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Expansion-free electromagnetic solutions of the Kerr–Schild class

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Starting with the general Kerr–Schild form of the metric tensor, $ds^2=\eta+l\otimes l$ (where l is null and η is flat space–time), a study is made for those solutions of the Einstein–Maxwell equations in which l is geodesic, shear-free, and expansion-free. It is shown that all resulting solutions must be of Petrov type [4] or type [−] and the Maxwell field must be null. Because of the expansion-free assumption there exist flat and conformally flat gauge conditions on all metrics in this class; i.e., there exist metrics of this Kerr–Schild form which are flat (or conformally flat) but are not Lorentz-related. A method is given for obtaining meaningful solutions to the field equations with the latter gauge equivalence class removed. A simple example of a radiative field of type [4] along a line singularity exhibits how solutions in this class may be generated.

1. INTRODUCTION

This work concerns itself primarily with radiation solutions of the Einstein—Maxwell equations for a metric in Kerr—Schild form. The assumption that the special null congruence is expansion-free bridges the gap left between those electromagnetic solutions of Debney, Kerr, and Schild[1] (hereafter called DKS) and the general expansion-free cases studied by Kundt.[2]

The original Kerr—Schild paper[3] concerned itself with vacuum space—time metrics which have the form (where η is the metric for flat space—time and l is tangent to a null congruence)

$$ds^2=\eta+l\otimes l. \qquad (1.1)$$

It assumed the Einstein vacuum field equations plus the condition that l have nonvanishing expansion ($\Leftrightarrow\rho\neq 0$, in Newman—Penrose notation,[4] hereafter called N—P). The general properties possessed by these vacuum space—times include: (a) They are all algebraically special; (b) l is a degenerate principal null direction for the Riemannian curvature tensor (and is both geodesic and shear-free); and (c) the Schwarzschild and Kerr[5] classes of solutions fall into this category.

Later, metrics of the same form satisfying the Einstein—Maxwell source-free equations were studied in DKS, again assuming the condition that l have nonvanishing expansion but also assuming l to be geodesic. The properties implied in general about these space—times turned out to be: (a) They are all algebraically special; (b) l is a principal null direction for the Weyl conformal tensor and l is shear-free; and (c) they contain the Reissner—Nordström and Kerr—Newman[6] classes of solutions.

Before the Kerr-Schild studies appeared, Kundt[2] considered all vacuum, and certain nonvacuum, space—times which possessed a geodesic and shear-free null congruence l with vanishing complex expansion. These fell into two general categories determined by whether the rotation (τ in N—P) of l vanishes or not.[7] It was concluded that such space—times fell into all algebraically special categories and the cases with vanishing rotation were the (type [4]) "pp waves." The general name of "expansion-free radiation fields" characterizes the whole expansion-free class.

It is the purpose of the present paper to examine in more detail the expansion-free constraint on the special null vector in the Kerr—Schild metrics and its implications in the context of Einstein—Maxwell theory. Such studies in vacuum cases have been treated by H. Urbantke[9] and by the author.[10]

Section 2 contains the algebraic preliminaries. Here, also, one assumes the expansion-free condition for l and its alignment with the electromagnetci field. Appendix A supplements this section with the computations to derive the field equations, proving along the way that the Petrov type must be [4] or [−].

Section 3 provides a better coordinate system (at least when the rotation $X\neq 0$) in which to solve the field equations. The work in Appendix B exhibits the flat and conformally flat "gauge" conditions on the metric, providing a way of obtaining type [4] solutions modulo these additional gauge terms. The problem here is that "the" flat background is not unique in these expansion-free cases: some of "$l\otimes l$" can go into "η" to form another flat background, not related to η by a Lorentz transformation. A method given for removing these solutions from the picture allows meaningful examples to be chosen. Section 4 exhibits, as an example of vanishing rotation, an electromagnetic field which falls off radially in cylindrical coordinates and propagates along the z axis; it possesses true singularities on this axis.

