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R. L. Bowden

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On the zeros of the dispersion function in particle transport theory

R. L. Bowden
Department of Physics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061

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The zeros of the dispersion function that arise in particle transport with anisotropic scattering are studied. An algebraic test for the number of zeros is presented.

I. INTRODUCTION

In treating particle transport in plane geometry with azimuthal symmetry, the transport equation of the particle density $\Psi(x,\mu)$ is often written in the form[1]

$$\mu\frac{\partial\Psi}{\partial x}+\Psi(x,\mu)=\frac{c}{2}\int_{-1}^{+1}f(\mu,\mu')\Psi(x,\mu')d\mu', \tag{1.1}$$

where c is the mean number of secondary particles per collision, x is the distance measured in mean free paths, and μ is the direction cosine of the angle between the x axis and the particle velocity. Here it is assumed that the scattering law is such that $f(\mu,\mu')$ can be adequately represented by a finite Legendre expansion, viz.,

$$f(\mu,\mu')=\sum_{n=0}^{N}(2n+1)f_nP_n(\mu)P_n(\mu'), \tag{1.2}$$

where $P_n(\mu)$ is the Legendre polynomial of order n and physical considerations require that $f_0=1$ and $|f_n|\leqslant 1$, $n>1$. For definitiveness it will be assumed that $f_N\neq 0$. The purpose of this paper is to reexamine the zeros of the dispersion function that arises in the solution to Eq. (1.1). In particular, Mika[2] showed over two decades ago that solutions of the form $\varphi_\nu(\mu)\exp(-x/\nu)$ yield the eigenvalue equation

$$(\nu-\mu)\varphi_\nu(\mu)=\frac{c}{2}\sum_{n=0}^{N}\nu P_n(\mu)h_{n,c}(\nu), \tag{1.3}$$

where

$$h_{n,c}(\nu)=\int_{-1}^{+1}P_n(\mu)\varphi_\nu(\mu)d\mu. \tag{1.4}$$

Further, Mika showed, using the orthogonality and recursion properties of Legendre polynomials, that $h_{n,c}(\mu)$ is a polynomial uniquely determined by the recursion formula

$$(n+1)h_{n+1,c}(\nu)+nh_{n-1,c}(\nu)=(2n+1)(1-cf_n)\nu h_{n,c}(\nu), \tag{1.5}$$

and the nonrestrictive requirement that $h_{-1,c}(\nu)=0$ and

$$h_{0,c}(\nu)=1. \tag{1.6}$$

The so-called discrete solutions of Eq. (1.1) are obtained by solving Eq. (1.3) for $\varphi_\nu(\mu)$ and using the normalization given by Eq. (1.6). The result is that discrete solutions occur for those values of ν in the complex plane $\mathbb{C}\setminus[-1,+1]$ that are zeros of the dispersion function

$$\Lambda_c(\nu)=1+\frac{c}{2}\int_{-1}^{+1}\frac{\nu g(\mu,\nu)}{\mu-\nu}d\mu, \tag{1.7}$$

where

$$g(\mu,\nu)=\sum_{n=0}^{N}(2n+1)f_nP_n(\mu)h_{n,c}(\nu). \tag{1.8}$$

The dispersion function obviously has a cut in the complex ν plane along $(-1,+1)$. The limit values $\Lambda_c^+(\mu)$ and $\Lambda_c^-(\mu)$ of $\Lambda_c(\nu)$ as ν approaches a value $\mu\in(-1,+1)$ from the upper and lower half complex planes, respectively, are given by

$$\Lambda_c^{\pm}(\mu)=1+\frac{c}{2}P\int_{-1}^{+1}\frac{\nu g(\eta,\mu)}{\eta-\mu}d\eta\pm\frac{i\pi c\mu\gamma_c(\mu)}{2}, \tag{1.9}$$

where P indicates the Cauchy principal value and

$$\gamma_c(\mu)=g(\mu,\mu). \tag{1.10}$$

Case[3] and Hangelbrook[4] have shown that $\Lambda_c^{\pm}(\mu)$ does not vanish for $-1<\mu<+1$ and Lekkerkerker[5] has shown that the same result is true for the end points ± 1. The limit value of $\Lambda_c(\nu)$ as $\nu\to\infty$ is given by[6]

$$\Lambda_c(\infty)=\prod_{n=0}^{N}(1-cf_n). \tag{1.11}$$

Other statements about the location and character of the zeros can be made. It is readily seen that the roots must occur in $\pm$ pairs. Further, Case[3] showed that if $c<1$ that the zeros of $\Lambda_c(\nu)$ are real. Moreover, Case showed that if $1-cf_n\geqslant 0$ for $n=1,3,5,\ldots$, the zeros are all simple and are either real or purely imaginary. However, the determination of the *number* of zeros of the dispersion function remains relatively primitive. The number of zeros $2M$ of $\Lambda_c(\nu)$ can be obtained from the argument principle. The contour C in Fig. 1 and a contour at infinity encloses the cut plane. Because $\Lambda_c(\infty)$ is a constant, the number of zeros of the dispersion function is given by the change in the argument of $\Lambda_c(\nu)$ along C as the contour is collapsed (with $\rho\to 0$) onto the real interval $(-1,+1)$. This procedure yields

$$M=(1/\pi)\Delta_C\operatorname{Arg}\Lambda_c^+(\mu), \tag{1.12}$$

where $\Delta_C\operatorname{Arg}\Lambda_c^+(\mu)$ represents the change in the argument of $\Lambda_c^+(\mu)$ as μ varies along the directed line from -1 to $+1$. Since the imaginary part of $\Lambda_c^+(\mu)$, $\mu\in(-1,+1)$, is a polynomial of at most degree $N+1$, then $M\leqslant N+1$. For linear anisotropic scattering ($N=1$), the number of pairs of zeros of $\Lambda_c(\nu)$ can be shown to be either one or two depending on the values of c and f_1. The proof of this last statement is essentially an algebraic one. As will be seen below, the enumeration of the pairs of zeros of

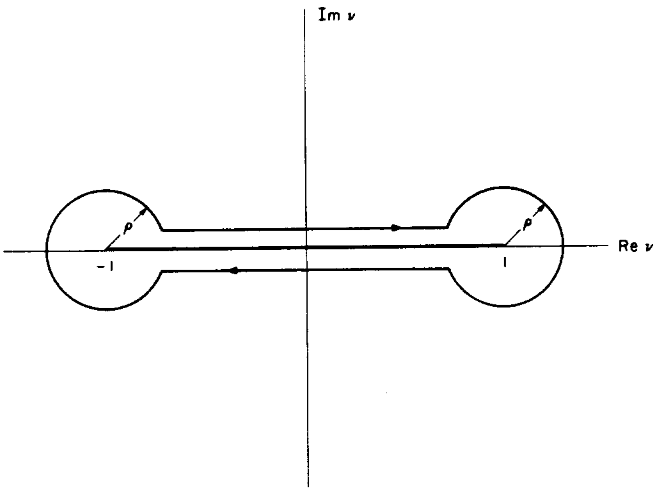

FIG. 1. Contour C.

$\Lambda_c(\nu)$ becomes more difficult as the order of the scattering increases. For $N>4$ and for a given value of c and a set of $\{f_n\}$, the enumeration of the pairs of zeros of the dispersion function in some kind of "closed form" is an unlikely possibility and resort to some sort of numerics is inevitable. The big problem with a numerical evaluation of the change in the argument of a function is that it is easy to lose track as the argument unfolds. Thus an independent evaluation of the number of pairs of zeros would be useful.

