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Resolvent integration techniques for generalized transport equations[a)]

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A generalized class of "transport type" equations is studied, including most of the known exactly solvable models; in particular, the transport operator K is a scalar type spectral operator. A spectral resolution for K is obtained by contour integration techniques applied to bounded functions of K. Explicit formulas are developed for the solutions of full and half range problems. The theory is applied to anisotropic neutron transport, yielding results which are proved to be equivalent to those of Mika.

I. INTRODUCTION

In 1973 Larsen and Habetler[1] introduced a technique for solving the one-speed neutron transport equation based on contour integration of the resolvent of the transport operator about its spectrum. (Although the transport operator, K^{-1} in their notation, is unbounded, its bounded inverse K can be treated by resolvent integration. This leads to an "eigenfunction expansion" in the sense of Titchmarsh, for K. To return to the neutron transport equation which involves K^{-1} rather than K, it is necessary to develop a functional calculus for K, after the manner von Neumann introduced into quantum mechanics[2]; this was accomplished in a later paper.[3])

The Larsen–Habetler method has been extended in the past two or three years to a number of more general forms of the neutron transport equation. Instead of listing these references here, the reader is referred to a recent comprehensive review article.[4] One special case should be noted, however, namely the so-called "critical" situation (which in one-speed theory corresponds to the situation $c = 1$). The orthodox resolvent integration technique cannot be applied in such a case, because K^{-1} is not invertible on its range. A modified and somewhat cumbersome technique can be used, however.[5] The idea is to restrict K^{-1} to a domain on which it is invertible, and proceed after the manner of Ref. 1, later extending results to the whole space.

Since the (linearized) equations describing electron oscillations in plasma and the kinetics of rarefied gases are similar in form to the neutron transport equation, it seems that resolvent integration techniques might be valuable in solving these equations also. However, they both pose difficult problems. For example, the unbounded operator describing gas kinetics is not invertible, and even if it is restricted to a domain in which it is invertible, its inverse is still unbounded. Thus it is not possible to integrate around the spectrum. The linearized Vlasov equation describing plasma oscillations is also unbounded, and although it is in general invertible, its inverse also is unbounded, so again straightforward resolvent integration techniques fail. (The relevant equations for these two physical problems are discussed in Ref. 6, Chap. 10.)

Very recently, a method suggested by Larsen[7] has been successfully applied to the Vlasov equation,[8] the gas kinetics equation (for a BGK model)[9,10] and also to conservative neutron transport.[11] For the first two cases this method was crucial to the solution; in the neutron transport case it merely simplified the previous somewhat cumbersome method described above. The basic idea was to transform the transport operator $K \rightarrow S = (K - \xi I)^{-1}$, where ξ is in the resolvent set of K. Then since S is a bounded operator with "thin" spectrum, the orthodox contour integration method can be applied to S to develop an eigenfunction expansion. Then a functional calculus is obtained along the lines of Ref. 3, so that the equation involving $K = S^{-1} + \xi I$ can be solved. (In more mathematical terms, a "constructive existence theorem" can be proved.)

In the present paper, we extend this technique, as developed in Refs. 8–11 to a general class of transport type equations of the form

$$\frac{\partial}{\partial x}\psi(x,\mu) = -(K\psi)(x,\mu) + q(x,\mu), \tag{1a}$$

where

$$(Kf)(\mu) = k(\mu)f(\mu) + \sum_{n=1}^{N} g_n(\mu)\int_A J_n(s)f(s)\,ds. \tag{1b}$$

Here $f \in \mathscr{L}^p(A,\sigma) = \mathscr{B}$, $A \subset \mathbb{R}$ is a directed Liaponov contour and $k(\mu)$ is a real valued, σ-measurable function on A. The functions k, g_n, and J_n are assumed to obey certain continuity and differentiability conditions which we enumerate later. In order to place Eq. (1) in perspective, we observe that the three equations discussed in Refs. 8–11 correspond to the following values of k, g_n, and J_n:

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1. One-speed conservative neutron transport (Ref. 11)

$k(\mu) = \frac{1}{\mu}$,

$g_n(\mu) = -\frac{1}{\mu}$,

$J_n(\mu) = \frac{1}{2}$,

$A = [-1,1]$,

$N = 1$.

2. BGK model for gas kinetics (Ref. 9),

$k(\mu) = \frac{1}{\mu}$,

$g_n(\mu) = -\frac{1}{\mu}$,

$J_n(\mu) = \frac{1}{\sqrt{\pi}} e^{-\mu^2}$,

$A = \mathbb{R}$,

$N = 1$.

3. Linearized Vlasov equation (Ref. 8),

$k(\mu) = \mu$,

$g_n(\mu) = \eta(\mu)$,

$J_n(\mu) = 1$,

$A = \mathbb{R}$,

$N = 1$.

[$\eta(\mu)$ is proportional to the derivative of the equilibrium electron distribution.]

4. One-speed neutron transport, anisotropic scattering (Ref. 6, p. 87),

$k(\mu) = \frac{1}{\mu}$,

$g_n(\mu) = \frac{-(2n-1)}{2}\frac{c}{\mu} f_{n-1} P_{n-1}(\mu)$,

$J_n(\mu) = P_{n-1}(\mu)$,

$A = [-1,1]$,

$N = N_0$.

(The f_n are the Legendre moments of the scattering kernel and P_n are Legendre polynomials.)

Other equations of transport type can be expected to occur in various areas, gas dynamics, radiative, and electron transport, etc., which basically involve the linearized Boltzmann equation. The solution to such equations can then be read off from our results. Although the smoothness restrictions which we place on coefficients in our transport-type equation are merely sufficient conditions, we believe that they are sufficiently general to encompass virtually all cases which may arise from physical application.

The plane of our paper is as follows. In Sec. II we compute $S = (K - \xi)^{-1}$ and the resolvent of S, $(zI - S)^{-1}$. We also obtain the spectrum of S. Then in Sec. III we perform the integration about the continuous spectrum of S, and in Sec. IV integrate about the point spectrum. These two results together give a "full-range" eigenfunction expansion for S. A similar "half-range" expansion is obtained in Sec. V. Then, in order to translate these into eigenfunction expansions for K, which are needed to solve Eq. 11, we first need to extend the results to Banach space. The analysis of Secs. III–V has been restricted to a dense subspace of Hölder continuous functions (since it was necessary to evaluate boundary values of Cauchy integrals). Once the extension is carried out in Sec. VI, a functional calculus for S can be developed; this is done in VII and so, as was explained earlier, an eigenfunction expansion for $K = S^{-1} + \xi I$ is thereby obtained. Sec. VIII presents some applications to boundary value problems.