2. PRELIMINARIES

The Kerr—Schild[3,1] form for the metric on a four-dimensional Lorentz manifold (C^∞) of signature (+++−) is stated simply as $ds^2=\eta+l\otimes l$, where η is the metric for a flat (Minkowski) background and l^μ is the tangent to a congruence of null curves. Notice that writing

$$g_{\mu\nu}=\eta_{\mu\nu}+l_\mu l_\nu \qquad (2.1)$$

tells us that l^μ is null with respect to both $g_{\mu\nu}$ and $\eta_{\mu\nu}$. The field equations for a Einstein—Maxwell space—time

$$R_{\mu\nu}=-8\pi T_{\mu\nu} \qquad (2.2)$$

plus the source-free Maxwell equations for the electromagnetic field $F_{\mu\nu}$

$$F^{\mu\nu}{}_{;\nu}=0=F_{[\mu\nu;\sigma]} \qquad (2.3)$$

must also be satisfied. [$R_{\mu\nu}\equiv R^\alpha{}_{\mu\nu\alpha}$ is the Ricci tensor,

$T_{\mu\nu} \equiv (\frac{1}{4}\pi)(F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta})$ is the electromagnetic stress—energy tensor, and the Riemannian curvature tensor $R_{\mu\nu\rho\alpha}$ satisfies $V_{\sigma;\mu\nu} - V_{\sigma;\nu\mu} = R^{\alpha}{}_{\sigma\mu\nu}V_{\alpha}$ for any vector field V.]

The approach used to find solutions to (2.1)—(2.3) makes use of a complex null tetrad. [This set of four independent vector fields forms a basis (or "frame") in which all geometric objects and equations may be written. Components with respect to such a basis are indicated by Latin indices, whereas components with respect to a coordinate basis are indicated by Greek indices.] The contravariant and covariant components of the tetrad are expressed through

$$e_a \equiv e_a{}^{\mu}\frac{\partial}{\partial x^{\mu}}, \quad \epsilon^b \equiv \epsilon^b{}_{\mu}dx^{\mu}, \tag{2.4}$$

respectively. Since the two systems of vector fields in (2.4) are vector space duals they satisfy, by definition,

$$e_a{}^{\mu}\epsilon^a{}_{\nu} = \delta^{\mu}_{\nu}, \quad e_a{}^{\mu}\epsilon^b{}_{\mu} = \delta^b_a$$

$$g_{\mu\nu} = \epsilon^a{}_{\mu}e_{a\nu}, \quad g_{ab} = e_{a\mu}e_b{}^{\mu}.$$

The "complex null" part comes from the additional relations resulting from a formal complexification, where "bar" denotes complex conjugation:

$$\mathbf{e}_2 = \bar{\mathbf{e}}_1, \quad \mathbf{e}_3 = \bar{\mathbf{e}}_3, \quad \mathbf{e}_4 = \bar{\mathbf{e}}_4.$$

In such a system the metric takes the form

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = 2\epsilon^1\epsilon^2 + 2\epsilon^3\epsilon^4 = g_{ab}\epsilon^a\epsilon^b. \tag{2.5}$$

If $\{x, y, z, t\} = \{x^{\mu}\}$ are Cartesian coordinates in the background Minkowski space, define complex null coordinates $\{\zeta, \bar{\zeta}, u, v\}$ by

$$\sqrt{2}\,\zeta \equiv x + iy, \quad \sqrt{2}\,\zeta \equiv x - iy,$$

$$\sqrt{2}\,u \equiv z + t, \quad \sqrt{2}\,v \equiv z - t.$$

Then $\eta_{\mu\nu}dx^{\mu}dx^{\mu} = 2d\zeta d\bar{\zeta} + 2dudv$. By letting h be an unknown scalar and $l^{\mu} \equiv (2h)^{1/2}k^{\mu}$ the metric (2.1) becomes

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = 2d\zeta d\bar{\zeta} + 2dudv + 2h(k_{\mu}dx^{\mu})^2. \tag{2.6}$$

A choice of tetrad is made in terms of these coordinates:

$$\epsilon^1 = d\zeta - Ydv, \quad \epsilon^2 = \bar{\epsilon}^1 = d\bar{\zeta} - \bar{Y}dv,$$

$$\epsilon^3 \equiv k_{\mu}dx^{\mu} = du + \bar{Y}d\zeta + Yd\bar{\zeta} - Y\bar{Y}dv, \tag{2.7}$$

$$\epsilon^4 = dv + h\epsilon^3.$$

As in DKS, the unknown complex function $Y(x^{\mu})$ may be introduced to express any null vector field $k_{\mu}dx^{\mu}$ in Minkowski space. The contravariant tetrad $\{\mathbf{e}_a\}$ is then computed to be

$$\mathbf{e}_1 = \partial_{\zeta} - \bar{Y}\partial_u, \quad \mathbf{e}_2 = \bar{\mathbf{e}}_1 = \partial_{\bar{\zeta}} - Y\partial_u,$$

$$\mathbf{e}_3 = \partial_u - h\mathbf{e}_4, \tag{2.8}$$

$$\mathbf{e}_4 = \partial_v + Y\partial_{\zeta} + \bar{Y}\partial_{\bar{\zeta}} - Y\bar{Y}\partial_u \doteq k^{\mu}\partial_{\mu}.$$

Denoting the operation of $\mathbf{e}_a$ on a scalar ϕ by $\mathbf{e}_a(\phi) \equiv \phi_{,a}$ it is clear from a study of Ricci rotation coefficients for k_{μ} that $Y_{,4} = 0 \Leftrightarrow k_{\mu}$ is geodesic and $Y_{,2} = 0 \Leftrightarrow k_{\mu}$ is shear-free. Furthermore $Y_{,1} = z$, the *complex expansion* of k_{μ} ("ρ" in Newman—Penrose[4]). (See Appendix A.)