The main result of this paper provides such an algebraic test for the number of pairs of zeros of the dispersion function. The proof of this test is based in part on the observation that the function $\gamma_c(1)$ can be regarded as a polynomial in c of order $N^* = N - K$, where K is the number of f_n, $0<n<N$, that are zero. It will be shown below that the N^* zeros of $\gamma_c(1)$ are all simple and real. Denote the nonpositive zeros of $\gamma_c(1)$ by c_p^-, $p = 1,...,P$, and the positive zeros by c_q^+, $q = 1,...,Q$, with $P + Q = N^*$. Order these zeros according to

$$c_P^- < c_{P-1}^- < \cdots < c_1^- < c_1^+ < c_2^+ < \cdots < c_Q^+. \qquad (1.13)$$

If $0 < c_{k-1}^+ < c < c_{k+1}^+$ for a given set of $\{f_n\}$, then the number of pairs of zeros of $\Lambda_c(\nu)$ is $k+1$. A similar idea was proposed by Dawn and Chen[7] but their analysis is not as complete as the one presented here.

The proof of the preceding test is contained in the remaining sections of this paper. It proves convenient in that proof to make the change of variables $c \to 1/s$. This change is made in Sec. II. The essential points of a mapping between the s plane and the ν plane are also made in that section. The proof of the test given above is contained in the main theorem proved in Sec. III. Concluding ancillary remarks about the character of the zeros of the dispersion function are made in Sec. IV.

II. MAPPING BETWEEN THE ν PLANE AND THE s PLANE

The dispersion function can also be written in the form

$$\Lambda_c(\nu) = R_c(\nu) - c\nu\gamma_c(\nu)Q_0(\nu), \qquad (2.1)$$

where here and in the subsequent analysis $Q_n(\nu)$ is the nth-order Legendre function of the second kind and $R_c(\nu)$ is a polynomial in ν and c. With the change of variables $c = 1/s$ an auxiliary dispersion function $\Lambda(\nu,s)$ is defined by

$$\begin{aligned}\Lambda(\nu,s) &= s^{N^*+1}\Lambda_{1/s}(\nu)\\ &= R(\nu,s) - \nu\gamma(\nu,s)Q_0(\nu)\\ &= s^{N^*+1} + s^{N^*}A_1(\nu) + \cdots + A_{N^*}(\nu),\end{aligned} \qquad (2.2)$$

where

$$\begin{aligned}R(\nu,s) &= s^{N^*+1}R_{1/s}(\nu)\\ &= s^{N^*+1} + s^{N^*}b_1(\nu) + \cdots + b_{N^*+1}(\nu),\end{aligned} \qquad (2.3)$$

and

$$\begin{aligned}\gamma(\nu,s) &= s^{N^*}\gamma_{1/s}(\nu)\\ &= s^{N^*}a_0(\nu) + s^{N^*-1}a_1(\nu) + \cdots + a_{N^*}(\nu).\end{aligned} \qquad (2.4)$$

Here $b_j(\nu)$ and $a_j(\nu)$ are even polynomials in ν only and $A_j(\nu)$ is an analytic function on $\nu\,\mathbb{C}\setminus[-1,+1]$. Obviously, $\Lambda(\nu,s)$ and $\Lambda_{1/s}(\nu)$ have the same zeros in the ν plane for $s\neq 0$ ($c\neq\infty$). The object is to consider $\Lambda(\nu,s)$ as a complex function of two complex variables and use the implicit function theorem to study $\Lambda(\nu,s) = 0$.

In particular, $\Lambda(\nu,s)$ for fixed ν can be regarded as a polynomial in s and its zeros can be investigated. For example, with $\nu = \infty$, Eq. (1.11) can be written in the present notation as

$$\Lambda(\infty,s) = \prod_{n=0}^{N} G_n(s), \qquad (2.5)$$

where

$$\begin{aligned}G_n(s) &= (s - f_n), &&\text{if } f_n \neq 0,\\ &= 1, &&\text{if } f_n = 0.\end{aligned} \qquad (2.6)$$

Thus the point $\nu = \infty$ maps by $\Lambda(\infty,s) = 0$ into $N^* + 1$ real points, the nonzero f_n in the s plane. These points are, of course, distinct if the f_n are all different. Consequently, it will be assumed for simplicity that all the nonzero f_n are distinct. However, since $\Lambda(\nu,s)$ and $\gamma(\nu,s)$ are also polynomials in the f_n, the main results obtained here also follow for nondistinct f_n by continuity. Other points in the ν plane also map into real points in the s plane. To this point consider the following.

Lemma 1: If $\nu_0 \in \mathbb{R}\setminus[-1,+1]$, then the roots of $\Lambda(\nu_0,s) = 0$ are all real.

Proof: The proof of this lemma follows from using the dispersion function in a form written by Inönü,[8]

$$\Lambda_c(\nu) = (N+1)[Q_{N+1}h_{N,c}(\nu) - Q_N(\nu)h_{N+1,c}(\nu)]. \qquad (2.7)$$

If $\kappa_n < n$ of the f_n are zero and

$$h_n(\nu,s) = s^{\kappa_n}h_{n,1/s}(\nu), \qquad (2.8)$$

the recursion formula for the $h_n(\nu,s)$ can be written as

$$\begin{aligned}&(n+1)h_{n+1}(\nu,s) + ns^{\delta_n}h_{n-1}(\nu,s)\\ &\quad = (2n+1)G_n(s)\nu h_n(\nu,s),\end{aligned} \qquad (2.9)$$

with

$$h_{-1}(\nu,s) = 0,\ h_0(\nu,s) = 1,$$

and

$$\delta_n = \kappa_{n+1} - \kappa_{n-1} \geqslant 0.$$

Thus the auxiliary dispersion function $\Lambda(\nu,s)$ takes the form

$$\Lambda(\nu,s) = (N+1)\left[Q_{N+1} s h_n(\nu,s) - Q_N(\nu) h_{N+1}(\nu,s)\right]. \tag{2.10}$$

Let $\nu_0 \in \mathbf{R} \setminus [-1,+1]$ be fixed and consider

$$\Lambda(\nu_0,s) = Q_{N+1}(\nu_0) s h_N(\nu_0,s) - Q_N(\nu_0) h_{N+1}(\nu_0,s) = 0. \tag{2.11}$$

Note that $h_N(\nu_0,s)$ and $h_{N+1}(\nu_0,s)$ cannot vanish for the same value of s, for if they did, then the recursion formula would yield $h_{N-1}(\nu_0,s) = 0$, which would imply $h_{N-2}(\nu_0,s) = 0$, etc. This would eventually lead to the contradiction $h_0(\nu_0,s) = 0$. It can be easily shown that

$$h_n(\nu,0) = \frac{1}{n!} \prod_{j=0}^{n-1} (2n+1)[-G_n(0)]\nu^j. \tag{2.12}$$

Thus $\Lambda(\nu_0,0)$ does not vanish for $\nu_0 \in \mathbf{R} \setminus [-1,+1]$. Now let s_1 and s_2 be nonzero roots of $\Lambda(\nu_0,s) = 0$. Equation (2.11) then yields

$$s_2 h_N(\nu_0,s_2) h_{N+1}(\nu_0,s_1) = s_1 h_N(\nu_0,s_1) h_{N+1}(\nu_0,s_2). \tag{2.13}$$

