, where $\psi_n^{(j)}(x_j,\theta_j)$ is the normalized eigenfunction for the eigenvalue problem (4.5). We extend these eigenfunctions to all of $\mathbf{R}$ by using the boundary conditions in (4.5). We also note that

$$\psi_n^{(j)}(x,\theta_j)=\overline{\psi_n^{(j)}(x,2\pi-\theta_j)}, \tag{4.6}$$

$$E_n^{(j)}(2\pi-\theta_j)=E_n^{(j)}(\theta_j), \tag{4.7}$$

and that for n even (resp., n odd) the functions $E_n^{(j)}(\theta_j)$ are strictly monotone increasing (resp., decreasing) on $[0,\pi]$ (Ref. 20, Theorem XIII.89). Moreover, near the edges of a gap in $\sigma(H_0^{(j)})$ we have [cf. (2.8)]

$$E_n^{(j)}(\theta_j)=E_n^{(j)}(0)+c_n^{(j)}\theta_j^2+O(\theta_j^4),\quad \theta_j\to 0, \tag{4.8}$$

$$E_n^{(j)}(\theta_j)=E_n^{(j)}(\pi)+d_n^{(j)}(\theta_j-\pi)^2+O([\theta_j-\pi]^4),\quad \theta_j\to\pi, \tag{4.9}$$

where $(-1)^n c_n^{(j)}>0$ and $(-1)^n d_n^{(j)}<0$. Without loss of generality we may assume that

$$E_0^{(1)}(0)=E_0^{(2)}(0)=0, \tag{4.10}$$

so that the first spectral band is given by $[0,E_0^{(1)}(\pi)+E_0^{(2)}(\pi)]$. We first consider absorption of an eigenvalue at the bottom of the spectrum, that is, at energy zero. For the Birman–Schwinger kernel we use $K_{0,E}$ as defined in (1.3), but to simplify the notation we omit the subscript 0 and just write K_E. For any interval Δ, let P^Δ denote the corresponding spectral projection for H_0. We first decompose K_E as follows: Pick $\delta>0$ and write

$$K_E=K_E^{[0,\delta]}+K_E^{(\delta,\infty)}, \tag{4.11}$$

where

$$K_E^{[0,\delta]}=W^{1/2}P^{[0,\delta]}(H_0-E)^{-1}P^{[0,\delta]}|W|^{1/2}, \tag{4.12}$$

$$K_E^{(\delta,\infty)}=W^{1/2}P^{(\delta,\infty)}(H_0-E)^{-1}P^{(\delta,\infty)}|W|^{1/2}. \tag{4.13}$$

Lemma 4.1: If $E<0$ and $\delta>0$, then $K_E^{(\delta,\infty)}$ is Hilbert–Schmidt and has an analytic continuation in the Hilbert–Schmidt norm to a complex neighborhood of $E=0$.

Proof: For $E<0$ write $K_E^{(\delta,\infty)}=ABC$ with $A=W^{1/2}(H_0+1)^{-1/2}$, $B=P^{(\delta,\infty)}(H_0+1)^{1/2}(H_0-E)^{-1}(H_0+1)^{1/2}P^{(\delta,\infty)}$, and $C=(H_0+1)^{-1/2}|W|^{1/2}$. Then A and C both lie in $\mathscr{T}_4$ and B is norm-analytic near $E=0$. Hence the product ABC is analytic in the Hilbert-Schmidt norm for E near 0. ■

In order to analyze the operator $K_E^{[0,\delta]}$ in (4.12) we use the representation of Ref. 20 (see Theorem XIII.98) which allows us to express the integral kernel of $K_E^{[0,\delta]}$ as

$$K_E^{[0,\delta]}(x,y)=W(x)^{1/2}\int_{\mathscr{D}_\delta}\left[\frac{\psi_0^{(1)}(x_1,\theta_1)\psi_0^{(2)}(x_2,\theta_2)\overline{\psi_0^{(1)}(y_1,\theta_1)\psi_0^{(2)}(y_2,\theta_2)}}{E_0^{(1)}(\theta_1)+E_0^{(2)}(\theta_2)-E}\,\frac{d^2\theta}{(2\pi)^2}\right]|W(y)|^{1/2}, \tag{4.14}$$

where $\theta=(\theta_1,\theta_2)$ and

$$\mathscr{D}_\delta=\{\theta\in[0,2\pi]^2:E_0^{(1)}(\theta_1)+E_0^{(2)}(\theta_2)\le\delta\}. \tag{4.15}$$

Let

$$S(x_1,x_2,y_1,y_2;\theta)=\psi_0^{(1)}(x_1,\theta_1)\psi_0^{(2)}(x_2,\theta_2)\overline{\psi_0^{(1)}(y_1,\theta_1)\psi_0^{(2)}(y_2,\theta_2)}, \tag{4.16}$$

$$G(x,y;\theta)=S(x_1,x_2,y_1,y_2;\theta)+S(y_1,x_2,x_1,y_2;\theta)+S(x_1,y_2,y_1,x_2;\theta)+S(y_1,y_2,x_1,x_2;\theta), \tag{4.17}$$

$$H(x,y;\theta)=G(x,y;\theta)-G(x,y;0). \tag{4.18}$$

Note that because $\psi_0^{(j)}(x_j,0)$ is real, we have

$$G(x,y;0)=4\psi_0^{(1)}(x_1,0)\psi_0^{(2)}(x_2,0)\psi_0^{(1)}(y_1,0)\psi_0^{(2)}(y_2,0). \tag{4.19}$$

Moreover, $G(x,y;\theta)$ can be written as

$$\begin{aligned}G(x,y;\theta)=4&[\mathrm{Re}(\psi_0^{(1)}(x_1,\theta_1))\mathrm{Re}(\psi_0^{(1)}(y_1,\theta_1))+\mathrm{Im}(\psi_0^{(1)}(x_1,\theta_1))\mathrm{Im}(\psi_0^{(1)}(y_1,\theta_1))]\\&\times[\mathrm{Re}(\psi_0^{(2)}(x_2,\theta_2))\mathrm{Re}(\psi_0^{(2)}(y_2,\theta_2))+\mathrm{Im}(\psi_0^{(2)}(x_2,\theta_2))\mathrm{Im}(\psi_0^{(2)}(y_2,\theta_2))].\end{aligned} \tag{4.20}$$

Lemma 4.2: For δ small enough, $K_E^{[0,\delta]}$ can be decomposed as

$$K_E^{[0,\delta]}=b(E;\delta)L+R_E^{[0,\delta]}, \tag{4.21}$$

where

$$b(E;\delta)=\frac{1}{\pi^2}\int_{\mathscr{E}_\delta}\frac{d^2\theta}{E_0^{(1)}(\theta_1)+E_0^{(2)}(\theta_2)-E},$$

$$\mathscr{E}_\delta=\{\theta\in[0,\pi]^2:E_0^{(1)}(\theta_1)+E_0^{(2)}(\theta_2)\leqslant\delta\},$$

$$L\cdot=\omega(\tilde{\omega},\cdot),$$

$$\omega(x)=W(x)^{1/2}\psi_0^{(1)}(x_1,0)\psi_0^{(2)}(x_2,0),\quad \tilde{\omega}=\omega\ \mathrm{sgn}\ W, \tag{4.22}$$

and $R_E^{[0,\delta]}$ is the integral operator with kernel

$$R_E^{[0,\delta]}(x,y)=W(x)^{1/2}\left[\int_{\mathscr{E}_\delta}\frac{H(x,y;\theta)}{E_0^{(1)}(\theta_1)+E_0^{(2)}(\theta_2)-E}\frac{d^2\theta}{(2\pi)^2}\right]|W(y)|^{1/2}. \tag{4.23}$$

Proof: We first notice that for δ sufficiently small, the set $\mathscr{D}_\delta$ in (4.15) consists of four disjoint pieces which are located near the corners of the square $[0,2\pi]^2$. The set $\mathscr{E}_\delta$ is that portion of $\mathscr{D}_\delta$ which contains the origin. By using (4.6), (4.7), (4.16), and (4.17), we can reduce the integral over $\mathscr{D}_\delta$ to one over $\mathscr{E}_\delta$, so that (4.14) becomes

$$K_E^{[0,\delta]}(x,y)=W(x)^{1/2}\left[\int_{\mathscr{E}_\delta}\frac{G(x,y;\theta)}{E_0^{(1)}(\theta_1)+E_0^{(2)}(\theta_2)-E}\frac{d^2\theta}{(2\pi)^2}\right]|W(y)|^{1/2}. \tag{4.24}$$

Inserting $G(x,y;\theta)=G(x,y;0)+H(x,y;\theta)$ in (4.24) and using (4.19) yields (4.21). Note that (4.3) implies $\omega\in L^2(\mathbf{R}^2)$ so that L is a rank-one operator on $L^2(\mathbf{R}^2)$. ■

Theorem 4.3: For $-\delta<E<0$ with $\delta>0$ sufficiently small we have the convergent expansion

$$\begin{aligned}b(E;\delta)=&-\frac{1}{2\pi^2}\left(\sum_{m=0}^{\infty}a_{2m}E^m\right)\ln(-E)+\frac{1}{2\pi^2}\left(\sum_{m=0}^{\infty}a_{2m}E^m\right)\ln(\delta-E)\\&+\frac{1}{2\pi^2}\sum_{k=1}^{\infty}\delta^{-k}\left(\sum_{m=k}^{\infty}\frac{a_{2m}\delta^{m+1}}{m-k+1}\right)E^{k-1},\end{aligned} \tag{4.25}$$

with suitable coefficients a_{2m} (to be defined below).

Proof: Define new variables v_1 and v_2 $(v_1,v_2\geqslant 0)$ by

$$v_1=\sqrt{E_0^{(1)}(\theta_1)},\quad v_2=\sqrt{E_0^{(2)}(\theta_2)}. \tag{4.26}$$

For sufficiently small v_1 and v_2 the inverse functions

$$\theta_1 = h_1(v_1), \quad \theta_2 = h_2(v_2), \tag{4.27}$$

exist and, by (4.8) and (4.10), are odd, analytic functions of v_1 and v_2, respectively. By means of the polar coordinates

$$r = \sqrt{v_1^2 + v_2^2}, \quad v_1 = r \cos \varphi, \quad v_2 = r \sin \varphi,$$

we can express $b(E;\delta)$ as

$$b(E;\delta) = \frac{1}{\pi^2} \int_0^{\sqrt{\delta}} \int_0^{\pi/2} \frac{h_1'(r \cos \varphi) h_2'(r \sin \varphi)}{r^2 - E} r dr d\varphi = \frac{1}{\pi^2} \int_0^{\sqrt{\delta}} \frac{u(r) r}{r^2 - E} dr, \tag{4.28}$$

$$u(r) = \int_0^{\pi/2} h_1'(r \cos \varphi) h_2'(r \sin \varphi) d\varphi. \tag{4.29}$$

The functions $h_1'(r \cos \varphi)$ and $h_2'(r \sin \varphi)$ are even functions of r and hence $u(r)$ can be expanded as

$$u(r) = \sum_{m=0}^{\infty} a_{2m} r^{2m}, \tag{4.30}$$

for $r < r_0$, where $r_0 > 0$ is less than the minimum of the radii of convergence for h_1 and h_2. Inserting (4.30) in (4.28) gives

$$b(E;\delta) = \frac{1}{\pi^2} \sum_{m=0}^{\infty} a_{2m} I_m(E;\delta),$$

$$I_m(E;\delta) = \int_0^{\sqrt{\delta}} \frac{r^{2m+1}}{r^2 - E} dr, \quad m = 0,1,2,\ldots . \tag{4.31}$$

Then

$$I_0(E;\delta) = -\tfrac{1}{2} \ln(-E) + \tfrac{1}{2} \ln(\delta - E),$$

and $I_m(E;\delta)$ is given recursively by

$$I_m(E;\delta) - E I_{m-1}(E;\delta) = \frac{\delta^m}{2m}, \quad m = 1,2,\ldots .$$

Thus

$$I_m(E;\delta) = -\frac{1}{2} E^m \ln(-E) + \frac{1}{2} E^m \ln(\delta - E) + \frac{1}{2} \sum_{k=1}^{m} \frac{\delta^{m-k+1}}{m-k+1} E^{k-1}, \tag{4.32}$$

as can be verified by induction on m or, alternately, by differentiation with respect to δ. Inserting (4.32) in (4.31) yields (4.25). It is easy to see that choosing $\delta < r_0^2$ ensures convergence of the various series on the right-hand side of (4.25) for $-\delta < E < 0$. ■

From (4.25) we infer that

$$b(E;\delta) = -\frac{a_0}{2\pi^2} \ln(-E) + c(\delta) - \frac{a_2}{2\pi^2} E \ln(-E) + O(E),$$

where

$$c(\delta) = \frac{1}{2\pi^2} \left[a_0 \ln \delta + \sum_{m=1}^{\infty} \frac{a_{2m} \delta^m}{m} \right].$$

By using (4.8), (4.26), and (4.27), we find $h_j'(0)=1/\sqrt{c_0^{(j)}}$ $(j=1,2)$ and thus, by (4.29) and (4.30),

$$a_0=\frac{\pi}{2\sqrt{c_0^{(1)}c_0^{(2)}}}. \tag{4.33}$$

We let

$$R_{E;\delta}=R_E^{[0,\delta]}+K_E^{(\delta,\infty)}=K_E-b(E;\delta)L, \tag{4.34}$$

and define the operator M_E by

$$K_E=-\frac{a_0}{2\pi^2}\ln(-E)L+M_E, \tag{4.35}$$

so that

$$M_E=\left[c(\delta)-\frac{a_2}{2\pi^2}E\ \ln(-E)+O(E)\right]L+R_{E;\delta}. \tag{4.36}$$

We now study the behavior of M_E as $E\to 0$. Note that, by (4.35), M_E is independent of δ. Also, note that the representation (4.35) is of the form (1.5) with $N=1$, $\lambda_0=0$, $d_{0;1}(E)=-a_0/(2\pi^2)$, $\omega_{0,E;1}=\omega$ as given by (4.22), and $R_{0,E}=M_E$. We remark that for two-dimensional perturbed periodic Schrödinger operators it has been noted earlier[35] that the Birman-Schwinger kernel diverges as $E\uparrow 0$, and this fact has been discussed in connection with the criticality of such operators.