The first assumption made for this system is that k_{μ} *is a principal null direction for* $F_{\mu\nu}$; i.e., $F_{\mu\nu}k^{\mu} = \alpha k_{\nu}$ for some scalar α. This is essentially the same as that made in DKS because k^{μ} is geodesic and shear-free if and only if it is a principal null direction for $F_{\mu\nu}$ (see Appendix A). Consequently, the scalar h in (2.6) may be chosen so that k^{μ} is tangent to an affinely parametrized congruence of null geodesics. The congruence is also shear-free.

The next assumption, $z = 0$, restricts the study to those Kerr—Schild electromagnetic solutions for which the vector l^{μ} in the metric is *expansion-free*. As shown in Appendix A, this produces the general theorem: *All source-free vacuum or Einstein—Maxwell fields of the Kerr—Schild class* $ds^2 = \eta + l \otimes l$, *where* l^{μ} *is an expansion-free principal null direction for the Maxwell field* $F_{\mu\nu}$ *are of Petrov type* [4]. Furthermore, the electromagnetic field $F_{\mu\nu}$ is *null*; i.e., $F_{\mu\nu}F^{\mu\nu} = 0 = F^{*}_{\mu\nu}F^{\mu\nu}$

3. NEW COORDINATES: SOLUTIONS OF THE FIELD EQUATIONS

It is shown in Appendix A that the vector fields $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4\}$ satisfy $[\mathbf{e}_1, \mathbf{e}_2] = [\mathbf{e}_1, \mathbf{e}_4] = [\mathbf{e}_2, \mathbf{e}_4] = 0$. Frobenius' theorem suggests that one may find new coordinates, say $\{\alpha, \bar{\alpha}, \rho, w\} = \{x^{\mu'}\}$, for which

$$\mathbf{e}_1 = \partial_{\alpha}, \quad \mathbf{e}_2 = \partial_{\bar{\alpha}}, \quad \mathbf{e}_4 = \partial_v. \tag{3.1}$$

Indeed this may be accomplished by setting

$$\alpha \equiv \zeta - Yv, \quad \bar{\alpha} \equiv \bar{\zeta} - \bar{Y}v, \quad w \equiv v,$$

$$\rho \equiv u + \bar{Y}\zeta + Y\bar{\zeta} - Y\bar{Y}v. \tag{3.2}$$

Notice that $\alpha\bar{\alpha} + \rho v = \zeta\bar{\zeta} + uv$ but that $\{x^{\mu}\} \to \{x^{\mu'}\}$ is not Lorentz except when Y is a constant. However $\rho = \eta_{\mu\nu}x^{\mu}k^{\nu} = g_{\mu\nu}x^{\mu}k^{\nu}$ so that $\rho = \mathbf{k}\cdot\mathbf{P}$, where $\mathbf{P}$ is a position vector in the original "background" Minkowski space.

The tetrads expressed in the new coordinates become[10]

$$\epsilon^1 = d\alpha + vX\epsilon^3, \quad \epsilon^2 = d\bar{\alpha} + v\bar{X}\epsilon^3,$$

$$\epsilon^3 = r^{-1}d\rho, \qquad \epsilon^4 = dv + h\epsilon^3 \tag{3.3}$$

(with rotation $X \equiv Y_{,3}$ and $r \equiv 1 + \alpha\bar{X} + \bar{\alpha}X$) and

$$\mathbf{e}_1 = \partial_{\alpha}, \quad \mathbf{e}_2 = \partial_{\bar{\alpha}}, \quad \mathbf{e}_4 = \partial_v,$$

$$\mathbf{e}_3 = -v(X\partial_{\alpha} + \bar{X}\partial_{\bar{\alpha}}) + r\partial_{\rho} - h\partial_v. \tag{3.4}$$

The field equations [Eqs. (A14a)—(A14d)] may be written as

$$h_{vv} = 0, \quad F_v = 0, \tag{3.5a}$$

$$h_v X - h_{\bar{\alpha}v} = 0, \tag{3.5b}$$

$$h_{\alpha\bar{\alpha}} - h_{\alpha}X - h_{\bar{\alpha}}\bar{X} = -2F\bar{F}, \tag{3.5c}$$

$$F_{\bar{\alpha}} - XF = 0. \tag{3.5d}$$

The geodesic, shear-free, and expansion-free conditions on k^{μ} clearly imply through (A4) and (A10) that $Y = Y(\rho)$.