II. RESOLVENTS

We wish to consider the integrodifferential Eq. (1). The operator K [Eq. (1b)] may be written in the obvious notation

$$Kf = kf + \mathbf{g}\cdot\int_A \mathbf{J}(s) f(s)\, ds, \qquad (2)$$

and where there is no confusion, we shall abbreviate $\int_A \mathbf{J}(f) f(s)\, ds = \mathbf{J}(s)$. K is not assumed to be bounded; in any case we shall for the most part restrict its domain to $D(K) = \{f \mid Kf \in \mathscr{B},\ f \text{ Hölder continuous on compacts}\}$. Finally, we shall let $(\mathbf{J}\otimes\mathbf{g})_{mn}(\mu) = J_m(\mu) g_n(\mu)$, and shall write $\mathscr{B}$ for the product of N copies of the Banach space $\mathscr{B}$. By a solution of Eq. (1), we demand a continuously differential function $\psi:\mathbb{R}\to\mathscr{B}$ satisfying specified boundary conditions (to be discussed later), where the inhomogeneous source term $q:\mathbb{R}\to\mathscr{B}$ is assumed to satisfy a uniform Hölder condition (on every compact subset of A).

Lemma 1: If there exists $\xi\in\mathbb{C}/\mathbb{R}$ such that the following are satisfied:

(i) $T_\xi = \int_A \frac{\mathbf{J}\otimes\mathbf{g}(s)}{k(s)-\xi}\, ds + I$ invertible on $\mathbb{C}^N$,

(ii) $\frac{1}{k-\xi}\,\mathbf{g}\in\mathscr{B}$,

(iii) $\mathscr{P}_\xi : f \to \mathbf{J}\left(\frac{1}{k-s} f\right) \in \mathscr{B}^*$,

then $S_\xi = (K - \xi I)^{-1}$ exists as a bounded operator on $\mathscr{B}$.

Proof: Letting $(K - \xi I) f = (k - \xi) f + \mathbf{g}\cdot\mathbf{J}(f) = h$, we obtain

$$f = \frac{1}{k-\xi} h - \frac{\mathbf{g}\cdot\mathbf{J}(f)}{k-\xi}$$

for f in the (dense) domain of K. This is valid in $\mathscr{B}$ by (ii) and the fact that $[1/(k-\xi)] : f \to [1/(k-\xi)] f \in L(\mathscr{B})$, the bounded operators on $\mathscr{B}$, since $\operatorname{ess\,sup}|1/(k-\xi)| \leqslant |\operatorname{Im}\xi|^{-1}$. Then, computing $\mathbf{J}(f)$ and utilizing (iii), after some rearranging we have

$$\sum_{n=1}^{N} J_n(f)\int_A \frac{J_m(s) g_n(s)\, ds}{k(s)-\xi} + \delta_{mn} = J_m\left(\frac{1}{k-\xi} h\right),$$

which may be written

$$\int_A \frac{\mathbf{J}\otimes\mathbf{g}(s)\, ds}{k(s)-\xi} + I\,\mathbf{J}(f) = \mathbf{J}\left(\frac{1}{k-\xi} h\right).$$

Inverting

$$\Lambda(\xi)=\int_A \frac{\mathbf{J}\otimes \mathbf{g}(s)\,ds}{k(s)-\xi}+I,$$

by virtue of (i), we obtain

$$(K-\xi I)^{-1}f=\left[\frac{1}{k-\xi}\right]f-\frac{1}{k-\xi}\,\mathbf{g}\cdot\Lambda^{-1}(\xi)\mathbf{J}\left(\frac{1}{k-\xi}f\right). \tag{3}$$

We have thus proved as well,

Lemma 2: With ξ defined as in the previous lemma, and

$$\Lambda(z)=\int_A \frac{\mathbf{J}(s)\otimes \mathbf{g}(s)\,ds}{k(s)-z}+I, \tag{4}$$

then

$$S_\xi f=(K-\xi I)^{-1}f=\left[\frac{1}{k-\xi}\right]f-\frac{1}{k-\xi}\,\mathbf{g}\cdot\Lambda^{-1}(\xi)\times\mathbf{J}\left(\frac{1}{k-\xi}f\right). \tag{5}$$

Lemma 3: The resolvent of S_ξ is given by

$$(S_\xi-zI)^{-1}f=\left[\frac{k-\xi}{1-z(k-\xi)}\right]f+\frac{1}{1-z(k-\xi)}\times\mathbf{g}\cdot\Lambda^{-1}\left(\xi+\frac{1}{z}\right)\mathbf{J}\left(\frac{1}{1-z(k-\xi)}f\right). \tag{6}$$

Proof: We compute

$$(S_\xi-zI)f=h,$$

as in Lemma 1, obtaining

$$f=\frac{k-\xi}{1-z(k-\xi)}h+\frac{1}{1-z(k-\xi)}\,\mathbf{g}\cdot\Lambda^{-1}(\xi)\times\mathbf{J}\left(\frac{1}{k-\xi}f\right);$$

thus,

$$\begin{aligned}\mathbf{J}\left(\frac{1}{k-\xi}f\right)&=\mathbf{J}\left(\frac{1}{1-z(k-\xi)}h\right)\\&+\sum_{i=1}^{N}\mathbf{J}\left(\frac{1}{(k-\xi)(1-z(k-\xi))}g_i\right)(\Lambda^{-1}(\xi)\mathbf{J})_i\left(\frac{1}{k-\xi}f\right)\\&=\left(I-\int_A\frac{\mathbf{J}\otimes\mathbf{g}(s)\,ds}{(k(s)-\xi)(1-z(k(s)-\xi))}\Lambda^{-1}(\xi)\right)^{-1}\times\mathbf{J}\left(\frac{1}{1-z(k-\xi)}h\right).\end{aligned}$$

We may then rewrite the expression for f as

$$f=\frac{k-\xi}{1-z(k-\xi)}h+\frac{1}{1-z(k-\xi)}\times\mathbf{g}\cdot\left(\Lambda(\xi)-\int_A\frac{\mathbf{J}\otimes\mathbf{g}(s)\,ds}{(k(s)-\xi)(1-z(k(s)-\xi))}\right)^{-1}\times\mathbf{J}\left(\frac{h}{k-z(k-\xi)}\right).$$

Let

$$\Lambda_\xi(z)=\Lambda(\xi)-\int_A\frac{\mathbf{J}\otimes\mathbf{g}(s)\,ds}{(k(s)-\xi)(1-z(k(s)-\xi))}.$$

Then we may write

$$\Lambda_\xi(z)=I+\int_A\mathbf{J}\otimes\mathbf{g}(s)\left(\frac{1}{k(s)-\xi}-\frac{1}{(k(s)-\xi)(1-z(k(s)-\xi))}\right)ds,$$

and utilizing

$$\frac{1}{k-\xi}-\frac{1}{(k-\xi)(1-z(k-\xi))}=\frac{1}{k-\xi-(1/z)},$$

obtain

$$\Lambda_\xi(z)=\Lambda(\xi+1/z),$$

which completes the proof.

