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Multigroup neutron transport. II. Half range*

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This paper accompanies a preceding one in which a functional analytic method was used to obtain the full-range expansion in multigoup neutron transport. In the present paper the analysis is extended to obtain the half-range expansion. The method used is an extension of the work of Larsen and Habetler for the one-group case. The results are given in terms of certain matrices which are solutions of coupled integral equations and which factor the dispersion matrix.

I. INTRODUCTION

In an accompanying paper,[1] hereafter referred as I, the full range eigenfunction expansion for the solution of a multigroup neutron transport equation is obtained from application of the resolvent operator technique of Larsen and Habetler.[2] We refer the reader to the introduction of I, in which we point out some of the advantages of the Larsen—Habetler technique over the usual Case orthogonal singular eigenmodes approach.[3] These advantages include mathematical rigor and simplified notation. But the major advantage is the subject of the present paper, namely that the technique can be applied just as easily to obtain the so-called half range eigenfunction expansion. Except for some special cases,[4,5] existence of the half range expansion, let alone explicit formulas, has never been demonstrated, due to a technicality in determining so called "partial indices."[6] (The only exception is a paper by Pahor and Suhadolc[7] whose ideas have some similarity to ours, but whose arguments are still based on the original Case approach. Like them, we are restricted by the condition $\|\Sigma^{-1}C\| < \frac{1}{2}$, which has been shown to be a condition that the infinite medium be subcritical.[8])

We should point out another major advantage of the present technique, namely its suitability to generalization. For example, the anisotropic case could be treated almost as easily as the isotropic case considered here. It is necessary in the analysis to write the transport equation in the form of an integral equation which, while tedious in high order anisotropy, is nonetheless feasible. Also, the condition on $\|\Sigma^{-1}C\|$ could be relaxed if the sufficient condition of Gohberg and Krein[9] for the convergence of Neumann series solutions to certain integral equations could be improved. (Our analysis is heavily based on a factorization theorem of T.W. Mullikin[10] which is, in turn, based on the work of Ref. 9.)

In the present paper we use the Larsen—Habetler technique and the results of the Mullikin theorem to obtain an explicit representation of the half-range expansion; the partial index question never arises. An outline of our presentation is as follows:

In Sec. III we find a projection operator E such that $(zI-K)^{-1}E\psi$ is analytic for $\mathrm{Re}\, z<0$, where ψ belongs to a certain function space and $(zI-K)^{-1}$ is the resolvent operator considered in I. It turns out that this operator contains certain noncanonical matrices X and Y which factor the dispersion matrix. Then, in Sec. III we obtain explicitly the half range eigenfunction expansion which can, in turn, be used to solve half-space transport problems in the usual way. The X and Y matrices are given in terms of the solutions of nonlinear, nonsingular integral equations, and they may be obtained numerically. This, in turn, will make it possible to obtain numerical solutions from the results of the present paper.

Appendix A is devoted to the conversion of the Mullikin results to the form which can be used in the present paper. In Appendix B a uniqueness theorem is proved for the matrices X and Y. Appendix C contains a solution to a half-space transport problem.

II. THE PROJECTION OPERATOR E

We consider the function space U' of N-dimensional column vectors ψ defined as

$$U'=\{\psi \mid \mu\psi_i \text{ is differentiable in } x\in(-\infty,\infty) \text{ and Hölder continuous in } \mu\in[0,1]\}.$$

The operator K defined by Eq. 4 of I is a bounded operator on the function space U defined in I. We closely follow the procedure of Ref. 2 and look for a projection operator E: $U'\to U$ with the two properties:

$$\text{(I)}\quad E\psi(\mu)=\psi(\mu),\quad 0<\mu\leq 1, \tag{1}$$

and

(II) $(zI-K)^{-1}E\psi(\mu)$ is analytic in z for $\mathrm{Re}\, z<0$,

where we use the notation defined in I to write for $f\in U$,

$$(zI-K)^{-1}f(\mu)=D(z,\mu)\left[f(\mu)+\Lambda^{-1}(z)\int_{-1}^{1} sD(z,s)f(s)\,ds\right]. \tag{2}$$