Hence, $Y_{,3}=X=r(dY/d\rho)=rY'$ so that

$$r^{-1}=1-\bar{Y}'\alpha-Y'\bar{\alpha}. \tag{3.6}$$

Equation (3.5a) implies that $F=F(\alpha,\bar{\alpha},\rho)$ and that

$$h=a(\alpha,\bar{\alpha},\rho)v+g(\alpha,\bar{\alpha},\rho), \tag{3.7a}$$

where a and g are real-valued functions of their arguments. Equation (3.5b) implies

$$a=rA(\rho), \tag{3.7b}$$

where A is arbitrary. The function a does not enter into (3.5c) so that (3.5c) and (3.5d) reduce to, respectively,

$$(gr^{-1})_{\alpha\bar{\alpha}}=-2r^{-1}F\bar{F}, \tag{3.7c}$$

$$(Fr^{-1})_{\bar{\alpha}}=0. \tag{3.7d}$$

Although it is not obvious, an investigation similar to that of Appendix B gives us that a conformally flat solution must necessarily have $X\equiv 0$, $F=F(\rho)$, and $g=D(\rho)\alpha\bar{\alpha}$. However, these only make up a proper subclass of solutions where $F=F(\rho)$. The latter solutions, even though $X=0$ is implied here too, are not all conformally flat since they admit a more general function $g(\alpha,\bar{\alpha},\rho)$ in the metric.

It is shown in Appendix B that any function (3.7a) having the form

$$\hat{h}=r[A(\rho)v+K(\rho)\alpha+\bar{K}(\rho)\bar{\alpha}+L(\rho)], \tag{3.8}$$

where A, K, L are arbitrary functions of ρ, results in the metric

$$ds^2=\eta+2\hat{h}(k\otimes k) \tag{3.9}$$

being a representation of flat space—time with no field F; i.e., any such $\hat{h}$ term in general makes no contribution to the curvature. Consequently, there exists a coordinate system in which (3.9) may be written manifestly as flat space. Instead of looking for this coordinate transformation we alternatively take the approach that solutions to (3.7c) and (3.7d) *not* containing $\hat{h}=av+\hat{g}$, where $\hat{g}=r[K\alpha+\bar{K}\bar{\alpha}+L]$ for $X\neq 0$ or $\hat{g}=D\alpha\bar{\alpha}+K\alpha+\bar{K}\bar{\alpha}+L$ for $X=0$, are to be regarded as meaningful for the present purposes.

4. A SPECIAL SOLUTION WITH $X=0$ (pp WAVE)

Since $X=0$ implies the system $\{\alpha,\bar{\alpha},\rho,v\}=\{\zeta,\bar{\zeta},u,v\}$, let

$$F_{31}=F=-\gamma(u)/\zeta, \tag{4.1}$$

where $\gamma(u)$ is arbitrary and real. Then by the discussion in Section 3 the metric is not conformally flat. Furthermore,

$$F\bar{F}=\gamma^2/\zeta\bar{\zeta}.$$

By solving (3.7c) the nonflat part of the metric becomes

$$g(\zeta,\bar{\zeta},u)=-2\gamma^2(u)|\ln(\zeta)|^2. \tag{4.2}$$

Hence, the metric is

$$ds^2=2d\zeta d\bar{\zeta}+2dudv-4\gamma^2|\ln(\zeta)|^2du^2, \tag{4.3}$$

with a curvature singularity at $\zeta=0$ (i.e., $x=y=0$) and possibly elsewhere if $\gamma(u)\to\infty$.

Let $\sqrt{2}\,\zeta=Re^{i\theta}$. Then $R=x^2+y^2$ and cylindrical coordinates $\{R,\theta,z,t\}$ are established. From the relation $F_{\mu\nu}dx^\mu\wedge dx^\nu=F_{ab}\epsilon^a\wedge\epsilon^b$ one obtains the electromagnetic field tensor

$$F_{\mu\nu}dx^\mu\wedge dx^\nu=\sqrt{2}\,(2\gamma/R)[dR\wedge dz+dR\wedge dt]. \tag{4.4}$$

Classically, the electric field **E** is along R and the magnetic field **H** is along θ; propagation takes place along the z axis and the intensity falls off as R^{-1} in the radial direction. The amplitude $\gamma(z+t)$ determines the longitudinal behavior of the wave.

