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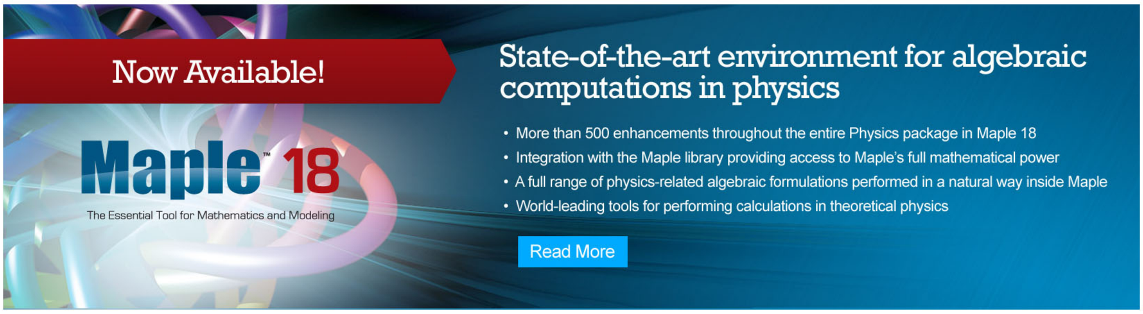

Time-dependent dissipation in nonlinear Schrödinger systems

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A coupled nonlinear Schrödinger–Poisson equation is considered which contains a time-dependent dissipation function as a specific model of dissipation effects in nonlinear quantum transport theory and other areas. The Wigner–Poisson equation associated with this system is derived. Using conservation and quasiconservation laws and certain growth assumptions for the nonlinearities and the dissipation function, global existence of solutions to the Cauchy problem of the time-dependent Schrödinger–Poisson system is shown both for small (attractive case) or arbitrary data (repulsive case). © *1995 American Institute of Physics.*

I. INTRODUCTION

In a recent article[1] the following nonlinear Schrödinger equation was introduced:

$$i\beta\phi_t+\frac{i}{2}\beta_t\phi=-\frac{1}{2}\Delta\phi+\frac{1}{2}\beta^2\phi+\alpha|\phi|^2\phi,\quad (x\in\mathbb{R}^3,t\in\mathbb{R}).\tag{1.1}$$

Here β is a specific real function of t only and α a given function of x. In Ref. 1, Eq. (1.1) models beam propagation in a nonlinear medium where the extra terms involving β are designed to take into account fast longitudinal field oscillations; moreover, t represents a time-dilated spatial variable, and Δ is the one-dimensional Lapacian. Now thinking of t as the time variable (with $x\in\mathbb{R}^3$), Eq. (1.1) can be interpreted as a cubic nonlinear Schrödinger equation modeling time-dependent dissipation (to be further explained in Sec. III).

In this article, we consider a generalized version of Eq. (1.1) which also includes a self-consistent potential V. In addition, we generalize the cubic nonlinearity to an arbitrary power and, for simplicity we take α to be a constant. Hence our equations become

$$i\beta\psi_t+\frac{i}{2}\beta_t\psi=-\frac{1}{2}\Delta\psi+V(\psi)\psi+g(|\psi|^2)\psi,\tag{1.2}$$

$$-\Delta V=|\psi|^2,\tag{1.3}$$

$$\psi(x,0)=\psi_0(x).\tag{1.4}$$

Here $\alpha\in\mathbb{R},p>0$, $x\in\mathbb{R}^3$, $t\in\mathbb{R}^+$, and β is a real function on $\mathbb{R}^+;g(s)=\alpha s^p(s\geqslant0)$. The term $\frac{1}{2}\beta^2\phi$ appearing in Eq. (1.1) has been omitted from Eq. (1.2) since it can be eliminated by use of the gauge transformation

$$\psi=e^{(i/2)\int_0^t\beta(s)\,ds}\tilde{\psi}.$$

Equations (1.2) and (1.3) may also be written in the following form:

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$$i\phi_t=-\frac{1}{2\beta}\Delta\phi+\frac{1}{\beta}V\left(\frac{\phi}{B}\right)\phi+\frac{1}{\beta}\cdot g\left(\frac{|\phi|^2}{\beta}\right)\phi, \tag{1.5}$$

$$-\Delta V=\frac{1}{\beta}|\phi|^2 \tag{1.6}$$

by use of the transformation

$$\phi=\sqrt{\beta}\psi.$$

The self-consistent term $V(\psi)\psi$ is familiar from quantum transport theory.[2–5] For the modified equation the introduction of β represents time-dependent dissipation; in Sec. III we show that the probability density, $\int_{\mathrm{R}^3}|\psi|^2\,dx$, for this system is proportional to $1/\beta$. A model for a constant dissipation rate has been considered in Ref. 6. This dissipation was implemented in a different manner from that presented here, namely, by adding certain complex terms to the Hamiltonian. Other models for dissipation in quantum mechanics have been treated in Refs. 7 and 8.

