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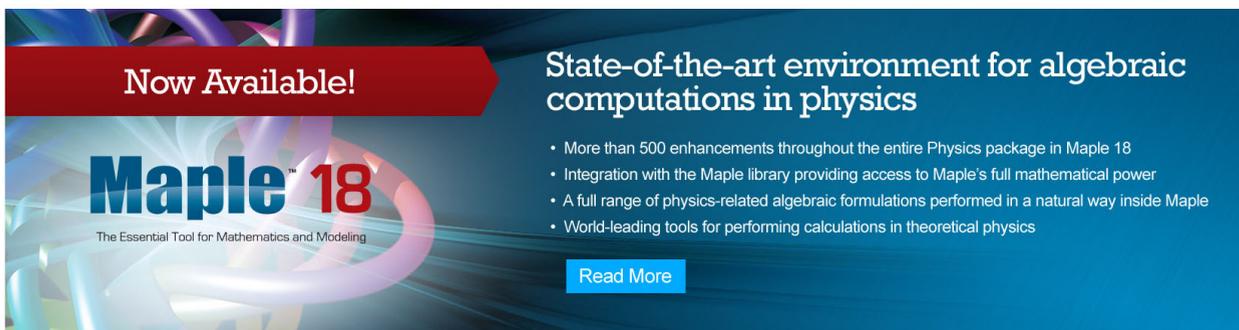
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On the Levinson theorem for Dirac operators

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For the Dirac equation with potential $V(r)$ obeying $\int_0^\infty (1+r)|V(r)|dr < \infty$ we prove a relativistic version of Levinson's theorem that relates the number of bound states in the spectral gap $[-m, m]$ to the variation of an appropriate phase along the continuous part of the spectrum. In the process, the asymptotic properties of the Jost function as $E \rightarrow \pm m$ are analyzed in detail. The connection with the nonrelativistic version of Levinson's theorem is also established.

I. INTRODUCTION

In this paper, we consider the Dirac equation for a particle moving in a central electrostatic potential $V(r)$. Separation of variables leads to the following systems of equations

$$H_\kappa(c)\psi = c \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \psi + \begin{pmatrix} mc^2 + V(r) & \kappa/r \\ \kappa/r & -mc^2 + V(r) \end{pmatrix} \psi = E\psi, \quad \psi = \begin{pmatrix} \psi_1(r) \\ \psi_2(r) \end{pmatrix} \quad (1.1)$$

on $0 < r < \infty$. Here, m is the mass of the particle, c is the velocity of light, E is the energy (in units where $\hbar = 1$), and κ is a nonzero integer. We assume that $V(r)$ satisfies

$$\int_0^\infty (1+r)|V(r)|dr < \infty. \quad (1.2)$$

This condition guarantees that the differential operator H is limit point at zero¹ (it is always limit point at infinity²) so that H_κ can be viewed as a self-adjoint operator in the Hilbert space of vector-valued functions ψ satisfying $\int_0^\infty (|\psi_1|^2 + |\psi_2|^2)dr < \infty$. The spectrum of H_κ is absolutely continuous on $(-\infty, -mc^2] \cup [mc^2, \infty)$ and consists of at most finitely many (simple) eigenvalues in the gap $[-mc^2, mc^2]$.

There is a deep connection between the continuous part and the discrete part of the spectrum. In the Schrödinger case, this is the content of Levinson's theorem.³ Here we study its relativistic analog. In order to facilitate the comparison with other authors, we make the substitution $\phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi$, which converts (1.1) into

$$\phi' = \begin{pmatrix} \kappa/r & mc + c^{-1}V(r) - c^{-1}E \\ mc - c^{-1}V(r) + c^{-1}E & -\kappa/r \end{pmatrix} \phi. \quad (1.3)$$

Henceforth, we will only consider $\kappa \geq 1$ which causes no loss of generality since on interchanging the components of ϕ the problem corresponding to κ, V, E is equivalent to that corresponding to $-\kappa, -V, -E$. We now also set $c = 1$ in this section and in Sec. II. Under assumption (1.2), Eq. (1.3) has a solution called the regular solution, which satisfies

$$\lim_{r \rightarrow 0} r^{-\kappa} \varphi_\kappa(E, r) = \begin{pmatrix} 1/(2\kappa - 1)!! \\ 0 \end{pmatrix}. \quad (1.4)$$

As $r \rightarrow \infty$ this solution behaves like

$$\varphi_\kappa(E, r) = k^{-\kappa} |F_\kappa(E)| \times \begin{pmatrix} \cos(kr - \kappa\pi/2 - \delta_\kappa(E)) \\ k/(E - m) \sin(kr - \kappa\pi/2 - \delta_\kappa(E)) \end{pmatrix} + o(1). \quad (1.5)$$

The parameter $k = \sqrt{E^2 - m^2}$ is defined by choosing a branch of k such that $k > 0$ for $E > m$ and $\text{Im } k > 0$ for $\text{Im } E > 0$. Then $k < 0$ corresponds to $E < -m$. This choice is different from that in Ref. 4 where $\text{Re } E > 0$ corresponds to $\text{Im } \kappa < 0$. Also, due to different conventions, our phase $\delta_\kappa(E)$ differs in sign from that in Refs. 4 or 5 for $E > m$ but agrees with it for $E < m$. Conceptually, the basic parameter for us is E and not k . The function $F_\kappa(E)$ is the analog of the Jost function in the Schrödinger case. It can be written as

$$F_\kappa(E) = |F_\kappa(E)| e^{i\delta_\kappa(E)} = 1 + \int_0^\infty (\varphi_\kappa(E, t))^T V(t) f_\kappa^0(E, t) dt, \quad (1.6)$$

where

$$f_\kappa^0(E, r) = k^\kappa \begin{pmatrix} [k^2 r / (E + m)] h_{\kappa-1}(kr) \\ kr h_\kappa(kr) \end{pmatrix}. \quad (1.7)$$

Here, $h_\kappa(kr) = n_\kappa(kr) + ij_\kappa(kr)$, where n_κ and j_κ denote spherical Bessel functions.⁴ Also, T denotes the ordinary transpose and a superscript 0 indicates a solution of the unperturbed ($V = 0$) problem. We recall that the zeros of $F_\kappa(E)$ are all simple, lie in the interval $[-m, m]$, and correspond to eigenvalues of H [see Ref. 4, Sec. 2, where $h_\kappa(k)$ corresponds to $F_\kappa(E)$]. The only exception occurs at $E = -m$ for $\kappa = 1$ when, if $F_1(-m) = 0$, the solution $\varphi_1(-m, r)$ is bounded but not square integrable at infinity [see (2.9), (2.11) below]. Then we say $E = -m$ is a half-bound state.

Theorem (1.1): Let $V(r)$ obey (1.2). Let N_κ ($\kappa \geq 1$) denote the number of eigenvalues of H_κ in $[-m, m]$. Then

$$N_\kappa = (1/\pi) \{ \delta_\kappa(-m) - \delta_\kappa(m) \}, \quad \kappa \geq 2, \quad (1.8)$$

$$N_\kappa = (1/\pi) \{ \delta_\kappa(-m) - \delta_\kappa(m) \} + (1/2\pi) \Delta, \quad \kappa = 1, \quad (1.9)$$

where

$$\Delta = \begin{cases} 0, & E = -m \text{ is not a half-bound state,} \\ -\pi, & E = -m \text{ is a half-bound state.} \end{cases} \quad (1.10)$$

It is important to add that the difference $\delta_\kappa(-m) - \delta_\kappa(m)$ may be viewed as the change of phase as we go, through *real* values, from $E = m$ to $+\infty$ and then from $E = -\infty$ to $E = -m$. Relativistic versions of the Levinson theorem have been studied before, by Barthélémy⁴ and more recently by Ni⁶ and Ma and Ni⁵ and also in Ref. 7. However, in Ref. 5, the authors point out that the results of Refs. 4 and 6 are not correct in general and they go on to derive a correct form of Levinson's theorem in the case where $V(r)$ has compact support. We will comment on the fallacy of Ref. 4 after the proof of Theorem (1.1). The hypothesis (1.2) is weaker than that in Ref. 4 where $\int_0^\infty |V(r)|r^n dr < \infty$ for $n = 0, 1$, or 2 was assumed and for Levinson's theorem $n = 2$ (and $n = 0$) were absolutely essential. The condition (1.2) is optimal for large r as far as moment-type conditions go since if $V \sim cr^{-2}$ as $r \rightarrow \infty$ then Levinson's theorem must be modified.⁸ This remark should answer to some extent a question in Ref. 5 concerning the proper assumptions on V which will insure that Levinson's theorem holds. Concerning the behavior as $r \rightarrow 0$, condition (1.2) excludes a r^{-1} singularity. It is conceivable to us that the methods used in this paper can be extended to include such a behavior. However, then the r^{-1} singularity cannot be treated perturbatively, which leads to some complications at the level of the unperturbed problem.

