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Derivation of an exact spectral density transport equation for a nonstationary scattering medium

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Within the framework of the quasioptical description and the pure Markovian random process approximation, an exact kinetic equation is derived for the spectral density function in the case of wave propagation in a nondispersive medium characterized by large-scale space–time fluctuations. Also, a quantity, called the *degree of coherence function*, is defined as a quantitative measure of the irreversible effects of randomness.

1. INTRODUCTION

Investigations of electromagnetic wave propagation in nonstationary random media are often based on the equations of classical radiation transport theory, the usual derivation[1,2] of which is based on considerations of energy balance, with no explicit "microscopic" interpretation given to the extinction and scattering coefficients entering into these equations. Moreover, use is frequently made of the random phase approximation which is valid only for incoherent waves (such as stellar radiation). Extensions to this approach introduced by Bugnolo,[3] Stott,[4] and Peacher and Watson[5] are applicable to partially coherent waves and account for multiple scattering effects.

In the past few years, primarily in connection with laser propagation, there has been considerable interest in the investigation of the transformation of the wave spectrum in media characterized by large-scale space—time random fluctuations. Recently reported studies along this direction[6,7] are confined to the quasistatic approximation, with the time dependence of the index of refraction entering parametrically, mostly via a constant or a variable (in the direction of propagation) transverse wind. Furthermore, authors who base their work on radiation transport theory often use uncritically the basic equations of Bugnolo and Peacher and Watson.

It is the intent in this paper to lift several of the aforementioned restrictions and systematically derive an exact spectral density kinetic equation for wave propagation in a nondispersive medium having large-scale space—time random fluctuations within the framework of the quasioptical description and the pure Markovian random process approximation.

2. THE QUASIOPTICAL DESCRIPTION

Ignoring depolarization effects, time-dependent electromagnetic wave propagation in a nondispersive medium with random space—time fluctuations of the refractive index is governed by the stochastic scalar wave equation,

$$\nabla^2 u(\mathbf{r},t) - \frac{1}{c^2}\epsilon_r(\mathbf{r},t)\frac{\partial^2}{\partial t^2}u(\mathbf{r},t)=0. \qquad (2.1)$$

Here, c is the velocity of light *in vacuo*, $\epsilon_r(\mathbf{r},t)$ is the relative permittivity which is assumed to be a real random function of space and time, and $u(\mathbf{r},t)$ is a scalar, real, random amplitude function.

For plane- or beam-wave propagation in the z direction, it is convenient to resort to the transformation

$$u(\mathbf{r},t)=\psi(\mathbf{r},t)\exp[ik(z-vt)]+\text{c.c.}, \qquad (2.2)$$

where $k=\omega_0/v$, $v=c/\langle\epsilon_r(\mathbf{r},t)\rangle^{1/2}$, and ω_0 is a reference (carrier) frequency. The ensemble average of the random relative permittivity, viz., $\langle\epsilon_r(\mathbf{r},t)\rangle$, is assumed to be constant.

In the quasioptical description, the slowly varying complex random amplitude function $\psi(\mathbf{r},t)$ obeys the nonstationary stochastic parabolic equation[8]

$$\frac{i}{k}\left(\frac{\partial}{\partial z}+\frac{1}{v}\frac{\partial}{\partial t}\right)\psi(\mathbf{x},t;z)=-\frac{1}{2k^2}\nabla_{\mathbf{x}}^2\psi(\mathbf{x},t;z) - \tfrac{1}{2}\epsilon_1(\mathbf{x},t;z)\,\psi(\mathbf{x},t;z), \quad z\geqslant 0, \qquad (2.3)$$

where $\mathbf{x}=(x,y)$ and

$$\epsilon_1(\mathbf{x},t;z)=[\epsilon_r(\mathbf{x},t;z)-\langle\epsilon_r(\mathbf{x},t;z)\rangle]/\langle\epsilon_r(\mathbf{x},t;z)\rangle \qquad (2.4)$$

is the normalized fluctuating part of the random relative permittivity. Equation (2.3) is rendered closed by specifying the boundary condition $\psi(\mathbf{x},t;0)=\psi_0(\mathbf{x},t)$.

