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Stochastic wave-kinetic theory in the Liouville approximation*

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The behavior of scalar wave propagation in a wide class of asymptotically conservative, dispersive, weakly inhomogeneous and weakly nonstationary, anisotropic, random media is investigated on the basis of a stochastic, collisionless, Liouville-type equation governing the temporal evolution of a phase-space Wigner distribution density function. Within the framework of the first-order smoothing approximation, a general diffusion-convolution-type kinetic or transport equation is derived for the mean phase-space distribution function containing generalized (nonlocal, with memory) diffusion, friction, and absorption operators in phase space. Various levels of simplification are achieved by introducing additional constraints. In the long-time, Markovian, diffusion approximation, a general set of Fokker-Planck equations is derived. Finally, special cases of these equations are examined for spatially homogeneous systems and isotropic media.

I. INTRODUCTION

A general wave-kinetic method has been developed by Besieris and Tappert,¹⁻⁵ which makes possible the systematic derivation of generalized transport (or kinetic) equations that are valid even for partially coherent waves in inhomogeneous, dispersive, anisotropic media. The theory has been extended to include also media which are slowly varying in time and weakly absorbing (asymptotically conservative).

In order to examine wave propagation by means of this technique, a phase-space description of the problem is developed first. Using the concept of a general analytic signal for wave fields, a phase-space distribution function is defined following Wigner's phase-space approach to quantum mechanical waves. An exact equation of motion of this Wigner distribution function is derived next (it is referred to as the Wigner or wave-kinetic equation) which is fully equivalent to the original field equations. The concept of Weyl transforms (related to pseudo-differential operators) also plays an important role in the rigorous derivation of the phase-space description of wave propagation.

Randomness is introduced by considering the wave-kinetic equations for an ensemble of inhomogeneous media with specified statistical properties. Equations are derived for the temporal evolution of the ensemble-averaged phase-space distribution function from which physically meaningful average quantities are obtainable by taking appropriate phase-space moments.

The stochastic wave-kinetic method has already been used with success to study the behavior of scalar wave propagation in a wide class of random media, with applications to radar, sonar, and other types of communication systems. It generalizes the geometric optics approximation to include coherent effects such as diffraction and random and dispersive spreading of wave-packets. It also provides a systematic basis for many

available classical transport or radiative transfer equations which have been formulated for the most part on the basis of ad hoc assumptions.

It is our intent in this paper to present a statistical analysis of the stochastic, collisionless Liouville equation

$$\frac{\partial}{\partial t} f(\mathbf{x}, \mathbf{p}, t; \alpha) = L f(\mathbf{x}, \mathbf{p}, t; \alpha), \quad (1.1a)$$

$$L f(\mathbf{x}, \mathbf{p}, t; \alpha) = \left(- \frac{\partial}{\partial \mathbf{p}} \omega_r(\mathbf{x}, \mathbf{p}, t; \alpha) \cdot \frac{\partial}{\partial \mathbf{x}} + \frac{\partial}{\partial \mathbf{x}} \omega_r(\mathbf{x}, \mathbf{p}, t; \alpha) \cdot \frac{\partial}{\partial \mathbf{p}} + 2\omega_i(\mathbf{x}, \mathbf{p}, t; \alpha) \right) f(\mathbf{x}, \mathbf{p}, t; \alpha). \quad (1.1b)$$

The medium is described by the linear dispersion relation $\omega(\mathbf{x}, \mathbf{p}, t; \alpha)$, and $\omega_r(\mathbf{x}, \mathbf{p}, t; \alpha)$ and $\omega_i(\mathbf{x}, \mathbf{p}, t; \alpha)$ are respectively the real and imaginary parts of $\omega(\mathbf{x}, \mathbf{p}, t; \alpha)$ (cf. Ref. 6). Within the framework of the wave-kinetic technique, $f(\mathbf{x}, \mathbf{p}, t; \alpha)$ is rigorously defined as the Weyl transform of the wave-analytic signal. Thus, $f(\mathbf{x}, \mathbf{p}, t; \alpha)$ is a Wigner distribution density function, and (1.1) follows from a systematic asymptotic expansion in the semiclassical (or correspondence-limit) approximation.

Equation (1.1) generalizes the Liouville-type stochastic partial differential equation arising in the geometric, ray-optical approach to random media.⁷⁻¹¹ Furthermore, when specialized to the one-species, linearized, Vlasov equation, it has been studied extensively¹² in connection with the stochastic acceleration of particles, a subject of importance in various areas such as cosmic rays, heating of thermonuclear plasma, turbulence of interstellar plasma, etc. Therefore, the results presented in this paper are also applicable to these fields.

In Sec. 2, a stochastic equation describing the evolution of the phase-space Wigner distribution function is derived and the conditions under which the Liouville approximation (1.1) is valid are discussed. In order for

the discussion to be self-contained, general equations for the mean and fluctuating parts of $f(\mathbf{x}, \mathbf{p}, t; \alpha)$ are derived in Sec. 3 using first a nonperturbative technique. These results are then specialized to the weak-coupling limit in order to arrive at the well-known first-order smoothing approximation. These findings are applied to the stochastic, collisionless Liouville equation (1.1) and a general diffusion-convolution-type kinetic or transport equation is given in Sec. 4. By using additional restrictions, various levels of approximation can be achieved. In Sec. 5, general Fokker-Planck equations for the mean phase-space distribution function are obtained. Finally, in Sec. 6, special cases of these general Fokker-Planck equations are considered for spatially homogeneous systems and isotropic media.

II. THE STOCHASTIC WAVE-KINETIC TECHNIQUE

A. The analytic signal

A large class of problems dealing with acoustic and electromagnetic scalar wave propagation in asymptotically conservative (cf. Lewis¹³), dispersive, weakly inhomogeneous and weakly nonstationary anisotropic media is governed by the general stochastic differential equation

$$\left\{ \epsilon^2 \left[\frac{\partial}{\partial t} - \Omega_i \left(\mathbf{x}, t, -i\epsilon \frac{\partial}{\partial \mathbf{x}}; \alpha \right) \right]^2 + \Omega_r^2 \left(\mathbf{x}, t, -i\epsilon \frac{\partial}{\partial \mathbf{x}}; \alpha \right) \right\} \times u(\mathbf{x}, t; \alpha) = 0, \quad t \geq 0, \quad \mathbf{x} \in R^3. \quad (2.1)$$

A distinguishing feature of this problem is the presence of the positive dimensionless parameter ϵ , which can be taken to be inversely proportional to the scale size of the spatial and temporal irregularities. As such, for a slowly varying medium, ϵ will be a small but finite quantity. In (2.1), Ω_r^2 is assumed to be a positive, self-adjoint, stochastic operator depending on a parameter $\alpha \in A$, A being a probability measure space. (All the fractional powers of Ω_r^2 are defined and are, themselves, positive self-adjoint operators.) On the other hand, Ω_i which arises solely from the dissipative properties of the medium, is assumed to be a Hermitian stochastic operator. In addition, $u(\mathbf{x}, t; \alpha)$, the real, scalar, random amplitude, is an element of an infinite-dimensional vector space. The problem is rendered closed by specifying Cauchy initial data and appropriate boundary conditions.

We shall be interested in the time evolution of observable quantities. In this sense, u and u^2 have little physical meaning. We may, however, consider the total wave energy and the total wave action which are given in terms of u , u_t and the operators Ω_r , Ω_i by integrals of the form

$$E = \frac{1}{2} \int_{R^3} F_1(u, u_t, \Omega_r, \Omega_i) d\mathbf{x}, \quad (2.2)$$

$$A = \frac{1}{2} \int_{R^3} F_2(u, u_t, \Omega_r, \Omega_i) d\mathbf{x}, \quad (2.3)$$

respectively. (In the absence of dissipation, $F_1 = u\Omega_r^2 u + \epsilon^2 u_t^2$ and $F_2 = u\Omega_r u + \epsilon^2 u_t \Omega_r^{-1} u_t$). In view of the assumption that the medium is time-dependent and dissipative,

neither of these quantities is conserved. The integrands in (2.2) and (2.3) are, respectively, the space wave energy and wave action density functions.