In Ref. 7, Dirac systems containing, in place of κ/r , a coefficient $p(r)$ such that $\int_0^\infty (1+r)|p(r)|dr < \infty$ were considered. In that case, $p(r)$ can be included in the perturbation which leads to some simplifications in the analysis. Conceivably, such a term $p(r)$ could be added to κ/r in (1.1) without essentially altering the analysis but we will not do so here. The Levinson theorem for the Schrödinger equation under assumption (1.2) was studied in Ref. 9. Although both Refs. 9 and 7 have provided us with some guidance for the present paper, we have encountered some unexpected complications in the case where $F_\kappa(\pm m) = 0$.

There exist several methods for proving the Levinson theorem in the relativistic and nonrelativistic case. In the relativistic case, the Green's function method was used in Ref. 5 and an approach based on the Sturm-Liouville theorem was used in Refs. 10 and 8. This latter approach was also used in the nonrelativistic case in Refs. 11 and 12. We follow Levinson's original proof for the Schrödinger equation³ which is based on a detailed study of the asymptotic properties of the Jost function whereby the main effort goes into analyzing the case where $F_\kappa(\pm m) = 0$. As in Ref. 9 but in contrast to Ref. 4 we do not work with the Jost solution at all, only with the regular solution $\varphi_\kappa(E, r)$ since the latter is better behaved as $E \rightarrow \pm m$ than the former [compare also Ref. 9, Corollary (3.31)].

Theorem (1.1) is proved in Sec. II. In Sec. III we discuss the nonrelativistic limit $c \rightarrow \infty$.

II. PROPERTIES OF $F_\kappa(E)$ AND PROOF OF THEOREM (1.1)

The solution $\varphi_\kappa(E, r)$ defined by (1.4) satisfies⁴

$$\varphi_\kappa(E, r) = \varphi_\kappa^0(E, r) + \int_0^r K(E; r, t) V(t) \varphi_\kappa(E, t) dt, \quad (2.1)$$

where

$$K(E; r, t) = \varphi_\kappa^0(E, r) (\psi_\kappa^0(E, t))^T - \psi_\kappa^0(E, r) (\varphi_\kappa^0(E, t))^T, \quad (2.2)$$

$$\varphi_\kappa^0(E, r) = k^{-\kappa} \begin{pmatrix} kr j_{\kappa-1}(kr) \\ [k/(E-m)] kr j_\kappa(kr) \end{pmatrix}, \quad (2.3)$$

$$\psi_\kappa^0(E, r) = k^\kappa \begin{pmatrix} (E-m) r n_{\kappa-1}(kr) \\ kr n_\kappa(kr) \end{pmatrix}. \quad (2.4)$$

We first collect some results concerning the solutions at $E = \pm m$ which will be used later on. From standard asymptotic analysis based on (2.1) it follows that

$$\varphi_\kappa(m, r) = \frac{1}{(2\kappa-1)!!} \times \begin{pmatrix} F_\kappa(m) r^\kappa + o(r^\kappa) \\ [2m/(2\kappa+1)] F_\kappa(m) r^{\kappa+1} + o(r^{\kappa+1}) \end{pmatrix}, \quad r \rightarrow \infty, \quad (2.5)$$

$$\varphi_\kappa(-m, r) = \frac{1}{(2\kappa-1)!!} \begin{pmatrix} F_\kappa(-m) r^\kappa + o(r^\kappa) \\ o(r^{\kappa-1}) \end{pmatrix}, \quad r \rightarrow \infty, \quad (2.6)$$

respectively, where [by (1.6)]

$$F_\kappa(m) = 1 + (2\kappa-1)!! \int_0^\infty \varphi_{\kappa,2}(m, t) V(t) t^{-\kappa} dt, \quad (2.7)$$

$$F_\kappa(-m) = 1 - 2m(2\kappa-3)!! \int_0^\infty \varphi_{\kappa,1}(-m, t) \times V(t) t^{-\kappa+1} dt + (2\kappa-1)!! \times \int_0^\infty \varphi_{\kappa,1}(-m, t) V(t) t^{-\kappa} dt. \quad (2.8)$$

The reason for having a term $o(r^{\kappa-1})$ in the second component of (2.6) is that $r^{-\kappa} \int_0^r \varphi_{\kappa,1}(-m, t) V(t) t^\kappa dt = o(r^{\kappa-1})$ which follows from the behavior of $\varphi_{\kappa,1}(-m, t)$ as $t \rightarrow \infty$ and (1.2). Besides the solution φ_κ , Eq. (1.3) has, for $E = m$, a second solution $\tilde{\varphi}$ satisfying

$$\tilde{\varphi}_\kappa(m, r) = \begin{pmatrix} o(r^{-\kappa-1}) \\ r^{-\kappa} + o(r^{-\kappa}) \end{pmatrix}, \quad r \rightarrow \infty. \quad (2.9)$$

Similarly, for $E = -m$, we have

$$\tilde{\varphi}_\kappa(-m, r) = \begin{pmatrix} [2m/(1-2\kappa)] r^{1-\kappa} + o(r^{1-\kappa}) \\ r^{-\kappa} + o(r^{-\kappa}) \end{pmatrix}, \quad r \rightarrow \infty. \quad (2.10)$$

Here, if we replace the o terms by zero we get exact solutions of the unperturbed ($V = 0$) problem which are bounded at infinity. By considering Wronskians, we see that φ_κ and $\tilde{\varphi}_\kappa$ are linearly dependent if and only if $F_\kappa(\pm m) = 0$. If that happens we set

$$\varphi_\kappa(\pm m, r) = -A_\kappa(\pm m) \tilde{\varphi}_\kappa(\pm m, r) \quad (2.11)$$

and deduce from (2.1) the representations

$$A_\kappa(m) = \int_0^\infty \varphi_{\kappa,1}(m, t) V(t) t^\kappa dt + \frac{2m}{2\kappa+1} \times \int_0^\infty \varphi_{\kappa,2}(m, t) V(t) t^{\kappa+1} dt, \quad (2.12)$$

$$A_\kappa(-m) = \int_0^\infty \varphi_{\kappa,1}(-m,t) V(t) t^\kappa dt. \quad (2.13)$$

If $E = m$ and $F_\kappa(m) = 0$, then, of course, there exists a second solution χ_κ of (1.3) such that $W(\varphi_\kappa, \chi_\kappa) = \varphi_\kappa \chi_{\kappa,2} - \chi_{\kappa,1} \varphi_{\kappa,2} = 1$. It satisfies

$$\chi_\kappa(m,r) = \frac{1}{A_\kappa(m)} \begin{pmatrix} r^\kappa + o(r^\kappa) \\ [2m/(2\kappa+1)]r^{\kappa+1} + o(r^{\kappa+1}) \end{pmatrix}, \quad r \rightarrow \infty \quad (2.14)$$

and

$$\chi_\kappa(m,r) = \begin{pmatrix} o(r^{-\kappa}) \\ (2\kappa-1)!!r^{-\kappa} + o(r^{-\kappa}) \end{pmatrix}, \quad r \rightarrow 0. \quad (2.15)$$

Similarly, at $E = -m$, if $F_\kappa(-m) = 0$, then a second solution χ_κ exists such that $[W(\varphi_\kappa, \chi_\kappa) = 1]$

$$\chi_\kappa(-m,r) = \frac{1}{A_\kappa(-m)} \begin{pmatrix} r^\kappa + o(r^\kappa) \\ o(r^{\kappa-1}) \end{pmatrix}, \quad r \rightarrow \infty, \quad (2.16)$$

$$\chi_\kappa(-m,r) = \begin{pmatrix} o(r^{-\kappa}) \\ (2\kappa-1)!!r^{-\kappa} + o(r^{-\kappa}) \end{pmatrix}, \quad r \rightarrow 0. \quad (2.17)$$

The solution χ_κ will be needed later. At the heart of our method is the following lemma whose proof we defer to the Appendix.