Lemma 4: Let $\Omega_\xi(z)=\det\Lambda(\xi+1/z)$ and N_p be the set of zeroes of Ω_ξ. Let $Q=\{z=1/(\omega-\xi)\in\mathbb{C}\,|\,\omega\in\mathrm{Ran}\,k\}$. Then

$$\sigma_p(S_\xi)=N_p,\quad \sigma_c(S_\xi)=Q.$$

Proof: This is an immediate consequence of Weyl's theorem and Eqs. (5) and (6).

III. CONTINUOUS SPECTRUM

Definition: We shall call the triple $\{k,g,J\}$ of transport type if k is one–one and differentiable and if for $\beta=1$ or else for $\beta=-1$, $s\to k(s)^{-\beta}$ is continuous and each of the functions $t\to k^{-1}(t^\beta)$, $s\to J(s)/k(s)^{-1/2(1-\beta)}k'(s)$, and $s\to J(s)\otimes g(s)/k(s)^{-1/2(1-\beta)}k'(s)$ is Hölder continuous on compact subsets of A.

Here and throughout we write $k(s)^{-1}$ for $1/k(s)$ and $k^{-1}(s)$ for the inverse function evaluated at s, e.g., $k^{-1}(s)=s$ if $k(s)=s$.

Lemma 5: Assume $\{k,g,J\}$ is of transport type, and for $f\in\mathscr{B}$ Hölder continuous with compact support, define

$$M_f(z)=\Lambda^{-1}(\xi+1/z)\mathbf{J}\left(\frac{1}{1-z(k-\xi)}f\right). \tag{7a}$$

Then the boundary values $M^{\pm}$ and $\Lambda^{\pm}$ are given by

$$M_f^{\pm}\left(\frac{1}{k(\omega)-\xi}\right)=\Lambda_\xi^{\mp1}(k(\omega))\pm\int_A\frac{\mathbf{J}(s)f(s)\,ds}{k(\omega)-k(s)}(k(\omega)-\xi)\pm\frac{i\pi\mathbf{J}(\omega)f(\omega)(k(\omega)-\xi)}{k'(\omega)}, \tag{7b}$$

$$\Lambda_\xi^\pm(k(\omega)) = I + \fint_A \frac{\mathbf{J}\otimes\mathbf{g}(s)\,ds}{k(s)-k(\omega)} \pm \frac{i\pi\mathbf{J}\otimes\mathbf{g}(\omega)}{k'(\omega)}$$

$$= \lambda(k(\omega)) \pm \frac{i\pi\mathbf{J}\otimes\mathbf{g}(\omega)}{k'(\omega)}, \qquad (7c)$$

where $+$ and $-$ refer to nontangential approach of z to the contour from the right and left, respectively.

Proof: With the substitution $t = k(s)^\beta$, the integral to be evaluated for $\Lambda^\pm M^\pm$ is

$$-\int \frac{\mathbf{J}(k^{-1}(t^\beta))f(k^{-1}(t^\beta))}{t^{1/2(1-\beta)}(t-[(1+z\xi)/z]^\beta)k'(k^{-1}(t))}\,dt.$$

The continuity of $s\rightarrow k(s)^{-\beta}$ assures that the integration may be restricted to a compact set. Then the Plemelj formulas may be applied by virtue of the required Hölder continuity, and (7b) follows. The computation of $\Lambda^\pm$ is similar.

Using Lemma 5, we may compute

$$\Lambda_\xi^+(k(\omega))\mathbf{M}^+\left(\frac{1}{k(\omega)-\xi}\right) - \Lambda_\xi^-(k(\omega))\mathbf{M}^-\left(\frac{1}{k(\omega)-\xi}\right)$$

$$= \frac{-2\pi i\mathbf{J}(\omega)f(\omega)(k(\omega)-\xi)}{k'(\omega)} \qquad (8a)$$

$$= \lambda(k(\omega))\left\{\mathbf{M}^+\left(\frac{1}{k(\omega)-\xi}\right) - \mathbf{M}^-\left(\frac{1}{k(\omega)-\xi}\right)\right\} + \pi i\frac{\mathbf{J}\otimes\mathbf{g}(\omega)}{k'(\omega)}\left[\mathbf{M}^+\left(\frac{1}{k(\omega)-\xi}\right) + \mathbf{M}^-\left(\frac{1}{k(\omega)-\xi}\right)\right] \qquad (8b)$$

Taking the inner product of Eqs. (8) with $J(\omega)$, denoting

$$\mathbf{J}(\omega)\cdot\mathbf{J}(\omega) = J^2(\omega),$$

and using the identity $\mathbf{J}\cdot\mathbf{J}\otimes\mathbf{g}\mathbf{M} = J^2\mathbf{g}\cdot\mathbf{M}$, we find

$$f(\omega) + \frac{1}{k(\omega)-\xi}\mathbf{g}(\omega)\cdot\tfrac{1}{2}\left[\mathbf{M}^+\left(\frac{1}{k(\omega)-\xi}\right) + \mathbf{M}^-\left(\frac{1}{k(\omega)-\xi}\right)\right]$$

$$= -\frac{1}{2\pi i}\frac{k'(\omega)}{J^2(\omega)}\frac{1}{k(\omega)-\xi}\mathbf{J}(\omega)\cdot\lambda(k(\omega))\left[\mathbf{M}^+\left(\frac{1}{k(\omega)-\xi}\right) - \mathbf{M}^-\left(\frac{1}{k(\omega)-\xi}\right)\right]. \qquad (9)$$

Further, we may compute the difference in boundary values of M from Lemma 5.

Utilizing

$$\Lambda^+(z)^{-1} \pm \Lambda^-(z)^{-1} = \Lambda^-(z)^{-1}\{\Lambda^-(z) \pm \Lambda^+(z)\}\Lambda^+(z)^{-1},$$

we obtain

$$M^+\left(\frac{1}{k(\omega)-\xi}\right) - M^-\left(\frac{1}{k(\omega)-\xi}\right)$$

$$= -\Lambda_\xi^-(k(\omega))^{-1}\mathbf{J}\otimes\mathbf{g}(\omega)\Lambda_\xi^+(k(\omega))^{-1}2\pi i\frac{(k(\omega)-\xi)}{k'(\omega)}\fint_A\frac{\mathbf{J}(s)f(s)\,ds}{k(\omega)-k(s)} - \Lambda_\xi^-(k(\omega))^{-1}\lambda(k(\omega))\Lambda_\xi^+(k(\omega))^{-1}$$

$$\times 2\pi i\frac{(k(\omega)-\xi)}{k'(\omega)}\mathbf{J}(\omega)f(\omega). \qquad (10)$$

We will now integrate the resolvent of S_ξ on a contour Γ about its spectrum,

$$f(\omega) = \frac{1}{2\pi i}\oint_{\Gamma(Q)}(zI-S_\xi)^{-1}f(\omega)\,dz + \frac{1}{2\pi i}\oint_{\Gamma(N_p)}(zI-S_\xi)^{-1}f(\omega)\,dz.$$

