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On the solutions to a class of nonlinear integral equations arising in transport theory

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Existence and uniqueness for the solutions to a class of nonlinear equations arising in transport theory are investigated in terms of a real parameter α which can take on positive and negative values. On the basis of contraction mapping and positivity properties of the relevant nonlinear operator, iteration schemes are proposed, and their convergence, either pointwise or in norm, is studied.

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I. INTRODUCTION

In this paper we shall consider the problem of solving the nonlinear integral equation

$$h(x) + \alpha h(x) \int_a^b K(x,y) h(y)\, dy = S(x), \quad x \in (a,b), \tag{1}$$

where $K(x,y)$ and $S(x)$ are real non-negative functions in the real domains $(a,b)x(a,b)$ and (a,b), respectively, and α is a real parameter. The class of nonlinear integral equations described in Eq. (1) includes the nonlinear particle transport equation when removal effects are dominant.[1,2] In that case the dependent variable x is the particle speed ranging from 0 to ∞, α is equal to unity, the known term $S(x)$ is the intensity of the external source, the unknown $h(x)$ is related to the particle distribution function $f(x)$ by

$$h(x) = G(x) f(x),$$

where G is the positive macroscopic removal collision frequency of the host medium, and finally the (symmetric) kernel $K(x,y)$ is given by

$$K(x,y) = \frac{1}{2xG(x)G(y)} \int_{|x-y|}^{x+y} u g(u)\, du,$$

where g is the microscopic removal collision frequency by the particles between themselves. On the other hand, Eq. (1) is also a generalization of a famous equation in transport theory, the so-called Chandrasekhar H-equation[3,4] in which x ranges from 0 to 1, $\alpha = -1$, $S(x) = 1$, h must be identified with the H-function, and

$$K(x,y) = x\psi(y)/(x+y)$$

for a non-negative characteristic function ψ. This latter equation has been widely studied in the literature,[5,6] and due to the analyticity properties of its kernel, it has been possible to determine exactly the number and properties of solutions, and even to write them out explicitly.

In this paper, after a preliminary investigation based on contraction mapping, we shall mainly employ positivity arguments to study the existence and uniqueness of solutions to Eq. (1), with particular emphasis on positive solutions, that for the applications mentioned before, are the only physical ones. The convergence of iterative schemes for the solution of Eq. (1) will also be demonstrated.

Throughout this paper we shall assume that $S \in L_p(a,b)$ for some p, with $1 \leqslant p \leqslant \infty$, and that the linear integral operator

$$(Tf)(x) = \int_a^b K(x,y) f(y)\, dy \tag{2}$$

is a bounded mapping of $L_p(a,b)$ into $L_\infty(a,b)$, with norm $\|T\|$. A sufficient condition for that would be[7]

$$\operatorname*{ess\,sup}_{x \in (a,b)} \int_a^b |K(x,y)|^q\, dy = M < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{3}$$

in which case $\|T\| \leqslant M^{1/q}$. Further conditions that will be needed in some occasions are the following.

Assumption 1: The function S is continuous and the kernel K satisfies the requirement

$$\lim_{x_1 \to x_2} \int_a^b |K(x_1,y) - K(x_2,y)|^q\, dy = 0, \quad x_1, x_2 \in (a,b) \tag{4}$$

for any $y \in (a,b)$.

Assumption 2: $(TS)(x)$ is bounded away from zero, namely,

$$\operatorname*{ess\,inf}_{x \in (a,b)} \int_a^b K(x,y) S(y)\, dy = \delta > 0. \tag{5}$$

We note that Eq. (1) may be rewritten either as

$$h = A(h), \quad A(h) = S - \alpha h\, Th, \tag{6a}$$

or as

$$h = B(h), \quad B(h) = S/(1 + \alpha Th), \tag{6b}$$

both operators A and B being nonlinear, with $A(0) = B(0) = S$.

Equations (1) can also be regarded as a particular Hammerstein equation; for example setting

$$h(x) = [1 + \alpha f(x)]^{-1} \tag{7}$$

one ends up with the equation of Hammerstein type[8]

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$$f(x) = \int_a^b K(x,y)\,\phi\,[\,y,f(y)]\,dy, \quad \phi(x,u) = \frac{1}{(1+\alpha u)},$$

where however the usual assumption of continuity of ϕ with respect to u is violated at $u = -1/\alpha$.

