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# The semiclassical limit of quantum dynamics. I. Time evolution

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The $\hbar \to 0$ limit of the quantum dynamics determined by the Hamiltonian $H(\hbar) = -(\hbar^2/2m)\Delta + V$ on $L^2(\mathbb{R}^n)$ is studied for a large class of potentials. By convolving with certain Gaussian states, classically determined asymptotic behavior of the quantum evolution of states of compact support is obtained. For initial states of class $C_0^1$ the error terms are shown to have $L^2$ norms of order $\hbar^{1/2-\epsilon}$ for arbitrarily small positive $\epsilon$.

## I. INTRODUCTION

Since the origin of quantum mechanics more than 60 years ago there has been much effort applied to the understanding of the relationship between the theory and its classical counterpart. Not only is such an understanding important from a purely theoretical viewpoint, but the mathematical techniques developed in order to study this question have provided physicists and chemists with useful and powerful computational tools. In this paper we study the relation between the classical and quantal descriptions of the dynamical evolution in the so-called semiclassical limit. We obtain results for the semiclassical time evolution, which, although basically being known from the work of Maslov[1] and Maslov–Fedoriuk,[2] are proved by new methods that we extend in a forthcoming paper[3] to the scattering theory. Our proofs rely heavily on the results of Hagedorn[4–6] concerning the semiclassical behavior of certain Gaussian initial states.

We now introduce the assumptions on the potential $V$ and state our main result. Our main theorem is concerned with the quantum evolution of certain initial states of compact support.

*Assumption 1.1:* We assume that $V \in C^{l+2}(\mathbb{R}^n,\mathbb{R})$ for some integer $l \geqslant 1 + n/2$ and that there exist positive constants $m_1$ and $m_2$ such that $|V(x)| \leqslant e^{m_1|x|^2}$ and $V(x) \geqslant -m_2$ for all $x \in \mathbb{R}^n$.

Under this assumption the quantum Hamiltonian $H(\hbar) = -(\hbar^2/2m)\Delta + V$ is essentially self-adjoint on the infinitely differentiable functions of compact support in $L^2(\mathbb{R}^n)$. The corresponding classical Hamiltonian is given by the function $H_{\text{cl}}(a,\eta) = (1/2m)|\eta|^2 + V(a)$ on $\mathbb{R}^{2n}$. For any initial condition $(a_0,\eta_0)$ in $\mathbb{R}^{2n}$ the system

$$\frac{\partial}{\partial t} a(t) = \frac{1}{m}\eta(t), \tag{1.1a}$$

$$\frac{\partial}{\partial t}\eta(t) = -\nabla V(a(t)) \tag{1.1b}$$

has a unique solution $(a(a_0,\eta_0,t),\ \eta(a_0,\eta_0,t))$ such that

$$(a(0),\eta(0)) = (a(a_0,\eta_0,0),\eta(a_0,\eta_0,0)) = (a_0,\eta_0).$$

The solution $(a(a_0,\eta_0,t),\eta(a_0,\eta_0,t))$ is bounded for all $t \in [0,T]$ for any finite $T > 0$ and is of class $C^{l+1}$ in the initial condition $(a_0,\eta_0)$ (see, e.g., Chaps. I and II of Lefschetz[7]).

**Theorem 1.2:** Suppose $V$ satisfies Assumption 1.1. Let $S_0 \in C^3(\mathbb{R}^n,\mathbb{R})$, $f \in C_0^1(\mathbb{R}^n,\mathbb{C})$, and $\lambda \in (0,\frac{1}{2})$. Define $Q(\cdot,t)$: $\mathbb{R}^n \mapsto \mathbb{R}^n$ by

$$Q(q_0,t) = a(q_0,\nabla S_0(q_0),t)$$

and assume that $t \geqslant 0$ is fixed and such that $\det[(\partial Q/\partial q_0)(x,t)] \neq 0$ for all $x \in \text{supp}(f)$. Then there exists $\delta > 0$ and a constant $C$ independent of $\hbar$ such that

$$\left\| e^{-itH(\hbar)/\hbar}(e^{iS_0/\hbar}f) - \sum_j e^{i(\mu_j + S_j/\hbar)} \left|\det\left[\frac{\partial Q}{\partial q_0}(x_j,t)\right]\right|^{-1/2} f(x_j) \right\| \leqslant C\hbar^\lambda,$$

for all $\hbar \leqslant \delta$. For fixed $x$, the summation is over all $j$ such that $Q(x_j,t) = x$ and hence is necessarily finite. Here $S_j(x)$ denotes the action

$$S(x_j) = S_0(x_j) + \int_0^t \left[\frac{1}{2m}|\eta(x_j,\nabla S_0(x_j),\tau)|^2 - V(a(x_j,\nabla S_0(x_j),\tau))\right] d\tau,$$

and $\mu_j$ is an integer multiple of $\pi/2$ defined explicitly in the proof.

*Remarks:* (1) The constants $C$ and $\delta$ are, in general, time dependent. The dependence of the rightmost function inside the norm [which is, of course, the norm of $L^2(\mathbb{R}^n,d^nx)$] on $x$ is given implicitly by $x = Q(x_j,t)$.

(2) The number $\mu_j$ is related to the Keller–Maslov index[1,8] of the path

$$\{(a(x_j,\nabla S_0(x_j),\tau),\eta(x_j,\nabla S_0(x_j),\tau)):\ 0 \leqslant \tau \leqslant t\}$$

and arises naturally in our proof in connection with the branch of the square root of $\det[(\partial Q/\partial q_0)(x,t)]$.

(3) The theorem is not a new result. Indeed, one can view our theorem as a slight generalization of the leading-order term in Theorem 12.3 of Maslov–Fedoriuk.[2] However, we prove our theorem by a different method and our proof extends to the scattering theory.

