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# Multigroup neutron transport. I. Full range*

R. L. Bowden and S. Sancaktar

*Department of Physics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061*

P. F. Zweifel†

*Department of Physics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061 and The Rockefeller University, New York, New York 10021*

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A functional analytic approach to the $N$-group, isotropic scattering, particle transport problem is presented. A full-range eigenfunction expansion is found in a particularly compact way, and the stage is set for the determination of the half-range expansion, which is discussed in a companion paper. The method is an extension of the work of Larsen and Habetler for the one-group case.

## I. INTRODUCTION

Ever since Case[1] successfully applied the method of singular eigenfunction expansion to the one-speed, one-dimensional neutron transport equation, there have been efforts to generalize the treatment to the multigroup case. The basic equation considered for $N$ groups is the following:

$$\mu \frac{\partial}{\partial x} \psi(x,\mu) + \Sigma\psi(x,\mu) = \int_{-1}^{1} C(\mu,s)\, \psi(x,s)\, ds. \qquad (1)$$

Here $\psi$ is a column vector whose elements $\psi_i$ represent the neutron angular density in the $i$th group, $1 \leq i \leq N$, and $\Sigma$ is a diagonal matrix whose $i$th element $\sigma_i$ is the neutron cross section in the $i$th group, ordered such that $\sigma_1 > \sigma_2 > \cdots > \sigma_n = 1$. The elements $C_{ij}$ of the transfer matrix $C$ represent the neutron scattering from group $j$ to group $i$.

The somewhat simplified isotropic case (in which $C$ is independent of $\mu$ and $s$) is treated in detail by Siewert and Zweifel[2] for a rather special situation physically relevant to radiative transfer, namely the determinant of every minor matrix of $C$ of rank $>1$ was assumed to vanish. The more general case, in which the determinant of $C$ was assumed not to vanish, has turned out to be rather difficult. Part of the problem is notational (a difficulty also encountered in Ref. 2), because the continuous spectrum is highly degenerate and because adjoint solutions must be introduced to calculate expansion coefficients. This leads to a bookkeeping task of no small magnitude. However, more fundamental is the difficulty that a satisfactory proof of half-range completeness of the singular eigenfunctions hinges on the signum of the so-called partial indices of the matrix Riemann—Hilbert problem, and these indices turn out to be difficult to pin down. A review of the partial index problem, and references to some attempts to deal with the problem have been given by Burniston *et al.*[3]

The notational problem alluded to above was solved in 1968 by Yoshimura and Katsuragi,[4] who also proved the relevant full-range completeness and orthogonality theorems. The more general case, in which $C$ is a function of angles, was discussed by Silvenionnen and Zweifel,[5] where the further notational difficulties are handled, and sufficient conditions for full-range completeness presented. Half-range completeness can be deduced, in fact, if $C$ is a symmetric matrix, as one obtains for example in thermal neutron problems with Maxwellian weights.

In 1973 a new development occurred in transport theory, namely the publication of a paper by Larsen and Habetler[6] in which the full- and half-range formulas originally obtained in Ref. 1 by heuristic arguments, were derived rigorously through functional analytic techniques. (A later paper by Larsen[7] extended these results to the anisotropic case, still, however, in one group theory.) These papers served not only to mollify the mathematicians who objected to Case and his disciples' cavalier treatment of the continuous spectrum, but also gave the hope of simplifying and generalizing the original results. For example, a paper[8] based on the results of Ref. 6 has extended the original expansion theorems of Case to an enormously larger class of functions. Basically, Ref. 6 made it possible to deal with linear operators in a Banach space, where previously one had to consider rather involved singular integral equations. Further, as we see by comparing the present paper with Ref. 4, the Larsen—Habetler technique is simpler and clearer than the standard technique of obtaining adjoint solutions, using Schmidt orthogonalization procedure, and calculating a large number of normalization integrals.

We propose to use the Larsen—Habetler technique to present an explicit solution to the half-range problem for a subcritical medium. (The extension to the more general case has not yet been found.) The solution will be given in terms of certain matrices $X$ and $Y$ which factor the dispersion matrix $\Lambda$.

It is well known, for example, in one-speed theory[1] that the matrix $X$ which solves the Riemann-Hilbert problem also factors the $\Lambda$ matrix [in Case's notation, $(1-c)(V_0^2 - Z^2)X(Z)X(-Z) = \Lambda(Z)$]. This "identity" as Case refers to it, in fact provides the basic connection between two apparently dissimilar methods for solving the same problem, i.e., the Case "eigenfunction expansion," leading to a Riemann—Hilbert problem, and the Wiener—Hopf method.