II. GENERAL RESULTS

We can obtain some quite general results by applying the contraction mapping theorem to the operator A in Eq. (6a). If h belongs to the closed ball of L_p with center at the origin and radius r, namely $\|h\| \leqslant r$, we get at once

$$|A(h)| \leqslant S + |\alpha|\,|h|\,|Th| \leqslant S + |\alpha|\,\|T\|\,\|h\|\,|h|$$

and then

$$\|A(h)\| \leqslant \|S\| + |\alpha|\,\|T\|\,\|h\|^2 \leqslant \|S\| + |\alpha|\,\|T\|r^2.$$

In order to insure that $\|A(h)\| \leqslant r$ we impose

$$|\alpha|\,\|T\|r^2 - r + \|S\| \leqslant 0,$$

which leads to the conditions

$$|\alpha| \leqslant 1/(4\|T\|\,\|S\|) \tag{8}$$

and

$$\frac{1-\sqrt{1-4|\alpha|\,\|T\|\,\|S\|}}{2|\alpha|\,\|T\|} \leqslant r \leqslant \frac{1+\sqrt{1-4|\alpha|\,\|T\|\,\|S\|}}{2|\alpha|\,\|T\|}. \tag{9a}$$

Further, for A to be a contraction, we note that

$$\begin{aligned}|A(h_1) - A(h_2)| &= |\alpha|\,|h_2(Th_2 - Th_1) + (h_2 - h_1)\,Th_1| \\ &\leqslant |\alpha|(\|T\|\,\|h_2 - h_1\|\,|h_2| \\ &\quad + \|T\|\,\|h_1\|\,|h_2 - h_1|)\end{aligned}$$

so that

$$\begin{aligned}\|A(h_1) - A(h_2)\| &\leqslant |\alpha|\,\|T\|(\|h_2\| + \|h_1\|)\|h_1 - h_2\| \\ &\leqslant 2r|\alpha|\,\|T\|\,\|h_1 - h_2\|,\end{aligned}$$

which leads to the restriction

$$r < 1/(2|\alpha|\,\|T\|). \tag{9b}$$

We can then state[9] the following.

Theorem 1: If α satisfies Eq. (8), there exists a unique solution h^* to Eq. (1) in any closed ball of $L_p(a,b)$ centered at the origin with radius r such that

$$\frac{1-\sqrt{1-4|\alpha|\,\|T\|\,\|S\|}}{2|\alpha|\,\|T\|} \leqslant r \leqslant \frac{1}{2|\alpha|\,\|T\|}. \tag{10}$$

Moreover, the iteration scheme

$$h_n = A(h_{n-1}) \tag{11}$$

converges in the L_p-norm to this unique solution if the initial guess h_0 is chosen in the ball.

This theorem is valid under the general hypotheses of the Introduction [$S(x)$ and $K(x,y)$ non-negative, $S \in L_p$, T is a bounded operator of L_p into L_∞]. In this theorem, however, the non-negativity of S and K plays no role, and can be dropped.

A trivial corollary of the theorem is

$$\|h^*\| \leqslant \frac{1-\sqrt{1-4|\alpha|\,\|T\|\,\|S\|}}{2|\alpha|\,\|T\|} \tag{12}$$

and that other solutions, if any, will have a norm not less than $1/(2|\alpha|\,\|T\|)$.

If now, in addition to non-negativity of S and K, we assume that α is positive, we can sharpen Theorem 1 to show the following.

Theorem 2: Let $\alpha > 0$ satisfy Eq. (8). Then Eq. (1) has at least one non-negative solution $h^* \in L_p$ with

$$\frac{\sqrt{1+4\alpha\|T\|\,\|S\|}-1}{2\alpha\|T\|} \leqslant \|h^*\| \leqslant \|S\|, \tag{13}$$

which is the unique L_p-solution with norm less than $1/(2\alpha\|T\|)$, and is the limit (in the L_p-norm) of the iterative scheme (11) for any initial guess h_0 with

$$\|h_0\| \leqslant \frac{1-\sqrt{1-4\alpha\|T\|\,\|S\|}}{2\alpha\|T\|}. \tag{14}$$

If in addition h_0 is non-negative and less than S, all h_n are also non-negative and less than S. [Whenever the context is clear, we will use the notation $g \leqslant f$ to mean $g(x) \leqslant f(x)$ a.e. for $x \in (a,b)$.]

