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A multiple-scales space–time analysis of a randomly perturbed one-dimensional wave equation[a)]

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An initial value problem for one-dimensional wave propagation is considered; the medium is assumed to be randomly perturbed as a function of both space and time. The stochastic perturbation theory of Papanicolaou and Keller [SIAM J. Appl. Math. **21**, 287 (1971)] is applied directly in the space–time regime to derive transport equations for the first and second moments of the solution. These equations are solved in special cases.

I. INTRODUCTION

The problem of characterizing long-range acoustic transmission in the ocean is essentially that of understanding the dynamics of wave propagation in a medium subjected to small random spatial and temporal perturbations. While the ocean problem is further complicated by a deterministic sound-speed profile (forming the SOFAR channel) and randomly irregular boundaries, understanding the effects of the medium itself represents a necessary prerequisite.

In this paper, a *model problem* involving one-dimensional wave propagation is considered. The properties of the medium are assumed to be randomly perturbed in both space and time. The stochastic perturbation theory of Papanicolaou and Keller[1] is applied directly in space–time; transport equations for the first and second moments of the solution emerge as necessary relations for the suppression of secular growth in the two characteristic directions of the unperturbed wave operator.

Dealing with the problem directly in space–time permits us to study the evolution of wavepackets in a spatially and temporally fluctuating environment. Fourier transforms are used, but only after the formal stochastic asymptotic analysis is complete. Only the infinite spatial domain is considered; however, the formalism can be developed as well for the semi-infinite spatial domain. It is hoped that subsequent analytis of this latter problem will provide insights into the nature of the boundary conditions that must accompany the limiting transport equations in cases (like the ocean) where boundaries exist.

In Sec. II the asymptotic formalism is developed while equations for the first moment of the solution are derived in Sec. III. In Sec. IV similar equations are derived for the second moment (i.e., the mutual coherence function). Sec. V deals with a specific example.

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II. DEVELOPMENT OF THE FORMALISM

The initial value problem that we shall consider is the following:

$$\frac{\partial^2}{\partial x^2}u(x,t,\omega,\epsilon)-\frac{\partial}{\partial t}\left(\bar{c}^{-2}(x,t,\omega,\epsilon)\frac{\partial}{\partial t}u(x,t,\omega,\epsilon)\right)=0,$$

$$-\infty<x<\infty,\quad t>0 \tag{1}$$

$$u(x,0,\omega,\epsilon)=f(x),\quad \frac{\partial u}{\partial t}(x,0,\omega,\epsilon)=g(x),\quad -\infty<x<\infty, \tag{2}$$

where ϵ is a small real parameter and ω is an element of some underlying probability space. We assume that

$$\bar{c}^{-2}(x,t,\omega,\epsilon)=c^{-2}(1+\epsilon\mu(x,t,\omega)), \tag{3}$$

i.e., the sum of a constant and a small randomly fluctuating quantity. The random field μ is assumed to be a zero mean wide-sense stationary function of both space and time; consequently, we have

$$\langle\mu(x,t,\omega)\rangle=0,$$
$$\langle\mu(x,t,\omega)\mu(x',t',\omega)\rangle=R(x-x',t-t'), \tag{4}$$
$$-\infty<x,x'<\infty,\quad 0\le t,t'<\infty,$$

where $\langle\cdot\rangle$ denotes expectation, i.e., integration with respect to the underlying probability measure. We shall further assume that the random field μ is mixing (Ref. 2) in the sense that as the space–time separation of (x_1,t_1) and (x_2,t_2) tends to infinity, the random variables $\mu(x_1,t_1,\omega)$ and $\mu(x_2,t_2,\omega)$ become asymptotically independent.

We are ultimately interested in $\langle u(x,t,\omega,\epsilon)\rangle$ and $\langle u(x,t,\omega,\epsilon)u(x',t',\omega,\epsilon)\rangle$ in the asymptotic limit where $\epsilon\to 0$ but where the space–time propagation paths (i.e., distances along the characteristics) tend to infinity. Prior work (Refs. 1, 2) has shown that since $\langle\mu\rangle=0$, interesting probabilistic effects will energe on ϵ^{-2} scales. Accordingly, we introduce the following slow spatial and temporal variables:

$$\xi\equiv\epsilon^2x,\quad \tau\equiv\epsilon^2t \tag{5}$$

and view the solution u as a function of $x,t,\xi,\tau,\epsilon,\omega$. The differential operators transform as follows:

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \epsilon^2 \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \epsilon^2 \frac{\partial}{\partial \tau} \tag{6}$$

and Eqs. (1) and (2) become

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial x^2} + 2\epsilon^2 \frac{\partial^2}{\partial x \partial \xi} + \epsilon^4 \frac{\partial^2}{\partial \xi^2}\right) u(x,t,\xi,\tau,\omega,\epsilon) - \left(\frac{\partial}{\partial t} + \epsilon^2 \frac{\partial}{\partial \tau}\right) \\
&\quad \times [c^{-2}(1+\epsilon\mu(x,t,\omega))\left(\frac{\partial}{\partial t} + \epsilon^2 \frac{\partial}{\partial \tau}\right) u(x,t,\xi,\tau,\omega,\epsilon)] \\
&\quad \equiv \left(\sum_{n=0}^{4} \epsilon^n \mathcal{L}_n\right) u = 0,
\end{aligned} \tag{7}$$

$$\begin{aligned}
&u(x,0,\xi,0,\omega,\epsilon) = f(x), \\
&\frac{\partial}{\partial t} u(x,0,\xi,0,\omega,\epsilon) + \epsilon^2 \frac{\partial}{\partial \tau} u(x,0,\xi,0,\omega,\epsilon) = g(x).
\end{aligned} \tag{8}$$

In particular:

$$\begin{aligned}
\mathcal{L}_0 &= \frac{\partial^2}{\partial x^2} - c^{-2}\frac{\partial^2}{\partial t^2}, \\
\mathcal{L}_1 &= -c^{-2}\frac{\partial}{\partial t}\left(\mu(x,t,\omega)\frac{\partial}{\partial t}(\cdot)\right), \\
\mathcal{L}_2 &= 2\left(\frac{\partial^2}{\partial x \partial \xi} - c^{-2}\frac{\partial^2}{\partial t \partial \tau}\right).
\end{aligned} \tag{9}$$

The solution u is expanded in a power series in ϵ,

$$u(x,t,\xi,\tau,\omega,\epsilon) \equiv \sum_{n=0}^{\infty} \epsilon^n u_n(x,t,\xi,\tau,\omega). \tag{10}$$

When (10) is substituted into (7), (8) and coefficients of the same power of ϵ are equated, we obtain the following hierarchy of problems:

(i) $\mathcal{L}_0 u_0 = 0, \quad u_0(x,0,\xi,0) = f(x), \quad \frac{\partial}{\partial t} u_0(x,0,\xi,0) = g(x),$

(ii) $\mathcal{L}_0 u_1 = -\mathcal{L}_1 u_0, \quad u_1(x,0,\xi,0,\omega) = 0,$

$\quad \frac{\partial}{\partial t} u_1(x,0,\xi,0,\omega) = 0,$

(iii) $\mathcal{L}_0 u_2 = -\mathcal{L}_1 u_1 - \mathcal{L}_2 u_0, \quad u_2(x,0,\xi,0,\omega) = 0,$

$$\frac{\partial}{\partial t} u_2(x,0,\xi,0,\omega) = -\frac{\partial}{\partial t} u_0(x,0,\xi,0), \tag{11}$$

$\vdots$

Stochastic effects in the actual solution gradually build up over long space–time propagation paths. Because of the assumed mixing property of the random field, the solution becomes essentially independent of the random field contained in any given space–time correlation cell. The stochastic perturbation formalism incorporates these features in the sense that computationally u_0 is a deterministic quantity and yet its dependence upon the slow variables ξ and τ will ultimately be dictated by properties of the random field.