(As in I, the x-dependence will be suppressed.) The idea of introducing the operator E is that the identity

$$\psi(\mu)=E\psi(\mu)=(1/2\pi i)\oint_\tau (zI-K)^{-1}E\psi(\mu)\,dz,\quad 0<\mu\leq 1, \tag{3}$$

where τ is a contour encircling the spectrum of the operator K, will reduce to the half range expansion of ψ for $0<\mu\leq 1$ because of property (II) above.

Let us now consider $\psi \in U'$ and write

$$(zI - K)^{-1}E\psi(\mu) = D(z, \mu)[E\psi(\mu) + G(z)], \quad (4)$$

where we have defined the column vector $G(z)$ by

$$G(z) = \Lambda^{-1}(z)\int_{-1}^{1} s\, D(z,s)E\psi(s)\, ds. \quad (5)$$

In order for $(zI-K)^{-1}E\psi(\mu)$ to be analytic in z for $\mathrm{Re}z < 0$, we require that:

(a) $G^+(\mu) = G^-(\mu) = G(\mu), \quad -1 < \mu < 0$

(b) $[G(\mu)]_i = -[E\psi(\sigma_i\mu)]_i, \quad -1 < \sigma_i\mu < 0, \quad i = 1, \ldots, N,$

and

(c) $G(-\nu_i) < \infty, \quad \mathrm{Re}\nu_i > 0, \quad i = 1, \ldots, n,$

where we recall that $\det\Lambda(\pm\nu_i) = \Omega(\pm\nu_i) = 0, \quad i = 1, \ldots, n.$

At this point, we introduce matrices $X(z)$ and $Y(z)$ which factor the dispersion matrix $\Lambda(z)$ according to

$$\Lambda(z) = Y(-z)X(z), \quad (6)$$

and which satisfy the following properties:

(i) $X(z)$ and $Y(z)$ are analytic in the complex z-plane cut along $[0,1]$,

(ii) $\det X(\nu_i) = \det Y(\nu_i) = 0$ and $\det X(-\nu_i) \neq 0$ and $\det Y$ $\det Y(-\nu_i) \neq 0$ for $\mathrm{Re}\nu_i > 0, \quad i = 1, 2, \ldots, n,$ and

(iii) $\lim_{|z|\to\infty} X(z) = \text{constant}$, and $\lim_{|z|\to\infty} Y(z) = = \text{constant}$.

The existence of the $X(z)$ and $Y(z)$ matrices with the properties listed above can be shown from the results of Mullikin.[10] This is done in Appendix A.

Let us find another representation of $G(z)$ by defining the operator E and the column vector function F so that the column vector function

$$Q(z) = \int_{-1}^{1} sD(z,s)E\psi(s)\, ds - Y(-z)\int_0^1 (1/z - s)F(s)\, ds, \quad (7)$$

is identically zero. Since $Q(z)$ vanishes as $|z| \to \infty$ and is analytic except perhaps for a cut along the interval $[-1,1]$, we need only to require that $Q(z)$ be continuous across that interval. Thus, using the Plemelj formulas and property (I) above, we find that $Q(z)$ is continuous across $[-1,1]$ if

$$F(\mu) = \mu Y^{-1}(-\mu)\Sigma^2\psi_\Sigma(\mu), \quad 0 \leq \mu \leq 1 \quad (8)$$

and

$$[E\psi]_i(\sigma_i\mu) = -[X^{-1}(\mu)\int_0^1 s(\mu - s)^{-1}\, Y^{-1}(-s)\Sigma^2\psi_\Sigma(s)\, ds]_i, \quad -1 \leq \sigma_i\mu < 0, \quad (9)$$

where we have used Eq. (8) in Eq. (9) and we have defined the column matrix $\psi_\Sigma(\mu)$ such that

$$[\psi_\Sigma(\mu)]_i = \begin{cases} \psi_i(\sigma_i\mu), & 0 \leq \sigma_i\mu \leq 1, \\ 0, & \text{otherwise}. \end{cases}$$

