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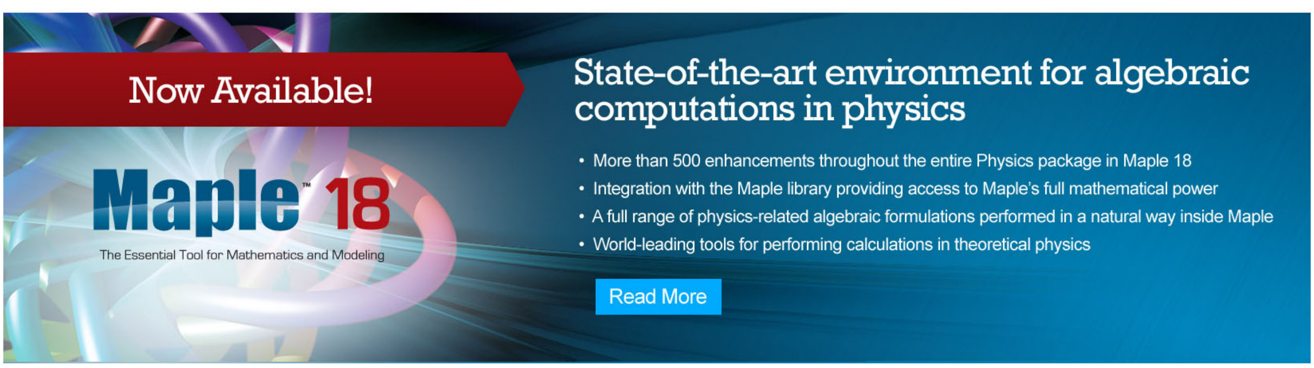

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On the solution of nonlinear matrix integral equations in transport theory

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The coupled nonlinear matrix integral equations for the matrices $X(z)$ and $Y(z)$ which factor the dispersion matrix $\Lambda(z)$ of multigroup transport theory are studied in a Banach space X. By utilizing fixed-point theorems we are able to show that iterative solutions converge uniquely to the "physical solution" in a certain sphere of X. Both isotropic and anisotropic scattering are considered.

I. INTRODUCTION

In a recent paper,[1] the Chandrasekhar H equation has been studied. In particular the following results were shown:

1. An iterative procedure, proved by Bittoni *et al*[2] to converge to a unique solution inside a certain region of the Banach space $L_1(0,1)$, actually converges to the "physical solution," i.e., the solution which is analytic in the right-half complex plane. (Alternatively, the "physical solution" is the one which obeys the so-called constraining equations.[3,4])

2. The iteration scheme of Bittoni *et al* can be extended to all values of $\|\psi\|$, provided $\psi(\mu)\geqslant 0$, $\mu\in[0,1]$, where $\psi(\mu)$ is the "characteristic function ($\|\psi\|=c/2$ in one-speed isotropic neutron transport)." In Ref. 2, only the case $\|\psi\|<1$ had been studied.

The advantage of these results is that in any "one-group" transport problem, the H functions can be calculated iteratively without the necessity of introducing constraining equations. Furthermore, the knowledge of the region of Banach space in which the solution exists is of considerable help in performing the numerics. In particular, we observe that if the initial estimate is chosen to be zero, the iterative procedure always converges to the "physical solution."

The purpose of this paper is to present a similar iteration scheme for solving the matrix versions of the Chandrasekhar H equations. The solution of these equations provides the Wiener—Hopf *matrix* factorization of the dispersion matrix Λ and is needed to construct the solution of half-space multigroup transport equations.[5,6] [In the one-speed or scalar case the H function is the Wiener—Hopf factorization of the dispersion function $\Lambda(z)$.]

For the multigroup problem it is necessary to consider coupled nonlinear nonsingular matrix equations which have been written in the form[6]

$$X(-z)=C^{-1}\Sigma-z\int_0^1 Y^{-1}(-s)\Sigma^2\Delta(s)\frac{ds}{s+z} \tag{1a}$$

and

$$Y(-z)=\Sigma-z\int_0^1 \Sigma^2\Delta(s)X^{-1}(-s)\frac{ds}{s+z}. \tag{1b}$$

Here Σ is the diagonal cross section matrix with elements $\delta_{ij}\sigma_i$, $\sigma_1\geqslant\sigma_2\geqslant\cdots\geqslant\sigma_N=1$, and C is the group-to-group scattering matrix, while Δ is a diagonal matrix with elements

$$\Delta_{ij}(s)=\delta_{ij}\theta(s-1/\sigma_i),$$

where θ is the Heavyside function

$$\theta(s-1/\sigma_i)=1,\quad s\leqslant 1/\sigma_i$$
$$=0,\quad s>1/\sigma_i.$$

