AIP | Journal of Mathematical Physics

Functional calculus for the symmetric multigroup transport operator

William Greenberg

Citation: Journal of Mathematical Physics **17**, 159 (1976); doi: 10.1063/1.522871
View online: http://dx.doi.org/10.1063/1.522871
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/17/2?ver=pdfcov
Published by the AIP Publishing

Functional calculus for the symmetric multigroup transport operator

William Greenberg

Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061
(Received 21 July 1975)

A rigorous treatment of the symmetric multigroup transport equation is given by developing the functional calculus for the transport operator. Von Neumann spectral theory is applied to nonorthogonal cyclic subspaces, and the isometries onto $C(N)$ are explicitly evaluated.

Hangelbroek and Larsen and Habetler have independently provided rigorous techniques for solving the time independent one speed linear transport equation.[1,2] While the Larsen—Habetler approach has the distinct advantage of demonstrating, as its central result, that the transport operator is spectral, when this result is already evident, as for example, with a self-adjoint kernel, the Hangelbroek approach appears to provide a most effective setting for understanding the underlying properties of the transport operator. In fact, Hangelbroek has succeeded in showing that the Wiener—Hopf factorization, which has been used as a basis for extending the solution of the full range problem to the half range, can be derived from a study of projections in the representation space of the full range theory.[3]

Recently, Zweifel has extended the Larsen—Habetler technique to the multigroup transport equation.[4,5] In this article we wish to show that for a symmetric kernel the functional calculus can be developed for the multigroup as by Hangelbroek for the one-speed equation.

Since it will be necessary to evaluate the isometries between subspaces of the solution space and the representation spaces explicitly, the von Neumann spectral theory will be applied to nonorthogonal cyclic subspaces. The subcritical case, $C < S$, is considered in detail first, with extensions in the last section to more general kernels.

1. THE ALGEBRA GENERATED BY $A^{-1}u$

We consider the Hilbert space $\mathcal{H} = \oplus_{i=1}^{n} \mathcal{L}^2(I)$, the direct sum of n copies of $\mathcal{L}^2(I)$, where I is the real interval $[-1,1]$. For $1 \leq i \leq n$, let $e_i \in \mathcal{H}$ be the zero function in each $\mathcal{L}^2(I)$ except the ith copy, where it is the unit constant function. A vector $\psi \in \mathcal{H}$ will be written $\psi = \{\psi_i\}_{i=1}^{n}$ with $(\psi, \phi)_{\mathcal{H}} = \sum_{i=1}^{n} \int_{-1}^{1} du\, \psi_i(u)\, \bar{\phi}_i(u)$.

Let S be a positive, diagonal $n \times n$ matrix, C a real symmetric matrix, and assume for simplicity that $S - C > 0$. Throughout we will write σ_i for S_{ii}. Define the orthogonal projection $P: \mathcal{H} \to \mathcal{H}$ by

$$P\phi = \tfrac{1}{2} \sum_{i=1}^{n} (\phi, e_i)_{\mathcal{H}}\, e_i,$$

and the bounded operators $A: \mathcal{H} \to \mathcal{H}$ and $M: \mathcal{H} \to \mathcal{H}$ by

$$A = S - CP,$$

$$(M\phi)(u) = u\phi(u), \quad u \in I.$$

Noting that P commutes with every constant matrix, the inverse of A is computed to be

$$A^{-1} = S^{-1} + S^{-1}C(S-C)^{-1}P.$$

Write $\mathcal{K}$ for the linear space $\mathcal{H}$ with inner product

$$\{\phi, \psi\} = (A\phi, \psi)_{\mathcal{H}}. \tag{1}$$

On $\mathcal{K}$, A is still positive, and additionally, $B = A^{-1}M$ is self-adjoint. Let $\mathcal{A}$ be the C^* algebra generated by B on $\mathcal{K}$, $\mathcal{C}(N)$ the C^* algebra of continuous, $\mathbb{C}$-valued functions on the spectrum $N = \sigma(B)$ of B with uniform norm, $P(\mathcal{H})$ [resp. $P(\mathcal{A})$, $P(\mathcal{C})$] the subspace of polynomials in each component in $\mathcal{K}$ [resp. $\mathcal{A}, \mathcal{C}$], and E the unit constant function in $\mathcal{C}(N)$. For each integer i, $1 \leq i \leq n$, define $M_i \subset \mathcal{K}$ by $M_i = P(\mathcal{A})e_i$.