The u_0 problem is solved by imposing the initial conditions upon the D'Alembert general solution; we obtain

$$u_0(x,t,\xi,\tau) = v_1(\xi,\tau,x-ct) + v_2(\xi,\tau,x+ct) \tag{12}$$

with

$$\begin{aligned}
v_1(\xi,0,x-ct) &= \tfrac{1}{2}f(x-ct) - \frac{1}{2c}\int_{-\infty}^{x-ct} g(\lambda)\,d\lambda, \\
v_2(\xi,0,x+ct) &= \tfrac{1}{2}f(x+ct) + \frac{1}{2c}\int_{-\infty}^{x+ct} g(\lambda)\,d\lambda.
\end{aligned} \tag{13}$$

Equations (13) will essentially provide the initial conditions for the resulting $\xi\tau$-transport equations. The equations themselves will emerge from the need to suppress secular growth in the expression for $\langle u_2 \rangle$.

We now solve the u_1 problem [cf. (11ii)], imposing the more stringent initial conditions $u_1(x,0,\xi,\tau,\omega) = (\partial/\partial t)u_1(x,0,\xi,\tau,\omega) = 0$ $(\tau \geq 0)$. Using Duhamel's Method[3] we obtain

$$\begin{aligned}
&u_1(x,t,\xi,\tau,\omega) \\
&\quad = -\frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \frac{\partial}{\partial s}\left(\mu(\lambda,s,\omega)\frac{\partial}{\partial s}u_0(\lambda,s,\xi,\tau)\right) d\lambda\, ds,
\end{aligned} \tag{14}$$

where u_0 is given by (12). In solving the u_2 problem (11iii) we again impose the more stringent initial conditions $u_2(x,0,\xi,\tau) = 0$, $(\partial/\partial t)u_2(x,0,\xi,\tau) = -(\partial/\partial\tau)u_0(x,0,\xi,\tau)$. Then, using superposition and Duhamel's Method, we obtain

$$\begin{aligned}
&u_2(x,t,\xi,\tau,\omega) \\
&\quad = -\frac{1}{2c}\int_0^{\tau} \int_{x-c(t-s)}^{x+c(t-s)} \left[\frac{\partial}{\partial s}\left(\mu(\lambda,s,\omega)\frac{\partial}{\partial s}u_1(\lambda,s,\xi,\tau,\omega)\right)\right. \\
&\qquad \left. -2\left(c^2\frac{\partial^2}{\partial \xi \partial \lambda} - \frac{\partial^2}{\partial \tau \partial s}\right) u_0(\lambda,s,\xi,\tau)\right] d\lambda\, ds \\
&\qquad -\frac{1}{2c}\int_{x-ct}^{x+ct} \frac{\partial}{\partial \tau} u_0(\lambda,0,\xi,\tau)\,d\lambda.
\end{aligned} \tag{15}$$

III. EQUATIONS FOR THE FIRST MOMENT

In this section we shall derive equations of evolution for v_1 and v_2 (as functions of ξ and τ). Taking expected values of the terms in (10), we have

$$\langle u \rangle = u_0 + \epsilon \langle u_1 \rangle + \epsilon^2 \langle u_2 \rangle + \cdots. \tag{16}$$

Recall that u_0 is a deterministic quantity. Therefore, noting (14) and the fact that $\langle \mu \rangle = 0$, a formal exchange of operations leads to $\langle u_1 \rangle = 0$. Therefore, $\langle u \rangle = u_0 + \epsilon^2 \langle u_2 \rangle + \cdots$. An examination of the terms comprising $\langle u_2 \rangle$ will reveal that some terms grow secularly with t; suppression of this growth, which is required to make the correction $\epsilon^2 \langle u_2 \rangle$ truly small on $O(1)$ $\xi\tau$ scales, will also determine the equations of evolution for v_1 and v_2 (i.e., u_0).

We are basically interested in the evolution of wavepackets in a statistically fluctuating environment. Thus we are tacitly assuming the initial data f and g to be such that $v_i(\xi,0,x)$, $i=1,2$, are suitably smooth with (essentially) compact support in x. [As (13) indicates, $v_i(\xi,0,x)$, $i=1,2$, are actually independent of ξ.] In the absence of random fluctuations, v_1 and v_2 would propagate undistorted along the characteristics. In the presence of a spatially and temporally fluctuating medium, however, the packets will become distorted and could also conceivably grow in strength (having energy "pumped in" by the medium) as they evolve. We shall assume, however, that any such accrual of energy occurs, on the average, at a sufficiently slow rate. Specifically, we shall assume that

$$\lim_{t\to\infty} \frac{1}{t} \int_{x-ct}^{x+ct} |v_i(\xi, \tau, \lambda)| d\lambda = 0, \quad i = 1, 2 \tag{17}$$

uniformly in ξ, τ, x. We shall also assume that similar averages of various partial derivatives of v_1 and v_2 vanish in the limit as $t \to \infty$.

Consider $\langle u_2 \rangle$, where u_2 is given by (15). Noting (12) the latter portion can be expressed as

$$\begin{aligned}
&\frac{1}{c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \left(c^2 \frac{\partial^2}{\partial\xi\partial\lambda} - \frac{\partial^2}{\partial\tau\partial s}\right) u_0(\lambda, s, \xi, \tau) d\lambda ds \\
&\quad - \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{\partial}{\partial\tau} u_0(\lambda, 0, \xi, \tau) d\lambda \\
&\quad = t\left[-\left(\frac{\partial}{\partial\tau} - c\frac{\partial}{\partial\xi}\right) v_1(\xi, \tau, x - ct)\right. \\
&\quad - \left(\frac{\partial}{\partial\tau} - c\frac{\partial}{\partial\xi}\right) v_2(\xi, \tau, x + ct) \\
&\quad \left. + \frac{1}{2t}\int_{x-ct}^{x+ct} \left(\frac{\partial}{\partial\xi} v_1(\xi, \tau, \lambda) - \frac{\partial}{\partial\xi} v_2(\xi, \tau, \lambda)\right) d\lambda\right].
\end{aligned} \tag{18}$$