  

In the multigroup case, it turned out, perhaps because two matrices $X$ and $Y$ are involved in the factorization, that the Wiener—Hopf approach was more practical. The existence of the Wiener—Hopf factorization has been proved by Mullikin,[9] and although explicit solutions have not been found except in some special cases[10] they can be determined as the solutions of a certain nonlinear, non-singular integral equation. Our approach is reminiscent of that used by Siewert, Burniston, and Kriese[11] for the two group problem, their work being based on earlier work of Siewert and Ishigura.[12] These authors introduce matrices $H$ and $H^*$ which are, in fact, closely related to our matrices $X^{-1}$ and $Y^{-1}$. (Along these lines, one should also note a paper by Burniston, Mullikin and Siewert.[13]) Perhaps even closer to our approach is that of Pahor and Suhadolc,[14] who uses a full range expansion to deduce the half-range formulas, again for subcritical media. These results are, however, based on heuristic arguments similar to Case's original work.

In the present paper, we will apply the Larsen—Habtler technique to the full-range, $N$ group problem, partially to demonstrate that the functional analysis can be carried through to this case, and to set the stage for the half-range analysis which will be published separately. Throughout both papers, we shall restrict our attention to the case that $C$ is a constant matrix, and $\det C \neq 0$. We work in the function space $U$ defined by

$U = \{\psi \,|\, \mu\psi_i$ is differentiable in $x \in (-\infty, \infty)$ and is Hölder continuous in $\mu \in [-1, 1]\}$.

The idea of the Larsen—Habetler technique is to rewrite Eq. (1) as

$$\frac{\partial}{\partial x}\,\psi + K^{-1}\psi = 0,$$

where the reduced transport operator $K^{-1}$, which acts only on $\mu$, is given by (for fixed $x$),

$$(K^{-1}\psi)(x,\mu) = (1/\mu)[\Sigma\psi(x,\mu) - C\int_{-1}^{1}\psi(x,s)\,ds], \quad \mu \neq 0. \tag{2}$$

The bounded inverse $K$ of $K^{-1}$ and the resolvent $(zI - K)^{-1}$ are then constructed. The identity

$$\psi(\mu) = (1/2\pi i)\oint_\tau (zI-K)^{-1}\psi(\mu)\,dz \tag{3}$$

is then used to obtain the full-range expansion. In Eq. (3), and otherwise as convenient, the dependence of $\psi$ on $x$ will be supressed.

In this paper, we study the properties of $K$ and $(zI - K)^{-1}$ in Sec. II and derive the full-range expansion in Sec. III. We outline the solution of a typical transport problem in the Appendix.

## II. THE RESOLVENT OPERATOR $(zI - K)^{-1}$

In this section we construct the bounded operator $K$ and study the properties of the resolvent operator $(zI - K)^{-1}$. In addition, the following notation will be used in the subsequent analysis:

$$B \equiv (\Sigma - 2C)C^{-1}\Sigma,$$

$$D(z,\mu) \equiv (zI - \mu\Sigma^{-1})^{-1},$$

and

$$g_n = \int_{-1}^{1} s^n g(s)\,ds,$$

where $g$ may be an element of $U$ or an $N\times N$ matrix.

The operator $K^{-1}$ in Eq. (2) can be easily inverted to yield $K$:

$$K\eta = \Sigma^{-1}\mu\eta + B^{-1}\eta_1. \tag{4}$$

Note that the diagonal terms of $K$ are one-speed operators, while the off-diagonal terms are of rank one. If it is assumed that $\det(\Sigma - 2C) \neq 0$, then $K$ will be a bounded operator of $U$. This determinant condition is related to the critical condition for an infinite medium[15] and would reduce in the one-speed limit to the condition $c \neq 1$, which we note is also required by the analysis of Ref. 6. (Note the definition of the matrix $C$; the factor $1/2$ which multiplies the scalar $c$ in the one-speed equation has been absorbed into $C$.)