Lemma 1: If $T \in P(\mathcal{A})$, then $Te_i \in P(\mathcal{K})$ for each i, and the degree satisfies:

$$\deg T = \deg(Te_i, e_i)_{\mathcal{H}},$$

$$\deg T \geq \deg(Te_i, e_j)_{\mathcal{H}} + 2, \quad j \neq i, \quad \deg T > 1.$$

Denoting by L.S. the linear manifold spanned, we have

$$M_i \cap \{\text{L.S.} \bigcup_{j \neq i} M_j\} = \phi,$$

$$\text{L.S.} \bigcup_{j=1}^{n} M_j = P(\mathcal{K}).$$

Proof: If $T \in P(\mathcal{A})$ and $\deg T = n$, then

$$BTe_i = \sigma_i^{-1} uTe_i + S^{-1}C(S-C)^{-1}P(uTe_i),$$

and the first part follows by induction, with

$$Be_i = u\sigma_i^{-1}e_i, \quad B^2 e_i = u^2\sigma_i^{-2}e_i + \tfrac{2}{3}S^{-1}C(S-C)^{-1}S^{-1}e_i.$$

For the last part it is sufficient that, for each i, all polynomials $\varphi \in \mathcal{K}$ with $(\varphi, e_j) = 0$ unless $j = i$ can be obtained. Assume, by induction, that all such polynomials for all i and $\deg\varphi \leq N$ can be obtained, and assume $\psi = (a_{N+1}u^{N+1} + P_N(u))e_i$. If

$$a_{N+1}\sigma_i^{N+1}B^{N+1}e_i = a_{N+1}u^{N+1}e_i + Q_N(u)e_i + \sum_{j \neq i} R_{N-1,j}(u)e_j,$$

let

$$\sum_j S_{N,j}(B)e_j = Q_N(u)e_i,$$

$$\sum_j T_{N-1,j}(B)e_j = \sum_{j \neq i} R_{N-1,j}(u)e_j,$$

$$\sum_j U_{N,j}(B)e_j = P_N(u)e_i,$$

for polynomials Q_N, $R_{N-1,j}$, $S_{N,j}$, $T_{N-1,j}$, and $U_{N,j}$ of indicated degree. Then

$$a_{N+1}\sigma_i^{N+1}B^{N+1}e_i+\sum_j[U_{N,j}(B)-S_{N,j}(B)-T_{N-1,j}(B)]e_j=\psi.$$

For each i, define $\pi_i:\mathcal{A}\to\bar{M}_i$ by π_i: $T\to Te_i$ and let κ: $\mathcal{A}\to\mathcal{C}(N)$ be the Gelfand isomorphism. If $T\in\mathcal{A}$ and $\pi_i(T)=0$, then $T\mathcal{A}e_i=\mathcal{A}Te_i=0$ so $T=0$ on $\mathcal{A}e_i$. Since $T=0$ on $M_i^\perp$, this proves:

Lemma 2: π_i is a bounded linear isomorphism of $\mathcal{A}$ onto $\mathcal{A}e_i$.

Define F_i: $\mathcal{K}\to\mathcal{C}(N)$ by $F_i=\kappa\pi_i^{-1}$ on $\mathcal{A}e_i$ and $F=0$ on $M_i^\perp$. Let $\mathcal{C}_i(N)$ be the linear space $\mathcal{C}(N)$ with inner product

$$(\varphi,\psi)_i=\{F_i^{-1}\varphi,F_i^{-1}\psi\}.$$

Then F_i is an isometric linear isomorphism on $\mathcal{A}e_i$, and $F_i(e_i)=E$.