In the context of our assumptions, the last term on the right-hand side of (18) will not contribute to secular growth in t. The remainder of (15) is evaluated using the expression for u_1 given by (14). For brevity, let ∂_i denote partial differentiation with respect to the ith argument. Then

$$\begin{aligned}
&-\frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \left\langle \frac{\partial}{\partial s}\left(\mu(\lambda, s, \omega)\frac{\partial}{\partial s} u_1(\lambda, s, \xi, \tau, \omega)\right)\right\rangle d\lambda ds \\
&\quad = t\left[\frac{1}{8c^2 t}\int_0^{2ct}\int_\sigma^{2ct-\sigma} [c^2 R(-\sigma, c^{-1}\sigma)(\partial_{33}^2 v_1(\xi, \tau, 2\sigma + x - ct)\right. \\
&\quad + \partial_{33}^2 v_2(\xi, \tau, \sigma + \eta + x - ct) + \partial_{33}^2 v_1(\xi, \tau, \sigma - \eta + x + ct) \\
&\quad + \partial_{33}^2 v_2(\xi, \tau, x + ct)) - c\partial_2 R(-\sigma, c^{-1}\sigma)(-\partial_3 v_1(\xi, \tau, 2\sigma \\
&\quad + x - ct) + \partial_3 v_2(\xi, \tau, \sigma + \eta + x - ct) \\
&\quad - \partial_3 v_1(\xi, \tau, \sigma - \eta + x + ct) + \partial_3 v_2(\xi, \tau, x + ct)) \\
&\quad + c^2 R(\sigma, c^{-1}\sigma)(\partial_{33}^2 v_1(\xi, \tau, x - ct) \\
&\quad + \partial_{33}^2 v_2(\xi, \tau, \eta - \sigma + x - ct) + \partial_{33}^2 v_1(\xi, \tau, -\sigma - \eta + x + ct) \\
&\quad + \partial_{33}^2 v_2(\xi, \tau, -2\sigma + x + ct)) - c\partial_2 R(\sigma, c^{-1}\sigma) \\
&\quad \times(-\partial_3 v_1(\xi, \tau, x - ct) + \partial_3 v_2(\xi, \tau, \eta - \sigma + x - ct) \\
&\quad - \partial_3 v_1(\xi, \tau, -\eta - \sigma + x + ct) \\
&\quad \left. + \partial_3 v_2(\xi, \tau, -2\sigma + x + ct))] d\eta d\sigma\right].
\end{aligned} \tag{19}$$

Note that $R(\pm\sigma, c^{-1}\sigma)$ corresponds to correlations along the two characteristic directions. We shall assume that both R and $\partial_2 R$ decrease rapidly as a function of σ in the sense that

$$\int_0^\infty [R(\pm\sigma, c^{-1}\sigma) + |\partial_2 R(\pm\sigma, c^{-1}\sigma)|]\sigma^n d\sigma < \infty \tag{20}$$

for $n = 0, 1$. In view of assumptions (17) and similar assumptions for the partial derivatives, contributions to the secular term will arise from those portions of the integrand of (19) which are independent of η. For brevity of notation, let

$$\alpha = x - ct, \quad \beta = x + ct. \tag{21}$$

Combining the secular terms in (19) with those of (18), we observe that

$$\begin{aligned}
&-\left(\frac{\partial}{\partial\tau} + c\frac{\partial}{\partial\xi}\right) v_1(\xi, \tau, \alpha) - \left(\frac{\partial}{\partial\tau} - c\frac{\partial}{\partial\xi}\right) v_2(\xi, \tau, \beta) \\
&\quad + \frac{c}{4}\int_0^{2ct} R(-\sigma, c^{-1}\sigma)\partial_{33}^2 v_1(\xi, \tau, 2\sigma + \alpha) d\sigma \\
&\quad + \frac{1}{4}\int_0^{2ct} \partial_2 R(-\sigma, c^{-1}\sigma)\partial_3 v_1(\xi, \tau, 2\sigma + \alpha) d\sigma \\
&\quad + \frac{c}{4}\partial_{33}^2 v_1(\xi, \tau, \alpha)\int_0^{2ct} R(\sigma, c^{-1}\sigma) d\sigma \\
&\quad + \frac{1}{4}\partial_3 v_1(\xi, \tau, \alpha)\int_0^{2ct} \partial_2 R(\sigma, c^{-1}\sigma) d\sigma \\
&\quad + \frac{c}{4}\partial_{33}^2 v_2(\xi, \tau, \beta)\int_0^{2ct} R(-\sigma, c^{-1}\sigma) d\sigma \\
&\quad - \frac{1}{4}\partial_3 v_2(\xi, \tau, \beta)\int_0^{2ct} \partial_2 R(-\sigma, c^{-1}\sigma) d\sigma \\
&\quad + \frac{c}{4}\int_0^{2ct} R(\sigma, c^{-1}\sigma)\partial_{33}^2 v_2(\xi, \tau, -2\sigma + \beta) d\sigma \\
&\quad - \frac{1}{4}\int_0^{2ct} \partial_2 R(\sigma, c^{-1}\sigma)\partial_3 v_2(\xi, \tau, -2\sigma + \beta) d\sigma
\end{aligned} \tag{22}$$

must necessarily vanish as $t \to \infty$ if $\epsilon^2\langle u_2 \rangle$ is to be genuinely small on $O(1)$ $\xi\tau$ scales. Recall that we have tacitly assumed initial data corresponding to wavepacket propagation. In the absence of random fluctuations, an initial localized disturbance would split into forward and backward propagating components; these components, in turn, would propagate undistorted along the characteristics.

In the randomly perturbed case, we shall assume that the same gross qualitative features exist, i.e., that the support of v_1 is concentrated on a family of characteristics $x - ct = \alpha =$ const while the support of v_2 remains concentrated on a family of characteristics $x + ct = \beta$ = const. Thus, while the packets may be distorted, smeared or otherwise affected by the random fluctuations, we assume that these fluctuations have not totally obliterated the packets.

Suppression of the secular term (22) as $t \to \infty$ reduces, therefore, to the suppression of secular growth in the two characteristic directions. Setting $\alpha \equiv$ const and letting $t \to \infty$, we obtain the equation

$$\begin{aligned}
&\left(\frac{\partial}{\partial\tau} + c\frac{\partial}{\partial\xi}\right) v_1(\xi, \tau, \alpha) \\
&\quad = \frac{c}{4}\int_0^\infty R(-\sigma, c^{-1}\sigma)\frac{\partial^2}{\partial\alpha^2} v_1(\xi, \tau, 2\sigma + \alpha) d\sigma \\
&\quad + \frac{1}{4}\int_0^\infty \partial_2 R(-\sigma, c^{-1}\sigma)\frac{\partial}{\partial\alpha} v_1(\xi, \tau, 2\sigma + \alpha) d\sigma \\
&\quad + \frac{c}{4}\frac{\partial^2}{\partial\alpha^2} v_1(\xi, \tau, \alpha)\int_0^\infty R(\sigma, c^{-1}\sigma) d\sigma \\
&\quad + \frac{1}{4}\frac{\partial}{\partial\alpha} v_1(\xi, \tau, \alpha)\int_0^\infty \partial_2 R(\sigma, c^{-1}\sigma) d\sigma.
\end{aligned} \tag{23a}$$

The partial derivatives with respect to the third argument have been interpreted as partial derivatives with respect to α. Since $\beta = \alpha + 2ct$, taking the limit $t \to \infty$ along a family of characteristics $\alpha = \text{const}$ will suppress the v_2 terms in (22); we are assuming that v_2 and its various partial derivatives with respect to β vanish as $\beta \to \infty$. [This added assumption is similar but more restrictive than the ones made in (17).]