## II. NOTATION AND DEFINITIONS

Throughout this paper $n$ denotes the space dimension, and $L^2(\mathbb{R}^n)$ is the Hilbert space of square integrable com-

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plex valued functions on $\mathbb{R}^n$ with inner product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$ is given by

$$\langle f,g \rangle = \int_{\mathbb{R}^n} \bar{f}(x) g(x) d^n x, \quad \|f\| = \langle f,f \rangle^{1/2}.$$

For $\Omega$ an open subset of $\mathbb{R}^n$, $\mathbb{E} = \mathbb{R}$ or $\mathbb{C}$, and $K$ a non-negative integer, $C^k(\Omega,\mathbb{E})$ denotes the linear space of $k$-times continuously differentiable functions mapping $\Omega$ into $\mathbb{E}$. Here $C_0^k(\Omega,\mathbb{E})$ is the subspace of functions in $C^k(\Omega,\mathbb{E})$ with support compact and contained in $\Omega$. We will denote the support of a function $f$ by $\mathrm{supp}(f)$. The quantum mechanical Hamiltonian $H(\hbar)$ is the operator $-(\hbar^2/2m)\Delta + V$ on $L^2(\mathbb{R}^n)$. Here, $\Delta$ is the $n$-dimensional Laplacian operator,

$$\Delta = \left(\frac{\partial}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial}{\partial x_n}\right)^2,$$

$\hbar$ is a small positive parameter (a dimensionless multiple of Planck's constant), $m$ is a positive constant, and $V$ is a real valued function on $\mathbb{R}^n$ viewed here as a multiplication operator on $L^2(\mathbb{R}^n)$. For potentials $V$ satisfying Assumption 1.1 the operator $H(\hbar)$ is essentially self-adjoint on the domain $C_0^\infty(\mathbb{R}^n,\mathbb{C})$.

A multi-index $\alpha$ is an ordered $n$-tuple $(\alpha_1,\alpha_2,\ldots,\alpha_n)$ of non-negative integers. The order of a multi-index $\alpha$ is given by $|\alpha| = \Sigma_{i=1}^n \alpha_i$ and the factorial of $\alpha$ by

$$\alpha! = \prod_{i=1}^n (\alpha_i!).$$

For $x = (x_1,x_2,\ldots,x_n) \in \mathbb{R}^n$ the symbol $x^\alpha$ is defined by

$$x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}.$$

Here $D^\alpha$ stands for the partial differential operator

$$D^\alpha = \left(\frac{\partial}{\partial x}\right)^\alpha = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1}(\partial x_2)^{\alpha_2}\cdots(\partial x_n)^{\alpha_n}}.$$

For $x \in \mathbb{R}^n$ or $\mathbb{C}^n$, $|x|$ denotes the Euclidean norm of $x$. We denote the usual inner product on $\mathbb{R}^n$ or $\mathbb{C}^n$ by

$$\langle u,v \rangle = \sum_{i=1}^n \bar{u}_i v_i$$

and let $\{e_i\}_{i=1}^n$ be the standard basis for $\mathbb{R}^n$ or $\mathbb{C}^n$. If $f \in C^1(\mathbb{R}^n,\mathbb{E})$, where $\mathbb{E}$ is $\mathbb{R}^n$ or $\mathbb{C}^n$, $\partial f/\partial x$ is the matrix $(\partial f_i/\partial x_j)$. For $f \in C^1(\mathbb{R}^n,\mathbb{R})$ we will usually write $f^{(1)}$ instead of $\nabla f$ to denote the gradient of $f$ and if $f \in C^2(\mathbb{R}^n,\mathbb{R})$ we will write $f^{(2)}$ to denote the Hessian matrix $(\partial^2 f/\partial x_i\, \partial x_j)$. We will not distinguish row and column vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$ in our formulas and hence matrix products must be interpreted in context. The symbol $\mathbb{1}$ will stand for the $n\times n$ identity matrix. For an $n\times n$ complex matrix $A$ we will use the symbol $|A|$ to denote the matrix $(AA^*)^{1/2}$, where $A^*$ is the adjoint (complex conjugate transpose) of $A$. The symbol $\mathscr{U}_A$ will denote the unique unitary matrix guaranteed by the polar decomposition theorem such that $A = |A|\,\mathscr{U}_A$, $|A| = (AA^*)^{1/2}$.

Following Hagedorn[6] we define generalized Hermite polynomials on $\mathbb{R}^n$ recursively as follows: We set $\tilde{\mathscr{H}}_0(x) = 1$ and $\tilde{\mathscr{H}}_1(v;x) = 2\langle v,x \rangle$, where $v$ is an arbitrary nonzero vector in $\mathbb{C}^n$. For $v_1,\ldots,v_m$ arbitrary nonzero vectors in $\mathbb{C}^n$ we set

$$\begin{aligned}\tilde{\mathscr{H}}_m(v_1,\ldots,v_m;x) &= 2\langle v_m,x \rangle \tilde{\mathscr{H}}_{m-1}(v_1,\ldots,v_{m-1};x) \\ &\quad - 2\sum_{i=1}^{m-1} \langle v_m,\bar{v}_i \rangle \\ &\quad \times \tilde{\mathscr{H}}_{m-2}(v_1,\ldots,v_{i-1},v_{i-2},\ldots,v_{m-1};x).\end{aligned}$$

The polynomials $\tilde{\mathscr{H}}_m$ are independent of the ordering of the vectors $v_1,\ldots,v_m$. Given a complex invertible $n\times n$ matrix $A$ and a multi-index $\alpha$ we define the polynomial

$$\begin{aligned}\mathscr{H}_\alpha(A;x) = \tilde{\mathscr{H}}_{|\alpha|}(&\mathscr{U}_A e_1,\ldots,\mathscr{U}_A e_1,\\ &\mathscr{U}_A e_2,\ldots,\mathscr{U}_A e_n;x),\end{aligned}$$

where the vector $\mathscr{U}_A e_i$ appears $\alpha_i$ times in the list of variables of $\tilde{\mathscr{H}}_{|\alpha|}$.

We will find it useful to consider complex $n\times n$ matrices $A$ and $B$ satisfying the following conditions:

$$A \text{ and } B \text{ are invertible}, \tag{2.1a}$$

$$BA^{-1} \text{ is symmetric}, \tag{2.1b}$$

$$\mathrm{Re}(BA^{-1}) \text{ is strictly positive definite}, \tag{2.1c}$$

$$(\mathrm{Re}(BA^{-1}))^{-1} = AA^*. \tag{2.1d}$$

[Here, symmetric means (real symmetric) $+ i$(real symmetric).]

