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The half-Hartley and the half-Hilbert transform

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Inversion formulas are obtained for the restrictions of the Hartley and Hilbert transforms to $\mathbb{R}^+$. Regularity results are derived, and an illustrative example presented.

I. INTRODUCTION

The Hartley transform arises in various areas of engineering, especially in signal processing;[1] it is defined by the following formula:

$$g_H(t)=\int_{-\infty}^{\infty} g(\omega)(\cos \omega t+\sin \omega t)d\omega, \quad t\in \mathbb{R}. \tag{1}$$

Its properties such as regularity, inversion formula, etc., can be deduced from Ref. 1. In this article we are interested not in Eq. (1) but in the half-Hartley transform which can arise in certain types of transport problems[2]

$$g_h(t)=\int_{0}^{\infty} g(\omega)(\cos \omega t+\sin \omega t)d\omega, \quad t\in \mathbb{R}^+. \tag{1.5}$$

Similarly, the Hilbert transform is defined by[3,4]

$$H_x(f(y))=\mathscr{P}\int_{-\infty}^{\infty} f(y)\,\frac{dy}{y-x}, \quad y\in \mathbb{R}. \tag{2}$$

As in the case of the Hartley transform, we shall be concerned with the half-Hilbert transform

$$h_x(f(y))=\mathscr{P}\int_{0}^{\infty} f(y)\,\frac{dy}{y-x}, \quad y>0. \tag{2.5}$$

The symbol "$\mathscr{P}$" indicates Cauchy principal value. The notation for the transforms "H" and "h" is adapted from Ref. 3.

The inversion formula for the (full-range) Hartley and Hilbert transforms (1) and (2) are well known

$$g(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} g_H(t)(\cos \omega t+\sin \omega t)dt, \tag{1^{-1}}$$

$$f(y)=-\frac{1}{\pi^2}\int_{-\infty}^{\infty} H_x(f)\,\frac{dy}{x-y}\,. \tag{2^{-1}}$$

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On the other hand it turns out that the inverse formulas for Eqs. (1.5) and (2.5) are not known, insofar as we are able to ascertain. The purpose of this article is to obtain those formulas. In fact, we show that the half-Hartley transform (1.5) can be reduced to the half-Hilbert transform (2.5), and determine the inversion formula for the latter.

Before proceeding, we quote (in part) two standard properties of Hilbert transforms which are needed in the subsequent analysis.

Proposition 1 (Ref. 5): Let $f \in L^p(\mathbb{R})$, $1<p<\infty$. *Then* $H_x(f) \in L^p(\mathbb{R})$.

Proposition 2 (Refs. 3, 6, and 7) (The Poincaré–Bertrand formula): For appropriate restrictions on the functions f and g

$$\mathscr{P}\int_I g(y)\,\frac{dy}{y-\omega}\;\mathscr{P}\int_I f(x)\,\frac{dx}{x-y} = -\pi^2 g(\omega)f(\omega) + \mathscr{P}\int_I f(x)dx\;\mathscr{P}\int_I g(y)\,\frac{dy}{(y-\omega)(x-y)}\,. \tag{3}$$

Here I may be an interval of $\mathbb{R}$ (Refs. 3,7) or an arc in $\mathbb{C}$ (Ref. 6). In Ref. 3, the "appropriate restrictions" are $g \in L^p(I)$, $f \in L^q(I)$, $(1/p)+(1/q)<1$, while in Ref. 6 it is required that f and g be (uniformly) Hölder continuous on I. (The assumptions on f and g made in Ref. 7 are not relevant to our problem, and will not be stated.)

We remind the reader that, as used in Ref. 6, the term "Hölder continous" is the same as "Lip-α", i.e., $\exists$ a constant C and a number $\alpha>0$ such that

$$|f(x)-f(y)| \leq C|x-y|^\alpha, \quad \forall x,y \in I. \tag{4}$$

The plan of this article is as follows. In Sec. II we reduce the half-Hartley transform (1.5) to the half-Hilbert transform (2.5); in Sec. III we obtain the inverse formula for the half-Hilbert transform. Section IV is devoted to a discussion of various regularity properties of the transforms and their inverses, and in Sec. V we present an illustrative example.