Setting $\beta \equiv \text{const}$ (so that $\alpha = \beta - 2ct$) and letting $t \to \infty$, we obtain the second equation,

$$\begin{aligned}\left(\frac{\partial}{\partial \tau} - c\frac{\partial}{\partial \xi}\right) v_2(\xi, \tau, \beta) &= \frac{c}{4}\frac{\partial^2}{\partial \beta^2} v_2(\xi, \tau, \beta) \int_0^\infty R(-\sigma, c^{-1}\sigma) d\sigma \\ &- \frac{1}{4}\frac{\partial}{\partial \beta} v_2(\xi, \tau, \beta) \int_0^\infty \partial_2 R(-\sigma, c^{-1}\sigma) d\sigma \\ &+ \frac{c}{4}\int_0^\infty R(\sigma, c^{-1}\sigma)\frac{\partial^2}{\partial \beta^2} v_2(\xi, \tau, -2\sigma + \beta) d\sigma \\ &- \frac{1}{4}\int_0^\infty \partial_2 R(\sigma, c^{-1}\sigma)\frac{\partial}{\partial \beta} v_2(\xi, \tau, -2\sigma + \beta) d\sigma. \end{aligned} \qquad (23b)$$

Together with Eq. (23), we have initial conditions (13), which can be recast as

$$v_1(\xi, 0, \alpha) = \tfrac{1}{2} f(\alpha) - \frac{1}{2c}\int_{-\infty}^{\alpha} g(s) ds, \qquad (24a)$$

$$v_2(\xi, 0, \beta) = \tfrac{1}{2} f(\beta) + \frac{1}{2c}\int_{-\infty}^{\beta} g(s) ds. \qquad (24b)$$

Both α and β range from $-\infty$ to $+\infty$ as x and t vary over the half-plane $-\infty < x < \infty$, $t \geq 0$. Introducing Fourier transforms greatly simplifies problems (23) and (24). Define

$$\hat{v}_i(\xi, \tau, \gamma) \equiv \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} v_i(\xi, \tau, z) e^{-i\gamma z} dz, \quad i = 1, 2. \qquad (25)$$

Then, (23) and (24) transform into the following pair of first order Cauchy problems:

$$\left(\frac{\partial}{\partial \tau} + c\frac{\partial}{\partial \xi}\right)\hat{v}_1(\xi, \tau, \gamma) = -\Gamma_1(\gamma)\hat{v}_1(\xi, \tau, \gamma), \qquad (26)$$
$$-\infty < \xi < \infty, \quad \tau \geq 0, \quad -\infty < \gamma < \infty$$

$$\begin{aligned}\Gamma_1(\gamma) \equiv &\frac{c\gamma^2}{4}\int_0^\infty R(-\sigma, c^{-1}\sigma) e^{i2\gamma\sigma} d\sigma + \frac{c\gamma^2}{4}\int_0^\infty R(\sigma, c^{-1}\sigma) d\sigma \\ &- \frac{i\gamma}{4}\int_0^\infty \partial_2 R(-\sigma, c^{-1}\sigma) e^{i2\gamma\sigma} d\sigma - \frac{i\gamma}{4}\int_0^\infty \partial_2 R(\sigma, c^{-1}\sigma) d\sigma,\end{aligned}$$

$$\left(\frac{\partial}{\partial \tau} - c\frac{\partial}{\partial \xi}\right)\hat{v}_2(\xi, \tau, \gamma) = -\Gamma_2(\gamma)\hat{v}_2(\xi, \tau, \gamma), \qquad (27)$$
$$-\infty < \xi < \infty, \quad \tau \geq 0, \quad -\infty < \gamma < \infty,$$

$$\begin{aligned}\Gamma_2(\gamma) \equiv &\frac{c\gamma^2}{4}\int_0^\infty R(-\sigma, c^{-1}\sigma) d\sigma + \frac{c\gamma^2}{4}\int_0^\infty R(\sigma, c^{-1}\sigma) e^{-i2\gamma\sigma} d\sigma \\ &+ \frac{i\gamma}{4}\int_0^\infty \partial_2 R(-\sigma, c^{-1}\sigma) d\sigma + \frac{i\gamma}{4}\int_0^\infty \partial_2 R(\sigma, c^{-1}\sigma) e^{-i2\gamma\sigma} d\sigma,\end{aligned}$$

where the initial data $\hat{v}_i(\xi, 0, \gamma)$, $i = 1, 2$, are obtained by the Fourier transformation of Eq. (24). Note that the initial data is actually independent of ξ; to make this point explicit, we shall set $\hat{v}_i(\xi, 0, \gamma) \equiv \Phi_i(\gamma)$, $i = 1, 2$. Then, the solutions of (26) and (27) are

$$\hat{v}_i(\xi, \tau, \gamma) = \Phi_i(\gamma)\exp[-\Gamma_i(\gamma)\tau], \quad i = 1, 2. \qquad (28)$$

The desired functions $v_1(\xi, \tau, \alpha)$ and $v_2(\xi, \tau, \beta)$ must then be determined by inverse Fourier transformation.

We shall conclude this section by considering an idealized special case for which the computations are particularly simple. Let

$$f(x) = \frac{e^{-x^2/2\aleph^2}}{(2\pi)^{1/2}\aleph}, \quad g(x) = -c\frac{df}{dx}(x). \qquad (29)$$

This choice of initial data corresponds to

$$v_1(\xi, 0, \alpha) = f(\alpha), \quad v_2(\xi, 0, \beta) = 0. \qquad (30)$$

Thus we consider the case of a right-propagating Gaussian pulse; in the absence of random fluctuations, this pulse would propagate undistorted. For simplicity, assume that the random field μ is independent of time and spatially delta-correlated, i.e.,

$$R(x - x', t - t') = S_0\delta(x - x'). \qquad (31)$$

Then, the inverse Fourier transformation of expressions (28) for this example leads to

$$\begin{aligned} v_1(\xi, \tau, x - ct) &= \frac{\exp[-(x - ct)^2/2(\aleph^2 + \frac{1}{2}cS_0\tau)]}{(2\pi)^{1/2}(\aleph^2 + \frac{1}{2}cS_0\tau)^{1/2}}, \\ v_2(\xi, \tau, x + ct) &= 0. \end{aligned} \qquad (32)$$