The difficulty of working with the complicated expressions (2.2) and (2.3) directly is circumvented by introducing the notion of the *complex analytic signal*. This quantity is defined by means of the relation

$$\psi(\mathbf{x}, t; \alpha) = 2^{-1/2} (\Omega_r^{1/2} u + i\epsilon \Omega_r^{-1/2} D_t u), \quad (2.4)$$

with the operator D_t given by $D_t = (\partial/\partial t) - \Omega_i[\mathbf{x}, t, -i\epsilon(\partial/\partial \mathbf{x}); \alpha]$. The total wave energy and wave action associated with (2.1) are given in terms of the analytic signal as follows:

$$E = \int_{R^3} \psi^* \Omega_r \psi d\mathbf{x}, \quad (2.5)$$

$$A = \int_{R^3} \psi^* \psi d\mathbf{x}. \quad (2.6)$$

Taking the time derivative of (2.4) and using (2.1), one has the formal relation

$$i\epsilon D_t \psi = \Omega_r \psi + \frac{1}{4} i\epsilon D_t (\log \Omega_r^{1/2}) \psi^*. \quad (2.7)$$

By neglecting nonadiabatic terms, the analytic signal obeys the closed equation

$$i\epsilon \frac{\partial}{\partial t} \psi = \Omega_r \psi + i\epsilon \Omega_i \psi. \quad (2.8)$$

(In the absence of dissipation, the total wave action is an adiabatic invariant to all orders in ϵ ; E , however, is not conserved because of the time dependence of the medium.)

B. The Wigner distribution function

The two-point, equal time density function is introduced next as follows in terms of the analytic signal:

$$\rho(\mathbf{x}_2, \mathbf{x}_1, t; \alpha) = \psi^*(\mathbf{x}_2, t; \alpha) \psi(\mathbf{x}_1, t; \alpha). \quad (2.9)$$

It obeys the von Neumann-like equation

$$i\epsilon \frac{\partial}{\partial t} \rho = -[\Omega_r, \rho]_- + i\epsilon [\Omega_i, \rho]_+, \quad (2.10)$$

where $[A, B]_{\mp} = AB \mp BA$ denote the usual commutation and anticommutation relations.

The phase-space analog of the density function is provided by the Wigner distribution function which is defined as follows¹⁴:

$$f(\mathbf{x}, \mathbf{p}, t; \alpha) = (2\pi\epsilon)^{-3} \int_{R^3} d\mathbf{y} \exp(i\mathbf{p} \cdot \mathbf{y}/\epsilon) \rho(\mathbf{x} + \frac{1}{2}\mathbf{y}, \mathbf{x} - \frac{1}{2}\mathbf{y}, t; \alpha). \quad (2.11)$$

This quantity is real, but not necessarily positive everywhere. In this sense, it does not qualify as a *bona fide* probability density function. One may introduce it in terms of other bilinear expressions of the analytic signal. The relation (2.11) has been chosen because of its relative simplicity and symmetry. The total wave energy and wave action can be written in terms of the Wigner distribution function as follows:

$$E = \int_{R^3} d\mathbf{x} \int_{R^3} d\mathbf{p} \omega_r(\mathbf{x}, \mathbf{p}, t; \alpha) f(\mathbf{x}, \mathbf{p}, t; \alpha), \quad (2.12)$$

$$A = \int_{R^3} d\mathbf{x} \int_{R^3} d\mathbf{p} f(\mathbf{x}, \mathbf{p}, t; \alpha). \quad (2.13)$$

Here, $\omega_r(\mathbf{x}, \mathbf{p}, t; \alpha)$ is the Weyl transform of the operator Ω_r . By virtue of (2.12), $\omega_r f$ can be interpreted as the wave energy density in phase space. Similarly, from (2.13), f can be thought of as the wave action density in phase space.

Given the definition of $f(\mathbf{x}, \mathbf{p}, t; \alpha)$ and using the von Neumann equation (2.10), it is found that the Wigner distribution function evolves according to the following equation¹⁵:

$$\begin{aligned} \frac{\partial}{\partial t} f(\mathbf{x}, \mathbf{p}, t; \alpha) &= \omega_r(\mathbf{x}, \mathbf{p}, t; \alpha) \frac{2}{\epsilon} \sin \left[\frac{\epsilon}{2} \left(\frac{\vec{\partial}}{\partial \mathbf{x}} \cdot \frac{\vec{\partial}}{\partial \mathbf{p}} - \frac{\vec{\partial}}{\partial \mathbf{p}} \cdot \frac{\vec{\partial}}{\partial \mathbf{x}} \right) \right] f(\mathbf{x}, \mathbf{p}, t; \alpha) \\ &+ \omega_i(\mathbf{x}, \mathbf{p}, t; \alpha) 2 \cos \left[\frac{\epsilon}{2} \left(\frac{\vec{\partial}}{\partial \mathbf{x}} \cdot \frac{\vec{\partial}}{\partial \mathbf{p}} - \frac{\vec{\partial}}{\partial \mathbf{p}} \cdot \frac{\vec{\partial}}{\partial \mathbf{x}} \right) \right] f(\mathbf{x}, \mathbf{p}, t; \alpha), \end{aligned} \quad (2.14)$$

where $\omega_i(\mathbf{x}, \mathbf{p}, t; \alpha)$ is the Weyl transform of the operator Ω_i . (Depending on the directions of the arrows, the differential operators on the right-hand side of (2.14) operate on ω_r , ω_i , or f .) We shall refer to the exact equation of evolution of f as the *stochastic Wigner equation*.

It is seen from (2.14) that in the "correspondence limit" ($\epsilon \rightarrow 0$), the Wigner distribution function obeys the simpler relation

$$\frac{\partial}{\partial t} f(\mathbf{x}, \mathbf{p}, t; \alpha) = L f(\mathbf{x}, \mathbf{p}, t; \alpha), \quad (2.15a)$$

$$\begin{aligned} L f(\mathbf{x}, \mathbf{p}, t; \alpha) &= \left(-\frac{\partial}{\partial \mathbf{p}} \omega_r(\mathbf{x}, \mathbf{p}, t; \alpha) \cdot \frac{\partial}{\partial \mathbf{x}} + \frac{\partial}{\partial \mathbf{x}} \omega_r(\mathbf{x}, \mathbf{p}, t; \alpha) \right. \\ &\left. \cdot \frac{\partial}{\partial \mathbf{p}} + 2\omega_i(\mathbf{x}, \mathbf{p}, t; \alpha) \right) f(\mathbf{x}, \mathbf{p}, t; \alpha) + O(\epsilon^2). \end{aligned} \quad (2.15b)$$

In the framework of this approximation, we shall refer to (2.15) as the *stochastic, collisionless Liouville equation*.

III. GENERAL EQUATIONS FOR THE MEAN AND FLUCTUATING FIELDS

In the first part of this section we derive general equations for the mean and fluctuating parts of the Wigner distribution function using a nonperturbative statistical approach. These results are then specialized to the weak-coupling limit in order to arrive at the well known first-order smoothing approximation. Further simplifications, required for the subsequent development, lead to the long-time and Markovian approximations. The discussion in this section is general and applies to both the stochastic Wigner equation [cf. Eq. (2.14)] and the stochastic, collisionless Liouville equation [cf. Eq. (2.15)].

Consider the linear, stochastic, partial differential equation¹⁶

$$\frac{\partial}{\partial t} f(\mathbf{x}, \mathbf{p}, t; \alpha) = L f(\mathbf{x}, \mathbf{p}, t; \alpha), \quad (3.1a)$$

$$f(\mathbf{x}, \mathbf{p}, 0; \alpha) = f_0(\mathbf{x}, \mathbf{p}; \alpha). \quad (3.1b)$$

The stochastic operator L is split into two parts as follows: $L = L_0 + L_1$. The selection of L_0 and L_1 is made

in such a way that they are linear operators in an infinite dimensional vector space H , corresponding, respectively, to "free" and "interaction" propagation.

The distribution function f is, in turn, decomposed abstractly into two mutually independent terms, viz., $f = Vf + Cf$ by means of the formal introduction of the two operators V and C .¹⁷ Vf is called the *mean* component, and Cf is the *fluctuating* part of the distribution function f .¹⁸ The uniqueness of the decomposition as well as the mutual independence of the two components are ensured by prescribing the properties $V + C = I$, $V^2 = V$, $C^2 = C$, $VC = 0$, $CV = 0$, where I is the identity operator. By virtue of these relations, V and C are called *projection operators*.

The interconnection between the decompositions for the operator L and the distribution function f is contained in the commutation relations $[L_0, V]_- = 0$ and $[L_0, C]_- = 0$ which constitute a mathematical statement of the fact that the fluctuating part of f is due only to the interaction part of the operator L . Therefore, L_0 must commute with V , and also, with $C = I - V$.