In Sec. II we derive the Wigner–Poisson equation associated with the Schrödinger–Poisson system (1.2)–(1.4) by utilizing the Wigner transform described in Ref. 4. In Sec. III we obtain a conservation law for the probability density and an associated evolution law for a Liapounov functional or "quasienergy" of system (1.2)–(1.4). In Sec. IV these laws are used to obtain global-in-time existence results for the Cauchy problem of the Schrödinger system (1.2)–(1.4); these results can be transferred to the associated Wigner equation along the same lines as in Ref. 3.

II. THE WIGNER EQUATION

The Wigner function, f_w, associated with the Schrödinger wave function ψ is given by (see Ref. 4)

$$f_w(x,v,t)=\left(\frac{1}{2\pi}\right)^3\int_{\mathrm{R}^3}e^{iv\eta}\bar{\psi}\left(x+\frac{\eta}{2},t\right)\psi\left(x-\frac{\eta}{2},t\right)d\eta. \tag{2.1}$$

Differentiating, and multiplying by $i\beta$ we get

$$i\beta\partial_t f_w(x,v,t)=\left(\frac{1}{2\pi}\right)^3 i\beta\int_{\mathrm{R}^3}e^{iv\eta}[\bar{\psi}_{+,t}\psi_-+\bar{\psi}_+\psi_{-,t}]d\eta, \tag{2.2}$$

where

$$\psi_\pm=\psi\left(x\pm\frac{\eta}{2},t\right). \tag{2.3}$$

For the time-derivative terms in Eq. (2.2) we now use the Schrödinger equation (1.2); the terms $-\frac{1}{2}\Delta\psi+V(\psi)\psi$ lead to the usual Wigner operator (Ref. 4) $-iW_0f_w$ on the right-hand side of Eq. (2.2)

$$W_0f_w=v\cdot\nabla_x f_w-i\Theta_0(V)f_w, \tag{2.4}$$

where $\Theta_0(V)$ is the usual pseudodifferential operator with symbol[1,2,8]

$$\mathrm{Sym}\ \Theta_0=V\left(x+\frac{\eta}{2},t\right)-V\left(x-\frac{\eta}{2},t\right), \tag{2.5}$$

i.e.,

$$(\Theta_0(V)f_w)(x,v,t)=\left(\frac{1}{2\pi}\right)^3\int_{\mathrm{R}^3_\eta} e^{iv\eta}\left[V\left(x+\frac{\eta}{2},t\right)-V\left(x-\frac{\eta}{2},t\right)\right]\hat{f}_w(x,\eta,t)d\eta \tag{2.6}$$

and

$$\hat{f}_w(x,\eta,t)=\int_{\mathrm{R}^3_{v'}} e^{-iv'\eta}f_w(x,v',t)dv' \tag{2.7}$$

(we always assume $\hbar=1$).

We introduce $\Theta_1(g)$ to represent the pseudodifferential operator with symbol

$$\mathrm{Sym}\ \Theta_1=g(n_+)-g(n_-), \tag{2.8}$$

where $n_\pm=|\psi_\pm|^2$ and $g(s)=\alpha s^p$. Using these definitions and Eq. (2.2), we arrive at

$$\partial_t(\beta f_w)+v\cdot\nabla_x f_w-i(\Theta_0(V)+\Theta_1(g))f_w=0; \tag{2.9}$$

this can also be written in terms of $f_w^\beta:=\beta f_w$

$$\partial_t f_w^\beta+\frac{1}{\beta}v\cdot\nabla_x f_w^\beta-i\left(\frac{1}{\beta^2}\Theta_0(V)+\frac{1}{\beta^{p+1}}\Theta_1(g)\right)f_w^\beta=0. \tag{2.10}$$

For the numerical computation of solutions, it is more convenient to deal with the Fourier transformed Wigner equation.[9] Using Eq. (2.7) this equation is

$$\partial_t(\beta\hat{f}_w)+i\nabla_\eta\cdot\nabla_x\hat{f}_w-i[V_+-V_-+g(n_+)-g(n_-)]\hat{f}_w=0, \tag{2.11}$$

where $V_\pm=V(x\pm(\eta/2),t)$.