Proof: The proof of this last theorem follows quickly from Theorem 1. The unique solution h^* in the ball defined by Eq. (10) is the limit of the sequence (11), and is independent of h_0, provided h_0 satisfies Eq. (14). We then take $0 \leqslant h_0 \leqslant S$, and show by induction that $0 \leqslant h_n \leqslant S$. Suppose $0 \leqslant h_{n-1} \leqslant S$; then $h_n \leqslant S$ follows immediately from Eq. (6a) for $\alpha > 0$ and $h_{n-1} \geqslant 0$. In addition,

$$\begin{aligned}h_n = A(h_{n-1}) &\geqslant S[1 - \alpha Th_{n-1}] \\ &\geqslant S[1 - \alpha\|T\|\,\|h_{n-1}\|] \geqslant 0,\end{aligned}$$

since $\|h_{n-1}\| < 1/(\alpha\|T\|)$ is guaranteed *a fortiori*, h_{n-1} being in the ball (10). This proves both inequalities for h_n, and thus for h^* in the limit for $n \to \infty$. Therefore, $0 \leqslant h^* \leqslant S$, and $\|h^*\| \leqslant \|S\|$ [which is of course stronger than Eq. (12)]. Finally, we note from Eq. (1) with $\alpha > 0$ and $h^* \geqslant 0$ that

$$\|S\| \leqslant \|h^*\| + \alpha\|h^*\|\,\|Th^*\| \leqslant \|h^*\| + \alpha\|T\|\,\|h^*\|^2$$

from which the other inequality in Eq. (13) follows directly. This completes the proof of the theorem.

From this proof we also note further that starting from a non-negative initial guess in the ball $\|h\| \leqslant \|S\|$ the iterative procedure is positivity preserving inside the same ball for

$$\alpha < 1/(\|T\|\,\|S\|), \tag{15}$$

which is weaker than the condition for convergence given by Eq. (8).

There of course may be other solutions of Eq. (1) than h^*. In this regard we write

Lemma 1: Let $\alpha > 0$, and suppose that a nonpositive solution $\hat{h}$ to Eq. (1) exists in L_p. Then

$$\|\hat{h}\| \geqslant \frac{1+\sqrt{1+4\alpha\|T\|\,\|S\|}}{2\alpha\|T\|}. \tag{16}$$

[Note that the right-hand side is greater than $1/(\alpha\|T\|)$ which is twice the lower bound in Theorem 1.]

Proof: To prove Eq. (16) it is sufficient to write $S = -\hat{h}[\alpha T(-\hat{h}) - 1]$, with S and $-\hat{h}$ non-negative, and $\alpha[T(-\hat{h})] > 1$, so that, taking norms, we find

$$\begin{aligned}\|S\| &\leqslant \|\hat{h}\|\ \text{ess sup}[\alpha T(-\hat{h}) - 1] \leqslant \|\hat{h}\|(\alpha\|T\|\,\|\hat{h}\| - 1) \\ &= \alpha\|T\|\,\|\hat{h}\|^2 - \|\hat{h}\|,\end{aligned}$$

which yields

$$\alpha\|T\|\,\|\hat{h}\|^2 - \|\hat{h}\| - \|S\| \geqslant 0$$

from which Eq. (16) easily follows.

Another general result can be established for positive α. We write

Lemma 2: Let $\alpha > 0$ and Assumption 1 hold. Then, if a non-negative solution $h \in L_p$ exists, this solution is necessarily continuous, and is positive if and only if S is positive.

Proof: By Eq. (4) T maps L_p functions into continuous functions, so that $1 + \alpha Th$ is continuous and bounded away from zero. Thus $B(h)$ is continuous since S is continuous and vanishes where S is zero, and only there.

We can complete Theorem 2 and Lemma 1 by writing

Theorem 3: If a solution h to Eq. (1) exists, and there exists an $\epsilon > 0$ for which

$$\|h\| \leqslant (1-\epsilon)/|\alpha| \, \|T\|, \tag{17}$$

then h is non-negative. It is also continuous when Assumption 1 holds.

Proof: The proof follows from $1 + \alpha Th > \epsilon$, which ensures that the denominator of $B(h)$ is positive and bounded away from zero (and continuous under Assumption 1).

Results somewhat similar to Theorems 2 and 3 can also be shown for the case $\alpha < 0$. It proves convenient to refer to Eq. (6a), and look for the fixed points of the operator A defined by

$$A(h) = S + \beta h Th, \quad \beta = -\alpha = |\alpha| > 0. \tag{18}$$

It is clear that now A has several nice properties. It is a continuous operator in L_p, since from

$$\begin{aligned}|A(f_1) - A(f_2)| &= \beta \, |(f_1 - f_2)\, Tf_1 + f_2 T(f_1 - f_2)| \\ &\leqslant \beta \, \|T\| (\|f_1\| \, |f_1 - f_2| + \|f_1 - f_2\| \, |f_2|)\end{aligned}$$

it is easily seen that for any $f_1, f_2 \in L_p$,

$$\|A(f_1) - A(f_2)\| \leqslant \beta \, \|T\| (\|f_1\| + \|f_2\|) \|f_1 - f_2\|. \tag{19}$$