For complex $n\times n$ matrices $A$ and $B$ satisfying conditions (2.1), vectors $a$ and $\eta \in \mathbb{R}^n$, multi-indices $\alpha$, and positive $\hbar$ we define

$$\begin{aligned}\phi_\alpha(A,B,\hbar,a,\eta,x) &= (\pi\hbar)^{-n/4}(2^{|\alpha|}\alpha!)^{-1/2}[\det(A)]^{-1/2} \\ &\quad \times \mathscr{H}_\alpha(A;\hbar^{-1/2}|A|^{-1}(x-a)) \\ &\quad \times \exp\{-(1/2\hbar)\langle (x-a),BA^{-1}(x-a)\rangle \\ &\quad + (i/\hbar)\langle \eta,(x-a)\rangle\}.\end{aligned}$$

Here $|A|$ is the matrix $(AA^*)^{1/2}$ and the branch of the square root of $\det(A)$ will be specified in the context in which the functions $\phi_\alpha$ are used. Whenever we write $\phi_\alpha(A,B,\hbar,a,\eta,x)$ we are assuming that the matrices $A$ and $B$ satisfy conditions (2.1). For fixed $A$, $B$, $\hbar$, $a$, and $\eta$ the functions $\phi_\alpha(A,B,\hbar,a,\eta,x)$ form an orthonormal basis of $L^2(\mathbb{R}^n)$.

## III. SOME PRELIMINARY LEMMAS

In this section we prove two rather technical lemmas on the small $\hbar$ asymptotics of certain integrals of a type we encounter frequently in Sec. IV. The reader may skip the roofs of these lemmas as the details are not needed in the sequel.

*Lemma 3.1:* Let $\Omega$ be an open subset of $\mathbb{R}^n$. Let $S \in C^3(\Omega,\mathbb{R})$ and let $g \in C_0^1(\mathbb{R}^n,\mathbb{C})$ be such that $\mathrm{supp}(g) \subset \Omega$. Suppose $T$ is a complex $n\times n$ matrix valued class $C^1$ function on $\Omega$ satisfying

(1) $T(x)$ is symmetric [(real symmetric) $+ i$(real symmetric)],

(2) $\mathrm{Re}(T(x))$ is strictly positive definite,

for all $x \in \Omega$. Define the square root of $\det[T(x) + iS^{(2)}(x)]$ for $x \in \Omega$ by analytic continuation along $\xi \in [0,1]$ of

$$(\det[\mathrm{Re}(T(x)) + \xi i(\mathrm{Im}(T(x)) + S^{(2)}(x))])^{1/2}$$

starting with a positive value for $\xi = 0$. For $\hbar > 0$ and $x \in \mathbb{R}^n$, let

$$F(\hbar,x) = \hbar^{-n/2} \int_\Omega g(q) \times \exp\left( -\frac{1}{2\hbar} \langle (x-q), T(q)(x-q) \rangle + \frac{i}{\hbar} [S(q) + \langle S^{(1)}(q), (x-q) \rangle] \right) d^n q$$

and

$$G(\hbar,x) = (2\pi)^{n/2} \times (\det[T(x) + iS^{(2)}(x)])^{-1/2} g(x) e^{iS(x)/\hbar}.$$

Then, given $\lambda \in (0,\frac{1}{2})$ there exists $\delta > 0$ and a constant $C$ independent of $\hbar < \delta$ such that

$$\|F(\hbar,\cdot) - G(\hbar,\cdot)\| < C\hbar^\lambda,$$

for all $\hbar \in (0,\delta)$.

*Proof:* We first note that the determinant of $\mathrm{Re}(T(x)) + \xi i(\mathrm{Im}(T(x)) + S^{(2)}(x))$ is nonzero for all $\xi \in [0,1]$ by virtue of the fact that $\mathrm{Re}(T(x))$ is strictly positive definite and $\mathrm{Im}(T(x)) + S^{(2)}(x)$ is real symmetric. We next reduce to a special case by noting that there exist finitely many open balls $\Omega_k$ such that

$$\mathrm{supp}(g) \subset \bigcup_{k=1}^{K} \Omega_k \subset \Omega.$$

By introducing a $C^1$ partition of unity $\{h_k\}_{k=1}^K$ satisfying $\mathrm{supp}(h_k) \subset \Omega_k$ and $\Sigma_{k=1}^K h_k = 1$ on $\mathrm{supp}(g)$ we see by the triangle inequality that it suffices to prove the lemma only for the special case in which $\Omega$ is an open ball. Assuming this, one can easily show the existence of a constant $\epsilon > 0$ and a closed ball $\mathscr{K} \subset \mathbb{R}^n$ such that

$$\mathrm{supp}(g) \subset \{q \in \mathbb{R}^n: \ \mathrm{dist}(q, \mathrm{supp}(g)) < \epsilon\} \subset \mathscr{K} \subset \Omega.$$

We set $\gamma = \frac{1}{3}(1+\lambda)$, $\delta = \epsilon^{1/\gamma}$, and choose $T_0$ such that

$$0 < T_0 < \inf_{q \in \mathscr{K}} \inf_{\|x\|=1} \langle x, T(q)x \rangle.$$

By the choice of $T_0$, $\mathrm{Re}(T(q)) > T_0 \mathbf{1}$, for all $q \in \mathscr{K}$.

We note that there exists a constant $C_\gamma$ independent of $\hbar \in (0,\delta)$ such that

$$\int_{|x| > \hbar^\gamma} \exp\left( -\frac{1}{2\hbar} T_0 |x|^2 \right) d^n x < C_\gamma \hbar^{(n+2)/2}.$$

The hypotheses on $S$, $T$, and $g$ along with Taylor's theorem imply the existence of constants $C_1$, $C_2$, and $C_3$ independent of $\hbar \in (0,\delta)$ such that

$$|S(q) + \langle S^{(1)}(q), (x-q) \rangle - S(x) + \tfrac{1}{2} \langle (x-q), S^{(2)}(x)(x-q) \rangle| \leqslant C_1 |x-q|^3,$$

for all $x,q \in \mathscr{K}$;

$$|\exp\{-(1/2\hbar)\langle (x-q), T(q)(x-q) \rangle\} - \exp\{-(1/2\hbar)\langle (x-q), T(x)(x-q) \rangle\}| \leqslant C_2 \hbar^{3\gamma-1} \exp\{-(1/2\hbar) T_0 |x-q|^2\},$$

for all $x,q \in \mathscr{K}$ with $|x-q| \leqslant \hbar^2$; and

$$\|g(x-y) - g(x)\|_{L^2(\mathbb{R}^n, d^n x)} \leqslant C_3 |y|,$$

for all $y \in \mathbb{R}^n$.