Lemma (2.1): Let $V(r)$ obey (1.2). Fix $\delta > 0$.

(i) If $F_\kappa(m) = 0, \kappa > 1$, then

$$|\varphi_{\kappa,j}(E,r) - \varphi_{\kappa,j}(m,r)| < Ck^2 [r/(1+kr)]^{\kappa+1}, \quad j = 1,2, \quad (2.18)$$

for $E \in [m, m + \delta]$ where C depends on δ but not on k and r .

(ii) If $F_\kappa(-m) = 0$ and $\kappa > 2$, then

$$|\varphi_{\kappa,1}(E,r) - \varphi_{\kappa,1}(-m,r)| < Ck^2 [r/(1+|k|r)]^\kappa, \quad (2.19)$$

$$|\varphi_{\kappa,2}(E,r) - \varphi_{\kappa,2}(-m,r)| < Ck^2 [(r/(1+|k|r))^{\kappa+1} + (r/(1+|k|r))^\kappa], \quad (2.20)$$

while if $\kappa = 1$, then

$$|\varphi_{\kappa,j}(E,r) - \varphi_{\kappa,j}(-m,r)| < C [(|k|r/(1+|k|r))^2 + k^2 r/(1+|k|r)], \quad j = 1,2, \quad (2.21)$$

for $E \in [-m - \delta, -m]$.

The pertinent properties of the Jost function are summarized in the next theorem. We denote the L^2 norm of a vector function by $\|\cdot\|$.

Theorem (2.2): Let $V(r)$ obey (1.2), then

(i) $F_\kappa(E)$ is analytic for $\text{Im } E > 0$ and has an analytic continuation into the half-plane $\text{Im } E < 0$. Moreover, the extended function $F_\kappa(E)$ assumes continuous boundary values as E approaches the real axis from either above or below.

(ii) As $|E| \rightarrow \infty, \text{Im } E > 0$,

$$F_\kappa(E) \rightarrow e^{i\int_0^\infty V(t)dt}. \quad (2.22)$$

(iii) If $F_\kappa(m) = 0$, then

$$F_\kappa(E) = c_\kappa k^2 + o(k^2), \quad (2.23)$$

$$c_\kappa = [(2\kappa-1)!!/2mA_\kappa(m)] \|\varphi_\kappa(m,\cdot)\|^2, \quad (2.24)$$

as $E \rightarrow m$ uniformly in $0 < \arg(E-m) < 2\pi$.

(iv) If $F_\kappa(-m) = 0$ and $\kappa > 2$, then

$$F_\kappa(E) = d_\kappa k^2 + o(k^2), \quad (2.25)$$

$$d_\kappa = -[(2\kappa-1)!!/2mA_\kappa(-m)] \|\varphi_\kappa(m,\cdot)\|^2, \quad (2.26)$$

while if $\kappa = 1$, then

$$F_\kappa(E) = d_\kappa k + o(k), \quad (2.27)$$

$$d_\kappa = 2miA_\kappa(-m), \quad (2.28)$$

as $E \rightarrow -m$ uniformly in $-\pi < \arg(E+m) < \pi$.

Proof: (i) It follows from Ref. 4 that there is a constant C such that for all E with $\text{Im } E > 0$:

$$|\varphi_{\kappa,1}(E,r)| < Ce^{(\text{Im } k)r} [r/(1+|k|r)]^\kappa \quad (2.29)$$

and

$$|\varphi_{\kappa,2}(E,r)| < C|E+m|e^{(\text{Im } k)r} [r/(1+|k|r)]^{\kappa+1}.$$

Also,

$$|f_{\kappa,1}^0(E,r)| < C(k^2/|E+m|)e^{-(\text{Im } k)r} \times [(1+|k|r)/r]^{\kappa-1}, \quad (2.30)$$

$$|f_{\kappa,2}^0(E,r)| < Ce^{-(\text{Im } k)r} [(1+|k|r)/r]^\kappa. \quad (2.31)$$

Hence,

$$|(\varphi_\kappa(E,r))^T V(r) f_\kappa^0(E,r)| < C|V(r)|(1+r), \quad |E \pm m| < \delta, \quad (2.32)$$

$$|(\varphi_\kappa(E,r))^T V(r) f_\kappa^0(E,r)| < C|V(r)|, \quad |E \pm m| > \delta, \quad (2.33)$$

for any $\delta > 0$ with an appropriate constant C . Since $\varphi_\kappa(E,r)$ is an entire function of E and $f_\kappa^0(E,r)$ is analytic for $\text{Im } E > 0$ and continuous for $\text{Im } E > 0$ the bounds (2.32), (2.33) insure that $F_\kappa(E)$ has the asserted analyticity and continuity properties. We get an analytic continuation into the lower half plane by the Schwarz reflection principle because $F_\kappa(E)$ is real for $-m < E < m$.

(ii) Since the construction of $\varphi_\kappa(E,r)$ only requires knowledge of V on $[0,r]$ the large E behavior of $\varphi_\kappa(E,r)$ can be inferred from Ref. 4,

$$\varphi_\kappa(E,r) = k^{-\kappa} \begin{pmatrix} \cos(kr - \kappa\pi/2 - \int_0^r V(t)dt) \\ \sin(kr - \kappa\pi/2 - \int_0^r V(t)dt) \end{pmatrix} + o(k^{-\kappa}), \quad (2.34)$$

and also

$$f_\kappa^0(E,r) = k^\kappa e^{i(kr - \kappa\pi/2)} \begin{pmatrix} i \\ 1 \end{pmatrix} + o(k^\kappa), \quad (2.35)$$

as $|E| \rightarrow \infty$ on $\{E: \text{Im } E > 0\}$. Owing to (2.29), (2.30), and (2.31) we may insert (2.34) and (2.35) in (1.6) and apply the Lebesgue Dominated Convergence Theorem. Then (2.22) follows.

