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On the spectrum of the linear transport operator*

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In this paper, spectral properties of the time-independent linear transport operator A are studied. This operator is defined in its natural Banach space $L_1(D\times V)$, where D is the bounded space domain and V is the velocity domain. The collision operator K accounts for elastic and inelastic slowing down, fission, and low energy elastic and inelastic scattering. The various cross sections in K and the total cross section are piecewise continuous functions of position and speed. The two cases $v_0>0$ and $v_0=0$ are treated, where v_0 is the minimum neutron speed. For $v_0=0$, it is shown that $\sigma(A)$ consists of a full half-plane plus, in an adjoining strip, point eigenvalues and curves. For $v_0>0$, $\sigma(A)$ consists just of point eigenvalues and curves in a certain half-space. In both cases, the curves are due to purely elastic "Bragg" scattering and are absent if this scattering does not occur. Finally the spectral differences between the two cases $v_0>0$ and $v_0=0$ are discussed briefly, and it is proved that A is the infinitesimal generator of a strongly continuous semigroup of operators.

I. INTRODUCTION

Since the pioneering work of Lehner and Wing,[1] the spectrum of the linear or "neutron" transport operator has been the subject of intensive study by mathematicians, physicists, and nuclear engineers.[2–37] Knowledge of this spectrum is, as has been stressed in Ref. 1, a necessary prerequisite to the calculation of eigenfunction expansions for time-dependent problems. (In some cases, due to the existence of a half plane of "continuum" spectrum, analysis has shown that such expansions are not feasible.) In addition, knowledge of the spectrum is important in the interpretation of pulsed neutron experiments.

It would be a formidable task to summarize the research embodied in the references (which listing should not be considered all-inclusive, incidentally, but only representative). We would like to make a few general comments, however, about their contents in order to motivate our own work.

A great deal of the above work involves specific models of the transport operator (i.e., of the collision term and the geometry). Thus Ref. 1 treats a one-speed, one-dimensional equation with isotropic scattering, while Ref. 12, for another example, deals with a three-dimensional ideal gas scattering model. As the years progressed, various authors attempted to treat increasingly general problems. For example, we find Mika[14] generalizing the results of Ref. 1 to anisotropic and energy dependent operators, still in one space dimension. As another example, Bednarz[13] obtained some fairly general results but specifically excluded purely elastic scattering at thermal neutron energies. An attempt was made to remove this restriction[17,27] but these results depend upon the assumptions that the minimum neutron speed v is zero, and $v\Sigma_0(v) < v\Sigma(v) - \lambda^*$. Here Σ_0 and Σ are the elastic and total cross sections, and λ^* is the minimum value of $v\Sigma(v)$, which is required to be at $v=0$. Some additional recent work[36] has treated elastic scattering involving a discontinuity in the total reaction rate (as a function of neutron speed), but it contains a technical error in the proof of Lemma 2.[38]

Most of the cited references employ a spectral analysis in Hilbert space similar to that in Ref. 1. However, other approaches have been used, two of which we shall mention here.

First, we refer to a paper by Jörgens,[2] where semigroup techniques are used to treat the case of a finite body in which the minimum neutron speed v_0 is positive. For this case Jörgens showed that the spectrum of the transport operator consists solely of isolated point spectrum. (Other authors, considering the case $v_0=0$, have asserted that the spectrum should contain a half-space $\mathrm{Re}\lambda \leq -\lambda^*$.)

Also, we refer to the homogeneous, infinite medium problems which have been treated[31] by taking a Fourier transform in the space variable and a Laplace transform in time. One obtains a problem involving both time and space eigenvalues in which a dispersion law, i.e., a functional relation between the two quantities, is sought.

The above brief overview of the previous work suggests the motivations for the present paper.

First, we employ a more general collision operator than has been considered previously. It consists of three terms: (i) a completely continuous portion, representing elastic slowing down, fission, and low energy inelastic scattering; (ii) a singular nilpotent portion, representing inelastic slowing down; and (iii) a singular portion, representing low energy elastic "Bragg" scattering.[39]

Secondly, our work is carried out in an L_1 space whereas nearly all previous work has been performed in L_2. We do this because L_1 is the natural space for the transport operator, since the integral of the angular density has physical significance. (In quantum mechanics, where $|\psi|^2$ carries physical meaning, L_2 is appropriate.) This point, incidentally, has been stressed by Case (private communications) and Ribaric,[37] who gives a lengthy discussion. See also Refs. 33 and 35.

Thirdly, we treat the two separate cases $v_0=0$ and $v_0>0$ where v_0 is the minimum neutron speed. (Thus our work for $v_0>0$ will generalize Jörgens' results.[2]) The differences in the spectrum for these cases is substantial, and we shall briefly discuss these differences in Sec. 6.

Finally, we feel that our methods have an advantage of being systematic. Subsequent generalizations, for example to problems of gas dynamics, should thus be made simpler.

The plan of this paper is as follows. In Sec. 2 we describe the transport operator and its domain, and we introduce various restrictions which we find necessary to impose on the collision terms mentioned above.

Section 3 is devoted to studying the spectrum of the streaming operator T, i.e., the transport operator minus collision terms. The results are embodied in Theorems 1 and 2, which state that $\sigma(T)$ (Ref. 40) consists of a half plane or only the point at infinity, depending respectively upon whether the minimum neutron speed v_0 is or is not zero. In Sec. 4 we study the one-speed transport operator denoted by $T+K_0$; i.e., K_0 is a collision operator which does not change the neutron speed. The analysis of this section depends heavily on the results of Sec. 3 and uses the concept of potentially compact operators[41] and a theorem of Gohberg,[42] sometimes called the "Smul'yan" theorem. The first conclusions (Theorem 3) are that the spectrum of $T+K_0$, for fixed v, is a pure isolated point spectrum of finite geometrical multiplicity restricted to a certain left half-space. (These results are also proved in Ref. 2. The lack of "continuum" spectrum is due to our considering a finite body; for an infinite slab, for example, the one-speed operator does have a "continuum" spectrum.[5]) Next (Theorem 5) we consider the "full" spectrum of $T+K_0$ by considering all admissable values of v. As v varies, the (point) spectrum of $T+K_0$ for fixed v shifts about to form curves; the full spectrum of $T+K_0$ consists of the closure of this set of curves plus $\sigma(T)$. We also prove (Theorem 6) that for $v_0=0$ and a sufficiently small body, all the spectrum is contained in the continuum, a result which has been argued heuristically[7,10] and proved for certain models.[12,13]

All of these results are, of course, of greater or lesser importance depending upon how meaningfully one takes a one-speed model of the transport operator. However, we use these results in Sec. 5, in which the total transport operator $T+K$ is considered and we prove that $\sigma(T+K)$ differs from $\sigma(T+K_0)$ only by the addition of point spectrum. Also, we make some estimates as to the location of this spectrum and we show that the low energy elastic scattering term K_0 can introduce lines of spectra.

Finally, we show, in Theorem 11, that the transport operator is the infinitesimal generator of a strongly continuous semigroup of operators. This theorem, in a sense, justifies this entire paper since it guarantees the existence of a semigroup which solves the initial value problem for the transport operator.

In Sec. 6 we discuss our results and indicate the direction in which future work might be aimed. We conclude with an Appendix to which some of the technical details of the proofs have been relegated.

II. DESCRIPTION OF THE TRANSPORT OEPRATOR

We take D to be an open, bounded, connected set of points $\mathbf{r}$ in three-dimensional configuration space. (If D is not convex then we require that neutrons emitted out of D be absorbed, so that the problem of reentering neutrons does not arise.) We take V to be the three-dimensional velocity space consisting of velocities $\mathbf{v}=v\mathbf{\Omega}$, with $|\mathbf{\Omega}|=1$ and $v_0\leq v\leq v_1<c$, where c is the speed of light. (Since we are dealing with a nonrelativistic equation, we require $v_1\ll c$. In a nuclear reactor, where the maximum neutron energy is about one percent of the neutron rest mass, this condition is certainly fulfilled.)

We define X as the Banach space of complex-valued, measurable functions $\psi(\mathbf{r},\mathbf{v})$, defined on $\bar{D}\times\bar{V}$, satisfying

$$\|\psi\|=\int_{\mathbf{r}\in D}\int_{\mathbf{v}\in V}|\psi(\mathbf{r},\mathbf{v})|\,d\mathbf{r}d\mathbf{v}<\infty.$$

Now we shall describe the transport operator A and its (dense) domain $X_0\subset X$.

We write A as the sum $A=T+K$ where T is the "streaming" operator and K is the "scattering" or "collision" operator.

The operator T is defined by[44]

$$(T\psi)(\mathbf{r},\mathbf{v})=-[\mathbf{v}\cdot\nabla+v\Sigma(\mathbf{r},v)]\psi(\mathbf{r},\mathbf{v}). \tag{2.1}$$

Here the gradient operator ∇ acts only on $\mathbf{r}$ and $v\Sigma(\mathbf{r},v)$ satisfies the following properties:

(a) $v\Sigma(\mathbf{r},v)$ is nonnegative, bounded, and piecewise continuous in $\mathbf{r}$ and v.