Therefore,

$$\langle u(x, t, \omega, \epsilon)\rangle \sim \frac{\exp[-(x - ct)^2/2(\aleph^2 + \frac{1}{2}\epsilon^2 cS_0 t)]}{(2\pi)^{1/2}(\aleph^2 + \frac{1}{2}\epsilon^2 cS_0 t)^{1/2}}. \qquad (33)$$

IV. EQUATIONS FOR THE MUTUAL COHERENCE FUNCTION

In this section we shall study the asymptotic behavior of $\langle u(x_1, t_1, \omega, \epsilon) u(x_2, t_2, \omega, \epsilon)\rangle$. We again introduce slow variables $\xi_i, \tau_i, i = 1, 2$, and develop the solution at each space–time point in an ϵ power series [cf. (10)]. For brevity, let $u^{(i)}$ and $u_n^{(i)}$ denote $u(x_i, t_i, \xi_i, \tau_i, \omega, \epsilon)$ and $u_n(x_i, t_i, \xi_i, \tau_i, \omega)$, respectively (where $i = 1, 2$ and $n = 0, 1, 2, \cdots$). Then

$$\begin{aligned} u^{(1)}u^{(2)} = u_0^{(1)}u_0^{(2)} &+ \epsilon(u_0^{(1)}u_1^{(2)} + u_1^{(1)}u_0^{(2)}) \\ &+ \epsilon^2(u_0^{(1)}u_0^{(2)} + u_1^{(1)}u_1^{(2)} + u_2^{(1)}u_0^{(2)}) + \cdots. \end{aligned} \qquad (34)$$

The product $u_0^{(1)}u_0^{(2)}$ is computationally a deterministic quantity. Noting (14) we obtain

$$\begin{aligned} \langle u^{(1)}u^{(2)}\rangle = u_0^{(1)}u_0^{(2)} &+ \epsilon^2(u_0^{(1)}\langle u_2^{(2)}\rangle + \langle u_1^{(1)}u_1^{(2)}\rangle \\ &+ \langle u_2^{(1)}\rangle u_0^{(2)}) + \cdots. \end{aligned} \qquad (35)$$

Recall that we are essentially interested in the evolution of wavepackets in a random environment. Thus, the spatial and temporal offsets of interest, i.e.,

$$\Delta x \equiv x_1 - x_2, \quad \Delta t \equiv t_1 - t_2 \tag{36}$$

will be $O(1)$, being limited basically by the support of the packet. Consequently, the case of interest will correspond to $O(\epsilon^2)$ offsets in the slow variables ξ and τ. Therefore, we shall be ultimately concerned (to leading order) with $\xi_1 = \xi_2 \equiv \xi$ and $\tau_1 = \tau_2 \equiv \tau$.

Note that the consideration of the mutual coherence function involves a "cross-coupling" term $\langle u_1^{(1)} u_1^{(2)} \rangle$ which will contribute to the ultimate equations of evolution. Let

$$\alpha_i = x_i - ct_i, \quad \beta_i = x_i + ct_i, \quad i = 1, 2,$$
$$X = \tfrac{1}{2}(x_1 + x_2), \quad T \equiv \tfrac{1}{2}(t_1 + t_2). \tag{37}$$

Again, we must identify the terms in the ϵ^2 coefficient of (35) which grow secularly as $T \to \infty$. Recall that the $v_1(\xi_i, \tau_i, \alpha_i)$ terms have their support located near $X - cT =$ const while the $v_2(\xi_i, \tau_i, \beta_i)$ terms have their support in β located near $X + cT =$ const. Therefore, as $T \to \infty$, two equations will emerge from the need to suppress secular growth along the two families of characteristics. Note that cross products of the form $v_1(\xi_i, \tau_i, \alpha_i) v_2(\xi_j, \tau_j, \beta_j)$, $i, j = 1, 2$, will tend to zero as T increases since the supports of the two terms forming the product become essentially disjoint.

Suppression of secular growth along the characteristic family $\alpha =$ const necessitates the vanishing of

$$-v_1(\xi_1, \tau_1, \alpha_1)\left(\frac{\partial}{\partial \tau_2} + c\frac{\partial}{\partial \xi_2}\right) v_1(\xi_2, \tau_2, \alpha_2) - v_1(\xi_2, \tau_2, \alpha_2)$$
$$\times \left(\frac{\partial}{\partial \tau_1} + c\frac{\partial}{\partial \xi_1}\right) v_1(\xi_1, \tau_1, \alpha_1) + \frac{c}{4}\int_0^\infty R(-\sigma, c^{-1}\sigma)$$
$$\times \left(v_1(\xi_1, \tau_1, \alpha_1)\frac{\partial^2}{\partial \alpha_2^2} v_1(\xi_2, \tau_2, 2\sigma + \alpha_2) + v_1(\xi_2, \tau_2, \alpha_2)\right.$$
$$\left.\times \frac{\partial^2}{\partial \alpha_1^2} v_1(\xi_1, \tau_1, 2\sigma + \alpha_1)\right) d\sigma$$
$$+ \frac{1}{4}\int_0^\infty \partial_2 R(-\sigma, c^{-1}\sigma)\left(v_1(\xi_1, \tau_1, \alpha_1)\frac{\partial}{\partial \alpha_2} v_1(\xi_2, \tau_2, 2\sigma + \alpha_2)\right.$$
$$\left. + v_1(\xi_2, \tau_2, \alpha_2)\frac{\partial}{\partial \alpha_1} v_1(\xi_1, \tau_1, 2\sigma + \alpha_1)\right) d\sigma$$
$$+ \frac{c}{4}\left(v_1(\xi_1, \tau_1, \alpha_1)\frac{\partial^2}{\partial \alpha_2^2} v_1(\xi_2, \tau_2, \alpha_2) + v_1(\xi_2, \tau_2, \alpha_2)\right.$$
$$\left. \times \frac{\partial^2}{\partial \alpha_1^2} v_1(\xi_1, \tau_1, \alpha_1)\right)\int_0^\infty R(\sigma, c^{-1}\sigma) d\sigma + \frac{1}{4}\left(v_1(\xi_1, \tau_1, \alpha_1)\right.$$
$$\left. \times \frac{\partial}{\partial \alpha_2} v_1(\xi_2, \tau_2, \alpha_2) + v_1(\xi_2, \tau_2, \alpha_2)\frac{\partial}{\partial \alpha_1} v_1(\xi_1, \tau_1, \alpha_1)\right)$$
$$\times \int_0^\infty \partial_2 R(\sigma, c^{-1}\sigma) d\sigma + \frac{c}{4}\frac{\partial}{\partial \alpha_1} v_1(\xi_1, \tau_1, \alpha_1)$$
$$\times \frac{\partial}{\partial \alpha_2} v_2(\xi_2, \tau_2, \alpha_2)\int_{-\infty}^\infty R(\sigma, c^{-1}(\sigma - \alpha_1 + \alpha_2)) d\sigma. \tag{38}$$