II. REDUCTION TO THE HALF-HILBERT TRANSFORM

Referring to Eq. (1.5) we define

$$g_h^c(k) = \int_0^\infty g_h(t)\cos kt\;dt \tag{4a}$$

and

$$g_h^s(k) = \int_0^\infty g_h(t)\sin kt\;dt. \tag{4b}$$

Assuming the existence of these two integrals, they can be computed by formal change of order of integration if we write

$$\int_0^\infty \cos kt \int_0^\infty g(\omega)\cos \omega t\;d\omega\;dt = \lim_{\epsilon\downarrow 0}\int_0^\infty g(\omega)d\omega \int_0^\infty e^{-\epsilon t}\cos kt\cos \omega t\;dt \tag{5}$$

and similarly for terms involving $\cos kt \sin \omega t$, $\sin kt \cos \omega t$, and $\sin kt \sin \omega t$.

Recalling[4]

$$\delta(x) = \lim_{\epsilon\downarrow 0}\frac{\epsilon}{\pi}\,\frac{1}{x^2+\epsilon^2}\,, \tag{6a}$$

$$\mathscr{P}\frac{1}{x}=\lim_{\epsilon\downarrow 0}\frac{x}{x^2+\epsilon^2} \tag{6b}$$

we find easily

$$g_h^c(k)=\frac{1}{2}\int_0^\infty g(\omega)\left[\pi\delta(\omega-k)+\frac{1}{\omega+k}+\mathscr{P}\frac{1}{\omega-k}\right]d\omega,\quad k>0 \tag{7a}$$

and

$$g_h^s(k)=\frac{1}{4}\int_0^\infty g(\omega)\left[2\pi\delta(\omega-k)+\frac{2}{\omega+k}-2\mathscr{P}\frac{1}{\omega-k}\right]d\omega,\quad k>0. \tag{7b}$$

Thus

$$g_h^c(k)-g_h^s(k)=\mathscr{P}\int_0^\infty g(\omega)\,\frac{d\omega}{\omega-k},\quad k>0. \tag{8}$$

This should be compared with Eq. (1^{-1}). The inversion of the half-Harley transform (1.5) is thus reduced formally to that of solving Eq. (8) for g, i.e., the inversion of the half-Hilbert transform.

III. INVERSION OF THE HALF-HILBERT TRANSFORM

Rewriting Eq. (8) in generic form

$$f(k)=\mathscr{P}\int_0^\infty g(\omega)\,\frac{d\omega}{\omega-k},\quad k>0 \tag{9}$$

we might attempt to solve for g by applying standard methods[6,8,9] for solving singular integral equations of the form

$$f(k)=\lambda(k)g(k)+\mathscr{P}\int_0^\infty \eta(\omega)g(\omega)\,\frac{d\omega}{\omega-k},\quad k>0. \tag{10}$$

Noting that $\lambda=0$, $\eta=1$, and referring to Refs. 6, 8, or 9, we see that the solution involves the solution $X(z)$ of the homogeneous Riemann–Hilbert problem

$$\frac{X^+(t)}{X^-(t)}=\frac{\lambda(t)+\pi i\eta(t)}{\lambda(t)-\pi i\eta(t)}=-1,\quad t\in[0,\infty), \tag{11a}$$

where

$$X^\pm=\lim_{\epsilon\downarrow 0}X(t\pm i\epsilon). \tag{11b}$$

The canonical solution of Eq. (11a) is[6,8,9]

$$X_0(t)=\exp\left\{\frac{1}{2}\int_0^\infty\frac{dt}{t-z}\right\}, \tag{12}$$

an integral which does not exist!