Rewriting Eq. (2.9) for $s = s_1$, $\nu = \nu_0$, and then for $s = s_2$, $\nu = \nu_0$, and combining the results in a familiar fashion yields

$$\begin{aligned}(N+1)&[s_2 h_N(\nu_0,s_2) h_{N+1}(\nu_0,s_1) \\ &- s_1 h_N(\nu_0,s_1) h_{N+1}(\nu_0,s_2)]/(s_1 s_2)^{N^*+1} \\ &= (s_1 - s_2) \sum_{n=0}^{N} \frac{\nu_0(2n+1) f_n h_n(\nu_0,s_1) h_n(\nu_0,s_2)}{(s_1 s_2)^{\kappa_n - 1}}.\end{aligned} \tag{2.14}$$

Because $h_n(\nu_0,s)$ for fixed $\nu_0 \in \mathbf{R} \setminus [-1,+1]$ is a polynomial in s with real coefficients, if s_1 is a zero of $\Lambda(\nu_0,s)$, then so is $\bar{s}_1$. Thus let $s_2 = \bar{s}_1$ and employ Eq. (2.13) to obtain

$$\operatorname{Im} s_1 \sum_{n=0}^{N} (2n+1) f_n \left| \frac{h_n(\nu_0,s_1)}{s_1^{\kappa_n}} \right|^2 = 0. \tag{2.15}$$

Hence, for example, if all of the f_n are non-negative, then the sum in the last expression is positive and therefore s_1 real.

To pin down the general situation consider the relation given by Bowden *et al.*,[9]

$$\begin{aligned}\Lambda_{1/s}&(\nu) P_n(\nu) \\ &= \frac{\nu}{2} \int_{-1}^{+1} \frac{P_n(\mu)}{\nu - \mu} \\ &\quad \times \sum_{m=0}^{N} \frac{(2m+1) f_m P_m(\mu) h_m(\nu,s)}{s^{\kappa_m+1}} \, d\mu \\ &\quad + h_n(\nu,s)/s^{\kappa_n}.\end{aligned} \tag{2.16}$$

Now let $\nu = \nu_0$ and $s = s_1$ be defined as above. Multiplying Eq. (2.16) by $(2n+1) f_n h_n(\nu_0,\bar{s}_1)/\bar{s}^{\kappa_n}$ and summing on n yields

$$\frac{\nu_0}{2s_1} \int_{-1}^{+1} \left| \sum_{n=0}^{N} \frac{(2n+1) f_n P_n(\mu) h_n(\nu_0,s_1)}{s_1^{\kappa_n}} \right|^2 \frac{d\mu}{\mu - \nu_0} + \sum_{n=0}^{N} (2n+1) f_n \left| \frac{h_n(\nu_0,s_1)}{s_1^{\kappa_n}} \right|^2 = 0. \tag{2.17}$$

Here the fact that $h_n(\nu_0,\bar{s}_1) = \bar{h}_n(\nu_0,s_1)$ for ν_0 real has been used. For $\nu_0 \in \mathbf{R} \setminus [-1,+1]$ the integral term in Eq. (2.17) will not vanish; thus Eqs. (2.15) and (2.17) state that $\operatorname{Im} s_1 = 0$, i.e., s_1 is real. This completes the proof of the lemma. To show that these zeros (for fixed ν_0) are simple, consider the following.

Lemma 2: If $\nu_0 \in \mathbf{R} \setminus [-1,+1]$ and $\Lambda(\nu_0,s_0) = 0$, then $\partial \Lambda(\nu_0,s_0)/\partial s \neq 0$.

Proof: Let $H_n(\nu,s) = s h_n(\nu,s)$. It follows from Eq. (2.10) that

$$\frac{\partial \Lambda(\nu,s)}{\partial s} = (N+1)\left[Q_{N+1}(\nu) \frac{\partial H_n(\nu,s)}{\partial s} - Q_N(\nu) \frac{h_{N+1}(\nu,s)}{\partial s}\right]. \tag{2.18}$$

If now both $\Lambda(\nu_0,s_0) = 0$ and $\partial \Lambda(\nu_0,s_0)/\partial s = 0$, then Eqs. (2.11) and (2.18) imply that

$$h_{N+1}(\nu_0,s_0) \frac{\partial H_N(\nu_0,s_0)}{\partial s} = H_N(\nu_0,s_0) \frac{\partial h_{N+1}(\nu_0,s_0)}{\partial s}. \tag{2.19}$$

Dividing both sides of Eq. (2.14) by $(s_1 - s_2)$ and taking the limit $s_2 \to s_1 = s_0$, where s_0 is a zero of $\Lambda(\nu_0,s)$, give

$$\begin{aligned}\frac{(N+1)}{s_0^{2N^*}}&\left[H_N(\nu_0,s_0) \frac{\partial h_{N+1}(\nu_0,s_0)}{\partial s} - h_N(\nu_0,s_0) \frac{\partial H_N(\nu_0,s_0)}{\partial s}\right] \\ &= \sum_{n=0}^{N} (2n+1) f_n \left| \frac{h_n(\nu_0,s_0)}{s_0^{\kappa_n}} \right|^2.\end{aligned} \tag{2.20}$$

Therefore from Eq. (2.19) a necessary condition for $\Lambda(\nu_0,s_0)$ and $\partial \Lambda(\nu_0,s_0)/\partial s$ to vanish is that the right-hand side of Eq. (2.20) also vanish. The proof of the lemma is completed by recalling from Lemma 1 that the right-hand side of Eq. (2.20) does not vanish for $\nu_0 \in \mathbf{R} \setminus [-1,+1]$.

There are $N^* + 1 = N + 1 - K$ nonvanishing roots of $\Lambda(\nu_0,s) = 0$ for $\nu_0 \in \mathbf{R} \setminus [-1,+1]$ that are real and simple. Denote these roots by $s_0^{(0)}, s_0^{(2)}, \ldots, s_0^{(N^*)}$. From the implicit function theorem there are neighborhoods, say $N(\nu_0)$ and $N_j(s_0^{(j)})$, such that the equation $\Lambda(\nu,s) = 0$ has a unique root $S_j(\nu)$ in $N_j(s_0^{(j)})$ for any ν in $N(\nu_0)$. Further, each function $S_j(\nu)$ is single valued and analytic on $N(\nu_0)$ and satisfies the condition $S_j(\nu_0) = s_0^{(j)}$.