In addition, A is positive [$f \geqslant 0$ implies $A(f) \geqslant S \geqslant 0$] and monotone in the cone of the non-negative functions, since for $f_1 \geqslant f_2$ we have

$$A(f_1) - A(f_2) = \beta (f_1 - f_2)\, Tf_1 + \beta f_2 T(f_1 - f_2) \geqslant 0. \tag{20}$$

We can prove

Lemma 3: Let $\beta = -\alpha > 0$ and

$$\beta \leqslant 1/(4\|T\| \, \|S\|). \tag{21}$$

Then the operator A maps the so-called conical segment[10] $\langle 0, cS \rangle$ into itself for any c such that

$$\frac{1 - \sqrt{1 - 4\beta \|T\| \, \|S\|}}{2\beta \|T\| \, \|S\|} \leqslant c \leqslant \frac{1 + \sqrt{1 - 4\beta \|T\| \, \|S\|}}{2\beta \|T\| \, \|S\|}. \tag{22}$$

Proof: If in fact h belongs to the conical segment $\langle 0, cS \rangle$ we get

$$A(h) \geqslant S \geqslant 0$$

and

$$A(h) = S + \beta h Th \leqslant S + \beta c^2 STS \leqslant S + \beta c^2 \|T\| \, \|S\| S.$$

Thus the requirement $A(h) \leqslant cS$ leads to the inequality

$$\beta \, \|T\| \, \|S\| c^2 - c + 1 \leqslant 0,$$

which is equivalent to Eqs. (21) and (22). Note that the left-hand side in Eq. (22) is always greater than unity.

From the continuity and monotonicity of A, from Lemma 3, and from the regularity of the cone of the non-negative functions in L_p, we can deduce immediately[10] the following.

Theorem 4: Under the assumptions of Lemma 3, Eq. (6a) has at least one non-negative solution h^* belonging to the conical segment $\langle 0, cS \rangle$, where c satisfies Eq. (22), and which is given by the limit (pointwise and in norm) of the monotonic successive approximations

$$h_{n+1} = A(h_n), \quad h_0 = S. \tag{23}$$

The conical segment to which h^* belongs can further be sharpened by observing that for any non-negative solution we have $h = A(h) \geqslant S$, so that we may write

$$S \leqslant h^* \leqslant \frac{1 - \sqrt{1 - 4\beta \|T\| \, \|S\|}}{2\beta \|T\| \, \|S\|} S. \tag{24}$$

It is worth noticing that Theorem 4 and Eq. (24) are in agreement with Theorem 1 and Eq. (12), and that Eq. (8) coincides with Eq. (21). The results are complementary and strengthen each other. The solution h^* of both theorems can be obtained starting from any initial guess allowed by Theorem 1, the solution is non-negative and satisfies Eq. (24) (Theorem 4), and there are no other solutions with norm less than $1/(2\beta \|T\|)$ (Theorem 1). Other solutions, possibly positive, might occur beyond the latter limit. When β is larger than $1/(4\|T\| \, \|S\|)$, a non-negative solution might even fail to exist.

The results obtained so far all require restrictions on the parameter α. We might search for conditions ensuring the existence and possibly the uniqueness of non-negative solutions to Eq. (1) without restrictions on the parameter α. However, the simple examples in the following section illustrate that this search might be fruitless for the case $\alpha < 0$.

III. SOME EXAMPLES

Equation (1) can be reduced to the solution of a system of transcendental equations for a finite number of scalar coefficients when the kernel $K(x,y)$ is degenerate (of finite rank N), namely,

$$K(x,y) = \sum_{n=1}^{N} X_n(x) Y_n(y) \tag{25}$$

with for instance $Y_n \in L_p$, $X_n \in L_\infty$, and of course $X_n \geqslant 0$, $Y_n \geqslant 0$. We will consider two very simple examples with the lowest possible order of degeneracy.

Example 1: $K(x,y) = \text{const} = k$. Setting

$$\xi = \int_a^b h(x)\, dx, \quad s = \int_a^b S(x)\, dx$$

we get at once

$$h(x) = S(x)/(1 + \alpha k \xi). \tag{26}$$

Now integrating over x, we obtain the second-degree algebraic equation for ξ

$$\alpha k \xi^2 + \xi - s = 0, \tag{27}$$

which always has two roots, with different features according to the values of α, k, and s. For $\alpha > 0$ there are always two real solutions

$$\xi = (-1 \pm \sqrt{1 + 4\alpha k s})/2\alpha k, \tag{28}$$

one positive, with the lower magnitude, and one negative, in agreement with Eqs. (13) and (16), respectively, but without any other restriction on α. When $\alpha < 0$, and if we are interested in real solutions only, there are no solutions for $|\alpha| > 1/(4ks)$ [see Eq. (8)], and there are two positive roots for ξ when $|\alpha| < 1/(4ks)$. Thus for the original unknown $h(x)$ we obtain one non-negative and one nonpositive solution (the latter with larger norm) for $\alpha > 0$, two non-negative solutions for $-1(4ks) < \alpha < 0$, and no solutions for $\alpha < -1/(4ks)$. In this example the solution is in fact reduced to the analysis of a quadratic algebraic equation and the solution depends merely on the sign of the numerical coefficients. Note that the factor $1 + \alpha k\xi$ never vanishes.