(b) If $v_0=0$, then

$$\operatorname*{ess\,inf}_{\mathbf{r}\in D}\lim_{v\to 0}v\Sigma(\mathbf{r},v)\equiv\lambda^* \tag{2.2}$$

exists, and

$$\frac{v\Sigma(\mathbf{r},v)-\lambda^*}{v}\geq -c_0,\quad \mathbf{r}\in\bar{D},\ 0<v\leq v_1, \tag{2.3}$$

where c_0 is a nonnegative constant. [If $v\Sigma(\mathbf{r},v)$ is monotone increasing in v for each $\mathbf{r}$, then c_0 automatically exists and can be taken to be 0.]

(c) For $v_0\geq 0$, we define the constant $\hat{\lambda}(v_0)$ by

$$\hat{\lambda}(v_0)=\operatorname*{ess\,inf}_{\substack{\mathbf{r}\in D\\ v_0\leq v\leq v_1}}v\Sigma(\mathbf{r},v). \tag{2.4}$$

We note that $\hat{\lambda}(0)\leq\lambda^*$, and that equality holds if $v\Sigma(\mathbf{r},v)$ is monotone increasing in v.[45]

Now we define X_0 to be the (dense) subspace of functions ψ such that $\psi(\mathbf{r},\mathbf{v})=0$ for $\mathbf{r}\in\partial D$ and $\mathbf{v}$ pointing into D, and $T\psi\in X$. T is a closed operator on X_0. Since K will be bounded on X, then $A=T+K$ will be a closed operator on X_0.

Next we discuss the scattering operator K. To do so, we write it as the sum

$$K=K_c+K_d+K_0, \tag{2.5}$$

where K_c, K_d, and K_0 are all bounded operators in X.

K_c represents "continuum" scattering and is described by an integral operator:

$$(K_c\psi)(\mathbf{r}, \mathbf{v}) = \int_{\mathbf{v}' \in V} k_c(\mathbf{r}, \mathbf{v}' \to \mathbf{v})\psi(\mathbf{r}, \mathbf{v}')d\mathbf{v}'. \tag{2.6}$$

The kernel k_c satisfies:

(d) k_c is nonnegative and piecewise continuous. Also, k_c is bounded except possibly for the case $v_0 = 0$, in which we allow

$$0 \le k_c(\mathbf{r}, \mathbf{v}' \to \mathbf{v}) \le M_c/v^2. \tag{2.7}$$

This "continuum" scattering corresponds physically to fission, high energy elastic slowing down, and thermal inelastic scattering.

K_d represents high energy inelastic scattering and is described by a "downshift" operator of the form

$$(K_d\psi)(\mathbf{r}, \mathbf{v}) = \sum_{m=1}^{M_0} (K_d^{(m)}\psi)(\mathbf{r}, \mathbf{v}),$$

$$(K_d^{(m)}\psi)(\mathbf{r}, \mathbf{v}) = \int_{\mathbf{v}' \in V} k_d^{(m)}(\mathbf{r}, v', \mathbf{\Omega}' \to \mathbf{\Omega}) \times \delta[v' - \omega_n(v)]\psi(\mathbf{r}, \mathbf{v}')d\mathbf{v}'. \tag{2.8}$$

Here the operator $K_d^{(m)}$ describes an event in which a discrete energy E_m is lost by a neutron at $\mathbf{r}$ with initial speed $\omega_m(v)$ and final speed v. $\omega_m(v)$ is defined by

$$E_m = \tfrac{1}{2}N\omega_m^2(v) - \tfrac{1}{2}Nv^2,$$

where N is the mass of a neutron. The kernels $k_d^{(m)}$ satisfy:

(e) $k_d^{(m)}$ are nonnegative, piecewise continuous, and bounded:

$$0 \le k_d^{(m)}(\mathbf{r}, v', \mathbf{\Omega}' \to \mathbf{\Omega}) \le M_d^{(m)}. \tag{2.9}$$

(f) There exists a threshold speed v_t such that, for all m, $k_d^{(m)}(\mathbf{r}, \mathbf{v}' \to \mathbf{v}) = 0$ for $v' < v_t$. (v_t is the threshold speed below which the high-energy inelastic scattering described by K_d cannot occur.)

It follows from the above description of K_d and from our assumption that neutron speeds are bounded above that K_d is nilpotent, i.e., there exists an integer M_1 such that

$$K_d^{M_1} = 0. \tag{2.10}$$

Physically, this means that after a maximum of $M_1 - 1$ consecutive high energy inelastic collisions, a neutron must have speed below v_t.

Finally, the operator K_0 in (2.5) is a "Bragg" scattering or one-speed operator for low energy neutrons described by

$$(K_0\psi)(\mathbf{r}, \mathbf{v}) = \int_{\mathbf{v}'} k_0(\mathbf{r}, v, \mathbf{\Omega}' \to \mathbf{\Omega})\delta(v' - v)\psi(\mathbf{r}, \mathbf{v}')d\mathbf{v}'. \tag{2.11}$$

The kernel k_0 satisfies:

(g) k_0 is nonnegative, piecewise continuous, and bounded except possibly for the case $v_0 = 0$, in which we allow

$$0 \le k_0(\mathbf{r}, v, \mathbf{\Omega}' \to \mathbf{\Omega}) \le M_0/v^2. \tag{2.12}$$

(h) $k_0(\mathbf{r}, v, \mathbf{\Omega}' \to \mathbf{\Omega}) = 0$ for $v > v_t$.

We note that (f) and (h) imply

$$K_d(\lambda I - T)^{-1}K_0 = 0, \quad \lambda \in \rho(T). \tag{2.13}$$

To end this section, we shall make some comments regarding the above assumptions.

First, the inequalities (2.7), (2.9), and (2.12) imply that $K = K_0 + K_c + K_d$ is a bounded operator. This means physically that for each neutron density ψ, the corresponding total rate of secondary neutron production $K\psi$ is uniformly bounded: $\|K\psi\| \le \|K\|\,\|\psi\|$.

Next, we note that the various kernels and cross sections have been assumed piecewise continuous. Physically, discontinuities in $\mathbf{r}$ correspond to boundaries between regions with different constituents while discontinuities in v correspond to threshold effects, either the Bragg scattering "cutoff" or the inelastic scattering threshold. To prove our results, we shall assume that "piecewise continuous" has one of the following two meanings:

(i) The various kernels and cross sections are continuous in all of their variables. (This corresponds to a body in which the constituents vary continuously with position, and no threshold effects occur in speed.)

(j) $\bar{D} = \cup_{n=1}^{M_2} \bar{D}_n$, where D_n are open sets. In D_n, the various kernels and cross sections are constant functions of $\mathbf{r}$, $k_c(\mathbf{r}, \mathbf{v}' \to \mathbf{v})$ is continuous in $\mathbf{v}'$ and $\mathbf{v}$; and $v\Sigma(\mathbf{r}, v)$, $k_0(\mathbf{r}, v, \mathbf{\Omega}' \to \mathbf{\Omega})$, and $k_d^{(m)}(\mathbf{r}, v, \mathbf{\Omega}' \to \mathbf{\Omega})$ are continuous in $\mathbf{\Omega}'$ and $\mathbf{\Omega}$ and piecewise continuous in v. For mathematical convenience, we take the values of these functions on ∂D_n to be the limiting value from either side of the boundary; we take Σ, k_0, and $k_d^{(m)}$ to be continuous from the right in v for $0 < v < v_1$ and continuous from the left for $v = v_1$; and require that the limits in v from the left exist for $0 < v < v_1$. (This corresponds to a composite body in which each part is homogeneous in position and threshold effects can occur in speed.)

More complicated discontinuities can be handled using our methods. However we shall not consider them here since the geometrical descriptions and proofs become very lengthy. Assumptions (i) and (j) will be explicitly needed only in Theorem 5 and in Lemmas 3 and 4 (Appendix).

III. THE STREAMING OPERATOR AND ITS SPECTRUM SPECTRUM

In this section we shall consider the operator T described in Sec. 2 and determine its spectrum for the two cases $v_0 > 0$ and $v_0 = 0$. First we consider the simpler case $v_0 > 0$.

Theorem 1: If $v_0 > 0$, then $\sigma(T) = \{\infty\}$.

Proof: For any λ we can formally solve the equation $(\lambda I - T)\phi = \psi$ for ϕ to obtain

$$(\lambda I - T)^{-1}\psi(\mathbf{r}, \mathbf{v}) = \frac{1}{v}\int_{t=0}^{d(\mathbf{r},\mathbf{\Omega})} \psi(\mathbf{r} - t\mathbf{\Omega}, \mathbf{v}) \times \exp\left(-\int_0^t \frac{\lambda + v\Sigma(\mathbf{r} - s\mathbf{\Omega}, v)}{v}\, ds\right) dt, \tag{3.1}$$

where $d(\mathbf{r}, \mathbf{\Omega})$ is the distance from $\mathbf{r}$ to ∂D in the direction $-\mathbf{\Omega}$.