Notice that many choices for the starting point (4.1) exist and it is possible to study many more type [4] cases when $X=0$ as long as $g_{\zeta\zeta}\neq 0$ and the conformally flat $\hat{g}=D\zeta\bar{\zeta}$ is avoided. Curvature singularities will most likely be determined by the singularities inherent in the choice of F.

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APPENDIX A: THE STRUCTURE EQUATIONS FOR THE KERR-SCHILD CLASS

The following discussion first makes use only of the Kerr—Schild form of the metric. Further assumptions come in later as: (a) l^μ is a principal null direction for the electromagnetic field $F_{\mu\nu}$ and (b) the complex expansion of l^μ vanishes (i.e., $z\equiv 0$), respectively.

The first structure equations are stated concisely as $d\epsilon^a=\Gamma^a{}_{bc}\epsilon^b\wedge\epsilon^c$, which are true for any torsion-free connection $\Gamma^a{}_{bc}$ and any tetrad (or "frame") $\{\epsilon^a\}$ locally throughout the space—time. The condition that the connection be the "metric connection" of Levi-Civita is the equivalent to stating $\Gamma_{bac}=-\Gamma_{abc}$, where $\Gamma_{abc}\equiv g_{am}\Gamma^m{}_{bc}$.

For the particular metric (2.1) and tetrad (2.7) the $d\epsilon^3$ equations are written as

$$\begin{aligned} d\epsilon^3 &= \tfrac{1}{2}(\Gamma_{4ab}-\Gamma_{4ba})\epsilon^a\wedge\epsilon^b \\ &= (Y_{,1}-\bar{Y}_{,2})\epsilon^1\wedge\epsilon^2+\bar{Y}_{,3}\epsilon^3\wedge\epsilon^1+Y_{,3}\epsilon^3\wedge\epsilon^2 \\ &\quad +\bar{Y}_{,4}\epsilon^4\wedge\epsilon^1+Y_{,4}\epsilon^4\wedge\epsilon^2. \end{aligned}$$

Equating coefficients and defining $z\equiv\Gamma_{241}$, $X\equiv Y_{,3}$, we obtain

$$\Gamma_{414}=-Y_{,4},\qquad \Gamma_{424}=-Y_{,4},\qquad \Gamma_{434}=0,$$

$$z-\bar{z}=Y_{,1}-\bar{Y}_{,2},\quad \Gamma_{423}=-X-\Gamma_{342},\quad \Gamma_{413}=-\bar{X}-\Gamma_{341}. \tag{A1}$$

The $d\epsilon^4$ equations become

$$d\epsilon^4 = \tfrac{1}{2}(\Gamma_{3ab} - \Gamma_{3ba})\epsilon^a \wedge \epsilon^b = dh \wedge \epsilon^3 + h d\epsilon^3$$
$$= h(z - \bar{z})\epsilon^1 \wedge \epsilon^2 + (-h_{,4})\epsilon^3 \wedge \epsilon^4 + (h\bar{X} - h_{,1})\epsilon^3 \wedge \epsilon^1$$
$$+ (hX - h_{,2})\epsilon^3 \wedge \epsilon^2 + h\bar{Y}_{,4}\epsilon^4 \wedge \epsilon^1 + hY_{,4}\epsilon^4 \wedge \epsilon^2.$$

Equating coefficients allows one to write

$$\Gamma_{342} - \Gamma_{324} = hY_{,4}, \quad \Gamma_{341} - \Gamma_{314} = h\bar{Y}_{,4}, \quad \Gamma_{343} = h_{,4},$$
$$\Gamma_{312} - \Gamma_{321} = h(z - \bar{z}), \quad \Gamma_{313} = h_{,1} - h\bar{X}, \quad \Gamma_{323} = h_{,2} - hX. \tag{A2}$$

Finally the $d\epsilon^2$ equations are written as

$$d\epsilon^2 = (1/2)(\Gamma_{1ab} - \Gamma_{1ba})\epsilon^a \wedge \epsilon^b$$
$$= -(\bar{X} + h\bar{Y}_{,4})\epsilon^3 \wedge \epsilon^4 + (-h\bar{Y}_{,1})\epsilon^3 \wedge \epsilon^1 + (-hY_{,2})\epsilon^3 \wedge \epsilon^2$$
$$+ \bar{Y}_{,1}\epsilon^4 \wedge \epsilon^1 + Y_{,2}\epsilon^4 \wedge \epsilon^2.$$