The immediate objective now is to continue the $S_j(\nu)$ to the right (left) half complex plane cut as described below. That each of these functions can be continued along any line in the ν plane that avoids the cut $[-1,+1]$ and zeros of the discriminant of Eq. (2.2) is clear. The discriminant of Eq. (2.2) can be written in the form

$$D(\nu) = \sum_{n=0}^{M_0} \beta_n(\nu)[\nu Q_0(\nu)]^n, \tag{2.21}$$

where M_0 is finite and $\beta_n(\nu)$ is an even polynomial with real coefficients. Thus $D(\nu)$ is analytic on the complex plane cut along $[-1,+1]$, has at most a finite-order pole at infinity, and has the limits

$$D^{\pm}(\mu)=\sum_{n=0}^{M_0}\beta_n(\mu)\left[\mu\tanh^{-1}\mu\mp\frac{i\pi\mu}{2}\right]^n \quad (2.22)$$

on the cut $(-1,+1)$. It is readily seen that the real and imaginary parts of $D^{\pm}(\mu)$ have only a finite number of zeros for $\mu\in(-1,+1)$. A straightforward argument principle calculation similar to the one about the contour C mentioned in Sec. I shows that the number of zeros of $D(\nu)$ is finite. Because of the assumption that nonzero f_n are distinct, $D(\nu)$ does not vanish at infinity. Further, since $D(\nu)=D(-\nu)$ and $D(\bar\nu)=\bar D(\nu)$, if $\nu=\nu'$ is a zero of $D(\nu)$ so are $\nu=-\nu'$ and $\nu=\bar\nu$. Let

$$\mathscr{D}=\{\zeta_i|D(\zeta_i)=0\}, \quad (2.23)$$

where $\pm\zeta_0, \pm\zeta_1,\ldots,\zeta_p=0$ are points on the imaginary axis with $|\zeta_0|>|\zeta_1|>\cdots>|\zeta_p|$ and $\pm\zeta_{p+1},\ldots,\pm\zeta_{p+q}, \pm\bar\zeta_{p+q}$ are the rest of the points of $\mathscr{D}$. (Note that p could be equal to zero.) Now cut the ν plane by joining $+\zeta_0,+\zeta_1,\ldots,\zeta_p$ in the upper half plane with a straight line, similarly joining $-\zeta_0,-\zeta_1,\ldots,\zeta_p$ in the lower half plane, joining $\zeta_p=0$, $\zeta_{p+1},\ldots,\zeta_{p+q}$ with a series of straight lines in the first quadrant, making similar joinings in the remaining quadrants, and finally adding the original cut along $(-1,+1)$.

Each of the $S_j(\nu)$ can be analytically continued to the right (left) half complex plane cut as described above so that, according to the monodromy theorem, each function will be single valued and analytic in the right (left) cut plane. Each function $S_j(\nu)$ can be continued from the right half plane to the left half plane by considering the regions $|\mathrm{Im}\,\nu|>|\zeta_0|$. Thus each $S_j(\nu)$ so continued has the property that $S_j(\nu)=S_j(-\nu)$. Since $S_j(\nu)$ is real for $\nu\in\mathbb{R}\setminus[-1,+1]$, the reflection principle yields the additional property that $S_j(\bar\nu)=\bar S_j(\nu)$. Since $S_j(\nu)$ is continuous across the imaginary axis for $|\mathrm{Im}\,\nu|>|\zeta_0|$, the two properties listed show that $S_j(\nu)$ is real if ν lies on the imaginary axis and $|\mathrm{Im}\,\nu|>|\zeta_0|$. Most importantly, of course, is the property that $\Lambda[\nu,S_j(\nu)]=0$ for every ν in the plane cut as described. The functions $S_j(\nu)$ will be labeled according to $\lim_{\nu\to\infty}S_j(\nu)=f_{n_j}$, where $f_{n_0}=f_0=1$ and f_{n_j}, $j=1,2,\ldots,N^*$, are the nonvanishing expansion coefficients.

To look at the behavior of the $S_j(\nu)$ on $[-1,+]$ it is helpful to consider the following lemmas.

Lemma 3: If $\nu_0\in\mathbb{R}\setminus(-1,+1)$, then the roots of $\gamma(\nu_0,s)=0$ are all real.

Proof: As demonstrated by Inönü,[8] the recursion formula for $P_n(\nu)$ and $h_n(\nu,s)$ can be used to write

$$\nu\gamma(\nu,s)=(N+1)[P_{N+1}(\nu)sh_N(\nu,s) - P_N(\nu)h_{N+1}(\nu,s)]. \quad (2.24)$$

This expression is entirely analogous to Eq. (2.10) with $Q_n(\nu)$ replaced by $P_n(\nu)$. Thus letting $\nu_0\in\mathbb{R}\setminus(-1,+1)$ be fixed, letting s_0 be a nonzero root of $\gamma(\nu_0,s)=0$, and following the proof of Lemma 1 yields

$$\mathrm{Im}\,s_0\sum_{n=0}^{N}(2n+1)f_n\left|\frac{h_n(\nu_0,s_0)}{s_0^{\kappa_n}}\right|^2=0. \quad (2.25)$$

Substituting $\nu=\nu_0$ and $s=s_0$ defined as above into Eq. (2.16), multiplying the resulting equation by $(2n+1)f_nh_n(\nu_0,\bar s_0)/\bar s_0^{\kappa_n}$, and summing on n gives

$$\frac{\nu_0}{2s_0}\int_{-1}^{+1}\left|\sum_{n=0}^{N}\frac{(2n+1)f_nP_n(\mu)h_n(\nu_0,s_0)}{s_0^{\kappa_n}}\right|^2\frac{d\mu}{\mu-\nu_0} + \sum_{n=0}^{N}(2n+1)f_n\left|\frac{h_n(\nu_0,s_0)}{s_0^{\kappa_n}}\right|^2=0. \quad (2.26)$$

Note that the integral in Eq. (2.26) is well defined for $\nu_0=1$ and for $\nu_0\in\mathbb{R}\setminus(-1,+1)$ the integral term does not vanish. Thus Eqs. (2.25) and (2.26) state that $\mathrm{Im}\,s_0=0$, i.e., s_0 is real.

Lemma 4: If $\nu_0\mathbb{R}\setminus(-1,+1)$ and s_0 is a nonzero root of $\Lambda(\nu_0,s_0)=0$, then $\partial\Lambda(\nu_0,s_0)/\partial s\neq0$.

The proof of this lemma is completely analogous to that of Lemma 3 and the details will be omitted.

It can be shown that $a_0(\nu)$ in Eq. (2.4) can be written as $\Sigma_{n=0}^{N}(2n+1)f_n[P_n(\nu)]^2$. It will be assumed that $a_0(1)\neq0$; this is equivalent to $f(1,1)>0$ in Eq. (1.2). However, again $a_0(\nu)$ is a polynomial in the f_n and the case $a_0(1)=0$ can be included by continuity.

The equation $\gamma(\nu_0,s)=0$ for $\nu_0\in\mathbb{R}\setminus(-1,+1)$ has N^* simple real nonvanishing roots. In particular denote the roots of $\gamma(1,s)=0$ by $\xi_1,\xi_2,\ldots,\xi_{N^*}$ with the ordering of the roots given by the following.

Lemma 5:

$$\lim_{\nu\to1}S_j(\nu)=\xi_j,\quad j=1,2,\ldots,N^*. \quad (2.27)$$

Proof: Let

$$\Lambda'(\nu,s)=R(\nu,s)/[\nu Q_0(\nu)]-\gamma(\nu,s). \quad (2.28)$$

For fixed $\nu\neq1$ the zeros of $\Lambda(\nu,s)$ and $\Lambda'(\nu,s)$ coincide. For $\nu=1$, it is obvious that $\Lambda'(\nu,s)$ vanishes at the zeros of $\gamma(1,s)$. Thus if $S_j'(\nu)$ is a zero of $\Lambda'(\nu,s)$ then

$$|S_j'(\nu)-S_j(\nu)|=|S_j'(\nu)-\xi_j+\xi_j-S_j(\nu)| =0,\quad \nu\neq1. \quad (2.29)$$

Therefore

$$|S_j(\nu)-\xi_j|=|S_j'(\nu)-\xi_j|,\quad \nu\neq1. \quad (2.30)$$

The proof is completed by noting that the right-hand side of the last equation vanishes in the limit $\nu\to1$. A similar calculation leads to the following.