Moreover, X and Y factor the Λ matrix,[6]

$$\Lambda(z)=(\Sigma-2C)C^{-1}\Sigma-\int_{-1}^{1}\mu[zI-\mu\Sigma^{-1}]^{-1}\,d\mu,$$

in the form

$$\Lambda(z)=Y(-z)X(z), \tag{2}$$

where $Y(z)$ and $X(z)$ are supposed to be analytic and nonsingular for $\operatorname{Re}z<0$. Because $Y(z)$ and $X(z)$ factor the dispersion matrix $\Lambda(z)$, the requirement that $Y(z)$ and $X(z)$ be analytic and nonsingular for $\operatorname{Re}z<0$ is equivalent to the constraints[6]

$$\det Y(+\nu_j)=\det X(+\nu_j)=0,$$
$$\operatorname{Re}\nu_j>0,\quad j=0,\ldots,d-1, \tag{3}$$

where $\pm\nu_j$, $j=0,\ldots,d-1$ are the $2d$ discrete Van Kampen—Case eigenvalues which obey

$$\Omega(\pm\nu_j)\equiv\det\Lambda(\pm\nu_j)=0.$$

The constraints in Eq. (3) are usually introduced to assure that the solution of Eqs. (1) (or comparable equations) is unique.[7] The solution of Eqs. (1) which obeys the constraints in (3) will be called the "physical solution." In the current analysis, uniqueness is guaranteed by restricting the solution to a certain sphere in Banach space. The resulting solution can then be shown to be the "physical solution."

The factorization of $\Lambda(z)$ (with a somewhat different notation) was originally obtained by Mullikin[8] and, as used in Ref. 6, was restricted to the case $\rho<\frac{1}{2}$, where ρ is the dominant eigenvalue of the nonnegative matrix

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$\Sigma^{-1}C$. The results of this paper are restricted to the more restrictive case $\int_0^1 \|\Delta C\|_M(s)\,ds < \frac{1}{2}$ for the case of isotropic scattering presented in Sec. II and a similar restriction for the case of anisotropic scattering presented in Sec. III. Here $\|\ \|_M$ represents the "matrix norm," e.g.,

$$\|A\|_M = \sup_i \sum_j |A_{ij}|. \tag{4}$$

Before presenting our analysis in the next section, we might remark that if C is a symmetric matrix, $C = C^t$, then $\Lambda = \Lambda^t$ and it can be shown quite easily that

$$Y = X^t C.$$

Then the two coupled equations (1) reduce to a simple equation, which after appropriate transformation becomes the "matrix H equation" considered by Siewert and co-workers.[8,9] Thus the equation they studied is a special case of ours.

II. BANACH SPACE ANALYSIS

Equations (1) can be transformed into a more convenient form by defining

$$U_1(z) = C^{-1}\Sigma X^{-1}(-z) \tag{5a}$$

and

$$U_2(z) = Y^{-1}(-z)\Sigma. \tag{5b}$$

For $z \in [0, 1]$, Eqs. (1) reduce then to the coupled nonlinear, nonsingular matrix integral equations

$$U_1(z) = I + z\int_0^1 U_1(z)U_2(s)\Sigma\Delta(s)\Sigma^{-1}C\frac{ds}{z+s} \tag{6a}$$

and

$$U_2(z) = I + z\int_0^1 \Sigma\Delta(s)\Sigma^{-1}CU_1(s)U_2(z)\frac{ds}{z+s}. \tag{6b}$$

We consider U_1 and U_2 as elements of a Banach space X_0 with norm[10]

$$\|U_i\|_{X_0} = \int_0^1 \|U_i\|_M(s)\,ds, \tag{7}$$

where $\|\ \|_M$ is the matrix norm already introduced.[11] Now consider the Banach space X, the Cartesian product of X_0 with itself,

$$\mathcal{U} = [U_1, U_2] \in X, \quad U_1, U_2 \in X_0, \tag{8a}$$

with norm

$$\|\mathcal{U}\|_X = \int_0^1 \max[\|U_1\|_M, \|U_2\|_M](s)\,ds. \tag{8b}$$

One can readily verify that $\|\ \|_X$ is a norm.