Combining Eqs. (5), (7), (8), and (9), we determine $G(z)$ to be given by

$$G(z) = X^{-1}(z)\int_0^1 s(z-s)^{-1}Y^{-1}(-s)\Sigma^2\psi_\Sigma(s)\, ds. \quad (10)$$

Note that $G(z)$ given by Eq. (10) satisfies requirements (a), (b) and (c) listed above. Therefore our desired projection operator E is defined by Eqs. (1) and (9).

Note Eq. (9) that $E\psi(\mu)$ is Hölder continuous for $\mu < 0$ (recalling that X is analytic). Thus we verify that E maps into U, in particular, we are justified in applying the full range expansion of I to $E\psi$.

III. HALF RANGE EXPANSION

We now can combine Eqs. (3), (4) and (10) to obtain the half range expansion in the form

$$\psi(\mu) = \sum_{i=1}^{n} \psi_{\Gamma_i} + \psi_\Gamma,$$

where

$$\psi_{\Gamma_i}(\mu) = (1/2\pi i)\oint_{\Gamma_i} D(z,s)[E\psi(\mu) + G(z)]\, dz,$$

and

$$\psi_\Gamma(\mu) = (1/2\pi i)\oint_\Gamma D(z,s)[E\psi(\mu) + G(z)]\, dz.$$

Each contour Γ_i consists of a small circle centered about ν_i, and the contour Γ encircles the interval $(0,1)$. Applying the residue theorem, we can calculate ψ_{Γ_i} to be

$$\psi_{\Gamma_i}(\mu) = [\Omega'(\nu_i)]^{-1}D(\nu_i, \mu)[\det Y(\nu_i)] \times X_c(\nu_i)\int_0^1 s(\nu_i - s)^{-1}Y^{-1}(-s)\Sigma^2\psi_\Sigma(s)\, ds, \quad (11)$$

where $X_c(\nu_i)$ is the transpose of the cofactor of $X(\nu_i)$.

Noting that each column of $X_c(\nu_i)$ is proportional to the eigenvector $\beta(\nu_i)$ of $\Lambda(\nu_i)$, as defined by Eq. (I-9) we can write

$$[\det Y(\nu_i)][X_c(\nu_i)]_{lm} = \alpha'_m\beta_l(\nu_i),$$

where α'_m is a constant. With the above definitions, we can express Eq. (11) as

$$\psi_{\Gamma_i}(\mu) = a'_i\phi_{\nu_i}(\mu), \quad (12)$$

where

$$a'_i = [\Omega'(\nu_i)]^{-1}\sum_{m=1}^{N}\alpha_m^1[\int_0^1 s(\nu_i - s)^{-1}Y^{-1}(-s)\Sigma^2\psi_\Sigma(s)\, ds]_m, \quad (13)$$

and $\phi_{\nu_i}(\mu)$ is the discrete eigenvector given in Eq. (I-10).

To find ψ_Γ, we apply the same integration technique to the contour Γ as was done in I, and we get

$$\psi_\Gamma(\mu) = \int_0^1 \Phi(\nu, \mu)A'(\nu)\, d\nu, \quad (14)$$

where

$$A'(\nu) = \frac{G^+(\nu) - G^-(\nu)}{2\pi i\nu}, \quad 0 \leq \nu \leq 1, \quad (15)$$

and $\Phi(\nu, \mu)$ is given by Eq. (I-16). Finally, we combine Eqs. (12) and (14) to write the half range expansion for $\psi \in U'$, $0 \leq \mu \leq 1$ as

$$\psi(\mu) = \sum_{i=1}^{N} a'_i\phi_{\nu_i}(\mu) + \int_0^1 \Phi(\nu, \mu)A'(\nu)\, d\nu,$$

where ϕ_{ν_i}, $\Phi(\nu, \mu)$, a'_i and $A'(\nu)$ are given by Eqs. (I-10), (I-16), (13), and (15), respectively.