Denoting the two integrals by $f_1(\omega), f_2(\omega)$, respectively, we have

$$f_1(\omega) = f(\omega) + \frac{1}{2\pi i}\mathbf{g}(\omega)\cdot\oint_{\Gamma(Q)}\frac{1}{z(k(\omega)-\xi)-1}\mathbf{M}(z)\,dz$$

$$= f(\omega) + \frac{1}{2\pi i}\mathbf{g}(\omega)\cdot\fint_R\frac{1}{(x-\xi)(x-k(\omega))}\left[\mathbf{M}^+\left(\frac{1}{x-\xi}\right) - \mathbf{M}^-\left(\frac{1}{x-\xi}\right)\right]dx + \frac{1}{2(k(\omega)-\xi)}$$

$$\times\mathbf{g}(\omega)\cdot\left[\mathbf{M}^+\left(\frac{1}{(k(\omega)-\xi)}\right) + \mathbf{M}^-\left(\frac{1}{(k(\omega)-\xi)}\right)\right],$$

where $R = \mathrm{Ran}k$ and we have utilized the analyticity and Hölder continuity of $\mathbf{M}(z)$. Using Eq. (9), this becomes

$$f_1(\omega) = \frac{1}{2\pi i}\,\mathbf{g}(\omega)\cdot\int_R \frac{1}{(x-\xi)(x-k(\omega))}\left[\mathbf{M}^+\!\left(\frac{1}{x-\xi}\right) - \mathbf{M}^-\!\left(\frac{1}{x-\xi}\right)\right]dx - \frac{1}{2\pi i}\,\frac{k'(\omega)}{J^2(\omega)}\,\frac{1}{(k(\omega)-\xi)}$$

$$\times\,\mathbf{J}(\omega)\cdot\lambda\,(k(\omega))\left[\mathbf{M}^+\!\left(\frac{1}{(k(\omega)-\xi)}\right) - \mathbf{M}^-\!\left(\frac{1}{(k(\omega)-\xi)}\right)\right].$$

Let us define

$$\mathbf{\Phi}_x(\omega) = P\,\frac{\mathbf{g}(\omega)}{x-k(\omega)} - \frac{k'(\omega)}{J^2(\omega)}\,\delta(x-k(\omega))\cdot\lambda^t(x)\mathbf{J}(\omega), \quad x\in R,\ \omega\in A \tag{11a}$$

$$\mathbf{A}(x) = -\Lambda^-(x)^{-1}\,\frac{\mathbf{J}\otimes\mathbf{g}(k^{-1}(x))}{k'(k^{-1}(x))}\,\Lambda^+(x)^{-1}\int_A \frac{\mathbf{J}(x)f(s)\,ds}{x-k(s)} - \Lambda^-(x)^{-1}\lambda\,(x)\cdot\Lambda^+(x)^{-1}\mathbf{J}(k^{-1}(x))f(k^{-1}(x))/k'(k^{-1}(x)), \tag{11b}$$

where t indicates the transpose. We have proved

Theorem 6: Suppose K satisfies the hypothesis of Lemma 5, and $f\in\mathscr{B}$ is Hölder continuous with compact support. Then

$$f_1(\omega) = \int_R \mathbf{\Phi}_x(\omega)\cdot\mathbf{A}(x)\,dx, \tag{11c}$$

where Φ_x and $\mathbf{A}$ are defined by Eqs. (11a) and (11b).

IV. POINT SPECTRUM

The contributions to $f(\omega)$ of zeroes of

$$\Omega\,(z) = \det\Lambda\left(\frac{1}{z}+\xi\right)$$

is a routine exercise in residue theory. We shall write

$$N_0 = \{z\in\mathbb{C} \mid \Omega\,(z)=0,\ z\notin Q\}, \qquad N_Q = \{z\in\mathbb{C} \mid \Omega(z)=0, z\in Q\}.$$

We will for simplicity assume that the zeroes in N_0 have multiplicity one, although for later applications, zeroes in N_Q of multiplicity one and two will be considered. Multiplicity of any order can be computed simply by using the residue formula for higher order poles. Finally, we assume

$$\frac{1}{k-\alpha_z}\,g\in\mathscr{B}, \quad \text{for } \alpha_z = k\left(\frac{1}{z}+\xi\right),\ z\in N_Q. \tag{12}$$

Theorem 7: If K satisfies the hypothesis of Lemma 5 and Eq. (12), and $f\in\mathscr{B}$, then $f_2(\omega)$ is a sum of contributions,

$$f_2(\omega) = \sum_{z\in N_0\cup N_Q} f_2^z(\omega), \tag{13}$$

where

(i) if $z\in N_0$ (multiplicity one),

$$f_2^z(\omega) = -\frac{1}{z(k(\omega)-\xi)-1}\,\mathbf{g}(\omega)\cdot\left(\frac{d\Omega}{dz}(z)\right)^{-1}\Lambda_c\left(\frac{1}{z}+\xi\right)\times\int_A \frac{\mathbf{J}(s)f(s)\,ds}{z(k(s)-\xi)-1}, \tag{14a}$$

(ii) if $z\in N_Q$ (multiplicity one),

$$f_2^z(\omega) = \tfrac{1}{2}\,\frac{1}{z(k(\omega)-\xi)-1}\,\mathbf{g}(\omega)\cdot\left\{\left[\left(\frac{d\Omega^+}{dz}(z)\right)^{-1}\Lambda_c^+\left(\frac{1}{z}+\xi\right) + \left(\frac{d\Omega^-}{dz}(z)\right)^{-1}\Lambda_c^-\left(\frac{1}{z}+\xi\right)\right]\int \frac{\mathbf{J}(s)f(s)\,ds}{z(k(s)-\xi)-1}\right.$$

$$\left. + \pi i\left[\left(\frac{d\Omega^+}{dz}\right)(z)^{-1}\Lambda_c^+\left(\frac{1}{z}+\xi\right) - \left(\frac{d\Omega^-}{dz}(z)\right)^{-1}\Lambda_c^-\left(\frac{1}{z}+\xi\right)\right]\mathbf{J}(\omega)f(\omega)\,\frac{(k(\omega)-\xi)}{k'(\omega)}\right\}, \tag{14b}$$