It seems reasonable to expect similar trends in general for Eq. (1). Of course the similarity cannot be interpreted literally, as shown by the following example.

Example 2: $h(x) + xh(x)\int_0^1 yh(y)\,dy = 1$. Setting

$$\xi = \int_0^1 xh(x)\,dx$$

we get at once $(1 + \xi x)\,h(x) = 1$, and then

$$h(x) = 1/(1 + \xi x), \tag{29}$$

where for an L_p-solution we discard all values of ξ less than or equal to -1. For $\xi = 0$ we would get $h = 1$ which contradicts the hypothesis $\xi = 0$. For $-1 < \xi < 0$ or $\xi > 0$ we multiply Eq. (29) by x and integrate, to get

$$\xi = 1/\xi - (1/\xi^2)\ln|1 + \xi|. \tag{30}$$

For $\xi > -1$, $\xi \neq 0$ all the roots of the transcendental equation (not algebraic any more, so that the number of roots is unknown *a priroi*) yield a positive solution h in L_p. It is easy to check that there is only one root $\xi_0 \neq 0$, with $0 < \xi_0 < 1$. Thus as in Example 1 (with $\alpha > 0$), there is a unique non-negative solution, but now no other solutions exist, neither negative, nor of changing sign.

Other well-known examples are available in the literature. The Chandrasekhar H-equation

$$H(x) - xH(x)\int_0^1 \frac{\psi(y)}{x+y}\,h(y)\,dy = 1, \tag{31}$$

with

$$\psi(x) \geqslant 0, \quad \psi_0 = \int_0^1 \psi(x)\,dx \leqslant \frac{1}{2},$$

($\alpha = -1$ in our notation) has one solution which is positive on $(0,1)$ for $\psi_0 = \frac{1}{2}$ and two positive solutions or just one positive solution for $\psi_0 < \frac{1}{2}$ depending on the particular form of $\psi(x)$.[5] In the former case the modified form of the same equation[4]

$$F(x) + \frac{F(x)}{\sqrt{1 - 2\psi_0}}\int_0^1 \frac{y\psi(y)F(y)}{x+y}\,dy = \frac{1}{\sqrt{1-2\psi_0}} \tag{32}$$

(with $\alpha > 0$ in our notation) has two solutions, one positive and one negative on $(0,1)$, and no other solutions.[11]

Example 1 above shows the non-negative solutions can actually fail to exist for some negative values of the parameter α, and that for other negative values of α two non-negative solutions can also actually occur, even in the simplest case where Eq. (6a) is amenable to a simple quadratic algebraic equation. It is then apparent that in general (i.e., without restrictions on α) existence and/or uniqueness theorems can not be formulated. However, for $\alpha > 0$ some very general results independent of the magnitude of α can be formulated. These results require only the very weak conditions on S and K outlined in the Introduction.

IV. THE CASE OF POSITIVE α

In this section we will look for non-negative solutions to Eq. (1) when $\alpha > 0$. In particular we look for non-negative fixed points of the operator B in Eq. (6b), which is more convenient to handle than A, since it is always positivity preserving. It is clear that the linear operator T is positive ($f \geqslant 0$ implies $Tf \geqslant 0$) and monotone[10] ($f_1 \geqslant f_2$ implies $Tf_1 \geqslant Tf_2$). B is also positive, but not montone, since $f_1 \geqslant f_2$ implies $B(f_1) \leqslant B(f_2)$. But just for this reason B^2 (still positive) is then monotone, i.e., $B^2(f_1) \geqslant B^2(f_2)$. Other properties of B with respect to the cone of the non-negative functions in L_p are $B(f) \leqslant S$ and $B(f) \geqslant S/(1 + \alpha\|T\|\,\|f\|)$ for $f \geqslant 0$, namely

$$\alpha_1(f)S \leqslant B(f) \leqslant \beta_1(f)S, \quad \alpha_1(f) = [1 + \alpha\|T\|\,\|F\|]^{-1},$$
$$\beta_1(f) = 1, \tag{33}$$

for any $f \geqslant 0$. Also, for any $f \geqslant 0$ and $\gamma \in (0,1)$ we have

$$\frac{B(\gamma f)}{B(f)} = \frac{1 + \alpha Tf}{1 + \gamma\alpha Tf}$$
$$= 1 + (1-\gamma)\frac{\alpha Tf}{1 + \gamma\alpha Tf} \geqslant 1 = \gamma + (1 - \gamma),$$
$$1 - \gamma = \eta > 0. \tag{34}$$