The expansion coefficients are given in terms of the matrices $X(z)$ and $Y(z)$. By rearranging the results of Mullikin,[10] we show in Appendix A that $X(z)$ and $Y(z)$ satisfy the functional equations

$$X(z) = C^{-1}\Sigma + \frac{z}{2\pi i}\int_0^1 \frac{Y^{-1}(-s)}{s(s-z)}[\Lambda^+(s) - \Lambda^-(s)]\,ds, \qquad (16)$$

and

$$Y(z) = \Sigma - \frac{z}{2\pi i}\int_0^1 \frac{\Lambda^+(s) - \Lambda^-(s)}{s(s-z)} X^{-1}(-s)\,ds. \qquad (17)$$

If z is restricted to the interval $[-1,0]$, these functional equations reduce to nonlinear, nonsingular coupled matrix integral equations for $Y(-\mu)$ and $X(-\mu)$, $0 \leq \mu \leq 1$. Using the results of Mullikin,[10] we show in Appendix A that a solution of this last set of equations exists if $\|\Sigma^{-1}C\| < \frac{1}{2}$. This is, of course, a sufficient condition and not a necessary condition. Furthermore, we show that any pair of matrices $X(z)$ and $Y(z)$ which satisfy the factorization to within a trivial factor, is unique (Mullikin has shown existence of at least one solution). Thus, one may proceed with confidence to evaluate X and Y numerically from Eqs. (16) and (17).

It is important to understand the difference between the present method and the "Case-type" approach. In the latter, one obtains the half range eigenfunction expansion as the solution of a singular integral equation on the line $[0,1]$, and requires a matrix L^{-1} analytic in the complex plane cut along $[0,1]$ such that $(L^-)^{-1}L^+ = (\Lambda^-)^{-1}\Lambda^+$. The existence of such a matrix has been proved[11] but its behavior at infinity (the so-called "partial indices") is crucial to the proof of completeness, i.e., the proof that the singular integral equation for the expansion coefficient possesses a unique solution. Unfortunately, in many cases the calculation of the partial indices is impossible. In the approach taken here here, where we deal with the complex plane as a whole, the factorization as described by Eq. (6) is relevant, and the behavior at infinity is known. The poles of X^{-1} and Y^{-1} at the discrete eigenvalues presented no problem as was seen.

The major advantage of the approach used here is that numerical methods may be applied to Eqs. (16) and (17) and thus the expansion coefficients may be numerically evaluated. Using the canonical approach, it seems that at best existence can be proved, since no explicit solutions of the canonical problems are known. (See, however, Ref. 7.)

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APPENDIX A: SOME PROPERTIES OF X AND Y MATRICES

T.W. Mullikin[10] has shown that if K_x is an $N\times N$ matrix operator

$$K_x f(x) = \int_0^\infty k(x,y) f(y)\,dy,$$

on vector functions f with norm

$$\|f\| = \sum_{i=1}^N \int_0^\infty |f_i(x)|\,dx,$$

and if $\|K_x\| < 1$, then there exists a Weiner–Hopf factorization

$$[I - \hat{k}(z)]H_r(z)H_l(-z) = I \quad \text{for } \operatorname{Im} z = 0,$$

where $H_r(z)$ and $H_l(z)$ are matrices, analytic for $\operatorname{Im} z > 0$, and continuous and non-singular in $\operatorname{Im} z \geq 0$. Here, $\hat{k}$ is the Fourier transform of k,

$$\hat{k}(z) = \int_{-\infty}^\infty k(x)\exp(-izx)\,dx.$$

Mullikin also showed that H_r and H_l satisfy

$$H_r^{-1}(z) = I + (1/2\pi i)\int_{-\infty}^\infty H_l(t)\hat{k}(-t)(t+z)^{-1}\,dt, \quad \operatorname{Im} z > 0, \qquad (A1)$$

and

$$H_l^{-1}(z) = I + (1/2\pi i)\int_{-\infty}^\infty \hat{k}(t)H_r(t)(t+z)^{-1}dt, \quad \operatorname{Im} z > 0. \qquad (A2)$$