(iii) if $z\in N_Q$ (multiplicity two)

$$f_2^z(\omega) = \frac{1}{\Omega''(z)^+}\,\frac{d}{dz}\left[\frac{1}{z(k(\omega)-\xi)-1}\,\mathbf{g}(\omega)\cdot\Lambda_c^+\left(\frac{1}{z}+\xi\right)\left(\int_A \frac{\mathbf{J}(s)f(s)\,ds}{z(k(s)-\xi)-1} + \frac{\pi i\mathbf{J}(\omega)f(\omega)(k(\omega)-\xi)}{k'(\omega)}\right)\right]$$

$$+ \frac{1}{\Omega''(z)}\,\frac{d}{dz}\left[\frac{1}{z(k(\omega)-\xi)-1}\,\mathbf{g}(\omega)\cdot\Lambda_c^-\left(\frac{1}{z}+\xi\right)\left(\int_A \frac{\mathbf{J}(s)f(s)\,ds}{z(k(s)-\xi)-1} - \pi i\,\frac{\mathbf{J}(\omega)f(\omega)(k(\omega)-\xi)}{k'(\omega)}\right)\right]$$

$$-\frac{2}{3}\frac{\Omega'''(z)}{(\Omega''(z))^2}\frac{1}{z(k(\omega)-\xi)-1}\mathbf{g}(\omega)\cdot\left\{\left[\Lambda_c^+\left(\frac{1}{z}+\xi\right)+\Lambda_c^-\left(\frac{1}{z}+\xi\right)\right]\int_A \frac{\mathbf{J}(s)f(s)\,ds}{z(k(s)-\xi)-1}\right.$$

$$\left.+\pi i\left[\Lambda_c^+\left(\frac{1}{z}+\xi\right)-\Lambda_c^-\left(\frac{1}{z}+\xi\right)\right]\frac{\mathbf{J}(\omega)f(\omega)(k(\omega)-\xi)}{k'(\omega)}\right\}. \tag{14c}$$

In these formulas, Λ_c indicates the cofactor matrix, $\Lambda^{-1}=\Omega\Lambda_c$.

V. CONSTRUCTION OF SOLUTIONS

In this section, we wish to establish a norm on $\mathscr{B}$ which will enable Eqs. (11) to be extended to the full Banach space. Because of certain technical difficulties in treating the problem when there are eigenvalues imbedded in the continuous spectrum, we will consider two cases.

Case (a): $\det\Omega(z)\neq 0$ for $z\subset R\subset\mathbb{R}$.

Let us define F: $\mathscr{B}\to\mathscr{B}'$ by

$$F(f)(x)=\mathbf{A}(x),$$

where $\mathbf{A}(x)$ is given by Eq. (11b). We wish to choose spaces $\mathscr{B}$ and $\mathscr{B}'$ such that F will be an invertible bounded transformation. We shall consider separately the terms

$$F_1(f)(x)=\Lambda^-(x)^{-1}\frac{\mathbf{J}\otimes\mathbf{g}(k^{-1}(x))}{k'(k^{-1}(x))}\Lambda^+(x)^{-1}\int_A\frac{J(s)f(s)\,ds}{x-k(s)} \tag{15a}$$

and

$$F_2(f)(x)=\Lambda^-(x)^{-1}\lambda(x)\Lambda^+(x)^{-1}\mathbf{J}(k^{-1}(x))f(k^{-1}(1)). \tag{15b}$$

The L_P estimates of the contributions to $\mathbf{A}(x)$ due to the zeros of Ω are trivial [by virtue of assumption (a) above] and are omitted.

Suppose f is Hölder continuous with compact support. Then, as a function of $\hat{x}=x^\beta$, $\hat{x}^{-\frac{1}{2}(1-\beta)}F_1(f)(\hat{x}^\beta)$ is Hölder continuous, and may be estimated in $L_p(\hat{x})$ norm $\|\;\|_{L_p(\hat{x})}$ by

$$\|\hat{x}^{-(1-\beta)/2}F_1(f)(\hat{x}^\beta)\|_{L_p(\hat{x})}$$
$$\leqslant\left|\left|\frac{\|\Lambda^-(\hat{x}^\beta)^{-1}\mathbf{J}\otimes\mathbf{g}(k^{-1}(\hat{x}^\beta))\Lambda^+(x^\beta)^{-1}\|}{k'(k^{-1}(\hat{x}^\beta))}\right|\right|_\infty$$
$$\cdot C_p\left|\left|\frac{\mathbf{J}(k^{-1}(t^\beta))f(k^{-1}(t^\beta))}{k'(k^{-1}(t^\beta))t^{(1-\beta)/2}}\right|\right|_{L_p^N(t^\beta)}$$

where C_p depends upon p only, and

$$\|\Lambda(\hat{x})\|_{L_p^N(\hat{x})}=\left\{\sum_{i=1}^N\int|\mathbf{A}_i(\hat{x})|^p d\hat{x}\right\}^{1/p}, \tag{16a}$$

$$\|\Lambda(\hat{x})\|_\infty=\sup_{\hat{x}\subset\mathbf{R}}\sup_{|ij|}|\Lambda_{ij}(\hat{x})|. \tag{16b}$$

Similarly, $\hat{x}^{-(1-\beta)/2}F_2(f)(\hat{x}^\beta)$ may be estimated by

$$\|\hat{x}^{-(1-\beta)/2}F_2(f)(\hat{x}^\beta)\|_{L_p^N(\hat{x})}$$
$$\leqslant\|\Lambda^-(\hat{x}^\beta)^{-1}\lambda(\hat{x}^\beta)\Lambda^+(\hat{x}^\beta)^{-1}\hat{x}^{(1-\beta)}k'(k^{-1}(\hat{x}^\beta))\|_\infty$$
$$\cdot\left|\left|\frac{\mathbf{J}(k^{-1}(\hat{x}^\beta))f(k^{-1}(\hat{x}^\beta))}{k'(k^{-1}(\hat{x}^\beta))\hat{x}^{(1-\beta)/2}}\right|\right|_{L_p^N(\hat{x})}.$$

Let $\mathscr{B}$ be the Banach space of real-valued measurable functions f on A such that

$$\|f\|_{\mathscr{B}}=\left|\left|\frac{\mathbf{J}(k^{-1}(\hat{x}^\beta))f(k^{-1}(\hat{x}^\beta))}{k'(k^{-1}(\hat{x}^\beta))\hat{x}^{(1-\beta)/2}}\right|\right|_{L_p^N(\hat{x})}<\infty,$$

and $\mathscr{B}'$ the Banach space of real-valued measurable functions A on R such that

$$\|A\|_{\mathscr{B}'}\equiv\|\hat{x}^{-(1-\beta)/2}\mathbf{A}(\hat{x}^\beta)\|_{L_p^N(\hat{x})}<\infty.$$

Call the triple $\{k,g,J\}$ smooth if

$$\Lambda^-(s^\beta)^{-1}\frac{\mathbf{J}\otimes\mathbf{g}(k^{-1}(s_\beta))}{k'(k^{-1}(s^\beta))},$$

$\Lambda^+(s^\beta)^{-1}$, and the functions $\Lambda^\pm(s^\beta)^{-1}\{s^{1-\beta}k'(k^{-1}(s^\beta))\}^\alpha$, $\alpha=\pm 1$, are bounded as $s\to\infty$.