(iii) Suppose $E > m$. We break the right-hand side of (1.6) into three parts, $F_\kappa(E) = I_1 + I_2 + I_3$, using $F_\kappa(m) = 0$, where

$$I_1 = \int_0^\infty (\varphi_\kappa(m,t))^T V(t) [f_\kappa^0(E,t) - f_\kappa^0(m,t)] dt, \quad (2.36)$$

$$I_2 = \int_0^\infty [(\varphi_\kappa(E,t))^T - (\varphi_\kappa(m,t))^T] V(t) f_\kappa^0(m,t) dt, \quad (2.37)$$

$$I_3 = \int_0^\infty [(\varphi_\kappa(E,t))^T - (\varphi_\kappa(m,t))^T] V(t) \times [f_\kappa^0(E,t) - f_\kappa^0(m,t)] dt. \quad (2.38)$$

Consider I_1 first. By (1.4), (2.9), and (2.11)

$$|\varphi_{\kappa,j}(m,r)| < C [r^\kappa / (1+r)^{2\kappa}], \quad j=1,2. \quad (2.39)$$

Moreover,

$$f_\kappa^0(E,r) = \begin{pmatrix} 0 \\ (2\kappa-1)!! r^{-\kappa} \end{pmatrix} + k^2 \begin{pmatrix} (2m)^{-1} (2m-3)!! r^{-\kappa+1} \\ (\frac{1}{2})(2\kappa-3)!! r^{-\kappa+2} \end{pmatrix} + o(k^2) \quad (2.40)$$

and

$$|f_{\kappa,2}^0(E,r) - f_{\kappa,2}^0(m,r)| < C k^2 [r^{2-\kappa} / (1+kr)^{2-\kappa}]. \quad (2.41)$$

From (2.39), (2.40), (2.41), and (2.30) we find

$$I_1 = a_\kappa k^2 + o(k^2) \quad \text{as } k \rightarrow 0 (E \downarrow m), \quad (2.42)$$

where

$$a_\kappa = \frac{(2\kappa-3)!!}{2m} \left(\int_0^\infty \varphi_{\kappa,1}(m,t) V(t) t^{-\kappa+1} dt + m \int_0^\infty \varphi_{\kappa,2}(m,t) V(t) t^{-\kappa+2} dt \right) \quad (2.43)$$

[with the convention $(-1)!! = 1$].

Now consider I_2 . If we expand

$$\varphi_\kappa(E,r) = \varphi_\kappa(m,r) + (E-m)u_\kappa(m,r) + O(k^4) \quad (2.44)$$

and use (2.18), then by dominated convergence

$$I_2 = b_\kappa k^2 + o(k^2) \quad \text{as } k \rightarrow 0, \quad (2.45)$$

where

$$b_\kappa = \frac{(2\kappa-1)!!}{2m} \int_0^\infty (u_{\kappa,2}(m,t))^T V(t) t^{-\kappa} dt. \quad (2.46)$$

For later use we need to know the asymptotic behavior of $u_\kappa(m,r)$ as $r \rightarrow 0$ and $r \rightarrow \infty$. Equation (1.3) for $E = m$ has the fundamental matrix

$$R_\kappa(r) = \begin{pmatrix} \varphi_{\kappa,1}(m,r) & \chi_{\kappa,1}(m,r) \\ \varphi_{\kappa,2}(m,r) & \chi_{\kappa,2}(m,r) \end{pmatrix}, \quad (2.47)$$

where χ_κ is the solution introduced before, see (2.14), (2.15). Applying the variation of parameters formula to (1.3), viewing $(E-m) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ as the nonhomogeneous term we find

$$u_\kappa(m,r) = \int_0^r R_\kappa(r) R_\kappa^{-1}(t) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \varphi_\kappa(m,t) dt. \quad (2.48)$$

Therefore, by (1.4), (2.9), (2.11), (2.14), and (2.15) we can say that

$$u_\kappa(m,r) = \begin{pmatrix} o(r^{\kappa+1}) \\ r^{\kappa+1} / (2\kappa+1)!! \end{pmatrix}, \quad r \rightarrow 0, \quad (2.49)$$

$$u_\kappa(m,r) = \|\varphi_\kappa(m,\cdot)\|^2 \chi_\kappa(m,r) (1 + o(1)), \quad r \rightarrow \infty. \quad (2.50)$$

Now, by (2.30), (2.41), (2.18), and (1.2) we have

$$|I_3| < C k^2 \int_0^\infty \left(\frac{kt}{1+kt} \right)^2 t |V(t)| dt = o(k^2). \quad (2.51)$$

Therefore, $F_\kappa(E) = (a_\kappa + b_\kappa)k^2 + o(k^2)$ and it only remains to reduce $c_\kappa = a_\kappa + b_\kappa$ to the expression (2.24). This can be accomplished by means of the following identities:

$$\int_0^\infty u_{\kappa,2}(m,t) V(t) t^{-\kappa} dt = \int_0^\infty t^{-\kappa} \varphi_{\kappa,2}(m,t) dt + A_\kappa^{-1}(m) \|\varphi_\kappa(m,\cdot)\|^2, \quad (2.52)$$

$$(2\kappa-1) \int_0^\infty \varphi_{\kappa,2}(m,t) t^{-\kappa} dt = \int_0^\infty \varphi_{\kappa,1}(m,t) (2m - V(t)) t^{-\kappa+1} dt, \quad (2.53)$$

$$-2 \int_0^\infty \varphi_{\kappa,1}(m,t) t^{-\kappa+1} dt = \int_0^\infty \varphi_{\kappa,2}(m,t) V(t) \times t^{-\kappa+2} dt. \quad (2.54)$$

To prove (2.52) we use the equation

$$u'_\kappa(m,r) = \begin{pmatrix} \kappa/r & V(r) \\ 2m - V(r) & -\kappa/r \end{pmatrix} u_\kappa(m,r) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \varphi_\kappa(m,r). \quad (2.55)$$

We multiply the equation $u'_{\kappa,1}(m,t) = (\kappa/t)u_{\kappa,1}(m,t) + V(t)u_{\kappa,2}(m,t) - \varphi_{\kappa,2}(m,t)$ by $t^{-\kappa}$ and integrate by parts. The boundary term is $u_{\kappa,1}(m,t)t^{-\kappa}|_0^\infty = A_\kappa^{-1}(m) \|\varphi_\kappa(m,\cdot)\|^2$ by virtue of (2.14), (2.49), and (2.50). Thus (2.52) follows. To prove (2.53) and (2.54) we use (1.3) and multiply the individual component equations by appropriate powers of t and integrate by parts. Now (2.52)–(2.54) are easily combined with (2.43) and (2.46) which yields (2.24). Since $F_\kappa(\bar{E}) = \overline{F_\kappa(E)}$, Eq. (2.23) also holds if $E \downarrow m$ along the lower edge of the cut $[m, \infty]$. Therefore, by a variant of the Phragmén–Lindelöf theorem¹³ (the same argument was used in Ref. 7) we obtain the desired uniformity in $\arg(E-m)$. Part (iii) is proved.

(iv) The proof is, of course, similar to that of (iii), but the case $\kappa = 1$ requires special attention. Also, the estimates are more tedious because the two components of $\varphi_\kappa(-m,r)$ must be controlled by separate bounds, namely,

$$|\varphi_{\kappa,1}(-m,r)| < C [r^\kappa / (1+r)^{2\kappa-1}], \quad (2.56)$$

$$|\varphi_{\kappa,2}(-m,r)| < C [r^\kappa / (1+r)^{2\kappa}], \quad (2.57)$$

and similarly for the difference $\varphi_\kappa(E,r) - \varphi_\kappa(-m,r)$ according to Lemma (2.1). Assume $\kappa \geq 2$ first. Then

$$f_\kappa^0(E,r) = \begin{pmatrix} -2m(2\kappa-3)!! r^{-\kappa+1} \\ (2\kappa-1)!! r^{-\kappa} \end{pmatrix} + \frac{(2\kappa-3)!!}{2} k^2 \times \begin{pmatrix} -r^{-\kappa+1}/m - [2m/(2\kappa-3)] r^{-\kappa+3} \\ r^{-\kappa+3} \end{pmatrix} + o(k^2) \quad (2.58)$$

and

$$|f_{\kappa,1}^0(E,r) - f_{\kappa,1}^0(-m,r)| \leq Ck^2[r^{1-\kappa}/(1+|k|r)^{1-\kappa}][1+[r/(1+|k|r)]^2], \quad (2.59)$$

$$|f_{\kappa,2}^0(E,r) - f_{\kappa,2}^0(-m,r)| \leq C(kr)^2(1+|k|r)^2. \quad (2.60)$$