Upon equating coefficients we obtain (through the use of complex conjugation)

$$\bar{Y}_{,1} = -\Gamma_{411}, \quad Y_{,2} = -\Gamma_{422}, \quad h\bar{Y}_{,1} = \Gamma_{311}, \quad hY_{,2} = \Gamma_{322},$$
$$\bar{Y}_{,2} = -\Gamma_{124} + \bar{z}, \quad Y_{,1} = \Gamma_{124} + z,$$
$$h\bar{Y}_{,2} = \Gamma_{123} + \Gamma_{312}, \quad hY_{,1} = -\Gamma_{123} + \Gamma_{321},$$
$$\Gamma_{121} = \Gamma_{122} = 0, \quad \Gamma_{413} = \Gamma_{314} - \bar{X} - h\bar{Y}_{,4},$$
$$\Gamma_{423} = \Gamma_{324} - X - hY_{,4}. \tag{A3}$$

Putting together the information in (A1)—(A3) results in

$$z = Y_{,1}, \quad \bar{z} = \bar{Y}_{,2}, \quad \Gamma_{321} = h\bar{z}, \quad \Gamma_{312} = hz,$$
$$\Gamma_{422}(\text{shear of } k^\mu) = -Y_{,2}, \quad \Gamma_{322} = hY_{,2},$$
$$\Gamma_{124} = \Gamma_{121} = \Gamma_{122} = 0, \quad \Gamma_{123} = -h(z - \bar{z}),$$
$$\Gamma_{424} = -Y_{,4}, \quad \Gamma_{342} = hY_{,4}, \quad \Gamma_{324} = 0 = \Gamma_{344},$$
$$-\Gamma_{423} = hY_{,4} + X, \quad \Gamma_{323} = h_{,2} - h\bar{X}. \tag{A4}$$

The second structure equations $d\Gamma^a{}_b + \Gamma^a{}_m \wedge \Gamma^m{}_b = \frac{1}{2}R^a{}_{bcd}\epsilon^c \wedge \epsilon^d$ contain implicitly the field equations $R_{ab} = -8\pi T_{ab}$ (where $\Gamma^a{}_b \equiv \Gamma^a{}_{bc}\epsilon^c$ are the connection 1-forms). However, one obtains in particular the relationship

$$-\tfrac{1}{2}R_{\mu\nu}k^\mu k^\nu = -\tfrac{1}{2}R_{44} = R_{4241} = -2h|Y_{,4}|^2. \tag{A5}$$

Hence, making the first assumption that k^μ is a principal null direction for $F_{\mu\nu}$(and therefore $T_{\mu\nu}$) implies $Y_{,4} \equiv 0$. But $Y_{,4} = 0$ implies $\Gamma_{424} = \Gamma_{414} = 0$, which is equivalent to stating that k^μ is a geodesic. Hence, k^μ is a principal null direction for $F_{\mu\nu}$ if and only if k^μ is a geodesic. Our assumption here (as in the earlier DKS paper) then produces the simplifications

$$\Gamma_{414} = \Gamma_{424} = \Gamma_{341} = \Gamma_{342} = 0, \quad \Gamma_{423} = -X, \quad \Gamma_{413} = -\bar{X}. \tag{A6}$$

Furthermore, this forces R_{ab} partly into a canonical form

$$R_{42} = R_{41} = R_{44} = 0 = R_{11} = R_{22} \tag{A7}$$

since $F_{14} = F_{24} = 0$ now, as well. The connection 1-forms of interest reduce to

$$\Gamma_{42} = -dY,$$
$$\Gamma_{12} + \Gamma_{34} = [h_{,4} - h(z - \bar{z})]\epsilon^3, \tag{A8}$$
$$\Gamma_{31} = h\bar{\sigma}\epsilon^1 + hz\epsilon^2 + (h_{,1} - h\bar{X})\epsilon^3,$$

where $\sigma \equiv -\Gamma_{422}$.

The $d\Gamma_{42} + \Gamma_{4m} \wedge \Gamma^m{}_2 = \frac{1}{2}R_{42ab}\epsilon^a \wedge \epsilon^b$ equation results in

$$R_{4242} = 0 \Rightarrow C^{(5)} = 0 \quad (\psi_0 \text{ in Newman—Penrose}),$$
$$R_{4212} = R_{4234} = 0 \Rightarrow C^{(4)} = 0 \quad (\psi_1 \text{ in N—P}),$$

implying that the space—time is algebraically special and that k^μ is a principal null direction for the Weyl conformal curvature tensor. Also $2R_{4231} = C^{(3)}$ (ψ_2 in N—P) so that

$$-C^{(3)} = 2z[h_{,4} - (z - \bar{z})h]. \tag{A9}$$

The field equation $-\frac{1}{2}R_{22} = R_{4232} = 0$ is not identically satisfied; it becomes

$$\sigma[h_{,4} - (z - \bar{z})h] = 0.$$

As in DKS, $\sigma \neq 0$ gives rise to a contradiction (i.e., vanishing electromagnetic field, algebraically special space—time, $\sigma \neq 0$ are incompatible). Therefore $\sigma = 0$ must result. Note that we have derived the relation that k^μ must be geodesic ($\Gamma_{424} = 0$) and shear-free ($\Gamma_{422} = 0$).

