The interaction function and lattice duals

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An interaction function is defined for lattice models in statistical mechanics. A correlation function expansion is derived, giving a direct proof of the duality relations for correlation functions.

A general theory of duality transformations between pairs of classical spin-½ lattice models has been developed by Gruber and Merlini and independently by Wegner. The theory of Gruber and Merlini is constructive, providing explicitly a family of "dual" lattices and Hamiltonians for any given spin-½ system. These duals are exact, all requisite boundary terms being provided for, which is necessary in considerations of correlation functions below criticality.

We define in this article the interaction functions $u_{A,B}$ of lattice duals $G$ and $G^*$, and express them in terms of correlation functions. This gives an easy derivation of the relationship between correlation functions of a lattice and its duals. The notation in this article, while somewhat different from Ref. 1 and some current usage, has the advantage, in addition to simplifying the derivation of generalizing to higher spin lattices.

The reader is referred to Ref. 1 for details on the construction of dual spin-½ lattices.

1. DUAL LATTICES

We suppose we are given a finite set $\Lambda$ of lattice sites in a $v$-dimensional space, along with a Hamiltonian $H$ defined on the configuration of $\Lambda$. It is convenient to take as the configuration space the group $P_{2}(\Lambda)$ of functions from $\Lambda$ to $Z_2$, the integers modulo 2, with group multiplication

$$fg(\lambda)=f(\lambda)+g(\lambda) \mod 2.$$ Consider $H$ as a function $H:P_{2}(\Lambda)\rightarrow C$, its Fourier decomposition

$$H(g)=\sum_{\lambda\in\Lambda} H_{\lambda} g(\lambda), \quad g \in P_{2}(\Lambda)$$
in terms of the elements of the character group $\hat{G}$ of $G = P_{2}(\Lambda)$ is just the usual decomposition of $H$ into a sum of products of spin matrices, since the characters of $G$ are products of characters of $Z_2$. Define the set of nonzero interactions

$$B=\{\sigma \in G | H_{\sigma} \neq 0\}.$$ Dual lattices are constructed with the set $B$. Defining $P_{2}(B)$ as the group of functions from $B$ to $Z_2$, let $p$ be the group homomorphism

$$p:P_{2}(B) \rightarrow G$$ by $$p(f)=\prod_{\sigma \in B} f(\sigma)$$ and denote its kernel by $K_{p}$. Suppose $X$ is any set which generates $K_{p}$ as a group. Then $X$ defines a dual of $\Lambda$, with configuration space $G^*=P_{2}(X)$ and dual Hamiltonian $H^*$ defined as follows. Let

$$q:B-B^* \subset G^* \text{ by } q(\sigma)=h-h(\sigma)$$ for $\sigma \in B$ and $h \in X \subset P_{2}(B)$. $q(\sigma)$ is indeed a character on $G^*$, and these $q(\sigma)$ are to be the nonzero interactions of the dual. The coefficients $H_{\sigma}(q)$ are given by

$$H_{\sigma}(q)=\frac{1}{2} \beta \log \prod_{q \in B} \tanh \beta H_{q},$$

$$H_{\sigma}^*(q)={\sum_{q \in B}} H_{\sigma}(q)q(\sigma),$$ except near the boundary, where (1) must be used.

The partition functions $Z(\beta H)={\sum_{g \in G}} \exp(-\beta H(g))$ of $G$ and $Z(\beta H^*)={\sum_{g \in G^*}} \exp(-\beta H^*(g))$ of its duals $G^*$ are related then by

$$Z(\beta H)=\frac{N(K^*)}{N(G)} \prod_{\sigma \in B} [\sinh(-\beta H_{\sigma}) \cosh(-\beta H_{\sigma})]^{1/2} Z(\beta H^*)$$ where $N(S)$ is the cardinality of $S$, and $K^*$ is defined after Eq. (2).

2. THE INTERACTION FUNCTION

The correlation functions $\rho(\sigma)$ of $G$ are defined by

$$\rho(\sigma)=Z(\beta H)^{-1} \sum_{g} \exp(-\beta H(g)) \delta(g, \sigma), \quad \sigma \in G$$

with $H^*$ replacing $H$ for the correlation functions $\rho(\sigma^*)$ of $G^*$, $\sigma^* \in G^*$. Note that $\rho(\sigma)=0$ if $\sigma$ is not a product of elements of $B$. 1

Define the characteristic projection $t:G^* \rightarrow P_{2}(B)$ by

$$t(g^*) \sigma = \frac{1}{2} (1-\sigma(g^*)), \quad \sigma \in B^*.$$ The support of $t(g^*)$ is precisely those characters $\sigma \in B^*$ whose value at $g^*$ is $-1$. Now if the kernel and range of $t$ are denoted, respectively, by $K_{t}^*$ and $R_{t}^*$, then the map $Q:K_{p}-P_{2}(B)$ given by

$$Q(f)(\sigma)=f(\sigma), \quad \sigma \in B,$$ is a group isomorphism $K_{p} \rightarrow R_{t}^*$. In particular, $f \in K_{p}$,

$$f(\sigma)=\begin{cases} 1, & \sigma \in S, \\ 0, & \sigma \in Y-S, \end{cases}$$

and only if $Q(f) \in R_{t}^*$,

$$Q(f)(\sigma)=\begin{cases} 1, & \sigma \in S, \\ 0, & \sigma \in Y-S, \end{cases}$$

and then
\[ \prod_{a \in \mathcal{F}_1(1)} \tanh(-\beta H_a) = \prod_{\mathcal{F}(1)} \exp(2\beta H_a). \]