For $x \in \mathbb{R}^n$ and $\hbar > 0$, define

$$F_1(\hbar,x) = \hbar^{-n/2} g(x) \times \int_{\mathscr{K}} \exp\left( -\frac{1}{2\hbar} \langle (x-q), T(q)(x-q) \rangle + \frac{i}{\hbar} [S(q) + \langle S^{(1)}(q), (x-q) \rangle] \right) d^n q,$$

$$F_2(\hbar,x) = \hbar^{-n/2} g(x) \times \int_{\mathscr{K}} \exp\left( -\frac{1}{2\hbar} \langle (x-q), T(q)(x-q) \rangle + \frac{i}{\hbar} \left[ S(x) - \frac{1}{2} \langle (x-q), S^{(2)}(x)(x-q) \rangle \right] \right) \times d^n q,$$

and

$$F_3(\hbar,x) = \hbar^{-n/2} g(x) \times \int_{\mathscr{K}} \exp\left( -\frac{1}{2\hbar} \langle (x-q), T(x)(x-q) \rangle + \frac{i}{\hbar} \left[ S(x) - \frac{1}{2} \langle (x-q), S^{(2)}(x)(x-q) \rangle \right] \right) \times d^n q.$$

Then,

$$|F(\hbar,x) - F_1(\hbar,x)| \leqslant \hbar^{-n/2} \int_{\mathscr{K}} |g(q) - g(x)| \times \exp\left\{ -\frac{1}{2\hbar} T_0 |x-q|^2 \right\} d^n q \leqslant \hbar^{-n/2} \int_{\mathbb{R}^n} |g(x-y) - g(x)| \times \exp\left\{ -\frac{1}{2\hbar} T_0 |y|^2 \right\} d^n y,$$

thus

$$\|F(\hbar,\cdot) - F_1(\hbar,\cdot)\| \leqslant \hbar^{-n/2} \int_{\mathbb{R}^n} \|g(x-y) - g(x)\|_{L^2(\mathbb{R}^n, d^n x)} \times \exp\left\{ -\frac{1}{2\hbar} T_0 |y|^2 \right\} d^n y \leqslant \hbar^{-n/2} C_3 \int_{\mathbb{R}^n} |y| \exp\left\{ -\frac{1}{2\hbar} T_0 |y|^2 \right\} d^n y = C_3' \hbar^{1/2},$$

where $C_3'$ is independent of $\hbar$. Similarly,

$$|F_1(\hbar,x) - F_2(\hbar,x)| \leqslant \hbar^{-n/2} |g(x)| \int_{\mathscr{K}} \exp\left\{ -\frac{1}{2\hbar} T_0 |x-q|^2 \right\}$$

$$\times \left| \exp\left\{\frac{i}{\hbar}[S(q)+\langle S^{(1)}(q),(x-q)\rangle]\right\} - \exp\left\{\frac{i}{\hbar}\left[S(x)-\frac{1}{2}\langle (x-q),S^{(2)}(x)(x-q)\rangle\right]\right\}\right| d^nq,$$

hence

$$\|F_1(\hbar,\cdot)-F_2(\hbar,\cdot)\|$$
$$\leqslant \hbar^{-n/2}\|g\| \sup_{x\in\mathrm{supp}(g)} \left[\int_{\mathscr{K}} \exp\left\{-\frac{1}{2\hbar}T_0|x-q|^2\right\} \left|\exp\left\{\frac{i}{\hbar}[S(q)+\langle S^{(1)}(q),(x-q)\rangle]\right\}\right.\right.$$
$$\left.\left. - \exp\left\{\frac{i}{\hbar}\left[S(x)-\frac{1}{2}\langle (x-q),S^{(2)}(x)(x-q)\rangle\right]\right\}\right| d^nq\right]$$
$$\leqslant \hbar^{-n/2}\|g\| \sup_{x\in\mathrm{supp}(g)} \left[\hbar^{-1}\int_{\mathscr{K}} \exp\left\{-\frac{1}{2\hbar}T_0|x-q|^2\right\}\left(\left|S(q)+\langle S^{(1)}(q),(x-q)\rangle\right.\right.\right.$$
$$\left.\left.\left. -S(x)+\frac{1}{2}\langle (x-q),S^{(2)}(x)(x-q)\rangle\right|\right) d^nq\right]$$
$$\leqslant \hbar^{-(n+2)/2}\|g\|C_1\int_{\mathbf{R}^n}|y|^3\exp\left\{-\frac{1}{2\hbar}T_0|y|^2\right\}d^ny = C_1'\hbar^{1/2},$$

where $C_1'$ is independent of $\hbar$. Moreover,

$$\|F_2(\hbar,\cdot)-F_3(\hbar,\cdot)\|$$
$$\leqslant \hbar^{-n/2}\|g\| \sup_{x\in\mathrm{supp}(g)} \left[\int_{\mathscr{K}} \left|\exp\left\{-\frac{1}{2\hbar}\langle (x-q),T(q)(x-q)\rangle\right\} - \exp\left\{-\frac{1}{2\hbar}\langle (x-q),T(x)(x-q)\rangle\right\}\right| d^nq\right]$$
$$\leqslant \hbar^{-n/2}\|g\| \sup_{x\in\mathrm{supp}(g)} \left[C_2\hbar^{3\gamma-1}\int_{|x-q|<\hbar^\gamma}\exp\left\{-\frac{1}{2\hbar}T_0|x-q|^2\right\}d^nq + 2\int_{|x-q|>\hbar^\gamma}\exp\left\{-\frac{1}{2\hbar}T_0|x-q|^2\right\}d^nq\right]$$
$$\leqslant \|g\|(C_2\hbar^\lambda + 2C_\gamma\hbar).$$

Now we observe that

$$G(\hbar,x)=\hbar^{-n/2}g(x)\int_{\mathbf{R}^n}\exp\left(-\frac{1}{2\hbar}\langle (x-q),(T(x)+iS^{(2)}(x))(x-q)\rangle + \frac{i}{\hbar}S(x)\right)d^nq$$