At this stage the work of DKS and that discussed here differ in that the assumption $z = 0$ (k^μ is expansion-free) is imposed. From (A9) it is clear that $C^{(3)} = 0$ so that Petrov types [3, 1], [4], or [−] are the only possibilities. We choose to exclude the conformally flat cases (type [−]) since these have been solved completely in the Einstein—Maxwell context (see, for example, Cahen and Leroy[11]).

The special relations imposed above imply the following relations from (A4) (omitting complex conjugates):

$$\Gamma_{122} = \Gamma_{121} = \Gamma_{123} = \Gamma_{124} = 0 = \Gamma_{421} = \Gamma_{422} = \Gamma_{424} = \Gamma_{414},$$
$$\Gamma_{311} = \Gamma_{312} = \Gamma_{314} = 0 = \Gamma_{341} = \Gamma_{344}, \tag{A10}$$
$$\Gamma_{343} = h_{,4}, \quad \Gamma_{313} = h_{,1} - h\bar{X}, \quad \Gamma_{423} = -X.$$

In tetrad form the Maxwell equations are written with $z = 0$:

$$(F_{12} + F_{34})_{,1} - 2F_{31,4} = 0,$$
$$(F_{12} + F_{34})_{,2} - 2(F_{12} + F_{34})X = 0,$$
$$(F_{12} + F_{34})_{,3} + 2F_{31,2} - 2F_{31}X = 0, \tag{A11}$$
$$(F_{12} + F_{34})_{,4} = 0.$$

Since F_{12} is pure imaginary and F_{34} is real, the field equations $R_{12}=-8\pi T_{12}$ and $R_{34}=-8\pi T_{34}$ become

$$|F_{12}|^2+|F_{34}|^2=0,$$

$$h_{,44}=-|F_{12}|^2-|F_{34}|^2$$

so that $h_{,44}=0=F_{12}=F_{34}$. Therefore the only nonzero components of F_{ab} are F_{31} and F_{32}. Study of canonical forms for F_{ab} reveals that F_{ab} is a *null* electromagnetic field. Furthermore, the only nonzero component of T_{ab} is $T_{33}=(4\pi)^{-1}F_{31}F_{32}=(4\pi)^{-1}|F_{31}|^2$. Hence the equations left are (letting $F_{31}\equiv F$)

$$h_{,44}=0=F_{,4}, \quad F_{,2}-XF=0. \tag{A12}$$

The equations $R_{31}=R_{32}=0$ come from the $d(\Gamma_{12}+\Gamma_{34})$ structure equations to give

$$h_{,4}\bar{X}-h_{,41}=C^{(2)} \quad (\psi_3 \text{ in N—P}),$$

$$h_{,4}X-h_{,42}=0.$$

The reality of h implies therefore that $C^{(2)}=0$, the Petrov type is [4], and radiation solutions are to be expected.

The $d\Gamma_{31}$ equation completes the set of field equations. These are given by

$$(hX-h_{,2})_{,4}=0,$$

$$h_{,1}X-hX\bar{X}+(h\bar{X})_{,2}-h_{,12}=\tfrac{1}{2}R_{33}=-2|F|^2,$$

$$h_{,1}\bar{X}-h\bar{X}^2+(h\bar{X})_{,1}-h_{,11}=\tfrac{1}{2}C^{(1)} \quad (\psi_4 \text{ in N—P}).$$