Lemma 4.4: For every $\alpha\in(0,1]$ and $\theta_j\geqslant 0$ sufficiently small there is a constant C_j such that

$$|\psi_0^{(j)}(x_j,\theta_j)-\psi_0^{(j)}(x_j,0)|\leqslant C_j\ (1+|x_j|)^{\alpha}\theta_j^{\alpha},\quad j=1,2. \tag{4.37}$$

Proof: Let $\psi_0^{(j,+)}(x_j,E_0^{(j)}(\theta_j))$ denote the solutions defined in (2.7) associated with the potential $V^{(j)}$ for $E=E_0^{(j)}(\theta_j)$. Thus

$$\psi_0^{(j)}(x_j,\theta_j)=\frac{\psi_0^{(j,+)}(x_j,E_0^{(j)}(\theta_j))}{N_0^{(j,+)}(\theta_j)}=\frac{e^{i\theta_j x_j/p_j}\xi_0^{(j,+)}(x_j,E_0^{(j)}(\theta_j))}{N_0^{(j,+)}(\theta_j)}, \tag{4.38}$$

where

$$N_0^{(j,+)}(\theta_j)=\left(\int_0^{p_j}|\xi_0^{(j,+)}(x_j,E_0^{(j)}(\theta_j))|^2dx_j\right)^{1/2}. \tag{4.39}$$

Recall that the functions $\psi_0^{(j)}(x_j,\theta_j)$ are normalized over one period. In the following estimates $\tilde{C}_j$ denotes a suitable constant. Similarly to (i) and (iv) of Lemma 2.3 we have

$$|\psi_0^{(j,+)}(x_j,E_0^{(j)}(\theta_j))|\leqslant\tilde{C}_j$$

and thus

$$|\psi_0^{(j,+)}(x_j,E_0^{(j)}(\theta_j))-\psi_0^{(j,+)}(x_j,0)|\leqslant 2\tilde{C}_j.$$

Moreover,

$$|\dot{\psi}_0^{(j,+)}(x_j,E_0^{(j)}(\theta_j))|\leqslant\tilde{C}_j(1+|x_j|),$$

where now the dot denotes differentiation with respect to θ_j. Therefore,

$$|\psi_0^{(j,+)}(x_j,E_0^{(j)}(\theta_j))-\psi_0^{(j,+)}(x_j,0)|\leqslant\tilde{C}_j\theta_j(1+|x_j|)$$

and hence, by interpolation,

$$|\psi_0^{(j,+)}(x_j,E_0^{(j)}(\theta_j))-\psi_0^{(j,+)}(x_j,0)|\leqslant 2^{1-\alpha}\tilde{C}_j\theta_j^\alpha(1+|x_j|)^\alpha$$

for any $\alpha\in(0,1]$. Furthermore,

$$|N_0^{(j,+)}(\theta_j)-N_0^{(j,+)}(0)|=O(\theta_j)$$

by (4.39). Now (4.37) follows from (4.38). ■

Theorem 4.5: There is a Hilbert–Schmidt operator M_0 and $\beta>0$ such that $\|M_E-M_0\|_{H.S.}=O(|E|^\beta)$ as $E\uparrow 0$.

Proof: In view of Lemma 4.1, (4.34), and (4.36), it suffices to show that $R_E^{[0,\delta]}$ converges in Hilbert–Schmidt norm to a limit $R_0^{[0,\delta]}$ as $E\uparrow 0$, and that $\|R_E^{[0,\delta]}-R_0^{[0,\delta]}\|_{H.S.}=O(|E|^\beta)$. We write

$$H(x,y;\theta)=[S(x_1,x_2,y_1,y_2;\theta)-\psi_0^{(1)}(x_1,0)\psi_0^{(2)}(x_2,0)\psi_0^{(1)}(y_1,0)\psi_0^{(2)}(y_2,0)]+\cdots, \quad (4.40)$$

where the right-hand side consists of four obvious terms coming from subtracting (4.19) from (4.17). Each of these terms can again be written as a sum of four terms, namely

$$\begin{aligned}S(x_1,x_2,y_1,y_2;\theta)&-\psi_0^{(1)}(x_1,0)\psi_0^{(2)}(x_2,0)\psi_0^{(1)}(y_1,0)\psi_0^{(2)}(y_2,0)\\&=[\psi_0^{(1)}(x_1,\theta_1)-\psi_0^{(1)}(x_1,0)]\psi_0^{(2)}(x_2,\theta_2)\overline{\psi_0^{(1)}(y_1,\theta_1)\psi_0^{(2)}(y_2,\theta_2)}+\cdots,\end{aligned} \quad (4.41)$$

where it is clear how the right-hand side is obtained. Applying Lemma 4.4 to each term on the right-hand side of (4.41) and then using the result in (4.40) gives

$$|H(x,y;\theta)|\leqslant C(1+|x|)^\alpha(1+|y|)^\alpha|\theta|^\alpha,\quad \alpha\in(0,1]. \quad (4.42)$$

Substituting (4.42) in (4.23) leads to θ-integrals that are bounded independently of E, since by using polar coordinates (ignoring irrelevant factors) we have

$$\int_{\mathscr{E}_\delta}\frac{|\theta|^\alpha}{E_0^{(1)}(\theta_1)+E_0^{(2)}(\theta_2)-E}\,d^2\theta\leqslant C\int_0^{\sqrt{\delta}}\frac{r^{\alpha+1}}{r^2-E}\,dr\leqslant C\,\frac{\delta^{\alpha/2}}{\alpha}.$$

Consequently,

$$|R_E^{[0,\delta]}(x,y)|\leqslant C|W(x)|^{1/2}(1+|x|)^\alpha(1+|y|)^\alpha|W(y)|^{1/2}. \quad (4.43)$$

So if we take α so that $0<\alpha\leqslant\min\{1,\gamma/2\}$, where γ is the constant in (4.3), then (4.43) provides us with the required uniform bound on $|R_E^{[0,\delta]}(x,y)|$ that allows us to conclude, as in the proof of Lemma 2.4(ii), that $R_E^{[0,\delta]}\to R_0^{[0,\delta]}$ in Hilbert–Schmidt norm as $E\uparrow 0$. Taking the derivative of $R_E^{[0,\delta]}$ with respect to E and using the estimate

$$\int_{\mathscr{E}_\delta}\frac{|\theta|^\alpha}{[E_0^{(1)}(\theta_1)+E_0^{(2)}(\theta_2)-E]^2}\,d^2\theta\leqslant C\int_0^{\sqrt{\delta}}\frac{r^{\alpha+1}}{(r^2-E)^2}\,dr\leqslant C|E|^{-1+\alpha/2}\int_0^\infty\frac{u^{\alpha+1}}{(u^2+1)^2}\,du,$$

we get

$$\left|\frac{dR_E^{[0,\delta]}}{dE}(x,y)\right|\leqslant C|E|^{-1+\alpha/2}|W(x)|^{1/2}(1+|x|)^\alpha(1+|y|)^\alpha|W(y)|^{1/2}. \quad (4.44)$$

Thus an integration gives $\|R_E^{[0,\delta]}-R_0^{[0,\delta]}\|_{H.S.}=O(|E|^{\alpha/2})$, and so we can choose any $\beta\leqslant\min\{1/2,\gamma/4\}$ to reach the conclusion of the theorem. ■

According to (1.8) and (4.35) the equation for the bound state reads

$$\lambda(\tilde{\omega},[1+\lambda M_E]^{-1}\omega)=\frac{2\pi^2}{a_0\ln(-E)}, \quad (4.45)$$

and we are looking for solutions obeying $E(\lambda)\uparrow 0$ as $\lambda\downarrow 0$. Since for $|E|$ small, the right-hand side of (4.45) is negative, such solutions can exist only if

$$(\tilde{\omega},\omega)=\int_{\mathbf{R}^2} W(x)\psi_0^{(1)}(x_1,0)^2\psi_0^{(2)}(x_2,0)^2 d^2x\leqslant 0.$$

The next lemma is the analog of Lemma 2.10(ii) in two dimensions.

Lemma 4.6: If $(\tilde{\omega},\omega)=0$ and $W\neq 0$, then $(\tilde{\omega},M_0\omega)>0$.

Proof: $(\tilde{\omega},\omega)=0$ is equivalent to $L\omega=0$, and so, by (4.35) and Theorem 4.5,

$$\lim_{E\uparrow 0}(\tilde{\omega},K_E\omega)=(\tilde{\omega},M_0\omega). \tag{4.46}$$

Let

$$\hat{K}_E=(\operatorname{sgn} W)K_E=|W|^{1/2}(H_0-E)^{-1}|W|^{1/2}, \quad E<0.$$

Then $\hat{K}_E$ is an increasing family of positive self-adjoint operators. So

$$(\tilde{\omega},K_E\omega)=(\omega,\hat{K}_E\omega)\geqslant 0,$$

and hence, by (4.46), $(\tilde{\omega},M_0\omega)\geqslant 0$. Furthermore, by (4.46), $(\tilde{\omega},M_0\omega)=0$ implies $(\omega,\hat{K}_E\omega)=0$ for all $E<0$. So $(H_0-E)^{-1/2}|W|^{1/2}\omega=0$ for all $E<0$; in particular,

$$(H_0+1)^{-1/2}|W|^{1/2}\omega=0, \tag{4.47}$$

and thus $|W|^{1/2}\omega=0$. Since $|W|^{1/2}\omega$ is generally not in $L^2(\mathbf{R}^2)$, the last conclusion requires an explanation. If we consider the scale of Hilbert spaces $\mathscr{H}_{+1}\subset L^2(\mathbf{R}^2)\subset\mathscr{H}_{-1}$, where $\mathscr{H}_{+1}=D([H_0+1]^{1/2})$ with the norm $\|f\|_{+1}=\|(H_0+1)^{1/2}f\|$ and $\mathscr{H}_{-1}$ is the completion of $L^2(\mathbf{R})$ in the norm $\|f\|_{-1}=\|(H_0+1)^{-1/2}f\|$, then $|W|^{1/2}\omega\in\mathscr{H}_{-1}$. Moreover, $(H_0+1)^{-1/2}$ maps $\mathscr{H}_{-1}$ to $L^2(\mathbf{R}^2)$ and has a trivial kernel. Since H_0+1 and $-\Delta+1$ have the same form domain, or, equivalently, by multiplying (4.47) from the left by $(-\Delta+1)^{-1/2}(H_0+1)^{1/2}$, we conclude that $(-\Delta+1)^{-1/2}|W|^{1/2}\omega=0$. Then, by using a Fourier transform, we obtain $W(x)=0$, a.e. Since this is not the case, $(\tilde{\omega},M_0\omega)>0$. ■

In the next theorem, $E_0(\lambda)$ denotes the lowest eigenvalue (ground state) of H_λ. Moreover, a_0 is the constant defined in (4.33) and β is as in Theorem 4.5.

Theorem 4.7: Assume (4.1)–(4.3). Then there is an eigenvalue $E_0(\lambda)$ of H_λ satisfying the following:

(i) $E_0(\lambda)$ is simple and the only eigenvalue converging to zero as $\lambda\downarrow 0$.

(ii) If $(\tilde{\omega},\omega)<0$, then

$$E_0(\lambda)=-e^{-h_1(\lambda)}[1+O(e^{-\rho_1/\lambda})], \quad \lambda\downarrow 0, \tag{4.48}$$

where $\rho_1=2\pi^2\gamma_{-1}\beta a_0^{-1}$ and

$$h_1(\lambda)=\frac{2\pi^2}{a_0}\left[\frac{\gamma_{-1}}{\lambda}+\gamma_0+\sum_{n=1}^{\infty}\gamma_n\lambda^n\right],$$

$$\gamma_{-1}=-\frac{1}{(\tilde{\omega},\omega)}, \quad \gamma_0=-\frac{(\tilde{\omega},M_0\omega)}{(\tilde{\omega},\omega)^2}, \tag{4.49}$$

with the series in (4.49) being convergent for λ small enough.