The transformation $r=x+(\eta/2)$, $s=x-(\eta/2)$ has been used[10] to prove the equivalence of the Wigner–Poisson and Schrödinger–Poisson systems (for $\beta\equiv1$, $\alpha=0$). Defining

$$z(r,s,t)=\hat{f}_w(x(r,s),\eta(r,s),t) \tag{2.12}$$

we get analogously for z

$$i\partial_t(\beta z)=(H_r'-H_s')z, \tag{2.13}$$

where

$$H_r'=-\tfrac{1}{2}\Delta_r+V(\psi(r,t))+g(n(r,t)). \tag{2.14}$$

III. CONSERVATION AND QUASICONSERVATION LAWS

In this section we consider (as a preparation for the existence results) some Liapounov-type functionals of (strong) local solutions ψ of system (1.2)–(1.4) (see Sec. IV for a definition of strong solutions). These functionals are important for the physical background of system (1.2) and (1.3). We define

$$P(t)=P(\psi;t)=\beta(t)\int_{\mathrm{R}^3}|\psi(x,t)|^2\,dx \tag{3.1}$$

and

$$Q(t)=Q(\psi;t)=\int_{\mathrm{R}^3}\{|\nabla\psi|^2+|\nabla V|^2+2h(|\psi|^2)\}dx, \tag{3.2}$$

where

$$h(s)=\int_0^s g(r)dr. \tag{3.3}$$

Let us remark that the potential term in Eq. (3.2) could be written more generally as

$$R(\psi)=\int_{\mathrm{R}^3}|\psi|^2(q*|\psi|^2)dx, \tag{3.4}$$

with $q(x)=1/|x|$, but we prefer to use the form given here, valid for a Coulomb potential, because it shows positivity directly.

In the following theorem, we assume ψ to be any local strong solution (see Sec. IV) of Eqs. (1.2)–(1.4) on a time interval $S_T=[0,T]$ $(T>0)$ such that $P(t)$ and $Q(t)$ exist for all t in S_T. Consequently, all relevant expressions used in the following proof exist. We also assume that $\beta\in C^1[0,T]$.

Theorem 3.1: *The following conservation or quasiconservation laws are valid:*

$$P(t)=\text{const}, \tag{3.5}$$

$$\partial_t(\beta Q(t))=\beta_t\left\{-\int_{\mathrm{R}^3}\left|\nabla V\right|^2 dx+2\int_{\mathrm{R}^3}[h(|\psi|^2)-g(|\psi|^2)|\psi|^2]dx\right\} \tag{3.6}$$

for all $t\in S_T$, and any local strong solution ψ on S_T.

Proof: For any local strong solution we have the identity

$$\partial_t(\beta|\psi|^2)=\mathrm{Im}(\Delta\bar{\psi}\cdot\psi). \tag{3.7}$$

This follows from

$$\begin{aligned}\partial_t(\beta|\psi|^2)&=\beta_t|\psi|^2+2\beta\ \mathrm{Re}(\psi_t\bar{\psi})=\beta_t|\psi|^2+2\beta\ \mathrm{Im}(i\psi_t\bar{\psi})\\&=\beta_t|\psi|^2-\beta_t|\psi|^2-\mathrm{Im}(\Delta\psi\bar{\psi})\\&=\mathrm{Im}(\Delta\bar{\psi}\psi);\end{aligned}$$

here we have used Eq. (1.2) as well as the fact that V and g are real functions.

From Eq. (3.7) we get

$$\partial_tP(t)=\int_{\mathrm{R}^3}\partial_t(\beta|\psi|^2)dx=\mathrm{Im}\int_{\mathrm{R}^3}(\Delta\bar{\psi})\psi\, dx=-\mathrm{Im}\int_{\mathrm{R}^3}|\nabla\psi|^2\, dx=0.$$

Furthermore, we calculate that

$$\partial_t(\beta Q(t))=\beta_tQ+\beta Q_t=\beta_tQ-2\ \mathrm{Im}\int_{\mathrm{R}^3}i\beta\psi_t\Delta\bar{\psi}\, dx+2\beta\int_{\mathrm{R}^3}g(|\psi|^2)|\psi|_t^2\, dx$$

$$+2\beta\int_{\mathrm{R}^3}V|\psi|_t^2\, dx$$

and using Eqs. (1.2), (1.3), (3.7), and some partial integration we arrive at

$$\partial_t(\beta Q(t))=\beta_t Q-\beta_t\int_{\mathbb{R}^3}|\nabla\psi|^2\,dx-2\,\mathrm{Im}\int_{\mathbb{R}^3}\psi\Delta\bar{\psi}V\,dx-2\,\mathrm{Im}\int_{\mathbb{R}^3}\psi\Delta\bar{\psi}g(|\psi|^2)dx$$
$$+2\beta\int_{\mathbb{R}^3}g(|\psi|^2)|\psi|_t^2\,dx+2\beta\int_{\mathbb{R}^3}V|\psi|_t^2\,dx$$
$$=-\beta_t\int_{\mathbb{R}^3}|\nabla V|^2\,dx+2\beta_t\int_{\mathbb{R}^3}[h(|\psi|^2)-g(|\psi|^2)|\psi|^2]dx.$$