Lemma 3: For a unique positive Lebesgue–Stieltzes measure σ_i,

$$(\varphi,\psi)_i=\int_N\varphi(\nu)\bar{\psi}(\nu)\,d\sigma_i(\nu),$$

for $\varphi,\psi\in P(\mathbb{C})$.

Proof: Defining the linear functional l_i: $\varphi\to(\varphi,E)_i$ for $\varphi\in P(\mathbb{C})$, the estimate

$$|(\varphi,E)_i|=|\{\kappa^{-1}\varphi e_i,e_i\}|\le\|\kappa^{-1}\varphi\|\{e_i,e_i\}$$
$$=\sup_{\nu\in N}|\varphi(\nu)|\{e_i,e_i\}$$

proves $l_i\in P(\mathbb{C})^*$. Hence,

$$(\varphi,\psi)_i=\{\varphi(B)e_i,\psi(B)e_i\}=\{\bar{\psi}(B)\varphi(B)e_i,e_i\}$$
$$=(\bar{\psi}\varphi,E)_i=\int_N\varphi(\nu)\bar{\psi}(\nu)\,d\sigma_i(\nu).$$

If φ is positive, $\varphi(B)$ is a positive operator, by the spectral theorem for self-adjoint operators, so

$$\int_N\varphi(\nu)\,d\sigma_i(\nu)=(\varphi,E)_i=\{\varphi(B)e_i,e_i\}\ge0.$$

By the Lemma, F_i extends to an isometric linear isomorphism of $\bar{M}_i$ onto $\mathcal{L}^2(N,\sigma_i)$. Write $\hat{M}$ for the bounded operator on $\mathcal{L}^2(N,\sigma_i)$,

$$(\hat{M}\varphi)(\nu)=\nu\varphi(\nu),\quad\varphi\in\mathcal{L}^2(N,\sigma_i).$$

Corollary: For any $\varphi\in\mathcal{K}$, $F_i(B\varphi)=\hat{M}F_i(\varphi)$.

For $F_i(B\varphi)=\nu\varphi(\nu)$ if $\varphi=\psi(B)e_i\in\bar{M}_i$, and $M_i^\perp$ is an invariant subspace of B.

2. SOLUTION OF THE EQUATION

Let $N_\pm$ represent the nonnegative/nonpositive subsets of N, and $P_\pm$ the orthogonal projections of $\mathcal{L}^2(N,\sigma)$ onto $\mathcal{L}^2(N_\pm,\sigma)$, viewed as subspaces of $\mathcal{L}^2(N,\sigma)$, $P_\pm\mathcal{L}^2(N_\mp,\sigma)=0$. The solution of the n-group isotropic nonhomogeneous linear transport equation is provided by the following theorem.

Theorem 1: Let $q:\mathbb{R}\to\mathcal{K}$ be uniformly Hölder continuous, and $(q(x),q(x))_H$ uniformly bounded. Consider $A^{-1}q=\sum_{i=1}^n q_i$ with $q_i:\mathbb{R}\to\bar{M}_i$. Define

$$\varphi_{i\pm}(x)=\int_{-\infty}^x\exp[-(x-\xi)\nu^{-1}](P_\pm F_iq_i)(\xi)\,d\xi.\tag{2}$$

Then

$$\psi(x)=M^{-1}A\sum_{i=1}^nF_i^{-1}(\varphi_{i+}+\varphi_{i-})$$

is the unique solution of the transport equation

$$\frac{d}{dx}M\psi(x)=A\psi(x)+q(x),\tag{3}$$

satisfying $(M\psi(x),\psi(x))_H$ uniformly bounded.