(iii) If $(\tilde{\omega},\omega)=0$, then

$$E_0(\lambda)=-e^{-h_2(\lambda)}[1+O(\lambda^{-2}e^{-\rho_2/\lambda^2})], \quad \lambda\downarrow 0, \tag{4.50}$$

where ρ_2 is any constant such that $0<\rho_2<2\pi^2\nu_{-2}\beta a_0^{-1}$, and

$$h_2(\lambda)=\frac{2\pi^2}{a_0}\left[\frac{\nu_{-2}}{\lambda^2}+\frac{\nu_{-1}}{\lambda}+\nu_0+\sum_{n=1}^{\infty}\nu_n\lambda^n\right],$$

$$\nu_{-2}=\frac{1}{(\tilde{\omega},M_0\omega)},\quad \nu_{-1}=\frac{(\tilde{\omega},M_0^2\omega)}{(\tilde{\omega},M_0\omega)^2},\quad \nu_0=\frac{(\tilde{\omega},M_0^2\omega)^2}{(\tilde{\omega},M_0\omega)^3}-\frac{(\tilde{\omega},M_0^3\omega)}{(\tilde{\omega},M_0\omega)^2}. \tag{4.51}$$

(iv) If $(\tilde{\omega},\omega)>0$, then λ_0 is not a c.c.th.

We remark that in the case $V\neq 0$ the coefficients γ_{-1}, γ_0, ν_{-2}, ν_{-1}, and ν_0 reduce to those given by Holden.[2]

Proof: (i) The existence and uniqueness of $E_0(\lambda)$ are obvious if we recall the approach based on eigenvalue perturbation theory.[1,3] However, for later use it is useful to have an independent argument based solely on (4.45). First, existence is obvious from (4.45), Theorem 4.5, and the intermediate value theorem. Uniqueness follows by taking the derivative with respect to E and showing that $[d/dE](\tilde{\omega},[1+\lambda M_E]^{-1}\omega)=O(|E|)^{-1+\alpha/2}$ as $E\uparrow 0$, uniformly in λ, with $0<\alpha\leqslant\min\{1,\gamma/2\}$. This follows from (4.44), which remains valid if we replace $R_E^{[0,\delta]}$ by M_E, and estimates similar to those used in the proof of Lemma 2.5. The restriction on α comes from the fact that the integral $\int_{\mathbf{R}^2}(1+|x|)^{2\alpha}|W(x)|d^2x$ must be finite. Further details are omitted. Since the derivative of the right-hand side of (4.45) is equal to $2\pi^2a_0^{-1}E^{-1}[\ln(-E)]^{-2}$, and thus dominates for small E, uniqueness follows.

To prove (ii) note that, by (4.45),

$$\ln(-E)=-\frac{2\pi^2}{a_0}\left[\frac{\gamma_{-1}}{\lambda}+\gamma_0+o(1)\right], \tag{4.52}$$

and so, by Theorem 4.5,

$$\|(1+\lambda M_E)^{-1}-(1+\lambda M_0)^{-1}\|_{H.S.}\leqslant C_1\lambda|E|^{\beta}\leqslant C_2\lambda e^{-\rho_1/\lambda}, \tag{4.53}$$

where $\rho_1=2\pi^2\gamma_{-1}\beta a_0^{-1}$. Hence we can write (4.45) as

$$\lambda(\tilde{\omega},(1+\lambda M_0)^{-1}\omega)+O(\lambda^2e^{-\rho_1/\lambda})=\frac{2\pi^2}{a_0\ln(-E)}. \tag{4.54}$$

Let

$$h_1(\lambda)=-\frac{2\pi^2}{a_0}\frac{1}{\lambda(\tilde{\omega},[1+\lambda M_0]^{-1}\omega)}. \tag{4.55}$$

Then (4.48) and (4.49) follow from (4.54) and (4.55) by expanding $(1+\lambda M_0)^{-1}$.

The proof of (iii) is similar. Instead of (4.52) we now have

$$\ln(-E)=-\frac{2\pi^2}{a_0}\left[\frac{\nu_{-2}}{\lambda^2}+\frac{\nu_{-1}}{\lambda}+\nu_0+o(1)\right],$$

and thus

$$\|(1+\lambda M_E)^{-1}-(1+\lambda M_0)^{-1}\|\leqslant C_1\lambda|E|^{\beta}\leqslant C_2\lambda e^{-\rho_2/\lambda^2},$$

with $0<\rho_2<2\pi^2\nu_{-2}\beta a_0^{-1}$, taking into account the possibility that ν_{-1} may be negative. Since $(\tilde{\omega},\omega)=0$, the function $h_1(\lambda)$ can be rewritten as

$$h_2(\lambda)=\frac{2\pi^2}{a_0}\frac{1}{\lambda^2(\tilde{\omega},M_0[1+\lambda M_0]^{-1}\omega)}.$$

Expanding the denominator leads to (4.51). In place of (4.54) we have from (4.45),

$$-\lambda^2(\tilde{\omega},M_0[1+\lambda M_0]^{-1}\omega)+O(\lambda^2e^{-\rho_2/\lambda^2})=\frac{2\pi^2}{a_0\ln(-E)}. \tag{4.56}$$

So, by solving (4.56) for E and putting $E=E_0(\lambda)$, we obtain (4.50).

Conclusion (iv) is obvious from (4.45) because the two sides have opposite signs when λ and E are small. ■

If we assume

$$\int_{\mathbf{R}^2} e^{c|x|}|W(x)|d^2x<\infty, \quad c>0, \tag{4.57}$$

then $E_0(\lambda)$ has a convergent expansion in terms of quantities containing factors of the form $e^{-d/\lambda}$ $(d>0)$ and powers of λ. First, note that from (2.3), (2.4), and (2.7), it follows that the normalization constants $N_0^{(j,+)}(\theta_j)$ $(j=1,2)$ in (4.39), originally defined for real θ_j, can be analytically continued to a complex neighborhood of zero and that $N_0^{(j,+)}(\theta_j)$ are even functions of θ_j. Consequently, the real part of $\psi_0^{(j)}(x_j,\theta_j)$ is an even function of θ_j and the imaginary part is an odd function of θ_j. Thus there is a $\tilde{\theta}>0$ such that for $|\theta_j|<\tilde{\theta}$ $(j=1,2)$ we have the convergent expansions

$$\psi_0^{(j)}(x_j,\theta_j)=\alpha_0^{(j)}(x_j)+i\beta_1^{(j)}(x_j)\theta_j+\alpha_2^{(j)}(x_j)\theta_j^2+i\beta_3^{(j)}(x_j)\theta_j^3+\ldots, \tag{4.58}$$

where

$$\alpha_0^{(j)}(x_j)=\frac{\psi_0^{(j,+)}(x_j,0)}{N_0^{(j,+)}(0)}, \quad \beta_1^{(j)}(x_j)=\frac{\phi_0^{(j)}(x_j,0)}{\phi_0^{(j)}(p_j,0)N_0^{(j,+)}(0)},$$

and $\alpha_n^{(j)}(x_j)$ and $\beta_n^{(j)}(x_j)$ are real. Also, recall that $\phi_0^{(j)}(p_j,0)\neq 0$, since $E_0^{(j)}(0)=0=\inf\sigma(H_0^{(j)})$. Equation (4.58) follows from (4.38), (2.3), and (2.4) with $q(E)=q(E_0^{(j)}(\theta_j))=\theta_j$, by expanding $\psi_0^{(j)}(x_j,\theta_j)$ in powers of θ_j. In view of (4.20) and (4.58) the function $H(x,y;\theta)$ in (4.18) is even in θ_1 and θ_2 separately, and hence can be expanded as

$$H(x,y;\theta)=\sum_{n,m=0}^{\infty}A_{nm}(x,y)\theta_1^{2n}\theta_2^{2m}, \tag{4.59}$$

where $A_{00}(x,y)=0$ because $H(x,y;0)=0$. From (4.16), (4.18), and (4.38) we obtain the inequality

$$|H(x,y;\theta)|\leq Ce^{|\theta_1||x_1|/p_1+|\theta_2||x_2|/p_2+|\theta_1||y_1|/p_1+|\theta_2||y_2|/p_2}. \tag{4.60}$$

Let $\tilde{p}=(p_1^{-2}+p_2^{-2})^{1/2}$ and assume $|\theta_j|<\tilde{\theta}(j=1,2)$. Using the Schwarz inequality gives

$$|H(x,y;\theta)|\leq Ce^{\tilde{\theta}\,\tilde{p}(|x|+|y|)}.$$

Thus, by Cauchy's inequality, we obtain

$$|A_{nm}(x,y)|\leq C\,\frac{e^{\tilde{\theta}\,\tilde{p}(|x|+|y|)}}{\tilde{\theta}^{2(n+m)}}. \tag{4.61}$$

Inserting (4.59) in (4.23) and using polar coordinates according to (4.26) and (4.27) gives rise to integrals of the form

$$\rho_{nm}(r)=\int_0^{\pi/2}h_1'(r\cos\varphi)h_2'(r\sin\varphi)\cos^{2n}\varphi\,\sin^{2m}\varphi\,d\varphi.$$

The functions $\rho_{nm}(r)$ can be expanded as

$$\rho_{nm}(r)=\sum_{s=0}^{\infty}c_{nm;2s}\,r^{2s},$$

where the series converges for $r<r_0$ with r_0 being the same as in (4.30). Again, by Cauchy's inequality, there is a constant $M_{r_0}>0$ (independent of n and m) such that

$$|c_{nm;2s}| \leqslant \frac{M_{r_0}}{r_0^{2s}}. \tag{4.62}$$

Assuming δ to be so small that $\delta < \min\{\tilde{\theta}^2, r_0^2\}$ and $\mathscr{E}_\delta \subset \{\theta \in [0,\pi]^2 : |\theta_j| < \tilde{\theta}, j=1,2\}$, for $-\delta < E < 0$ we have

$$R_E^{[0,\delta]}(x,y) = \frac{W(x)^{1/2}}{4\pi^2}\left[\sum_{n,m,s=0}^{\infty} c_{nm;2s} A_{nm}(x,y) I_{n+m+s}(E;\delta)\right] |W(y)|^{1/2}, \tag{4.63}$$

where $I_{n+m+s}(E;\delta)$ is given by (4.32). Inserting (4.32) in (4.63) gives

$$R_E^{[0,\delta]}(x,y) = \left[\sum_{n=0}^{\infty} D_n^{(1)}(x,y) E^n\right] E \ln(-E) + \sum_{n=0}^{\infty} D_n^{(2)}(x,y) E^n, \tag{4.64}$$

where the coefficients $D_n^{(1)}(x,y)$ and $D_n^{(2)}(x,y)$ obey

$$|D_n^{(j)}(x,y)| \leqslant C |W(x)|^{1/2} e^{\tilde{\theta}\tilde{p}(|x|+|y|)} |W(y)|^{1/2}, \quad j=1,2.$$

Hence the kernels $D_n^{(j)}(x,y)$ are Hilbert–Schmidt if we assume that $\tilde{\theta} \leqslant c/(2\tilde{p})$, where c is the constant in (4.57). By (4.36), and since $K_E^{(\delta,\infty)}$ is analytic at $E=0$, the expansion (4.64) carries over to $R_{E;\delta}$ and to M_E. As a result, (4.45) can be written in the form

$$\ln(-E) = \begin{cases} -\dfrac{2\pi^2}{a_0}\left[\dfrac{\gamma_{-1}}{\lambda} + \gamma_0 + \displaystyle\sum_{\substack{n,m,s=0\\ n+m+s\geqslant 1}} e_{nms} E^n [E \ln(-E)]^m \lambda^s\right], & (\tilde{\omega},\omega) < 0, \\ -\dfrac{2\pi^2}{a_0}\left[\dfrac{\nu_2}{\lambda^2} + \dfrac{\nu_{-1}}{\lambda} + \nu_0 + \displaystyle\sum_{\substack{n,m=0\\ s\geqslant -2}} f_{nms} E^n [E \ln(-E)]^m \lambda^s\right], & (\tilde{\omega},\omega) = 0, \end{cases} \tag{4.65}$$

where $f_{00-2} = f_{00-1} = f_{000} = 0$. Let

$$v = \lambda^{-1} e^{-2\pi^2 \gamma_{-1}/(a_0\lambda)}, \quad w = \lambda^{-4} e^{-(2\pi^2/a_0)[\nu_{-2}/\lambda^2 + \nu_{-1}/\lambda]}. \tag{4.66}$$

Theorem 4.8: Assume (4.57). Then the following holds.

(i) If $(\tilde{\omega},\omega) < 0$, then for λ sufficiently small, $E_0(\lambda)$ has the uniformly and absolutely convergent expansion

$$E_0(\lambda) = \sum_{n,m=1}^{\infty} c_{nm} v^n \lambda^m, \tag{4.67}$$

where $c_{11} = -e^{-2\pi^2 \gamma_0/a_0}$.

(ii) If $(\tilde{\omega},\omega) = 0$, then

$$E_0(\lambda) = \sum_{\substack{n=1\\ m=4}}^{\infty} c_{nm} w^n \lambda^m, \tag{4.68}$$

where $c_{14} = -e^{-2\pi^2 \nu_0/a_0}$.