□

Remark: Theorem 3.1 implies that

$$\int_{\mathbb{R}^3}|\psi(x,t)|^2\,dx=\frac{C}{\beta(t)} \tag{3.8}$$

on any time interval S_T where ψ exists as a local strong solution. If we assume β to be a positive increasing function, Eq. (3.8) implies that the probability density of the system decreases with time, illustrating the dissipation in the model. Moreover, Eq. (3.6) implies an *a priori* bound for the terms comprising $Q(t)$ when the right-hand side of Eq. (3.6) is nonpositive. The last assertion is true if, e.g., $\alpha\geqslant 0$, $p\geqslant 1$, and $\beta_t\geqslant 0$. This follows from

$$2\beta_t\int_{\mathbb{R}^3}[h(|\psi|^2)-g(|\psi|^2)|\psi|^2]dx=2\alpha\beta_t\,\frac{1-p}{1+p}\int_{\mathbb{R}^3}|\psi|^{2(p+1)}\,dx.$$

We note that $\psi\in L^{2(p+1)}(\mathbb{R}^3)$ for strong solutions ψ (see Sec. 4).

Let us remark further that the conservation and quasiconservation laws could also be proven by multiplying Eq. (1.2) by ψ (and taking the imaginary part), and then multiplying Eq. (1.2) by $\beta\psi_t+\frac{1}{2}\beta_t\psi$ (and taking the real part).

IV. GLOBAL EXISTENCE

Now we present some global existence results for the system (1.2)–(1.4). These results follow partially from the theory for the nonlinear Schrödinger equation (without the potential V and for the case $\beta\equiv 1$) and partially from results of the Wigner–Poisson system (for the case $\beta\equiv 1$ and $\alpha=0$) (see Refs. 3 and 11). To proceed we first need some notation and definitions.

Let $T(t)$ $(t\in\mathbb{R})$ be the group generated by $(i/2)\Delta$ on $L^2(\mathbb{R}^3)$, and

$$U(t,s)=T\left(\int_s^t\frac{dr}{\beta(r)}\right) \tag{4.1}$$

be the evolution operator for the linear equation

$$i\phi_t=-\frac{1}{2\beta}\,\Delta\phi, \tag{4.2}$$

which belongs to Eq. (1.2). Here $\beta:\mathbb{R}^+\to\mathbb{R}$ is any continuously differentiable positive real function. By a *strong solution* (H^2 solution) of Eqs. (1.2) and (1.3) on a finite interval $S_T=[0,T]$ we mean a function $\psi\in C(S_T,H^2(\mathbb{R}^3))\cap C^1(S_T,L^2(\mathbb{R}^3))$ such that Eqs. (1.2) and (1.3) are fulfilled in the L^2 sense [for a given $\psi_0\in H^2(\mathbb{R}^3)$] and $V=G*|\psi|^2\in H^2(\mathbb{R}^3)$ with $G(x)=(1/4\pi)\cdot(1/|x|)$. A strong solution is *global* if it exists on any time interval $S_T(T>0)$.

We would like to prove that Eqs. (1.2)–(1.4) have a unique global strong solution for a given $\psi_0\in H^2(\mathbb{R}^3)$. To reach this result we need following assumptions:

$$\alpha>0, \quad 0\leqslant p<2, \tag{P}$$

$$\alpha<0, \quad 0\leqslant p<\tfrac{2}{3}, \tag{N$_1$}$$

$$\alpha<0, \quad \tfrac{2}{3}\leqslant p<2, \quad \beta_t\geqslant 0, \quad \text{on} \quad \mathbb{R}^+. \tag{N$_2$}$$

Remark: In (N_2) the assumption $\beta_t\geqslant 0$ is made to simplify the proof of this case in the following theorem. Furthermore, we note that one could formulate other more complicated conditions on β such that the global existence result again would be true.