3. THE SPECTRAL DENSITY

A two- (transverse) point, two-time field density function is next introduced as follows in terms of the wavefunction:

$$\rho(\mathbf{x}_2,\mathbf{x}_1,t_2,t_1;z)=\psi^*(\mathbf{x}_2,t_2;z)\,\psi(\mathbf{x}_1,t_1;z). \qquad (3.1)$$

It obeys the equation

$$\frac{i}{k}\frac{\partial}{\partial z}\rho(\mathbf{x}_2,\mathbf{x}_1,t_2,t_1;z) = \left[-\frac{i}{k}\frac{1}{v}\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}\right)-\frac{1}{2k^2}\nabla_{\mathbf{x}_1}^2+\frac{1}{2k^2}\nabla_{\mathbf{x}_2}^2 - \tfrac{1}{2}\epsilon_1(\mathbf{x}_1,t_1;z)+\tfrac{1}{2}\epsilon_1(\mathbf{x}_2,t_2;z)\right]\rho(\mathbf{x}_2,\mathbf{x}_1,t_2,t_1;z), \quad z\geqslant 0, \qquad (3.2\text{a})$$

$$\rho(\mathbf{x}_2,\mathbf{x}_1,t_2,t_1;0)=\rho_0(\mathbf{x}_2,\mathbf{x}_1,t_2,t_1). \qquad (3.2\text{b})$$

The "phase-space" analog of the density function is provided by the field spectral density which is defined as follows:

$$f(\mathbf{x},\mathbf{p},t,w;z)=\left(\frac{k}{2\pi}\right)^3\int_{R^2}d\mathbf{y}\int_{R^1}d\tau\,\exp[ik(\mathbf{p}\cdot\mathbf{y}-w\tau)] \times\rho(\mathbf{x}+\tfrac{1}{2}\mathbf{y},\mathbf{x}-\tfrac{1}{2}\mathbf{y},t+\tfrac{1}{2}\tau,t-\tfrac{1}{2}\tau;z). \qquad (3.3)$$

This quantity is real, but not necessarily positive everywhere.[9] It will be shown, however, later on in the exposition, that appropriate moments of the spectral density are physical observables.

Using the definition of $f(\mathbf{x}, \mathbf{p}, t, w; z)$ in conjunction with (3.1) and (2.3), it is found that the spectral density evolves according to the equation

$$\frac{\partial}{\partial z} f(\mathbf{x}, \mathbf{p}, t, w; z) = L f(\mathbf{x}, \mathbf{p}, t, w; z), \quad z \geq 0, \tag{3.4a}$$

$$f(\mathbf{x}, \mathbf{p}, t, w; 0) = f_0(\mathbf{x}, p, t, w), \tag{3.4b}$$

$$L f(\mathbf{x}, \mathbf{p}, t, w; z) = -\left(\frac{1}{v}\frac{\partial}{\partial t} + \mathbf{p}\cdot\frac{\partial}{\partial \mathbf{x}}\right) f(\mathbf{x}, \mathbf{p}, t, w; z) + \theta f(\mathbf{x}, \mathbf{p}, t, w; z). \tag{3.4c}$$

The following representation of the permittivity-dependent term on the right-hand side of (3.4c) will prove useful in the sequel[10]:

$$\theta f(\mathbf{x}, \mathbf{p}, t, w; z) = \left(\frac{i}{k}\right)^{-1}\left(\frac{2\pi}{k}\right)^{-3}\int_{R^2} d\mathbf{y}\int_{R^1} d\tau \exp[ik(\mathbf{p}\cdot\mathbf{y} - w\tau)] \times \rho(\mathbf{x}+\tfrac{1}{2}\mathbf{y}, \mathbf{x}-\tfrac{1}{2}\mathbf{y}, t+\tfrac{1}{2}\tau, t-\tfrac{1}{2}\tau; z) \times[\tfrac{1}{2}\epsilon_1(\mathbf{x}+\tfrac{1}{2}\mathbf{y}, t+\tfrac{1}{2}\tau; z) - \tfrac{1}{2}\epsilon_1(\mathbf{x}-\tfrac{1}{2}\mathbf{y}, t-\tfrac{1}{2}\tau; z)]. \tag{3.5}$$