The proof of the theorem is an immediate consequence of the following two lemmas.

Lemma 4: $\psi_i:\mathbb{R}\to\mathcal{K}$ is a solution of

$$B\frac{d}{dx}\psi_i(x)=-\psi_i(x)+q_i(x),\tag{4}$$

satisfying $(M\psi_i(x),\psi_i(x))_H$ uniformly bounded if and only if $\varphi_i(x)=F_iA^{-1}M\psi_i$ is a solution of

$$\frac{d}{dx}\varphi_i(x)=-\hat{M}^{-1}\varphi_i(x)+F_iq_i(x),\tag{5}$$

satisfying $(\varphi_i(x),\varphi_i(x))_i$ uniform bounded.

Lemma 5[2]: Suppose $g:\mathbb{R}\to X$ is a uniformly Hölder continuous function from $\mathbb{R}$ to the Banach space X, and $\|g(x)\|$ is uniformly bounded. If $\beta<0$ and $-(T+\beta)$ is the generator of a bounded holomorphic semigroup,[6] then

$$\varphi(x)=\int_{-\infty}^x\left\{\frac{1}{2\pi i}\int\Gamma\exp[\lambda(x-\xi)]\frac{1}{-T+\lambda}\,d\lambda\right\}g(x)\,dx$$

for a contour Γ about $\sigma(T)$, is continuously differentiable in $\mathbb{R}$, and is the unique solution of

$$\frac{d}{dx}\varphi(x)=-T\varphi(x)+g(x),$$

satisfying $\|\varphi(x)\|$ uniformly bounded.

Lemma 4 is established immediately by the isomorphisms F_i, and Lemma 5, the generalization of a result from Hille-Yosida semigroup theory, is proved in Ref. 2.

If $\{\psi_i\}$ satisfies Eq. (4), then $\psi=\sum_{i=1}^n\psi_i$ is clearly a solution of the transport equation (3), so the problem reduces to a consideration of Eq. (5). Since $\mathcal{L}^2(N_\pm,\sigma)$ are invariant subspaces of $\hat{M}$, it is sufficient to solve the equations on each of the subspaces. But on $\mathcal{L}^2(N_\pm,\sigma)$, the restrictions of $\hat{M}^{-1}$ are semi-bounded self-adjoint operators, and Lemma 5 is applicable. Equation (2) results trivially from evaluating the contour integral in Lemma 5 for $T=1/\nu$.

3. EVALUATION OF THE ISOMORPHISMS F_i

It remains to compute the maps F_i and F_i^{-1} explicitly in order to apply Theorem 1, and it is desirable as well to derive the measure σ.

Lemma 6: Let ν_l denote the poles of $\Lambda(\lambda)^{-1}$ for $l=1,\ldots,m$ and $R(\nu_l)$ the residues. If $\varphi\in P(\mathbb{C})$, then

$$(F_i^{-1}\varphi)(\mu)=\sum_{l=1}^m\varphi(\nu_l)(\nu_lI-\mu S^{-1})^{-1}R(\nu_l)e_i+(1/2\pi i)$$
$$\times P\int_{-1}^1d\nu\varphi(\nu)(\nu I-\mu S^{-1})^{-1}(\Lambda^{-1}(\nu)^--\Lambda^{-1}(\nu)^+)e_i$$
$$+\tfrac{1}{2}\tau_S(\varphi\Lambda^{-1+}+\varphi\Lambda^{-1-})(\mu)e_i,$$

where $\Lambda^{-1\pm}$ are the boundary values of Λ^{-1} along the cut I, and for any function $W:\mathbb{R}\to\mathcal{L}(\mathcal{K})$ from $\mathbb{R}$ to bounded operators on $\mathcal{K}$,

$(e_k, (\tau_S W)(x)\xi)_H = (e_k, W(x/\sigma_k)\xi)_H,$

for all $1 \le k \le n$ and $\xi \in \mathcal{K}$.