Proof: (i) Write

$$E_0(\lambda) = e^{-(2\pi^2/a_0)[\gamma_{-1}/\lambda + \gamma_0]}[1+\eta],$$

and substitute it in the first equation in (4.65). After some simplifications we get an equation of the form

$$F(\eta,v,\lambda)=0, \tag{4.69}$$

where F is analytic in all three variables in a neighborhood of zero, $F(0,0,0)=0$, and $[\partial F/\partial\eta](0,0,0)\neq 0$. Thus, by the implicit function theorem for analytic functions, (4.69) has a unique solution $\eta=\eta(v,\lambda)$ given by a convergent power series in the variables v and λ; this implies (4.67). The proof of (ii) is similar: We insert

$$E_0(\lambda)=e^{-(2\pi^2/a_0)[\nu_{-2}/\lambda^2+\nu_{-1}/\lambda+\nu_0]}[1+\eta]$$

in the second equation in (4.65) to obtain an equation of the form $F(\eta,w,\lambda)=0$ whose solution $\eta=\eta(w,\lambda)$ is analytic in both variables for $|w|$ and $|\lambda|$ small. Thus (4.68) follows. ■

There is a slight discrepancy between our expansion for $E_0(\lambda)$ and the corresponding expansion of Ref. 2 in the case $V=0$ in that the quantity w used here contains the factor λ^{-4} in contrast to the factor λ^{-3} in Ref. 2. The reason for having the factors λ^{-1} and λ^{-4} in front of the exponentials in (4.66) is due to the terms $e_{010}\,E\,\ln(-E)$ and $f_{01-2}E\,\ln(-E)\lambda^{-2}$ in (4.65), respectively. These terms arise from expanding $(\tilde{\omega},M_E\omega)$ for E small [cf. (4.45)].

This concludes our discussion of the behavior of the ground state. We now investigate what happens in the first gap, that is in the interval

$$(E_0^{(1)}(\pi)+E_0^{(2)}(\pi),\min\{E_1^{(1)}(\pi),E_1^{(2)}(\pi)\}),$$

provided

$$\min\{E_1^{(1)}(\pi),E_1^{(2)}(\pi)\}>E_0^{(1)}(\pi)+E_0^{(2)}(\pi). \tag{4.70}$$

We only consider the case $\lambda_0=0$ and content ourselves with obtaining the leading terms in the expansion of the eigenvalues. For this purpose we divide our analysis into two parts.

Case 1: Absorption at $E^*=E_0^{(1)}(\pi)+E_0^{(2)}(\pi)$. In this case we decompose K_E as follows:

$$K_E=K_E^{[E^*-\delta,E^*]}+K_E^{\mathbf{R}\setminus[E^*-\delta,E^*]},$$

where the notation [cf. (4.11)–(4.13)] is self-explanatory. The integral associated with $K_E^{[E^*-\delta,E^*]}$ [i.e., the analog of (4.14)] extends over the region $\{\theta\in[0,2\pi]^2{:}E^*-\delta\leq E_0^{(1)}(\theta_1)+E_0^{(2)}(\theta_2)\leq E^*\}$. It contains the point (π,π) and is symmetric with respect to the lines $\theta_1=\pi$ and $\theta_2=\pi$. Using this symmetry and (4.6) and (4.7), the integration can be reduced to one over that portion of the region on which $\theta_1\leq\pi$ and $\theta_2\leq\pi$. It follows that we can write

$$K_E=\frac{b_0}{2\pi^2}\ln|E-E^*|L+M_E^{(1)}, \tag{4.71}$$

where now $b_0=2^{-1}\pi[d_0^{(1)}d_0^{(2)}]^{-1/2}$, with $d_0^{(1)}$ and $d_0^{(2)}$ being the constants in (4.9), $L\cdot=\omega(\tilde{\omega},\cdot)$, and $\tilde{\omega}=\omega\ \mathrm{sgn}\ W$, where

$$\omega(x)=W(x)^{1/2}\psi_0^{(1)}(x_1,\pi)\psi_0^{(2)}(x_2,\pi).$$

As in Theorem 4.5 one shows that $M_E^{(1)}\to M_{E^*}^{(1)}$ in the Hilbert–Schmidt norm as $E\downarrow E^*$. Note the sign change in the first term in (4.71) as compared to (4.35). Therefore, an eigenvalue can get absorbed at E^* only if

$$(\tilde{\omega},\omega)=\int_{\mathbf{R}^2}W(x)\psi_0^{(1)}(x_1,\pi)^2\psi_0^{(2)}(x_2,\pi)^2d^2x\geq 0.$$

Theorem 4.9: Suppose (4.1)–(4.3) and (4.70) hold. Then the following holds:
(i) If $(\tilde{\omega},\omega)>0$, then there is a unique, simple eigenvalue $E(\lambda)$ such that

$$E(\lambda)-E^*=e^{-(2\pi^2/b_0)[\mu_{-1}/\lambda+\mu_0+o(1)]},\quad \lambda\downarrow 0. \tag{4.72}$$

(ii) If $(\tilde{\omega},[M_{E*}^{(1)}]^k\omega)=0$ for $k=0,1,...,n-1$ and $(-1)^n(\tilde{\omega},[M_{E*}^{(1)}]^n\omega)>0$, then there are constants $c_0,\ldots,c_{n+1}$ such that

$$E(\lambda)-E^*=e^{c_0/\lambda^{n+1}+c_1/\lambda^n+\cdots+c_{n+1}+o(1)},\quad \lambda\downarrow 0,$$

where

$$c_0=\frac{2\pi^2(-1)^{n+1}}{b_0(\tilde{\omega},[M_{E*}^{(1)}]^n\omega)}<0.$$

(iii) If $(\tilde{\omega},[M_{E*}^{(1)}]^k\omega)=0$ for $k=0,1,...,n-1$ and $(-1)^n(\tilde{\omega},[M_{E*}^{(1)}]^n\omega)<0$, then λ_0 is not a c.c.th.

Proof: (i) is a special case of (ii). In analogy to (4.45), the equation for $E(\lambda)$ now reads

$$-\lambda(\tilde{\omega},[1+\lambda M_E^{(1)}]^{-1}\omega)=\frac{2\pi^2}{b_0\ \ln|E-E^*|}. \tag{4.73}$$

Then (ii) follows from (4.73) by proceeding as in the proof of Theorem 4.7(ii). The coefficients c_n are obtained by expanding $[1+\lambda M_{E*}^{(1)}]^{-1}$, noting that the difference $[1+\lambda M_E^{(1)}]^{-1}-[1+\lambda M_{E*}^{(1)}]^{-1}$ has no effect on these coefficients because it is controlled by an estimate similar to that in (4.53). Uniqueness and simplicity of the eigenvalue follow as in the proof of Theorem 4.7. ■

Of course, we could be more explicit about the remainder terms in (4.72) and write it in a form like (4.48). If $(\tilde{\omega},\omega)=0$, then in general nothing can be said about the sign of $(\tilde{\omega},M_{E*}^{(1)}\omega)$; that is, there is no analog of Lemma 4.6. The reason is that the proof of Lemma 4.6 uses in an essential way the fact that we are at the bottom of the spectrum.

Case 2: Absorption at $E^*=\min\{E_1^{(1)}(\pi),E_1^{(2)}(\pi)\}$. This case will be divided further into three cases. As we will see, in one of these cases it is possible that two eigenvalues approach E^* as $\lambda\downarrow 0$. We first describe the decompositions of K_E that characterize each case. Then, in Theorem 4.10, we collect the information about the behavior of the eigenvalues that converge to E^* as $\lambda\downarrow 0$.

(a) $E^*=E_1^{(1)}(\pi)$ and $E_1^{(2)}(\pi)>E^*$: In this case, for sufficiently small δ, the relevant region for the θ-integration is given by $\{\theta\in[0,2\pi]^2:E^*\leqslant E_1^{(1)}(\theta_1)+E_0^{(2)}(\theta_2)\leqslant E^*+\delta\}$. This region has two components containing the points $(\pi,0)$ and $(\pi,2\pi)$, respectively. The singular contribution to K_E again has rank one and we have the decomposition

$$K_E=-\frac{e_0}{2\pi^2}\ln|E-E^*|L+M_E^{(2)},$$

where $e_0=2^{-1}\pi[d_1^{(1)}c_0^{(2)}]^{-1/2}$, $L\cdot=\omega(\tilde{\omega},\cdot)$, $\tilde{\omega}=\omega\ \mathrm{sgn}\ W$, and

$$\omega(x)=W(x)^{1/2}\psi_1^{(1)}(x_1,\pi)\psi_0^{(2)}(x_2,0).$$

Moreover, $M_E^{(2)}\to M_{E*}^{(2)}$ in the Hilbert–Schmidt norm as $E\uparrow E^*$.

(b) $E^*=E_1^{(2)}(\pi)$ and $E_1^{(1)}(\pi)>E^*$: This case is analogous to case (a). We have

$$K_E=-\frac{f_0}{2\pi^2}\ln|E-E^*|L+M_E^{(3)},$$

where $f_0=2^{-1}\pi[d_1^{(2)}c_0^{(1)}]^{-1/2}$, $L\cdot=\omega(\tilde{\omega},\cdot)$, $\tilde{\omega}=\omega\ \mathrm{sgn}\ W$, and

$$\omega(x)=W(x)^{1/2}\psi_0^{(1)}(x_1,0)\psi_1^{(2)}(x_2,\pi).$$

Moreover, $M_E^{(3)}\to M_{E*}^{(3)}$ in the Hilbert–Schmidt norm as $E\uparrow E^*$.

(c) $E^*=E_1^{(1)}(\pi)=E_1^{(2)}(\pi)$: In this case the critical region of integration is $\{\theta\in[0,2\pi]^2:E^*\leqslant E_0^{(1)}(\theta_1)+E_1^{(2)}(\theta_2)\leqslant E^*+\delta\}\cup\{\theta\in[0,2\pi]^2:E^*\leqslant E_1^{(1)}(\theta_1)+E_0^{(2)}(\theta_2)\leqslant E^*+\delta\}$ and thus has four components located near $(\pi,0)$, $(2\pi,\pi)$, $(\pi,2\pi)$, and $(0,\pi)$. Let

$$\omega_1=W(x)^{1/2}\psi_0^{(1)}(x_1,0)\psi_1^{(2)}(x_2,\pi),\quad \omega_2=W(x)^{1/2}\psi_1^{(1)}(x_1,\pi)\psi_0^{(2)}(x_2,0). \tag{4.74}$$

Then K_E is of the form

$$K_E=d_1(E)\omega_1(\tilde{\omega}_1,\cdot)+d_2(E)\omega_2(\tilde{\omega}_2,\cdot)+R_E,$$

where

$$d_1(E)=\frac{1}{\pi^2}\int_{\mathscr{E}_{1,\delta}}\frac{d^2\theta}{E_0^{(1)}(\theta_1)+E_1^{(2)}(\theta_2)-E},$$

$$\mathscr{E}_{1,\delta}=\{\theta\in[0,\pi]^2:E^*\leqslant E_0^{(1)}(\theta_1)+E_1^{(2)}(\theta_2)\leqslant E^*+\delta\},$$

$$d_2(E)=\frac{1}{\pi^2}\int_{\mathscr{E}_{2,\delta}}\frac{d^2\theta}{E_1^{(1)}(\theta_1)+E_0^{(2)}(\theta_2)-E},$$

$$\mathscr{E}_{2,\delta}=\{\theta\in[0,\pi]^2:E^*\leqslant E_1^{(1)}(\theta_1)+E_0^{(2)}(\theta_2)\leqslant E^*+\delta\}.$$

Also, $R_E\to R_{E^*}$ in the Hilbert–Schmidt norm as $E\uparrow E^*$. Since ω_1 and ω_2 are linearly independent, the operator $d_1(E)\omega_1(\tilde{\omega}_1,\cdot)+d_2(E)\omega_2(\tilde{\omega}_2,\cdot)$ has rank two. The coefficients $d_1(E)$ and $d_2(E)$ diverge as $E\uparrow E^*$ according to

$$d_1(E)=-\beta_1\ \ln|E-E^*|+O(1),$$

$$d_2(E)=-\beta_2\ \ln|E-E^*|+O(1),$$

where $\beta_1=(4\pi)^{-1}[d_1^{(2)}c_0^{(1)}]^{-1/2}$, $\beta_2=(4\pi)^{-1}[d_1^{(1)}c_0^{(2)}]^{-1/2}$. This follows from a result analogous to (4.25). Thus (1.8) reads

$$\det\begin{bmatrix}1+\lambda d_1(E)(\tilde{w}_1,[1+\lambda R_E]^{-1}w_1) & \lambda d_2(E)(\tilde{w}_1,[1+\lambda R_E]^{-1}w_2)\\ \lambda d_1(E)(\tilde{w}_2,[1+\lambda R_E]^{-1}w_1) & 1+\lambda d_2(E)(\tilde{w}_2,[1+\lambda R_E]^{-1}w_2)\end{bmatrix}=0. \tag{4.75}$$

The next theorem gives information about the behavior of the eigenvalues for each of the above situations.

Theorem 4.10: Suppose (4.1)–(4.3) and (4.70) hold. Then we have the following:

(a) Suppose $E^*=E_1^{(1)}(\pi)$ and $E_1^{(2)}(\pi)>E^*$: If $(\tilde{\omega},\omega)<0$, then there is a unique, simple eigenvalue obeying

$$E^*-E(\lambda)=e^{-(2\pi^2/e_0)[\sigma_{-1}/\lambda+\sigma_0+o(1)]},\quad \lambda\downarrow 0,$$

where

$$\sigma_{-1}=-\frac{1}{(\tilde{\omega},\omega)},\quad \sigma_0=-\frac{(\tilde{\omega},M_{E^*}^{(2)}\omega)}{(\tilde{\omega},\omega)^2}.$$

If $(\tilde{\omega},\omega)=0$, then a result analogous to Theorem 4.9(ii) holds. If $(\tilde{\omega},\omega)>0$, then $\lambda_0=0$ is not a c.c.th.