4. SPECTRAL DENSITY TRANSPORT EQUATION IN THE PURE MARKOVIAN RANDOM PROCESS APPROXIMATION

We consider in this section a statistical analysis of the stochastic equation (3.4). Specifically, we shall derive an exact kinetic equation for the mean spectral density $\langle f(\mathbf{x}, \mathbf{p}, t, w; z)\rangle$ in the pure Markovian random process approximation.

Averaging both sides of (3.1) yields

$$\left(\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t} + \mathbf{p}\cdot\frac{\partial}{\partial \mathbf{x}}\right)\langle f(\mathbf{x}, \mathbf{p}, t; w; z)\rangle = \Theta\langle f(\mathbf{x}, \mathbf{p}, t, w; z)\rangle, \tag{4.1a}$$

$$\Theta\langle f(\mathbf{x}, \mathbf{p}, t, w; z)\rangle = \left(\frac{i}{k}\right)^{-1}\left(\frac{2\pi}{k}\right)^{-3}\int_{R^2} d\mathbf{y}\int_{R^1} d\tau \times \exp[ik(\mathbf{p}\cdot\mathbf{y} - w\tau)]\langle \rho(\mathbf{x}+\tfrac{1}{2}\mathbf{y}, \mathbf{x}-\tfrac{1}{2}\mathbf{y}, t+\tfrac{1}{2}\tau, t-\tfrac{1}{2}\tau; z) \times[\tfrac{1}{2}\epsilon_1(\mathbf{x}+\tfrac{1}{2}\mathbf{y}, t+\tfrac{1}{2}\tau; z) - \tfrac{1}{2}\epsilon_1(\mathbf{x}-\tfrac{1}{2}, t-\tfrac{1}{2}\tau; z)]\rangle. \tag{4.1b}$$

We assume that $\epsilon_1(\mathbf{x}, t; z)$ is a δ correlated (in z), homogeneous, wide-sense stationary Gaussian process specified completely by the correlation function

$$\langle \epsilon_1(\mathbf{x}_2, t_2; z_2)\,\epsilon_1(\mathbf{x}_1, t_1; z_1)\rangle = \frac{2\pi}{k}\gamma(\mathbf{x}_2 - \mathbf{x}_1, t_2 - t_1)\delta(z_2 - z_1). \tag{4.2}$$

Then, on the basis of the Furutsu—Novikov[11,12] functional formalism, we have

$$\langle \rho(\mathbf{x}_2, \mathbf{x}_1, t_2, t_1; z)[\epsilon_1(\mathbf{x}_2, t_2; z) - \epsilon_1(\mathbf{x}_1, t_1, z]\rangle$$
$$= \int_{R^2} d\mathbf{x}_2'\int_{R^2} d\mathbf{x}_1'\int_{R^1} dt_2'\int_{R^1} dt_1'\int_{R^1} dz'\,\langle[\epsilon_1(\mathbf{x}_2, t_2; z) - \epsilon_1(\mathbf{x}_1, t_1; z)][\epsilon_1(\mathbf{x}_2', t_2'; z') - \epsilon_1(\mathbf{x}_1', t_1'; z')]\rangle \times\langle \delta\rho(\mathbf{x}_2, \mathbf{x}_1, t_2, t_1; z)/\delta[\epsilon_1(\mathbf{x}_2', t_2'; z') - \epsilon_1(\mathbf{x}_1', t_1'; z')]\rangle$$
$$= \left(\frac{i}{k}\right)^{-1}\left(\frac{2\pi}{k}\right)[\gamma(\mathbf{x}_2 - \mathbf{x}_1, t_2 - t_1) - \gamma(0,0)] \times\langle \rho(\mathbf{x}_2, \mathbf{x}_1, t_2, t_1; z)\rangle. \tag{4.3}$$

[The symbol $\delta(\cdot)$ denotes a functional derivative.] The last equality follows readily from the equation of evolution of the density function [cf. (3.2)] and an extension of the procedure followed by Tatarskii[13] in connection with the time-independent stochastic parabolic equation.