We can analytically extend the matrices $H_r(z)$ and $H_l(z)$ to the lower half of the z plane by defining

$$H^*(z) = \begin{cases} H_r(z), & \operatorname{Im} z \geq 0, \\ [I - \hat{k}(z)]H_l^{-1}(-z), & \operatorname{Im} z < 0, \end{cases} \qquad (A3)$$

and

$$H(z) = \begin{cases} H_l(z) & \operatorname{Im} z \geq 0 \\ H_r^{-1}(-z)[I - \hat{k}(z)]^{-1} & \operatorname{Im} z < 0. \end{cases} \qquad (A4)$$

Now, $H_r(z)$ is analytic for $\operatorname{Im} z > 0$ and $[I - \hat{k}(z)]^{-1}[H_l(-z)]^{-1}$ is analytic for $\operatorname{Im} z < 0$ except for a branch cut along $[-i, -i\infty)$, due to $[I - k(z)]^{-1}$ and poles at the zeros of $\det[I - \hat{k}(z)]$. Since $H_r(z) = [I - \hat{k}(z)]\,[H_l(-z)]^{-1}$, $\operatorname{Im} z = 0$, $H^*(z)$ is analytic everywhere in the complex plane except for the cut along $[-i, -i\infty)$, and poles at the zeros of $\det[I - \hat{k}(z)]$ in the lower half-plane.

Similar arguments follow for the matrix $H(z)$. Using Eqs. (A3) and (A4), one may easily show by direct substitution that

$$[I - \hat{k}(z)]H^*(z)H(-z) = I, \qquad (A5)$$

is valid for all z.

To link Mullikin's results to the X and Y matrices used in this paper, we note that the multigroup transport equation with source Q in half-space problems may be reduced to an equation for the density ρ (cf. Eq. I-A7),

$$\rho(x) = \int_0^\infty k(|x - x'|)\rho(x')\,dx' + \bar{Q}(x)$$

where

$$k(|x - x'|) = E_1(\Sigma|x - x'|)C,$$

with

$$[E_1(\Sigma|x - x'|)]_{ij} = \delta_{ij}\int_0^1 (1/\mu)\exp[-\sigma_i|x - x'|/\mu]\,d\mu.$$

Let us call K_x the operator with kernel k. Using this particular k, we calculate $\hat{k}$ and find that it is related to the dispersion matrix Λ by the relationship

$$\Lambda(z) = \Sigma[I - \hat{k}(-i/z)]C^{-1}\Sigma. \qquad (A6)$$

From Eqs. (A5) and (A6) we obtain

$$\Lambda(z) = \Sigma[H(i/z)]^{-1}[H^*(-i/z)]^{-1}C^{-1}\Sigma. \qquad (A7)$$

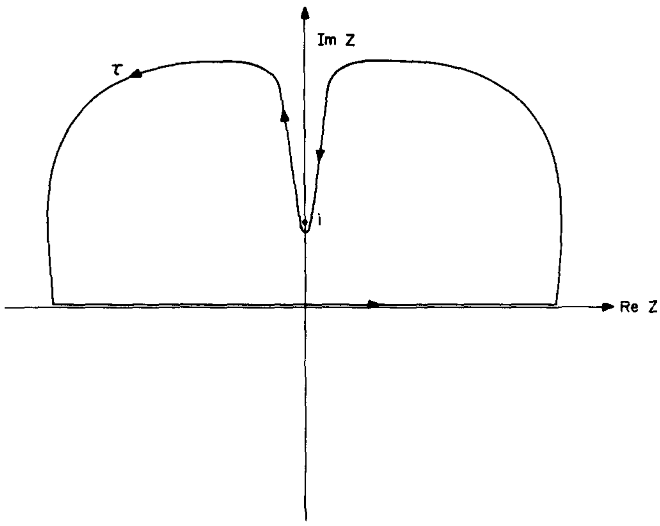

FIG. 1. The contour τ used for relating $\hat{k}(\omega)$ and $\Lambda(z)$.