Proof: Since $B - \lambda I = \Lambda(\lambda)(S^{-1}u - \lambda I)$, F_i^{-1} can be evaluated on analytic functions by a contour integral of the resolvent applied to the constant vectors e_i, i.e.,

$$(F_i^{-1}\varphi)(\mu) = (\varphi(B)e_i)(\mu) = (1/2\pi i)\int_\Gamma d\lambda\, \varphi(\lambda)(\lambda I - S^{-1}\mu)^{-1}\Lambda^{-1}(\lambda)e_i. \quad (7)$$

Computation of the contour integral is routine, contributions rising from the same terms as in the one group case.

For $\varphi, \psi \in \mathcal{K}$, let

$$[\varphi, \psi] = \sum_{k=1}^n \varphi_k \psi_k.$$

Lemma 7: If $\varphi \in P(\mathcal{A})e_i$ and $\Omega: \mathbb{C} \to \mathcal{L}(\mathcal{K})$ satisfies

(a) $[\Lambda^{-1}(\lambda)e_i, (\tau_S^{-1}\Omega(\lambda))e_i] = 1$ on a neighborhood of N,

(b) $[e_k, \Omega(\lambda)e_i]$ analytic on the complement of I,

then

$$(F_i\varphi)(\nu) = (1/2\pi i)\int_\Gamma d\lambda [\varphi(\lambda), (\lambda I - \nu S)^{-1}\Omega(\lambda)e_i], \quad \nu \in N$$

for a closed contour Γ about N.

Proof: If Γ' is a contour about N inside Γ,

$$\begin{aligned}\hat\varphi(\nu) &= (1/2\pi i)\int_{\Gamma'} d\lambda' [\varphi(\lambda'), (\lambda' I - \nu S)^{-1}\Omega(\lambda')e_i] \\ &= (1/2\pi i)\int_\Gamma d\lambda (F\varphi)(\lambda)(1/2\pi i)\sum_{k=1}^n \int_{\Gamma'} d\lambda'(1/\sigma_k\lambda - \lambda') \\ &\quad \times [S\Lambda^{-1}(\lambda)e_i, e_k][(\lambda' I - \nu S)^{-1}\Omega(\lambda')e_i, e_k],\end{aligned}$$

using Lemma 6. From property (b) and the known analytic behavior of $\Lambda(\lambda)$, the Γ' integral can be evaluated

$$\begin{aligned}\hat\varphi(\nu) &= (1/2\pi i)\sum_{k=1}^n \int_\Gamma d\lambda (F\varphi)(\lambda)[S\Lambda^{-1}(\lambda)e_i, e_k] \\ &\quad \times [(S\lambda - \nu S)^{-1}(\tau_S^{-1}\Omega)(\lambda)e_i, e_k] \\ &= (1/2\pi i)\int_\Gamma d\lambda (F\varphi)(\lambda)(1/\lambda - \nu)\sum_{k=1}^n [\Lambda^{-1}(\lambda)e_i, e_k] \\ &\quad \times [(\tau_S^{-1}\Omega)(\lambda)e_i, e_k) = (F\varphi)(\nu).\end{aligned}$$

Corollary: $\Omega(\lambda)$ defined by

$$\Omega = (\tau_S \Lambda)^t,$$

satisfies Lemma 7.

Proof: For any $\lambda \in \mathbb{C}$,

$$\begin{aligned}[\tau_S^{-1}(\tau_S\Lambda)^t(\lambda)e_i, e_k] &= [(\tau_S\Lambda)^t(\lambda\sigma_k)e_i, e_k] \\ &= [\tau_S\Lambda(\lambda\sigma_k)e_k, e_i] = [\Lambda(\lambda)e_k, e_i].\end{aligned}$$