Let us therefore attempt to solve Eq. (4) by studying the equation

$$f(k)=\mathscr{P}\int_0^L g(\omega)\,\frac{d\omega}{\omega-k},\quad k\in(0,L). \tag{13}$$

The solution of Eq. (13) is given, for example, in Ref. 3 [or the methods of Refs. 6, 8, or 9 can be used to solve Eq. (13)]. In either case one finds

$$g(k)=-\frac{1}{\pi^2\sqrt{k(L-k)}}\left\{\mathscr{P}\int_0^\infty \sqrt{\omega(L-\omega)}f(\omega)\,\frac{d\omega}{\omega-k}+C\right\},\quad k\in(0,L), \tag{14}$$

where C is an arbitrary constant [which comes from the fact that the index of Eq. (13) is $+1$]. Letting $L\to\infty$, we obtain the putative solution

$$g(k)=-\frac{1}{\pi^2\sqrt{k}}\,\mathscr{P}\int_0^\infty \sqrt{\omega}f(\omega)\,\frac{d\omega}{\omega-k},\quad k>0. \tag{15}$$

It remains to be proven that Eq. (15) does provide a solution of Eq. (9). In point of fact, Eq. (15) is not the most general solution; we can state

Theorem 3: *Let f obey the conditions*

(i) $\sqrt{\omega}f$ *is Hölder continuous on* $[0,\infty)$,
(ii) $\sqrt{\omega}f\in L^p(\mathrm{R}^+)$ *for some p,* $1<p<\infty$,
(iii) $(1/\sqrt{k})h_k(\sqrt{\omega}f(\omega))\in L^q(\mathrm{R}^+)$ *for some q,* $1<q<\infty$.

Then a general solution to Eq. (9) is

$$g(k)=-\frac{1}{\pi^2\sqrt{k}}\left\{\mathscr{P}\int_0^\infty \sqrt{\omega}f(\omega)\,\frac{d\omega}{\omega-k}+C\right\}, \tag{16}$$

where C is an arbitrary constant.

The proof of Theorem 3 is accomplished by applying two lemmata.

Lemma 4:

$$I(t)=\mathscr{P}\int_0^\infty \frac{1}{\sqrt{k}}\,\frac{dk}{k-t}=0,\quad t>0. \tag{17}$$

Proof: Let $F(z)=(1/\sqrt{z})=(1/\sqrt{|z|})e^{i\theta/2}$, $0<\theta<2\pi$. Then $F(z)$ is analytic in $\mathrm{C}\setminus\mathrm{R}^+$ with boundary values $F^{\pm}(k)=\pm 1/\sqrt{k}$, $k>0$. Similarly[4]

$$\mathscr{P}\frac{dk}{k-t}=\lim_{\epsilon\downarrow 0}\frac{dk}{k\pm i\epsilon-t}\pm\pi i\,\delta(k-t)\,dk. \tag{18$\pm$}$$

Substituting into Eq. (17) we have

$$I(k)=\frac{1}{2}\left(\int_0^\infty F^+(k)\,\frac{dk}{k+i\epsilon-t}-\int_0^\infty F^-(k)\,\frac{dk}{k-i\epsilon-t}\right), \tag{19}$$

the terms arising from integration over the delta function canceling. By Cauchy's theorem

$$I(k)=\frac{1}{2}\oint_{C_\epsilon}F(z)\,\frac{dz}{z-k}-\frac{1}{2}\oint_{C_\infty}F(z)\,\frac{dz}{z-k},$$

where C_ϵ is an infinitesimal contour surrounding the point $z=0$ and C_∞ is a circle at infinity. Because $F(z)=1/\sqrt{z}$, both integrals vanish, proving the lemma.

Remark. The technique used to prove the lemma, i.e., expressing $\sqrt{k}$ as the difference of boundary values and replacing the Cauchy denominator as in Eq. (18), will be used subsequently for a number of other integrals.