Lemma 6:

$$\lim_{\nu\to1}S_0(\nu)=\lim_{\nu\to1}\xi_0(\nu), \quad (2.31)$$

where

$$\xi_0(\nu)=-b_1(\nu)-a_0(\nu)\nu Q_0(\nu) +a_1(\nu)/a_0(\nu). \quad (2.32)$$

with the polynomials $a_n(\nu)$ and $b_n(\nu)$ given by Eqs. (2.3) and (2.4).

Proof: Let

$$\Lambda''(\nu,s)=s+\sum_{n=0}^{N^*}\frac{b_{n+1}(\nu)a_n(\nu)\nu Q_0(\nu)}{s^n}, \quad (2.33)$$

and note that for $\nu\neq1$, the zeros of $\Lambda(\nu,s)$ and $\Lambda''(\nu,s)$ coincide. Let $S_0''(\nu)$ be a zero of $\Lambda''(\nu,s)$, i.e., $\Lambda''[\nu,S_0''(\nu)]=0$, and note that

$$|S_0(\nu) - S_0''(\nu)|$$
$$= |S_0(\nu) - \xi_0(\nu) + \xi_0(\nu) - S_0''(\nu)|$$
$$= 0. \tag{2.34}$$

Using the same argument as in Lemma 5 completes the proof. Note, for example, that as $\nu \to 1$ along the real axis that $\xi_0(\nu) \to \infty$. Further if $\xi_0^+(\mu)$ and $\xi_0^-(\mu)$ are the limits of $\xi_0(\nu)$ as $\nu \to \mu \in (-1, +1)$ from the upper and lower complex plane, respectively, then

$$\lim_{\mu \to 1} \xi_0^{\pm}(\mu) = +\infty \mp i\pi a_0(1)/2 \equiv \xi_0^{\pm}. \tag{2.35}$$

Let $\Lambda^+(\mu,s)$ and $\Lambda^-(\mu,s)$ be the limits of $\Lambda(\nu,s)$ as $\nu \to \mu \in (-1, +1)$ from the upper and lower half complex ν plane, respectively, and consider for fixed μ the roots of $\Lambda^{\pm}(\mu,s) = 0$. There are $N^* + 1$ such roots, some of which may be multiple roots if μ is a zero of the discriminant of $\Lambda^{\pm}(\mu,s)$. Let $S_j^{\pm}(\mu)$, $j = 0,1,...,N^*$, be the functions generated by such roots as μ takes on values along $(-1, +1)$.

Lemma 7: Each function $S_j^{\pm}$ is continuous on $(-1, +1)$.

Proof: The proof will be illustrated for $S_j^+(\mu)$. The proof for $S_j^-(\mu)$ follows in an analogous manner. Let s_0 be a root of $\Lambda^+(\mu_0,s) = 0$, where $\mu_0 \in (-1, +)$ is not a zero of the discriminant of $\Lambda^+(\mu,s)$. Further, let K_ϵ be a circle of radius $\epsilon > 0$ centered on s_0 so small that $\Lambda^+(\mu_0,s)$ contains no zero except at the point s_0 itself. Since $\Lambda^+(\mu_0,s)$ is analytic inside of K_ϵ, let $\eta > 0$ be the minimum of $|\Lambda^+(\mu_0,s)|$ on K_ϵ. For fixed s, $\Lambda^+(\mu,s)$ is a continuous function of μ on $(-1, +1)$. Therefore, choose a real interval Δ so small that $|\Lambda^+(\mu_0,s) - \Lambda^+(\mu,s)| < \eta$ for all $\mu \in \Delta$. Thus according to Rouche's theorem

$$\Lambda^+(\mu,s) = \Lambda^+(\mu_0,s) + [\Lambda^+(\mu,s) - \Lambda^+(\mu_0,s)] \tag{2.36}$$

has only one zero inside K_ϵ for any fixed but arbitrary $\mu \in \Delta$. If $\Lambda^+(\mu_0,s) = 0$ has a k-fold multiple root, then repeating the argument above shows that the circle K_ϵ encloses k zeros of $\Lambda^+(\mu,s)$ for $\mu \in \Delta$. Thus each $S_j^+(\mu)$ is continuous on $(-1, +1)$ and at each zero of the discriminant of $\Lambda^+(\mu,s)$ that corresponds to a k-fold multiple root of $\Lambda^+(\mu,s) = 0$ (e.g., $\mu = 0$) k of the functions $S_j^+(\mu)$ take on the same value. The labeling of the functions $S_j^+(\mu)$ is given by the following.

Lemma 8: The limits of $S_j(\nu)$, $j = 0,...,N^*$, as $\nu \to \mu \in (-1, +1)$ from the upper and lower complex plane are $S_j^+(\mu)$ and $S_j^-(\mu)$, respectively.

Proof: The proof of this lemma is similar to Lemma 7 and again the proof will be illustrated for $S_j^+(\mu)$. The proof for $S_j^-(\mu)$ follows in an analogous manner. As in Lemma 7, let s_0 be a root of $\Lambda^+(\mu_0,s) = 0$, where $\mu_0 \in (-1, +1)$ is not a zero of the discriminant of $\Lambda^+(\mu,s)$. Again let K_ϵ be a circle of radius $\epsilon > 0$ centered on s_0 so small that $\Lambda^+(\mu_0,s)$ encloses only the zero at s_0 itself. Let $\eta > 0$ be the minimum of $|\Lambda^+(\mu_0,s)|$ on K_ϵ. Finally let K_δ be a circle centered on μ_0 so small that $|\Lambda^+(\mu_0,s) - \Lambda(\nu,s)| < \eta$ for any ν with $\operatorname{Re}\nu > 0$ inside K_δ. Thus again from Rouche's theorem

$$\Lambda(\nu,s) = \Lambda^+(\mu_0,s) + [\Lambda(\nu,s) - \Lambda^+(\mu_0,s)] \tag{2.37}$$

has only one zero inside K_ϵ for any fixed but arbitrary point ν in K_δ with $\operatorname{Re}\nu > 0$. If $\Lambda^+(\mu_0,s) = 0$ has a k-fold multiple root at s_0, then the circle K_ϵ will contain k roots of $\Lambda(\nu,s)$.

III. MAIN THEOREM

Consider the contours generated by $s = S_j(\nu)$, $j = 0,1,...,N^*$, as ν varies along the contour C of Fig. 1 as that contour is collapsed (with $\rho \to 0$) onto the real interval $(-1, +1)$. These contours are in fact the contours Γ_j generated parametrically by $s = S_j^{\pm}(\mu)$, $j = 0,1,...,N^*$, as μ varies along the real interval $(-1, +1)$. Note that $S_j^+(-\mu) = \overline{S}_j^+(\mu)$, $S_j^+(\mu) = S_j^-(-\mu)$, and that each of the contours begins and ends at the limit points given by Lemmas 5 and 6. Thus the contour Γ_j starts, say, at ξ_j, varies continuously in the s plane as μ varies from -1 to 0 along the top of the cut, passes through zero at $\mu = 0$, traces out its complex conjugate as μ continues to vary from 0 to $+1$ along the top of the cut, and finally retraces itself as μ varies from $+1$ to -1 along the bottom of the cut. That the contours do not cross the real s plane axis except at $s = 0$ and $s = \xi_j$, $j > 0$, is clear. For if $S_j^+(\mu_0) = s_0 \in \mathbb{R}$ for some value of $\mu_0 \in (-1, +1)$, that would imply that $\Lambda^+(\mu_0,s_0) = 0$ in contradiction to the results cited in Sec. I.