Theorem 4.1: *Let $\psi_0\in H^2(\mathbb{R}^3),\beta\in C^1(\mathbb{R}^+,\mathbb{R})$ be real and positive and either $\alpha=0$ or one of the conditions (P) or (N_1) hold. Then there exists a unique global strong solution ψ of Eqs. (1.2)–(1.4). If (N_2) is true, then there exists a $\delta_0>0$ such that the same result follows if either*

$$|\alpha| \quad or \quad \|\psi_0\|_{L^2} \quad or \quad \|\nabla\psi_0\|_{L^2}+\|\nabla V_0\|_{L^2}\leqslant\delta_0. \tag{4.3}$$

Proof (of Theorem 4.1): We first sketch a proof of the existence of a *local* strong solution on a small time interval $S_{T_0}=[0,T_0]$. This proof uses a technique developed in Ref. 11 and 3 (see also Ref. 12). To show the existence of a local strong H^2 solution one must first prove the existence of a unique local (weak) H^1 solution on the small time interval S_{T_0}; this is a function $\psi\in C(S_{T_0},H^1(\mathbb{R}^3))\cap C^1(S_{T_0},H^{-1}(\mathbb{R}^3))$ satisfying Eqs. (1.2)–(1.4) in the (weak) H^1 sense. This follows from a slight variation and combination of the proofs of Theorem 3.10 of Ref. 3 and Theorem 4.3.1 of Ref. 11; first one writes Eq. (1.2) in the form

$$i\phi_t=-\frac{1}{2\beta}\Delta\phi+\mathscr{T}(\phi), \tag{4.4}$$

where the nonlinearity $\mathscr{T}$ is given by

$$\mathscr{T}(\phi)=\frac{1}{\beta}V\left(\frac{\phi}{\sqrt{\beta}}\right)\phi+\frac{1}{\beta}g\left(\frac{|\phi|^2}{\beta}\right)\phi, \tag{4.5}$$

with $V(\psi)=G*|\psi|^2$. The H^1-Lipschitz properties of $\mathscr{T}$ is proven as in Ref. 11 or Ref. 3 by estimating the time-dependent factors by a constant C_{T_0} on the time interval S_{T_0}. For the proof of the Lipschitz property of the term $\alpha|\psi|^{2p}\psi$ one needs the assumption $0\leqslant p<2$ (see Ref. 11). To get the local H^1-solution one applies Banach's fixed-point theorem and the Lipschitz properties of $\mathscr{T}$ to the "mild" version of Eq. (4.4), namely,

$$\phi(t)=U(t,0)\phi_0-i\int_0^t U(t,s)\mathscr{T}(\phi(s))ds. \tag{4.6}$$

Next, to get the local strong H^2-solution [for $\psi_0\in H^2(\mathbb{R}^3)$] one needs to show that for $u\in H^2(\mathbb{R}^3)$ with $\|u\|_{H^2}\leqslant M$

$$\|\mathscr{T}(u)\|_{L^q}\leqslant C(M)(1+\|u\|_{H^2}) \tag{4.7}$$

for some $q>2$ (see Theorem 5.5.1 of Ref. 11). This is true for $\mathscr{T}$ by remarks 5.2.9 of Ref. 11 and the fact that $G\in L^r(\mathbb{R}^3)+L^\infty(\mathbb{R}^3)$ for any $r<3$; also one needs the boundedness of $1/\beta$ and β_t on any finite time interval. A crucial estimate essential to proving the H^2-bound in Theorem 5.5.1 is the following well-known decay property of the group $T(t)$ on $L^2(\mathbb{R}^n)$

$$\|T(t)\|_{B(L^q(\mathbb{R}^n),L^p(\mathbb{R}^n))}\leqslant(4\pi|t|)^{n((1/p)-(1/2))}$$

($2\leqslant p\leqslant\infty, t\in\mathbb{R}\setminus\{0\}, (1/p)+(1/q)=1$, see Ref. 11, Prop. 3.2.1). This is applied to values of the form $t-s$ ($0\leqslant s\leqslant t$) of the time variable. In our case this is true analogously on any finite time interval $S_T=[0,T]$, since on S_T one has $0<\beta(t)\leqslant\beta_T$ for some $\beta_T>0$, and thus for $0\leqslant s<t$

$$\|U(t,s)\|_{B(L^q(\mathbb{R}^3),L^p(\mathbb{R}^3))}\leqslant\left(4\pi\int_s^t\frac{d\tau}{\beta(\tau)}\right)^{3((1/p)-(1/2))}\leqslant\left(\frac{\beta_T}{4\pi(t-s)}\right)^{3((1/2)-(1/p))},$$

which is enough to give the desired result in our case.