Let us now define $U_1' \in X_0$ and $U_2' \in X_0$ by

$$U_1'(s) = \Sigma\Delta(s)\Sigma^{-1}CU_1(s), \quad s \in [0, 1] \tag{9a}$$

and

$$U_2'(s) = U_2(s)\Sigma\Delta(s)\Sigma^{-1}C, \quad s \in [0, 1]. \tag{9b}$$

We can then write from Eqs. (6) the single equation for

$$\mathcal{U}' = [U_1', U_2'] \in X, \quad \mathcal{U}' = \mathcal{J} + A(\mathcal{U}', \mathcal{U}'), \tag{10a}$$

where

$$\mathcal{J} = [\Sigma\Delta\Sigma^{-1}C, \Sigma\Delta\Sigma^{-1}C] \in X \tag{10b}$$

and A is the bilinear form

$$A(\mathcal{U}, \mathcal{V})(z) = \left[z\int_0^1 V_1(z)U_2(s)\frac{ds}{s+z},\ z\int_0^1 U_1(s)V_2(z)\frac{ds}{s+z}\right]. \tag{10c}$$

The following lemma which is proved in Ref. 2 (and restated in Ref. 1) is vital to the subsequent analysis:

Lemma I: Let Y be a Banach space with norm $\|\ \|_Y$ and $B(u, v)$ a bilinear map: $Y \times Y \to Y$ with "norm"

$$\|B\| = \sup\{\|B(u, v)\|_Y : \|u\|_Y = 1 \text{ and } \|v\|_Y = 1\}.$$

Then for $2\|B + B^*\|_Y \|f\|_Y < 1$, the equation

$$u = Tu \equiv f + B(u, u), \quad f \in Y$$

has one and only one solution in the ball

$$S = \{u \in Y : \|u - f\|_Y < \tfrac{1}{2}\}.$$

Furthermore, $TS \subset S$. [Here $B^*(u, v) \equiv B(v, u)$.]

Corollary: For every $u_0 \in S$, $\lim_{n\to\infty} T^n u_0$ converges in Y to the unique solution of the equation $u = Tu$.

We now prove

Lemma II: If A is the bilinear form given by Eq. (10c) and $A^*(\mathcal{U}, \mathcal{V}) \equiv A(\mathcal{V}, \mathcal{U})$, then $\|A + A^*\| = 1$.

Proof: By direct calculation we find

$$\begin{aligned}
&\|A(\mathcal{U}, \mathcal{V}) + A^*(\mathcal{U}, \mathcal{V})\|_X \\
&= \int_0^1 dx \max\left[\left\|\int_0^1 ds\, V_1(x)U_2(s)\frac{x}{s+x} + \int_0^1 ds\, U_1(x)V_2(s)\frac{x}{s+x}\right\|_M, \left\|\int_0^1 ds\, U_1(s)V_2(x)\frac{x}{s+x} + \int_0^1 ds\, V_1(s)U_2(x)\frac{x}{s+x}\right\|_M\right] \\
&\le \int_0^1 dx \max\left[\int_0^1 dx\{\|V_1\|_M(x)\|U_2\|_M(s) + \|U_1\|_M(x)\|V_2\|_M(s)\}\frac{x}{s+x},\ \int_0^1 ds\{\|U_1\|_M(s)\|V_2\|_M(x) + \|V_1\|_M(s)\|U_2\|_M(x)\}\frac{x}{s+x}\right] \\
&\le \int_0^1 dx\int_0^1 ds\Big\{\max[\|V_1\|_M, \|V_2\|_M](x) \times \max[\|U_1\|_M, \|U_2\|_M](s)\frac{x}{s+x} + \max[\|V_1\|_M, \|V_2\|_M](s) \times \max[\|U_1\|_M, \|U_2\|_M](x)\frac{x}{s+x}\Big\} \\
&\le \int_0^1 dx\int_0^1 ds \max[\|V_1\|_M, \|V_2\|_M](x) \times \max[\|U_1\|_M, \|U_2\|_M](s) \\
&\le \|\mathcal{U}\|_X \cdot \|\mathcal{V}\|_X.
\end{aligned}$$

(In going from the third to the fourth relation, the change of variable $x \to s$ has been made.) The above calculations show that $\|A + A^*\| \le 1$. Equality is obtained by setting $\mathcal{U} = \mathcal{V} = [I, I]$. This completes the proof of the lemma.