(b) Suppose $E^*=E_1^{(2)}(\pi)$ and $E_1^{(1)}(\pi)>E^*$: If $(\tilde{\omega},\omega)<0$, then there is a unique, simple eigenvalue obeying

$$E^*-E(\lambda)=e^{-(2\pi^2/f_0)[\rho_{-1}/\lambda+\rho_0+o(1)]},\quad \lambda\downarrow 0,$$

where

$$\rho_{-1} = -\frac{1}{(\tilde{\omega},\omega)}, \quad \rho_0 = -\frac{(\tilde{\omega}, M_{E^*}^{(3)}\omega)}{(\tilde{\omega},\omega)^2}.$$

If $(\tilde{\omega},\omega)=0$, then a result analogous to Theorem 4.9(ii) holds. If $(\tilde{\omega},\omega)>0$, then $\lambda_0=0$ is not a c.c.th.

(c) Suppose $E^*=E_1^{(1)}(\pi)=E_1^{(2)}(\pi)$: Let

$$c_{\pm} = -\beta_1(\tilde{\omega}_1,\omega_1)-\beta_2(\tilde{\omega}_2,\omega_2)\pm\sqrt{[\beta_1(\tilde{\omega}_1,\omega_1)-\beta_2(\tilde{\omega}_2,\omega_2)]^2+4\beta_1\beta_2(\tilde{\omega}_1,\omega_2)^2}.$$

If $c_{\pm}>0$, then there are two eigenvalues (counting multiplicities) approaching E^* as $\lambda\downarrow 0$ given by

$$E^*-E(\lambda)=e^{-2/(c_{\pm}\lambda)+O(1)}, \quad \lambda\downarrow 0. \tag{4.76}$$

If $c_{\pm}$ are nonzero and of opposite signs, then there is only one eigenvalue, the one corresponding to c_+, that converges to E^* as $\lambda\downarrow 0$, and for it (4.76) holds.

Proof: The proofs in cases (a) and (b) are by now familiar. The result in case (c) follows by expanding the left-hand side of (4.75) for λ small and E near E^*, and using an estimate like (4.53). Further details are omitted. ■

We see from (4.75) that if $W\leqslant 0 (W\neq 0)$, then $c_{\pm}>0$ by the Schwarz inequality, $(\tilde{\omega}_1,\omega_2)^2 \leqslant (\tilde{\omega}_1,\omega_1)^2(\tilde{\omega}_2,\omega_2)^2$. Thus there are two eigenvalues (counting multiplicities) that converge to E^* as $\lambda\downarrow 0$. As one would expect, if V and W have certain symmetry properties, then the two eigenvalues may degenerate into one eigenvalue of multiplicity two. For example, this happens if $V^{(1)}$ and $V^{(2)}$ in (4.1) are given by identical functions, $V^{(1)}(\xi)=V^{(2)}(\xi)$ $(\xi\in\mathbf{R})$, W is symmetric, $W(x_1,x_2)=W(x_2,x_1)$, and even in x_1 and x_2 separately, and $E_1^{(1)}(\pi)$ is the first Dirichlet eigenvalue of the operator $H_0^{(1)}$ on the interval $[0,p]$. To see this, let $\mathscr{U}$ be the unitary operator $(\mathscr{U}\psi)(x_1,x_2)=\psi(x_2,x_1)$ and let $\mathscr{M}_1=\{\psi\in L^2(\mathbf{R}^2): \psi(x_1,x_2)=\psi(-x_1,x_2)=-\psi(x_1,-x_2)\}$ and $\mathscr{M}_2=\{\psi\in L^2(\mathbf{R}^2): \psi(x_1,x_2)=\psi(x_1,-x_2)=-\psi(-x_1,x_2)\}$. Then $\mathscr{M}_1$ and $\mathscr{M}_2$ are orthogonal, invariant subspaces for H_λ (for any $\lambda\geqslant 0$) such that $\mathscr{U}\mathscr{M}_1=\mathscr{M}_2$ and $\mathscr{U}\mathscr{M}_2=\mathscr{M}_1$. Moreover, $\mathscr{M}_1$ and $\mathscr{M}_2$ are invariant subspaces for K_E. By (4.74) and because of the symmetries of the potentials,

$$\omega_1=W(x)^{1/2}\psi_0^{(1)}(x_1,0)\psi_1^{(1)}(x_2,\pi), \quad \omega_2=W(x)^{1/2}\psi_1^{(1)}(x_1,\pi)\psi_0^{(1)}(x_2,0).$$

Since $\psi_0^{(1)}(\cdot,0)$ is even and $\psi_1^{(1)}(\cdot,\pi)$ is odd, $\omega_1\in\mathscr{M}_1$ and $\omega_2\in\mathscr{M}_2$. It follows that the restrictions of K_E to the invariant subspaces $\mathscr{M}_1$ and $\mathscr{M}_2$ are unitarily equivalent; in fact $K_E\restriction\mathscr{M}_1 = \mathscr{U}(K_E\restriction\mathscr{M}_2)\mathscr{U}$. Thus the eigenvalue in (4.76) is degenerate and of multiplicity two. Furthermore, $(\tilde{\omega}_1,\omega_2)=0$, $(\tilde{\omega}_1,\omega_1)=(\tilde{\omega}_2,\omega_2)$, $\beta_1=\beta_2$, and thus $c_+=c_-=-2\beta_1(\tilde{\omega}_1,\omega_1)$.

If $W\geqslant 0(W\neq 0)$, then $c_{\pm}<0$, and there are no eigenvalues that converge to E^* as $\lambda\downarrow 0$. If one or both of $c_{\pm}$ are zero, then a more detailed investigation is necessary.

We do not discuss here in detail eigenvalues in higher gaps. We only remark that the singular part of K_E can have any rank. The following example shows how rank three might arise. Assume that $E_0^{(1)}=E_0^{(2)}(0)=0$, $E_0^{(1)}(\pi)=E_0^{(2)}(\pi)=1$, $E_1^{(1)}(\pi)=E_1^{(2)}(\pi)=5$, $E_1^{(1)}(0)=E_1^{(2)}(0)=6$, $E_2^{(1)}(0)=E_2^{(2)}(0)=7$, $E_2^{(1)}(\pi)=E_2^{(2)}(\pi)=8$, and $E_3^{(1)}(\pi)=E_3^{(2)}(\pi)=10$. Then the lowest two (open) gaps of the two-dimensional problem are (2,5) and (9,10) (the point {7} represents a closed gap). The singular part of K_E has rank 1 at $E=2$, rank 2 at $E=5$, rank 1 at $E=9$, and rank 3 at $E=10$. In the last instance we have $\omega_1(x)=W(x)^{1/2}\psi_0^{(1)}(x_1,0)\psi_3^{(2)}(x_2,\pi)$ $\omega_2(x) =W(x)^{1/2}\psi_3^{(1)}(x_1,\pi)\psi_0^{(2)}(x_2,0)$, and $\omega_3(x)=W(x)^{1/2}\psi_2^{(1)}(x_1,\pi)\psi_2^{(2)}(x_2,\pi)$ because $E_0^{(1)}(0)+E_3^{(2)}(\pi)=E_3^{(1)}(\pi)+E_0^{(2)}(0)=E_1^{(1)}(\pi)+E_1^{(2)}(\pi)=10$.

V. THREE DIMENSIONS

We briefly discuss the three-dimensional problem under the assumptions that

$$V(x)=V^{(1)}(x_1)+V^{(2)}(x_2)+V^{(3)}(x_3), \quad x=(x_1,x_2,x_3), \tag{5.1}$$

with W, $V^{(j)}$ real-valued, $V^{(j)}\in L^1_{loc}(\mathbf{R})$, and

$$V^{(j)}(x_j+p_j)=V(x_j), \quad p_j>0, \quad j=1,2,3, \tag{5.2}$$

$$W\in L^{3/2}(\mathbf{R}^3), \quad \int_{\mathbf{R}^3}(1+|x|)^{\gamma}|W(x)|d^3x, \tag{5.3}$$

for a suitable $\gamma>0$. As in one and two dimensions, H_0 can be defined by the forms method or as the closure of a suitable minimal operator (cf. Section IV). The condition $W\in L^{3/2}(\mathbf{R}^3)$ implies that $W\in\mathscr{R}$, the class of Rollnick potentials (Ref. 18, p. 170; Ref. 26). Such potentials are relatively form compact with respect to $-\Delta$ and so also with respect to H_0. We only consider eigenvalue absorption at the bottom of the continuous spectrum which we assume to be at $E=0$. As in the two-dimensional case we may suppose that the spectrum of each $H_0^{(j)}$ also begins at zero. The Birman–Schwinger kernel is given by

$$K_E=W^{1/2}(H_0-E)^{-1}|W|^{1/2}, \quad E<0,$$

and K_E is Hilbert–Schmidt because $W\in\mathscr{R}$. To shorten the notation we will write

$$\Psi_0(x,\theta)=\psi_0^{(1)}(x_1,\theta_1)\psi_0^{(2)}(x_2,\theta_2)\psi_0^{(3)}(x_3,\theta_3), \quad \theta=(\theta_1,\theta_2,\theta_3).$$

We also let

$$G(x,y;\theta)=\Psi_0(x,\theta)\overline{\Psi_0(y,\theta)}+\cdots, \tag{5.4}$$

where the dots indicate terms that are obtained from the first one by interchanging any number of elements between the ordered triples $\{x_1,x_2,x_3\}$ and $\{y_1,y_2,y_3\}$, respecting the order of the elements as in (4.17).

Theorem 5.1: Suppose (5.1)–(5.3) hold. Then K_E has the expansion

$$K_E=K_0+(-E)^{1/2}B+o(|E|^{1/2}), \quad E\uparrow 0, \tag{5.5}$$

where the symbol o refers to the Hilbert-Schmidt norm and B is a rank-one operator with kernel

$$B(x,y)=-\frac{1}{4\pi\sqrt{c_0^{(1)}c_0^{(2)}c_0^{(3)}}}W(x)^{1/2}\Psi_0(x,0)\Psi_0(y,0)|W(y)|^{1/2}. \tag{5.6}$$

We remark that if $V=0$, then $E_0^{(k)}(\theta_k)=\theta_k^2/p_k^2$, where the periods p_k are arbitrary positive numbers, so that $c_0^{(k)}=1/p_k^2$, $\Psi_0(x,0)=1/\sqrt{p_1p_2p_3}$. Hence

$$B(x,y)=-\frac{1}{4\pi}W(x)^{1/2}|W(y)|^{1/2},$$

which is a familiar result [see, e.g., Ref. 3, Eq. (2.7)].

Proof: In order to shorten the notation we define

$$E_0(\theta)=E_0^{(1)}(\theta_1)+E_0^{(2)}(\theta_2)+E_0^{(3)}(\theta_3).$$

(i) In place of (4.24) we now have

$$K_E^{[0,\delta]}(x,y)=W(x)^{1/2}\left[\int_{\mathscr{E}_\delta}\frac{G(x,y;\theta)}{E_0(\theta)-E}\frac{d^3\theta}{(2\pi)^3}\right]|W(y)|^{1/2}, \tag{5.7}$$

where $\mathscr{E}_\delta=\{\theta\in[0,\pi]^3:E_0(\theta)\leqslant\delta\}$. We now write (5.7) as

$$K_E^{[0,\delta]}(x,y)=X_E^{[0,\delta]}(x,y)+Y_E^{[0,\delta]}(x,y), \tag{5.8}$$

$$X_E^{[0,\delta]}(x,y)=W(x)^{1/2}G(x,y;0)|W(y)|^{1/2}\left[\int_{\mathscr{E}_\delta}\frac{d^3\theta}{(2\pi)^3[E_0(\theta)-E]}\right], \tag{5.9}$$

$$Y_E^{[0,\delta]}(x,y)=W(x)^{1/2}\left[\int_{\mathscr{E}_\delta}\frac{H(x,y;\theta)}{E_0(\theta)-E}\frac{d^3\theta}{(2\pi)^3}\right]|W(y)|^{1/2},\tag{5.10}$$

$$H(x,y;\theta)=G(x,y;\theta)-G(x,y;0).\tag{5.11}$$

First consider the integral on the right-hand side of (5.9). Following (4.26) and (4.27) we set $v_j=\sqrt{E_0^{(j)}(\theta_j)}$, $\theta_j=h_j(v_j)$, for $j=1,2,3$, and $r=\sqrt{v_1^2+v_2^2+v_3^2}$, $v_1=r\sin\chi\cos\varphi$, $v_2=r\sin\chi\sin\varphi$, $v_3=r\cos\chi$. Note that the domain of integration lies in the first octant. By using these spherical coordinates we obtain

$$\int_{\mathscr{E}_\delta}\frac{d^3\theta}{E_0(\theta)-E}=\int_0^{\sqrt{\delta}}\frac{g(r)r^2}{r^2-E}\,dr,\tag{5.12}$$

where

$$g(r)=\int_0^{\pi/2}\left(\int_0^{\pi/2}h_1'(r\sin\chi\cos\varphi)h_2'(r\sin\chi\sin\varphi)d\varphi\right)h_3'(r\cos\chi)\sin\chi\,d\chi.$$