Lemma 5: Let ϕ obey conditions (i)–(iii) of Theorem 3. Then

$$\mathscr{P}\int_0^\infty \frac{1}{\sqrt{k}}\frac{dk}{k-t}\ \mathscr{P}\int_0^\infty \sqrt{\omega}\phi(\omega)\ \frac{d\omega}{\omega-k}=-\pi^2\phi(t)+\mathscr{P}\int_0^\infty \sqrt{\omega}\phi(\omega)d\omega\ \mathscr{P}\int_0^\infty \frac{1}{\sqrt{k}}\frac{1}{\omega-k}\frac{1}{k-t}\ dk \tag{20a}$$

$$=-\pi^2\phi(t). \tag{20b}$$

Remark: This is the Poincaré–Bertrand formula, Proposition 2, except that none of the "appropriate conditions" quoted there is satisfied (in particular, $1/\sqrt{k}$ is in no L^p space, nor is it Hölder continuous near zero).

Proof: Equation (20b) follows from Eq. (20a) by partial fraction decomposition and Lemma 4. To prove Eq. (20a), we adapt the proof of Muskhilishvili, Ref. 6, pp. 57–60. Note by hypothesis plus Proposition 1, the left-hand side of Eq. (20a) exists, as does the right-hand side by Lemma 4. The basic idea, then, is to replace the real variable t by the complex variable z. Then if the order of integration can be interchanged for Im $z\neq 0$ the result follows from simple application of the Plemelj formulas (18±)—we do not repeat the details here. We do, below, give the crucial proof that for $t\in\mathbb{C}\setminus\mathbb{R}^+$, the order of the integrals in Eq. (20a), can indeed be reversed. Following Muskhilishvili's ideas (Ref. 6. pp. 59–60) it is sufficient to show for $t\in\mathbb{C}\setminus\mathbb{R}^+$ that

$$I_1=\int_0^\infty \frac{1}{\sqrt{k}}\frac{dk}{k-t}\ \lim_{\epsilon\downarrow 0}\mathscr{P}\int_{\alpha(k)}^{k+\epsilon} \sqrt{\omega}\phi(\omega)\ \frac{d\omega}{\omega-k}=0 \tag{21a}$$

and

$$I_2=\mathscr{P}\int_0^\infty \sqrt{\omega}\phi(\omega)d\omega\ \lim_{\epsilon\downarrow 0}\mathscr{P}\int_\omega^{\omega+\epsilon} \frac{1}{\sqrt{k}}\frac{dk}{(\omega-k)(k-t)}=0, \tag{21b}$$

where

$$\alpha(t)=\max(t-\epsilon,0). \tag{21c}$$

The fact that $I_1=0$ follows as in Ref. 6 since $\sqrt{t}\phi$ is Hölder continuous on $\mathbb{R}^+$. So we need to prove $I_2=0$. Decomposing by partial fractions

$$I_2=\int_0^\infty \sqrt{\omega}\phi(\omega)\ \frac{d\omega}{\omega-t}\ (-J_1(\omega)+J_2(\omega)), \tag{22a}$$

where

$$J_1(\omega)=\lim_{\epsilon\downarrow 0}\mathscr{P}\int_{\alpha(\omega)}^{\omega+\epsilon}\frac{1}{\sqrt{k}}\frac{dk}{k-\omega}, \tag{22b}$$

$$J_2(\omega)=\lim_{\epsilon\downarrow 0}\int_{\alpha(\omega)}^{\omega+\epsilon}\frac{1}{\sqrt{k}}\frac{dk}{k-t}. \tag{22c}$$

We show the result for J_1; J_2 is similar. Since $1/\sqrt{k}$ is Hölder continuous on the range $[\delta,\infty)$, $\delta>0$, we need only consider

$$I_2'=\int_0^\delta \sqrt{\omega}\phi(\omega)J_1(\omega)\,\frac{d\omega}{\omega-t}.$$