The specific realization of the projection operators V and C which will be used in the ensuing work is the following: $Vf \rightarrow E\{f\}$, $Cf \rightarrow \delta f$, where $E\{f\}$ and δf are, respectively, the ensemble average and fluctuating part of the random distribution $f(\alpha)$. Within the framework of this specific realization, the aforementioned commutation relations signify that L_0 is a deterministic operator and L_1 is a generally noncentered random operator.

A. Equations for the mean and fluctuating distribution functions; first-order smoothing approximation

Operating on (3.1) with the projection operators V , C yields the equations

$$\frac{\partial}{\partial t} E\{f(t)\} = VLE\{f(t)\} + VL\delta f(t), \quad (3.2a)$$

$$\frac{\partial}{\partial t} \delta f(t) = CL\delta f(t) + CLE\{f(t)\}, \quad (3.2b)$$

respectively. Equation (3.2b) can be solved for $\delta f(t)$ in terms of the mean field and the initial value of the fluctuating part of the distribution:

$$\delta f(t) = U_V(t, 0) \delta f(0) + \int_0^t dt_1 U_V(t, t_1) CL(t_1) E\{f(t_1)\}. \quad (3.3)$$

The propagator U_V is defined as the solution of the initial value problem

$$\frac{\partial}{\partial t} U_V(t, 0) = CL(t) U_V(t, 0), \quad U_V(0, 0) = 1. \quad (3.4)$$

Substituting (3.3) into (3.2a) results in the equation¹⁹

$$\begin{aligned} \frac{\partial}{\partial t} E\{f(t)\} &= L_0 E\{f(t)\} + VL_1 VE\{f(t)\} + VL_1 U_V(t, 0) C \delta f(0) \\ &+ \int_0^t dt_1 VL_1(t) U_V(t, t_1) CL_1(t_1) E\{f(t_1)\}. \end{aligned} \quad (3.5)$$

This formal expression for the mean field is valid for both weak and strong fluctuations in the randomly vary-

ing inhomogeneities of the medium. It should also be noted that no restriction whatsoever has been imposed on the random operator L_1 and the initial value of the distribution function.

Equation (3.3) indicates that the fluctuating part of the field can be calculated by quadratures once the mean distribution function has been determined separately.

B. The first-order smoothing approximation

Balescu and Misguich²⁰ have shown recently that

$$U_V(t, 0) = \sum_{n=0}^{\infty} \left[\int_0^t dt_1 W(t, t_1) CL_1(t_1) \right]^n W(t, 0), \quad (3.6)$$

with the propagator W defined as the solution of the initial value problem

$$\frac{\partial}{\partial t} W(t, 0) = L_0(t) W(t, 0), \quad W(0, 0) = 1. \quad (3.7)$$

The *first-order smoothing approximation* is determined by introducing in (3.5) the weak-coupling limit approximation

$$U_V(t, 0) C \rightarrow CW(t, 0). \quad (3.8)$$

For the sake of simplicity, we also impose the restriction that L_1 is a centered random operator. This condition is stated mathematically as $VL_1V = 0$. Furthermore, $f(0)$ is taken to be deterministic so that $C\delta f(0) = 0$. We have, then, in the place of (3.5)

$$\frac{\partial}{\partial t} E\{f(t)\} \quad (3.9)$$

$$= L_0 E\{f(t)\} + \int_0^t dt_1 VL_1(t) W(t, t_1) L_1(t_1) E\{f(t_1)\}.$$

The first-order smoothing approximation (cf. also Refs. 9, 10, 21–23) is essentially a perturbational method applicable for weak random variations. The mean field $E\{f(t)\}$, which is determined by successive iterations of (3.9), is found to be a partial summation of the exact, infinite, conventional perturbation series solution. This subseries, besides yielding results consistent with physically imposed constraints, enables one to circumvent certain time and space secularities which are characteristic of solutions consisting of only a finite number of terms of the infinite perturbation series (e.g., Born approximation and its various modifications).

In general, the solution of (3.7) for the propagator W needed in (3.9) has the form

$$W(t, 0) = X \exp\left\{ \int_0^t dt_1 L_0(t_1) \right\}, \quad (3.10)$$

where X is a time-ordering operator. In the following, we shall assume that L_0 is time-independent. In this case (3.10) simplifies to

$$W(t, 0) = \exp(tL_0). \quad (3.11)$$

The approximation (3.9) together with the assumption (3.11) will be used in the following section to derive a diffusion-convolution-type kinetic equation for the average part of the Wigner distribution function.

C. Long-time Markovian approximation

Let $E\{\hat{f}(t)\}$ denote the asymptotic limit of the field $E\{f(t)\}$ as $t \rightarrow \infty$. Then, given that $U_V(t, 0)C \rightarrow 0$ as $t \rightarrow \infty$, it can be established that (3.5) assumes the simpler form^{17,24}

$$\begin{aligned} \frac{\partial}{\partial t} E\{\hat{f}(t)\} \\ = L_0 E\{\hat{f}(t)\} + \int_0^\infty dt_1 VL_1(t) U_V(t, t_1) CL_1(t_1) E\{\hat{f}(t_1)\}. \end{aligned} \quad (3.12)$$

We shall call this relation the *long-time approximation* corresponding to (3.5). It is interesting to note that the term in (3.5) containing the fluctuating part of the initial distribution is asymptotically null in this approximation. No restriction need, therefore, be imposed on $f(0)$. It should, further, be pointed out that the initial mean distribution $E\{\hat{f}(0)\}$ required for the complete specification of the initial value problem (3.12) must be chosen so that the solution of (3.12) will coincide, for large times, with the asymptotic value of the solution of the “exact” equation (3.5) with the initial value $E\{f(0)\}$.

In the first-order smoothing approximation and under the assumption that the background deterministic medium is time-independent, (3.12) simplifies to

$$\begin{aligned} \frac{\partial}{\partial t} E\{\hat{f}(t)\} = L_0 E\{\hat{f}(t)\} \\ + \int_0^\infty dt_1 VL_1(t) W(t_1) L_1(t-t_1) E\{\hat{f}(t-t_1)\}. \end{aligned} \quad (3.13)$$

This relation can be rewritten in the equivalent, purely differential form

$$\frac{\partial}{\partial t} E\{\hat{f}(t)\} = KE\{\hat{f}(t)\}, \quad (3.14)$$

K being the solution of the nonlinear integral equation

$$K = L_0 + \int_0^\infty dt_1 VL_1(t) W(t_1) L_1(t-t_1) \exp(-t_1 K). \quad (3.15)$$

The last expression can be solved for K by the method of successive substitutions. If only the first two terms of the expansion are retained, one obtains the *long-time Markovian approximation*

$$\frac{\partial}{\partial t} E\{\hat{f}(t)\} = L_0 E\{\hat{f}(t)\} + \int_0^\infty dt_1 VL_1(t) W(t_1) L_1(t-t_1) E\{\hat{f}(t)\}. \quad (3.16)$$

This simplified integro-differential equation for the mean field will be used in Sec. 5 to derive a general Fokker–Planck equation.

IV. KINETIC EQUATION FOR THE MEAN DISTRIBUTION FUNCTION

In this section we specialize our findings in the previous section to the case of the stochastic Liouville-type operator

$$L = \frac{\partial}{\partial \mathbf{x}} \omega_r(\mathbf{x}, \mathbf{p}; \alpha) \cdot \frac{\partial}{\partial \mathbf{p}} - \frac{\partial}{\partial \mathbf{p}} \omega_r(\mathbf{x}, \mathbf{p}; \alpha) \cdot \frac{\partial}{\partial \mathbf{x}} + 2\omega_i(\mathbf{x}, \mathbf{p}; \alpha) \quad (4.1)$$

introduced earlier [cf. (2.15)] in connection with the stochastic wave-kinetic technique. For simplicity, L is assumed in the sequel to be independent of time. This restriction is not a serious drawback since it can be

easily lifted (cf. Ref. 5). Thus, L is translationally invariant with respect to time, and its free and interaction parts are given simply by

$$L_0 = \frac{\partial}{\partial \mathbf{x}} E\{\omega_r(\mathbf{x}, \mathbf{p}; \alpha)\} \cdot \frac{\partial}{\partial \mathbf{p}} \quad (4.2a)$$

$$- \frac{\partial}{\partial \mathbf{p}} E\{\omega_r(\mathbf{x}, \mathbf{p}; \alpha)\} \cdot \frac{\partial}{\partial \mathbf{x}} + 2E\{\omega_i(\mathbf{x}, \mathbf{p}; \alpha)\},$$

$$L_1 = \frac{\partial}{\partial \mathbf{x}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha) \cdot \frac{\partial}{\partial \mathbf{p}} \quad (4.2b)$$

$$- \frac{\partial}{\partial \mathbf{p}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha) \cdot \frac{\partial}{\partial \mathbf{x}} + 2\delta\omega_i(\mathbf{x}, \mathbf{p}; \alpha)$$

in terms of the mean and the fluctuating parts of ω_r and ω_i .