We next determine the resolvent operator $(zI - K)^{-1}$ from Eq. (4) by a straightforward calculation and write it in the form

$$(zI-K)^{-1}\,\psi(\mu) = D(z,\mu)[\psi(\mu) + \Lambda^{-1}(z)\,(D\psi)_1(z)],$$

where the dispersion matrix $\Lambda$ is defined by

$$\Lambda(z) \equiv B - D_1(z). \tag{5}$$

The function $(zI - K)^{-1}\psi$ is analytic in $z$ except for a branch cut along $[-1, +1]$ due to the branch cuts in the element of $\Lambda$ and poles at the zeros of the function

$$\Omega(z) \equiv \det\Lambda(z).$$

A simple calculation shows that the zeros of $\Omega$ are eigenvalues of $K$.

The above definition of $\Lambda$ differs from that, for example, of Ref. 4 by an immaterial multiplicative factor which does not affect the condition $\Omega(z) = 0$. In fact, $\Lambda$

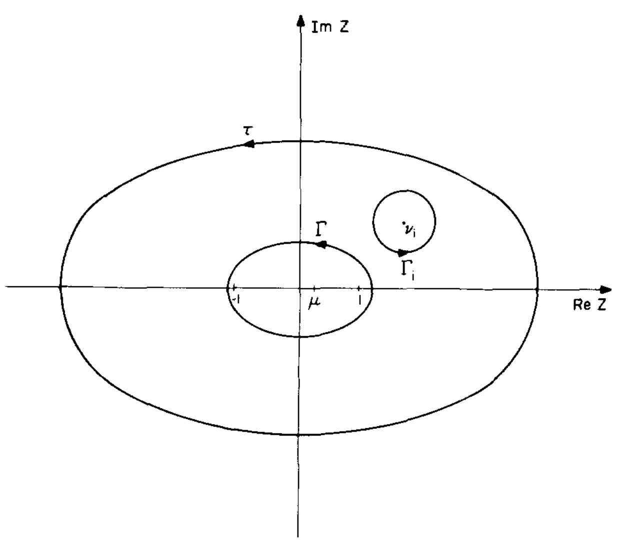


FIG. 1. The contour $\tau$, surrounding the spectrum of $K$ is deformed into the contour $\Gamma$ surrounding the continuum plus circles $\Gamma_i$ about each eigenvalue.

above is related to $\Lambda$ usually encountered in the literature by $\Lambda_{\text{usual}} = C\Sigma^{-1}\Lambda\Sigma^{-1}$. The elements of $\Lambda(z)$ are even real functions of $z$. Thus the eigenvalues of $K$ occur in pairs; if $\nu_i$ is an eigenvalue, so is $-\nu_i$. We shall label, for convenience, $-\nu_i = \nu_{i+n}$, where $n$ is the number of pairs of eigenvalues. For simplicity, we will assume that each $\nu_i$ is a simple zero of $\Omega$ and lies outside the branch cut $[-1,1]$. Although these restrictions are not necessary for the validity of the subsequent analysis, they simplify the paperwork. These restrictions can be relaxed and with some modifications the analysis would follow in a similar manner.

## III. THE FULL-RANGE EXPANSION

We can use the identity in Eq. (3) to write for $|\mu| \le 1$,

$$\psi(\mu) = (1/2\pi i) \oint_\tau D(z,\mu)[\psi(\mu) + \Lambda^{-1}(z)(D\psi)_1(z)]\,dz,$$

where $\tau$ encircles the spectrum of $K$ which consists of the point spectrum given by the points $\{\nu_i\}$, $i = 1,\ldots,2n$, and the rest of the spectrum (the union of the continuous and residual spectrums) given by the interval $[-1,1]$.[16] We now deform the contour $\tau$, as shown in Fig. 1, into small circles $\Gamma_i$ centered about each eigenvalue $\nu_i$, plus a contour $\Gamma$ about the branch cut $[-1, 1]$ to obtain,

$$\psi(\mu) = \sum_{i=1}^{2n} \psi_{\nu_i}(\mu) + \psi_\Gamma(\mu),$$

where

$$\psi_{\nu_i}(\mu) \equiv (1/2\pi i) \oint_{\Gamma_i} D(z,\mu)[\psi(\mu) + \Lambda^{-1}(z)(D\psi)_1(z)]\,dz, \quad (6)$$

and

$$\psi_\Gamma(\mu) \equiv (1/2\pi i) \oint_\Gamma D(z,\mu)[\psi(\mu) + \Lambda^{-1}(z)(D\psi)_1(z)]\,dz.$$