Rewrite

$$J_1(\omega)=\lim_{\epsilon\downarrow 0}\int_{\alpha(\omega)}^{\omega+\epsilon}\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{\omega}}\right)\frac{dk}{k-\omega}+\lim_{\epsilon\downarrow 0}\frac{1}{\sqrt{\omega}}\,\mathscr{P}\int_{\alpha(\omega)}^{\omega+\epsilon}\frac{dk}{k-\omega}. \tag{23}$$

In Eq. (23), the second term tends uniformly to zero, just as in Ref. 6 [note the cancellation of the $\sqrt{\omega}$ factors when it is substituted into Eq. (22a)]. The first term yields

$$I_2'=-\int_0^\delta \phi(\omega)\,\frac{d\omega}{\omega-t}\,\lim_{\epsilon\downarrow 0}\int_{\alpha(\omega)}^{\omega+\epsilon}\frac{dk}{k+\sqrt{k\omega}}=-2\int_0^\delta \phi(\omega)\,\frac{d\omega}{\omega-t}\,\lim_{\epsilon\downarrow 0}\ln\frac{\sqrt{\omega+\epsilon}+\sqrt{\omega}}{\sqrt{\alpha(\omega)}+\sqrt{\omega}}. \tag{24}$$

Rewriting this integral as

$$I_2'=-2\lim_{\epsilon\downarrow 0}\left(\int_0^\epsilon \phi(\omega)\,\frac{d\omega}{\omega-t}\,\ln\frac{\sqrt{\omega+\epsilon}+\sqrt{\omega}}{\sqrt{\omega}}+\int_\epsilon^\delta \phi(\omega)\,\frac{d\omega}{\omega-t}\,\ln\frac{\sqrt{\omega+\epsilon}+\sqrt{\omega}}{\sqrt{\omega-\epsilon}+\sqrt{\omega}}\right) \tag{25}$$

we see that both terms tend to zero as $\epsilon\to 0$, proving the Lemma.

Proof of Theorem 3: Substitute the putative solution (16) into Eq. (9) to obtain the conjecture

$$f(k)=\mathscr{P}\int_0^\infty -\frac{1}{\pi^2\sqrt{\omega}}\left[\mathscr{P}\int_0^\infty \sqrt{\omega_1}f(\omega_1)\,\frac{d\omega_1}{\omega_1-\omega}+C\right]\frac{d\omega}{\omega-k},\quad k>0. \tag{26}$$

The term proportional to the constant C vanishes by virtue of Lemma 4. By Lemma 5, the iterated integral term reduces simply to $f(k)$, and Eq. (26) becomes the tautology $f(k)=f(k)$, proving the theorem.

IV. REGULARITY

The question we address in this section is, given some L^p property on f in Eq. (9), what L^q property or properties are induced for g. Since $1/\sqrt{k}$ is in no L^q space over $\mathbb{R}^+$, this makes it convenient to set $C=0$ in Eq. (16). We repeat for convenience, the formulas for the half-Hilbert transform and its inverse

$$f(k)=\mathscr{P}\int_0^\infty g(\omega)\,\frac{d\omega}{\omega-k},\quad k>0, \tag{27a}$$

$$g(\omega)=-\frac{1}{\pi^2\sqrt{\omega}}\,\mathscr{P}\int_0^\infty \sqrt{k}f(k)\,\frac{d\omega}{k-\omega},\quad k>0 \tag{27b}$$

and state

Proposition 6: Let $f\in L^p(\mathbb{R}^+)\cap L^r(\mathbb{R}^+)$, $2\leq r<p$. Then $(1/\sqrt{k})f\in L^q(\mathbb{R}^+)$ for every q satisfying $2r/(r+2)<q<\min(4r/(r+2),2p/(p+2))$.