Now to arrive at the global strong H^2-solution, we need an *a priori* H^2-*norm* estimate for the local strong H^2-solution on any finite time interval $S_T=[0,T]$. By the proof of Theorem 3.1 it is clear that any local strong H^2-solution satisfies the conservation and quasiconservation laws (3.5) and (3.6). Note that $\psi\in L^{2(p+1)}(\mathbb{R}^3)$ if $\psi\in H^2(\mathbb{R}^3)$; this follows from the Sobolev embedding $H^2(\mathbb{R}^3)\subset L^{2(p+1)}(\mathbb{R}^3)$ ($\forall p\geqslant 0$). From Eq. (3.5) we know that the L^2-norm of any local strong solution is bounded, i.e.,

$$\|\psi(t)\|_{L^2}\leqslant C(T)(\forall t\in S_T), \tag{4.8}$$

where $C(T)$ depends continuously on T.

First we consider an H^1-bound on the solution ψ. For this bound we consider the different signs of α, noting that for $\alpha=0$ we just have a special case of Theorem 3.10 of Ref. 3. For assumption (P) ($\alpha>0$, $0\leqslant p<2$) we have by integrating Eq. (3.6)

$$\begin{aligned}Q(t)&\leqslant\frac{\beta(0)}{\beta(t)}Q(0)+\frac{1}{\beta(t)}\int_0^t\beta_t(s)\left\{-\int|\nabla V|^2\,dx+2\alpha\frac{1-p}{1+p}\int|\psi|^{2(p+1)}\,dx\right\}ds\\&\leqslant C_T\beta(0)Q(0)+C_T\int_0^t Q(s)ds.\end{aligned} \tag{4.9}$$

Hence by Gronwall's Lemma this implies the H^1-bound on S_T

$$\|\psi(t)\|_{H^1}\leqslant C(T). \tag{4.10}$$

Next considering (N_1), we again integrate Eq. (3.6); now using the Gagliardo–Nirenberg inequality (see Ref. 13)

$$\int|\psi|^{2(p+1)}\,dx\leqslant C\left\{\int|\psi|^2\,dx\right\}^{1-p/2}\left\{\int|\nabla\psi|^2\,dx\right\}^{3p/2} \tag{4.11}$$

along with the quantity derived from Eq. (3.5)

$$\int|\psi|^2\,dx=\frac{\beta(0)}{\beta(t)}\int|\psi_0|^2\,dx \tag{4.12}$$

leads to [with $\beta_0:=\beta(0)$]

$$\begin{aligned}\beta(t)Q(t)\leqslant\beta_0Q(0)+\int_0^t|\beta_t(s)|\Bigg\{\int|\nabla V|^2\,dx+2|\alpha|\frac{1-p}{1+p}C\left[\frac{\beta_0}{\beta(s)}\int|\psi_0|^2\,dx\right]^{1-(p/2)}\\\times\left[\int|\nabla\psi|^2\,dx\right]^{3p/2}\Bigg\}ds.\end{aligned} \tag{4.13}$$

Since $3p/2<1$, we can use Young's inequality [$ab\leqslant\epsilon a^r+C_\epsilon b^{r'}; r,r'\geqslant 1, (1/r)+(1/r')=1$] with $r=2/3p$ to get from Eqs. (4.11)–(4.13) the following:

$$\int \{|\nabla\psi|^2+|\nabla V|^2\}dx \leq C_{T,p,\alpha}\left(1+\epsilon\|\psi_0\|_{L^2}^{\sigma}\int |\nabla\psi|^2\,dx\right)+\tilde{C}_{T,p,\alpha}\|\psi_0\|_{L^2}^{\sigma}$$

$$\times\int_0^t\int \{|\nabla\psi|^2+|\nabla V|^2\}dx\,ds. \tag{4.14}$$

Here $\sigma=2(2-p)/(2-3p)$. Once again Gronwall's lemma gives that Eq. (4.14) implies Eq. (4.10) if ϵ is chosen small enough.

Finally for (N_2), we consider the two cases: $2/3\leq p\leq 1$ and $1<p<2$ (for both $\alpha<0$). But before we can proceed, we need the following local version of Gronwall's lemma:

Lemma 4.2 (Local Gronwall Lemma): Let $\phi\in C[0,T)(0<T\leq\infty),\phi(t)\geq 0$, *and let there be positive constants* A,B,γ *such that*

$$\phi(t)\leq A+B\phi(t)^{\gamma},\quad (\forall t\in[0,T)).$$

If one of the following conditions:

(i) $0\leq\gamma<1$,

(ii) $\gamma=1$, $B<1$,

(iii) $\gamma>1$, *and there is an* $\epsilon_0\in(0,1)$ *such that*

$$B\phi(0)^{\gamma-1}<\epsilon_0<1, BA^{\gamma-1}<\epsilon_0(1-\epsilon_0)^{\gamma-1}$$

is valid, then there exists a constant $M>0$ *such that*

$$\phi(t)\leq M,\quad (\forall t\in[0,T)).$$

Proof of lemma: (See Appendix.)