Using the coordinate transformation $\mathbf{x}_{2,1} \to \mathbf{x} \pm \frac{1}{2}\mathbf{y}$, $t_{2,1} \to t \pm \frac{1}{2}\tau$ in (4.3) and introducing the result into the statistically averaged equation (4.1), we obtain

$$\left(\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t} + \mathbf{p}\cdot\frac{\partial}{\partial \mathbf{x}}\right)\langle f(\mathbf{x}, \mathbf{p}, t, w; z)\rangle = \left(\frac{\pi k}{2}\right)\left(\frac{k}{2\pi}\right)^3\int_{R^2} d\mathbf{y}\int_{R^1} d\tau \exp[ik(\mathbf{p}\cdot\mathbf{y} - w\tau)][\gamma(\mathbf{y}, \tau) - \gamma(0,0)]\langle \rho(\mathbf{x}+\tfrac{1}{2}\mathbf{y}, \mathbf{x}-\tfrac{1}{2}\mathbf{y}, t+\tfrac{1}{2}\tau, t-\tfrac{1}{2}\tau; z)\rangle. \tag{4.4}$$

This equation simplifies considerably upon introducing the spectrum of the space—time correlation function, viz.,

$$\hat{\gamma}(\mathbf{p}, w) = \left(\frac{k}{2\pi}\right)^3\int_{R^2} d\mathbf{y}\int_{R^1} d\tau \exp[-ik(\mathbf{p}\cdot\mathbf{y} - w\tau)]\gamma(\mathbf{y}, \tau), \tag{4.5a}$$

$$\gamma(\mathbf{y}, \tau) = \int_{R^2} d\mathbf{p}\int_{R^1} dw \exp[ik(\mathbf{p}\cdot\mathbf{y} - w\tau)\hat{\gamma}(\mathbf{p}, w). \tag{4.5b}$$

Bearing in mind the definition of the spectral density [cf. (3.3)], (4.4) changes to the simple, convolution-type transport equation

$$\left(\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t} + \mathbf{p}\cdot\frac{\partial}{\partial \mathbf{x}} + \frac{\pi k}{2}\gamma(0,0)\right)\langle f(\mathbf{x}, \mathbf{p}, t, w; z)\rangle = \frac{\pi k}{2}\int_{R^2} d\mathbf{p}'\int_{R^1} dw'\,\hat{\gamma}(\mathbf{p}-\mathbf{p}', w - w')\langle f(\mathbf{x}, \mathbf{p}', t, w'; z)\rangle. \tag{4.6}$$

It follows from (4.5b) that

$$\gamma(0,0) = \int_{R^2} d\mathbf{p}\int_{R^1} dw\,\hat{\gamma}(\mathbf{p}, w). \tag{4.7}$$

The spectrum $\hat{\gamma}(\mathbf{p}, w)$, however, is real, nonnegative, and even in both arguments. By virtue of the last property, it is seen that

$$\gamma(0,0) = \int_{R^2} d\mathbf{p}'\int_{R^1} dw'\,\hat{\gamma}(\mathbf{p}-\mathbf{p}', w - w'), \tag{4.8}$$

and Eq. (4.6) can be recast into the form

$$\left(\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t} + \mathbf{p}\cdot\frac{\partial}{\partial \mathbf{x}}\right)\langle f(\mathbf{x}, \mathbf{p}, t, w; z)\rangle = \int_{R^2} d\mathbf{p}'\int_{R^1} dw'\,W(\mathbf{p}, \mathbf{p}', w, w') \times[\langle f(\mathbf{x}, \mathbf{p}', t, w'; z) - \langle f(\mathbf{x}, \mathbf{p}, t, w; z)\rangle], \tag{4.9a}$$

$$W(\mathbf{p}, \mathbf{p}', w, w') = \frac{\pi k}{2}\hat{\gamma}(\mathbf{p}-\mathbf{p}', w - w'). \tag{4.9b}$$