For each $v_0 > 0$ and λ, there exists a positive constant $M(\lambda, v_0)$ satisfying

$$\left|\frac{1}{v}\exp\left(-\int_0^t \frac{\lambda+v\Sigma(\mathbf{r}-s\mathbf{\Omega},v)}{v}\,ds\right)\right| \leq M(\lambda,v_0),$$

$$\mathbf{r}\in D,\quad v_0\leq v\leq v_1,$$

$$0\leq t\leq d(\mathbf{r},\mathbf{\Omega}).$$

Then by (3.1),

$$|(\lambda I-T)^{-1}\psi(\mathbf{r},\mathbf{v})|\leq M(\lambda,v_0)\int_{t=0}^{d(\mathbf{r},\mathbf{\Omega})}|\psi(\mathbf{r}-t\mathbf{\Omega},\mathbf{v})|\,dt.$$

Integrating this inequality over $\mathbf{r}$ and $\mathbf{v}$, we obtain

$$\|(\lambda I-T)^{-1}\psi\|\leq lM(\lambda,v_0)\|\psi\|,$$

where l is the length of the longest straight line in D. Thus $\lambda\in\rho(T)$ for every finite λ. Since T is unbounded, then $\sigma(T)$ must consist solely of the point at ∞. QED

Theorem 2: If $v_0=0$, then $\sigma(T)=\{\lambda\,|\,\mathrm{Re}\,\lambda\leq-\lambda^*\}$. Further, for each $\lambda\in\sigma(T)$, there exists a sequence $\{\psi_n\}\subset X_0$ such that $\|\psi_n\|=1$ and $\lim_{n\to\infty}(\lambda I-T)\psi_n=0$.

Proof: For each λ, the operator $(\lambda I-T)^{-1}$ exists on $R(\lambda I-T)$ (Ref. 40) and is given by (3.1). Thus $\lambda\in\rho(T)$ iff $(\lambda I-T)^{-1}$ is a bounded operator defined on X.

First we consider $\mathrm{Re}\lambda>-\lambda^*$. Then

$$\left|\frac{1}{v}\exp\left(-\int_0^t \frac{\lambda+v\Sigma(\mathbf{r}-s\mathbf{\Omega},v)}{v}\,ds\right)\right|$$

$$=\left|\frac{1}{v}\exp\left(-t\,\frac{\lambda+\lambda^*}{v}\right)\exp\left(-\int_0^t\frac{v\Sigma(\mathbf{r}+s\mathbf{\Omega},v)-\lambda^*}{v}ds\right)\right|$$

$$\leq\frac{1}{v}\exp[-t(\mathrm{Re}\lambda+\lambda^*)/v]\exp(lc_0),\qquad(3.2)$$

where λ^* is defined by (2.2), c_0 by (2.3), and l is the length of the longest straight line in D. By (3.1) and (3.2),

$$|(\lambda I-T)^{-1}\psi(\mathbf{r},\mathbf{v})|\leq\exp(lc_0)\int_{t=0}^{d(\mathbf{r},\mathbf{\Omega})}|\psi(\mathbf{r}-t\mathbf{\Omega},\mathbf{v})|\times\frac{\exp[-t(\mathrm{Re}\lambda+\lambda^*)/v]}{v}\,dt.$$

We integrate this inequality along the line L: $\mathbf{r}_0+s\mathbf{\Omega}$, $\mathbf{r}_0\in\partial D$, $0\leq s\leq d(\mathbf{r}_0,-\mathbf{\Omega})$, to obtain

$$\int_0^{d(\mathbf{r}_0,-\mathbf{\Omega})}|(\lambda I-T)^{-1}\psi(\mathbf{r}_0+s\mathbf{\Omega},\mathbf{v})|\,ds$$

$$\leq\frac{\exp(lc_0)}{\mathrm{Re}\lambda+\lambda^*}\int_0^{d(\mathbf{r}_0,-\mathbf{\Omega})}\times|\psi(\mathbf{r}_0+t\mathbf{\Omega},\mathbf{v})|\,dt.$$

Now we integrate over the remaining two space directions and $\mathbf{v}$ to get

$$\|(\lambda I-T)^{-1}\psi\|\leq\frac{\exp(lc_0)}{\mathrm{Re}\lambda+\lambda^*}\|\psi\|.$$

Thus $\lambda\in\rho(T)$, and

$$\|(\lambda I-T)^{-1}\|\leq\frac{\exp(lc_0)}{\mathrm{Re}\lambda+\lambda^*},\quad \mathrm{Re}\lambda>-\lambda^*.\qquad(3.3)$$

Next we consider $\mathrm{Re}\lambda<-\lambda^*$. Then there exists a point $\mathbf{r}_0\in D$ and positive numbers ϵ_0, ϵ_1, and ϵ_2 such that

$$\mathrm{Re}\lambda+v\Sigma(\mathbf{r},v)<-\epsilon_0$$

for

$$|\mathbf{r}-\mathbf{r}_0|<\epsilon_1\ \text{ and }\ 0\leq v\leq\epsilon_2.$$

For n such that $0<1/n<\epsilon_2/2$, we define

$$\phi_n(\mathbf{r},\mathbf{v})=\left(\frac{3}{4\pi}\right)^2\frac{n^3}{7\epsilon_1^3}H_n(\mathbf{r},v)\exp\left(-i\,\frac{\mathrm{Im}\lambda}{v}\,d(\mathbf{r},\mathbf{\Omega})\right)$$

where Im denotes "imaginary part" and

$$H_n(\mathbf{r},v)=\begin{cases}1, & |\mathbf{r}-\mathbf{r}_0|<\epsilon_1\ \text{ and }\ 1/n<v<2/n\\ 0, & \text{otherwise.}\end{cases}$$

Then $\phi_n\in X$ and $\|\phi_n\|=1$. From (3.1) we can easily verify that $(\lambda I-T)^{-1}\phi_n\in X$; also, we obtain the estimate

$$|(\lambda I-T)^{-1}\phi_n(\mathbf{r},\mathbf{v})|\geq\left(\frac{3}{4\pi}\right)^2\frac{n^3}{7\epsilon_1^3}\,\frac{1}{\epsilon_0}\left[\exp\left(\frac{\epsilon_0\epsilon_1}{4}\,n\right)-1\right]$$

$$\text{for }\frac{1}{n}<v<\frac{2}{n},\quad|\mathbf{r}-\mathbf{r}_0|<\epsilon_1/2.$$

Therefore,

$$\|(\lambda I-T)^{-1}\phi_n\|\geq\frac{1}{8\epsilon_0}\left[\exp\left(\frac{\epsilon_0\epsilon_1}{4}n\right)-1\right].$$

Since this becomes unbounded as $n\to\infty$, then $\lambda\in\sigma(T)$. Thus $\{\lambda\,|\,\mathrm{Re}\lambda<-\lambda^*\}\subset\sigma(T)$, and since the spectrum is a closed set, then $\sigma(T)$ is as described in the statement of the theorem.

Next we let $\mathrm{Re}\lambda<-\lambda^*$ and take ϕ_n to be as defined above. We define ψ_n by

$$\psi_n=(\lambda I-T)^{-1}\phi_n/\|(\lambda I-T)^{-1}\phi_n\|.$$

Then $\|\psi_n\|=1$ and $\lim_{n\to\infty}(\lambda I-T)\psi_n=0$. For $\mathrm{Re}\lambda=-\lambda^*$, there exist sequences $\{\lambda_n\}$ with $\mathrm{Re}\lambda_n<-\lambda^*$ and $\lambda_n\to\lambda$, and $\{\psi_{n,m}\}$ with $\|\psi_{n,m}\|=1$ and $(\lambda_nI-T)\psi_{n,m}\to0$ as $m\to\infty$. We can thus construct a sequence $\psi_n=\psi_{n,m_n}$ such that $(\lambda_nI-T)\psi_n\to0$. Then ψ_n satisfies

$$\lim_{n\to\infty}(\lambda I-T)\psi_n=\lim_{n\to\infty}[(\lambda-\lambda_n)\psi_n+(\lambda_nI-T)\psi_n]=0.$$

This proves the second half of the theorem. QED

Thus the finite spectrum exists and is a half-space only if neutrons can exist with arbitrarily small speeds. Also the description of $\sigma(T)$ (i.e., of λ^*) depends only on the limiting value of $v\Sigma(\mathbf{r},v)$ as $v\to0$ and is insensitive to discontinuities or nonmonotonicity of $v\Sigma(\mathbf{r},v)$.

Finally, we add that for $v_0>0$, the calculations leading to (3.3) can be modified to yield the useful inequality

$$\|(\lambda I-T)^{-1}\|\leq\frac{1}{\mathrm{Re}\lambda+\hat{\lambda}(v_0)},\qquad\mathrm{Re}\lambda>-\hat{\lambda}(v_0),\qquad(3.4)$$

where $\hat{\lambda}(v_0)$ is defined by (2.4).

IV. THE ONE SPEED OPERATOR $T+K_0$ AND ITS SPECTRUM

In this section we shall consider the operator $T+K_0$ and determine the basic properties of its spectrum. $T+K_0$ is a one-speed operator in the sense that it commutes with functions of v alone. Thus we define an auxiliary Banach space X^0 as follows. We let S be the unit sphere in V and define X^0 as the set of all complex-valued functions $\psi(\mathbf{r},\mathbf{\Omega})$ defined on $\bar{D}\times S$, satisfying

$$\|\psi\|^0=\int_{\mathbf{r}\in D}\int_{|\mathbf{\Omega}|=1}|\psi(\mathbf{r},\mathbf{\Omega})|\,d\mathbf{r}d\mathbf{\Omega}.$$

Then the operators $T=T(v)$, $K_0=K_0(v)$, and $T+K_0=A(v)$ (which, as we have indicated, depend parametrically upon v) are defined on the subspace $X_0^0\subset X^0$ of functions such that $T(v)\psi \in X^0$ and $\psi(\mathbf{r}, \mathbf{\Omega}) = 0$ for $\mathbf{r} \in \partial D$ and $\mathbf{\Omega}$ pointing into D. Clearly, X_0^0 is independent of v.