For $r<r_1$ with $r_1>0$ sufficiently small, we can expand $g(r)$ as

$$g(r)=\sum_{m=0}^{\infty}b_{2m}r^{2m},$$

and insert it in (5.12). Note that [cf. (4.33)]

$$b_0=\frac{\pi}{2\sqrt{c_0^{(1)}c_0^{(2)}c_0^{(3)}}}.\tag{5.13}$$

This gives

$$\int_{\mathscr{E}_\delta}\frac{d^3\theta}{E_0(\theta)-E}=\sum_{m=0}^{\infty}b_{2m}J_m(E;\delta),\tag{5.14}$$

where

$$J_m(E;\delta)=\int_0^{\sqrt{\delta}}\frac{r^{2m+2}}{r^2-E}\,dr=(-1)^m(-E)^{m+1/2}\left[-\frac{\pi}{2}+\arctan\sqrt{-\frac{E}{\delta}}\right]+\sum_{k=1}^{m+1}\frac{\delta^{m-k+3/2}}{2m-2k+3}E^{k-1}.\tag{5.15}$$

Note that in contrast to (4.32) the right-hand side of (5.15) does not contain any logarithmic terms. From (5.14), (5.15), and (5.9), we see that the term $(-E)^{1/2}B$ in (5.5) comes from the first term on the right-hand side of (5.15) for $m=0$ and by using (5.13) and $G(x,y;0)=8\Psi_0(x,0)\Psi_0(y,0)$ which follows from (5.4) and the reality of $\Psi_0(x,0)$. The other terms on the right-hand side of (5.15) only affect either the term $X_0^{[0,\delta]}(x,y)$ or the corrections of order $O(|E|)$. Thus

$$X_E^{[0,\delta]}(x,y)=X_0^{[0,\delta]}(x,y)+(-E)^{1/2}B(x,y)+O(|E|).\tag{5.16}$$

Note that $X_E^{[0,\delta]}(x,y)$ is a Hilbert–Schmidt kernel because $W\in L^1(\mathbf{R}^3)$. Now we turn to the term $Y_E^{[0,\delta]}(x,y)$ in (5.10) which we write as $Y_E^{[0,\delta]}(x,y)=Y_0^{[0,\delta]}(x,y)+[Y_E^{[0,\delta]}(x,y)-Y_0^{[0,\delta]}(x,y)]$, and note that

$$Y_E^{[0,\delta]}(x,y)-Y_0^{[0,\delta]}(x,y)=EW(x)^{1/2}\left[\int_{\mathscr{E}_\delta}\frac{H(x,y;\theta)}{[E_0(\theta)-E]E_0(\theta)}\frac{d^3\theta}{(2\pi)^3}\right]|W(y)|^{1/2}.\tag{5.17}$$

As in (4.42) we have the estimate $|H(x,y;\theta)|\leqslant C(1+|x|)^\alpha(1+|y|)^\alpha|\theta|^\alpha(\alpha\in(0,1])$ which we insert in (5.17). From the θ-integration we get

$$\int_0^{\sqrt{\delta}} \frac{r^\alpha}{r^2-E}\, dr = |E|^{(\alpha-1)/2}\int_0^{\sqrt{\delta/|E|}} \frac{u^\alpha}{u^2+1}\, du.$$

If we pick any $\alpha\in(0,1)$, then the integral on the right-hand side stays bounded as $E\uparrow 0$ and thus the right-hand side is $O(|E|^{(\alpha-1)/2})$. Hence, by (5.17),

$$|Y_E^{[0,\delta]}(x,y)-Y_0^{[0,\delta]}(x,y)|\leqslant C|E|^{(\alpha+1)/2}|W(x)|^{1/2}(1+|x|)^\alpha(1+|y|)^\alpha|W(y)|^{1/2}. \tag{5.18}$$

So, by (5.3), if we choose α such that $0<\alpha<\min\{1,\gamma/2\}$, then $(1+|x|)^\alpha|W(x)|^{1/2}\in L^2(\mathbf{R}^3)$, which says that the right-hand side of (5.18) is Hilbert–Schmidt and of order $o(|E|^{1/2})$. Thus

$$Y_E^{[0,\delta]}(x,y)=Y_0^{[0,\delta]}(x,y)+o(|E|^{1/2}). \tag{5.19}$$

Since $K_E^{(\delta,\infty)}=K_0^{(\delta,\infty)}+O(|E|)$, by adding (5.16) and (5.19) and using $K_0=X_0^{[0,\delta]}(x,y)+Y_0^{[0,\delta]}(x,y)+K_0^{(\delta,\infty)}$, we obtain (5.5). ∎

Recall that λ_0 is a c.c.th. at $E=0$ if and only λ_0K_0 has eigenvalue -1, resp., K_0 has eigenvalue $-\lambda_0^{-1}$. Hence $\lambda_0=0$ cannot be a c.c.th. in three dimensions for the class of potentials considered here. The following lemma will help us classify the possible behaviors of the eigenvalues $E(\lambda)$ that converge to zero as $\lambda\downarrow 0$.

Lemma 5.2: Assume (5.1), (5.2), and (5.3) with $\gamma>1$. Suppose that $K_0f_0=-\lambda_0^{-1}f_0$, $f_0\in L^2(\mathbf{R}^3)$, $f_0\neq 0$, and $Bf_0=0$. Then zero is an eigenvalue of H_{λ_0}. If $-\lambda_0^{-1}$ is an eigenvalue of K_0 of geometric multiplicity k with geometric eigenspace $\mathscr{N}_0$ and $Bf=0$ for all $f\in\mathscr{N}_0$, then zero is an eigenvalue of H_{λ_0} of multiplicity k.

Proof: We will show that

$$\chi_0=H_0^{-1}|W|^{1/2}f_0 := s-\lim_{E\uparrow 0}(H_0-E)^{-1}|W|^{1/2}f_0 \tag{5.20}$$

is an eigenfunction of H_{λ_0} for the eigenvalue zero. To do this we start from the decomposition

$$(H_0-E)^{-1}|W|^{1/2}f_0=(H_0-E)^{-1}P^{[0,\delta]}|W|^{1/2}f_0+(H_0-E)^{-1}P^{(\delta,\infty)}|W|^{1/2}f_0, \tag{5.21}$$

and first establish the strong convergence as $E\uparrow 0$ of each term on the right-hand side. For the second term on the right-hand side of (5.21) this is clear from

$$(H_0-E)^{-1}P^{(\delta,\infty)}|W|^{1/2}f_0=[(H_0-E)^{-1}(H_0+1)^{1/2}P^{(\delta,\infty)}][(H_0+1)^{-1/2}|W|^{1/2}f_0],$$

since $(H_0+1)^{-1/2}|W|^{1/2}$ is a bounded operator and $\lim_{E\uparrow 0}(H_0-E)^{-1}(H_0+1)^{1/2}P^{(\delta,\infty)}$ exists in norm by the spectral theorem. As for the first term on the right-hand side of (5.21), our task is to show that

$$s-\lim_{E\uparrow 0}\int_{\mathbf{R}^3}\left[\int_{\mathscr{E}_\delta}\frac{G(x,y;\theta)}{E_0(\theta)-E}\frac{d^3\theta}{(2\pi)^3}\right]|W(y)|^{1/2}f_0(y)d^3y\in L^2(\mathbf{R}^3). \tag{5.22}$$

To do this we split the terms that make up $G(x,y;\theta)$ as follows; we indicate it for the first term on the right-hand side of (5.7):

$$\Psi_0(x,\theta)\overline{\Psi_0(y,\theta)}=\Psi_0(x,\theta)\Psi_0(y,0)+\Psi_0(x,\theta)[\overline{\Psi_0(y,\theta)}-\Psi_0(y,0)]. \tag{5.23}$$

Notice that when (5.23) is substituted in (5.22), the first term in (5.23) gives no contribution because of the assumption $Bf_0=0$ and (5.6). The integral arising from the second term is dealt with as follows. We set $\hat{x}=(x_1/p_1,x_2/p_2,x_3/p_3)$ so that

$$\Psi_0(x,\theta)=e^{i\theta\cdot\hat{x}}\frac{\xi_0^{(1,+)}(x_1,E_0^{(1)}(\theta_1))}{N_0^{(1,+)}(\theta_1)}\frac{\xi_0^{(2,+)}(x_2,E_0^{(2)}(\theta_2))}{N_0^{(2,+)}(\theta_2)}\frac{\xi_0^{(3,+)}(x_3,E_0^{(3)}(\theta_3))}{N_0^{(3,+)}(\theta_3)}, \tag{5.24}$$

from which, by expanding the functions $\xi_0^{(+)}(x_j,E_0^{(j)}(\theta_j))$ in powers of θ_j, we obtain

$$\Psi_0(x,\theta)=\sum_{m,n,s=0}^{\infty} a_{mns}(x)\,\theta_1^m\theta_2^n\theta_3^s, \tag{5.25}$$

where the coefficients $a_{mns}(x)$ obey

$$|a_{mns}(x)|\leq M\tilde{\theta}^{-m-n-s}, \tag{5.26}$$

for some $M>0$ and $\tilde{\theta}>0$, and where $|\theta_j|<\tilde{\theta}$. As a result, from (5.22)–(5.25) we obtain

$$\int_{\mathbf{R}^3}\left[\int_{\mathscr{E}_\delta}\frac{G(x,y;\theta)}{E_0(\theta)-E}\,\frac{d^3\theta}{(2\pi)^3}\right]|W(y)|^{1/2}f_0(y)d^3y=\sum_{m,n,s=0}^{\infty} a_{mns}(x)\int_{\mathscr{E}_\delta}e^{i\theta\cdot\hat{x}}g_{mns}(\theta,E)d^3\theta+\cdots, \tag{5.27}$$

where

$$g_{mns}(\theta,E)=\frac{\theta_1^m\theta_2^n\theta_3^s}{(2\pi)^3[E_0(\theta)-E]}\int_{\mathbf{R}^3}\Delta(y;\theta)|W(y)|^{1/2}f_0(y)d^3y,$$

$$\Delta(y;\theta)=\overline{\Psi_0(y;\theta)}-\Psi_0(y;0),$$

and where the dots in (5.27) indicate further terms similar to the one displayed [cf. (5.4)]. By Lemma 4.4,

$$|\Delta(y;\theta)|\leq C(1+|y|)^{\alpha}|\theta|^{\alpha},\quad \alpha\in(0,1],$$

and thus using the Schwarz inequality gives

$$\left|\int_{\mathbf{R}^3}\Delta(y;\theta)|W(y)|^{1/2}f_0(y)d^3y\right|\leq C|\theta|^{\alpha}\|W\|_{2\alpha}^{1/2}\ \|f_0\|,$$

where $\|W\|_{2\alpha}=\int_{\mathbf{R}^3}(1+|y|)^{2\alpha}|W(y)|d^3y$. Now, since (5.3) holds with $\gamma>1$, we may choose α such that $1/2<\alpha<\min\{1,\gamma/2\}$. This guarantees that

$$\frac{|\theta|^{m+n+s+\alpha}}{E_0(\theta)-E}\in L^2(\mathscr{E}_\delta),$$

for all $E\leq 0$ with the L^2-norm being bounded independently of E. Therefore

$$\|g_{mns}(\theta,E)\|_{L^2(\mathscr{E}_\delta)}\leq C\delta^{(m+n+s+\alpha)/2-1/4}\|W\|_{2\alpha}^{1/2}\|f_0\|,$$

for all $E\leq 0$. By the Plancherel theorem and (5.26), we conclude that

$$\left\|a_{mns}(x)\int_{\mathscr{E}_\delta}e^{i\theta\cdot\hat{x}}g_{mns}(\theta,E)d^3\theta\right\|_{L^2(\mathbf{R}^s)}\leq\tilde{C}\delta^{\alpha/2-1/4}\left(\frac{\sqrt{\delta}}{\tilde{\theta}}\right)^{m+n+s}\|W\|_{2\alpha}^{1/2}\|f_0\|.$$

Thus, if we choose $\sqrt{\delta}<\tilde{\theta}$, then the series on the right-hand side of (5.27) converges absolutely in the L^2-norm. This implies that $\|(H_0-E)^{-1}P^{[0,\delta]}|W|^{1/2}f_0\|$ is bounded independently of E for $E<0$. Moreover, this norm is increasing in E and hence converges as $E\uparrow 0$. In addition, $(H_0-E)^{-1}P^{[0,\delta]}|W|^{1/2}f_0$ converges weakly as $E\uparrow 0$ because $(\varphi,(H_0-E)^{-1}P^{[0,\delta]}|W|^{1/2}f_0)$ converges for any φ from the dense set $\{\varphi\in L^2:\ \varphi\in \mathrm{Ran}\ P^{[\epsilon,\infty)}$ for some $\epsilon>0\}$. These facts imply (5.22), and hence we have shown that $\chi_0=H_0^{-1}|W|^{1/2}f_0\in L^2(\mathbf{R}^3)$. It remains to show that $H_{\lambda_0}\chi_0=0$. We argue as follows. Let $\psi\in\mathscr{D}(H_{\lambda_0})$. Thus $\psi\in\mathscr{D}(H_0^{1/2})$ and so

$$
\begin{aligned}
(\chi_0,H_{\lambda_0}\psi)&=(H_0^{-1}|W|^{1/2}f_0,H_{\lambda_0}\psi)\\
&=\lim_{E\uparrow 0}([H_0-E]^{-1}|W|^{1/2}f_0,H_{\lambda_0}\psi)\\
&=\lim_{E\uparrow 0}([H_0+1]^{-1/2}H_{\lambda_0}[H_0-E]^{-1}|W|^{1/2}f_0,[H_0+1]^{1/2}\psi)\\
&=\lim_{E\uparrow 0}([H_0+1]^{-1/2}|W|^{1/2}(1+\lambda_0K_E)f_0,[H_0+1]^{1/2}\psi)=0,
\end{aligned}
$$

because $K_E\to K_0$ in norm and $-\lambda_0^{-1}$ is an eigenvalue of K_0 with eigenvector f_0. Thus $\chi_0\in\mathscr{D}(H_{\lambda_0}^*)=\mathscr{D}(H_{\lambda_0})$ and $H_{\lambda_0}\chi_0=0$. To prove the last assertion, suppose that $f_1,\dots,f_k$ are linearly independent eigenvectors of K_0 for the eigenvalue $-\lambda_0^{-1}$. Multiplying the equation $c_1H_0^{-1}|W|^{1/2}f_1+\cdots+c_kH_0^{-1}|W|^{1/2}f_k=0$ from the left by $W^{1/2}$ and using that $\lambda_0W^{1/2}H_0^{-1}|W|^{1/2}f_j=-f_j$ $(j=1,\dots,k)$ implies $c_1=\cdots=c_k=0$. ■

From now on we assume

$$W(x)\leqslant 0 \quad \text{a.e.,} \tag{5.28}$$

which implies that $K_E=-|W|^{1/2}(H_0-E)^{-1}|W|^{1/2}$ is self-adjoint. Assumption (5.28) is made here only for convenience, in order to avoid discussion of certain technical points that arise when we apply perturbation theory to K_E. The case where W changes sign has been discussed in Ref. 3 (Section 8). The next theorem is similar to Theorem 2.3 in Ref. 3 and to Theorem 2.1 in Ref. 9.