The contours Γ_j, $j = 1,...,N^*$, are closed. (The contour Γ_0 can be regarded as closed if it is regarded as being closed at infinity.) The contours Γ_j have positive (counterclockwise) orientation. Since $\operatorname{Im} S_j^+(\mu) \neq 0$ for $0 < |\mu| < 1$, it is sufficient to show positive orientation of the Γ_j by demonstrating for some μ_0 with $0 < \mu_0 < 1$ that

$$\begin{aligned} \operatorname{Im} S_j^+(\mu_0) &< 0, \quad \text{if } \xi_j > 0, \\ &> 0, \quad \text{if } \xi_j < 0. \end{aligned} \tag{3.1}$$

Note first that $\operatorname{Im} S_0^+(\mu) > 0$, since

$$\lim_{\mu \to 1} \operatorname{Im} S_0^+(\mu) = -a_0(1)\pi/2, \tag{3.2}$$

with $a_0(1) > 0$. If $\Lambda^+(\mu,s)$ is evaluated from Eq. (2.2), it is easy to see that

$$\lim_{\mu \to 1} \operatorname{Im} \Lambda^+(\mu,s) = \gamma(1,s)\pi/2. \tag{3.3}$$

Now order the zeros of $\gamma(1,s)$ according to

$$\xi_{m_1} > \xi_{m_2} > \cdots > \xi_{m_Q} > 0 > \xi_{m_{Q+1}} > \cdots > \xi_{m_{N^*}}, \tag{3.4}$$

and choose $1 > \mu_0 > 0$ so that either

$$\begin{aligned} &\xi_{m_{q+1}} < \operatorname{Re} S_{m_q}^+(\mu_0) < \xi_{m_q}, \quad \text{if } 1 \leqslant q < Q, \\ &0 < \operatorname{Re} S_{m_q}^+(\mu_0) < \xi_{m_Q}, \quad \text{if } q = Q, \\ &\xi_{m_{Q+1}} < \operatorname{Re} S_{m_q}^+(\mu_0) < 0, \quad \text{if } q = Q+1, \text{ or} \\ &\xi_{m_q} < \operatorname{Re} S_{m_q}^+(\mu_0) < \xi_{m_{q-1}}, \quad \text{if } N^* \geqslant q > Q+1. \end{aligned} \tag{3.5}$$

If $s = \operatorname{Re} S_{m_q}^+(\mu_0)$, $q = 1,...,N^*$, then

$$\Lambda^+[\mu,\mathrm{Re}\,S^+_{m_q}(\mu_0)] = -\mathrm{Im}\,S^+_{m_q}(\mu_0)[\mathrm{Re}\,S^+_{m_q}(\mu_0) - \mathrm{Re}\,S^+_0(\mu)] \times\left\{\prod_{\substack{j=1\\ j\neq m_q}}^{N^*}[\mathrm{Re}\,S^+_{m_q}(\mu_0) - \mathrm{Re}\,S^+_j(\mu)] + T(\mu,\mu_0)\right\}, \tag{3.6}$$

where $T(\mu,\mu_0)$ is a function such that $T(\mu,\mu_0)\to 0$ as $\mu\to 1$. Thus for μ sufficiently close to 1, Eqs. (3.3) and (3.6) yield

$$\begin{aligned}\mathrm{sgn}(\Lambda^+[\mu,\mathrm{Re}\,S^+_{mq}(\mu_0)])&\\ =\mathrm{sgn}\Big[-\mathrm{Im}\,S^+_{m_q}(\mu_0)&\\ \times\prod_{\substack{j=0\\ j\neq m_q}}^{N^*}(\mathrm{Re}\,S^+_{m_q}(\mu_0) - \mathrm{Re}\,S^+_j(\mu))\Big]&\\ =\mathrm{sgn}(\gamma[1,\mathrm{Re}\,S^+_{m_q}(\mu_0)]).&\end{aligned} \tag{3.7}$$

Moreover, since $\lim_{s\to\infty}\gamma(1,s)\to\infty$, then $\mathrm{sgn}[\gamma(1,s)] = \mathrm{sgn}[(-1)^q]$ if $\xi_{m_{q+1}} < s < \xi_{m_q}$. Thus if $\mathrm{Re}\,S^+_{m_q}(\mu_0)$ is chosen by Eq. (3.5), then Eq. (3.7) gives

$$\begin{aligned}\mathrm{sgn}[-(-1)^q\,\mathrm{Im}\,S^+_{m_q}(\mu_0)]&\\ =\mathrm{sgn}[(-1)^q],\quad &\text{if } \xi_{m_q}>0,\\ =\mathrm{sgn}[(-1)^{q+1}],\quad &\text{if } \xi_{m_q}<0.\end{aligned} \tag{3.8}$$

Theorem: Let $I(\Gamma_j)$ and $E(\Gamma_j)$ represent the interior and exterior of the contours Γ_j, $j=0,1,\ldots,N^*$, respectively, and let

$$s\in\bigcap_{j=0}^{P-1} I(\Gamma_{m_j})\bigcap_{j=P}^{N^*} E(\Gamma_{m_j}). \tag{3.9}$$

In other words, let s lie in the interior of P of the contours Γ_j and the exterior to all the other Γ_j. The number of roots of $\Lambda(\nu,s)=0$ is $\Sigma_{j=0}^{P-1} N_{m_j}$, where N_{m_j} is the index of s with respect to Γ_j. Further, if s is real and satisfies Eq. (3.9) then $N_{m_j}=1$ and $M=P+1$, i.e., just equal to the number of contours Γ_j in which s lies.

Proof: As indicated in Sec. I, $\Lambda(\infty,s)$ is a constant and the number of zeros of $\Lambda(\nu,s)$ is given by the change in the argument of $\Lambda(\nu,s)$ along the contour C in Fig. 1 as the contour is collapsed (with $\rho\to 0$) onto the real interval $(-1,+1)$. This procedure yields [cf. Eq. (1.12)]

$$M=(1/\pi)\Delta_C\,\mathrm{Arg}\,\Lambda^+(\mu,s), \tag{3.10}$$

where $\Delta_C\,\mathrm{Arg}\,\Lambda^+(\mu,s)$ represents the change in the argument along the directed line from -1 to $+1$. Thus

$$\begin{aligned}M&=\Delta_C\,\mathrm{Arg}\prod_{j=0}^{N^*}[s-S^+_j(\mu)]\\ &=\sum_{j=0}^{N^*}\Delta_C\,\mathrm{Arg}[s-S^+_j(\mu)]=\sum_{j=0}^{P-1}N_{m_j}.\end{aligned} \tag{3.11}$$