Let the symbol \( \sum_{Y, T} \) with \( Y, T \subset \mathcal{B} \) indicate that the summation [over \( f \in \mathcal{P}_2(\mathcal{B}) \), say] is to be restricted to \( f \) satisfying \( f(\sigma) = 0 \) if \( \sigma \in S \), \( f(\sigma) = 1 \) if \( \sigma \in T \). Then the interaction function \( u(\mathcal{A}, \mathcal{C}) \) is given by
\[ u(\mathcal{A}, \mathcal{C}) = \sum_{f \in \mathcal{P}_2(\mathcal{B})} \prod_{\mathcal{F}(1)} \exp(2\beta H_a) \]
for \( \mathcal{A}, \mathcal{C} \subset \mathcal{B}^* \).

We wish to evaluate \( u(\mathcal{A}, \mathcal{C}) \) in terms of the correlation functions of \( \mathcal{G}^* \). Thus, suppose \( \mathcal{Y} \) and \( \mathcal{W} \) are any disjoint subsets of \( \mathcal{B}^* \). Writing \( \mathcal{Y} \) for \( \mathcal{Y} = \{ \mathcal{Y} \} \), etc., obtain from (2):
\[ \left( \prod_{\mathcal{A}, \mathcal{B}} \sum_{\mathcal{F}(1)} \prod_{\mathcal{F}(1)} \exp(2\beta H_a) \right)^{-1} \left( \prod_{\mathcal{A}, \mathcal{B}} \sum_{\mathcal{F}(1)} \prod_{\mathcal{F}(1)} \exp(2\beta H_a) \right) = \sum_{\mathcal{F}(1)} \prod_{\mathcal{F}(1)} \exp(2\beta H_a) \]
for \( \mathcal{A} \subset \mathcal{B} \).

Now, expanding the product
\[ \prod_{\mathcal{A}, \mathcal{B}} \left( \prod_{\mathcal{F}(1)} \exp(2\beta H_a) \right) = \sum_{\mathcal{F}(1)} \prod_{\mathcal{F}(1)} \exp(2\beta H_a) \]
and similarly with \( \prod_{\mathcal{A}, \mathcal{B}} \left( \prod_{\mathcal{F}(1)} \exp(2\beta H_a) \right) = \sum_{\mathcal{F}(1)} \prod_{\mathcal{F}(1)} \exp(2\beta H_a) \), this becomes
\[ N(K^*) \sum \mathcal{L} \mathcal{C} = \sum_{\mathcal{F}(1)} \prod_{\mathcal{F}(1)} \exp(2\beta H_a) \]
for \( \mathcal{A}, \mathcal{B} \subset \mathcal{B} \).

Therefore, with a change in summation variable,
\[ \left( \prod_{\mathcal{A}, \mathcal{B}} \sum_{\mathcal{F}(1)} \prod_{\mathcal{F}(1)} \exp(2\beta H_a) \right)^{-1} \left( \prod_{\mathcal{A}, \mathcal{B}} \sum_{\mathcal{F}(1)} \prod_{\mathcal{F}(1)} \exp(2\beta H_a) \right) = \sum_{\mathcal{F}(1)} \prod_{\mathcal{F}(1)} \exp(2\beta H_a) \]
for all \( \mathcal{G} \subset \mathcal{B} \).

From the one-one correspondence between \( f \in \mathcal{P}_2(\mathcal{B}) \) with \( f(\sigma) = \bar{f}(f) \) and \( f' \in \mathcal{P}_2(\mathcal{B}) \) with
\[ f'(\sigma) = \begin{cases} f(\sigma), & \text{if } \sigma \notin \mathcal{Y}, \\ f(\sigma) + 1, & \text{if } \sigma \in \mathcal{Y}, \end{cases} \]
the expansion can be written as
\[ \left( \prod_{\mathcal{A}, \mathcal{B}} \sum_{\mathcal{F}(1)} \prod_{\mathcal{F}(1)} \exp(2\beta H_a) \right)^{-1} \left( \prod_{\mathcal{A}, \mathcal{B}} \sum_{\mathcal{F}(1)} \prod_{\mathcal{F}(1)} \exp(2\beta H_a) \right) = \sum_{\mathcal{F}(1)} \prod_{\mathcal{F}(1)} \exp(2\beta H_a) \]
for any \( \mathcal{Y} \subset \mathcal{B} \) such that \( \bar{\mathcal{Y}} = \mathcal{Y} \).

In the event that the duality map \( q \) is one-one, Eq. (4) simplifies to the path formula of Kadanoff and Ceva.\(^1\)

Injectivity of \( q \) is equivalent to requiring that the elements of \( K_\mathcal{G} \) separate the bonds \( \sigma \) of \( \mathcal{B} \), and is satisfied, for example, by a hexagonal Ising lattice with periodic boundary conditions, or with an external field at the boundary, but is not satisfied by this lattice with open boundary conditions.