(see, e.g., Theorem B, Sec. 1a of Bargmann[9]). Hence

$$|G(\hbar,x)-F_3(\hbar,x)| \leqslant \hbar^{-n/2}|g(x)|\int_{\mathbf{R}^n/\mathscr{K}}\exp\left(-\frac{1}{2\hbar}\langle (x-q),\mathrm{Re}(T(x))(x-q)\rangle\right)d^nq$$
$$\leqslant \hbar^{-n/2}|g(x)|\int_{|x-q|>\hbar^\gamma}\exp\left(-\frac{1}{2\hbar}T_0|x-q|^2\right)d^nq \leqslant C_\gamma\hbar|g(x)|$$

and therefore

$$\|G(\hbar,\cdot)-F_3(\hbar,\cdot)\|\leqslant C_\gamma\hbar\|g\|.$$

The triangle inequality completes the proof of the lemma. ■

*Lemma 3.2:* Let $\Omega$, $S$, $g$, and $T$ be as in Lemma 3.1. Suppose $c(\hbar,\cdot)\in C(\Omega,\mathbb{C})$ and $P\in C(\Omega\times\mathbf{R}^n,\mathbb{C})$ are such that there exist $r,\beta>0$ and constants $C_r$ and $C_\beta$ such that

$$|c(\hbar,q)|\leqslant C_r\hbar^{r/2}, \quad \text{for all } q\in\Omega$$

and

$$|P(q,x)|\leqslant C_\beta|x-q|^\beta, \quad \text{for all } q\in\Omega, \quad x\in\mathbf{R}^n.$$

Let

$$F(\hbar,x)=\hbar^{-n/2}\int_\Omega g(q)c(\hbar,q)P(q,x)$$
$$\times\exp\left(-\frac{1}{2\hbar}\langle (x-q),T(q)(x-q)\rangle\right.$$
$$\left.+\frac{i}{\hbar}[S(q)+\langle S^{(1)}(q),(x-q)\rangle]\right)d^nq.$$

Then, there exists $\delta>0$ and a constant $C$ independent of $\hbar$ such that

$$\|F(\hbar,\cdot)\|\leqslant C\hbar^{(r+\beta)/2},$$

for all $\hbar<\delta$.

*Proof:* Choose $T_0$ as in the proof of Lemma 3.1 and let $C_1$ be such that

$$\|g(x-y)-g(x)\|_{L^2(\mathbf{R}^n,d^nx)}\leqslant C_1|y|,$$

for all $y\in\mathbf{R}^n$. Define

$$F_1(\hbar,x)=\hbar^{-n/2}g(x)$$
$$\times\int_\Omega c(\hbar,q)P(q,x)$$
$$\times\exp\left(-\frac{1}{2\hbar}\langle (x-q),T(q)(x-q)\rangle\right.$$

$$+\frac{i}{\hbar}[S(q)+\langle S^{(1)}(q),(x-q)\rangle]\Big)d^nq$$

and

$$F_2(\hbar,x)=\hbar^{-n/2}g(x)c(\hbar,x)P(x,x)$$
$$\times\int_\Omega \exp\Big(-\frac{1}{2\hbar}\langle(x-q),T(q)(x-q)\rangle$$
$$+\frac{i}{\hbar}[S(q)+\langle S^{(1)}(q),(x-q)\rangle]\Big)d^nq.$$

Note that $P(x,x)$ [and hence $F_2(\hbar,x)$] is zero if $\beta\neq 0$. Then,

$$|F(\hbar,x)-F_1(\hbar,x)|<C_rC_\beta\hbar^{(r-n)/2}$$
$$\times\int_{\mathbf{R}^n}|g(x-y)-g(x)|\,|y|^\beta$$
$$\times\exp\Big(-\frac{1}{2\hbar}T_0|y|^2\Big)d^ny.$$

Thus

$$\|F(\hbar,\cdot)-F_1(\hbar,\cdot)\|$$
$$<C_rC_\beta C_1\hbar^{(r-n)/2}\int_{\mathbf{R}^n}|y|^{\beta+1}\exp\Big(-\frac{1}{2\hbar}T_0|y|^2\Big)d^ny$$
$$=C_1'\hbar^{(r+\beta+1)/2},$$

where $C_1'$ is independent of $\hbar$. Since

$$|c(\hbar,q)P(q,x)-c(\hbar,x)P(x,x)|<2C_rC_\beta\hbar^{r/2}|x-q|^\beta,$$

for all $q\in\Omega$ we obtain

$$|F_1(\hbar,x)-F_2(\hbar,x)|<C_2\hbar^{(r+\beta)/2}|g(x)|,$$

for some $C_2$ independent of $\hbar$, and hence

$$\|F_1(\hbar,\cdot)-F_2(\hbar,\cdot)\|<C_2\hbar^{(r+\beta)/2}\|g\|.$$

The proof is completed by noticing that there is a constant $C_3$ independent of $\hbar$ such that

$$\|F_2(\hbar,\cdot)\|<C_3\hbar^{r/2},$$

if $\beta=0$, and

$$\|F_2(\hbar,\cdot)\|=0,$$

if $\beta\neq 0$.

## IV. PROOF OF THE THEOREM

In this section we prove Theorem 1.2. Suppose $V$ satisfies Assumption 1.1 and let $S_0\in C^3(\mathbf{R}^n,\mathbf{R})$. Given $T>0$ and $a_0,\eta_0\in\mathbf{R}^n$, the system of ordinary differential equations,

$$\frac{\partial}{\partial t}a(t)=\frac{1}{m}\eta(t), \tag{4.1a}$$

$$\frac{\partial}{\partial t}\eta(t)=-V^{(1)}(a(t)), \tag{4.1b}$$

$$\frac{\partial}{\partial t}A(t)=\frac{i}{m}B(t), \tag{4.1c}$$

$$\frac{\partial}{\partial t}B(t)=iV^{(2)}(a(t))A(t), \tag{4.1d}$$

$$\frac{\partial}{\partial t}S(t)=\frac{1}{2m}|\eta(t)|^2-V(a(t)), \tag{4.1e}$$

subject to the initial conditions $a(0)=a_0$, $\eta(0)=\eta_0$, $A(0)=\mathbb{1}$, $B(0)=\mathbb{1}$, and $S(0)=S_0(a_0)$, has a unique bounded solution for $t\in[0,T]$. We denote this solution by $[a(a_0,\eta_0,t),\eta(a_0,\eta_0,t),A(a_0,\eta_0,t),B(a_0,\eta_0,t),S(a_0,\eta_0,t)]$. By considering the system (4.1a) and (4.1b) we find that the functions $a$ and $\eta$ are of class $C^{l+1}$ in $a_0$ and $\eta_0$. These facts are standard results from the theory of ordinary differential equations.[7] By Theorem 1.1 of Hagedorn[4] the matrices $A(a_0,\eta_0,t)$ and $B(a_0,\eta_0,t)$ satisfy conditions (2.1) for all $t\in[0,T]$ and are given by

$$A(a_0,\eta_0,t)=\frac{\partial}{\partial a_0}a(a_0,\eta_0,t)+i\frac{\partial}{\partial\eta_0}a(a_0,\eta_0,t), \tag{4.2a}$$

$$B(a_0,\eta_0,t)=\frac{\partial}{\partial\eta_0}\eta(a_0,\eta_0,t)-i\frac{\partial}{\partial a_0}\eta(a_0,\eta_0,t). \tag{4.2b}$$

From (4.2) and the remarks above it follows that $A$ and $B$ are of class $C^l$ in $a_0$ and $\eta_0$.