To proceed we need the following theorem, which is due to Gohberg.[42]

Theorem (Gohberg): Let $L(\lambda)$ be an operator-valued function, holomorphic in an open connected set G, and compact for $\lambda \in G$. Then for all points $\lambda \in G$, with the possible exception of certain isolated points, the number $\alpha(\lambda)$ of linearly independent solutions of the equation

$$[I - L(\lambda)]\phi = 0$$

is constant:

$$\alpha(\lambda) = n;$$

at the isolated points mentioned,

$$\alpha(\lambda) > n.$$

Henceforth we shall denote all quantities pertaining to X^0 by a "zero" superscript. We can now prove

Theorem 3: Let $\tilde{\lambda}(v) = \inf_{\mathbf{r}} v\Sigma(\mathbf{r}, v)$. Then $\{\lambda \mid \mathrm{Re}\lambda > -\tilde{\lambda}(v) + \|K_0(v)\|^0\} \subset \rho^0[A(v)]$. Also, $\sigma^0[A(v)]$ consists entirely of isolated eigenvalues of finite geometrical multiplicity.

Proof: The proof of Theorem 1 can be modified to show that $\sigma^0[T(v)] = \{\infty\}$. Also, the calculations leading to (3.3) can be modified to give

$$\|[\lambda I - T(v)]^{-1}\|^0 \leq \frac{1}{\mathrm{Re}\lambda + \tilde{\lambda}(v)} \quad \text{for } \mathrm{Re}\lambda > -\tilde{\lambda}(v),$$

and thus

$$\|[\lambda I - T(v)]^{-1}K_0(v)\|^0 \leq \frac{\|K_0(v)\|^0}{\mathrm{Re}\lambda + \tilde{\lambda}(v)} < 1 \quad \text{for } \mathrm{Re}\lambda > -\tilde{\lambda}(v) + \|K_0(v)\|^0. \tag{4.1}$$

For such λ, $\lambda I - A(v) = [\lambda I - T(v)]\{I - [\lambda I - T(v)]^{-1}K_0(v)\}$ is invertible, yielding

$$\|[\lambda I - A(v)]^{-1}\| \leq \frac{1}{\mathrm{Re}\lambda + \tilde{\lambda}(v) - \|K_0(v)\|^0}, \quad \mathrm{Re}\lambda > -\tilde{\lambda}(v) + \|K_0(v)\|^0. \tag{4.2}$$

This proves the first half of the theorem.

To prove the remainder of the Theorem, we shall consider the operator $Q(\lambda, v) \equiv K_0(v)[\lambda I - T(v)]^{-1}K_0(v)$. In the Appendix, we shall prove that Q is a compact operator on X^0. (See Lemma 3.) Therefore, $Q(\lambda, v)[\lambda I - T(v)]^{-1} = \{K_0(v)[\lambda I - T(v)]^{-1}\}^2$ is a compact, operator-valued function of λ which is holomorphic in the entire complex plane. Also, by (4.1), $I - \{K_0(v)[\lambda I - T(v)]^{-1}\}^2$ is invertible for $\mathrm{Re}\lambda > -\tilde{\lambda}(v) + \|K_0(v)\|^0$. Thus by Gohberg's Theorem there exists at most a set of isolated values of λ in the complex plane such that $1 \in P\sigma^0\{[K_0(v)[\lambda I - T(v)]^{-1}]^2\}$, and at these points 1 is an eigenvalue of finite geometrical multiplicity. At all other values of λ, $1 \in \rho^0\{[K_0(v)[\lambda I - T(v)]^{-1}]^2\}$.

Now since $\{K_0(v)[\lambda I - T(v)]^{-1}\}^2$ is compact, then $K_0(v)[\lambda I - T(v)]^{-1}$ is potentially compact[41] and its spectrum, except possibly for the point 0, consists entirely of point spectrum.

Thus by the spectral mapping theorem, only for the above set of isolated values of λ can we have $1 \in P_\sigma^0\{K_0(v)[\lambda I - T(v)]^{-1}\}$. At such a λ value the equation $0 = \{I - K_0(v)[\lambda I - T(v)]^{-1}\}\phi = [\lambda I - A(v)][\lambda I - T(v)]^{-1}\phi$ has a finite number of solutions. Consequently, λ is an eigenvalue of $A(v)$ of finite geometrical multiplicity. At all other values of λ, $1 \in \rho^0\{K_0(v)[\lambda I - T(v)]^{-1}\}$ and for such λ the operator

$$I - K_0(v)[\lambda I - T(v)]^{-1} = [\lambda I - A(v)][\lambda I - T(v)]^{-1}$$

has a bounded inverse defined on X^0. Taking this inverse, we find

$$[\lambda I - A(v)]^{-1} = [\lambda I - T(v)]^{-1}\{I - K_0(v)[\lambda I - T(v)]^{-1}\}^{-1},$$

and consequently $\lambda \in \rho^0[A(v)]$. This proves the theorem.

QED

The next theorem is based on a result of Vidav[22]:

Theorem 4: Let $\tilde{\lambda}_0(v) \equiv \sup_{\lambda \in \sigma^0[A(v)]} \mathrm{Re}\lambda$, and let $\tilde{\lambda}(v) < \tilde{\lambda}_0(v)$. Then $\tilde{\lambda}_0(v)$ is an eigenvalue of $A(v)$, corresponding to which is a positive eigenfunction.

Proof: A simple modification of the proof of Theorem 3, Ref. 22 yields the result. QED

Thus, the picture of $\sigma^0[A(v)]$ which emerges can be graphically described by Fig. 1. In the Appendix, we show that $\|K_0(v)\|^0 = \sup_{\mathbf{r}} v\Sigma_0(\mathbf{r}, v)$, where $\Sigma_0(\mathbf{r}, v)$ is the cross section for low energy elastic collisions at speed v. Thus there is no spectrum to the right of the line

$$\mathrm{Re}\lambda = v[\sup_{\mathbf{r}} \Sigma_0(\mathbf{r}, v) - \inf_{\mathbf{r}} \Sigma(\mathbf{r}, v)].$$

We note that the above results hold for v fixed; consequently, the eigenvalues depend parametrically upon v. In general, as v varies between v_0 and v_1, some of the eigenvalues will remain stationary and some will shift and trace out curves. Thus, the set

$$\mathcal{S} \equiv \bigcup_{v_0 \leq v \leq v_1} \sigma^0[A(v)] \tag{4.3}$$

will consist of isolated points and curved lines. Using $\mathcal{S}$, we have:

Theorem 5: Let $\mathcal{S}$ be defined by (4.3). Then, for $v_0 \geq 0$,

$$\sigma(T + K_0) = \sigma(T) \cup \overline{\mathcal{S}} \tag{4.4}$$

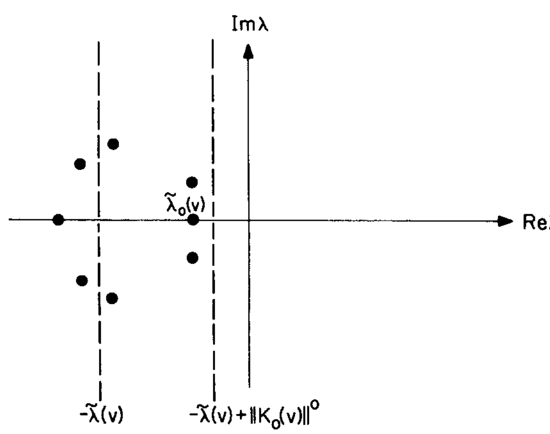

FIG. 1. $\sigma^0[A(v)]$, $\tilde{\lambda}(v) < \tilde{\lambda}_0(v)$.

and

$$\{\lambda \mid \mathrm{Re}\lambda > -\hat{\lambda}(v_0) + \|K_0\|\} \subset \rho(T+K_0). \tag{4.5}$$

Furthermore, for each $\lambda \in \sigma(T+K_0)$, there exists a sequence $\{\psi_n\} \subset X$ such that $\|\psi_n\| = 1$, $\psi_n(\mathbf{r}, \mathbf{v}) = 0$ for $v > v_t$, and $\|(\lambda I - T - K_0)\psi_n\| \to 0$.

Proof: First we show that $\sigma(T) \cup \mathcal{S} \subset \sigma(T+K_0)$. To do this, let $\lambda \in \sigma(T)$. Then by Theorem 2, there exists a sequence $\{\phi_n\}$ with $\|\phi_n\| = 1$ such that $(\lambda I - T)\phi_n \to 0$. By Lemma 1 (see the Appendix) there exists a sequence of integers M_n such that, with

$$\psi_n(\mathbf{r}, \mathbf{v}) = \exp(iM_n \mathbf{\Omega} \cdot \mathbf{\Omega}_0)\, \phi_n(\mathbf{r}, \mathbf{v}), \tag{4.6}$$

we have

$$\|K_0 \psi_n\| \to 0.$$

Thus,

$$\|(\lambda I - T - K_0)\psi_n\| \le \|(\lambda I - T)\phi_n\| + \|K_0\psi_n\| \to 0,$$

and so if $(\lambda I - T - K_0)^{-1}$ exists, it must be unbounded. Consequently, $\lambda \in \sigma(T+K_0)$, and so $\sigma(T) \subset \sigma(T+K_0)$.