For the first case ($\alpha<0$, $\beta_t\geq 0$, $\frac{2}{3}\leq p\leq 1$) we deduce from Eqs. (3.6), (4.9), and (4.11) that

$$\beta Q(t)\leq\beta_0 Q(0). \tag{4.15}$$

Since $\beta(0)\leq\beta(t)(\forall t\in[0,T])$, Eq. (4.15) implies for $\phi(t):=\int\{|\nabla\psi|^2+|\nabla V|^2\}\,dx$ that

$$\phi(t)\leq\phi(0)+\frac{2|\alpha|}{p+1}\int |\psi|^{2(p+1)}\,dx\leq\phi(0)+\frac{2|\alpha|}{p+1}\|\psi_0\|_{L^2}^{\sigma_0}\left\{\int |\nabla\psi|^2\,dx\right\}^{3p/2}, \tag{4.16}$$

where $\sigma_0=2-p$. From Eq. (4.16) it follows that

$$\phi(t)\leq\phi(0)+\frac{2|\alpha|}{p+1}\|\psi_0\|_{L^2}^{\sigma_0}\phi(t)^{3p/2}.$$

Hence, case (iii) of Lemma 4.2 implies Eq. (4.12) if $|\alpha|$ or $\|\psi_0\|_{L^2}$ or $\|\nabla\psi_0\|_{L^2}+\|\nabla V_0\|_{L^2}$ is sufficiently small [the values being proportional to the constants A, B, and $\psi(0)$ in Lemma 4.2(iii)].

In the second case ($\alpha<0$, $\beta_t\geq 0$ and $1<p<2$), we proceed similarly as in Eqs. (4.14) and (4.16) again using Eq. (4.11). Let $\tilde{\phi}(t)=\sup_{0\leq s\leq t}\phi(t)$. From Eq. (3.6) we get (since $\beta_t\geq 0$)

$$\phi(t)\leq\phi(0)+\frac{2|\alpha|}{p+1}\|\psi_0\|_{L^2}^{\sigma_0}\phi(t)^{3p/2}+2|\alpha|\frac{p-1}{p+1}\|\psi_0\|_{L^2}^{\sigma_0}\frac{1}{\beta(t)}\int_0^t\beta_t(s)\phi(s)^{3p/2}\,ds. \tag{4.17}$$

Now let $0\leq t\leq\tau\leq T$, τ arbitrary; then from Eq. (4.17) we arrive at

$$\phi(t)\leqslant\phi(0)+\frac{2|\alpha|}{p+1}\,\|\psi_0\|_{L^2}^{\sigma_0}\left\{1+(p-1)\,\frac{1}{\beta(t)}\int_0^t\beta_t(s)ds\right\}\tilde{\phi}(\tau)^{3p/2}$$

for all $t\in[0,\tau]$. This implies

$$\tilde{\phi}(\tau)\leqslant\phi(0)+2|\alpha|\,\frac{p}{p+1}\,\|\psi_0\|_{L^2}^{\sigma_0}\tilde{\phi}(t)^{3p/2},$$

which again by Lemma 4.2(iii) gives the desired estimate (4.10) if $|\alpha|$ or $\|\psi_0\|_{L^2}$ is small enough.

The existence of an *a priori* H^2-bound

$$\|\psi(t)\|_{H^2}\leqslant C(T)$$

on any time interval $[0,T]$ where the local strong solution ψ exists, follows from Ref. 3 (for the $V\cdot\psi$ term) and Ref. 11 (for the $\alpha|\psi|^{2p}\psi$ term) by just applying the Laplacian Δ to the right-hand side of the mild version (4.6) of (1.2)–(1.4) and then estimating as in Ref. 3 or Ref. 11's proof of Theorem 5.2.1, Remark 4.3.2 [here again $G\in L^q(\mathrm{R}^3)+L^\infty(\mathrm{R}^3)$ for any $q\in[1,3)$]. □

Remarks:

(1) The proof of Theorem 4.1 shows that in some cases one has the estimate

$$\int\{|\nabla\psi|^2+|\nabla V|^2\}dx\leqslant\frac{C}{\beta(t)},\quad(\forall t\in\mathrm{R}^+),$$

e.g., in the case $\alpha>0$, $\beta_t\geqslant0$, $1\leqslant p<2$, which includes the repulsive (or defocusing) cubic nonlinear Schrödinger equation ($p=1$).