Let $Q$ be the mapping from $\mathbf{R}^n\times(0,T)$ into $\mathbf{R}^n$ defined by

$$Q(q_0,t)=a(q_0,S_0^{(1)}(q_0),t).$$

By the hypothesis on $S_0$ and the remarks above, $Q$ is of class $C^2$ in the variable $q_0\in\mathbf{R}^n$. Let $f\in C^1(\mathbf{R}^n,\mathbf{C})$ have compact support. We now fix $t\in(0,T)$ and assume

$$\det\Big[\frac{\partial Q}{\partial q_0}(x,t)\Big]\neq 0, \tag{4.3}$$

for all $x\in\operatorname{supp}(f)$. Since $t$ is fixed, we will omit reference to $t$ where possible but it should be remembered that all estimates obtained in this section are $t$ dependent.

Under the assumption (4.3), for each $x\in\operatorname{supp}(f)$ there exists an open ball $\mathcal{N}_x\ni x$ such that the mapping $Q\restriction\mathcal{N}_x$ is a class $C^2$ diffeomorphism of $\mathcal{N}_x$ onto $Q[\mathcal{N}_x]$. Since $\operatorname{supp}(f)$ is compact, some finite subcollection $\{\mathcal{N}_k: k=1,\ldots,K\}$ of the family $\{\mathcal{N}_x: x\in\operatorname{supp}(f)\}$ covers $\operatorname{supp}(f)$. We denote by $Q_k$ the diffeomorphism $Q\restriction\mathcal{N}_x$ and define, for $q\in Q_k[\mathcal{N}_k]$,

$$p_k(q)=\eta(Q_k^{-1}(q),S_0^{(1)}(Q_k^{-1}(q)),t),$$
$$A_k(q)=A(Q_k^{-1}(q),S_0^{(1)}(Q_k^{-1}(q)),t),$$
$$B_k(q)=B(Q_k^{-1}(q),S_0^{(1)}(Q_k^{-1}(q)),t),$$
$$S_k(q)=S(Q_k^{-1}(q),S_0^{(1)}(Q_k^{-1}(q)),t).$$

It is well known and not difficult to show that

$$p_k(q)=\frac{\partial}{\partial q}S_k(q). \tag{4.4}$$

Note that the functions $p_k$, $A_k$, and $B_k$ are of class $C^2$ in $q\in Q_k[\mathcal{N}_k]$ while $S_k$ is of class $C^3$. Moreover, since $A_k(q)A_k(q)^*$ is strictly positive definite, the operational calculus shows that the matrix $|A_k(q)|$ is continuously differentiable with respect to $q\in Q_k[\mathcal{N}_k]$.

*Proposition 4.1:* Let $q\in Q_k[\mathcal{N}_k]$. Then

$$\det[B_k(q)+iS_k^{(2)}(q)A_k(q)]$$
$$=\det\Big[\frac{\partial}{\partial q}Q_k^{-1}(q)\Big]\cdot\det[\mathbb{1}+iS_0^{(2)}(Q_k^{-1}(q))].$$

*Proof:* First note that, by differentiating the expression defining $Q$,

$$\frac{\partial}{\partial q} Q_k^{-1}(q) = \left[\frac{\partial a}{\partial a_0} + \frac{\partial a}{\partial \eta_0} S_0^{(2)}\right]^{-1},$$

and, differentiating (4.4),

$$S_k^{(2)}(q) = \left[\frac{\partial \eta}{\partial a_0} + \frac{\partial \eta}{\partial \eta_0} S_0^{(2)}\right] \frac{\partial}{\partial q} Q_k^{-1}(q),$$

where functions of $a_0$ and $\eta_0$ are assumed evaluated at $a_0 = Q_k^{-1}(q)$ and $\eta_0 = S_0^{(1)}(Q_k^{-1}(q))$. By (4.2) and partitioning of determinants,[10]

$$\det[B_k(q) + iS_k^{(2)}(q)A_k(q)] = \det\left[\left(\frac{\partial \eta}{\partial \eta_0} - i\frac{\partial \eta}{\partial a_0}\right) + i\left(\frac{\partial \eta}{\partial a_0} + \frac{\partial \eta}{\partial \eta_0} S_0^{(2)}\right)\left(\frac{\partial}{\partial q} Q_k^{-1}(q)\right)\left(\frac{\partial a}{\partial a_0} + i\frac{\partial a}{\partial \eta_0}\right)\right]$$

$$= \det\left[\frac{\partial}{\partial q} Q_k^{-1}(q)\right]\cdot\det\begin{bmatrix} \frac{\partial \eta}{\partial \eta_0} - i\frac{\partial \eta}{\partial a_0} & \frac{\partial \eta}{\partial a_0} + \frac{\partial \eta}{\partial \eta_0} S_0^{(2)} \\ \frac{\partial a}{\partial \eta_0} - i\frac{\partial a}{\partial a_0} & \frac{\partial a}{\partial a_0} + \frac{\partial a}{\partial \eta_0} S_0^{(2)} \end{bmatrix}$$

$$= \det\left[\frac{\partial}{\partial q} Q_k^{-1}(q)\right]\cdot\det\begin{bmatrix} \frac{\partial \eta}{\partial \eta_0} & \frac{\partial \eta}{\partial a_0} \\ \frac{\partial a}{\partial \eta_0} & \frac{\partial a}{\partial a_0} \end{bmatrix}\cdot\det\begin{bmatrix} 1 & S_0^{(2)} \\ -i1 & 1 \end{bmatrix}$$

$$= \det\left[\frac{\partial}{\partial q} Q_k^{-1}(q)\right]\cdot\det[1 + iS_0^{(2)}(Q_k^{-1}(q))].$$