Next, we let $\lambda \in \mathcal{S}$. Then by Theorem 3 there exists a function $\tilde{\phi}(\mathbf{r}, \mathbf{\Omega})$ such that $\|\tilde{\phi}\|^0 = 1$ and $(\lambda I - T - K_0)\tilde{\phi}(\mathbf{r}, \mathbf{\Omega}) = 0$ for $v = \tilde{v}$. If $\tilde{v} < v_1$, we define the sequence ψ_n by

$$\psi_n(\mathbf{r}, \mathbf{v}) = \begin{cases} n/v^2 & \phi(\mathbf{r}, \mathbf{\Omega}), \quad \tilde{v} \le \tilde{v} \le v + 1/n \\ 0, & \text{otherwise.} \end{cases} \tag{4.7}$$

Then $\|\psi_n\| = 1$. For $\tilde{v} \le v \le \tilde{v} + 1/n$,

$$\begin{aligned}(\lambda I - T - K_0)\psi_n(\mathbf{r}, \mathbf{v}) &= n\, \frac{v\Sigma(\mathbf{r}, v) - \tilde{v}\Sigma(\mathbf{r}, \tilde{v})}{v^2}\, \phi(\mathbf{r}, \mathbf{\Omega}) \\ &\quad + n\, \frac{v - \tilde{v}}{v^2}\, \mathbf{\Omega} \cdot \nabla \phi(\mathbf{r}, \mathbf{\Omega}) \\ &\quad - \frac{n}{v^2} \int_{|\mathbf{\Omega}'|=1} [k_0(\mathbf{r}, v, \mathbf{\Omega}' \to \mathbf{\Omega}) \\ &\quad - k_0(\mathbf{r}, \tilde{v}, \mathbf{\Omega}' \to \mathbf{\Omega})]\phi(\mathbf{r}, \mathbf{\Omega}')\, d\mathbf{\Omega}',\end{aligned}$$

while for $v < \tilde{v}$ or $v > \tilde{v} + 1/n$, $(\lambda I - T - K_0)\psi_n = 0$. Thus, integrating over $\mathbf{r}$ and $\mathbf{v}$, we obtain

$$\begin{aligned}\|(\lambda I - T - K_0)\psi_n\| &\le n \int_{v=\tilde{v}}^{\tilde{v}+1/n} |v\Sigma(\mathbf{r}, v) - \tilde{v}\Sigma(\mathbf{r}, \tilde{v})|\, dv \\ &\quad + \frac{1}{2n} \int_{\mathbf{r}\in D} \int_{|\mathbf{\Omega}|=1} |\mathbf{\Omega} \cdot \nabla \phi(\mathbf{r}, \mathbf{\Omega})|\, d\mathbf{\Omega}\, d\mathbf{r} \\ &\quad + n \int_{v=\tilde{v}}^{\tilde{v}+1/n} \int_{\mathbf{r}\in D} \int_{|\mathbf{\Omega}|=1} \int_{|\mathbf{\Omega}'|=1} \\ &\quad \times |k_0(\mathbf{r}, v, \mathbf{\Omega}' \to \mathbf{\Omega}) - k_0(\mathbf{r}, \tilde{v}, \mathbf{\Omega}' \to \mathbf{\Omega})| \\ &\quad \times |\phi(\mathbf{r}, \mathbf{\Omega}')|\, d\mathbf{\Omega}'\, d\mathbf{\Omega}\, d\mathbf{r}\, dv.\end{aligned}$$

Since $v\Sigma(\mathbf{r}, v)$ and $k_0(\mathbf{r}, v, \mathbf{\Omega}' \to \mathbf{\Omega})$ are continuous from the right in v (see Sec. 2), then each of the above integrals will tend to zero as $n \to \infty$. Consequently, $\|(\lambda I - T - K_0)\psi_n\| \to 0$, and so $\lambda \in \sigma(T+K_0)$. This result holds if $\tilde{v} < v_1$. If $\tilde{v} = v_1$, we define the sequence ψ_n as in (4.7), except that we take ψ_n to be nonzero over the interval $v_1 - 1/n \le v \le v_1$. Then since $v\Sigma(\mathbf{r}, v)$ and k_0 are continuous from the left at v_1, the above procedure will apply. Thus we have $\mathcal{S} \subset \sigma(T+K)$.

Hence $\sigma(T) \cup \bar{\mathcal{S}} \subset \sigma(T+K_0)$, since the spectrum is closed. To prove inclusion the other way, we shall consider the equivalent inclusion

$$C[\sigma(T) \cup \bar{\mathcal{S}}] \subset \rho(T+K_0), \tag{4.8}$$

where C means "complement." Thus, we let $\lambda \in C[\sigma(T) \cup \bar{\mathcal{S}}]$. Then the equation

$$(\lambda I - T - K_0)\phi(\mathbf{r}, \mathbf{v}) = \psi(\mathbf{r}, \mathbf{v})$$

has a solution ϕ such that $\|\phi(\mathbf{r}, \mathbf{v})\|^0 \le \mathrm{const}\|\psi(\mathbf{r}, \mathbf{v})\|^0$ for $v_0 \le v \le v_1$, even if $v_0 = 0$. Consequently, $\|\phi\| = \|(\lambda I - T - K_0)^{-1}\psi\| < \infty$ for each $\psi \in X$. By the results in Sec. 2, T is closed and K is bounded, so $T + K_0$ is closed in X. Consequently, $(\lambda I - T - K_0)^{-1}$ is closed, and so by the closed graph theorem is bounded. Thus $\lambda \in \rho(T+K_0)$. This proves (4.8), and also (4.4).

Next we use (3.4) and we repeat the same arguments which led to (4.2) to obtain

$$\|(\lambda I - T - K_0)^{-1}\| \le \frac{1}{\mathrm{Re}\lambda + \hat{\lambda}(v_0) - \|K_0\|},$$

$$\mathrm{Re}\lambda > -\hat{\lambda}(v_0) + \|K_0\|. \tag{4.9}$$

This proves (4.5).

Finally, we let $\lambda \in \sigma(T+K_0)$. (With no change in the spectrum, we can decrease v_1 so that $v_1 = v_t$.) If $\lambda \in \sigma(T)$, then the sequence described by (4.6) satisfies $\|\psi_n\| = 1$ and $\|(\lambda I - T - K_0)\psi_n\| \to 0$. If $\lambda \in \mathcal{S}$ then the sequence described by (4.7) satisfies these conditions. If $\lambda \in \bar{\mathcal{S}}$, then there exist sequences $\{\psi_n\} \subset \mathcal{S}$ with $\lambda_n \to \lambda$ and $\{\psi_{n,m}\}$ with $\|\psi_{n,m}\| = 1$ and $\|(\lambda_n I - T - K_0)\psi_{n,m}\| \to 0$ as $m \to \infty$. We can thus construct a sequence $\psi_n \equiv \psi_{n,m_n}$ such that $(\lambda_n I - T - K_0)\psi_n \to 0$. Then

$$(\lambda I - T - K_0)\psi_n = (\lambda - \lambda_n)\psi_n + (\lambda_n I - T - K_0)\psi_n \to 0.$$

These results hold for $v_1 = v_t$. For $v_1 > v_t$, we extend the above functions ψ_n by defining them to be zero for $v_1 \ge v > v_t$; this yields a sequence which satisfies all the conditions of the theorem.

This completes the proof of Theorem 5. QED

We remark that $\sigma(T+K_0)$ is described graphically by Fig. 2 for $v_0 > 0$ and by Fig. 3 for $v_0 = 0$. Also, as in the case of $\sigma^0[A(v)]$, $\|K_0\| = \sup_{\mathbf{r},v} v\Sigma_0(\mathbf{r}, v)$ (see Appendix). Thus there can exist no spectrum to the right of the line $\mathrm{Re}\lambda = \sup_{\mathbf{r},v} v\Sigma_0(\mathbf{r}, v) - \inf_{\mathbf{r},v} v\Sigma(\mathbf{r}, v)$. Note that the de-

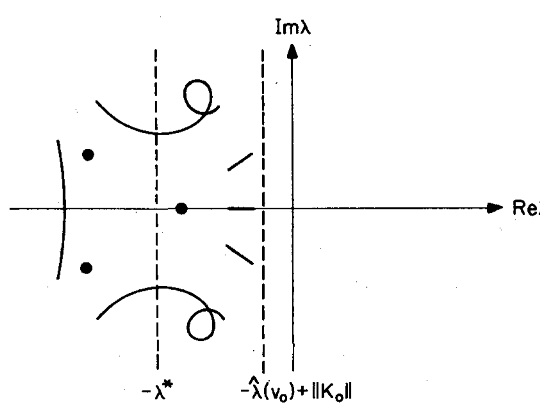

FIG. 2. $\sigma(T+K_0)$, $v_0 > 0$.

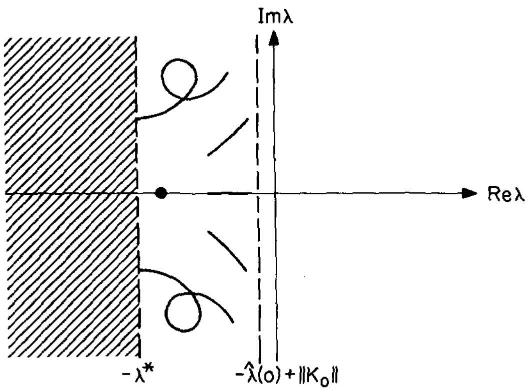

FIG. 3. $\sigma(T+K_0)$, $v_0=0$.

finition of $\Sigma_0(\mathbf{r}, v)$ is such that it may actually exceed $\Sigma(\mathbf{r}, v)$. In fact, in the extreme one-speed case,[44] $v\Sigma(\mathbf{r}, v)=1$ and $v\Sigma_0(\mathbf{r}, v)=c$.