This expression has the form of a radiation transport equation. [More precisely, if (4.9a) is integrated over w, it becomes a Boltzmann equation for waves (quasi-particles in phase space).] It extends the kinetic equation reported by Klyatskin and Tatarskii[14] in connection with the stationary stochastic parabolic equation, and, in the quasistatic case, it provides a rigorous basis for the work of Fante (cf. Ref. 7).

From what was said earlier about $\hat{\gamma}(\mathbf{p}, w)$, it follows that the *transition probability* (or *scattering indicatrix*) $W(\mathbf{p}, \mathbf{p}', w, w')$ is real, nonnegative, and obeys the (detailed balance) property $W(\mathbf{p}', \mathbf{p}, w', w) = W(\mathbf{p}, \mathbf{p}', w, w')$. The *scattering rate* (also called the *extinction coefficient* or *collision frequency*) is defined in general by

$$\nu(\mathbf{p}, w) = \int_{R^2} d\mathbf{p}' \int_{R^1} dw' W(\mathbf{p}, \mathbf{p}', w, w'). \tag{4.10}$$

In the case under consideration here, the scattering rate is independent of $\mathbf{p}$ and w and is given by

$$\nu = (\pi k/2)\gamma(0, 0). \tag{4.11}$$

5. PHYSICAL OBSERVABLES

Having established an expression for the mean spectral density by solving the kinetic equation (4.9), the following physically meaningful averaged quantities can be obtained by straightforward integration: (i) the mutual space—time coherence $\langle \rho(\mathbf{x}+\frac{1}{2}\mathbf{y}, \mathbf{x}-\frac{1}{2}\mathbf{y}, t+\frac{1}{2}\tau, t-\frac{1}{2}\tau; z)\rangle = \int d\mathbf{p} \int dw \exp[-ik(\mathbf{p}\cdot\mathbf{y} - w\tau)]\langle f(\mathbf{x}, \mathbf{p}, t, w; z)\rangle$; (ii) the mean intensity density $\langle \psi^*(\mathbf{x}, t; z)\psi(\mathbf{x}, t; z)\rangle = \int d\mathbf{p} \int dw \times \langle f(\mathbf{x}, \mathbf{p}, t, w; z)\rangle$; (iii) the intensity density in momentum space $\langle \hat{\rho}(\mathbf{p}, \mathbf{p}, w, w; z)\rangle = \int d\mathbf{p} \int dt \langle f(\mathbf{x}, p, t, w; z)\rangle$, where $\hat{\rho}(\mathbf{p}, \mathbf{p}, w, w; z)$ is the momentum representation of the intensity density; (iv) the mean intensity flux density $\langle \mathbf{J}(\mathbf{x}, t; z)\rangle = \int d\mathbf{p} \int dw \mathbf{p} \langle f(\mathbf{x}, p, t, w; z)\rangle$, where $\mathbf{J}(\mathbf{x}, t; z) = (i/2k)[(\nabla_\mathbf{x}\psi^*)\psi - \psi^*(\nabla_\mathbf{x}\psi)]$ is the intensity flux density. Furthermore, denoting the total mean intensity, viz., $\int d\mathbf{x} \langle \psi^*(\mathbf{x}, t; z)\psi(\mathbf{x}, t; z)\rangle$ by $I(t; z)$, the following two averaged quantities are important in connection with the propagation of spatially bounded beams: (i) the mean "center of gravity" of the beam $\mathbf{x}_c(t; z) = [\int d\mathbf{p} \int dw \int d\mathbf{x}\mathbf{x} \times \langle f(\mathbf{x}, \mathbf{p}, t, w; z)\rangle]/I(t; z)$; (ii) spread of a beam $\frac{1}{2}\sigma^2(t; z) = [\int d\mathbf{p} \int dw \int d\mathbf{x}(\mathbf{x}-\mathbf{x}_c)^2 \langle f(\mathbf{x}, \mathbf{p}, t, w; z)\rangle]/I(t; z)$.