Theorem 5.3: Suppose V and W satisfy (5.1), (5.2), (5.3) with $\gamma>1$, and (5.28). Let λ_0 be a c.c.th. at $E=0$. Suppose that m $(m\geqslant 1)$ eigenvalues approach zero as $\lambda\downarrow\lambda_0$. Then we have the alternatives:

(i) 0 is not an eigenvalue of H_{λ_0}, $m=1$, and there is a unique eigenvalue obeying $E(\lambda)=-c(\lambda-\lambda_0)^2+o([\lambda-\lambda_0]^2)$ with $c\neq 0$.

(ii) 0 is an eigenvalue of H_{λ_0} in which case at most one eigenvalue is in case (i) while the other eigenvalues obey $E(\lambda)=-c(\lambda-\lambda_0)+o(\lambda-\lambda_0)$ with $c\neq 0$. Moreover, 0 is either an eigenvalue of H_{λ_0} of multiplicity $m-1$ or m, depending on whether or not there is an eigenvalue in case (i).

Proof: By the Birman–Schwinger principle, $-\lambda_0^{-1}$ is an eigenvalue of K_0 of multiplicity m. Let $\mathscr{N}_0$ denote the corresponding eigenspace. For $E<0$ there are m eigenvalues $\tau_j(E)$ $(j=1,\dots,m)$ of K_E converging to $-\lambda_0^{-1}$ and the $\tau_j(E)$ are strictly decreasing functions of E. We can obtain $\tau_j(E)$ from (5.5) by using eigenvalue perturbation theory and then solve $\lambda\tau_j(E)=-1$ for $E(\lambda)$ to get the eigenvalue branch $E_j(\lambda)$.

Considering (i), since 0 is not an eigenvalue of H_{λ_0}, by Lemma 5.2, we have that $Bf\neq 0$ for all nonzero $f\in\mathscr{N}_0$. However, since B has rank one, this can only happen if $m=1$. Let $f_0\in\mathscr{N}_0$ with $\|f_0\|=1$. Then, by (5.6) and (5.28), $(f_0,Bf_0)>0$. From perturbation theory[17] and (5.5), we infer that

$$\tau(E)=-\lambda_0^{-1}+(f_0,Bf_0)(-E)^{1/2}+o(|E|^{1/2}),$$

which leads to $E(\lambda)=-c(\lambda-\lambda_0)^2+o([\lambda-\lambda_0]^2)$ with $c=\lambda_0^{-4}(f_0,Bf_0)^{-2}$; so (i) is proved.

To prove (ii), notice that B either vanishes on all of $\mathscr{N}_0$ or on a subspace of dimension $m-1$ of $\mathscr{N}_0$. Thus, by Lemma 5.2, 0 is an eigenvalue of H_{λ_0} of multiplicity m or $m-1$, respectively. Now the number of eigenvalues that behave like $-c(\lambda-\lambda_0)+o(\lambda-\lambda_0)$ with $c>0$ as $\lambda\downarrow\lambda_0$ is equal to the multiplicity of 0 as eigenvalue of H_{λ_0}. To see this, let P_0 denote the orthogonal projection onto the kernel of H_{λ_0}. Then

$$P_0WP_0=-\lambda_0^{-1}P_0H_0P_0\leqslant 0,$$

so that, since H_0 has a trivial kernel, P_0WP_0 also has a trivial kernel on Ran P_0. Thus P_0WP_0, which is self-adjoint and negative, has $p=\dim[\text{Ran }P_0]$ strictly negative eigenvalues, and

$p=m-1$ or m. Then Theorem 2.4(ii) in Ref. 10 implies that the number of eigenvalues that approach 0 as $\lambda\downarrow\lambda_0$ and which, for λ sufficiently close to λ_0, obey an estimate of the form

$$-a(\lambda-\lambda_0)<E(\lambda)<-b(\lambda-\lambda_0), \tag{5.29}$$

with $a>b>0$, is equal to p. However, from the proof of that theorem one also infers that there is a $c>0$ such that for any $\epsilon>0$ we can choose a and b such that $c-\epsilon<b<a<c+\epsilon$ and so that (5.29) is satisfied as $\lambda\downarrow\lambda_0$ ($-c$ is just one of the p nonzero eigenvalues of P_0WP_0). Since $\epsilon>0$ is arbitrary, it follows that we can replace (5.29) by the stronger statement

$$E(\lambda)=-c(\lambda-\lambda_0)+o(\lambda-\lambda_0), \quad \lambda\downarrow\lambda_0. \tag{5.30}$$

If $p=m$, then all m eigenvalues obey (5.30), with the coefficient c depending on the eigenvalue, of course. If $p=m-1$, then there is exactly one eigenvalue that does not obey (5.29). Then B does not vanish on all of $\mathscr{N}_0$ and so, by perturbation theory, there is one eigenvalue that is in case (i), and this must be the eigenvalue that does not obey (5.29). ■

APPENDIX: PROOFS OF LEMMAS 2.3–2.7

1. Proof of Lemma 2.3

We first prove the entire lemma when $\lambda=0$. Obviously, (i) and (iv) immediately follow from (2.7). To prove (ii) and (iii) we note that from (2.3), (2.4), (2.17), and (2.30), we have

$$\phi_0(x,E)=-d_0(E)[\psi_0^{(+)}(x,E)-\psi_0^{(-)}(x,E)]. \tag{A1}$$

In order to simplify the notation we define

$$g(x,\mu)=\xi_0^{(+)}(x,E(\tilde{n}\pi/p+i\mu)), \tag{A2}$$

where we now assume that an endpoint E_n of the gap $\mathscr{G}_{\tilde{n}}$, together with the proper branch of $E(\tilde{n}\pi/p+i\mu)$, have been selected. We temporarily use the shorthand notation $\tilde{\psi}_0^{(\pm)}(x,\mu)=\psi_0^{(\pm)}(x,E(\tilde{n}\pi/p+i\mu))$ and note that, by (2.3), (2.4), and (2.17),

$$\tilde{\psi}_0^{(+)}(x,\mu)=\tilde{\psi}_0^{(-)}(x,-\mu).$$

This implies, by (2.7),

$$\xi_0^{(-)}(x,E(\tilde{n}\pi/p+i\mu))=e^{2i\tilde{n}\pi x/p}g(x,-\mu), \tag{A3}$$

and thus

$$\tilde{\psi}_0^{(\pm)}(x,\mu)=e^{i\tilde{n}\pi x/p}e^{\mp\mu z}g(x,\pm\mu). \tag{A4}$$

Thus, from (A2)–(A4) it follows that [again using $E=E(\tilde{n}\pi/p+i\mu)$]

$$\begin{aligned}\psi_0^{(+)}(x,E)-\psi_0^{(-)}(x,E)&=e^{i\tilde{n}\pi x/p}[e^{-\mu x}g(x,\mu)-e^{\mu x}g(x,-\mu)]-e^{i\tilde{n}\pi x/p}\sinh\mu x[g(x,\mu)\\&\quad+g(x,-\mu)]+e^{i\tilde{n}\pi x/p}\cosh\mu x[g(x,\mu)-g(x,-\mu)].\end{aligned} \tag{A5}$$

Next we note that, by (2.30),

$$d_0(E)=\frac{\alpha}{\mu}+\beta\mu+O(\mu^3), \tag{A6}$$

where α and β are constants; in particular,

$$\alpha=\frac{(-1)^{\tilde{n}}\phi_0(p,E_n)}{2p}.$$

Moreover, $d_0(E)$ is an odd function of μ. Furthermore (a dot means $d/d\mu$),

$$g(x,\mu)+g(x,-\mu)=2\xi_0^{(+)}(x,E_n)+O(\mu^2), \tag{A7}$$

$$g(x,\mu)-g(x,-\mu)=2\dot{\xi}_0^{(+)}(x,E_n)\mu+O(\mu^3). \tag{A8}$$

Therefore, on inserting (A5)–(A8) in (A1) we get

$$\phi_0(x,E)=2\alpha\xi_0^{(+)}(x,E_n)e^{i\tilde{n}\pi x/p}\mu^{-1}\sinh\mu x-2\alpha\dot{\xi}_0^{(+)}(x,E_n)e^{i\tilde{n}\pi x/p}\cosh\mu x+O(\mu)\sinh\mu x$$
$$+O(\mu^2)\cosh\mu x. \tag{A9}$$

Now (ii) follows from (A9) and the estimates

$$|\sinh\mu x|\leqslant C\,\frac{\mu|x|}{1+\mu|x|}\,e^{\mu|x|},\quad \cosh\mu x\leqslant Ce^{\mu|x|}. \tag{A10}$$

Similarly, on differentiating (A9) with respect to μ and using (A10), along with the estimate

$$|\sinh\mu x-\mu x\cosh\mu x|\leqslant C\,\frac{\mu^3|x|^3}{(1+\mu|x|)^2}\,e^{\mu|x|}, \tag{A11}$$

and assuming $\mu<1$, we obtain (iii). Thus the lemma has been proved when $\lambda=0$. Now suppose that $\lambda>0$. In view of (2.19) and (2.21) it suffices to prove (i) and (iv) for $F_\lambda^{(\pm)}(x,E)$ instead of $\psi_\lambda^{(\pm)}(x,E)$. By using (2.11), (2.17), (A2), and (A4), we obtain

$$A_0(x,y;E)=2\alpha e^{(i\tilde{n}\pi/p)(x+y)}\xi_0^{(+)}(x,E_n)\xi_0^{(+)}(y,E_n)\mu^{-1}\sinh\mu(x-y)-2\alpha e^{(i\tilde{n}\pi/p)(x+y)}[\dot{\xi}_0^{(+)}$$
$$\times(x,E_n)\xi_0^{(+)}(y,E_n)-\xi_0^{(+)}(x,E_n)\dot{\xi}_0^{(+)}(y,E_n)]\cosh\mu(x-y)+O(\mu)\sinh\mu(x-y)$$
$$+O(\mu^2)\cosh\mu(x-y).$$

Thus, by using (A10), (A11), and obvious variants of them, we obtain the bounds

$$|A_0(x,y;E)|\leqslant C(1+|x-y|)e^{\mu|x-y|}, \tag{A12}$$

$$|\dot{A}_0(x,y;E)|\leqslant C\,\frac{\mu(1+|y-x|)^3}{(1+\mu|y-x|)^2}\,e^{\mu|y-x|}, \tag{A13}$$

To prove (i), use (A12) and (i) with $\lambda=0$ in (2.10). This yields, for $x>0$,

$$|F_\lambda^{(+)}(x,E)|\leqslant Ce^{-\mu x}+C\lambda e^{-\mu x}\int_x^\infty(1+y)|W(y)|e^{\mu y}|F_\lambda^{(+)}(y,E)|dy. \tag{A14}$$

Multiplying (A14) by $e^{\mu x}$ and applying Gronwall's inequality proves (i) for $F_\lambda^{(+)}(x,E)$. The proof for $F_\lambda^{(-)}(x,E)$ is similar. Considering (iv) we start from

$$\dot{F}_\lambda^{(+)}(x,E)=\dot{\psi}_0(x,E)-\lambda\int_x^\infty\dot{A}_0(x,y;E)W(y)F_\lambda^{(+)}(y,E)dy-\lambda\int_x^\infty A_0(x,y;E)W(y)\dot{F}_\lambda^{(+)}(y,E)dy.$$

Thus, by (A12), (A13), and (i) for $\lambda=0$, we conclude that

$$|\dot{F}_\lambda^{(+)}(x,E)|\leqslant C(1+x)e^{-\mu x}+C\lambda e^{-\mu x}\int_x^\infty\frac{\mu(1+y)}{(1+\mu y)^2}\,(1+y)^2|W(y)|dy+C\lambda e^{-\mu x}$$
$$\times\int_x^\infty|W(y)|(1+y)^2\,\frac{e^{\mu y}|\dot{F}_\lambda^{(+)}(y,E)|}{1+y}\,dy. \tag{A15}$$