In the last step we have used the fact that the mapping

$$(a_0,\eta_0) \mapsto (a(a_0,\eta_0,t),\eta(a_0,\eta_0,t))$$

is a canonical transformation.[11] ■

Let $q \in Q_k[\mathcal{N}_k]$. Define the branch of the square root of $\det[A_k(q)]$ by analytic continuation along $\tau \in [0,t]$ of $(\det[A(Q_k^{-1}(q),S_0^{(1)}(Q_k^{-1}(q)),\tau)])^{1/2}$ starting with a value of 1 for $\tau = 0$. We determine the branches of

$$(\det[1 + iS_0^{(2)}(Q_k^{-1}(q))])^{1/2}$$

and

$$(\det[B_k(q)A_k(q)^{-1} + iS_k^{(2)}(q)])^{1/2}$$

by analytic continuation of

$$(\det[1 + \xi iS_0^{(2)}(Q_k^{-1}(q))])^{1/2}$$

and

$$(\det[\mathrm{Re}(B_k(q)A_k(q)^{-1}) + \xi i(\mathrm{Im}(B_k(q)A_k(q)^{-1}) + S_k^{(2)}(q))])^{1/2},$$

respectively, along $\xi \in [0,1]$ starting with positive values for $\xi = 0$. This determines the branch of

$$(\det[B_k(q) + iS_k^{(2)}(q)A_k(q)])^{1/2}$$

and the branch of $(\det[(\partial/\partial q)Q_k^{-1}(q)])^{1/2}$ is then determined by Proposition 4.1. Let $\mu_k(q)$ be the positive integer (mod 4) such that

$$\left(\det\left[\frac{\partial Q}{\partial q_0}(Q_k^{-1}(q))\right]\right)^{1/2} = e^{i(\pi/2)\mu_k(q)}\left|\det\left[\frac{\partial Q}{\partial q_0}(Q_k^{-1}(q))\right]\right|^{1/2}.$$

By continuity and the fact that $Q_k[\mathcal{N}_k]$ is pathwise connected the index $\mu_k(q)$ is independent of the choice of $q \in Q_k[\mathcal{N}_k]$. Hence to each $\mathcal{N}_k$ we can assign a unique (mod 4) index $\mu_k$. Moreover, in light of the facts above we see that if $\mathcal{N}_k \cap \mathcal{N}_j$ is nonempty then $\mu_k = \mu_j$. We now introduce a partition of unity $\{h_k\}_{k=1}^K$ on $\cup_{k=1}^K \mathcal{N}_k$ satisfying $h_k \in C_0^1(\mathbb{R}^n,\mathbb{C})$, $\mathrm{supp}(h_k) \subset \mathcal{N}_k$, $\Sigma_{k=1}^K h_k = 1$ on $\mathrm{supp}(f)$ and set $f_k = fh_k$. The following proposition allows us to restrict our attention to a single $\mathcal{N}_k$ and $f_k$.

*Proposition 4.2:* Let $x \in \mathbb{R}^n$ be fixed. Let $\{x_j\}_{j=1}^J$ be the set of points in $\cup_{k=1}^K \mathcal{N}_k$ such that $Q(x_j) = x$ for $j = 1,2,...,J$ and let $\mu_j'$ be defined to be equal to $\mu_k$ if $x_j \in \mathcal{N}_k$. Define

$$S_j'(x) = S_0(x_j) + \int_0^t \left[\frac{1}{2m}|\eta(x_j,S_0^{(1)}(x_j),\tau)|^2 - V(a(x_j,S_0^{(1)}(x_j),\tau))\right]d\tau$$

and let $\chi_k$ denote the characteristic function of $Q_k[\mathcal{N}_k]$. Then

$$\sum_{k=1}^K e^{iS_k(x)/\hbar + \mu_k\pi/2}\left|\det\left[\frac{\partial Q}{\partial q_0}(Q_k^{-1}(x))\right]\right|^{-1/2} \times f_k(Q_k^{-1}(x))\chi_k(x) = \sum_{j=1}^J e^{iS_j'(x)/\hbar + i\mu_j'\pi/2}\left|\det\left[\frac{\partial Q}{\partial q_0}(x_j)\right]\right|^{-1/2} f(x_j).$$

*Proof:* We note that $\mu_j'$ is well-defined by the remarks preceding the proposition. Moreover, if $x_j \in \mathcal{N}_{k_1} \cap \mathcal{N}_{k_2}$, then $S_{k_1}(x) = S_{k_2}(x) = S_j'(x)$. The proposition follows by changing variables from $Q_k^{-1}(x)$ to $x_j$ in the first summation. ■

The next lemma is a version of Theorem 1.2 for the functions $f_k$. The proof requires the functions $\phi_\alpha(A,B,\hbar,a,\eta,x)$ defined in Sec. II. This lemma, Proposition 4.2, and the triangle inequality complete the proof of Theorem 1.2.

*Lemma 4.3:* Given $\lambda\in(0,\frac{1}{2})$, there exist a number $\delta>0$ and a constant $C$ independent of $\hbar<\delta$ such that

$$\Bigg\| e^{-itH(\hbar)/\hbar}(e^{iS_0/\hbar}f_k) - e^{iS_k/\hbar + i\mu_k\pi/2}\left|\det\left[\frac{\partial Q}{\partial q_0}(Q_k^{-1}(\cdot))\right]\right|^{-1/2} \times f_k(Q_k^{-1}(\cdot))\chi_k\Bigg\| \leqslant C\hbar^\lambda,$$

for all $\hbar\in(0,\delta)$.