6. CONSERVATION OF THE MEAN INTENSITY; DEGREE OF COHERENCE

By virtue of the self-adjointness of the operator

$$H_{\text{op}}\left(-\frac{i}{k}\frac{\partial}{\partial \mathbf{x}}, \mathbf{x}, t; z\right) = -\frac{1}{2k^2}\nabla_\mathbf{x}^2 - \tfrac{1}{2}\epsilon_1(\mathbf{x}, t; z)$$

appearing on the right-hand side of (2.3), the intensity density function $|\psi(\mathbf{x}, t; z)|^2$ obeys the conservation law[15]

$$\left(\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t}\right)|\psi(\mathbf{x}, t; z)|^2 + \nabla_\mathbf{x}\cdot\mathbf{J}(\mathbf{x}, t; z) = 0, \tag{6.1}$$

where $\mathbf{J}(\mathbf{x}, t; z)$ is the intensity flux density (cf. previous section).

It was pointed out in the previous section that $\langle |\psi(\mathbf{x}, t; z)|^2\rangle = \int d\mathbf{p} \int dw \langle f(\mathbf{x}, \mathbf{p}, t, w; z)\rangle$ and $\langle \mathbf{J}(\mathbf{x}, t; z)\rangle = \int d\mathbf{p} \int dw \mathbf{p} \langle f(\mathbf{x}, \mathbf{p}, t, w; z)\rangle$. Bearing in mind these relationships and integrating both sides of (4.9) over $\mathbf{p}$ and w results in the following conservation law for the mean intensity:

$$\left(\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t}\right)\langle |\psi(\mathbf{x}, t; z)|^2\rangle + \nabla_\mathbf{x}\cdot\langle \mathbf{J}(\mathbf{x}, t; z)\rangle = 0. \tag{6.2}$$

Integration of this equation over the entire transverse observation plane yields the relation

$$\left(\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t}\right)\left[\int_{R^2} d\mathbf{x} \langle |\psi(\mathbf{x}, z)|^2\rangle\right] = 0. \tag{6.3}$$

The quantity $D(\mathbf{x}, \mathbf{p}, t, w; z) = \langle f(\mathbf{x}, \mathbf{p}, t, w; z)\rangle^2$ is defined next as the *phase-space degree of coherence density*. Integrating this quantity over $\mathbf{p}$- and w-space we obtain the *configuration-space degree of coherence density* $d(\mathbf{x}, t; z) = \int d\mathbf{p} \int dw D(\mathbf{x}, \mathbf{p}, t, w; z)$. Both sides of this last relation are operated on next with $[\partial/\partial z + (1/v)(\partial/\partial t)]$ and use is made of the transport equation (4.9):

$$\left(\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t}\right) d(\mathbf{x}, t; z) + \nabla_\mathbf{x}\cdot\mathbf{K}(\mathbf{x}, t; z)$$
$$= 2\int_{R^2} d\mathbf{p} \int_{R^2} d\mathbf{p}' \int_{R^1} dw \int_{R^1} dw' W(\mathbf{p}, \mathbf{p}', w, w')$$
$$\times[\langle f(\mathbf{x}, \mathbf{p}', t, w'; z)\rangle \langle f(\mathbf{x}, \mathbf{p}, t, w; z)\rangle - \langle f(\mathbf{x}, \mathbf{p}, t, w; z)\rangle^2], \tag{6.4}$$

where

$$\mathbf{K}(\mathbf{x}, t; z) = \int_{R^2} d\mathbf{p} \int_{R^1} dw\, \mathbf{p} D(\mathbf{x}, \mathbf{p}, t, w; z) \tag{6.5}$$

is the *configuration-space degree of coherence flux*.