Equations (33) and (34) guarantee that B is S-concave.[10]

Moreover B is a continouous operator with respect to the cone of non-negative functions in L_p. We have in fact, for $f_1, f_2 \in L_p(a,b)$, f_1 and f_2 non-negative, $\alpha > 0$

$$|B(f_1) - B(f_2)| = \frac{\alpha S\,|Tf_2 - Tf_1|}{(1 + \alpha Tf_1)(1 + \alpha Tf_2)}$$
$$\leqslant \alpha\|T\|\,\|f_1 - f_2\|S$$

and thus

$$\|B(f_1) - B(f_2)\| \leqslant \alpha\|T\|\,\|S\|\,\|f_1 - f_2\|, \tag{35}$$

which proves the continuity, and shows that Eq. (15) above is a condition for B to be a contraction.

As for the problem of non-negative solution to Eq. (6b), we remark that if h is such a solution, then $h \leqslant S$ and $h = B(h) \geqslant B(S)$. All non-negative solutions are then located in the strip.

$$\frac{S}{1 + \alpha\|T\|\,\|S\|} \leqslant \frac{S}{1 + \alpha TS} \leqslant h \leqslant S. \tag{36}$$

With these results at hand we can now draw the following conclusions.

Lemma 4: Let $\alpha > 0$. Consider the successive approximation scheme

$$h_n = B(h_{n-1}), \quad h_0 = 0, \tag{37}$$

where obviously $0 \leqslant h_n \leqslant S$. Then (i) the even subsequence $\{h_n\}, n = 0, 2, 4,\ldots$, is monotonically nondecreasing and converges pointwise and in norm to a non-negative limit $h^e \leqslant S$ belonging to L_p; (ii) the odd subsequence $\{h_n\}, n = 1, 3, 5,\ldots$, is monotonically nonincreasing and converges point-wise and in norm to a non-negative limit $h^o \leqslant S$ belonging to L_p; (iii) any approximation from the even subsequence is less than or equal to any approximation from the odd subsequence (no overlapping); and finally (iv) the even and odd limits are such that

$$h^e \leqslant h^0, \quad h^0 - h^e = \alpha(h^e T h^o - h^0 T h^e). \tag{38}$$

Proof: Note first that if $h_{n-1} \leqslant h_n$, then $B(h_{n-1}) \geqslant B(h_n)$. Thus we have $h_n \geqslant h_{n+1}$, $h_{n+1} \leqslant h_{n+2}$, and so on (reversed at any step). Now if $h_{n-1} \leqslant h_{n+1}$, then by analogous argument we see that $h_n \geqslant h_{n+2}$, $h_{n+1} \leqslant h_{n+3}$, and so on (conserved for terms of the same parity). Because we have taken $h_0 = 0$, we have $h_0 \leqslant h_1 = S$ and $h_0 \leqslant h_2 = S[1+\alpha TS]^{-1}$. Thus the oscillating behavior of the sequence, the monotonicity of subsequences, and the non-overlapping follow by induction. Because the subsequences are bounded above and below by S and 0, respectively, they converge pointwise. Further because the cone of non-negative functions in L_p is regular,[10] the even and odd subsequences also converge in L_p norm to $h^e \in L_p$ and $h^o \in L_p$, respectively, both limits lying in the strip (36). If $h^e = h^0$, Eq. (38) is obviously true. If $h^e \neq h^o$ the two limits must be related by

$$h^e = \frac{S}{1+\alpha T h^0} = B(h^0), \quad h^0 = \frac{S}{1+\alpha T h^e} = B(h^e) \tag{39}$$

from which Eq. (38) follows.

Sufficient conditions for the convergence of the iterative scheme (37) are given by the following.

Lemma 5: Let $\alpha > 0$ and suppose that either Eq. (15) holds or the kernel K is symmeric, i.e., $K(y,x) = K(x,y)$. Then the even and odd limits of Lemma 4 coincide, and the common value h^* is a non-negative solution of Eq. (6b).

Proof: If Eq. (15) holds then the operator B is a contraction. If then $h^e \neq h^o$ we see from Eq. (39) that the contradiction

$$\|h^o - h^e\| = \|B(h^e) - B(h^o)\| < \|h^e - h^o\|$$

would follow. On the other hand, if the kernel K is symmetric, integration of Eq. (38) over (a,b) gives at once

$$\int_a^b [h^0(x) - h^e(x)]\, dx = 0,$$

where the integrand is non-negative. Thus we have $h^o = h^e = h^*$ and from Eq. (39) it follows finally that $h^* = B(h^*)$ which completes the proof.