If we now define

$$X(z)=[H^*(-i/z)]^{-1}C^{-1}\Sigma, \tag{A8}$$

and

$$Y(z)=\Sigma[H(-i/z)]^{-1}, \tag{A9}$$

we get Eq. (6). We recall from I that $\|k\|<1$ if $\|\Sigma^{-1}C\| < \frac{1}{2}$.

To determine Eqs. (16) and (17) for the X and Y matrices, we consider the contour τ given in Fig. 1 and note that the integrands in Eqs. (A1) and (A2) are analytic inside τ and have a branch cut $[i, i\infty)$ due to $\hat{k}$. Since $\hat{k}$ vanishes as $|z|\to\infty$, we can write the integral in Eq. (A1) as

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{-R}^{R}H_l(t)\hat{k}(-t)\frac{dt}{t+z}$$
$$=\lim_{R\to\infty}\left[\int_{iR}^{i}H_l(\omega)\frac{\hat{k}^+(-\omega)\,d\omega}{\omega+z}+\int_{i}^{iR}H_l(\omega)\frac{\hat{k}^-(-\omega)}{\omega+z}\,d\omega\right]. \tag{A10}$$

Using Eq. (A10), we can write Eq. (A1) as

$$H_r^{-1}(-i/z)=I+\frac{z}{z\pi i}\int_0^1\frac{H_l(i/s)}{s(s+z)}[\hat{k}^+(i/s)-\hat{k}^-(i/s)]\,ds. \tag{A11}$$

We calculate $k^+(i/s)-k^-(i/s)$, $0<s\leqslant 1$ from Eq. (A6) and substitute into Eq. (A11) to get

$$H_r^{-1}(-i/z)=I+\frac{z}{2\pi i}\int_0^1\frac{ds\,H_l(i/s)}{s(s+z)}\Sigma^{-1}[\Lambda^+(s)-\Lambda^-(s)]\Sigma^{-1}C. \tag{A12}$$

Identifying $X(z)$ and $Y(z)$ from Eqs. (A8) and (A9), we get Eq. (16). A similar analysis on Eq. (A2) yields Eq. (17).

APPENDIX B: UNIQUENESS OF THE X AND Y MATRICES

The factorization of Λ, Eq. (16) [along with conditions (i)–(iii)] is unique up to right multiplication of Y by a constant invertible matrix R and left multiplication of X by R^{-1}. [From Eq. (9) we observe that such a transformation does not affect E, and hence leaves the solution of the transport equation unchanged.) In addition, solutions of the nonlinear equations, (16) and (17) always factor Λ. Furthermore, these equations "normalize" X and Y. To be quite specific, we state these results as lemmas.

Lemma 1: Any pair of matrices X and Y which satisfy Eqs. (16) and (17) provide a factorization Eq. (6) of Λ.

Remark: Only the converse of this has been proved by Mullikin.

Proof: Eqs. (16) and (17) may be combined to give

$$[Y(-z)-\Sigma][X(z)-C^{-1}\Sigma]=\frac{-z^2}{(2\pi i)^2}\int_0^1\int_0^1\frac{[\Lambda^+(s)-\Lambda^-(s)]}{s(s+z)}\times X^{-1}(-s)\frac{Y^{-1}(t)[\Lambda^+(t)-\Lambda^-(t)]}{t(t-z)}\,ds\,dt. \tag{B1}$$

If the right hand side of (B1) is expanded by a partial fraction decomposition and common terms are cancelled, we obtain

$$Y(-z)X(z)=\Sigma C^{-1}\Sigma+(z/z\pi i)\times\int_{-1}^{1}[\Lambda^+(s)-\Lambda^-(s)/s(s-z)]\,ds=\Lambda(z), \tag{B2}$$

proving the lemma. (The definition of Λ, Eq. (I-5), has been utilized.)