The general relationships $\phi_{,ab}-\phi_{,ba}=\phi_{,m}(\Gamma^m{}_{ab}-\Gamma^m{}_{ba})$ are true in any tetrad system for any scalar ϕ. These correspond to the Lie brackets $[\mathbf{e}_a,\mathbf{e}_b]=C^m{}_{ab}\mathbf{e}_m$ on the basis fields $\{\mathbf{e}_a\}$. Applying this to $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_4$ and using (A10), we find that

$$[\mathbf{e}_1,\mathbf{e}_2]=[\mathbf{e}_1,\mathbf{e}_4]=[\mathbf{e}_2,\mathbf{e}_4]=0; \tag{A13}$$

i.e., $\phi_{,12}=\phi_{,21}$, $\phi_{,14}=\phi_{,41}$, and $\phi_{,24}=\phi_{,42}$. In further applying this to the function Y in the metric one obtains algorithms for derivatives of X:

$$X_{,1}=X\bar{X}, \quad X_{,2}=X^2, \quad X_{,4}=0.$$

The simplifications above reduce the set of field equations for the expansion-free Kerr—Schild case with electromagnetism to

$$h_{,44}=0, \tag{A14a}$$

$$h_{,4}X-h_{,24}=0=h_{,4}\bar{X}-h_{,14}, \tag{A14b}$$

$$h_{,12}-h_{,1}X-h_{,2}X=-2F\bar{F}, \tag{A14c}$$

$$F_{,2}-XF=0=F_{,4}, \tag{A14d}$$

with the [−] case excluded through the constraint $C^{(1)}\neq 0$; i.e.,

$$h_{,11}-2h_{,1}\bar{X}\neq 0. \tag{A15}$$

The only relationships to come from the Bianchi identities turn out to be

$$C^{(1)}{}_{,2}-C^{(1)}X+R_{33,1}-R_{33}\bar{X}=0,$$

$$C^{(1)}{}_{,4}=0=R_{33,4},$$

which are all identically satisfied.

APPENDIX B: THE GAUGE CONDITIONS FOR FLAT SPACE

In the $\{\alpha,\bar{\alpha},\rho,v\}$ coordinates the general solution for the function h in the metric is given by the real-valued functions

$$h=av+g, \quad a=a(\alpha,\bar{\alpha},\rho), \quad g=g(\alpha,\bar{\alpha},\rho). \tag{B1}$$

Furthermore, $a=rA(\rho)$, where $A(\rho)$ is arbitrary and $r^{-1}=1-\bar{Y}'\alpha-Y'\bar{\alpha}$. Thus far the relations (A14a) and (A14b) are the ones satisfied.

Consider the cases with *no* electromagnetism present ($F=0$). Then the equation (A14c) is the condition that

$$(gr^{-1})_{\alpha\bar{\alpha}}=0. \tag{B2}$$

Hence, $gr^{-1}=S(\alpha,\rho)+\bar{S}(\bar{\alpha},\rho)$, where S is arbitrary. However, there will be a large amount of repetition of solutions because of a gauge condition implicit in (A15); i.e., the set of all functions $\hat{g}$ for which

$$\hat{g}_{\alpha\alpha}-2\hat{g}_{\alpha}\bar{X}=0 \quad (\Leftrightarrow C^{(1)}=0) \tag{B3}$$

give a solution $h=av+\hat{g}$ which is necessarily flat space, completely independent of "a". [When inserting $h=av+g$ into (A15) one finds that the "av" terms cancel each other identically in all cases, leaving only a constraint on g.] It would make things much simpler if somehow one might "divide out" these flat-space solutions.

The condition(B3) is the gauge condition for flat space. The general solution for all such functions $\hat{g}$ satisfying (B2) and (B3) is

$$\hat{g}=r[K(\rho)\alpha+\bar{K}(\rho)\bar{\alpha}+L(\rho)], \tag{B4}$$

where K, L are arbitrary and L is real. This works in a vacuum ($F=0$), but it also holds true in the present electromagnetic case. This is most easily seen by observing that (A14c) is the differential equation

$$(gr^{-1})_{\alpha\bar{\alpha}}=-2F\bar{F}. \tag{B5}$$

Hence, any function $\hat{g}$ added to g will contribute nothing to the field equation (B5) since $(\hat{g}r^{-1})_{\alpha\bar{\alpha}}=0$.

In summary we have shown that the metric

$$ds^2=2\epsilon^1\epsilon^2+2\epsilon^3\epsilon^4$$

$$=2(d\alpha+vXr^{-1}d\rho)(d\bar{\alpha}+v\bar{X}r^{-1}d\rho)+2r^{-1}d\rho dv$$

$$+2r^{-1}(Av+K\alpha+\bar{K}\bar{\alpha}+L)d\rho^2$$

is a representation of *flat space* (where A, K, L are arbitrary functions of the coordinate ρ). It illustrates a peculiarity of these expansion-free wave solutions in that "the" flat background is by no means unique. In fact, a comparison of results in a vacuum with the linearized theory, even in the $X=0$ case, is doomed to frustration until one writes his particular metric in the correct coordinate system. (Such a coordinate system is usually *not* manifestly $\eta+l\otimes l$. See, for example, Misner, Thorne, and Wheeler[12].)

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