Suppression of secular growth along the characteristic family $\beta =$ const leads to the vanishing of

$$-v_2(\xi_1, \tau_1, \beta_1)\left(\frac{\partial}{\partial \tau_2} - c\frac{\partial}{\partial \xi_2}\right) v_2(\xi_2, \tau_2, \beta_2) - v_2(\xi_2, \tau_2, \beta_2)$$
$$\times \left(\frac{\partial}{\partial \tau_1} - c\frac{\partial}{\partial \xi_1}\right) v_2(\xi_1, \tau_1, \beta_1) + \frac{c}{4}\left(v_2(\xi_1, \tau_1, \beta_1)\frac{\partial^2}{\partial \beta_2^2}\right.$$
$$\left. \times v_2(\xi_2, \tau_2, \beta_2) + v_2(\xi_2, \tau_2, \beta_2)\frac{\partial^2}{\partial \beta_1^2} v_2(\xi_1, \tau_1, \beta_1)\right)$$
$$\times \int_0^\infty R(-\sigma, c^{-1}\sigma) d\sigma - \frac{1}{4}\left(v_2(\xi_1, \tau_1, \beta_1)\frac{\partial}{\partial \beta_2} v_2(\xi_2, \tau_2, \beta_2)\right.$$
$$\left. + v_2(\xi_2, \tau_2, \beta_2)\frac{\partial}{\partial \beta_1} v_2(\xi_1, \tau_1, \beta_1)\right)\int_0^\infty \partial_2 R(-\sigma, c^{-1}\sigma) d\sigma$$
$$+ \frac{c}{4}\int_0^\infty R(\sigma, c^{-1}\sigma)\left(v_2(\xi_1, \tau_1, \beta_1)\frac{\partial^2}{\partial \beta_2^2} v_2(\xi_2, \tau_2, \beta_2 - 2\sigma)\right.$$
$$\left. + v_2(\xi_2, \tau_2, \beta_2)\frac{\partial^2}{\partial \beta_1^2} v_2(\xi_1, \tau_1, \beta_1 - 2\sigma)\right) d\sigma$$
$$- \frac{1}{4}\int_0^\infty \partial_2 R(\sigma, c^{-1}\sigma)\left(v_2(\xi_1, \tau_1, \beta_1)\frac{\partial}{\partial \beta_2} v_2(\xi_2, \tau_2, \beta_2 - 2\sigma)\right.$$
$$\left. + v_2(\xi_2, \tau_2, \beta_2)\frac{\partial}{\partial \beta_1} v_2(\xi_1, \tau_1, \beta_1 - 2\sigma)\right) d\sigma$$
$$+ \frac{c}{4}\frac{\partial}{\partial \beta_1} v_2(\xi_1, \tau_1, \beta_1)\frac{\partial}{\partial \beta_2} v_2(\xi_2, \tau_2, \beta_2)$$
$$\times \int_{-\infty}^\infty R(\sigma, c^{-1}(\beta_1 - \beta_2 - \sigma)) d\sigma. \tag{39}$$

Recall that we are interested in the case where $\xi_1 = \xi_2 \equiv \xi$ and $\tau_1 = \tau_2 \equiv \tau$; define

$$w_1(\xi, \tau, \alpha_1, \alpha_2) \equiv v_1(\xi, \tau, \alpha_1) v_1(\xi, \tau, \alpha_2),$$
$$w_2(\xi, \tau, \beta_1, \beta_2) \equiv v_2(\xi, \tau, \beta_1) v_2(\xi, \tau, \beta_2). \tag{40}$$

Then, (38) and (39) can be recast as the following equations for w_1 and w_2:

$$\left(\frac{\partial}{\partial \tau} + c\frac{\partial}{\partial \xi}\right) w_1(\xi, \tau, \alpha_1, \alpha_2)$$
$$= \frac{c}{4}\int_0^\infty R(-\sigma, c^{-1}\sigma)\left(\frac{\partial^2}{\partial \alpha_1^2} w_1(\xi, \tau, 2\sigma + \alpha_1, \alpha_2)\right.$$
$$\left. + \frac{\partial^2}{\partial \alpha_2^2} w_1(\xi, \tau, \alpha_1, 2\sigma + \alpha_2)\right) d\sigma$$
$$+ \frac{1}{4}\int_0^\infty \partial_2 R(-\sigma, c^{-1}\sigma)\left(\frac{\partial}{\partial \alpha_1} w_1(\xi, \tau, 2\sigma + \alpha_1, \alpha_2)\right.$$
$$\left. + \frac{\partial}{\partial \alpha_2} w_1(\xi, \tau, \alpha_1, 2\sigma + \alpha_2)\right) d\sigma$$
$$+ \frac{c}{4}\left(\frac{\partial^2}{\partial \alpha_1^2} + \frac{\partial^2}{\partial \alpha_2^2}\right) w_1(\xi, \tau, \alpha_1, \alpha_2)\int_0^\infty R(\sigma, c^{-1}\sigma) d\sigma$$
$$+ \frac{1}{4}\left(\frac{\partial}{\partial \alpha_1} + \frac{\partial}{\partial \alpha_2}\right) w_1(\xi, \tau, \alpha_1, \alpha_2)\int_0^\infty \partial_2 R(\sigma, c^{-1}\sigma) d\sigma$$
$$+ \frac{c}{4}\frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} w_1(\xi, \tau, \alpha_1, \alpha_2)\int_{-\infty}^\infty R(\sigma, c^{-1}(\sigma - \alpha_1 + \alpha_2)) d\sigma, \tag{41}$$

$$\left(\frac{\partial}{\partial\tau}-\frac{c\partial}{\partial\xi}\right)w_2(\xi,\tau,\beta_1,\beta_2)$$
$$=\frac{c}{4}\left(\frac{\partial^2}{\partial\beta_1^2}+\frac{\partial^2}{\partial\beta_2^2}\right)w_2(\xi,\tau,\beta_1,\beta_2)\int_0^\infty R(-\sigma,c^{-1}\sigma)d\sigma$$
$$-\frac{1}{4}\left(\frac{\partial}{\partial\beta_1}+\frac{\partial}{\partial\beta_2}\right)w_2(\xi,\tau,\beta_1,\beta_2)\int_0^\infty \partial_2R(-\sigma,c^{-1}\sigma)d\sigma$$
$$+\frac{c}{4}\int_0^\infty R(\sigma,c^{-1}\sigma)\left(\frac{\partial^2}{\partial\beta_1^2}w_2(\xi,\tau,\beta_1-2\sigma,\beta_2)\right.$$
$$\left.+\frac{\partial^2}{\partial\beta_2^2}w_2(\xi,\tau,\beta_1,\beta_2-2\sigma)\right)d\sigma-\frac{1}{4}\int_0^\infty \partial_2R(\sigma,c^{-1}\sigma)$$
$$\times\left(\frac{\partial}{\partial\beta_1}w_2(\xi,\tau,\beta_1-2\sigma,\beta_2)+\frac{\partial}{\partial\beta_2}w_2(\xi,\tau,\beta_1,\beta_2-2\sigma)\right)d\sigma$$
$$+\frac{c}{4}\frac{\partial^2}{\partial\beta_1\beta_2}w_2(\xi,\tau,\beta_1,\beta_2)\int_{-\infty}^\infty R(\sigma,c^{-1}(\beta_1-\beta_2-\sigma))d\sigma. \tag{42}$$