Diffusion-convolution-type kinetic equation

We commence with the first-order smoothing approximation of the Dyson-Schwinger equation for the mean Wigner distribution function [cf. Eq. (3.9)]. In order to write our explicitly the second part on the right-hand side of (3.9), we use the definition of L_1 given in (4.2b). We have, then

$$\begin{aligned} \left(\frac{\partial}{\partial t} - L_0\right) E\{f(\mathbf{x}, \mathbf{p}, t; \alpha)\} = & \frac{\partial}{\partial \mathbf{p}} \cdot \left(\int_0^t dt_1 E\left\{\frac{\partial}{\partial \mathbf{x}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha) \exp(t_1 L_0) \frac{\partial}{\partial \mathbf{x}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha)\right\} \cdot \frac{\partial}{\partial \mathbf{p}} E\{f(\mathbf{x}, \mathbf{p}, t-t_1; \alpha)\}\right) \\ & - \frac{\partial}{\partial \mathbf{p}} \cdot \left(\int_0^t dt_1 E\left\{\frac{\partial}{\partial \mathbf{x}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha) \exp(t_1 L_0) \frac{\partial}{\partial \mathbf{p}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha)\right\} \cdot \frac{\partial}{\partial \mathbf{x}} E\{f(\mathbf{x}, \mathbf{p}, t-t_1; \alpha)\}\right) \\ & - \frac{\partial}{\partial \mathbf{x}} \cdot \left(\int_0^t dt_1 E\left\{\frac{\partial}{\partial \mathbf{p}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha) \exp(t_1 L_0) \frac{\partial}{\partial \mathbf{x}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha)\right\} \cdot \frac{\partial}{\partial \mathbf{p}} E\{f(\mathbf{x}, \mathbf{p}, t-t_1; \alpha)\}\right) \\ & + \frac{\partial}{\partial \mathbf{x}} \cdot \left(\int_0^t dt_1 E\left\{\frac{\partial}{\partial \mathbf{p}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha) \exp(t_1 L_0) \frac{\partial}{\partial \mathbf{p}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha)\right\} \cdot \frac{\partial}{\partial \mathbf{x}} E\{f(\mathbf{x}, \mathbf{p}, t-t_1; \alpha)\}\right) \\ & + \frac{\partial}{\partial \mathbf{p}} \cdot \left(\int_0^t dt_1 E\left\{\frac{\partial}{\partial \mathbf{x}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha) \exp(t_1 L_0) 2\delta\omega_i(\mathbf{x}, \mathbf{p}; \alpha)\right\} E\{f(\mathbf{x}, \mathbf{p}, t-t_1; \alpha)\}\right) \\ & - \frac{\partial}{\partial \mathbf{x}} \cdot \left(\int_0^t dt_1 E\left\{\frac{\partial}{\partial \mathbf{p}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha) \exp(t_1 L_0) 2\delta\omega_i(\mathbf{x}, \mathbf{p}; \alpha)\right\} E\{f(\mathbf{x}, \mathbf{p}, t-t_1; \alpha)\}\right) \\ & + \int_0^t dt_1 E\left\{2\delta\omega_i(\mathbf{x}, \mathbf{p}; \alpha) \exp(t_1 L_0) \frac{\partial}{\partial \mathbf{x}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha)\right\} \cdot \frac{\partial}{\partial \mathbf{p}} E\{f(\mathbf{x}, \mathbf{p}, t-t_1; \alpha)\} \\ & - \int_0^t dt_1 E\left\{2\delta\omega_i(\mathbf{x}, \mathbf{p}; \alpha) \exp(t_1 L_0) \frac{\partial}{\partial \mathbf{p}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha)\right\} \cdot \frac{\partial}{\partial \mathbf{x}} E\{f(\mathbf{x}, \mathbf{p}, t-t_1; \alpha)\} \\ & + \int_0^t dt_1 E\{2\delta\omega_i(\mathbf{x}, \mathbf{p}; \alpha) \exp(t_1 L_0) 2\delta\omega_i(\mathbf{x}, \mathbf{p}; \alpha)\} E\{f(\mathbf{x}, \mathbf{p}, t-t_1; \alpha)\}. \end{aligned} \quad (4.3)$$

This rather formidable integro-differential relation, which will be referred to as the *kinetic equation* for the mean distribution function, generalizes previous equations of this type (cf. Refs. 7-12). It applies to media with inhomogeneous deterministic background, and constitutes a uniform approximation, valid for any value of time, from which short and long time limiting cases can be considered. (The latter will be dealt with in detail in the following section). The right-hand side of (4.3) contains generalized diffusion operators (nonlocal, with memory) in phase space, and, also, generalized friction and absorption operators.

For a spatially homogeneous background, (4.3) is a generalization of the kinetic equation for random geometrical optics obtained by Frisch (cf. Ref. 10). It is a *diffusion-type equation* with respect to space and wave vector (momentum) coordinates, and a *convolution equation* with respect to position and time. (Under various special conditions, e.g., homogeneous and isotropic randomness and spatially homogeneous systems, it may be possible to arrive at an exact analytical solution.)

V. GENERAL FOKKER-PLANCK EQUATION

A. Diffusion, friction, and absorption coefficients

By imposing additional restrictions, various levels of simplification of (4.3) can be obtained. The long-time, Markovian approximation yields the expression

$$\begin{aligned} \left(\frac{\partial}{\partial t} + v(\mathbf{p}) \cdot \frac{\partial}{\partial \mathbf{x}}\right) E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\} \\ = \frac{\partial}{\partial \mathbf{p}} \cdot \left[\mathbf{D}_{pp}^{rr} \cdot \frac{\partial}{\partial \mathbf{p}} E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\}\right] \\ - \frac{\partial}{\partial \mathbf{p}} \cdot \left[\mathbf{D}_{px}^{rx} \cdot \frac{\partial}{\partial \mathbf{x}} E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\}\right] \end{aligned}$$

$$\begin{aligned} - \frac{\partial}{\partial \mathbf{x}} \cdot \left[\mathbf{D}_{xp}^{rr} \cdot \frac{\partial}{\partial \mathbf{p}} E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\}\right] \\ + \frac{\partial}{\partial \mathbf{x}} \cdot \left[\mathbf{D}_{xx}^{rr} \cdot \frac{\partial}{\partial \mathbf{x}} E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\}\right] \\ + \frac{\partial}{\partial \mathbf{p}} \cdot \left[\mathbf{F}_p^{ri} E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\}\right] \\ - \frac{\partial}{\partial \mathbf{x}} \cdot \left[\mathbf{F}_x^{ri} E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\}\right] \\ + \mathbf{F}_p^{ir} \cdot \frac{\partial}{\partial \mathbf{p}} E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\} - \mathbf{F}_x^{ir} \cdot \frac{\partial}{\partial \mathbf{x}} E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\} \\ + A^{ii} E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\} + A^i E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\}, \end{aligned} \quad (5.1)$$

with the dyadic (\mathbf{D}), vector (\mathbf{F}), and scalar operator coefficients defined by

$$\mathbf{D}_{pp}^{rr} = \int_0^\infty d\tau E \left\{ \frac{\partial}{\partial \mathbf{x}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha) \right. \quad (5.2a)$$

$$\left. \times \exp \left(-\tau \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \frac{\partial}{\partial \mathbf{x}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha) \right\}$$

$$\mathbf{D}_{px}^{rr} = \int_0^\infty d\tau E \left\{ \frac{\partial}{\partial \mathbf{x}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha) \right. \quad (5.2b)$$

$$\left. \times \exp \left(-\tau \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \frac{\partial}{\partial \mathbf{p}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha) \right\},$$