Because of the simple pole of the elements of $\Lambda^{-1}(z)$ at $z = \nu_i$, $i = 1,\ldots, 2n$, we can use the residue theorem to write

$$\psi_{\nu_i}(\mu) = [\Omega'(\nu_i)]^{-1} D(\nu_i,\mu)\Lambda_c(\nu_i)(D\psi)_1(\nu_i), \quad (7)$$

where $\Omega'(\nu_i) = [d\Omega/dz]_{z=\nu_i}$ and $\Lambda_c(z)$ is the transpose of the cofactor matrix of $\Lambda(z)$. We note that $\psi_{\nu_i}$ is an eigenvector of $K$ since from Eqs. (4), (5), and (7) we have

$$[\nu_i I - K]\psi_{\nu_i} = [\Omega'(\nu_i)]^{-1}[I - B^{-1}D_1(\nu_i)] \times \Lambda_c(\nu_i)(D\psi)_1(\nu_i) = 0. \quad (8)$$

Equation (7) can be expressed as the product of an eigenfunction of $K$, $\psi_{\nu_i}$, times an expansion coefficient $\alpha_i$ as follows. Note that each column of $\Lambda_c(\nu_i)$ is proportional to the eigenvector of $\Lambda(\nu_i)$, i.e.,

$$[\Lambda_c(\nu_i)]_{lm} = \alpha_m \beta_l(\nu_i), \quad (9a)$$

where

$$\Lambda(\nu_i)\,\beta(\nu_i) = 0, \quad (9b)$$

and $\alpha_m$, $m = 1,\ldots,N$, are constants depending on the elements of $\Lambda_c(\nu_i)$. Then we can write

$$\psi_{\nu_i}(\mu) = a_i \phi_{\nu_i}(\mu),$$

where

$$\phi_{\nu_i}(\mu) = D(\nu_i,\mu)\,\beta(\nu_i) \quad (10)$$

is the discrete eigenvector obtained in Ref. 4, and

$$a_i = \sum_{m=1}^{N} \alpha_m[(D\psi)_1(\nu_i)]_m\,[\Omega'(\nu_i)]^{-1} \quad (11)$$

is the expansion coefficient.

We now turn to the continuum term, $\psi_\Gamma$ of the eigenfunction expansion. Define

$$M(z) \equiv \Lambda^{-1}(z)(D\psi)_1(z). \quad (12)$$

Then we can write

$$\psi_\Gamma(\mu) = (1/2\pi i) \oint_\Gamma D(z,\mu)[\psi(\mu) + M(z)]\,dz. \quad (13)$$

An application of the Cauchy integral theorem reduces the first term on the rhs of Eq. (13) to $\psi(\mu)$. To evaluate the remaining term, we deform the contour $\Gamma$ as shown in Fig. 2. The contribution from $\Gamma_{-1}$ and $\Gamma_{+1}$ can be shown to vanish (see, for example, Ref. 6). The contribution from $\Gamma'$ and $\Gamma_\mu$ give the first and second terms on the rhs of the following equation:

$$\begin{aligned}[(1/2\pi i)\oint D(z,\mu)M(z)\,dz]_i &= (1/2\pi i)\Big\{\int_{-1}^{1} D(\nu,\mu) \\ &\quad\times[M^-(\nu) - M^+(\nu)]\,d\nu\Big\}_i \\ &\quad+ \tfrac{1}{2}[M_i^+(\mu/\sigma_i) + M_i^-(\mu/\sigma_i)], \\ &\quad i = 1,\ldots,N, \end{aligned} \quad (14)$$

where we use the notation that $M^+(\nu)$ and $M^-(\nu)$ are the boundary values of $M(\nu \pm i\epsilon)$ as $\epsilon \to 0$ respectively. The integral on the rhs of Eq. (14) is understood to be the Cauchy principal value. If we further denote

$$[\Delta(\nu,\mu)]_{ml} = \delta_{ml}\delta(\sigma_m\nu - \mu),$$

and

$$[\psi_D(\mu)]_i = \begin{cases} \psi_i(\sigma_i\mu) & |\sigma_i\mu| \le 1, \\ 0 & \text{otherwise}, \end{cases}$$

where $\delta_{ml}$ and $\delta(\sigma_m\nu - \mu)$ are the kronecker delta function and dirac delta distribution, respectively, we can