Finally, let us consider $v_0=0$. We note that K_0 is a bounded operator on X but that the operator $v^{-1}K_0$ is in general unbounded. However, if k_0 satisfies an inequality of the form $k_0(\mathbf{r}, v, \mathbf{\Omega}' \to \mathbf{\Omega}) \le M_0/v$ [instead of (2.12)], then $v^{-1}K_0$ will be bounded. For cases of this type, we can prove the following theorem:

Theorem 6: Let $v_0=0$ and $v^{-1}K_0$ be a bounded operator on X. If the maximum diameter l of D satisfies

$$l\exp(lc_0)\|v^{-1}K_0\| < 1, \tag{4.10}$$

then

$$\{\lambda \mid \mathrm{Re}\lambda > -\lambda^*\} \subset \rho(T+K_0). \tag{4.11}$$

Proof: Let (4.10) be satisfied and let $\mathrm{Re}\lambda > -\lambda^*$. Then by (3.1) and (3.2) we have

$$\begin{aligned}|(\lambda I - T)^{-1}K_0\psi(\mathbf{r}, \mathbf{v})| &\le \int_{t=0}^{d(\mathbf{r},\mathbf{\Omega})} |(v^{-1}K_0)\psi(\mathbf{r}-t\mathbf{\Omega}, \mathbf{v})| \\ &\quad\times \exp\left(-t\,\frac{\mathrm{Re}\lambda+\lambda^*}{v}\right)\exp(lc_0)\,dt \\ &\le \exp(lc_0)\int_{t=0}^{d(\mathbf{r},\mathbf{\Omega})} |(v^{-1}K_0)\psi(\mathbf{r}-t\mathbf{\Omega}, \mathbf{v})|\,dt.\end{aligned}$$

Integrating over $\mathbf{r}$ and $\mathbf{v}$, we obtain

$$\|(\lambda I - T)^{-1}K_0\| \le l\exp(lc_0)\|v^{-1}K_0\| < 1.$$

Therefore, the operators

$$(\lambda I - T)^{-1}(\lambda I - T - K_0) \subset I - (\lambda I - T)^{-1}K_0$$

have bounded inverses defined on X, and

$$(\lambda I - T - K_0)^{-1} = [I - (\lambda I - T)^{-1}K_0]^{-1}(\lambda I - T)^{-1}$$

is bounded, proving that $\lambda \in \rho(T+K_0)$. This verifies the inclusion (4.11). QED

Thus if the hypotheses of Theorem 6 are satisfied, then all the spectrum of A is imbedded in the "continuum" $\mathrm{Re}\lambda \le -\lambda^*$. We shall comment on this in Sec. 6.

V. THE TRANSPORT OPERATOR $T+K$ AND ITS SPECTRUM

In this section, we shall prove that $\sigma(T+K)$ differs from $\sigma(T+K_0)$ only by the addition of point spectrum, and we shall give estimates on the location of this spectrum. If $\rho(T+K_0)$ is a connected set, then the added spectrum consists of isolated, discrete points of finite geometrical multiplicity. If $\rho(T+K_0)$ is not a connected set, as in Fig. 2 and 3, then certain connected components of $\rho(T+K_0)$ can become wholly or partially filled with point spectrum. We shall prove these results in the following theorems.

Theorem 7: $\sigma(T+K_0) \subset \sigma(T+K)$.

Proof: Let $\lambda \in \sigma(T+K_0)$. Then by Theorem 4, there exists a sequence $\{\phi_n\}$ such that $\|\phi_n\|=1$, $\phi_n(\mathbf{r}, \mathbf{v})=0$ for $v > v_t$, and $(\lambda I - T - K_0)\phi_n \to 0$. By Lemma 2 (Appendix), there exists a sequence of integers $\{M_n\}$ such that, with

$$\psi_n(\mathbf{r}, \mathbf{v}) \equiv \phi_n(\mathbf{r}, \mathbf{v})\exp(iM_n v),$$

we have $\|\psi_n\|=1$ and $K_c\psi_n \to 0$. Furthermore, by condition (f) of Sec. 2, $K_d\psi_n=0$. Therefore, by (2.5),

$$\|(\lambda I - T - K)\psi_n\| \le \|(\lambda I - T - K_0)\phi_n\| + \|K_c\psi_n\| \to 0.$$

This proves the theorem. QED

To state the next theorem, we write $\rho(T+K_0)$ as the union of its connected components:

$$\rho(T+K_0) = \bigcup_{-\alpha \le n \le \alpha} S_n,$$

where α is a nonnegative integer or ∞. Each S_n is a connected, open set and is the reflection of S_{-n} across the $\mathrm{Re}\lambda$ axis. S_0 is the "largest" of these sets and contains the right half plane $\mathrm{Re}\lambda > -\hat{\lambda}(v_0) + \|K\|$. (See Theorem 9.) Figures 2 and 3 illustrate this situation. If the lines generated by $\sigma(T+K_0)$ do not form closed loops, then $\rho(T+K_0)$ is connected and is equal to S_0.

Now we can state the theorem.

Theorem 8: The set $\sigma(T+K) \cap \rho(T+K_0)$ is described by:

(i) $[\sigma(T+K) \cap \rho(T+K_0)] \subset P\sigma(T+K)$.

(ii) $\sigma(T+K) \cap S_n$ consists of eigenvalues of finite geometrical multiplicity.

(iii) $\sigma(T+K) \cap S_0$ consists of isolated points.

Proof: We define $K_1 \equiv K_c + K_d$. Then the operator $Q(\lambda) \equiv K_1(\lambda I - T - K_0)^{-1}$ is a holomorphic, operator-valued function of λ in S_n. Simple algebraic manipulations allow us to rewrite $Q(\lambda)$ in the form

$$Q(\lambda) = K_1(\lambda I - T)^{-1} + K_1(\lambda I - T)^{-1}K_0(\lambda I - T - K_0)^{-1}. \tag{5.1}$$

In the Appendix we shall show that for $\lambda \in \rho(T)$, $K_1(\lambda I - T)^{-1}K_0$ and $[K_1(\lambda I - T)^{-1}]^{M_1+1}$ are compact, where M_1 satisfies (2.10). Then $Q^{M_1+1}(\lambda)$ is a holomorphic, compact operator-valued function of λ in S_n. It follows from Gohberg's theorem that either $1 \in P\sigma[Q^{M_1+1}(\lambda)]$ for all $\lambda \in S_n$, or there exist at most isolated values of λ for which $1 \in P\sigma[Q^{M_1+1}(\lambda)]$ and $1 \in \rho[Q^{M_1+1}(\lambda)]$ for all other values of $\lambda \in S$. In either case the eigenvalue 1 has finite geometrical multiplicity.

Since $Q^{M_1+1}(\lambda)$ is compact, then $Q(\lambda)$ is potentially compact and its spectrum, except possibly for the point 0, consists entirely of point spectrum. Thus by the

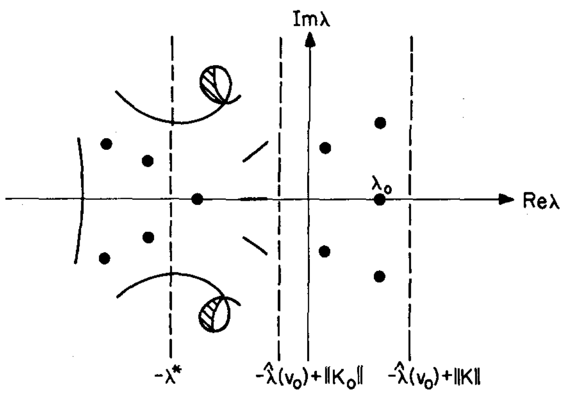

FIG. 4. $\sigma(T+K)$, $v_0>0$, $-\hat{\lambda}(v_0)<\sup\tilde{\lambda}_0(v)<\lambda_0$.

spectral mapping theorem, we must have for $\lambda\in S_n$ either $1\in\rho[Q(\lambda)]$ or $1\in P\sigma[Q(\lambda)]$. If 1 is in the point spectrum, it has finite geometrical multiplicity.

If $1\in\rho[Q(\lambda)]$, then

$$I-Q(\lambda)=(\lambda I-T-K)(\lambda I-T-K_0)^{-1}$$

has a bounded inverse defined on X, and hence

$$(\lambda I-T-K)^{-1}=(\lambda I-T-K_0)^{-1}[I-Q(\lambda)]^{-1}.$$

Thus $\lambda\in\rho(T+K)$.

If $1\in P\sigma[Q(\lambda)]$, then the equation

$$0=[I-Q(\lambda)]\phi=(\lambda I-T-K)[(\lambda I-T-K_0)^{-1}\phi]$$

has a finite number of solutions. Hence $\lambda\in P\sigma(T+K)$ and the geometrical multiplicity of λ is finite. This proves claims (i) and (ii).

To prove claim (iii), we note from (5.1) and (4.9) that $Q(\lambda)\to 0$ as $\mathrm{Re}\lambda\to\infty$. Hence by Gohberg's theorem there exist at most isolated values of $\lambda\in S_0$ for which $1\in P\sigma[Q^{M_1+1}(\lambda)]$. By the spectral mapping theorem, only for these λ values can $1\in P\sigma[Q(\lambda)]$, and as we showed above, only such λ can be in $\sigma(T+K)$. QED

The next theorem provides estimates on the location of $\sigma(T+K)$.