Lemma 2: Let $X(z)$ and $Y(z)$ satisfy Eqs. (16) and (17) plus the conditions (i)–(iii) following Eq. (6). Let $X'(z)$ and $Y'(z)$ satisfy the same equations and the same conditions. Then

$$X(z)=X'(z)\quad\text{and}\quad Y(z)=Y'(z)$$

Remark: In a sense, this result is unimportant, since we already know that a factorization can be computed from (16) and (17) and, unique or not, it will provide a solution to the transport equation. However, it is interesting to note the constraints on the solutions do not require more than verifying conditions (i)–(iii) mentioned above.

Proof: The matrices

$$D_1(z)=[Y'(z)]^{-1}Y(z)\text{ and }D_2(z)=X(z)[X'(z)]^{-1}$$

are analytic everywhere in the complex plane except perhaps for a cut along (0,1) and poles at $\{+\nu_i\}$, $i=1, \cdots, n$. Also, because X, Y and X', Y' both satisfy (10) and (17),

$$\lim_{|z|\to\infty}D_1(z)=\lim_{|z|\to\infty}D_2(z)=I. \tag{B3}$$

Now calculate

$$D_1(-z)D_2(z)=[Y'(-z)]^{-1}Y(-z)X(z)[X'(z)]^{-1}=I, \tag{B4}$$

where *Lemma* 1 has been invoked. Similarly,

$$D_1(z)D_2(-z)=I. \tag{B5}$$

These equations are valid for all z. Since $D_2(z)$ is analytic in the left half plane, it follows from Eq. (B4), that $D_1(-z)$ is also analytic in the left half plane, i.e, that $D_1(z)$ is analytic in the right half plane. Similarly, from (B5) we conclude that D_2 is analytic in the right half plane. Thus, D_1 and D_2 are analytic everywhere and approach I at infinity. Hence,

$$D_1(z) = D_2(z) = I$$

or

$$[Y'(z)]^{-1} Y(z) = I \text{ and } X(z)[X'(z)]^{-1} = I.$$

Similarly, by redefining D_1 and D_2, we can show

$$Y(z)[Y'(z)]^{-1} = I \text{ and } [X'(z)]^{-1}X(z) = I.$$

So

$$X(z) = X'(z) \quad \text{and} \quad Y(z) = Y'(z).$$

APPENDIX C: SOLUTION OF HALF-SPACE TRANSPORT PROBLEMS

The solution of half space problems is similar to the infinite medium case considered in I. Consider the Albedo problem. That is, we seek solutions of the transport equation in the source free half space subject to

$$\psi(x, \mu) \to 0 \quad \text{as } |x| \to \infty, \tag{C1}$$

and

$$\psi(x, \mu) = \psi_0(\mu), \quad \mu > 0, \tag{C2}$$

where ψ_0 represents the incident distribution. As in I, we expand ψ_0, in a "half-range expansion"

$$\psi_0(\mu) = \sum_{i=1}^{n} a'_i \phi_{\nu_i}(\mu) + \int_0^1 \Phi(\nu, \mu) A'(\nu)\, d\nu. \tag{C3}$$

Then

$$\psi(x, \mu) = \sum_{i=1}^{n} a'_i \exp(-x/\nu_i \Phi_{\nu_i}(\mu) + \\ + \int_0^1 \Phi(\nu, \mu) \exp(-x/\nu) A'(\nu)\, d\nu. \tag{C4}$$

Equation (C4) is the (unique) solution because, as in I, it satisfied the transport equation and obeys the boundary conditions (C1) and (C2).

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