Noting that

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$\|\mathcal{J}\|_X = \int_0^1 \|\Delta C\|_M(s)\,ds,$

we combine Lemmas I and II to obtain

Lemma III: If $\int_0^1 \|\Delta C\|_M(s)\,ds < \frac{1}{2}$, Eq. (10) has a unique solution $\hat{\mathcal{U}}$ in the ball

$$S_1 = \{\mathcal{U}' \in X : \|\mathcal{U}' - \mathcal{J}\|_X < \tfrac{1}{2}\}.$$

Furthermore, the iteration procedure defined by

$$\mathcal{U}_n = \mathcal{J} + A(\mathcal{U}_{n-1}, \mathcal{U}_{n-1})$$

converges to $\hat{\mathcal{U}}$ for every $\mathcal{U}_0 \in S_1$.

The convergence can easily be seen to be uniform and pointwise (see Lemma III of Reference 1). We omit the details here.

We now know that we can solve Eq. (10a) iteratively to obtain $\hat{\mathcal{U}}$. To recover $X(z)$ and $Y(z)$ from Eqs. (5) we must first obtain U_1 and U_2 from U_1' and U_2' [Eqs. (9)]. Unfortunately, $\Delta(s)$ is not an invertible matrix. Therefore, we describe below the scheme which can be used. At the same time, this scheme provides the analytic continuation of $\mathcal{U}$ to the rest of the complex plane.

In other words, we wish to show that the solution of Eq. (10) referred to in Lemma III can be used to obtain the matrices $U_1(z)$ and $U_2(z)$ satisfying Eqs. (6). Moreover we shall prove that these matrices are analytic for $\mathrm{Re}z \geq 0$. To this end let us now state

Lemma IV: If $\hat{\mathcal{U}} = [\hat{U}_1, \hat{U}_2]$ is the unique solution to Eq. (9) in the ball S_1 for $\int_0^1 \|\Delta C\|_M(s)\,ds < \frac{1}{2}$, then for $z \in \mathbb{C}$

$$\det\left[I - \int_0^1 \hat{U}_i(s)\frac{z}{z+s}\,ds\right] \neq 0, \quad \mathrm{Re}z \geq 0, \quad i = 1, 2. \quad (11)$$

Proof: Since $\hat{\mathcal{U}} \in S_1$ and $\|\mathcal{J}\|_X < \frac{1}{2}$, we have

$$\tfrac{1}{2} > \|\hat{\mathcal{U}} - \mathcal{J}\|_X > \big|\,\|\hat{\mathcal{U}}\|_X - \|\mathcal{J}\|_X\big|.$$

Thus we have

$$\|\hat{\mathcal{U}}\| < 1.$$

Hence

$$\|U_i\|_{X_0} < 1, \quad i = 1, 2.$$

Now let $z = \alpha + i\beta$, $\alpha \geq 0$. Suppose for some value of z

$$\det\left(I - \int_0^1 \hat{U}_i(s)\frac{z}{z+s}\,ds\right) = 0, \quad \mathrm{Re}z \geq 0, \quad i = 1, 2.$$

This would imply that there exists a nonzero vector Ψ such that

$$\|\Psi\| \leq \left\| \int_0^1 \hat{U}_i(s)\frac{\alpha + i\beta}{\alpha + s + i\beta}\,ds\right\|_M \cdot \|\Psi\|,$$

where here $\|\Psi\|$ is a vector norm consistent with $\|\ \|_M$. This last relation yields

$$\begin{aligned}
1 &\leq \int_0^1 \|\hat{U}_i\|_M(s)\left|\frac{\alpha + i\beta}{\alpha + s + i\beta}\right| ds \\
&\leq \int_0^1 \|\hat{U}_i\|_M(s)\left(\frac{\alpha^2 + \beta^2}{(\alpha + s)^2 + \beta^2}\right)^{1/2} ds \\
&\leq \int_0^1 \|\hat{U}_i\|_M(s)\,ds \ (\text{for } \alpha \geq 0) \\
&= \|\hat{U}_i\|_{X_0} < 1
\end{aligned}$$

which is a contradiction. Thus the inequality (11) must hold. This completes the proof of the lemma.