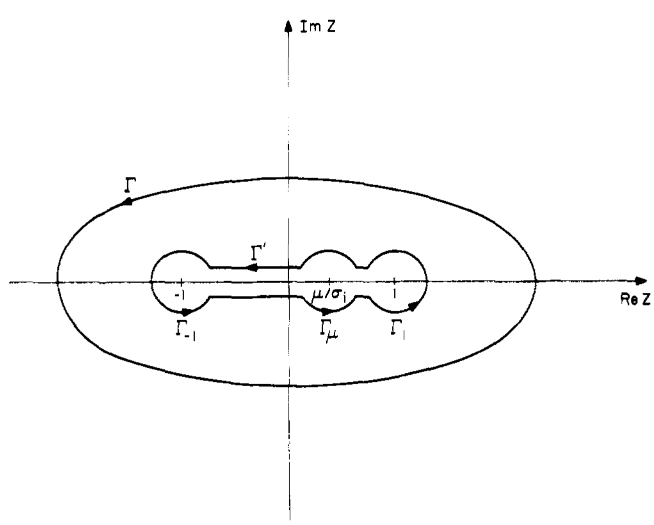


FIG. 2. The contour $\Gamma$ about the continuum is squeezed down to the contour $\Gamma' \cup \Gamma_{-1} \cup \Gamma_{+1} \cup \Gamma_\mu$

write

$$\psi_\Gamma(\mu) = (1/2\pi i)\int_{-1}^{1}[D(\nu,\mu)\{M^-(\nu) - M^+(\nu)\}$$
$$+ \pi i \Sigma\Delta(\nu,\mu)(2\psi_\Sigma(\nu) + \{M^-(\nu) + M^+(\nu)\})]\,d\nu.$$

Noting from Eqs. (5) and (12) that

$$\psi_\Sigma(\mu) = (1/2\pi i\nu)\Sigma^{-1}[\Lambda\ (\nu) + \Lambda\ (\nu)]$$
$$\times[M^-(\nu) - M^+(\nu)] - \tfrac{1}{2}[M^+(\nu) + M^-(\nu)],$$

we can write

$$\psi_\Gamma(\mu) = \int_{-1}^{1} \Phi(\nu,\mu)A(\nu)\,d\nu, \tag{15}$$

where

$$\Phi(\nu,\mu) \equiv \nu D(\nu,\mu) + \tfrac{1}{2}\Delta(\nu,\mu)\,\Sigma^{-1}[\Lambda^+(\nu) + \Lambda^-(\nu)], \tag{16}$$

and

$$A(\nu) \equiv (1/2\pi i\nu)[M^+(\nu) - M^-(\nu)]. \tag{17}$$

(Note that $\psi_\Gamma(\mu)$ is Holder continuous on $[-1,1]$.)

It has become customary to express formulas like Eq. (15) in terms of so-called continum eigenvectors multiplied by an expansion coefficient integrated on $\nu$. [In fact, the columns of $\Phi(\nu,\mu)$ are exactly the continuum eigenvectors of Ref. 4.] Although the mathematical justification of this custom is not exact, it is instructive to note that the columns of the matrix $\Phi$ are indeed generalized eigenvectors of the operator $K$. By this we mean that

$$K\psi_\Gamma(\mu) = \int_{-1}^{1} \nu\Phi(\nu,\mu)\,\Lambda(\nu)\,d\nu, \qquad |\nu| \le 1. \tag{18}$$

To prove Eq. (18), we utilize the identity

$$(zI-K)^{-1}K\psi(\mu) = z(zI-K)^{-1}\ \psi(\mu) - \psi(\mu). \tag{19}$$

Then the analysis leading to Eq. (18) is carried out for $Kf$ rather than $f$, using the rhs of Eq. (19). An almost identical calculation leads to Eq. (18).

## IV. CONCLUSION

As a result of the analysis of the previous sections, we have established the following eigenfunction expansion for $\psi$, which we state as a theorem.

*Theorem* 1: Let $\psi \in U$. Then for a fixed $x$, $\psi$ can be expanded as

$$\psi(x,\mu) = \sum_{i=1}^{2n} a_i(x)\,\phi_{\nu_i}(\mu) + \int_{-1}^{1} \Phi(\nu,\mu)A(\nu)\,d\nu, \tag{20}$$

where $\phi_{\nu_i}$, $a_i$, $\phi$ and $A$ are given by Eqs. (10), (11), (16), and (17) respectively. Furthermore, the $\phi_{\nu_i}$ are eigenvectors of $K$ and $\Phi(\nu,\mu)$ is a generalized eigenvector in the sense of Eq. (19).