$$\mathbf{D}_{xp}^{rr} = \int_0^\infty d\tau E \left\{ \frac{\partial}{\partial \mathbf{p}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha) \right. \quad (5.2c)$$

$$\left. \times \exp \left(-\tau \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \frac{\partial}{\partial \mathbf{x}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha) \right\},$$

$$\mathbf{D}_{xx}^{rr} = \int_0^\infty d\tau E \left\{ \frac{\partial}{\partial \mathbf{p}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha) \right. \quad (5.2d)$$

$$\left. \times \exp \left(-\tau \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \frac{\partial}{\partial \mathbf{p}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha) \right\},$$

$$\mathbf{F}_p^{ri} = \int_0^\infty d\tau E \left\{ \frac{\partial}{\partial \mathbf{x}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha) \right. \quad (5.2e)$$

$$\left. \times \exp \left(-\tau \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \right) 2\delta\omega_i(\mathbf{x}, \mathbf{p}; \alpha) \right\},$$

$$\mathbf{F}_x^{ri} = \int_0^\infty d\tau E \left\{ \frac{\partial}{\partial \mathbf{p}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha) \right. \quad (5.2f)$$

$$\left. \times \exp \left(-\tau \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \right) 2\delta\omega_i(\mathbf{x}, \mathbf{p}; \alpha) \right\},$$

$$\mathbf{F}_p^{ir} = \int_0^\infty d\tau E \left\{ 2\delta\omega_i(\mathbf{x}, \mathbf{p}; \alpha) \right. \quad (5.2g)$$

$$\left. \times \exp \left(-\tau \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \frac{\partial}{\partial \mathbf{x}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha) \right\},$$

$$\mathbf{F}_x^{ir} = \int_0^\infty d\tau E \left\{ 2\delta\omega_i(\mathbf{x}, \mathbf{p}; \alpha) \right. \quad (5.2h)$$

$$\left. \times \exp \left(-\tau \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \frac{\partial}{\partial \mathbf{p}} \delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha) \right\},$$

$$A^{ii} = \int_0^\infty d\tau E \left\{ 2\delta\omega_i(\mathbf{x}, \mathbf{p}; \alpha) \right. \quad (5.2i)$$

$$\left. \times \exp \left(-\tau \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \right) 2\delta\omega_i(\mathbf{x}, \mathbf{p}; \alpha) \right\},$$

$$A^i = 2 E \{ \omega_i(\mathbf{x}, \mathbf{p}; \alpha) \} \quad (5.2j)$$

and the vector quantity $\mathbf{v}(\mathbf{p})$ given as follows:

$$\mathbf{v}(\mathbf{p}) = \frac{\partial}{\partial \mathbf{p}} E \{ \omega_r(\mathbf{x}, \mathbf{p}; \alpha) \}. \quad (5.3)$$

In writing down (5.1) we have resorted to the following simplifying assumptions:

(i) The deterministic background medium is spatially homogeneous. (This implies that both $E \{ \omega_r(\mathbf{x}, \mathbf{p}; \alpha) \}$ and $E \{ \omega_i(\mathbf{x}, \mathbf{p}; \alpha) \}$ are independent of position.)

(ii) The quantity $E \{ \omega_i(\mathbf{x}, \mathbf{p}; \alpha) \}$ is a slowly varying function of momentum so that its first- and higher-order derivatives with respect to \mathbf{p} can be neglected by comparison to the quantity itself.

In the following, the translational effects of the operator $\exp[-\tau \mathbf{v} \cdot (\partial/\partial \mathbf{x})]$ on the mean distribution function $E \{ \hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha) \}$ will be ignored. This simplification is known as the *diffusion approximation*. Within the confines of

this approximation, (5.1) becomes a Fokker-Planck equation. The coefficients \mathbf{D} are called the *dyadic diffusion coefficients*, the quantities \mathbf{F} are the *vector friction coefficients*, and, finally, the A 's are the *scalar absorption coefficients*.

B. Anisotropic, dissipative and/or dispersive systems; uniform, homogeneous, and isotropic fluctuations

We shall derive here explicit expressions for the diffusion, friction, and absorption coefficients (5.2) in the case of a general anisotropic, dissipative and/or dispersive medium characterized by uniform, spatially homogeneous and isotropic fluctuations of the randomly varying functions $\delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha)$ and $\delta\omega_i(\mathbf{x}, \mathbf{p}; \alpha)$. We assume, furthermore, the conditions (i) and (ii) specified in the previous subsection. Thus, the background medium is independent of the space coordinates, and $E \{ \omega_i(\mathbf{x}, \mathbf{p}; \alpha) \}$ is a slowly varying function of \mathbf{p} .

In many physically important problems $\delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha)$ can be written as a product of a random function $\delta\chi_r(\mathbf{x}; \alpha)$, and a deterministic function $g_r(\mathbf{p})$, viz.,

$$\delta\omega_r(\mathbf{x}, \mathbf{p}; \alpha) = \delta\chi_r(\mathbf{x}; \alpha) g_r(\mathbf{p}). \quad (5.4)$$

Similarly, $\delta\omega_i(\mathbf{x}, \mathbf{p}; \alpha)$ is written as a product of a random function of position, $\delta\chi_i(\mathbf{x}; \alpha)$, and a deterministic function, $g_i(\mathbf{p})$, viz.,

$$\delta\omega_i(\mathbf{x}, \mathbf{p}; \alpha) = \delta\chi_i(\mathbf{x}; \alpha) g_i(\mathbf{p}). \quad (5.5)$$

For spatially homogeneous and isotropic fluctuations, we define the following two-point correlation functions:

$$\Gamma_{rr}(y) = E \{ \delta\chi_r(\mathbf{x}; \alpha) \delta\chi_r(\mathbf{x} - \mathbf{y}; \alpha) \}, \quad (5.6a)$$

$$\Gamma_{ri}(y) = E \{ \delta\chi_r(\mathbf{x}; \alpha) \delta\chi_i(\mathbf{x} - \mathbf{y}; \alpha) \}, \quad (5.6b)$$

$$\Gamma_{ir}(y) = E \{ \delta\chi_i(\mathbf{x}; \alpha) \delta\chi_r(\mathbf{x} - \mathbf{y}; \alpha) \}, \quad (5.6c)$$

$$\Gamma_{ii}(y) = E \{ \delta\chi_i(\mathbf{x}; \alpha) \delta\chi_i(\mathbf{x} - \mathbf{y}; \alpha) \}. \quad (5.6d)$$

Furthermore, for a uniform, spatially homogeneous and isotropic model, we specify the relationships:

$$\frac{\Gamma_{rr}(y)}{\Gamma_{rr}(0)} = \frac{\Gamma_{ri}(y)}{\Gamma_{ri}(0)} = \frac{\Gamma_{ir}(y)}{\Gamma_{ir}(0)} = \frac{\Gamma_{ii}(y)}{\Gamma_{ii}(0)} = \rho(y), \quad (5.7a)$$

$$\rho(0) = 1. \quad (5.7b)$$

The quantity $\rho(y)$ will be referred to as the correlation coefficient.

While, admittedly, more complicated choices for the correlation functions may more closely resemble the true situations in the real world, the relatively simple forms (5.6) and (5.7) contain the essential behavior of the random fluctuations.

With $y = v\tau$, where $v = |\mathbf{v}|$, the first dyadic diffusion coefficient becomes

$$\mathbf{D}_{pp}^{rr} = -\frac{1}{v(\mathbf{p})} g_r^2(\mathbf{p}) \Gamma_{rr}(0) \int_0^\infty dy \left[\left(\mathbf{1} - \frac{\mathbf{v}\mathbf{v}}{v^2} \right) \frac{1}{y} \frac{\partial}{\partial y} \rho(y) \right. \quad (5.8)$$

$$\left. - \frac{\mathbf{v}\mathbf{v}}{v^2} \frac{\partial^2}{\partial y^2} \rho(y) \right],$$

which, in turn, simplifies to

$$\mathbf{D}_{pp}^{rr} = \frac{1}{v(\mathbf{p})} g_r^2(\mathbf{p}) \left(\mathbf{1} - \frac{\mathbf{v}\mathbf{v}}{v^2} \right) \Gamma_{rr}(0) B, \quad (5.9a)$$

$$B = - \int_0^\infty dy \frac{1}{y} \frac{\partial}{\partial y} \rho(y) \quad (5.9b)$$

provided that

$$\frac{\partial}{\partial y} \rho(y) \xrightarrow[y \rightarrow \infty]{y \rightarrow 0} 0. \quad (5.10)$$

$\mathbf{1}$ in (5.8) is the unit dyadic, and B in (5.9) is a quantity which will be related to the correlation length of the random inhomogeneities at the end of this subsection.