Stronger results follow from the next theorem which gives the main result concerning existence of non-negative solutions.

Theorem 5: Let $\alpha > 0$, and consider the iterative scheme

$$h_{n+1} = B\left(\frac{1}{n}\sum_{j=1}^{n} h_j\right), \quad 0 \leqslant h_1 \leqslant S. \tag{40}$$

Then the sequence h_n converges pointwise and in the L_p-norm to a limit h^* in the strip (36) which is a non-negative solution of Eq. (6b).

Proof: All h_n, starting from h_2, are in the strip (36). For any fixed n, and p with $1 \leqslant p < n$ we have

$$\begin{aligned}
|h_{n+1} - h_{n-p+1}| &= \left| B\left(\frac{1}{n}\sum_{j=1}^{n} h_j\right) - B\left(\frac{1}{n-p}\sum_{j=1}^{n-p} h_j\right)\right| \\
&\leqslant \alpha S \left| \left(\frac{1}{n-p} - \frac{1}{n}\right)\sum_{j=1}^{n-p} T h_j - \frac{1}{n}\sum_{j=n-p+1}^{n} T h_j \right| \\
&\leqslant \alpha \|T\| S \left[\frac{p}{n(n-p)}\sum_{j=1}^{n-p} \|h_j\| + \frac{1}{n}\sum_{j=n-p+1}^{n} \|h_j\|\right] \\
&\leqslant 2\frac{p}{n}\alpha \|T\|\, \|S\| S.
\end{aligned}$$

Thus for any fixed p we have (since $S \in L_p$)

$$\lim_{n\to\infty} |h_{n+1} - h_{n-p+1}| = 0 = \lim_{n\to\infty} \|h_{n+1} - h_{n-p+1}\|$$

so that a pointwise and L_p-limit h^* exist, and is in the strip (36). We now show that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} h_j = h^* \tag{41}$$

in the sense of both pointwise and L_p-convergence. Let $\epsilon > 0$ be fixed, and n_ϵ be the corresponding index for which $|h^* - h_j| < \epsilon$ for any $j > n_\epsilon$. We may then write for $n > n_\epsilon$

$$\begin{aligned}
\left|\frac{1}{n}\sum_{j=1}^{n} h_j - h^*\right| &\leqslant \frac{1}{n}\sum_{j=1}^{n_\epsilon} (|h^*| + |h_j|) \\
&\quad + \frac{1}{n}\sum_{j=n_\epsilon+1}^{n} |h^* - h_j| \\
&\leqslant 2\frac{n_\epsilon}{n} S + \left(1 - \frac{n_\epsilon}{n}\right)\epsilon
\end{aligned}$$

and therefore the left-hand side can be made smaller than any fixed positive number, provided n is large enough. This proves Eq. (41) for the pointwise convergence. The proof of convergence in the L_p-norm proceeds in a similar manner with moduli replaced by norms. (It is understood, here and elsewhere that pointwise properties are "almost everywhere"; thus S could even diverge on a set of zero measure.) Note from Eq. (41) that the nth approximation is, when n is large, just the arithmetic average of the preceding ones. Now from Eq. (40) and the continuity of B it follows that $h^* = B(h^*)$, and the existence theorem is complete. We note in passing that the pointwise convergence $B((1/n)\Sigma_{j=1}^{n} h_j)$ to $B(h^*)$ can be proved even without continuity of B if we assume that T maps sequences in L_p converging in norm into pointwise converging sequences.

We now consider the problem of uniqueness of non-negative solutions. To this end note the following.

Lemma 6: Let $\alpha > 0$ and Assumption 2 hold. Then B^2 is an S-concave operator.[10]

Proof: From Eq. (33) we get directly, for any $h \in L_p$, $h \geqslant 0$

$$\alpha_2(h)S \leqslant B^2(h) \leqslant \beta_2(h)S, \quad \alpha_2(h) = \alpha_1[B(h)],$$
$$\beta_2(h) = \beta_1[B(h)]. \tag{42}$$

We then consider, for a fixed non-negative h in L_p, and for a fixed γ with $0<\gamma<1$, the quantity $B^2(\gamma h)$; it can be shown that there exists a number $\eta = \eta(\gamma,h) > 0$ such that

$$B^2(\gamma h) \geqslant (1+\eta)\gamma B^2(h) \tag{43}$$

and therefore, by Eqs. (42) and (43), B^2 is S-concave. Proving Eq. (43) is equivalent to proving

$$1+\alpha TB(h) \geqslant (1+\eta)\gamma[1+\alpha TB(\gamma h)],$$

which is satisfied if

$$0<\eta \leqslant \frac{1-\gamma+\alpha T[B(h)-\gamma B(\gamma h)]}{\gamma[1+\alpha TB(\gamma h)]}.$$