In an accompanying paper, the so-called half-range problem in which an eigenfunction expansion of $\psi$ corresponding to the part of the spectrum containing elements $z$ with re $z \ge 0$ will be considered. The ground work for that paper has been laid by the present paper.

Also, the authors expect an extension of the domain of Eq. (1) to a larger class of function spaces, namely $X_p = \{\psi\,|\,\psi_i$ differentiable in $x \in (-\infty,\infty)$ and $\mu\psi_i \in L_p[-1,1]$ in $\mu$, $p > 1$, in a similar manner to Ref. 8 which extended the results of Ref. 6. The restriction of the present paper to the smaller space $U$ was made so that equations like (14) could be written down.

Finally, in the authors' opinion, the major benefit of the above analysis, other than its mathematical rigor, is the fact that the cumbersome calculations of the so-called degenerate continuum eigenfunctions in various regions of the branch cut (see Ref. 4) are avoided. Also, the analysis is suitable for more generalized cases, namely the anisotropic multigroup transport equation.

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## APPENDIX A: REMARKS ON SOLUTIONS OF THE MULTIGROUP TRANSPORT EQUATION

Larsen and Habetler[6] have used the Case eigenfunction expansion to solve the transport equation by asserting that any solution which was "sufficiently smooth" could be expanded, and its expansion coefficients related to those of the source. The trouble with this approach is that it requires, *ex post facto*, a verification that the solution is indeed "sufficiently smooth." Here we follow a different approach, more in accord with Case's original work, of constructing a solution which satisfies the boundary conditions, and then relying on a uniqueness theorem to guarantee that there is no other solution.

Since, in the context of a full-range expansion, only infinite medium problems are really relevant, we consider the infinite medium Green's function, that is we seek solutions of the homogeneous equation

$$\frac{\partial}{\partial x}\psi + K^{-1}\psi = 0, \tag{A1}$$

subject to the conditions

$$\lim_{|x|\to\infty} \psi(x,\mu) = 0, \tag{A2}$$

$$\lim_{\substack{\epsilon\to 0\\ \epsilon>0}} \{\psi(\epsilon,\mu) - \psi(-\epsilon,\mu)\} = \frac{\varphi(\mu)}{\mu} \equiv Q_o(\mu), \tag{A3}$$

where $Q_o$ is assumed Hölder continuous. ($Q$ is the source strength vector.)

Let us expand $Q_o(\mu)$ according to Theorem 1, rewritten for convenience in the following form:

$$\varphi_0(\mu) = \sum_{i=1}^{n} a_i\phi_{\nu_i}(\mu) + \sum_{k=1}^{n} a_k\phi_{-\nu_k}(\mu)$$
$$+ \int_0^1 \Phi(\nu,\mu)A(\nu)\,d\nu + \int_{-1}^{0} \phi(\nu,\mu)A(\nu)\,d\nu. \tag{A4}$$

Then we make the following

*Assertion*: The solution of Eqs. (A1), (A2), and (A3) is given by

$$\psi(x,\mu)=\sum_{i=1}^{n} a_i \exp(-x/\nu_i)\,\phi_{\nu_i}(\mu)+\int_0^1 \Phi(\nu,\mu)$$

$$\times\exp(-x/\nu)A(\nu)\,d\nu,\quad x>0, \tag{A5a}$$

$$=-\sum_{k=1}^{n} a_k \exp(x/\nu_k)\,\phi_{-\nu_k}(\mu)-\int_{-1}^0 \Phi(\nu,\mu)$$

$$\times\exp(-x/\nu)A(\nu)\,d\nu,\quad x<0. \tag{A5b}$$

*Justification of Assertion*: The assertion can be proved rigorously by developing a full functional calculus for $K$, as has been done[8] for the one-speed case. This could easily be carried out, but requires some detailed estimates which have not yet been made. A simpler procedure involves rewriting the transport equation in the form

$$\left(K\frac{\partial\psi}{\partial x}\right)(x,\mu)+\psi(x,\mu)=0. \tag{A6}$$

This is justified so long as $\mu\,\partial\psi/\partial x$ is Hölder continuous in $\mu$ for every $x$. (Even this is not really required since $K$ is a bounded operator which could, by continuity, be extended to a complete function space, say $L_p[-1,1]$.) Then from Eqs. (8) and (19) we observe that $\psi(x,\mu)$ obeys the transport equation, and the boundary conditions (A2) and (A3) and is, hence, the unique solution.[17]

For the one speed problem, uniqueness is well known.[18] For the energy dependent transport equation, a number of uniqueness theorems have been shown,[18,19] and the multigroup case considered here can, with a little effort, be shown to be a special case of some of these treated there. However, we note that uniqueness for the one-dimensional case is more or less trivial; we sketch the proof because the argument involves the calculation of the norm of an integral operator and the same calculation is involved in a different way in the construction of the half-range eigenfunction expansion (accompanying paper).