So, if we split $F_\kappa(E)$ in analogy to (2.36)–(2.38) with respect to E near $-m$, we have that

$$I_1 = \alpha_\kappa k^2 + o(k^2), \quad k \rightarrow 0 (E \uparrow -m), \quad (2.61)$$

where

$$\begin{aligned} \alpha_\kappa = & -(2\kappa - 5)!! m \int_0^\infty \varphi_{\kappa,1}(-m,t) V(t) t^{-\kappa+3} dt \\ & + \frac{(2\kappa - 3)!!}{2} \int_0^\infty \varphi_{\kappa,2}(-m,t) V(t) t^{-\kappa+2} dt \\ & - \frac{(2\kappa - 3)!!}{2m} \int_0^\infty \varphi_{\kappa,1}(-m,t) V(t) t^{-\kappa+1} dt \end{aligned} \quad (2.62)$$

and

$$I_2 = \beta_\kappa k^2 + o(k^2), \quad k \rightarrow 0, \quad (2.63)$$

where

$$\begin{aligned} \beta_\kappa = & (2\kappa - 3)!! \int_0^\infty u_{\kappa,1}(-m,t) V(t) t^{-\kappa+1} dt \\ & - \frac{(2\kappa - 1)!!}{2m} \int_0^\infty u_{\kappa,2}(-m,t) V(t) t^{-\kappa} dt. \end{aligned} \quad (2.64)$$

Here, $u_\kappa(-m,r)$ obeys $\varphi_\kappa(E,r) = \varphi_\kappa(-m,r) + (E-m)u_\kappa(-m,r) + O(k^4)$. By using dominated convergence $I_3 = o(k^2)$ so that $F_\kappa(E) = d_\kappa k^2 + o(k^2)$ with $d_\kappa = \alpha_\kappa + \beta_\kappa$ and we must reduce this coefficient to the form (2.26). To this end we use the following identities, the proof of which is similar to that of (2.52)–(2.54) and is therefore omitted:

$$\begin{aligned} & \int_0^\infty u_{\kappa,2}(-m,t) V(t) t^{-\kappa} dt \\ & = \int_0^\infty \varphi_{\kappa,2}(-m,t) t^{-\kappa} dt \\ & \quad - 2m \int_0^\infty u_{\kappa,2}(-m,t) t^{-\kappa} dt \\ & \quad + A_\kappa^{-1}(-m) \|\varphi_\kappa(-m, \cdot)\|^2, \end{aligned} \quad (2.65)$$

$$\begin{aligned} & \int_0^\infty u_{\kappa,1}(-m,t) V(t) t^{-\kappa+1} dt \\ & = \int_0^\infty \varphi_{\kappa,1}(-m,t) t^{-\kappa+1} dt - (2\kappa - 1) \\ & \quad \times \int_0^\infty u_{\kappa,2}(-m,t) t^{-\kappa} dt, \end{aligned} \quad (2.66)$$

$$\begin{aligned} & \int_0^\infty \varphi_{\kappa,2}(-m,t) t^{-\kappa} dt \\ & = \frac{-1}{2\kappa - 1} \int_0^\infty \varphi_{\kappa,1}(-m,t) V(t) t^{-\kappa+1} dt, \quad (2.67) \\ & - 2 \int_0^\infty \varphi_{\kappa,1}(-m,t) t^{-\kappa+1} dt \\ & = 2m \int_0^\infty \varphi_{\kappa,2}(-m,t) t^{-\kappa+2} dt \end{aligned}$$

$$+ \int_0^\infty \varphi_{\kappa,2}(-m,t) V(t) t^{-\kappa+2}, \quad (2.68)$$

$$\begin{aligned} & (2\kappa - 3) \int_0^\infty \varphi_{\kappa,2}(-m,t) t^{-\kappa+2} dt \\ & = - \int_0^\infty \varphi_{\kappa,1}(-m,t) V(t) t^{-\kappa+3} dt. \end{aligned} \quad (2.69)$$

Finally, if $\kappa = 1$, then I_2 and I_3 both are $o(k)$ on account of (2.21). Since $f_{\kappa,1}^0(E,r) - f_{\kappa,1}^0(-m,r) = -2mikr + O(k^2)$ and $f_{\kappa,2}^0(E,r) - f_{\kappa,2}^0(-m,r) = O(k^2)$ we get

$$\begin{aligned} F_\kappa(E) & = -2mik \int_0^\infty \varphi_{\kappa,1}(-m,t) V(t) t dt + o(k) \\ & = -2mik A_\kappa(-m) + o(k). \end{aligned} \quad (2.70)$$

The uniform validity in $\arg(E+m)$ of (2.25) and (2.27) again follows from a Phragmén–Lindelöf type argument. This completes the proof of Theorem (2.2).

Proof of Theorem (1.1): As in Ref. 7, we choose a contour in the closed upper half plane consisting of a semicircle of radius R , two line segments $[-R, -m-\epsilon]$ and $[m+\epsilon, R]$ and two semicircles about $\pm m$ of radius ϵ . Then we extend the contour into the lower half plane by reflection and assign a counterclockwise orientation. For ϵ sufficiently small, all zeros of $F_\kappa(E)$ except possibly those at $\pm m$ lie inside the contour. By the argument principle, the change in $\delta_\kappa(E)$ on this contour equals $2\pi \tilde{N}_\kappa$, where \tilde{N}_κ denotes the number of eigenvalues that lie in $(-m, m)$. Since $F_\kappa(\bar{E}) = \overline{F_\kappa(E)}$ the change in $\delta_\kappa(E)$ on the top half of the contour is the same as that on the bottom half. For $\kappa > 2$, the change on the small circles centered at $\pm m$, respectively, approaches, as $\epsilon \rightarrow 0$, the value

$$\eta_\pm = \begin{cases} -2\pi, & \text{if } F_\kappa(\pm m) = 0, \\ 0, & \text{if } F_\kappa(\pm m) \neq 0. \end{cases} \quad (2.71)$$

Thus

$$\tilde{N}_\kappa = (1/\pi)(\delta_\kappa(-m) - \delta_\kappa(m)) + (1/2\pi)(\eta_+ + \eta_-). \quad (2.72)$$

Since N_κ also counts the eigenvalues at $\pm m$ if there are any we get (1.8) from (2.72) by dropping the term $(1/2\pi)(\eta_+ + \eta_-)$. Equation (2.71) also holds when $\kappa = 1$ with respect to $E = m$. If $\kappa = 1$ and $F_\kappa(-m) = 0$ then the change on the small circle centered at $-m$ is $-\pi$ if there is a half-bound state at $-m$ and 0 otherwise. This establishes (1.8) and (1.9) with respect to the above contour where now $\epsilon = 0$ but R is still finite. Of course, we can let $R \rightarrow \infty$ by using (2.22) so that $\delta_\kappa(-m) - \delta_\kappa(m)$ can be viewed as the change of phase over the continuous spectrum of H . Theorem (1.1) is proved.

The version of Levinson's theorem in Ref. 14 can easily be seen to agree with ours because $\delta_\kappa(\pm m) = 0 \pmod{\pi}$ except for $\delta_1(-m)$ which equals $\pi/2 \pmod{\pi}$ when $F_1(-m) = 0$. In connection with Ref. 4 we recall that there the concern was to find a relationship between the phase and the number of eigenvalues in $[0, m]$ and $[-m, 0]$, respectively. Let the former be denoted by N_κ^+ , the latter by N_κ^- . Suppose $E = 0$ is not an eigenvalue. Then again by a contour argument (take a contour which lies in $\{E: \operatorname{Re} E > 0\}$ such that it coincides with our previous contour for $\operatorname{Re} E > 0$ and

consists of a vertical segment joining iR to $-iR$) we have that

$$N_{\kappa}^{+} = (1/\pi)(\delta_{\kappa}(0) - \delta_{\kappa}(m)), \quad \kappa > 1. \quad (2.73)$$

A similar formula holds for N_{κ}^{-} if we replace $\delta_{\kappa}(m)$ by $\delta_{\kappa}(0)$ in (1.8) and (1.9). The discrepancy with Ref. 4 is that the term $\delta_{\kappa}(0)$ is missing from the formula corresponding to (2.73). In Ref. 4 (p. 146) the phase changes over two line segments along the imaginary axis were said to cancel, but in our setting these segments correspond precisely to the segments from iR to 0 and from 0 to $-iR$, so the phase changes add, giving rise to the term $\pi^{-1}\delta_{\kappa}(0)$.