The analysis of (41) and (42) is again greatly facilitated by the use of Fourier transforms; define

$$\hat{w}_i(\xi,\tau,\gamma_1,\gamma_2)\equiv\frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty w_i(\xi,\tau,z_1,z_2)\times\exp[-i(\gamma_1z_1+\gamma_2z_2)]dz_1dz_2,\quad i=1,2. \tag{43}$$

Note that if the random field varies solely as a function of space or time, the equations for $\hat{w}_1$ and $\hat{w}_2$ will be first order linear constant coefficient equations. The general case, however, will not be so simple; the "cross-product" terms [i.e., the last terms in (41) and (42)] become convolution integrals in the transform domain. Let

$$\rho_1(z)\equiv\int_{-\infty}^\infty R(\sigma,c^{-1}(\sigma-z))d\sigma,$$
$$\rho_2(z)\equiv\int_{-\infty}^\infty R(\sigma,c^{-1}(z-\sigma))d\sigma \tag{44}$$

and let $\hat{\rho}_i$, $i=1,2$, denote the corresponding Fourier transforms. Then, a Fourier transformation of (41) and (42) leads to

$$\left(\frac{\partial}{\partial\tau}+c\frac{\partial}{\partial\xi}\right)\hat{w}_1(\xi,\tau,\gamma_1,\gamma_2)$$
$$=-\Lambda(\gamma_1,\gamma_2)\hat{w}_1(\xi,\tau,\gamma_1,\gamma_2)-\frac{c}{4(2\pi)^{1/2}}\int_{-\infty}^\infty\hat{\rho}_1(\gamma)\times(\gamma_1-\gamma)(\gamma_2+\gamma)\hat{w}_1(\xi,\tau,\gamma_1-\gamma,\gamma_2+\gamma)d\gamma,$$
$$\Lambda_1(\gamma_1,\gamma_2)\equiv\frac{c}{4}\left(\gamma_1^2\int_0^\infty R(-\sigma,c^{-1}\sigma)\exp(i2\gamma_1\sigma)d\sigma+\gamma_2^2\int_0^\infty R(-\sigma,c^{-1}\sigma)\exp(i2\gamma_2\sigma)d\sigma+(\gamma_1^2+\gamma_2^2)\int_0^\infty R(\sigma,c^{-1}\sigma)d\sigma\right)$$
$$-\frac{i}{4}\left(\gamma_1\int_0^\infty\partial_2R(-\sigma,c^{-1}\sigma)\exp(i2\gamma_1\sigma)d\sigma+\gamma_2\int_0^\infty\partial_2R(-\sigma,c^{-1}\sigma)\exp(i2\gamma_2\sigma)d\sigma+(\gamma_1+\gamma_2)\int_0^\infty\partial_2R(\sigma,c^{-1}\sigma)d\sigma\right) \tag{45}$$

and

$$\left(\frac{\partial}{\partial\tau}-c\frac{\partial}{\partial\xi}\right)\hat{w}_2(\xi,\tau,\gamma_1,\gamma_2)$$
$$=-\Lambda_2(\gamma_1,\gamma_2)\hat{w}_2(\xi,\tau,\gamma_1,\gamma_2)-\frac{c}{4(2\pi)^{1/2}}\times\int_{-\infty}^\infty\hat{\rho}_2(\gamma)(\gamma_1-\gamma)(\gamma_2+\gamma)\hat{w}_2(\xi,\tau,\gamma_1-\gamma,\gamma_2+\gamma)d\gamma, \tag{46}$$
$$\Lambda_2(\gamma_1,\gamma_2)\equiv\frac{c}{4}\left((\gamma_1^2+\gamma_2^2)\int_0^\infty R(-\sigma,c^{-1}\sigma)d\sigma+\gamma_1^2\int_0^\infty R(\sigma,c^{-1}\sigma)\exp(-i2\gamma_1\sigma)d\sigma+\gamma_2^2\int_0^\infty R(\sigma,c^{-1}\sigma)\exp(-i2\gamma_2\sigma)d\sigma\right)$$
$$+\frac{i}{4}\left((\gamma_1+\gamma_2)\int_0^\infty\partial_2R(\sigma,c^{-1}\sigma)d\sigma+\gamma_1\int_0^\infty\partial_2R(\sigma,c^{-1}\sigma)\exp(-i2\gamma_1\sigma)d\sigma+\gamma_2\int_0^\infty\partial_2R(\sigma,c^{-1}\sigma)\exp(-i2\gamma_2\sigma)d\sigma\right).$$

We conclude this section by considering again the example discussed at the end of Sec. III; we consider a right-propagating Gaussian pulse in a random field that is time independent and spatially delta-correlated [cf. (29)–(31)]. In this case, (45) and (46) simplify to

$$\left(\frac{\partial}{\partial\tau}+\frac{c\partial}{\partial\xi}\right)\hat{w}_1=-\frac{c}{4}S_0(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)\hat{w}_1,$$
$$\hat{w}_1(\xi,0,\gamma_1,\gamma_2)=\frac{e^{-\aleph^2(\gamma_1^2+\gamma_2^2)/2}}{2\pi},$$
$$\left(\frac{\partial}{\partial\tau}-\frac{c\partial}{\partial\xi}\right)\hat{w}_2=-\frac{c}{4}S_0(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)\hat{w}_2,$$
$$\hat{w}_2(\xi,0,\gamma_1,\gamma_2)=0. \tag{47}$$

Solving these equations for $\hat{w}_i$, $i=1,2$, and taking inverse Fourier transforms lead to

$$\langle u(x_1,t_1,\omega,\epsilon)u(x_2,t_2,\omega,\epsilon)\rangle\sim\exp\left(-\frac{(\Delta x-c\Delta t)^2}{4(\aleph^2+\frac{1}{4}\epsilon^2cS_0T)}-\frac{(X-cT)^2}{(\aleph^2+\frac{3}{4}\epsilon^2cS_0T)}\right)\Big/\left[2\pi(\aleph^2+\tfrac{1}{4}\epsilon^2cS_0T)^{1/2}(\aleph^2+\tfrac{3}{4}\epsilon^2cS_0T)^{1/2}\right]. \tag{48}$$