*Proof:* We will omit the subscript $k$ at the risk of confusion with previously defined quantities. The branches of square roots appearing in the proof are determined according to the discussion following Proposition 4.2. For $x\in\mathbb{R}^n$ let

$$\mathscr{F}_\hbar(x) = (4\pi\hbar)^{-n/4} \times\int_{\mathscr{N}}(\det[1+S_0^{(2)}(q_0)])^{1/2}f(q_0)e^{iS_0(q_0)/\hbar} \times\phi_0(1,1,\hbar,q_0,S_0^{(1)}(q_0),x)d^Nq_0.$$

By definition of $\phi_0$ and Lemma 3.1, there exist $\delta_1$ and $C_1$ independent of $\hbar$ such that

$$\|e^{-itH(\hbar)/\hbar}(e^{iS_0/\hbar}f) - e^{-itH(\hbar)/\hbar}\mathscr{F}_\hbar\|\leqslant C_1\hbar^\lambda, \tag{4.5}$$

for all $\hbar\in(0,\delta_1)$ and $t\geqslant 0$. By Theorem 1.1 of Hagedorn,[6] for any $q_0\in\mathbb{R}^n$ there exists a constant $C_2$ such that

$$\Bigg\| e^{itH(\hbar)/\hbar}(e^{iS_0/\hbar}\phi_0(1,1,\hbar,q_0,S_0^{(1)}(q_0),\cdot)) - e^{iS(q_0,S_0^{(1)}(q_0),t)/\hbar}\sum_{|\alpha|=0}^{3(l-1)}c_\alpha(\hbar,q_0,t) \times\phi_\alpha(A(q_0,S_0^{(1)}(q_0),t),B(q_0,S_0^{(1)}(q_0),t), \hbar,a(q_0,S_0^{(1)}(q_0),t),\eta(q_0,S_0^{(1)}(q_0),t),\cdot)\Bigg\| < C_2\hbar^{l/2}, \tag{4.6}$$

where $c_\alpha(\hbar,q_0,\tau)$ is the unique solution of the system of ordinary differential equations

$$\frac{\partial}{\partial\tau}c_\alpha(\hbar,q_0,\tau) = \sum_{|\beta|=0}^{3(l-1)}\sum_{|\mu|=3}^{l+1} -i\hbar^{(|\mu|-2)/2}(\mu!)^{-1}[D^\mu V] \times(a(q_0,S_0^{(1)}(q_0),\tau))b_{\alpha\beta\mu}(q_0,\tau)c_\beta(\hbar,q_0,\tau), \tag{4.7}$$

subject to the initial conditions $c_0(\hbar,q_0,0)=1$ and $c_\alpha(\hbar,q_0,0)=0$ for $|\alpha|=0$. The quantities $b_{\alpha\beta\mu}$ are defined by

$$b_{\alpha\beta\mu}(q_0,\tau) = \langle\phi_\alpha(A(q_0,S_0^{(1)}(q_0),\tau),B(q_0,S_0^{(1)}(q_0),\tau),1,0,0,\cdot), x^\mu\phi_\beta(A(q_0,S_0^{(1)}(q_0),\tau),B(q_0,S_0^{(1)}(q_0),\tau),1,0,0,\cdot)\rangle. \tag{4.8}$$

By the argument of Lemma 2.5 of Hagedorn[5] and the remark following that lemma there exists a constant $C'$ such that the functions $c_\alpha$ satisfy

$$|c_0(\hbar,q_0,\tau)-1|\leqslant C'\hbar^{1/2}, \tag{4.9a}$$

$$|c_\alpha(\hbar,q_0,\tau)|\leqslant C'\hbar^{r/2}, \tag{4.9b}$$

for $3(r-1)<|\alpha|\leqslant 3r$ and for all $\tau\in[0,t]$.

From the proofs[4-6] of these facts we conclude that (4.6) and (4.9) hold uniformly for $q_0$ in a compact subset of $\mathbb{R}^n$. From (4.7), (4.8), and the differentiability properties discussed above we conclude that $c_\alpha$ is of class $C^1$ in the variable $q_0\in\mathbb{R}^n$. Define

$$c_\alpha(\hbar,q) = c_\alpha(\hbar,Q^{-1}(q),t)$$

and

$$\Phi_\alpha(\hbar,x,t) = (4\pi\hbar)^{-n/4}\int_{\mathscr{N}}(\det[1+S_0^{(2)}(q_0)])^{1/2}f(q_0) \times e^{iS(q_0,S_0^{(1)}(q_0),t)/\hbar}c_\alpha(\hbar,q_0,t) \times\phi_\alpha(A(q_0,S_0^{(1)}(q_0),t),B(q_0,S_0^{(1)}(q_0),t),\hbar, a(q_0,S_0^{(1)}(q_0),t),\eta(q_0,S_0^{(1)}(q_0),t),x)d^nq_0.$$

By (4.6),

$$\Bigg\| e^{-itH(\hbar)/\hbar}\mathscr{F}_\hbar - \sum_{|\alpha|=0}^{3(l-1)}\Phi_\alpha(\hbar,\cdot,t)\Bigg\|\leqslant C_2\hbar^{1/2}. \tag{4.10}$$

Changing the variable of integration in the definition of $\Phi_\alpha$,

$$\Phi_\alpha(\hbar,x,t) = (4\pi\hbar)^{-n/4}\int_{Q[\mathscr{N}]}(\det[1+S_0^{(2)}(Q^{-1}(q))])^{1/2} \times f(Q^{-1}(q))e^{iS(q)/\hbar} \times c_\alpha(\hbar,q)\phi_\alpha(A(q),B(q),\hbar,q,p(q),x) \times\left|\det\left(\frac{\partial Q^{-1}}{\partial q}(q)\right)\right| d^nq. \tag{4.11}$$

We now note that the functions $\mathscr{H}_\alpha$ appearing in the definition of $\phi_\alpha(A(q),B(q),\hbar,q,p(q),x)$ are of class $C^1$ in $q\in Q[\mathscr{N}]$ by virtue of the fact that the matrices $\mathscr{U}_{A(q)}$ and $|A(q)|$ are continuously differentiable with respect to $q\in Q[\mathscr{N}]$. Hence we can apply Lemmas 3.1 and 3.2 to each term in each of the $\Phi_\alpha$'s [the function $(f\circ Q^{-1})_{\chi_{Q[\mathscr{N}]}}$ is of class $C_0^1$ on $\mathbb{R}^n$] and conclude, by (4.4), (4.9), (4.11), and Proposition 4.1 that there exist $\delta_3$ and $C_3$ independent of $\hbar\in(0,\delta_3)$ such that

$$\Bigg\|\sum_{|\alpha|=0}^{3(l-1)}\Phi_\alpha(\hbar,\cdot,t) - e^{iS(\cdot)/\hbar + i\mu\pi/2} \times\left|\det\left(\frac{\partial Q}{\partial q_0}(Q^{-1}(\cdot))\right)\right|^{-1/2}f(Q^{-1}(\cdot))\chi\Bigg\| < C_3\hbar^\lambda, \tag{4.12}$$

for all $\hbar\in(0,\delta_3)$. Equations (4.5), (4.10), (4.12), and the triangle inequality complete the proof of the lemma. ■


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