The subspace spanned by $\{e_j\}_{j=1}^n$ is an invariant subspace of $\Lambda(\lambda)$, and the restriction $\Lambda_c(\lambda)$ of Λ is a constant matrix. Then for $\lambda \notin N$,

$$\sum_k [\Lambda(\lambda)^{-1}e_i, e_k][\Lambda(\lambda)e_k, e_i] = \sum_k \Lambda_c(\lambda)_{ik}\Lambda_c(\lambda)^{-1}_{ki} = I_{ii}.$$

Also,

$$\begin{aligned}\Lambda_c(\lambda)_{ik} &= \left[\left(I + D\int_{-1}^1 d\mu \frac{\sigma_k\mu}{\mu - \lambda\sigma_k}\right)e_k, e_i\right] \\ &= \delta_{ik} + D_{ik}\int_{-1}^1 d\mu \frac{\sigma_k\mu}{\mu - \lambda\sigma_k}\end{aligned}$$

where $D = S^{-1}C(S - C)^{-1}$, so

$$(\Lambda_c\ (\lambda/\sigma_k)^t)_{ki} = \delta_{ik} + \sigma_k D_{ik}\int_{-1}^1 (\mu/\mu - \lambda)\, d\mu,$$

which is analytic for $\lambda \in \mathbb{C}/I$.

Corollary: If $\varphi \in P(\mathcal{A})e_i$, then

$$\begin{aligned}(F_i\varphi)(\nu) &= \tfrac{1}{2}[\tau_S^{-1}\varphi(\nu), \tau_S^{-1}(\Omega^+(\nu) + \Omega^-(\nu))e_i]\delta_{\nu\ne\nu_l} \\ &\quad + (1/2\pi i)P\int_{-1}^1 d\lambda[\varphi(\lambda), (\lambda I - \nu S)^{-1}(\Omega^-(\lambda) - \Omega^+(\lambda))e_i],\end{aligned}$$

where

$$\delta_{\nu\ne\nu_l} = \begin{cases} 1, & -1 < \nu < 1 \\ 0, & \nu = \nu_l \end{cases}$$

and $(e_k, \tau_S^{-1}\varphi(\mu)e_i) = \varphi_k(\sigma_k\nu)$.

To compute σ conveniently, it is helpful to collect some properties of Λ.

Lemma 8: For $\lambda \in \mathbb{C}$,

$$\Lambda_c(\lambda)^- - \Lambda_c(\lambda)^+ = 2\pi i\lambda DSW_\lambda, \quad (8)$$

where

$$(W_\lambda)_{kk} = \begin{cases} 1, & |\lambda| \le 1/\sigma_k \\ 0, & |\lambda| > 1/\sigma_k \end{cases}$$

is diagonal. For $\lambda \notin N$,

$$\Lambda^{-1}(\lambda) = I - (I + DT_\lambda)^{-1}DPM(S^{-1}M - \lambda I)^{-1}, \quad (9)$$

where

$$(T_\lambda)_{kk} = \tfrac{1}{2}\int_{-1}^1 (\sigma_k\mu/\mu - \lambda\sigma_k)\, d\mu$$

is diagonal. Hence,

$$\Lambda^{-1}(0)^\pm = I - S^{-1}CP. \quad (10)$$

Proof: Since $\Lambda(\lambda) = I + DP\mu(S^{-1}\mu - \lambda I)^{-1}$, $\Lambda_c(\lambda) = I + DT_\lambda$ and $I - DT_\lambda$ is invertible if $\lambda \notin N$. Multiplying Eq. (9) by Λ and suitably collecting terms proves the operator inversions:

$$\begin{aligned}\Lambda\Lambda^{-1} &= I - \{(I + DT_\lambda)^{-1}D + D - DT_\lambda(I + DT_\lambda)^{-1}D\}P_\mu(S^{-1}\mu - \lambda I)^{-1} \\ &= I - \{-(I + DT_\lambda)(I + DT_\lambda)^{-1} + I\}DP\mu(S^{-1}\mu - \lambda I)^{-1} \\ &= I = \Lambda^{-1}\Lambda.\end{aligned}$$