Lemma 8: If $\{k,g,J\}$ is smooth of transport type, then, F:$\mathscr{B}\to\mathscr{B}'$ extends to a bounded linear transformation.

Likewise, we define

$$F_1'(A)(\omega)=\mathbf{g}(\omega)\cdot\int\frac{\mathbf{A}(x)\,dx}{x-k(\omega)} \tag{17a}$$

and

$$F_2'(A)(\omega)=-\frac{1}{J^2(\omega)}\Lambda'(k(\omega))\mathbf{J}(\omega)\cdot\mathbf{A}(k(\omega)). \tag{17b}$$

Let $k(\omega)=t^\beta$ and $x=\hat{x}^\beta$. Then

$$\frac{F_1'(A)(k^{-1}(t^\beta))\mathbf{J}(k^{-1}(t^\beta))}{k'(k^{-1}(t^\beta))t^{(1-\beta)/2}}$$
$$=\frac{\mathbf{J}(k^{-1}(t^\beta))\mathbf{g}(k^{-1}(t^\beta))}{k'(k^{-1}(t^\beta))}\cdot\int\frac{\mathbf{A}(\hat{x}^\beta)d\hat{x}}{(\hat{x}-t)\hat{x}^{(1-\beta)/2}}$$

and

$$\|F_1'(A)\|_{\mathscr{B}}\leqslant\left|\left|\frac{\|\mathbf{J}(k^{-1}(t^\beta))\otimes\mathbf{g}(k^{-1}(t^\beta))\|}{k'(k^{-1}(t^\beta))}\right|\right|_\infty\|A\|_{\mathscr{B}'}.$$

The second term may be estimated by

$$\|F_2'(A)\|_{\mathscr{B}}\leqslant\left|\left|\frac{\|\mathbf{J}(k^{-1}(t^\beta))\cdot\lambda'(t^\beta)\mathbf{J}(k^{-1}(t^\beta))\|}{J^2(k^{-1}(t^\beta))t^{(1-\beta)}k'(k^{-1}(t^\beta))}\right|\right|_\infty\|A\|_{\mathscr{B}'}.$$

Lemma 9: If $\{k,g,J\}$ is smooth of transport type, then F':$\mathscr{B}'\to\mathscr{B}$ extends to a bounded linear transformation and $F'=F^{-1}$.

We may now obtain a resolution of the identity corresponding to K. For $\lambda\in R$ let us define

$$E(\lambda)f(\omega)=\int_{-\infty}^{\lambda}\mathbf{\Phi}_x(\omega)\cdot\mathbf{A}(x)\,dx, \tag{18}$$

where $\mathbf{A}$ and $\mathbf{\Phi}$ are given by Eqs. (11). To obtain the discrete eigenprojections, we note that $\Lambda_c(1/z+\xi)$ may be written

$$A_c\left(\frac{1}{z}+\xi\right)=\gamma_z\otimes\alpha_z, \tag{19}$$

for $z\in N_p$, where

$$A\left(\frac{1}{z}+\xi\right)\gamma_z=0.$$

Then $f_2^z(\omega)$ may be expanded as

$$f_2^z(\omega)=\Phi_z(\omega)A_z, \tag{20a}$$

with

$$\Phi_z(\omega)=\frac{\mathbf{g}(\omega)\cdot\gamma_z}{z(k(\omega)-\xi)-1}, \tag{20b}$$

an eigenvector of K, and

$$A_z=-\frac{1}{\Omega'(z)}\alpha_z\cdot\int\frac{\mathbf{J}(s)f(s)\,ds}{z(k(s)-\xi)-1}. \tag{20c}$$

Defining $E(\lambda)$ for $1/(\lambda-\xi)=z\in N_p$ by

$$E(\lambda)f(\omega)=\Phi_z(\omega)A_z, \tag{21}$$

we may follow Ref. 3 to prove that the family of projections $E(\lambda)$ is a resolution of the identity, and

$$K=\int_R\lambda\,dE(\lambda)+\sum_{(1/(\lambda-\xi)\in N_p}\lambda E(\lambda). \tag{22}$$

We state this as

Theorem 10: The family $E(\lambda)$ is a resolution of the identity for K. The solution of Eq. (1) satisfying the boundedness condition $\lim_{x\to\infty}\|\psi(x)\|_{\mathscr{B}}=0$ is given by

$$\begin{aligned}\psi(x,\nu)=&\int_{-\infty}^{x}d\xi\int_0^{\infty}e^{-(x-\xi)\lambda}d(E(\lambda)q)(\xi,\nu)\\&-\int_x^{\infty}d\xi\int_{-\infty}^{0}e^{-(x-\xi)\lambda}d(E(\lambda)q)(\xi,\nu)\\&+\sum_{1/(\lambda-\xi)\in N_p}e^{x(1-\lambda)}E(\lambda)q.\end{aligned} \tag{23}$$

Case (b): $\det\Omega(z)=0$ for $z\in R$.

When there are eigenvalues imbedded in the continuous spectrum, the L^p estimates of the previous paragraphs are not valid. It is nevertheless possible to verify that the expression in Theorem 10 is indeed the solution of the boundary value problem. To see this we may rederive the eigenfunction expression, Eq. (11c), for Sf, with $f\in\mathscr{B}$ Hölder continuous, obtaining

$$Sf=\int\frac{1}{x-\xi}\mathbf{\Phi}_x(\omega)\cdot\mathbf{A}_f(x)\,dx+\sum_{Sf}, \tag{24a}$$

where $\mathbf{A}_f$ is used to denote the transform A given by Eq. (11b), and Σ_{Sf} denotes the discrete terms, Eq. (13). Now let us choose $h\in\mathscr{D}(S^{-1})$, whence

$$h=\int\mathbf{\Phi}_x(\omega)\cdot\mathbf{A}_h(x)\,dx+\sum_g=Sf, \tag{24b}$$

and therefore

$$\frac{1}{x-\xi}A_f(x)=A_h(x). \tag{25}$$

We have used a Liouville theorem argument to go from Eqs. (24) to Eq. (25). For analogous use of this argument, see Refs. 12 and 13.

Thus, we may write

$$S^{-1}h=\int(x-\xi)\mathbf{\Phi}_x(\omega)\cdot\mathbf{A}_h(x)\,dx+\sum_{S^{-1}h}-l_h. \tag{26}$$

We have then immediately Kh, and we may substitute the expression in Theorem 10 into Eq. (1) to obtain

Corollary 11: The solution of Eq. (11) satisfying the boundedness conditions $\lim_{x\to\infty}\|\psi(x)\|_{\mathscr{B}}=0$ is given by Eq. (23).

VI. HALF RANGE

The eigenfunction expansion developed in the previous three sections can be used to solve so called "full range" problems involving Eq. (1). The terminology "full range" means we are interested in solutions for $x\in\mathbb{R}$, i.e., infinite media problems. Of more practical interest is the case $x\in\mathbb{R}^+$, i.e., half-space problems; typically one needs an eigenfunction expansion on the so-called "half-range," $\mu\in A^+=\{\mu\in A\,|\mu\geqslant 0\}$. (A detailed discussion of this point may be found in Ref. 1, for one-speed neutron transport.)