The proof is based on the following result:[10]

Lemma 7 (Carlson's inequality): Let $\tilde{p}$, $\tilde{q}$, λ, and μ be positive constants with $\lambda<\tilde{p}+1$, $\mu<\tilde{q}+1$. Then

$$\left(\int_0^\infty \tilde{f} dx\right)^{\mu+\lambda} \leq K\left(\int_0^\infty x^{\tilde{p}-\lambda}\tilde{f}^{\tilde{p}+1}\right)^\mu\left(\int_0^\infty x^{\tilde{q}+\mu}\tilde{f}^{\tilde{p}+1}\right)^\lambda \tag{28}$$

for some constant K.

Proof of Proposition 6: Let $\tilde{f}=x^{-q/2}|f|^q$, $(\tilde{p}+1)q=p$, and $(\tilde{q}+1)=r$ in Eq. (28). Then the conditions on $\tilde{p}$, $\tilde{q}$, λ, and μ give the stated result.

From Proposition 1, Proposition 6, and Theorem 3 we have

Theorem 8: *Let $\sqrt{k}f\in L^p(\mathbb{R}^+)\cap L^r(\mathbb{R}^+), 2\leq r<p\leq\infty$ and obey (i)–(iii) of Theorem 3. Then Eq. (27b) is the inverse formula to Eq. (27a) and $g\in L^q(\mathbb{R}^+)$ for every q satisfying $2r/(r+2)<q<\min(4r/(r+2),2p/(p+2))$.*

Corollary 9: A necessary condition that Eq. (27b) hold is that $f\in L^q(\mathbb{R}^+)$ for some $q, 1<q<2$.

The proof follows either from Proposition 1 or Proposition 6. The following result is actually a corollary to the proof of Theorem 3.

Corollary 10: Let g_h [Eq. (1.5)] be such that $\sqrt{\omega}(g_h^c-g_h^s)$ [Eq. (8)] satisfies the conditions of Theorem 8. Then the inverse half-Hartley transform g exists, and is given by Eq. (27b) with $f=g_h^c-g_h^s$. Further, $g\in L^p(\mathbb{R}^+)$ for every q satisfying $2r/(r+2)<q<\min(4r/(r+2),2p/(p+2))$.

Remark: Recalling Proposition 6 as well as the fact that the Fourier transform maps L^p into its conjugate space,[11] one sees that the inverse of the half-Hartley transform has been obtained only if g_h, Eq. (1.5), is an element of some L^p space, $p>2$.

Corollary 11: Let ϕ be Hölder continuous on $[0,\infty)$ and let the integral

$$Y=\int_0^\infty \frac{1}{\sqrt{k}}\, dk\, \mathscr{P}\int_0^\infty \phi(t)\,\frac{dt}{t-k}$$

exist. Then Y=0.

Proof: If the order of integration can be reversed, the result follows from Lemma 4. The proof of interchange of order is similar to (but simpler than) the proof of Theorem 3.

From the corollary we have immediately

Proposition 12: Let $f=h_x(g)$ for some g. Then if

$$J=\int_0^\infty \frac{1}{\sqrt{x}}\, f(x)dx$$

exists, J=0.

This imposes a useful constraint on the data (i.e., the left-hand side) of the half-Hilbert transform which is well known in the case of the finite-range Hilbert transform.[3,12]

V. AN EXAMPLE

We solve Eq. (1.5) for $g_h(t)=e^{-\alpha t}$, $\alpha>0$, $t\in\mathbb{R}^+$. Then [Eq. (4)]

$$g_h^c(k)=\frac{\alpha}{\alpha^2+k^2}, \quad g_h^s(k)=\frac{k}{\alpha^2+k^2}.$$

Thus $g(\omega)$ is given by Eq. (27b) with

$$f(k)=g_h^c(k)-g_h^s(k)=\frac{\alpha-k}{\alpha^2+k^2}.$$

It is easy to see that $f(k)$ obeys the hypotheses of Theorems 3 and 8.