III. NONRELATIVISTIC LIMIT

The Jost function associated with (1.1) when c is no longer equal to one can be obtained from (1.6) by making the replacements $E \rightarrow c^{-1}E$, $m \rightarrow mc$, $V \rightarrow c^{-1}V$ [cf. (1.3)] so that

$$F_{\kappa}(E, c) = 1 + c^{-1} \int_0^{\infty} (\varphi_{\kappa}(E, c, t))^T \times V(t) f_{\kappa}^0(E, c, t) dt, \quad (3.1)$$

where

$$f_{\kappa}^0(E, c, t) = k_c^{\kappa} \begin{pmatrix} [k_c ct / (E + mc^2)] h_{\kappa-1}(k_c t) \\ k_c t h_{\kappa}(k_c t) \end{pmatrix}, \quad (3.2)$$

with $k_c = c^{-1}\sqrt{E^2 - m^2c^4}$, and where we have modified our notation in an obvious manner in order to exhibit the c dependence. We are interested in the nonrelativistic limit $c \rightarrow \infty$ of $F_{\kappa}(E, c)$ and its phase $\delta_{\kappa}(E, c)$ because by taking this limit we should be able to connect the relativistic Levinson theorem with the nonrelativistic one. Recall that if $c \rightarrow \infty$, then the Dirac equation goes over into a Schrödinger equation in a sense that has been made precise by several authors, see Hunziker,¹⁴ Gesztesy *et al.*¹⁵ (and the references quoted therein). The main goal of these papers was to develop the perturbation theory of eigenvalues and eigenfunctions in powers of c^{-1} . Some aspects of the scattering theory (convergence of wave operators) in the nonrelativistic limit were studied by Yajima.¹⁶ These authors admit general, not necessarily spherically symmetric potentials. The only paper we are aware of which specifically considers the spherically symmetric case in a rigorous way is the old paper by Titchmarsh.¹⁷ There it is shown that the solution $\varphi_{\kappa}(E, c, r)$ has a convergent expansion in powers of c^{-1} although under the strong restriction that V is a bounded function. But it has been pointed out in Ref. 17 and is not hard to verify that locally the integrability of V is the only requirement for the results of Ref. 17 to go through. In order to formulate our results we need some notation. Put $F_{\kappa}^{+}(e, c) = F_{\kappa}(E, c)$ if $E = mc^2 + e$ and $F_{\kappa}^{-}(e, c) = F_{\kappa}(E, c)$ if $E = -mc^2 - e$ where in both cases $e > 0$. Let L_{κ}^{+} denote the Schrödinger operators

$$L_{\kappa}^{\pm} y = - (1/2m)y'' + [\kappa(\kappa \pm 1)/2mr^2]y \pm Vy = ey \quad (3.3)$$

[with $y(0) = 0$ when $\kappa = 1$] and let $\tilde{F}_{\kappa}^{\pm}(e)$ denote the cor-

responding Jost functions.⁹ Also, put $\delta_{\kappa}^{\pm}(e, c) = \arg F_{\kappa}^{\pm}(e, c)$ and $\tilde{\delta}_{\kappa}^{\pm}(e) = \arg \tilde{F}_{\kappa}^{\pm}(e)$. Then we have

Theorem (3.1): (i) As $c \rightarrow \infty$, $F_{\kappa}^{+}(e, c) \rightarrow \tilde{F}_{\kappa}^{+}(e)$ and $F_{\kappa}^{-}(e, c) \rightarrow \tilde{F}_{\kappa}^{-}(e)$ uniformly on $e > 0$.

(ii) If $\tilde{F}_{\kappa}^{+}(0) \neq 0$, then $\delta_{\kappa}^{+}(e, c) \rightarrow \tilde{\delta}_{\kappa}^{+}(e)$ uniformly on $e > 0$ as $c \rightarrow \infty$, while if $\tilde{F}_{\kappa}^{+}(0) = 0$, then $\delta_{\kappa}^{+}(e, c) \rightarrow \tilde{\delta}_{\kappa}^{+}(e)$ uniformly on $e > \epsilon > 0$ for any ϵ . Analogous statements hold for $F_{\kappa}^{-}(e, c)$ with the difference that $\delta_{\kappa}^{-}(e, c) \rightarrow -\tilde{\delta}_{\kappa}^{-}(e)$ [by (i)].

(iii) Let n_{κ}^{\pm} denote the number of negative eigenvalues of L_{κ}^{\pm} and let $N_{\kappa}(c)$ be the number of eigenvalues of $H_{\kappa}(c)$ in $[-mc^2, mc^2]$. Suppose $\tilde{F}_{\kappa}^{\pm}(0) \neq 0$. Then $N_{\kappa}(c) = n_{\kappa}^{+} + n_{\kappa}^{-}$ for c sufficiently large.

Proof: We omit the suffix κ from the solutions φ_{κ} and f_{κ}^0 for this proof. Consider $F_{\kappa}^{+}(e, c)$. By (2.29) and (2.30) we have

$$c^{-1} |\varphi_1(E, c, t) V(t) f_1^0(E, c, t)| \leq C [k_c^2 / (e + 2mc^2)] [|V(t)| t / (1 + k_c t)] \leq C c^{-1} |V(t)|, \quad (3.4)$$

so this contribution to (3.1) vanishes as $c \rightarrow \infty$ uniformly on $e > 0$ [here we are also using that the constants C in the estimates (2.29), (2.30) can be chosen to be independent of c ; this follows from their derivation in Ref. 4]. Regarding the second component we note the bound

$$c^{-1} |\varphi_2(E, c, t) V(t) f_2^0(E, c, t)| \leq C [(e + 2mc^2) / c^2] \times [|V(t)| t / (1 + k_c t)] \leq C (e/c^2 + 2m) |V(t)| t. \quad (3.5)$$

This shows that the theorem on dominated convergence is applicable to (3.1). Alternatively, the middle term in (3.4) can be estimated by

$$C(1/c + \sqrt{2m}/\sqrt{e}) |V(t)|, \quad (3.6)$$

(3.5) and (3.6) together imply that in order to prove $F_{\kappa}^{+}(e, c) \rightarrow \tilde{F}_{\kappa}^{+}(e)$ uniformly on $e > 0$ it suffices to prove

$$|c^{-1} \int_0^R \varphi_2(E, c, t) V(t) f_2^0(E, c, t) dt - \int_0^R \tilde{\varphi}_2(e, t) V(t) \tilde{f}_2^0(e, t) dt| \rightarrow 0 \quad (3.7)$$

uniformly on every bounded interval $0 < e < e_0$. Here,

$$\tilde{\varphi}_2(E, t) = \lim_{c \rightarrow \infty} c^{-1} \varphi_2(E, c, t) \quad \text{and} \quad \tilde{f}_2^0(e, t) = \lim_{c \rightarrow \infty} f_2^0(e, c, t).$$

This is so because the difference $|c^{-1} \int_R^{\infty} \dots - \int_R^{\infty} \dots|$ can be made arbitrarily small uniformly in e by choosing R sufficiently large and letting $c \rightarrow \infty$ [use (3.5) for $e \in [0, 1]$ and (3.6) for $e \in (1, \infty)$]. Another appeal to (3.6) then shows that the difference in (3.7) can be made arbitrarily small uniformly in e for $e > e_0$ by choosing e_0 large enough and taking $c \rightarrow \infty$. To prove (3.7) for a finite energy interval we estimate separately the integrals

$$\int_0^R (c^{-1} \varphi_2(E, c, t) - \tilde{\varphi}_2(e, t)) V(t) f_2^0(E, c, t) dt \quad (3.8)$$

and

$$\int_0^R \tilde{\varphi}_2(e,t) V(t) (f_2^0(E,c,t) - \tilde{f}_2^0(e,t)) dt. \quad (3.9)$$