It is well known [see Ref. 7, Sec. (3.6)] that the one speed transport equation can be written as an integral equation by introducing the Green's function. In exactly the same way, the multigroup transport equation with source $Q$ (which may be a distributed source or a plane source as in the Green's function problem) is equivalent to an equation for the density $\rho$:

$$\rho(x)=\int_{-\infty}^{\infty} G(|x-x'|)\,\rho(x')\,dx'+\overline{Q}(x), \tag{A6a}$$

where

$$\overline{Q}(x)=\int_{-\infty}^{\infty}dx'\int_{-1}^{1}d\mu\,\frac{\exp(-\Sigma|x-x'|/|\mu|)}{|\mu|}\,CQ_0(x',\mu), \tag{A6b}$$

with

$$G(|x-x'|)=E(|x-x'|)\delta_{ij}, \tag{A7a}$$

and

$$E_{ij}\equiv E_1(\sigma_i|x-x'|)\delta_{ij}. \tag{A7b}$$

$E_1$ being defined in Ref. 17, Appendix E.

Now, a Fredholm equation like (A-6) is known to possess a unique solution if the norm of its kernel $G$ is less than unity. In the next appendix we compute this norm in an $L_1$ space and conclude that our solution given by Eqs. (A5) is indeed unique if the infinite medium under consideration is subcritical. This is precisely the condition at which one arrives in one-speed theory, i.e., $c<1$.

## APPENDIX B: CALCULATION OF $\|G\|$

We work with the Banach space

$$L=\bigoplus_1^N L_1,$$

with norm

$$\|f\|=\sum_{i=1}^{N}\int_{-\infty}^{\infty}|f_i|\,dx.$$

(We note the solution obtained in Appendix A is an element of $L$.) The operator norm is

$$\|G\|=\sup_{f\in L}\|Gf\|/\|f\|.$$

It is trivial to show that the integral operator $G$, with kernel $G(|x-x'|)$ is a map of $L$ into itself. In particular, if $A$ is a matrix of constants then

$$\|A\|=\sup_k\sum_{j=1}^{N}|A_{jk}|\equiv\|A\|_M.$$

Now, if $K$ is a matrix of operators, then each matrix element $K_{ij}$ has an $L_1$ norm which is denoted by $\|K_{ij}\|_1$. By writing the operator $G=EC=(E\Sigma)(\Sigma^{-1}C)$ we conclude that ($E$ and $\Sigma$ are diagonal)

$$\|G\|\le\|E\Sigma\|\cdot\|\Sigma^{-1}C\|$$

$$\le\sup_i\|E_{ii}\Sigma_{ii}\|\cdot\|\Sigma^{-1}C\|_M.$$

To compute $\|E\Sigma\|$, we may use Kato's criterion[20] which, for a difference kernel reduce to

$$\|E_{ii}\Sigma_{ii}\|\le\sup_x\int_{-\infty}^{\infty}\sigma_i E_1(\sigma_i|x-x'|)\,dx'=2.$$

Thus,

$$\|G\|\le 2\,\|\Sigma^{-1}C\|_M,$$

and the solution obtained in Appendix A will be unique if

$$2\|\Sigma^{-1}C\|_M\le 1.$$

This result is exactly one of those derived by Bowden[15] as the subcriticality condition for an infinite medium.

We shall find that this norm result plays a crucial role in the half-space problem discussed in the accompanying paper. We point out that for half-space problems, the range of integration is $(0,\infty)$ rather than $(-\infty,\infty)$. However, the norm calculated is identical [evaluated with the limits $(0,\infty)$].

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[17]Mathematical purists (only) will object to this "justification" on several grounds, one being the necessity of a delicate proof that

$$K\frac{\partial\psi}{\partial x}(x,\mu)=\frac{\partial}{\partial x}(K\psi)(x,\mu).$$

The complete functional calculus, as in Ref. 8, could be developed to satisfy this question, but the question then arises as to how much effort "purity" is worth.
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