The second dyadic diffusion coefficient can be rewritten as follows:

$$\mathbf{D}_{px}^{rr} = - \int_0^\infty d\tau E \left\{ \frac{\partial}{\partial \mathbf{x}} \delta \omega_r(\mathbf{x}, \mathbf{p}; \alpha) \frac{\partial}{\partial \mathbf{p}} \delta \omega_r(\mathbf{x} - \mathbf{v}\tau, \mathbf{p}; \alpha) \right\}. \quad (5.11)$$

Bearing in mind (5.4), (5.6a), and (5.7), this becomes

$$\mathbf{D}_{px}^{rr} = -g_r(\mathbf{p}) \frac{\partial}{\partial \mathbf{p}} g_r(\mathbf{p}) \frac{\mathbf{v}(\mathbf{p})}{v^2(\mathbf{p})} \Gamma_{rr}(0) \int_0^\infty dy \frac{\partial}{\partial y} \rho(y), \quad (5.12)$$

which, upon integration, changes to

$$\mathbf{D}_{px}^{rr} = -g_r(\mathbf{p}) \frac{\partial}{\partial \mathbf{p}} g_r(\mathbf{p}) \frac{\mathbf{v}(\mathbf{p})}{v^2(\mathbf{p})} \Gamma_{rr}(0) \quad (5.13)$$

if, in addition to (5.10), one specifies that $\rho(y) \rightarrow 0$ as $y \rightarrow \infty$.

By virtue of the definition (5.2a), it is easily seen that

$$\mathbf{D}_{xp}^{rr} = -\mathbf{D}_{px}^{rr}. \quad (5.14)$$

It also develops that the last diffusion coefficient is given by

$$\mathbf{D}_{xx}^{rr} = \frac{1}{v(\mathbf{p})} \left(\frac{\partial}{\partial \mathbf{p}} g_r(\mathbf{p}) \right)^2 \Gamma_{rr}(0) C, \quad (5.15a)$$

$$C = \int_0^\infty dy \rho(y). \quad (5.15b)$$

A physical interpretation of C will be presented later on in this subsection.

We shall turn next to the evaluation of the friction coefficients. The first one [cf. Eq. (5.2e)] is rewritten as

$$\mathbf{F}_p^{ri} = \int_0^\infty d\tau E \left\{ \frac{\partial}{\partial \mathbf{x}} \delta \omega_r(\mathbf{x}, \mathbf{p}; \alpha) 2\delta \omega_i(\mathbf{x} - \mathbf{v}\tau, \mathbf{p}; \alpha) \right\}. \quad (5.16)$$

Using (5.4), (5.5), and (5.6b), and (5.7), this yields

$$\mathbf{F}_p^{ri} = -2g_r(\mathbf{p})g_i(\mathbf{p}) \frac{\mathbf{v}(\mathbf{p})}{v^2(\mathbf{p})} \Gamma_{ri}(0). \quad (5.17)$$

An examination of (5.2g) shows that

$$\mathbf{F}_p^{ir} = -\mathbf{F}_p^{ri}. \quad (5.18)$$

The remaining two friction coefficients are found to be equal:

$$\mathbf{F}_x^{ri} = \mathbf{F}_x^{ir} = 2g_i(\mathbf{p}) \frac{\partial}{\partial \mathbf{p}} g_r(\mathbf{p}) \frac{1}{v(\mathbf{p})} \Gamma_{ri}(0) C. \quad (5.19)$$

Finally, the scalar absorption coefficients, A^{ii} and A^i , are given simply by

$$A^{ii} = 4g_i^2(\mathbf{p}) \frac{1}{v(\mathbf{p})} \Gamma_{ii}(0) C \quad (5.20)$$

and

$$A^i = 2E\{\omega_i(\mathbf{x}, \mathbf{p}; \alpha)\}. \quad (5.21)$$

The parameters B and C , introduced earlier in this subsection as special integrals of the correlation coefficient, can be considered as measures of the correlation distance of the random processes $\delta\chi_r(\mathbf{x}; \alpha)$ and $\delta\chi_i(\mathbf{x}; \alpha)$, without any reference to specialized correlation functions. Such a general definition reduces considerably the mathematical complexity of having to work with specific correlation functions chosen from within an already plethoric set of physically meaningful ones, without at the same time detracting much from the physical content of the ensuing results. The motivation for this interpretation is given in the first part of Appendix A.

In the following, we shall use the convention

$$B = 1/l, \quad C = \lambda. \quad (5.22)$$

The parameters l and λ will be referred to in the sequel as the *correlation lengths* of the random process. An interpretation of these quantities in terms of the spectral correlation function is provided in the second part of Appendix A.

We now summarize the main results of this section. Introducing (5.9a), (5.13), (5.14), and (5.17)–(5.22) in (5.1), we obtain the general *Fokker–Planck equation*

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + \mathbf{v}(\mathbf{p}) \cdot \frac{\partial}{\partial \mathbf{x}} \right) E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\} \\ &= \frac{\partial}{\partial \mathbf{p}} \cdot \left[g_r^2(\mathbf{p}) \frac{\Gamma_{rr}(0)}{l} \frac{1}{v(\mathbf{p})} \left(\mathbf{1} - \frac{\mathbf{v}\mathbf{v}}{v^2} \right) \cdot \frac{\partial}{\partial \mathbf{p}} E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\} \right] \\ &+ \frac{\partial}{\partial \mathbf{p}} \cdot \left(g_r(\mathbf{p}) \frac{\partial}{\partial \mathbf{p}} g_r(\mathbf{p}) \Gamma_{rr}(0) \frac{\mathbf{v}(\mathbf{p})}{v^2(\mathbf{p})} \cdot \frac{\partial}{\partial \mathbf{x}} E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\} \right) \\ &- \frac{\partial}{\partial \mathbf{x}} \cdot \left(g_r(\mathbf{p}) \frac{\partial}{\partial \mathbf{p}} g_r(\mathbf{p}) \Gamma_{rr}(0) \frac{\mathbf{v}(\mathbf{p})}{v^2(\mathbf{p})} \cdot \frac{\partial}{\partial \mathbf{p}} E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\} \right) \\ &+ \frac{\partial}{\partial \mathbf{x}} \cdot \left[\left(\frac{\partial g_r(\mathbf{p})}{\partial \mathbf{p}} \right)^2 \Gamma_{rr}(0) \frac{\lambda}{v(\mathbf{p})} \cdot \frac{\partial}{\partial \mathbf{x}} E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\} \right] \\ &- \frac{\partial}{\partial \mathbf{p}} \cdot \left(2g_r(\mathbf{p})g_i(\mathbf{p})\Gamma_{ri}(0) \frac{\mathbf{v}(\mathbf{p})}{v^2(\mathbf{p})} E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\} \right) \\ &- \frac{\partial}{\partial \mathbf{x}} \cdot \left(2g_i(\mathbf{p}) \frac{\partial}{\partial \mathbf{p}} g_r(\mathbf{p}) \Gamma_{ri}(0) \frac{\lambda}{v(\mathbf{p})} E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\} \right) \\ &+ 2g_r(\mathbf{p})g_i(\mathbf{p})\Gamma_{ir}(0) \frac{\mathbf{v}(\mathbf{p})}{v^2(\mathbf{p})} \cdot \frac{\partial}{\partial \mathbf{p}} E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\} \\ &- 2g_i(\mathbf{p}) \frac{\partial}{\partial \mathbf{p}} g_r(\mathbf{p}) \Gamma_{ir}(0) \frac{\lambda}{v(\mathbf{p})} \cdot \frac{\partial}{\partial \mathbf{x}} E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\} \\ &+ 4g_i^2(\mathbf{p})\Gamma_{ii}(0) \frac{\lambda}{v(\mathbf{p})} E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\} \\ &+ 2E\{\omega_i(\mathbf{x}, \mathbf{p}; \alpha)\} E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\}. \end{aligned} \quad (5.23)$$

This equation should be augmented by the initial mean distribution function $E\{\hat{f}(\mathbf{x}, \mathbf{p}, 0; \alpha)\}$.