(2) As in Theorem 4.2 of Ref. 3 the results of Theorem 4.1 can be transferred to the Wigner–Poisson system to arrive at the existence of global solutions to the Cauchy problem of Eqs. (2.9) and (2.10); we omit the details.

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APPENDIX: PROOF OF LOCAL GRONWALL LEMMA

We prove Lemma 4.2 in the case when (iii) is true only, since the case (ii) is trival and in case (i) the assertion follows by a straightforward application of Young's inequality. Let (iii) hold. Since ϕ is continuous and $B\phi(0)^{\gamma-1}<\epsilon_0$ there is a $t_1>0$ such that $B\phi(t)^{\gamma-1}<\epsilon_0$ for $0\leqslant t\leqslant t_1$. For $t\in[0,t_1]$ we then have

$$\phi(t)\leqslant A+B\phi(t)^{\gamma-1}\phi(t)<A+\epsilon_0\phi(t),$$

which implies $\phi(t)<M:=A/(1-\epsilon_0)$. Thus one has that

$$Z:=\{t|0\leqslant t<T,\phi(t)<M\}\neq\emptyset.$$

Let $t^*=\sup Z$. We show that $t^*=T$ (from which the assertion follows). Assume $t^*<T$; then there exists a sequence $t_n\in Z$ such that $t_n\to t^*$. This means that $\phi(t^*)\leqslant M$. If $\phi(t^*)<M$ there would be a $\delta>0$ with $\phi(t^*+\delta)<M$ contrary to the definition of t^*. Thus $\phi(t^*)=M$, and $t^*<T$ implies

$$M=\phi(t^*)\leqslant A+B\phi(t^*)^{\gamma-1}\phi(t^*)=A+B\left(\frac{A}{1-\epsilon_0}\right)^{\gamma-1}M<A+\epsilon_0M=(1-\epsilon_0)M+\epsilon_0M=M,$$

which is a contradiction.

Remark: The maximal value of the function $t(1-t)^{\gamma-1}$ in $(0,1)$ is attained for $t=1/\gamma$; thus, the best ϵ_0 possible in condition (iii) of Lemma 4.2 is $\epsilon_0=1/\gamma$. The second inequality then reads

$$BA^{\gamma-1}<(\gamma-1)^{\gamma-1}\gamma^{-\gamma}.$$

In this context, the condition $\phi(t)<A+B\phi^\gamma$ of Lemma 4.2 is implied by assuming that at $t=0$ the function $\phi(t)$ satisfies the estimates which should hold for all t, namely,

$$\phi(t)\leq\frac{\gamma A}{\gamma-1},$$

with $\epsilon_0=1/\gamma$.

[1] N. Akhmediev, A. Ankiewicz, and J. M. Soto-Crespo, Opt. Lett. **18**, 411 (1993); N. Akhmediev and J. M. Soto-Crespo, Opt. Commun. **101**, 223 (1993).
[2] F. Brezzi and P. A. Markowich, Math. Methods Appl. Sci. **14**, 35 (1991).
[3] R. Illner, H. Lange, and P. F. Zweifel, Math. Methods Appl. Sci. **17**, 349 (1994).
[4] P. F. Zweifel, Trans. Theor. Stat. Phys. **22**, 459 (1993).
[5] P. Markowich, C. Ringhofer, and C. Schmeiser, *Semiconductor Equations* (Springer-Verlag, New York, 1990).
[6] H. Lange and P. F. Zweifel, "Dissipation in Wigner–Poisson systems," J. Math Phys. **35**, 1513 (1994).
[7] J. Messer, Act. Phys. Austriaca **50**, 75 (1979).
[8] H. Lange, Nonlinear Analysis TMA **9**, 1115 (1985).
[9] A. Arnold and C. Ringhofer, "An operator splitting method for the Wigner–Poisson problem," Preprint 174 Center for Appl. Math., Purdue Univ., 1992.
[10] P. A. Markowich, Math. Methods Appl. Sci. **11**, 459 (1989).
[11] T. Cazenave, *An Introduction to Nonlinear Schrödinger Equations*, Textos de Métodos Matemáticos 22 (Univ. Fed. do Rio de Janeiro, CCMN, Instituto de Matemática, 1989).
[12] I. E. Segal, Ann. Math. **2**, 339 (1963); A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations* (Springer-Verlag, Berlin, 1983).
[13] A. Friedman, *Partial Differential Equations* (Rinehart and Winston, New York, 1969).