We can also check Proposition 11. Since

$$J=\int_0^\infty \frac{(\alpha-k)}{\alpha^2+k^2}\frac{dk}{\sqrt{k}}$$

exists, we expect $J=0$. J can be evaluated explicitly by contour integration as in Lemma 4, i.e., we express $1/\sqrt{k}$ as the difference of the boundary values of $F(z)=1/\sqrt{z}$ and close the contour as in Lemma 4. Evidently, J is given by the residues at $\pm i\alpha$

$$J=\frac{2\pi i}{2}\left[\frac{\alpha-i\alpha}{2i\alpha}\frac{1}{\sqrt{\alpha}}-\frac{\alpha+i\alpha}{2i\alpha}\frac{1}{\sqrt{-\alpha}}\right]=0.$$

Explicitly

$$g(\omega)=-\frac{1}{\pi^2\sqrt{\omega}}\,\mathscr{P}\int_0^\infty \sqrt{k}\,\frac{(\alpha-k)}{\alpha^2+k^2}\frac{dk}{k-\omega}. \tag{29}$$

This integral can be evaluated by contour integration in a manner similar to that used in proving $J=0$ above, that is, we write

$$\sqrt{k}=\tfrac{1}{2}(F^+(k)-F^-(k)), \tag{30a}$$

where

$$F(z)=\sqrt{z} \tag{30b}$$

and use Eq. (18±) to express $\mathscr{P}(dk/(k-\omega))$. Closing the contour around $\mathbb{R}^+$ and at infinity, we obtain (taking into account the residues at $\pm i\alpha$),

$$g(\omega)=\frac{1}{\pi}\frac{\sqrt{2\alpha\omega}}{\alpha^2+\omega^2}, \tag{31}$$

that is,

$$e^{-\alpha t}=\frac{1}{\pi}\sqrt{2\alpha}\int_0^\infty \frac{\sqrt{\omega}}{\alpha^2+\omega^2}(\cos\omega t+\sin\omega t)d\omega, \quad t\geqslant 0. \tag{32}$$

As a check, we might integrate this equation over t from zero to infinity. Changing the order of the double integral as in Eq. (5) we have, using Eqs. (6)

$$\frac{1}{\alpha}=\frac{1}{\pi}\sqrt{2\alpha}\int_0^\infty \frac{\sqrt{\omega}}{\alpha^2+\omega^2}\left(\pi\delta(\omega)+\mathscr{P}\frac{1}{\omega}\right)d\omega \tag{33a}$$

$$=\frac{1}{\pi}\sqrt{2\alpha}\int_0^\infty \frac{1}{\alpha^2+\omega^2}\frac{d\omega}{\sqrt{\omega}}. \tag{33b}$$

This integral can be evaluated in the usual manner, i.e., by converting to a contour integral, and we verify easily that Eq. (33b) is a tautology.

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[1] R. N. Bracewell, *The Hartley Transform* (Oxford University, London, 1968).
[2] S. Paveri-Fontana and P. F. Zweifel, Trans. Theor. Stat. Phys. (to be published).
[3] F. G. Tricomi, *Integral Equations* (Interscience, New York, 1967).
[4] B. W. Roos, *Analytic Functions and Distributions in Physics and Engineering* (Wiley, New York, 1969).
[5] E. C. Titchmarsh, *Eigenfunction Expansions* (Oxford University, London, 1946).
[6] N. I. Muskhilishvili, *Singular Integral Equations* (Noordhoff, Groningen, 1953).
[7] F. G. Tricomi, Rend. Lincei **18**, 3 (1955).
[8] R. Estrada and R. P. Kanwal, SIAM Rev. **29**, 263 (1987).
[9] C. V. M. van der Mee and P. F. Zweifel, J. Int. Eqs. Appl. **2**, 185 (1990).
[10] E. F. Beckenbach and R. Bellman, *Inequalities* (Springer-Verlag, Berlin, 1961), p. 176.
[11] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II. Fourier Analysis, Self Adjointness* (Academic, New York, 1975), p. 32.
[12] N. Mohankumar, A. Natarajan, and P. V. Gopinath, Trans. Theor. Stat. Phys. **20**, 307 (1991).