The idea, as introduced in Ref. 1, is to define a map E with certain properties which guarantee that the "half-range" expansion of f is given by the full range expansion of Ef. We define E: $D(E)\to D(K)$, with

$$D(E)=\{f\in L_p(A^+,\sigma)\mid f\text{ is Hölder continuous with compact support}\}$$

as

$$\begin{aligned}E(f)(\mu)&=f(\mu),\quad\mu\in A^+\\&=\mathbf{g}(\mu)\cdot X^{-1}(\mu)\int_{A^+}\frac{Y^{-1}(-s)\mathbf{J}(s)f(s)\,ds}{k(s)-z},\quad\mu\in A^-.\end{aligned} \tag{27}$$

Here the matrices X and Y are supposed to provide the Wiener–Hopf factorization of the matrix A, i.e.,

$$A(z)=Y(-z)X(z),$$

where X and Y are analytic in z for $\mathrm{Re}z+\mathrm{Im}z<0$ and $\lim_{|z|\to\infty}Y(-z)$ and $\lim_{|z|\to\infty}X(z)$ exist. The sufficient conditions that such a factorization exist have been discussed by Mullikin[14] and Victory[15] (see also Ref. 4, Sec. V and VI). The existence of such factorization is crucial to the analysis of the present section. (For a slightly different approach, see Ref. 16.)

We now state

Theorem 12: Let E be defined by Eq. (25). Then $(S_\xi-zI)^{-1}Ef$ is analytic in z for $\mathrm{Re}z+\mathrm{Im}z<0$.

Proof: Writing

$$(zI - S_\xi)^{-1}Ef = \frac{k(\mu)-\xi}{z(k(\mu)-\xi)-1}\Big(Ef(\mu) - \frac{\mathbf{g}(\mu)}{k(\mu)-\xi}\cdot\mathbf{G}(z)\Big),$$

where $\mathbf{G}$ is given by

$$\mathbf{G}(z) = \Lambda^{-1}\Big(\xi + \frac{1}{z}\Big)\int_A \frac{\mathbf{J}(s)Ef(x)\,ds}{1 - z(k(s)-\xi)}$$

$$= \Lambda^{-1}\Big(\xi + \frac{1}{z}\Big)\mathbf{T}\Big(\xi + \frac{1}{z}\Big),$$

we demand

$$\frac{\mathbf{g}(\mu)}{k(\mu)-\xi}\cdot\mathbf{G}^+\Big(\frac{1}{k(\mu)-\xi}\Big) = \frac{\mathbf{g}(\mu)}{k(\mu)-\xi}\cdot\mathbf{G}^-\Big(\frac{1}{k(\mu)-\xi}\Big) = Ef(\mu),$$

for $\mu \in A^-$. In fact, let us assume

$$G^+\Big(\frac{1}{k(\mu)-\xi}\Big) = G^-\Big(\frac{1}{k(\mu)-\xi}\Big)$$

on A; whence

$$Y^+(-k(\mu))^{-1}\mathbf{T}^+(k(\mu)) = Y^-(-k(\mu))^{-1}\mathbf{T}^-(k(\mu)). \quad (28)$$

Define

$$\mathbf{Q}(z) = Y\Big(-\xi - \frac{1}{z}\Big)^{-1}\mathbf{T}\Big(\xi + \frac{1}{z}\Big) - \int_{A^*}\frac{Y^{-1}(-k(s))\mathbf{J}(s)f(s)\,ds}{1 - z(k(s)-\xi)}.$$

Then

$$Q^+\Big(\frac{1}{\nu-\xi}\Big) = Q^-\Big(\frac{1}{\nu-\xi}\Big)$$

for $\nu \in \mathbb{R}^-$ by virtue of Eq. (28), and on $\mathbb{R}^+$ by a direct computation. Since Q is bounded near N_p and $Q(t) \to 0$ at infinity, we conclude

$$Y\Big(-\xi - \frac{1}{z}\Big)\int_{A^*}\frac{Y^{-1}(-k(s))\mathbf{J}(s)f(s)\,ds}{1 - z(k(s)-\xi)} = \int_A \frac{\mathbf{J}(s)\,Ef(s)\,ds}{1 - z(k(s)-\xi)}.$$

By evaluating at limits at $z = 1/[k(\mu) - \xi]$, taking a scalar product with $\mathbf{J}(z)$, and computing $\mathbf{Y}^+ - \mathbf{Y}^- = (\Lambda^+ - \Lambda^-)\mathbf{X}^{-1}$, this implies that

$$Ef(\mu) = \frac{\mathbf{g}(\mu)}{k(\mu)-\xi}\cdot X^{-1}(k(\mu))\int_{A^*}\frac{Y^{-1}(-k(s))\mathbf{J}(s)f(s)\,ds}{1 - (k(s)-\xi)/(k(\mu)-\xi)}, \quad \mu \in A^-.$$

Finally, a straightforward computation gives

$$Ef(\mu) = \frac{\mathbf{g}(\mu)}{k(\mu)-\xi}\cdot\mathbf{G}\Big(\frac{1}{k(\mu)-\xi}\Big)$$

For $\omega \in A^+$, the half range expansion of $f \in L^p$ is given by

$$f(\omega) = \int_{R^*}\mathbf{\Phi}_x(\omega)\cdot\mathbf{A}_x\,dx + \sum_f^+, \quad (29a)$$

where $\mathbf{\Phi}_x$ is defined by Eq. (11a), and

$$\mathbf{A}(x) = -\Lambda^-(x)^{-1}\mathbf{J}\otimes\mathbf{g}(k^{-1}(x))\Lambda^+(x)^{-1}Y(-x) \times \int_{A^*}\frac{Y(-k(s))\mathbf{J}(s)f(s)\,ds}{x - k(x)} - \Lambda^-(x)^{-1}\lambda(x)\Lambda^+(x)^{-1}\mathbf{J}(k^{-1}(x))f(k^{-1}(x)). \quad (29b)$$

(Note this is a "half range expansion", as the negative spectrum does not enter.)

The contribution of isolated eigenvalues Σ_f^+ from the appropriate half space is carried out as in Sec. IV. We omit details.

VII. ANISOTROPIC NEUTRON TRANSPORT

In this section we present a quick illustration of the full and half-range expansions obtained above. The illustration that we have in mind is the neutron transport equation with anisotropic scattering. In particular, if we assume the scattering function can be expanded as a finite series of Legendre polynomials, we obtain the triple (k,g,J) as indicated in example 4 of Sec. I.