Theorm 9: $\{\lambda\mid\mathrm{Re}\lambda>-\hat{\lambda}(v_0)+\|K\|\}\subset\rho(T+K)$. Also, if $v_0=0$, if $v^{-1}K$ is a bounded operator on X, and if the maximum diameter l of D satisfies

$$l\exp(lc_0)\|v^{-1}K\|<1, \tag{5.3}$$

then

$$\{\lambda\mid\mathrm{Re}\lambda>-\lambda^*\}\subset\rho(T+K).$$

Proof: Using (3.4) and repeating the argument which led to (4.2), we obtain

$$\|(\lambda I-T-K)^{-1}\|\leq\frac{1}{\mathrm{Re}\lambda+\hat{\lambda}(v_0)-\|K\|},\quad \mathrm{Re}\lambda>-\hat{\lambda}(v_0)+\|K\|. \tag{5.4}$$

This proves the first part of the theorem. To prove the second part, we simply repeat the proof of Theorem 6 with K replacing K_0. QED

The next result generalizes Theorem 4, from which we borrow some notation.

Theorem 10: Let $\lambda_0\equiv\sup_{\lambda\in\sigma(A)}\mathrm{Re}\lambda$. If $\lambda_0>\sup_v\tilde{\lambda}_0(v)$ and $\lambda_0>-\hat{\lambda}(v_0)$ then λ_0 is an eigenvalue of A, corresponding to which is a positive eigenfunction.

Proof: As in Theorem 4, we refer to the proof of Theorem 3 in Ref. 22. QED

Thus, $\sigma(T+K)$ is described by Fig. 4 for $v_0>0$ and by Fig. 5 for $v_0=0$. These figures differ from Figs. 2 and 3 respectively only by the addition of point spectrum. We note that if "loops" exist, as shown in Figs. 2 and 3, then we cannot exclude the possibility that their interiors (the sets S_n with $n=0$) become partially or wholly filled with point spectrum. Also, $\|K\|$ is given in the Appendix and as in the cases of $\sigma^0[A(v)]$ and $\sigma(T+K_0)$, we deduce an absolute limit to the real part of $(T+K)$ as $\sup_{\mathbf{r},v}v\Sigma_s(\mathbf{r},v)-\inf_{\mathbf{r},v}v\Sigma(\mathbf{r},v)$.

We conclude the main body of this paper with the following result:

Theorem 11: The transport operator $A=T+K$ is the infinitesimal generator of a strongly continuous semigroup of operators.

Proof: By the results in Sec. 2, T is a closed, densely defined operator and K is bounded. Thus $T+K$ is a closed, densely defined operator satisfying (5.4), and so the conditions of the Hille—Yosida—Phillips theorem[46] are met. QED

It follows that the semigroup $\Upsilon(t)=\exp(At)$ exists and enables us to solve the initial value problem

$$\frac{\partial\psi}{\partial t}=A\psi$$

$$\psi|_{t=0}=\psi_0.$$

The behavior of the solution of this problem thus depends on the location and classification of $\sigma(A)$. In this paper we have described many of the basic properties of this spectrum for arbitrary bounded domains and, we hope, realistic and general transport operators.

VI. DISCUSSION

If we refer to the results of Sec. 5, as exhibited schematically in Figs. 4 and 5, we see that the spectrum of $T+K$ can have a rather complicated structure. This is due in part to the curves and loops which $\sigma(T+K)$ inherits from $\sigma(T+K_0)$ (Figs. 2 and 3). At present,

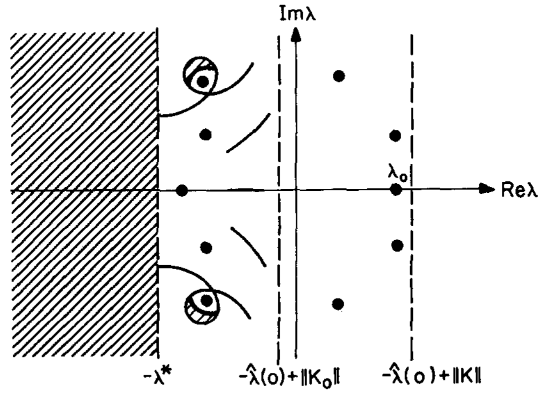

FIG. 5. $\sigma(T+K)$, $v_0=0$, $-\hat{\lambda}(0)<\sup\tilde{\lambda}_0(v)<\lambda_0$.

little is known about the spectrum of this one-speed operator. However, for a restrictive case (isotropic scattering, spatial independence of all cross sections and kernels, and analysis in L_2) it is known that the portion of the spectrum of one-speed operators which lies to the right of the line $\mathrm{Re}\lambda = -v\Sigma(v)$ is real [where $\Sigma(v)$ is the total cross section]. If this is true in general, then the curves in Figs. 2 through 5 to the right of the line $\mathrm{Re}\lambda = -\hat{\lambda}(v_0)$ will consist of line segments on the real axis, and the spectral picture for $T+K$ will simplify considerably.

Another question of obvious importance concerns the existence of a simple dominant eigenvalue, i.e., a simple eigenvalue whose real part is larger than any other λ in the spectrum, and whose eigenfunction is nonnegative. Physically one expects that such an eigenvalue will exist but this remains to be proved. Our results indicate only that under the conditions of Theorem 10, a "semidominant" real eigenvalue λ_0 with real eigenfunction exists. At present we cannot show that complex eigenvalues with real parts equal to λ_0 do not exist, nor can we say anything about the algebraic multiplicity of λ_0 or any other eigenvalue. (Vidav's proof[22] that the eigenvalues out of the continuum have finite multiplicity is valid only for geometrical multiplicity, and his proof that λ_0 is a simple eigenvalue in L_p, $p>1$, only shows that λ_0 has geometrical multiplicity one.)

Finally, we note (Theorem 9) that if $v_0=0$ and $v^{-1}K$ is a bounded operator, then for sufficiently small bodies the spectrum to the right of the line $\mathrm{Re}\lambda=-\lambda^*$ disappears. This famous "disappearance of the point spectrum into the continuum" was first predicted on a heuristic basis by Nelkin[7] and has become a part of the folklore of neutron transport theory. It turns out that this effect has never been observed experimentally.[47] Our results suggest that this is due to the absence of the continuum to the left of $\mathrm{Re}\lambda=-\lambda^*$ for $v_0>0$. In other words the case $v_0>0$ corresponds more closely to physical reality. This is hardly surprising; the Boltzman equation considered here treats the neutrons as classical particles and cannot be expected to be valid for neutron speeds so low that the neutron wavelength becomes comparable to a mean free path. To consider realistically the case $v_0=0$, another equation should be studied. The experimental evidence suggests strongly that equation would not predict the continuous spectrum we find here for the case $v_0=0$.

APPENDIX

Here we shall prove certain results which were needed earlier. We shall state these results as lemmas.

Lemma 1: Let $\psi\in X$ and let $\mathbf{\Omega}_0$ be fixed. Then $\psi_n(\mathbf{r},\mathbf{v}) \equiv \psi(\mathbf{r},\mathbf{v})\exp(in\mathbf{\Omega}\cdot\mathbf{\Omega}_0)\in X$, $\|\psi_n\|=\|\psi\|$, and $K_0\psi_n\to 0$.

Lemma 2: Let $\psi\in X$. Then $\psi_n(\mathbf{r},\mathbf{v})\equiv\psi(\mathbf{r},\mathbf{v})\exp(inv)\in X$, $\|\psi_n\|=\|\psi\|$, and $K_c\psi_n\to 0$.

Proofs: By the Riemann–Lebesgue Lemma,[48] the sequence $K_0\psi_n$ of Lemma 1 satisfies $\lim_{n\to\infty}(K_0\psi_n)(\mathbf{r},\mathbf{v})=0$ for almost every $\mathbf{r}$ and $\mathbf{v}$. Since, by (2.11) and (2.12),

$$|(K_0\psi_n)(\mathbf{r},\mathbf{v})| \le M_0\int_{\Omega'}\psi(\mathbf{r},v\mathbf{\Omega}')d\Omega' \equiv g(\mathbf{r},v),$$

and since $\|g\|<\infty$, then it follows from the Lebesgue dominated convergence theorem[48] that $\|K_0\psi_n\|\to 0$. Lemma 2 is proved in a similar way. QED

Lemma 3: $K_0(v)[\lambda I - T(v)]^{-1}K_0(v)$ is, for fixed v, a compact operator in X^0.

Lemma 4: For $\lambda\in\rho(T)$, the operators $K_1(\lambda I-T)^{-1}K_0$, $K_c(\lambda I-T)^{-1}K_c$, and $K_d(\lambda I-T)^{-1}K_c$ are compact in X.

Proofs: The proofs that each of the operators of Lemmas 3 and 4 is compact are virtually identical. Thus we shall single out $K_1(\lambda I-T)^{-1}K_0$ and prove the result only for this operator, and for the more difficult case $v_0=0$.

Since $K_1=K_c+K_d$, then by (2.13) $K_1(\lambda I-T)^{-1}K_0 = K_c(\lambda I-T)^{-1}K_0$. Thus we need to show only that $K_c(\lambda I-T)^{-1}K_0\equiv L$ is compact for $\mathrm{Re}\lambda>-\lambda^*$.