VI. SPECIAL CASES OF THE GENERAL FOKKER-PLANCK EQUATION

We present in this section two simplifications of the

general Fokker–Planck equation (5.23) corresponding to spatially homogeneous system and isotropic media.

A. Spatially homogeneous systems

Besides the assumptions (i) and (ii) made in the previous section we must also impose in this case the condition

$$\frac{\partial}{\partial \mathbf{x}} E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\} = 0. \quad (6.1)$$

It follows, then, that the only nonvanishing of the coefficients (5.2) are \mathbf{D}_{pp}^{rr} , \mathbf{F}_p^{ir} , \mathbf{F}_p^{ir} , A^{ii} , A^i , and (5.23) reduces to the following relaxation equation in momentum space:

$$\begin{aligned} & \frac{\partial}{\partial t} E\{\hat{f}(\mathbf{p}, t; \alpha)\} \\ &= \frac{\partial}{\partial \mathbf{p}} \cdot \left[g_r^2(p) \frac{\Gamma_{rr}(0)}{l} \frac{1}{v(p)} \left(\mathbf{1} - \frac{\mathbf{v}\mathbf{v}}{v^2} \right) \cdot \frac{\partial}{\partial \mathbf{p}} E\{\hat{f}(\mathbf{p}, t; \alpha)\} \right] \\ & - \frac{\partial}{\partial \mathbf{p}} \cdot \left(2g_r(p)g_i(p)\Gamma_{ir}(0) \frac{\mathbf{v}(p)}{v^2(p)} E\{\hat{f}(\mathbf{p}, t; \alpha)\} \right) \\ & + 2g_r(p)g_i(p)\Gamma_{ir}(0) \frac{\mathbf{v}(p)}{v^2(p)} \cdot \frac{\partial}{\partial \mathbf{p}} E\{\hat{f}(\mathbf{p}, t; \alpha)\} \\ & + 4g_i^2(p)\Gamma_{ii}(0) \frac{\lambda}{v(p)} E\{\hat{f}(\mathbf{p}, t; \alpha)\} \\ & + 2E\{\omega_i(\mathbf{x}, \mathbf{p}; \alpha)\} E\{\hat{f}(\mathbf{p}, t; \alpha)\}. \end{aligned} \quad (6.2)$$

When written in spherical coordinates in \mathbf{p} -space, (6.2) is a generalization of the Fokker–Planck equation obtained by Chernov⁸ for the probability, $P(\theta, \phi, s)$, of ray directions (θ, ϕ are spherical coordinates) of arc length s in the Markovian approximation. It is also related to the expression for the coherent distribution function, $E\{f(\mathbf{v}, t; \alpha)\}$, in the problem of stochastic acceleration of uniformly distributed particles under the action of time-independent electric and magnetic fields.

B. Isotropic, dissipative and/or dispersive systems²⁵

As a consequence of the isotropy of the medium, $\mathbf{v}(\mathbf{p}) = v(p)\hat{\mathbf{p}}$, $g_r(\mathbf{p}) = g_r(p)$, $g_i(\mathbf{p}) = g_i(p)$, $(\partial/\partial \mathbf{p})g_r(\mathbf{p}) = (\partial/\partial p)g_r(p)\hat{\mathbf{p}}$, and $E\{\omega_i(\mathbf{x}, \mathbf{p}; \alpha)\} = E\{\omega_i(\mathbf{x}, p; \alpha)\}$, where $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$. Under these conditions, the general Fokker–Planck equation (5.23) simplifies to

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + V(p)\hat{\mathbf{p}} \cdot \frac{\partial}{\partial \mathbf{x}} \right) E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\} \\ &= D_x(p) \left(\hat{\mathbf{p}} \cdot \frac{\partial}{\partial \mathbf{x}} \right)^2 E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\} \\ & + D_p(p) \left(\mathbf{p} \times \frac{\partial}{\partial \mathbf{p}} \right)^2 E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\} \\ & + 2\nu_{\text{eff}}(p) E\{\hat{f}(\mathbf{x}, \mathbf{p}, t; \alpha)\}, \end{aligned} \quad (6.3a)$$

with

$$\begin{aligned} V(p) &= v(p) + \frac{1}{2} \frac{\partial}{\partial p} \left[\left(\frac{\partial}{\partial p} g_r^2(p) \right) \frac{1}{v(p)} \right] \Gamma_{rr}(0) \\ & + \frac{1}{2} \left(\frac{\partial}{\partial p} g_r^2(p) \right) \frac{1}{v(p)} \Gamma_{rr}(0) \frac{\partial}{\partial \mathbf{p}} \cdot \hat{\mathbf{p}} \end{aligned} \quad (6.3b)$$

$$- 4g_i(p) \left(\frac{\partial}{\partial p} g_r(p) \right) \frac{\lambda}{v(p)} \Gamma_{ri}(0),$$

$$D_x(p) = \left(\frac{\partial}{\partial p} g_r(p) \right)^2 \frac{\lambda}{v(p)} \Gamma_{rr}(0), \quad (6.3c)$$

$$D_p(p) = \frac{1}{p^2} \frac{g_r^2(p)}{v(p)} \frac{\Gamma_{rr}(0)}{l}, \quad (6.3d)$$

$$\begin{aligned} \nu_{\text{eff}}(p) &= E\{\omega_i(\mathbf{x}, p; \alpha)\} + \frac{1}{2} \left\{ - \frac{\partial}{\partial p} \left(2g_r(p)g_i(p) \frac{1}{v(p)} \right) \Gamma_{ri}(0) \right. \\ & - \left. \left(2g_r(p)g_i(p) \frac{1}{v(p)} \right) \Gamma_{ri}(0) \frac{\partial}{\partial \mathbf{p}} \cdot \hat{\mathbf{p}} \right. \\ & \left. + 4g_i^2(p) \frac{\lambda}{v(p)} \Gamma_{ii}(0) \right\}. \end{aligned} \quad (6.3e)$$

In (6.3a), the left-hand side describes convection with modified group velocity $V(p)$, the first term on the right-hand side describes longitudinal spatial diffusion, the second term arises because of angular diffusion in momentum space, and, finally, the last term designates the effective absorption in the random medium.

In Appendix B we shall discuss the specific forms of Eq. (6.3a), as well as the coefficients $V(p)$, $D_x(p)$, $D_p(p)$, and $\nu_{\text{eff}}(p)$, for three-, two-, and one-dimensional problems.

APPENDIX A: PHYSICAL INTERPRETATION OF THE PARAMETERS l AND λ

In order to motivate our interpretation in Sec. 5 of the quantities l and λ as general measures of the correlation length of the random process, without recourse to specialized correlation functions, we refer to two special, but widely used correlation coefficients:

$$(i) \rho(y) = \exp(-y^2/L^2), \quad l = L/\sqrt{\pi}, \quad \lambda = (\sqrt{\pi}/2)L, \quad (A1)$$

$$(ii) \rho(y) = [1 + (y/L)^2]^{-2}, \quad l = (4/3\pi)L, \quad \lambda = (\pi/8)L. \quad (A2)$$

In these two examples, L denotes the correlation length of the random process under consideration.

In the second part of this appendix, we provide a physical interpretation of the parameters l and λ with the aid of the spectral correlation function.

In Sec. 5, the correlation lengths l and λ were defined in terms of the correlation coefficient $\rho(y)$ as follows:

$$\frac{1}{l} = - \int_0^\infty dy \frac{1}{y} \frac{\partial}{\partial y} \rho(y), \quad (A3)$$

$$\lambda = \int_0^\infty dy \rho(y). \quad (A4)$$

The spectral correlation function is introduced next by means of the integral

$$S(p) = (2\pi)^{-3} \int_{R^3} d\mathbf{y} \rho(y) \exp(-i\mathbf{p} \cdot \mathbf{y}) \quad (A5)$$

Integrating (A5) over \mathbf{p} yields

$$\int_{R^3} d\mathbf{p} S(p) = 1 \quad (A6)$$

since $\rho(0) = 1$. An inversion of the Fourier transformation (A5) results in the expression

$$\rho(y) = (4\pi/y) \int_0^\infty dp p (\sin py) S(p). \quad (A7)$$

Introducing this result in (A3) and (A4) gives rise to the relations

$$\frac{1}{l} = -4\pi \int_0^\infty dp p S(p) \left[\int_0^\infty dy \frac{1}{y} \frac{\partial}{\partial y} \left(\frac{\sin py}{y} \right) \right], \quad (A8)$$

$$\lambda = 4\pi \int_0^\infty dp p S(p) \left[\int_0^\infty dy \left(\frac{\sin py}{y} \right) \right], \quad (A9)$$

However, the definite integrals over y appearing in (A8) and (A9) can be carried out explicitly, viz.,

$$\int_0^\infty dy \frac{1}{y} \frac{\partial}{\partial y} \left(\frac{\sin py}{y} \right) = -\frac{\pi}{4} p^2, \quad (A10)$$

$$\int_0^\infty dy \frac{\sin py}{y} = \frac{\pi}{2}. \quad (A11)$$

Therefore, one finally has

$$1/l = (\pi/4) \int_0^\infty dp p^3 S(p), \quad (A12)$$

$$\lambda = (\pi/2) \int_0^\infty dp p S(p). \quad (A13)$$

Similar expressions can be written for the two-dimensional and the one-dimensional case.