If $s\in\mathbf{R}\subset I(\Gamma_{m_j})$ then $N_{m_j}=1$ since Γ_{m_j} does not cross the real axis for $0<s<\xi_{m_j}$. Of course if s does not lie inside of any

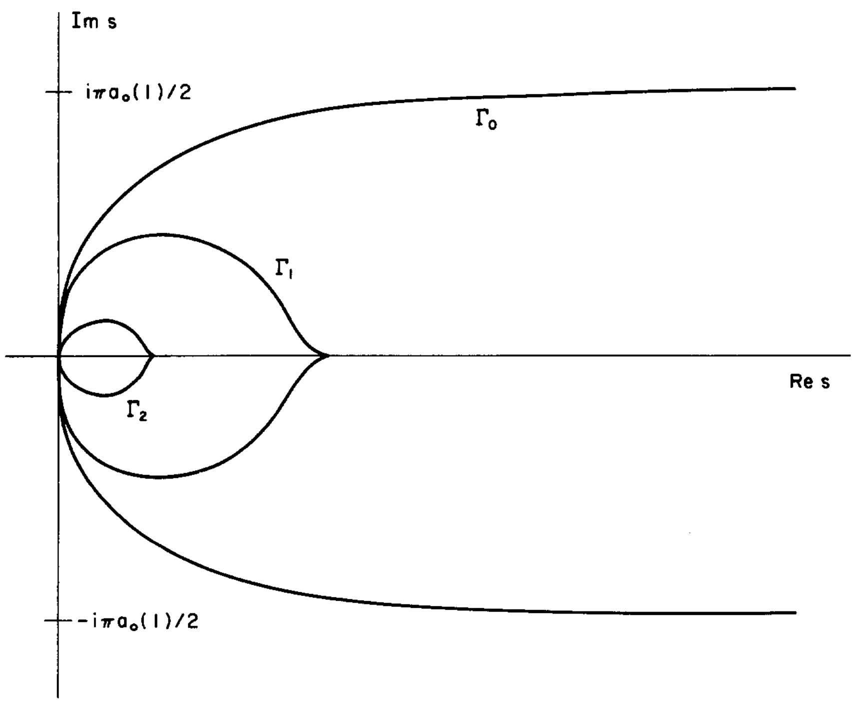

FIG. 2. Contours Γ_0, Γ_1, and Γ_2 for $f_1=0.2$ and $f_2=0.05$. Scale of Γ_0 reduced by factor of 6 and scale of Γ_2 enlarged by factor of 2.

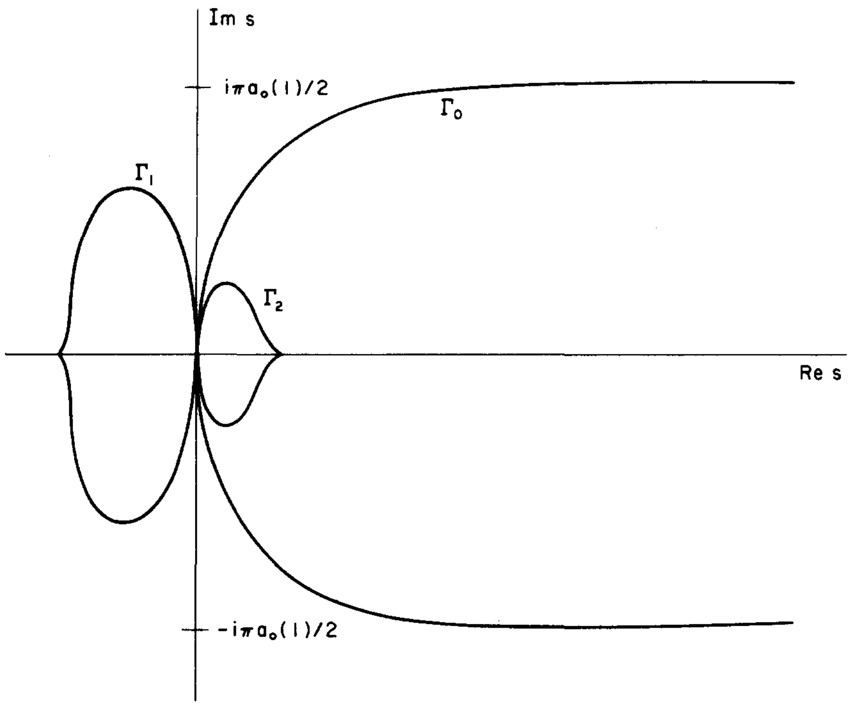

FIG. 3. Contours Γ_0, Γ_1, and Γ_2 for $f_1 = -0.1$ and $f_2 = 0.05$. Scale of Γ_0 reduced by factor of 5 and scale of Γ_2 enlarged by factor of 2.

of the contours then $M = 0$, i.e., $\Lambda(\nu,s)$ has no zeros. The number of zeros of $\Lambda(\nu,s)$ is intimately connected to the zeros of $\gamma(1,s)$, i.e., to the ξ_j, $j = 1,...,N^*$.

Corollary: If $s\in\mathbf{R}$ satisfies $0 < \xi_{m_{j+1}} < s < \xi_{m_j}$, where the ξ_{m_j} are ordered according to Eq. (3.4), then the number of pairs of zeros of $\Lambda(\nu,s)$ is $j + 1$. Further, if $c_q^+ = 1/\xi_{m_j}$, then the test of Sec. I follows directly.

For a numerical illustration of the mappings $s = S_j^+(\mu)$, $j = 0,1,...,N^*$, consider Figs. 2–5. These curves were generated for the case $N = 2$ by solving $\Lambda^+(\mu,s) = 0$ for s as μ varies from 0 to $+1$. For a numerical illustration of the roots of $\gamma(1,s)$, consider Table I. In this calculation

$$f(\mu,\mu') = \sum_{n=0}^{N} (2n+1) f_n^J P_n(\mu) P_n(\mu'), \qquad (3.12)$$

where the expansion coefficients are given by the recursion relation[9]

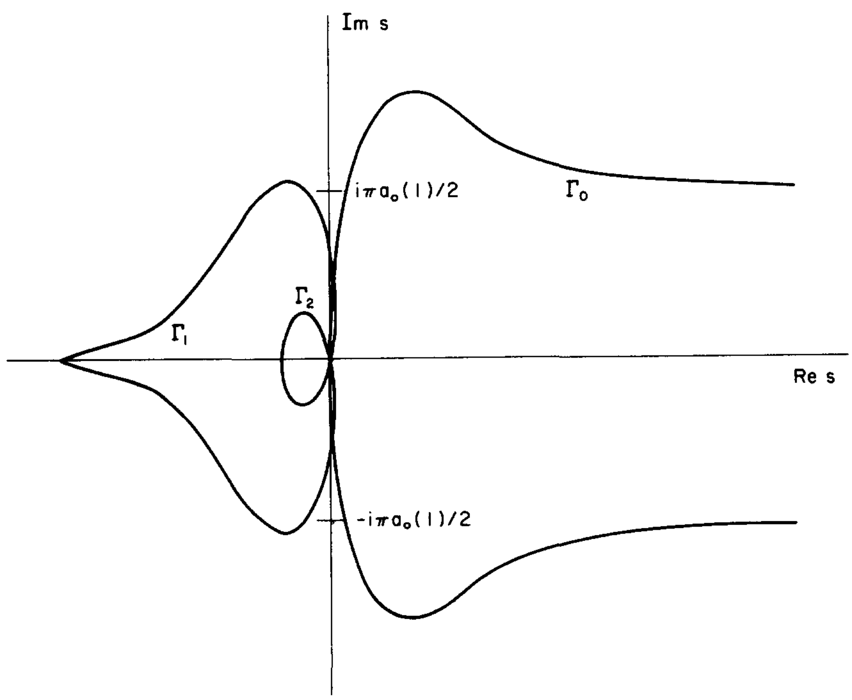

FIG. 4. Contours of Γ_0, Γ_1, and Γ_2 for $f_1 = -0.1$ and $f_2 = -0.05$. Scale of Γ_0 reduced by factor of 3 and scale of Γ_2 enlarged by factor of 4.