Therefore, from Eq. (4), we can write

$$\Big[\Lambda\Big(\frac{1}{\omega}\Big)\Big]_{lm} = \delta_{lm} + \frac{c}{2}\omega\int_{-1}^{1}\frac{(2m+1)f_m P_m(s)P_l(s)}{s-\omega}\,ds.$$

We have then, from Eq. (11),

$$f_1(\omega) = \int_{-1}^{+1}\mathbf{\Phi}_y(\omega)\cdot\mathbf{A}(y)\,dy,$$

where for this problem,

$$[\mathbf{\Phi}_y(\omega)]_l = P\frac{(c/2)(2l+1)yf_lP_l(\omega)}{y-\omega} + \tfrac{1}{2}\sum_n\frac{[\Lambda_{nl}^+(1/y) + \Lambda_{nl}^-(1/y)]}{\Sigma_k P_K^2(\omega)}P_n(\omega)\delta(y-\omega),$$

and

$$[A(y)]_l = -\frac{1}{y}\sum_{\mu,k,m}\Lambda_{l\mu}^-\Big(\frac{1}{y}\Big)^{-1}\frac{c}{2}(2k+1) \times f_kP_k(y)P_\mu(y)\Lambda_{km}^+\Big(\frac{1}{y}\Big)^{-1}\int_{-1}^{1}\frac{P_m(s)f(s)}{1/y - 1/s}\,ds + \tfrac{1}{2}\sum_n\Big[\Lambda_{lm}^-\Big(\frac{1}{y}\Big)^{-1} + \Lambda_{ln}^+\Big(\frac{1}{y}\Big)^{-1}\Big]P_nf(y).$$

However let us note the identity[17]

$$\Lambda(1/\omega)\mathbf{h}(\omega) = \Omega(1/\omega)\mathbf{P}(\omega),$$

where

$$[\mathbf{P}(\omega)]_n = P_n(\omega)$$

and[18]

$$\Omega(1/\omega) = \det\Lambda(1/\omega) = 1 + \omega \int_{-1}^{+1} \frac{K(\omega,x)}{x-\omega}\, dx.$$

Here

$$K(\omega,x) = \frac{c}{2}\sum_{n=0}^{N}(2n+1)f_n P_n(x) h_n(\omega),$$

and

$$[\mathbf{h}(\omega)]_n = h_n(\omega),$$

where the polynomials $h_n(\omega)$ are defined recursively by[19]

$$(n+1)h_{n+1}(\omega) + n h_{n-1}(\omega) = \omega[(2n+1) - cf_n]h_n(\omega),$$

$$h_0(\omega) = 1, \quad h_1(\omega) = (1-c)\omega.$$

Use of this identity and a modest amount of algebra allows us to write

$$[\mathbf{A}(y)]_l = \frac{h_l(y)}{\Omega^+(1/y)\Omega^-(1/y)} \int_{-1}^{+1} \frac{s K(y,s) f(s)}{y-s}\, ds + \bar{\lambda}(y) f(y),$$

where

$$\bar{\lambda}(y) = \tfrac{1}{2}[\Omega^+(1/y) + \Omega^-(1/y)].$$

Using this result and Eqs. (11) we obtain

$$\int_{-1}^{+1} \mathbf{\Phi}_y(\omega)\cdot\mathbf{A}(y)dy = \int_{-1}^{+1} \varphi_y(\omega)\mathscr{A}(y)\, dy,$$

where

$$\varphi_y(\omega) = \frac{Pyk(\omega,y)}{y-\omega} + \bar{\lambda}(y)\delta(y-\omega) \tag{30a}$$

and

$$\mathscr{A}(y) = [y(\Omega^+(1/y)\Omega^-(1/y)]^{-1} \int_{-1}^{+1} s f(s)\varphi_y(s)\, ds. \tag{30b}$$

This is the same result obtained by Mika using the singular eigenfunction technique.[19]

For half-range problems it is necessary to use the factorization

$$\Lambda(1/z) = Y(-1/z)X(1/x).$$

For the problem under consideration, this factorization has been shown by Mullikin[20] for those cases when $c < 1$. More precisely, the matrices $X(1/\omega)$ and $Y(1/\omega)$ can be written in the form[20]

$$[Y(1/\omega)]_{nk} = \delta_{nk} + \frac{c\omega}{2}\int_0^1 \frac{(2k+1)f_k \Psi_k(s) P_n(s)}{s+\omega}\, ds, \tag{31a}$$

and

$$[X(1/\omega)]_{nk} = \delta_{nk} + \frac{(-1)^{n+k-1}c\omega}{2} \times \int_0^1 \frac{(2k+1)f_k \Psi_k(s) P_n(s)}{s+\omega}\, ds, \tag{31b}$$

where the functions Ψ_k satisfy the nonlinear equations

$$\Psi_l(s) = P_l(s) + \frac{cs}{2}\sum_{k=0}^{N}(-1)^{l+k}(2k+1)f_k \times \int_0^1 \frac{\Psi_k(s)\Psi_k(t)P_l(t)}{t+s}\, dt, \tag{31c}$$

plus certain analyticity constraints. Mullikin has shown the existence of the solution to Eqs. (31) and recent results indicate that Eq. (31c) or a variant of it is a likely candidate for solution by iteration.

Using the same techniques as in the full-range we find from Eq. (29b) that

$$[\mathbf{A}(y)]_l = \frac{h_l(y)}{\Omega^+(1/y)\Omega^-(1/y)} \int_0^1 \frac{sG(y,s)f(s)}{y-s}\, ds + \bar{\lambda}(y)f(y),$$

where

$$G(y,s) = \frac{c}{2}\sum_{n,l}(2n+1)f_n h_n(y) Y_{nl}(-1/y) \times Y_{lm}^{-1}(-1/s)P_m(s).$$

Substituting this into Eq. (29a), we will obtain the contribution to the half-range expansion from the continuum to be

$$\bar{f}_1(\omega) = \int_0^1 \mathbf{\Phi}_y(\omega)\cdot\mathbf{A}(y)\, dy = \int_0^1 \varphi_y(\omega)\bar{\mathscr{A}}(y)\, dy,$$

where $\varphi_y(\omega)$ is given by Eq. (30) and

$$\bar{\mathscr{A}}(y) = [y\Omega^+(1/y)\Omega^-(1/y)]^{-1} \int_0^1 \frac{sy\, G(y,s) f(s)\, ds}{y-s} + \bar{\lambda}(y)f(y).$$

The contribution to the expansion from the discrete roots can be easily, if laboriously, worked out, using Theorem 7(i). Again, Mika's results[19] are reproduced. (Recall[21] that for the subcritical case considered here the discrete roots fall on the real line outside $[-1,1]$.) The singular eigenfunction method has the advantage that the matrix Λ need not be factored; the method presented here however, is somewhat simpler, granted the matrix factorization known, and gives the results in somewhat simpler form. In any event, this section has been included as an illustration of our technique and to connect our results with the (seemingly) different formulas in the literature, rather than to obtain new results on this particular application.

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