It is then sufficient to choose (since T is linear and positive)

$$0<\eta \leqslant \frac{1-\gamma}{\gamma}\frac{\alpha}{1+\alpha\|T\|\,\|S\|}\frac{\alpha_1(\gamma h)}{1+\alpha\|T\|\,\|h\|}TS$$
$$\leqslant \frac{1-\gamma}{\gamma}\frac{\alpha}{1+\alpha\|T\|\,\|h\|}\frac{TB(\gamma h)}{1+\alpha TB(\gamma h)}$$

or by Eq. (5)

$$0<\eta \leqslant \frac{1-\gamma}{\gamma}\frac{\delta}{1+\alpha\|T\|\,\|S\|}\frac{\alpha_1(\gamma h)}{1+\alpha\|T\|\,\|h\|}, \tag{44}$$

where the right-hand side is positive by Assumption 2.

Our final theorem gives sufficient conditions for uniqueness of non-negative solutions.

Theorem 6: Let $\alpha>0$ and suppose either α satisfies Eq. (15) or Assumption 2 holds. Then Eq. (6b) has a unique non-negative solution h^* given by Theorem 5.

Proof: If Eq. (15) is satisfied, B is a contraction and h^* is necessarily the only non-negative solution, for if another non-negative solution $\hat{h}$ existed, with $\hat{h} \neq h^*$, we would immediately get the contradiction

$$\|h^* - \hat{h}\| = \|B(h^*) - B(\hat{h})\| < \|h^* - \hat{h}\|.$$

If Assumption 2 holds, then by the previous lemma, B^2 is S-concave, and as such it has at most one non-negative fixed point.[10] Since all fixed points of B are also fixed points of B^2, and one non-negative fixed point h^* of B exists, then h^* is necessarily the unique non-negative solution of Eq. (6b).

Corollary 1: Let the hypotheses of Theorem 6 be satisfied. Then the iterative scheme (40) converges to the same limit h^* whatever the initial guess h_1 in (40), and further the iterative scheme (37) also converges to h^* for any initial guess h_0, with $0 \leqslant h_0 \leqslant S$.

Proof: The first part of the corollary is a direct consequence of uniqueness. The second part follows from noting that the even subsequence is just the whole sequence for the equation $h = B^2(h)$ with initial guess h_0, and converges thus monotonically[10] to h^*, the unique non-negative fixed point of B^2 and B. The same occurs to add subsequence, which is the whole sequence for $h = B^2(h)$ with initial guess h_1.

V. CONCLUSIONS

We have given existence and uniqueness theorems for the solution of the nonlinear Eq. (1) under assumption of non-negativity and a couple of very mild smoothness requirements for the known term S and linear kernel K. Most of the theorems are constructive, since they provide the way to evaluate the solution. Particular emphasis has been given to the existence of non-negative solutions. For $\alpha>0$ a non-negative solution always exists, and is unique under a very weak positivity assumption on K and S. In any case it is unique if α is small enough. On the other hand, for $\alpha<0$ there is at least one non-negative solution as long as the magnitude of α is small enough, but it is in general not unique. Moreover, for $\alpha<0$ and $|\alpha|$ large, non-negative solutions can fail to exist.

The analysis of bifurcation with respect to the parameter α deserves further investigation.

Another tool to get more insight into the problem would be a generalization of the analyticity argument used in Ref. 5 for the Chandrasekhar H-equation. It is always possible to rewrite Eq. (1) as

$$[h(x)]^{-1} = 1+\alpha\int_a^b K(x,y)\,h(y)\,dy \tag{45}$$

and assuming that the new kernel satisfies

$$K(x,y)K(-x,z) = K(y,z)\,K(x,y) + K(z,y)\,K(-x,z), \tag{46}$$

one gets the factorization

$$[h(x)h(-x)]^{-1} = T(x)$$
$$\equiv 1+\alpha\int_a^b [K(x,y)+K(-x,y)]\,dy, \tag{47}$$

which could allow some explicit results depending on the analyticity properties of the "dispersion function" $T(x)$. An example of a kernel satisfying Eq. (46) is provided by the straightforward generalization of Eq. (32)

$$K(x,y) = \frac{\phi(x)\psi(y)}{\phi(x)+\phi(y)} \tag{48}$$

with $\phi(-x) = -\phi(x)$, for which

$$T(x) = 1+2\alpha\phi^2(x)\int_a^b \frac{\psi(y)}{\phi^2(x)-\phi^2(y)}\,dy.$$

This approach will be, hopefully, a matter for further research.

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