Since t is restricted to a finite interval we can use the methods of Ref. 17 (and also Ref. 1) to show that $|c^{-1}\varphi_2(E,c,t) - \tilde{\varphi}_2(e,t)| < Cc^{-2}t^\kappa$. Since the techniques are standard we omit the details. Inserting this estimate along with (2.31) in (3.8) shows that the integral is $O(c^{-2})$ uniformly on $[0, e_0]$. Moreover, it follows that $\tilde{\varphi}_2(e,r)$ is a solution of $L_\kappa^+ y = ey$ such that $r^{-\kappa-1}\tilde{\varphi}_2(e,r) \rightarrow 2m/(2\kappa+1)!!$ as $r \rightarrow 0$. Considering (3.9) we have to estimate the difference $f_2^0(E,c,t) - \tilde{f}_2^0(c,t) = k_c^* \hat{h}_\kappa(k_c t) - \tilde{k}^* \hat{h}_\kappa(\tilde{k}t) = (k_c^* - \tilde{k}^*) \hat{h}_\kappa(k_c t) + \tilde{k}^* (\hat{h}_\kappa(k_c t) - \hat{h}_\kappa(\tilde{k}t))$ where $\hat{h}_\kappa(kt) = kth_\kappa(kt)$ and $\tilde{k} = \lim_{c \rightarrow \infty} k_c = \sqrt{2me}$. Clearly, $k_c^* - \tilde{k}^* = O(c^{-2})$ uniformly on $[0, e_0]$. Furthermore, $|\hat{h}_\kappa(k_c t) - \hat{h}_\kappa(\tilde{k}t)| < (k_c - \tilde{k})t \sup |\hat{h}'_\kappa(\xi)|$ where the sup is over $\xi \in [\tilde{k}t, k_c t]$. Now $\hat{h}'_\kappa(\xi) = [(\kappa+1)/\xi] \hat{h}_\kappa(\xi) - \hat{h}_{\kappa+1}(\xi)$ and $|\hat{h}_\kappa(\xi)| < C\xi^{-\kappa}(1+\xi)^\kappa$ so that

$$\tilde{k}^* |\hat{h}_\kappa(k_c t) - \hat{h}_\kappa(\tilde{k}t)| < C(k_c - \tilde{k})(\tilde{k}/k_c)^\kappa t(1+k_c t/t)^\kappa. \quad (3.10)$$

Since $|\tilde{\varphi}_2(e,t)| < Ct^{\kappa+1}$ we see that the integral (3.9) is $O(c^{-2})$. This proves the uniform convergence of $F^+(e,c)$ to $\tilde{F}_\kappa^+(e) = 1 + \int_0^\infty \tilde{\varphi}_2(e,t) V(t) \tilde{f}_2^0(e,t) dt$. The identification of $\tilde{F}_\kappa^+(e)$ as the Jost function for L_κ^+ is a straightforward computation using the relations $n_\kappa(kr) = (-1)^\kappa \times (\pi/2kr)^{1/2} J_{-\kappa-1/2}(kr)$ and $j_\kappa(kr) = (\pi/2kr)^{1/2} \times J_{\kappa+1/2}(kr)$ [and keeping in mind that $\tilde{\varphi}_2(e,r)$ is $(2\kappa+1)!!/2m$ times the standard solution y of $L_\kappa^+ y = ey$ with $r^{-\kappa-1}y \rightarrow 1$ as $r \rightarrow 0$]. In a similar manner, one proves the statements about $F_\kappa^-(e,c)$. One has to remember that since $E < -mc^2$ then $k_c < 0$ and hence one also has to use relations like $J_\nu(k_c r) = e^{i\pi\nu} J_\nu(|k_c|r)$ in the process. Thus part (i) is proved.

The statements in (ii) immediately follow from the uniform convergence of the Jost functions and the fact that $F_\kappa^\pm(e,c)$ does not vanish for $e > 0$, and also not for $e = 0$ if c is sufficiently large and $\tilde{F}_\kappa^\pm(0) \neq 0$.

The assertion in (iii) is a consequence of Theorem (1.1) since

$$N_\kappa(c) = (1/\pi)(\delta_\kappa^+(\infty, c) - \delta_\kappa^+(0, c)) + (1/\pi)(\delta_\kappa^-(0, c) - \delta_\kappa^-(\infty, c)) \quad (3.11)$$

$$\begin{aligned} \Delta\varphi_\kappa(E,r) &= \Delta\varphi_\kappa^0(E,r) + \Delta\varphi_\kappa^0(E,r) \int_0^r (\psi_\kappa^0(m,t))^T V(r) \varphi_\kappa(m,t) dt + \varphi_\kappa^0(E,r) \int_0^r (\Delta\psi_\kappa^0(E,t))^T V(t) \varphi_\kappa(m,t) dt \\ &\quad - \Delta\psi_\kappa^0(E,r) \int_0^r (\varphi_\kappa^0(E,t))^T V(t) \varphi_\kappa(m,t) dt - \psi_\kappa^0(m,r) \int_0^r (\Delta\varphi_\kappa^0(E,t))^T V(t) \varphi_\kappa(m,t) dt \\ &\quad + \int_0^r [\varphi_\kappa^0(E,t) (\psi_\kappa^0(E,t))^T - \psi_\kappa^0(E,r) (\varphi_\kappa^0(E,t))^T] V(t) \Delta\varphi_\kappa(E,t) dt. \end{aligned} \quad (A1)$$

We denote the six terms on the right-hand side by A_1 through A_6 . The idea is to estimate these terms so that Gronwall's inequality can be used at the end. Suppose now that $E \in [m, m + \delta]$ for some $\delta > 0$. We use C to denote a generic constant which depends on δ and κ but not on k and r . We

so that on letting $c \rightarrow \infty$ the first term tends to $(1/\pi)(\delta_\kappa^+(\infty) - \delta_\kappa^+(0)) = n_\kappa^+$ by the nonrelativistic Levinson theorem³ (remember our phase convention) and the second term tends to $(1/\pi)(\delta_\kappa^-(\infty) - \delta_\kappa^-(0)) = n_\kappa^-$ [since $F_\kappa^-(e,c) \rightarrow \tilde{F}_\kappa^-(e)$]. Theorem (3.1) is thus proved.

In closing, we remark that the weaker statement $N_\kappa(c) > n_\kappa^+ + n_\kappa^-$ for c sufficiently large can also be deduced directly from the results in Refs. 15 and 14 where it is shown (under somewhat different conditions on V) that

$$(H_\kappa(c) - m^2c - z)^{-1} \rightarrow \begin{pmatrix} (L_\kappa^+ - z)^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$(-H_\kappa(c) - m^2c - z)^{-1} \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & (L_\kappa^- - z)^{-1} \end{pmatrix}$$

in norm as $c \rightarrow \infty$ ($\text{Im } z \neq 0$). By some additional arguments one can also obtain equality [i.e., $N_\kappa(c) = n_\kappa^+ + n_\kappa^-$] and one can extend the result to nonspherically symmetric operators [the condition $\tilde{F}_\kappa^\pm(0) \neq 0$ then becomes a condition on the absence of zero-energy resonances, resp., zero-energy bound states, for the Schrödinger operators $-(2m)^{-1}\Delta \pm V$]. Moreover, note that if $\kappa = 1$ and $\tilde{F}_1^-(0) = 0$ then $\delta_1^-(e) \rightarrow (\pi/2) \pmod{\pi}$ as $e \rightarrow 0$ (Ref. 9, Theorem 4.1, case $l = 0$). In general, one will have $F_1^-(0,c) \neq 0$ for c large enough and so, since $F_1^-(0,c)$ is real, $\delta_1^-(0,c) = 0 \pmod{\pi}$. Thus in this case $\delta_1^-(0,c)$ does not converge to $-\delta_1^-(0)$ as $c \rightarrow \infty$ explaining the restriction $e > \epsilon > 0$ in (ii).