Dividing both sides by $(1+x)e^{-\mu x}$ and applying Gronwall's inequality yields (iv). Note that condition (H2) enters through the second and third term on the right-hand side of (A15). To prove (ii) we use the fact that $\phi_\lambda(x,E)$ is a solution of the integral equation

$$\phi_\lambda(x,E)=\phi_0(x,E)+\lambda\int_0^x A_0(x,y;E)W(y)\phi_\lambda(y,E)dy. \tag{A16}$$

By using (ii) with $\lambda=0$ and (A12) we obtain (assuming $x>0$)

$$|\phi_\lambda(x,E)|=C(1+x)e^{\mu x}+C(1+x)e^{\mu x}\int_0^x e^{-\mu y}W(y)|\phi_\lambda(y,E)|dy. \tag{A17}$$

Dividing (A17) by $(1+x)e^{\mu x}$ and using Gronwall's inequality yields (ii) for $x>0$. When $x<0$, the argument is similar, so (ii) is proved. The proof of (iii) is similar to the proof of (iv). We differentiate (A16) with respect to μ and use (ii) and (iii) for $\lambda=0$, together with (A12), (A13), and the monotonicity of the function $y\to\mu(1+y)^2/[1+\mu y]^2$ (assuming $\mu<1$). This gives (for $x>0$)

$$\frac{e^{-\mu x}(1+\mu x)^2|\dot{\phi}_\lambda(x,E)|}{\mu(1+x)^3}\leqslant C+C\int_0^x(1+y)|W(y)|\ \frac{e^{-\mu y}(1+\mu y)^2|\dot{\phi}_\lambda(x,E)|}{\mu(1+y)^3}\ dy.$$

An application of Gronwall's inequality yields the desired estimate when $x>0$, and an analogous estimate holds when $x<0$. Note that only condition (H1) is needed here. ■

2. Proof of Lemma 2.4

(i) The bound immediately follows from Lemma 2.3(i), (ii), and (2.28). To prove (ii) use (2.26), (2.28), part (i), and the dominated convergence theorem, to conclude that

$$\|R_{\lambda,E}-R_{\lambda,E_n}\|_{H.S.}^2=\int_{\mathbf{R}^2}|W(x)||G_{\lambda,E}^D(x,y)-G_{\lambda,E_n}^D(x,y)|^2|W(y)|dxdy\to 0,$$

as $E\to E_n$. Finally, (iii) follows by differentiating (2.28) and applying Lemma 2.3. ■

3. Proof of Lemma 2.5

Let $C_1>0$ be a constant such that

$$|\omega_\lambda(x,E)|\leqslant C_1|W(x)|^{1/2},\quad |\dot{\psi}_\lambda(x,E)|\leqslant C_1|W(x)|^{1/2}(1+|x|). \tag{A18}$$

These estimates follow from Lemma 2.3(i), (iv), and (2.27). Let

$$\sigma=\int_{-\infty}^{\infty}|W(x)|(1+|x|)dx.$$

Then using (2.27), Lemma 2.4(i), and

$$\int_{-\infty}^{\infty}(1+\min\{|x|,|z|\})|W(z)|(1+\min\{|z|,|y|\})dz\leqslant\sigma(1+\min\{|x|,|y|\}),$$

we deduce by induction that

$$|(R_{\lambda,E}^s)(x,y)|\leqslant C^s\sigma^{s-1}|W(x)|^{1/2}[1+\min\{|x|,|y|\}]|W(y)|^{1/2},\quad s=1,2,...,$$

and thus, by using (A18),

$$|(R_{\lambda,E}^s\omega_{\lambda,E})(x)|\leqslant C_1C^s\sigma^s|W(x)|^{1/2},\quad s=0,1,2,...\ . \tag{A19}$$

Now (i) follows from (A18) and (A19) which give

$$|(\tilde{\omega}_{\lambda,E},R^{s}_{\lambda,E}\omega_{\lambda,E})|\leq C_1^2C^s\sigma^{s+1}. \tag{A20}$$

So choosing $\tilde{C}=C_1^2\sigma$, $r=C\sigma$ proves estimate (i). To prove (ii) we formally differentiate $(\tilde{\omega}_{\lambda,E},R^{s}_{\lambda,E}\omega_{\lambda,E})$:

$$\frac{d(\tilde{\omega}_{\lambda,E},R^{s}_{\lambda,E}\omega_{\lambda,E})}{d\mu}=(\tilde{\omega}_{\lambda,E},R^{s}_{\lambda,E}\dot{\omega}_{\lambda,E})+(\dot{\tilde{\omega}}_{\lambda,E},R^{s}_{\lambda,E}\omega_{\lambda,E})+\sum_{j=0}^{s-1}(\tilde{\omega}_{\lambda,E},R^{j}_{\lambda,E}\dot{R}_{\lambda,E}R^{s-j-1}_{\lambda,E}\omega_{\lambda,E}). \tag{A21}$$

Then we note that

$$|(R^{*s}_{\lambda,E}\tilde{\omega}_{\lambda,E})(x)|\leq C_1C^s\sigma^s|W(x)|^{1/2}, \tag{A22}$$

where the * denotes the adjoint, and so

$$|\dot{\tilde{\omega}}(x,E)(R^{s}_{\lambda,E}\omega_{\lambda,E})(x)|\leq C_1^2C^s\sigma^s|W(x)|(1+|x|), \tag{A23}$$

$$|\dot{\omega}_{\lambda}(x,E)(R^{*s}_{\lambda,E}\tilde{\omega}_{\lambda,E})(x)|\leq C_1^2C^s\sigma^s|W(x)|(1+|x|), \tag{A24}$$

$$|(R^{*j}_{\lambda,E}\tilde{\omega}_{\lambda,E})(x)\dot{R}_{\lambda,E}(x,y)(R^{s-j-1}_{\lambda,E}\omega_{\lambda,E})(y)|\leq C_1^2C^s\sigma^{s-1}|W(x)|(1+|x|)(1+|y|)|W(y)|. \tag{A25}$$

For the last inequality we used (A20), (A22), and (2.34). Since the various scalar products in (A25) are multiple integrals involving the functions $R_{\lambda,E}(x,y)$, $\omega_{\lambda}(x,E)$, $\tilde{\omega}_{\lambda}(x,E)$, and their derivatives with respect to μ, the estimates (A22)–(A25) provide us with the necessary L^1-bounds on the integrands to justify the formal differentiation leading to (A21) (see, e.g., Ref. 36, Theorem 16.8). From (A23)–(A25) we obtain

$$|(\tilde{\omega}_{\lambda,E},R^{s}_{\lambda,E}\dot{\omega}_{\lambda,E})|\leq C_1^2C^s\sigma^{s+1},\quad |(\dot{\tilde{\omega}}_{\lambda,E},R^{s}_{\lambda,E}\omega_{\lambda,E})|\leq C_1^2C^s\sigma^{s+1}, \tag{A26}$$

$$|(\tilde{\omega}_{\lambda,E},R^{j}_{\lambda,E}\dot{R}_{\lambda,E}R^{s-j-1}_{\lambda,E}\omega_{\lambda,E}|\leq C_1^2C^s\sigma^{s+1}. \tag{A27}$$

Inserting (A26) and (A27) in (A21) gives

$$\left|\frac{d(\tilde{\omega}_{\lambda,E},R^{s}_{\lambda,E}\omega_{\lambda,E})}{d\mu}\right|\leq(2+s)C_1^2C^s\sigma^{s+1},$$

and so assertion (ii) follows on setting $r=C\sigma$, $\tilde{C}=(2+s)C_1^2\sigma$. ■

4. Proof of Lemma 2.6

In proving (i)–(iii) it suffices to consider the case $a_N(0)>0$ and, in (iii), to find the solution $z_+(\mu)$; the modifications needed in the other cases are obvious. We always assume that $0\leq\mu\leq\mu_0$ and $0\leq z<r^{-1}$ so that the series on the left-hand side of (2.35) converges absolutely, and uniformly in μ. We begin by writing (2.35) in the form

$$a_N(\mu)z^N\left(1+\sum_{j=1}^{\infty}\frac{a_{N+j}(\mu)}{a_N(\mu)}z^j\right)=\dot{c}(0)\mu\left(1-\sum_{j=1}^{N-1}\frac{a_j(\mu)}{\mu\dot{c}(0)}z^j+o(1)\right), \tag{A28}$$

where the $o(1)$-term is a function of μ and comes from $c(\mu)=\dot{c}(0)\mu[1+o(1)]$. Note that the infinite sum on the left of (A28) is $O(z)$ uniformly in μ and that the finite sum on the right is also $O(z)$ uniformly in μ because $\mu^{-1}a_j(\mu)$ is bounded by assumption (b). Therefore, for any $\delta\in(0,1)$ there is $z_1>0$ and $\mu_1>0$ $(z_1<r^{-1},\mu_1\leq\mu_0)$ such that

$$\begin{aligned}a_N(\mu)z^N(1-\delta)&\leq\dot{c}(0)\mu(1+\delta),\\ a_N(\mu)z^N(1+\delta)&\geq\dot{c}(0)\mu(1-\delta),\end{aligned} \tag{A29}$$

for $0 \le z \le z_1$ and $0 \le \mu \le \mu_1$. Define

$$h(\mu,z) = \sum_{j=1}^{\infty} a_j(\mu) z^j - c(\mu), \tag{A30}$$

so that (2.35) becomes

$$h(\mu,z) = 0. \tag{A31}$$

Now choose any two numbers b_1 and b_2 such that $0 < b_1 < [\dot{c}(0)/a_N(0)]^{1/N} < b_2$. Since δ in (A29) is arbitrary, we conclude that for μ_1 and z_1 sufficiently small, every pair $\{\mu,z\}$ satisfying (A31), $0 \le \mu \le \mu_1$, and $0 < z \le z_1$, also satisfies $b_1 \mu^{1/N} < z < b_2 \mu^{1/N}$. Moreover, a simple calculation using (b) and (c) gives

$$\lim_{\mu \to 0} \mu^{-1} h(\mu, b_1 \mu^{1/N}) = a_N(0) b_1^N - \dot{c}(0) < 0,$$

$$\lim_{\mu \to 0} \mu^{-1} h(\mu, b_2 \mu^{1/N}) = a_N(0) b_2^N - \dot{c}(0) > 0.$$

Therefore, if μ is sufficiently small, the function $z \mapsto h(\mu,z)$ has a zero $z(\mu)$ in the interval $[b_1 \mu^{1/N}, b_2 \mu^{1/N}]$. This proves the existence of solutions to (2.35). To prove uniqueness we consider the partial derivative $h_z(\mu,z)$ for fixed μ and $z \in [b_1 \mu^{1/N}, b_2 \mu^{1/N}]$. We have

$$h_z(\mu,z) = \sum_{j=1}^{N-1} j a_j(\mu) z^{j-1} + N a_N(\mu) z^{N-1} + \sum_{j=N+1}^{\infty} j a_j(\mu) z^{j-1}.$$

The first sum is $O(\mu)$ by assumption (b), the term in the middle is bounded from below by $N[a_N(0)/2] b_1^{N-1} \mu^{1-1/N}$ for μ small, and the last sum is $O(\mu)$. Hence the middle term dominates as $\mu \to 0$, and so $h_z(\mu,z) > 0$ on $[b_1 \mu^{1/N}, b_2 \mu^{1/N}]$. Thus the solution $z(\mu)$ is unique. The continuity of $z(\mu)$ follows from the continuity of $h(\mu,z)$ in both variables.

Assertions (iv) and (v) follow directly from (2.35) which becomes inconsistent. First, if N is even and $a_N(0) < 0$, then the two sides of (2.35) have different signs for μ and z small enough. The sign of z does not matter. Secondly, if $a_j(0) = 0$ for all j with the specified bound on $|a_j(\mu)|$, the left-hand side of (2.35) is bounded by $C_1 r_1 (1 - r_1|z|)^{-1} |z| \mu$ provided $|z| < \min\{r^{-1}, r_1^{-1}\}$. This bound is incompatible with the right-hand side and the assumption that $\dot{c}(0) > 0$. Thus Lemma 2.6 is proved. ■

5. Proof of Lemma 2.7

(i) The existence of a solution $\mu(z)$ follows from (A28) arguing similarly as in the proof of Lemma 2.6. Considering only the case where $a_N(0) > 0$ and assuming $z \ge 0$, we see that the solution $\mu(z)$ is located in an interval of the form $[\tilde{b}_1 z^N, \tilde{b}_2 z^N]$, where $0 < \tilde{b}_1 < a_N(0)/\dot{c}(0) < \tilde{b}_2$. The uniqueness is a consequence of the fact that $h(\mu,z)$ is differentiable with respect to μ if $|z| < \min\{r^{-1}, r_2^{-1}\}$ and $h_\mu(\mu,z) = -\dot{c}(0) + o(1)$ on the interval $[\tilde{b}_1 z^N, \tilde{b}_2 z^N]$. The proof of (iii) proceeds by substituting $\mu(z) = [a_N(0)/\dot{c}(0)] z^N [1 + \varphi(z)]$ in (2.35) and using $a_j(\mu) = \dot{a}_j(0)\mu + o(\mu)$ for $j = 1,...,N-1$, and the given expansion for $c(\mu)$, to find the leading contribution of order $O(z)$ to $\varphi(z)$. The result is (2.37). ■

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