Now define

$$U_1(z) = \left(I - \int_0^1 \hat{U}_2(s)\frac{z}{z+s}\,ds\right)^{-1} \quad (12a)$$

and

$$U_2(z) = \left(I - \int_0^1 \hat{U}_1(s)\frac{z}{z+s}\,ds\right)^{-1}, \quad (12b)$$

where $\hat{\mathcal{U}} = [\hat{U}_1, \hat{U}_2]$ is the unique solution of Eq. (10) referred to in Lemma IV. The matrices $U_1(z)$ and $U_2(z)$ are analytic in the complex z plane cut along $[-1, 0]$ with (possible) poles at those values of z for which

$$\det\left[I - \int_0^1 \hat{U}_i(s)\frac{z}{z+s}\,ds\right] = 0, \quad i = 1, 2.$$

In particular we observe from Lemma IV that $U_1(z)$ and $U_2(z)$ are analytic in the complex z plane for $\mathrm{Re}z \geq 0$ $(z \neq 0)$. Furthermore, we have

Lemma V: The matrices $U_1(z)$ and $U_2(z)$ defined by Eqs. (12) satisfy Eqs. (6).

Proof: For those values of z such that

$$\det\left(I - \int_0^1 \hat{U}_i(s)\frac{z}{z+s}\,ds\right) \neq 0,$$

$U_1(z)$ and $U_2(z)$ satisfy

$$U_1(z) = I + \int_0^1 U_1(z)\hat{U}_2(s)\frac{z}{z+s}\,ds \quad (13a)$$

and

$$U_2(z) = I + \int_0^1 \hat{U}_1(s)U_2(z)\frac{z}{z+s}\,ds. \quad (13b)$$

We then need only to prove that

$$\Sigma\Delta(s)\Sigma^{-1}CU_1(s) = \hat{U}_1(s), \quad s \in [0, 1]$$

and

$$U_2(s)\Sigma\Delta(s)\Sigma^{-1}C = \hat{U}_2(s), \quad s \in [0, 1].$$

However, from Lemma IV, we note that Eqs. (12) are well defined for $z \in [0, 1]$ and

$$\begin{aligned}
\Sigma\Delta(z)\Sigma^{-1}CU_1(z) &= \Sigma\Delta(z)\Sigma^{-1}C\left(I - z\int_0^1 \hat{U}_2(s)\frac{ds}{z+s}\right)^{-1} \\
&= \hat{U}_1(z), \quad z \in [0, 1]
\end{aligned}$$

and

$$\begin{aligned}
U_2(z)\Sigma\Delta(z)\Sigma^{-1}C &= \left(I - z\int_0^1 \hat{U}_1(s)\frac{ds}{z+s}\right)^{-1}\Sigma\Delta(z)\Sigma^{-1}C \\
&= \hat{U}_2(z), \quad z \in [0, 1].
\end{aligned}$$

This completes the proof.

The matrices $U_1(z)$ and $U_2(z)$ are analytic in the left-half complex z plane except for a cut along $[-1, 0]$ and (possible) poles at those values of z for which $\det U_1(z)$ and $\det U_2(z)$ vanish. In this regard, we have

Lemma VI: If $U_1(z)$ and $U_2(z)$ are defined by Eqs. (12), then

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$$\det U_1^{-1}(-\nu_j) = \det U_2^{-1}(-\nu_j) = 0,$$
$$\mathrm{Re}\,\nu_j \geq 0, \quad j = 0, \ldots, d-1, \tag{14}$$

where we recall that $\pm\nu_j$, $j = 0, \ldots, d-1$ are the zeros of $\det\Lambda(z)$.