Note that if $x_1=x_2=ct_1=ct_2$, the effect of the random field upon the mutual coherence function reduces to an attenuation due to the demoninator term. In general, the fluctuations in the solution are given by

$$\langle u^2(x,t,\omega,\epsilon)\rangle - \langle u(x,t,\omega,\epsilon)\rangle^2$$

$$\sim \exp\left(\frac{-(x-ct)^2}{(\aleph^2+\frac{3}{4}\epsilon^2 cS_0 t)}\right)\Big/\left[2\pi(\aleph^2+\tfrac{1}{4}\epsilon^2 cS_0 t)^{1/2}\;(\aleph^2+\tfrac{3}{4}\epsilon^2 cS_0 t)^{1/2}\right] - \exp\left(\frac{-(x-ct)^2}{(\aleph^2+\frac{1}{2}\epsilon^2 cS_0 t)}\right)\Big/\,2\pi(\aleph^2+\tfrac{1}{2}\epsilon^2 cS_0 t)\,. \tag{49}$$

V. A SPECIAL CASE

If the random field fluctuates solely as a function of space or time, the functions $\hat{w}_i(\xi,\tau,\gamma_1,\gamma_2)$, $i=1,2$, are readily determined as solutions of first-order linear equations. The mutual coherence function is then obtained by inverse Fourier transformation. In this section, we consider the case of a spatially fluctuating Gaussian random field; let

$$R(x,t)\equiv R(x)=\exp(-x^2/2\sigma^2)/(2\pi)^{1/2}\sigma\,. \tag{50}$$

We again adopt initial data corresponding to a right-propagating Gaussian pulse [cf. (29)]. Then, (45) and (46) [with $\hat{\rho}_1(\gamma)=\hat{\rho}_2(\gamma)=(2\pi)^{1/2}\delta(\gamma)$] imply

$$\hat{w}_1(\xi,\tau,\gamma_1,\gamma_2)$$

$$=\frac{1}{2\pi}\exp\Big[-\frac{\gamma_1^2}{2}\Big(\aleph^2+\frac{c\tau}{4}[1+\exp(-2\sigma^2\gamma_1^2)]\Big)$$

$$-\frac{\gamma_2^2}{2}\Big(\aleph^2+\frac{c\tau}{4}[1+\exp(-2\sigma^2\gamma_2^2)]\Big)-\frac{c\tau}{4}\gamma_1\gamma_2-\frac{ic\tau}{4(2\pi)^{1/2}}$$

$$\times\Big(\gamma_1^2\exp(-2\sigma^2\gamma_1^2)\int_0^{\sigma\gamma_1}\exp(\lambda^2/2)d\lambda+\gamma_2^2$$

$$\times\int_0^{\sigma\gamma_2}\exp(\lambda^2/2)d\lambda\Big)\,, \tag{51}$$

$$w_2(\xi,\tau,\gamma_1,\gamma_2)=0\,.$$

For simplicity, we shall study the parametric dependence of $w_1(\xi,\tau,0,0)$ (i.e., $x_1=x_2=ct_1=ct_2$) upon σ. Noting (51) we have

$$w_1(\xi,\tau,0,0)$$

$$=\frac{1}{2\pi^2}\int_0^\infty\int_{-\infty}^\infty\exp\Big[-\frac{\gamma_1^2}{2}\Big(\aleph^2+\frac{c\tau}{4}[1+\exp(-2\sigma^2\gamma_1^2)]\Big)$$

$$-\frac{\gamma_2^2}{2}\Big(\aleph^2+\frac{c\tau}{4}[1+\exp(-2\sigma^2\gamma_2^2)]\Big)-\frac{c\tau}{4}\gamma_1\gamma_2\Big]$$

$$\times\cos\Big(\frac{c\tau}{4(2\pi)^{1/2}}[\gamma_1^2\exp(-2\sigma^2\gamma_1^2)\int_0^{\sigma\gamma_1}\exp(\lambda^2/2)d\lambda$$

$$+\gamma_2^2\exp(-2\sigma^2\gamma_2^2)\int_0^{\sigma\gamma_2}\exp(\lambda^2/2)d\lambda]\Big)d\gamma_1 d\gamma_2 \tag{52}$$

and

$$\frac{\langle u^2(x,t,\omega,\epsilon)\rangle|_{x=ct}}{\langle u^2(0,0,\omega,\epsilon)\rangle}\sim 2\pi\aleph^2 w_1(\xi,\tau,0,0)\,. \tag{53}$$

The variation of this ratio as a function of $c\tau$ ($=\epsilon^2 ct$) is shown in Fig. 1 for several values of σ. The case $\sigma=0$ corresponds to the delta-correlated case with $S_0=1$.

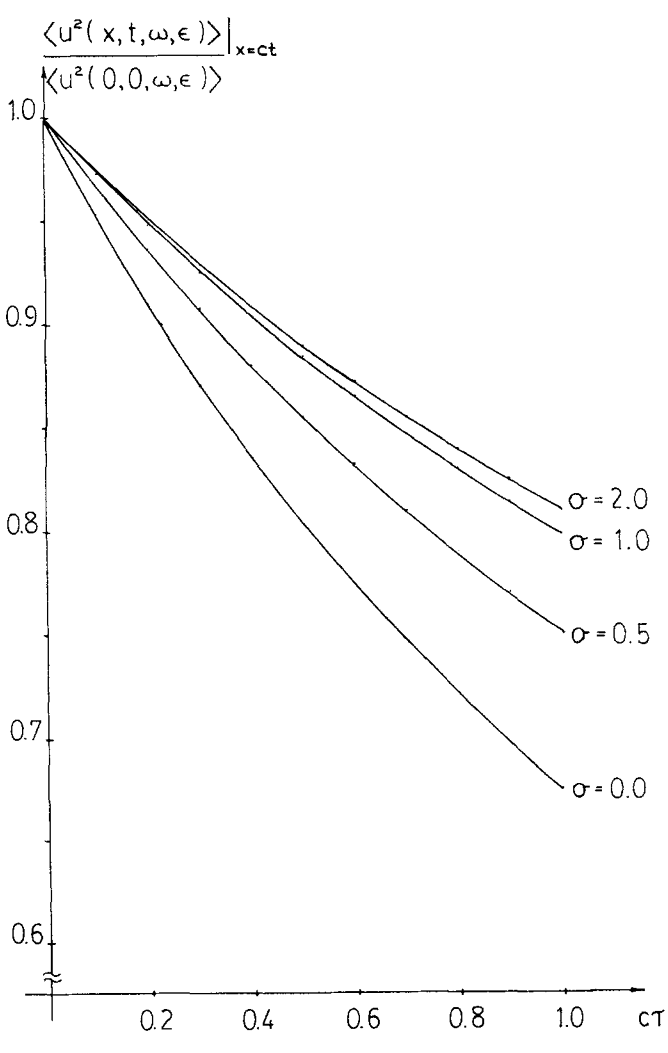

FIG. 1.

[1] G. C. Papanicolaou and J. B. Keller, SIAM J. Appl. Math. 21, 287 (1971).

[2] G. C. Papanicolaou and S. R. S. Varadhan, Commun. Pure Appl. Math. 26, 497 (1973).

[3] F. John, *Partial Differential Equations* (Springer, New York, 1975), 2nd ed.