Using (2.6), (2.10), and (3.1) we write L in the explicit form

$$(L\psi)(\mathbf{r},\mathbf{v})=\int_{\mathbf{r}'}\int_{\mathbf{v}'}G(\mathbf{r}',\mathbf{v}',\mathbf{r},\mathbf{v})\psi(\mathbf{r}',\mathbf{v}')d\mathbf{v}'\,d\mathbf{r}'$$

where $]^{m_1+1}$,

$$G(\mathbf{r}',\mathbf{v}',\mathbf{r},\mathbf{v})=\frac{v'}{|\mathbf{r}-\mathbf{r}'|^2}\,k_0\left(\mathbf{r}',v',\mathbf{\Omega}'\to\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\right)\times k_c\left(\mathbf{r},v'\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\to\mathbf{v}\right)\times\exp\left\{-\int_{s=0}^{|\mathbf{r}-\mathbf{r}'|}\frac{1}{v'}\left[\lambda+v'\Sigma\left(\mathbf{r}-s\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|},v'\right)\right]ds\right\}.$$

Now for each $(\mathbf{r}',\mathbf{v}')\in\bar{D}\times V$, we define the function $\psi_{\mathbf{r}',\mathbf{v}'}$ by

$$\psi_{\mathbf{r}',\mathbf{v}'}(\mathbf{r},\mathbf{v})\equiv G(\mathbf{r}',\mathbf{v}',\mathbf{r},\mathbf{v}).$$

Then by the "Dunford–Pettis" theorem,[49] L is a compact operator if $\Psi\equiv\{\psi_{\mathbf{r}',\mathbf{v}'}|(\mathbf{r}',\mathbf{v}')\in\bar{D}\times V\}$ is a "compact" subset of X. Equivalently, L is compact if $\Psi\subset X$ and every infinite sequence in Ψ possesses a Cauchy subsequence.

First we show that $\Psi\subset X$. Using (2.7), (2.12), and (3.2), we obtain

$$|\psi_{\mathbf{r}',\mathbf{v}'}(\mathbf{r},\mathbf{v})|\le\frac{M_0M_c\exp(lc_0)\exp\{-|\mathbf{r}-\mathbf{r}_0|[(\mathrm{Re}\lambda+\lambda^*)/v']\}}{v'|\mathbf{r}-\mathbf{r}'|^2v^2} \tag{A1}$$

Now we integrate over $\mathbf{r}$ and $\mathbf{v}$ to get

$$\|\psi_{\mathbf{r}',\mathbf{v}'}\|\le\frac{M_0M_c\exp(lc_0)}{v'}4\pi v_1\times\int_{\Omega}\int_{t=0}^{d(\mathbf{r},\mathbf{\Omega})}\exp\left(-t\frac{\mathrm{Re}\lambda+\lambda^*}{v'}\right)dt\,d\Omega\le(4\pi)^2\frac{v_1M_0M_c\exp(lc_0)}{\mathrm{Re}\lambda+\lambda^*}.$$

Thus $\Psi\subset X$.

Next, we consider an infinite sequence in Ψ. If the various kernels and cross sections are continuous, as described by (i) of Sec. 2, then we select a subsequence $\psi_n\equiv\psi_{\mathbf{r}'_n,\mathbf{v}'_n}$ such that $(\mathbf{r}_n',\mathbf{v}_n')\to(\mathbf{r}_0',\mathbf{v}_0')$. Letting $\psi_0\equiv\psi_{\mathbf{r}_0,\mathbf{v}_0}$, we have, for any $\epsilon>0$,

$$\|\psi_0 - \psi_n\| = \int_{|\mathbf{r}-\mathbf{r}_0|\le\epsilon} \int_{\mathbf{v}} |\psi_0 - \psi_n| d\mathbf{v} d\mathbf{r} + \int_{|\mathbf{r}-\mathbf{r}_0|>\epsilon} \int_{\mathbf{v}} |\psi_0 - \psi_n| d\mathbf{v} d\mathbf{r}.$$

By (A1), the first integral on the right side of this equation is $O(\epsilon)$. By the continuity of the various kernels in G, the second integral can be made $O(\epsilon)$ by requiring n to be sufficiently large. Thus $\|\psi_0 - \psi_n\| \to 0$, proving that L is compact if condition (i) of Sec. 2 holds.

If condition (j) of Sec. 2 holds, then from any infinite sequence in Ψ we select an infinite subsequence $\psi_n \equiv \psi_{\mathbf{r}_n', \mathbf{v}_n'}$ such that $(\mathbf{r}_n', \mathbf{v}_n') \to (\mathbf{r}_0, \mathbf{v}_0) \in \bar{D} \times V$; for some k, $\mathbf{r}_n' \in D_K$, all n; and v_n' is either a decreasing or an increasing sequence. Then, as a simple modification of the above proof for condition (i) will show, the sequence ψ_n is a Cauchy sequence because of the conditions imposed on the kernels and cross sections. Thus, the operator L is again compact. This proves the lemma for cases (i) and (j) of Sec. 2. QED

Lemma 5: For $\lambda \in \rho(T)$, $[K_1(\lambda I - T)^{-1}]^{M_1+1}$ is compact.

Proof: We write

$$\begin{aligned}[K_1(\lambda I - T)^{-1}]^{M_1+1} &= [K_c(\lambda I - T)^{-1} + K_d(\lambda I - T)^{-1}]^{M_1+1} \\ &= [K_c(\lambda I - T)^{-1}]^{M_1+1} + \cdots + K_c(\lambda I - T)^{-1} \\ &\quad \times [K_d(\lambda I - T)^{-1}]^{M_1} + [K_d(\lambda I - T)^{-1}]^{M_1+1}, \end{aligned} \quad \text{(A2)}$$

where the dots refer to operators, all of which are products of $K_c(\lambda I - T)^{-1}$ and $K_d(\lambda I - T)^{-1}$, and all of which contain the product $K_d(\lambda I - T)^{-1} K_c$, which by Lemma 4 is compact. Also by Lemma 4, the first term on the right side of (A2) is compact. By Eq. (2.10) and the fact that $(\lambda I - T)^{-1}$ is a one-speed operator, $[K_d(\lambda I - T)^{-1}]^{M_1} = 0$ so that the last two terms on the right side of (A2) are zero. Therefore, $[K_1(\lambda I - T)^{-1}]^{M_1+1}$ is a sum of compact operators, and hence is itself compact. QED

Lemma 6: $\|K\| = \sup_{\mathbf{r},v} v\Sigma_s(\mathbf{r}, v)$, where $\Sigma_s(\mathbf{r}, v)$ is the cross section for scattering plus fission (i.e., for collisions in which secondary neutrons are emitted).

Proof: In terms of the differential cross section $\Sigma_s(\mathbf{r}, \mathbf{v}' \to \mathbf{v})$, we can write

$$(K\psi)(\mathbf{r}, \mathbf{v}) = \int_{\mathbf{v}'} v' \Sigma_s(\mathbf{r}, \mathbf{v}' \to \mathbf{v}) \psi(\mathbf{r}, \mathbf{v}') d\mathbf{v}',$$

and, by definition,

$$\Sigma_s(\mathbf{r}, v') = \int_{\mathbf{v}} \Sigma_s(\mathbf{r}, \mathbf{v}' \to \mathbf{v})\, d\mathbf{v}.$$

We combine these equations to obtain

$$\|K\psi\| \le \int_{\mathbf{r}} \int_{\mathbf{v}} v\Sigma_s(\mathbf{r}, v) |\psi(\mathbf{r}, \mathbf{v})| d\mathbf{r} d\mathbf{v}, \quad \text{(A3)}$$

where equality holds for $\psi \ge 0$. From this equation, we get

$$\|K\| \le \sup_{\mathbf{r},v} v\Sigma_s(\mathbf{r}, v). \quad \text{(A4)}$$

However, equality holds in (A4) as can be seen by taking in (A3) a nonnegative sequence ψ_n which "converges" to the delta function at the point where $v\Sigma_s(\mathbf{r}, v)$ "attains" its maximum. QED

Corollary: $\|K_0\| = \sup_{\mathbf{r},v} v\Sigma_0(\mathbf{r}, v)$, where $\Sigma_0(\mathbf{r}, v)$ is the cross section for low energy elastic collisions. Similarly, for fixed v, $\|K_0\|^0 = \sup_{\mathbf{r}} v\Sigma_0(\mathbf{r}, v)$.

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[38]In Lemma 2 of Ref. 36 it is essentially claimed that if

$$\int_{0\leq|\mathbf{r}|<\infty} |f(\mathbf{r})|^2 d\mathbf{r} < \infty,$$

then

$$0=\lim_{R\to\infty} \int_{|\mathbf{r}|=R} |f(\mathbf{r})|^2(\mathbf{r}/|\mathbf{r}|)\, d\sigma.$$

However, if $\mathbf{r}=(r,\Theta,\phi)$ where Θ is the polar angle, then a counterexample is given by

$$f(r,\Theta,\phi)=\begin{cases}1 & n\leq r\leq n+1/n^3,\ n=1,2,\ldots;\ \text{and } 0\leq\Theta\leq\pi/2\\ 0 & \text{otherwise.}\end{cases}$$

[39]The problem of true Bragg scattering, i.e., an elastic scattering kernel involving delta functions in angle, has never been treated, nor do we consider it in the present paper [cf. Eq. (2.11)]. We should point out, however, that our analysis does not assume that the elastic scattering rate is a continuous function of v. Then, our results are valid for the polycrystalline case discussed in Ref. 36.

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