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APPENDIX: PROOF OF LEMMA (2.1)

The method of proof is similar to that used in the Schrödinger case to prove a corresponding result (see Ref. 9, Appendix) but as already mentioned, there are several complications which need to be dealt with carefully. We give a detailed proof of part (i) and then only indicate the changes that are needed for part (ii).

(i) Let $\Delta\varphi_\kappa(E,r) = \varphi_\kappa(E,r) - \varphi_\kappa(m,r)$ and define $\Delta\varphi_\kappa^0(E,r)$ and $\Delta\psi_\kappa^0(E,r)$ analogously.

Then using (2.2) we may write

will frequently use the following bounds.

$$|krj_\kappa(kr)| < C [kr/(1+kr)]^{\kappa+1}, \quad (A2)$$

$$|krn_\kappa(kr)| < C [(1+kr)/kr]^\kappa, \quad (A3)$$

$$|krj_\kappa(kr) - (kr)^{\kappa+1}/(2\kappa+1)!!| < C(kr)^{\kappa+3}/(1+kr)^2, \quad (A4)$$

$$|krm_\kappa(kr) - (kr)^{-\kappa}(2\kappa-1)!!| < C[kr/(1+kr)]^{2-\kappa}. \quad (A5)$$

In the following if we have a vector $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and estimates $|f_1| < a_1, |f_2| < a_2$, then we use the notation $|f| < \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ to denote this fact. We also set $L(kr) = kr/(1+kr)$:

$$|\varphi_\kappa^0(E,r)| < Ck^{-\kappa}L^\kappa(kr) \begin{pmatrix} 1 \\ r/(1+kr) \end{pmatrix}, \quad (A6)$$

$$|\psi_\kappa^0(E,r)| < Ck^\kappa L^{-\kappa}(kr) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (A7)$$

$$|\Delta\varphi_\kappa^0(E,r)| < Ck^2 \frac{r^{\kappa+1}(1+r)}{(1+kr)^2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (A8)$$

$$|\Delta\psi_\kappa^0(E,r)| < Ck^{1+\kappa}L^{1-\kappa}(kr) \begin{pmatrix} 1 \\ r/(1+kr) \end{pmatrix}. \quad (A9)$$

Combining A_1 and A_2 , using $\psi_\kappa^0(m,r) = (2\kappa-1)!! \begin{pmatrix} 0 \\ r^{-\kappa} \end{pmatrix}$ and that the right-hand side of (2.7) is zero we get

$$A_1 + A_2 = -\Delta\varphi_\kappa^0(E,r) \int_r^\infty (\psi_\kappa^0(m,t))^T V(t) \varphi_\kappa(m,t) dt, \quad (A10)$$

so that by elementary estimates

$$|A_1 + A_2| < Ck^{1-\kappa}L^{\kappa+1}(kr) \int_r^\infty \frac{|V(t)|}{(1+t)^\kappa} dt \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (A11)$$

Estimating the third term in (A1) yields

$$|A_3| < Ck^{1-\kappa}L^{\kappa+1}(kr) \int_0^r \frac{|V(t)|}{(1+t)^{\kappa-1}} dt \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (A12)$$

Similarly, for A_4 and A_5 we get

$$|A_4| < Ck^{1-\kappa}L^{\kappa+1}(kr) \int_0^r \frac{|V(t)|}{(1+t)^{\kappa-1}} dt \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (A13)$$

$$|A_5| < Ck^{1-\kappa}L^{\kappa+1}(kr) \int_0^r \frac{|V(t)|}{(1+t)^\kappa} dt \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (A14)$$

The entries of the matrix $\varphi_\kappa^0(E,r)(\psi_\kappa^0(E,t))^T - \psi_\kappa^0(E,r)(\varphi_\kappa^0(E,t))^T$ are each bounded in magnitude by

$$CL^{\kappa+1}(kr)L^{-\kappa-1}(kt)(1+t). \quad (A15)$$

So if we set

$$u(E,r) = (|\Delta\varphi_{\kappa,1}(E,r)| + |\Delta\varphi_{\kappa,2}(E,r)|)L^{-\kappa-1}(kr)k^{\kappa-1} \quad (A16)$$

and combine (A11)–(A15), then we arrive at the inequality

$$u(E,r) < C + C \int_0^r |V(t)|(1+t)u(E,t) dt. \quad (A17)$$

Hence by Gronwall's inequality $u(E,r) < C$ which is equivalent to (2.18). Part (i) is proved.

(ii) Here, $E \in [-m-\delta, -m], k < 0$. It turns out that the quantity $k^2L^{\kappa+1}(|k|r)$ is not sufficient to control the difference $\Delta\varphi_\kappa(E,r) = \varphi_\kappa(E,r) - \varphi_\kappa(-m,r)$, we must also use $k^{2-\kappa}L^\kappa(|k|r)$. So we introduce

$$h_\kappa(r) = k^{1-\kappa}L^{\kappa+1}(|k|r) + k^{2-\kappa}L^\kappa(|k|r). \quad (A18)$$

Proceeding as in (i) we can then show that

$$\begin{aligned} & |\Delta\varphi_{\kappa,1}(E,r)| \\ & < Ck^{2-\kappa}L^\kappa(|k|r) + Ck^{-1}L^{-\kappa}(|k|r) \int_0^r L^{1-\kappa}(|k|t) \\ & \quad \times |V(t)| |\Delta\varphi_{\kappa,1}(E,t)| dt + CL^\kappa(|k|r) \\ & \quad \times \int_0^r L^{-\kappa}(|k|t) |V(t)| |\Delta\varphi_{\kappa,2}(E,t)| dt, \end{aligned} \quad (A19)$$

for $\kappa > 2$, and

$$\begin{aligned} & |\Delta\varphi_{\kappa,1}(E,r)| < CL^2(|k|r) + Ck^{-1}L(|k|r) \\ & \quad \times \int_0^r |V(t)| |\Delta\varphi_{\kappa,1}(E,t)| dt + CL(|k|r) \\ & \quad \times \int_0^r L^{-1}(|k|t) |V(t)| |\Delta\varphi_{\kappa,2}(E,t)| dt, \end{aligned} \quad (A20)$$

for $\kappa = 1$. For all $\kappa > 1$ we have

$$\begin{aligned} & |\Delta\varphi_{\kappa,2}(E,r)| < Ch_\kappa(r) + CL^\kappa(|k|r) \\ & \quad \times \int_0^r L^{-\kappa}(|k|t) |V(t)| |\Delta\varphi_{\kappa,1}(E,t)| dt \\ & \quad + CkL^{\kappa+1}(|k|r) \\ & \quad \times \int_0^r L^{-\kappa}(|k|t) |V(t)| |\Delta\varphi_{\kappa,2}(E,t)| dt. \end{aligned} \quad (A21)$$

Now when $\kappa = 1$ we set $u(E,r) = (|\Delta\varphi_{\kappa,1}(E,r)| + |\Delta\varphi_{\kappa,2}(E,r)|)/h_\kappa(r)$ and when $\kappa > 2$ we set $u(E,r) = |\Delta\varphi_{\kappa,1}(E,r)|/(k^{2-\kappa}L^\kappa(|k|r)) + |\Delta\varphi_{\kappa,2}(E,r)|/h_\kappa(r)$. Then $u(E,r)$ is seen to obey an inequality of the form (A17) and hence (2.19), (2.20), and (2.21) follow immediately. This concludes the proof of Lemma (2.1).

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