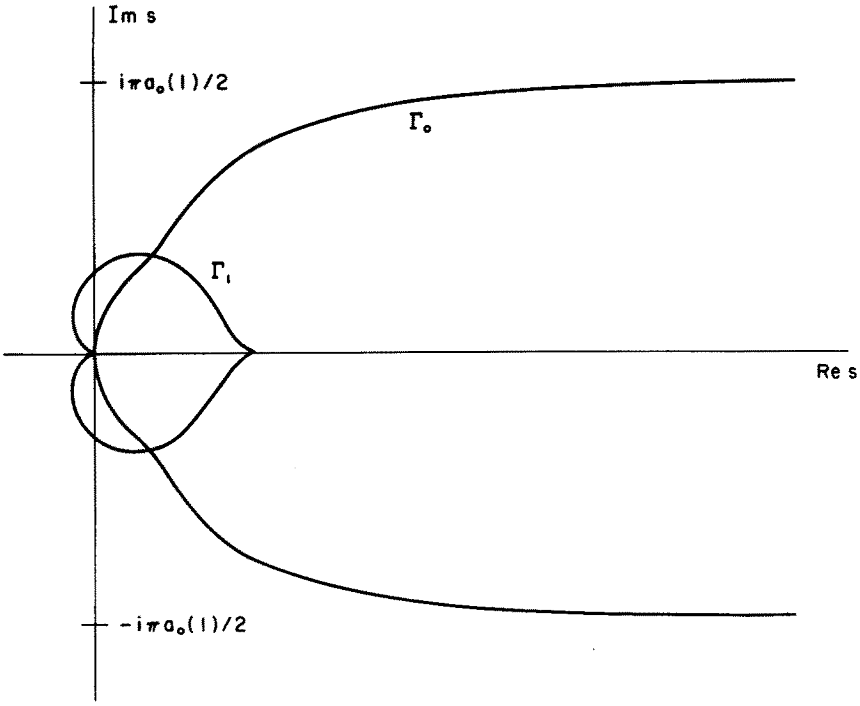

FIG. 5. Contours Γ_0 and Γ_1 for $f_1 = 0$ and $f_2 = 0.1$. Scale of Γ_0 reduced by factor of 3.

$$f_n^j = \frac{j+1}{2j(2n+1)}\left[\frac{n}{(2n-1)} f_{n-1}^{j-1} + f_n^{j-1} + \frac{n+1}{2n+3} f_{n+1}^{j-1}\right], \qquad (3.13)$$

with $f_0^j = 1$, $j = 0,1,\ldots,$ and $f_n^j = 0$ if $n > j$. The calculations in Table I were made with $J = 50$. Other numerical results agree with the azimuthally symmetric results reported by Shultis and Hill.[11]

IV. CONCLUDING REMARKS

It seems appropriate to conclude with a couple of remarks about the nature of the zeros of $\Lambda(\nu,s)$ and $\gamma(\nu,s)$. Several years ago Kuščer[12] pointed out that for the case $N = 2$ that the zeros of $\Lambda_c(\nu)$ could be complex. The advantage of the present analysis is that it points out that the zeros of $\Lambda(\nu,s)$ become complex (even for real s) whenever the discriminant $D(\nu)$ has zeros on the imaginary axis. Stated somewhat differently, the zeros of $\Lambda(\nu,s)$ are mapped via the $S_j(\nu)$ from the ν plane to the s plane and that map is conformal as long as the path in the ν plane avoids the cuts as described in Sec. II. In particular, the imaginary axis in the ν plane is conformally mapped to the real axis in the s plane as ν marches in from infinity. This conformal mapping is broken if a zero in the discriminant of $\Lambda(\nu,s)$ is encountered, resulting with complex zeros of $\Lambda(\nu,s)$. One can quickly show that this is just the situation for the special case considered by Kuščer.

TABLE I. Zeros of $\gamma(1,s)$. The last row is the reported number of pairs of zeros of $\Lambda_c(\nu)$ for $c = 0.95$ (see Ref. 10).

Order of scattering N				
4	6	8	10	15
3.0765	5.0607	6.8243	8.0824	9.2031
1.0388	1.6213	2.1600	2.5665	2.9677
0.6067	0.8576	1.1052	1.2990	1.5039
0.4645	0.5854	0.7131	0.8201	0.9402
	0.4683	0.5405	0.5986	0.6710
	0.3396	0.4422	0.4893	0.5332
		0.3265	0.3920	0.4457
		0.2057	0.2842	0.3491
			0.1860	0.2554
			0.1047	0.1762
				0.1147
				0.0701
				0.0399
				0.0206
				0.0091
2	3	4	4	4

Somewhat similar related remarks can be made about the zeros of $\gamma(\nu,s)$. It has been shown that the number of zeros of $\Lambda(\nu,s)$ are related to the zeros of $\gamma(1,s)$. If the number of pairs of zeros of $\gamma(\nu,s)$ (for fixed s) that lie in the interval $(-1,+1)$ is denoted by α, the discussion in Sec. I indicates that the number of pairs of zeros M of $\Lambda(\nu,s)$ must be bounded $M \leqslant \alpha + 1$. Further, numerical calculation with real s not too small (c not too large) suggest that M can be, in fact, just equal to $\alpha + 1$. To see the reason for this consider the fact that $\gamma(\nu,s) = 0$ generates an algebraic function, say $\nu(s)$, each branch of which conformally maps the appropriately cut s plane to the ν plane. Note that $\nu(\xi_j) = 1$, $j = 1,2,\ldots,N^*$. The number of zeros of $\gamma(\nu,s)$ must always be sufficient to satisfy the main theorem. Thus there is always a certain branch of $\nu(s)$ that maps the interval $(\xi_j,0)$ in the s plane to the real interval $(-1,+1)$ in the ν plane, and that mapping will be conformal (and thus one-to-one) if the discriminant of $\gamma(\nu,s)$ does not vanish on the interval $(\xi_j,0)$. Therefore, if the set of expansion coefficients $\{f_n\}$ is such that the discriminant of $\gamma(\nu,s)$ does not vanish on any of the intervals $(\xi_j,0)$, $j = 1,2,\ldots,N^*$, in the s plane, then indeed $M = \alpha + 1$. This is certainly the case for $N = 0$ and $N = 1$. However, one can show quite easily that for the case $N = 2$, $f_1 < 0$, and $f_2 < 0$ that the discriminant does vanish for s small enough. However, it is apparent that there always exist values of s greater than the largest zero of the discriminant of $\gamma(\nu,s)$ for which the number of pairs of zeros of $\Lambda(\nu,s)$ is always given by $M = \alpha + 1$.

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