Since

$$\Lambda^{-1}(\lambda) = I - (I + DT_\lambda)^{-1}DSP - (I + DT_\lambda)^{-1}DS\lambda P(S^{-1}\mu - \lambda I)^{-1}$$

and $T_0 = S$,

$$\begin{aligned}\Lambda^{-1}(0) &= I - (I + S^{-1}C(S - C)^{-1}S)^{-1}S^{-1}C(S - C)^{-1}SP \\ &= I - (S^{-1}(S - C)C^{-1}S + I)^{-1}P.\end{aligned}$$

Finally,

$$\Lambda(\lambda) = I + DPSS^{-1}\mu(S^{-1}\mu - \lambda I)^{-1} = I + DSP(S^{-1}\mu - \lambda I)^{-1},$$

and thus,

$$\Lambda_c(\lambda)^{\pm}_{jk} = (I + DS + \lambda DSP \int_{-1}^{1} d\mu (S^{-1}\mu - \lambda I)^{-1} \pm \lambda DS\pi i \delta(|\lambda| \leq 1/\sigma_k))_{jk}$$

Theorem 9: For each $1 \leq j \leq n$, and $1 \leq i \leq n$,

$$\overline{(F_i e_j)}(\nu)\, d\sigma(\nu) = \begin{cases} \left(\frac{1}{\nu}(S-C)CS^{-1}D(\nu)e_i, e_j\right)_{H}, & \nu = \nu_l, \\ ((S-C)C^{-1}S\Lambda^{-1}(\nu)^{-}DSW_\nu\Lambda^{-1}(\nu)^{+}e_i, e_j)_{H}\, d\nu, & -1 < \nu < 1. \end{cases}$$

Proof: Since F_i^{-1} is an isometry, for all $\varphi \in P(\mathbb{C})$,

$$\int_N \varphi(\nu) F_i e_j(\nu)\, d\sigma(\nu) = (\varphi, F_i e_j)_N = \{F_i^{-1}\varphi, F_i^{-1}F_i e_j\}$$
$$= \{\varphi(B)e_i, e_j\}$$
$$= (1/2\pi i) \int_\Gamma d\lambda\, \varphi(\lambda)\{(\lambda I - B)^{-1}e_i, e_j\}.$$

Rewrite Λ as in the proof of Lemma 8 to obtain:

$$\{(\lambda I - B)^{-1}e_i, e_j\} = \{(\lambda I - S^{-1}\mu)^{-1}\Lambda(\lambda)^{-1}e_i, e_j\}$$
$$= ((\lambda I - S^{-1}\mu)^{-1}\Lambda(\lambda)^{-1}e_i, Ae_j)_{H}$$
$$= (1/\lambda)(P\lambda(\lambda I - S^{-1}\mu)^{-1}\Lambda(\lambda)^{-1}e_i, (S-C)e_j)$$
$$= (1/\lambda)((C^{-1}S - I + P - (C^{-1}S - I)\Lambda(\lambda))\Lambda(\lambda)^{-1}e_i, (S-C)e_j)_{H}$$
$$= (1/\lambda)((S-C)C^{-1}S\Lambda(\lambda)^{-1}e_i, e_j) - (1/\lambda)((S-C)(C^{-1}S - I)e_i, e_j).$$

Therefore, the integral can be expanded. The contour integration is completely analogous to that in the proof of Lemma 7, i.e.,