APPENDIX B: THREE-, TWO-, AND ONE-DIMENSIONAL FOKKER-PLANCK EQUATIONS

When the Fokker—Planck equation (6.3a) for an isotropic, dissipative and/or dispersive medium is considered in a three-dimensional Euclidean space, it is convenient to introduce a spherical polar coordinate system in momentum space: $\mathbf{p} = (p, \theta, \phi)$. Then we use (6.3a) with

$$\left(\mathbf{p} \times \frac{\partial}{\partial \mathbf{p}} \right)^2 = \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right). \quad (B1)$$

The coefficients $V(p)$, $D_x(p)$, ..., are given as in (6.3b), (6.3c), ..., with $(\partial/\partial \mathbf{p}) \cdot \hat{\mathbf{p}} = (2/p)$.

In the two-dimensional case, we introduce a polar coordinate system in momentum space: $\mathbf{p} = (p, \phi)$. Then, in examining specific problems, we must use (6.3a) with

$$\left(\mathbf{p} \times \frac{\partial}{\partial \mathbf{p}} \right)^2 = \frac{\partial^2}{\partial \phi^2}, \quad \frac{\partial}{\partial \mathbf{p}} \cdot \hat{\mathbf{p}} = \frac{1}{p}. \quad (B2)$$

In the one-dimensional case one has

$$\left(\mathbf{p} \times \frac{\partial}{\partial \mathbf{p}} \right)^2 = 0, \quad \frac{\partial}{\partial \mathbf{p}} \cdot \hat{\mathbf{p}} = 0. \quad (B3)$$

Equation (6.3a) assumes the simpler form

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + V(p) \frac{\partial}{\partial x} \right) E\{\hat{f}(x, p, t; \alpha)\} \\ & = D_x(p) \frac{\partial^2}{\partial x^2} E\{\hat{f}(x, p, t; \alpha)\} \\ & \quad + 2\nu_{\text{eff}}(p) E\{\hat{f}(x, p, t; \alpha)\}, \end{aligned} \quad (B4a)$$

and the coefficients $V(p)$, $D_x(p)$, and $\nu_{\text{eff}}(p)$ are modified as follows:

$$V(p) = v(p) - \frac{1}{2} \left\{ \frac{\partial}{\partial p} \left(\frac{\partial}{\partial p} g_r^2(p) \frac{1}{v(p)} \right) \Gamma_{rr}(0) \right. \quad (B4b)$$

$$\left. - 4g_i(p) \frac{\partial}{\partial p} g_r(p) \frac{\lambda}{v(p)} \Gamma_{ri}(0) \right\},$$

$$D_x(p) = \left(\frac{\partial}{\partial p} g_r(p) \right)^2 \frac{\lambda}{v(p)} \Gamma_{rr}(0), \quad (B4c)$$

$$\nu_{\text{eff}}(p) = E\{\omega_i(x, p; \alpha)\}$$

$$+ \left\{ - \frac{\partial}{\partial p} \left(2g_r(p) g_i(p) \frac{1}{v(p)} \right) \Gamma_{ri}(0) \right. \quad (B4d)$$

$$\left. + 4g_i^2(p) \frac{\lambda}{v(p)} \Gamma_{ii}(0) \right\}.$$

The transport equation (B4) has been used to study the problem of wave packet spreading on a random transmission line (cf. Ref. 3) and the propagation of frequency-modulated pulses in a randomly stratified plasma (cf. Ref. 4).

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²F. D. Tappert, SIAM Rev. (Chronicle) **13**, 281 (1971).

³F. D. Tappert and I. M. Besieris, "Stochastic Wave Kinetic Equation and its Application to Wave Packet Spreading," URSI Intern. Symp. on Electromagnetic Wave Theory, Tbilisi, USSR, Izdatel'stro Nauka, Sibirskoe Otdelenie, Novosibirsk, SSR, pp. 230-34 (1971).

⁴I. M. Besieris and F. D. Tappert, J. Appl. Phys. **44**, 2119 (1973).

⁵I. M. Besieris and F. D. Tappert, J. Math. Phys. **14**, 704 (1973).

⁶I. M. Besieris and F. D. Tappert, J. Math. Phys. **14**, 1829 (1973).

⁷H. Bremmer, Radio Sci. **8**, 511 (1973).

⁸V. Y. Kharanen, Dokl. Akad. Nauk SSSR **88**, 253 (1953).

⁹L. A. Chernov, *Wave Propagation in a Random Medium* (McGraw-Hill, New York, 1960).

¹⁰J. B. Keller, Proc. Symp. Appl. Math. **13**, 227 (1962).

¹¹U. Frisch, "Wave Propagation in Random Media," in *Probabilistic Methods in Applied Mathematics*, Vol. I, edited by A. T. Bharucha-Reid (Academic, New York, 1968).

¹²In the geometric, ray-optical approach to random media, a linear, Liouville-type, partial differential equation is set up for $P(\mathbf{x}, \mathbf{u}, s) = \delta(\mathbf{x} - \mathbf{x}(s)) \delta(\mathbf{u} - \mathbf{u}(s))$, where $\mathbf{x}(s)$ and $\mathbf{u}(s)$ denote respectively the position and direction of a ray of arclength s .

The ensemble average of $P(\mathbf{x}, \mathbf{u}, s)$ gives the simultaneous probability of position and ray direction.

- ¹²R. Kubo, *J. Math. Phys.* **4**, 174 (1963); S. Orszag and R. Kraichnan, *Phys. Fluids* **10**, 1720 (1967); A.N. Kaufman, *Phys. Fluids* **11**, 326 (1968); E.C. Roelof, *Can. J. Phys.* **46**, S990 (1969); C.E. Newman, P.A. Sturrock, and E. Tadamaru, *Astrophys. Space Sci.* **10**, 102 (1971); G. Papanicolau, *J. Math. Phys.* **12**, 1493 (1971); J.R. Jokipii, *Astrophys. J.* **172**, 319 (1972).
- ¹³R.M. Lewis, *Arch. Ratl. Mech. Anal.* **20**, 191 (1965).
- ¹⁴E. Wigner, *Phys. Rev.* **40**, 749 (1932).
- ¹⁵J.E. Moyal, *Proc. Cambridge Phil. Soc.* **45**, 99 (1949); B. Leaf, *J. Math. Phys.* **9**, 65, 769 (1968).
- ¹⁶The space and momentum coordinates \mathbf{x} , \mathbf{p} , as well as the parameter α appearing in the argument of the distribution function will occasionally be suppressed for convenience in this section.

¹⁷R. Balescu, *Physica* **38**, 98 (1968); **42**, 464 (1969).

¹⁸I.M. Besieris, *J. Math. Phys.* **13**, 358 (1972).

¹⁹J. Weinstock, *Phys. Fluids* **12**, 1045 (1969); **13**, 2308 (1970).

²⁰R. Balescu and J.H. Misguich, *J. Plasma Phys.* **11**, 357, 377 (1974); **13**, 33, 53 (1974).

²¹W.C. Meecham, "On Radiation in a Randomly Inhomogeneous Medium," Space Tech. Lab. Rpt. BSD-TR-61-36, Los Angeles, California (1961).

²²R.C. Bourret, *Nuovo Cimento* **26**, 1 (1962); *Can. J. Phys.* **40**, 782 (1962).

²³J.B. Keller, *Proc. Symp. Appl. Math.* **16**, 145 (1964).

²⁴In writing down (3.12), it has been assumed that the operator L_1 has zero mean.

²⁵The main results in this section have been used in Ref. 2 in order to examine the problem of spreading of planar wave-packets having Gaussian envelopes.