The right-hand side of (6.4) can be rewritten in the more useful form

$$-\int_{R^2} d\mathbf{p} \int_{R^2} d\mathbf{p}' \int_{R^1} dw \int_{R^1} dw' W(\mathbf{p}, \mathbf{p}', w, w')$$
$$\times[\langle f(\mathbf{x}, \mathbf{p}, t, w; z)\rangle - \langle f(\mathbf{x}, \mathbf{p}', t, w'; z)\rangle]^2 \le 0 \tag{6.6}$$

on using the following two properties of the transition probability: (i) $W(\mathbf{p}', \mathbf{p}, w', w) = W(\mathbf{p}, \mathbf{p}', w, w')$ (detailed balance); (ii) $W(\mathbf{p}, \mathbf{p}', w, w') \ge 0$ (nonnegativity). Using, then, (6.6) in conjunction with (6.4), it is seen that

$$\left(\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t}\right) d(\mathbf{x}, t; z) + \nabla_\mathbf{x}\cdot\mathbf{K}(\mathbf{x}, t; z) \le 0. \tag{6.7}$$

Integrating this relation over $\mathbf{x}$ results in the inequality

$$\left(\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t}\right)\left[\int_{R^2} d\mathbf{x}\, d(\mathbf{x}, t; z)\right] \le 0 \tag{6.8}$$

which exhibits the monotonic decrease of the total degree of coherence as it is convected along the z direction with the constant velocity v.

It should be noted that inequality (6.8) is analogous to Boltzmann's H theorem is statistical mechanics. In the latter case, the configuration -space degree of coherence density (related to the entropy) would be defined as $d(\mathbf{x}, t, z) = -\int d\mathbf{p} \int dw \langle f\rangle \ln\langle f\rangle$. It has been pointed out, however, that $\langle f\rangle$ can assume negative values; hence, the need for the alternative approach presented in this section.

7. CONCLUDING REMARKS

The transport equation for the spectral density derived in Sec. 4 is an integrodifferential equation of the convolution type which can be integrated formally, i.e., $\langle f(\mathbf{x}, \mathrm{p}, t, w; z)\rangle$ can be expressed in terms of the initial distribution $\langle f(\mathbf{x}, \mathrm{p}, t, w; 0)\rangle$, by a technique analogous to that suggested by Dolin[16] in the case of a stationary scattering medium. This formal solution can then be examined for specific fluctuation spectra (cf. Refs. 17 and 18), in particular, those arising from a constant or a space-dependent (in the z direction) transverse wind (cf. Refs. 6 and 7). It should be noted, however, that the formulation presented in this paper is general enough, and it allows also the investigation of stochastic wave propagation in a space–time-dependent medium to and from moving sources. The latter subject has been recently examined by Strobehn[19] who used a quasistatic approximation and Rytov's method of smooth perturbations.

The discussion in this paper is confined to the mean spectral density, or, equivalently, to the space–time mutual coherence (cf. Sec. 5). This work, however, can be extended in several directions. For example, within the quasioptical assumption and the pure Markovian random process approximation, one can examine longitudinal (in the z direction) correlations, as well as transverse correlations for higher moments. In particular, a kinetic equation for the fourth moment would be important because of its relationship with the physical phenomenon of scintillation.

The main results presented in this paper, as well as the various extensions outlined in the previous paragraph, although interesting by virtue of the fact that they extend the corresponding results for the case of a stationary scattering medium, are, nonetheless, restricted in scope because of the following three underlying assumptions: (i) quasioptical approximation; (ii) nondispersive medium; (iii) pure Markovian random process approximation. Attempts are presently being made towards relaxing these serious restrictions.

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