$$\int_N \varphi(\nu)\overline{F_i e_j}(\nu)\, d\sigma(\nu)$$
$$= \frac{1}{2\pi i}\int_\Gamma d\lambda \frac{\varphi(\lambda)}{\lambda}((S-C)C^{-1}S\,\Lambda^{-1}(\lambda)e_i, e_j)$$
$$- \frac{1}{2\pi i}\int_\Gamma d\lambda \frac{\varphi(\lambda)}{\lambda}((S-C)(C^{-1}S - I)e_i, e_j)$$
$$= \sum_l \frac{\varphi(\nu_l)}{\nu_l}((S-C)C^{-1}SR(\nu_l))_{ji} + \frac{1}{2\pi i}P\int_{-1}^{1} d\nu \frac{\varphi(\nu)}{\nu}$$
$$\times ((S-C)C^{-1}S(\Lambda^{-1}(\nu)^{-} - \Lambda^{-1}(\nu)^{+}))_{ji}$$
$$+ \tfrac{1}{2}\varphi(0)((S-C)C^{-1}S(\Lambda^{-1}(0)^{+}$$
$$+ \Lambda^{-1}(0)^{-})_{ji} - ((S-C)(C^{-1}S - I))_{ji}\varphi(0). \quad (11)$$

Equation (10) of Lemma 8 gives

$$\tfrac{1}{2}(S-C)C^{-1}S(\Lambda^{-1}(0)^{+} + \Lambda^{-1}(0)^{-}) = (S-C)(C^{-1}S - I)$$

so the last two terms of Eq. (11) cancel. Moreover,

$$\Lambda^{-1}(\nu)^{-} - \Lambda^{-1}(\nu)^{+} = -\Lambda^{-1}(\nu)^{-}(\Lambda(\nu)^{-} - \Lambda(\nu)^{+})\Lambda^{-1}(\nu)^{+}$$
$$= 2\pi i\nu\Lambda^{-1}(\nu)^{-}DSW_\nu\Lambda^{-1}(\nu)^{+}$$

from Eq. (8). Since $P(\mathbb{C}) \subset \mathcal{L}^2(N, \sigma)$ densely, the integrands are equal.

Note that Theorem 9 gives $\sigma(\nu)$, since $F_i e_i = e_i \equiv 1$.

4. EXTENSION TO SELF-ADJOINT KERNEL

The functional calculus approach which has been developed above can be extended in toto to the case of the general symmetric kernel: S diagonal, $C = C^*$, $\det(S - C) \neq 0$ (or C similar to a self-adjoint matrix, S invariant), by considering H as a Pontrjagin space with indefinite metric defined by Eq. (1). More precisely, for every pair of imaginary eigenvalues, a two-dimensional invariant subspace is split off from H, and for each real eigenvalue such that $(A\varphi, \varphi)_H < 0$, a one-dimensional eigenspace is removed. On the remainder, B is similar to a self-adjoint operator. This decomposition is due in general to Krein's Invariant Subspace theorem.

In the case of the 2-group, the assumption $C = C^*$ can also be dropped. For it is evident that C can always be symmetrized by a similarity transformation which leaves the diagonal matrix S invariant.

For details on these Pontrjagin techniques, see Refs. 3 and 7.

ACKNOWLEDGMENT

I am indebted to J. C. T. Pool, Associate Director of the Applied Mathematics Division, Argonne National Laboratory, for his kind hospitality during my visit, at which time part of this research was carried out.

[1] E.W. Larsen, and G.J. Habetler, Commun. Pure Appl. Math. **26**, 525 (1973).

[2] R. Hangelbroek, "A Functional Analytic Approach to the Linear Transport Equation," thesis, Groningen (1973).

[3] R. Hangelbroek, "Hilbert Space Approach to the Linear Transport Equation for Multiplying Media in Slab Geometry," in *The Fourth Conference in Transport Theory*, to appear.

[4] S. Sancaktar and P. Zweifel, "Multigroup Neutron Transport: Full-range," preprint.

[5] R. Bowden, S. Sancaktar, and P. Zweifel, "Multigroup Neutron Transport: Half-range," preprint.

[6] T. Kato, *Perturbation Theory For Linear Operators* (Springer, New York, 1966).

[7] W. Greenberg, "Pontrjagin Structure of the Transport Operator," in *The Fourth Conference in Transport Theory*, to appear.