Proof: From Lemma V, $U_1(z)$ and $U_2(z)$ satisfy Eqs. (6), but by considering

$$[U_2^{-1}(z) - I][U_1^{-1}(-z) - I] = z^2 \int_0^1 ds \int_0^1 dt\, \Sigma\Delta(s)\Sigma^{-1}CU_1(s) \times U_2(t)\Sigma\Delta(t)\Sigma^{-1}C[(z+s)(z-t)]^{-1}$$

one can show that $U_2^{-1}(z)$ form the Wiener—Hopf factorization of $\Lambda(z)$ (cf. Ref. 6),

$$U_2^{-1}(z)U_1^{-1}(-z) = \Sigma^{-1}\Lambda(z)\Sigma^{-1}C. \tag{15}$$

Since $\Lambda(z)$ is even in z, we must have

$$\det U_2^{-1}(\nu_j)\det U_1^{-1}(-\nu_j) = 0, \quad j = 0, \ldots, d-1, \tag{16a}$$

and

$$\det U_2^{-1}(-\nu_j)\det U_1^{-1}(+\nu_j) = 0, \quad j = 0, \ldots, d-1. \tag{16b}$$

The lemma now follows from Lemma IV.

We note that from Eqs. (14) and (16) if ν_j is purely imaginary, then

$$\det U_i^{-1}(+\nu_j) = \det U_i^{-1}(-\nu_j)^* = 0,$$

in contradiction to Lemma IV. We thus have

Corollary to proof of Lemma VI: If $\int_0^1 \|\Delta C\|_M(s)\, ds < \frac{1}{2}$, then there are no purely imaginary zeros of $\det\Lambda(z)$.

We summarize the results of this section with

Theorem I: If $\int_0^1 \|\Delta C\|_M(s)\, ds < \frac{1}{2}$, then the matrices $U_1(z)$ and $U_2(z)$ given by Eqs. (12) satisfy Eqs. (6) with $\hat{\mathcal{U}} = [U_1(s), \hat{U}_2(s)]$, $s \in [0, 1]$, being the unique solution to Eq. (10) in the ball S_1. Furthermore, $U_1(z)$ and $U_2(z)$ are analytic in the complex z plane cut along $[-1, 0]$ except for poles at $-\nu_j$, $j = 0, \ldots, d-1$ and factor the dispersion matrix $\Lambda(z)$ according to Eq. (15).

III. ANISOTROPIC SCATTERING

The procedure presented in the preceding section can easily be generalized to the case of anisotropic scattering. The transport equation for a degenerate scattering kernel of the form

$$C(\mu, \mu') = \sum_{i=1}^{M} A_i(\mu)B_i(\mu')$$

has been studied by Larsen and Zweifel.[12] The nonlinear integral equations were written in this reference as

$$X(-z) = I - z \int_0^1 Y^{-1}(-s)\sum_{j=1}^{N} B(s\sigma_j)I_jA(s\sigma_j)\frac{ds}{z+s} \tag{17a}$$

and

$$Y(-z) = I - z \int_0^1 \sum_{j=1}^{N} B(\sigma s_j)I_jA(s\sigma_j)X^{-1}(-s)\frac{ds}{z+s}. \tag{17b}$$

Here, X and Y are $NM\times NM$ matrices (N is the number of groups and M is the order of anisotropy), A is an $N\times NM$ matrix defined by

$$A = (A_1A_2\cdots A_M),$$

and B is the $NM\times N$ matrix defined by

$$B^t = (B_1^tB_2^t\cdots B_M^t).$$

Also, I_j is an $N\times N$ matrix for which the element in the jth row and jth column is unity and all other elements are zero. (We are discussing only the solution of the X and Y equations in this paper; the reader curious as to the reason for the introduction of such a cumbersome structure should consult Ref. 12.) For technical reasons, it is convenient in the anisotropic scattering case to define the Λ matrix slightly differently from that used in isotropic scattering. Specifically the matrix is defined by

$$\Lambda(z) = I - z\int_{-1}^{1} B(s)\Sigma^{-1}(zI - \Sigma^{-1}s)^{-1}A(s)\, ds.$$

Then the X and Y matrices which satsify Eqs. (17) factor $\Lambda(z)$ according to Eq. (2).

The procedure followed in Sec. II can equally well be applied to Eqs. (17). In particular, if we define

$$X^{-1}(-z) = V_1(z) \quad\text{and}\quad Y^{-1}(-z) = V_2(z),$$

Eqs. (17) can be written as

$$V_1(z) = I + z\int_0^1 V_1(z)V_2(s)R(s)\frac{ds}{z+s} \tag{18a}$$

and

$$V_2(z) = I + z\int_0^1 R(s)V_1(s)V_2(z)\frac{ds}{z+s}, \tag{18b}$$

where we have defined

$$R(s) = \sum_{j=0}^{N} B(s\sigma_j)I_jA(s\sigma_j).$$

If we now make the transformation

$$V_1'(s) = R(s)V_1(s), \quad s \in [0, 1]$$

and

$$V_2'(s) = V_2(s)R(s), \quad s \in [0, 1]$$

we can write the single equation for $\mathcal{V}' = [V_1', V_2'] \in X$,

$$\mathcal{V}' = \mathcal{J}' + A(\mathcal{V}', \mathcal{V}'), \tag{19a}$$

where

$$\mathcal{J}' = [R, R] \in X, \tag{19b}$$

and A is the bilinear form given by Eq. (10c). By the analysis of Sec. II we see that if $\|\mathcal{J}'\|_X < \frac{1}{2}$, i.e., if

$$\int_0^1 \|R\|_M(s)\, ds < \tfrac{1}{2},$$

then Eq. (19) has a unique solution $\hat{\mathcal{V}}$ in the ball S_2 given by

$$S_2 = \{\mathcal{V}' \in X;\ \|\mathcal{V}' - \mathcal{J}'\|_X < \tfrac{1}{2}\}.$$

We now define

$$V_1(z) = \left(I - \int_0^1 \hat{V}_2(s)\frac{z}{z+s}\, ds\right)^{-1} \tag{20a}$$

and

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$$V_2(z) = \left(I - \int_0^1 \hat{V}_1(s) \frac{z}{z+s}\, ds\right)^{-1} \tag{20b}$$

and state from the results of Sec. II

Theorem II: If $\int_0^1 \|R\|_M(s)\, ds < \frac{1}{2}$, then the matrices $V_1(z)$ and $V_2(z)$ given by Eqs. (20) satisfy Eqs. (18) with $\hat{V} = [V_1(s), V_2(s)]$, $s \in [0, 1]$ being the unique solution of Eq. (19) in the ball S_2. Furthermore, $V_1(z)$ and $V_2(z)$ are analytic in the complex z plane cut along $[-1, 0]$, except for poles at $-\nu_j$, $j = 0, \ldots, d-1$, and factor the dispersion matrix $\Lambda(z)$ according to

$$V_2(z) V_1(-z) \Lambda(z) = I. \tag{21}$$

IV. DISCUSSION

In Sec. II, the transformation from the set $[U_1, U_2]$ to $[U_1', U_2']$ is made. This is a technical convenience, and one could just as well work with Eqs. (6) for $[U_1, U_2]$. However, in Sec. III, where anisotropic scattering is considered, we have not discovered a convenient way to work with the unprimed quantities. The transformation almost seems unavoidable in that case.

We note, further, that in the solution to either Eqs. (6) for U or Eqs. (10) for U' one need only obtain the solution for the ith row of $U_1(s)$ and the ith column of $U_2(s)$ for $0 \leq s\sigma_i \leq 1$. If the solution is desired for the entire range of s, $0 \leq s \leq 1$, or for that matter in the remainder of the complex plane, one only needs to carry out the analytic continuation according to Eqs. (12). However, for the solution of the transport equation,[6] one needs only the values of U_1 and U_2 for the restricted range of $[0, 1]$ described above and at the discrete eigenvalues $-\nu_j$, $j = 0, \ldots, d-1$.

Finally, we address ourselves to the question of generalizing our results. If ρ is the dominant eigenvalue of the nonnegative matrix $\Sigma^{-1}C$, the inequality $\rho < \frac{1}{2}$ is the condition that the infinite medium be subcritical.[13] However, we note that

$$\rho \leq \left\| \int_0^1 \Sigma\Delta(s)\Sigma^{-1} C\, ds \right\|_M \leq \int_0^1 \|\Delta C\|_M(s)\, ds.$$

If we wish to discuss the general case of infinite medium subcriticality for the isotropic scattering case, then the norm condition in Theorem I is too strong, since there may be some systems which obey the infinite medium subcriticality condition but not the norm inequality in Theorem I. A similar argument also applies to Theorem II in the case of anisotropic scattering. Although it might be possible, by appropriately defining norms, to extend the results of Sections II and III to all subcritical parameters, a more fruitful procedure seems to be indicated. That is to try to find a transformation similar to that introduced in Ref. 1 to extend our results to all systems, supercritical, critical, and subcritical. That is the problem that we are currently pursuing.

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