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Time-ordered operators and Feynman–Dyson algebras

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An approach to time-ordered operators based upon von Neumann's infinite tensor product Hilbert spaces is used to define Feynman–Dyson algebras. This theory is used to show that a one-to-one correspondence exists between path integrals and semigroups, which are integral operators defined by a kernel, the reproducing property of the kernel being a consequence of the semigroup property. For path integrals constructed from two semigroups, the results are more general than those obtained by the use of the Trotter–Kato formula. Perturbation series for the Feynman–Dyson operator calculus for time evolution and scattering operators are discussed, and it is pointed out that they are "asymptotic in the sense of Poincaré" as defined in the theory of semigroups, thereby giving a precise formulation to a well-known conjecture of Dyson stated many years ago in the context of quantum electrodynamics. Moreover, the series converge when these operators possess suitable holomorphy properties.

I. INTRODUCTION

It has long been an open question as to what mathematical meaning can be given to the Feynman–Dyson time-ordered operator calculus, which was developed in the 1950's for the study of quantum electrodynamics. In this paper we define Feynman–Dyson algebras and show that they give a natural algebraic framework which allows for the replacement of the *noncommutative structure* of quantum theory with a uniquely defined commutative structure in the time-ordered sense. This approach is analogous to the well-known method in the study of Lie algebras wherein the use of the universal enveloping algebra allows the replacement of a nonassociative structure with a uniquely defined *associative structure* for the development of a coherent representation theory.[1]

The use of this tensor algebra framework allows us to improve upon the customary formal approach to time-ordered operators based upon product integration.

In Sec. II we discuss infinite tensor product Hilbert spaces V and V_ϕ modeled on an arbitrary separable Hilbert space $\mathscr{H}$ and discuss the relationship between algebras of bounded linear operators on these two types of spaces. *It is shown that V_ϕ may be assumed separable with no loss in generality* (see also Sec. IV).

In Sec. III we apply these considerations to the discussion of time-ordered integral operators and discuss how this approach leads to unique solutions to the Cauchy problem for the Schrödinger equation with time-dependent Hamiltonians. The use of infinite tensor product Hilbert spaces requires the introduction of a new topology, and so we discuss how uniqueness in the Cauchy problem is to be understood in this framework.

In Sec. IV we discuss the relationship between various algebras of bounded linear operators on infinite tensor product Hilbert spaces and give a mathematically rigorous treatment of algebras of time-ordered operators on these spaces. The latter algebras, called Feynman–Dyson algebras, provide a mathematical treatment of Feynman's operator calculus.[2] Our use of infinite tensor product Hilbert spaces in this connection can be seen to be the mathematical embodiment of the method of Fujiwara[3] in the implementation of Feynman's approach. The definition of these so-called "expansional" operators has been discussed in a Banach algebraic framework different from that of the present paper by Miranker and Weiss[4] and Araki.[5] Related discussions of time-ordered operators have been given by Nelson[6] and Maslov.[7]

In Sec. V we apply our theory of time-ordered operators to the discussion of path integrals of the type first envisioned by Feynman.[8] We show that there exists a one-to-one correspondence between path integrals and semigroups which are integral operators defined by a kernel. In this situation, the reproducing property of the kernel follows from the semigroup property. In this section, path integrals are written for more general Hamiltonians than perturbations of Laplacians by making use of some results of Maslov and Shishmarev[9,10] on hypoelliptic pseudodifferential operators. In those cases in which one is dealing with two semigroups, it is not necessary to assume that the sum of the generators is a generator of a third semigroup. In particular, it is not necessary to assume that one of the two generators is small in some sense relative to the other.

In Sec. VI we discuss perturbation expansions for time-evolution operators. It is shown that these expansions generally do not converge, but are "asymptotic in the sense of Poincaré" as this term is used in the theory of semigroups.[11] This nonconvergence of the perturbation expansions was conjectured in the special case of the renormalized perturbation expansions of quantum electrodynamics in a well-known paper by Dyson.[12] We also prove that these series converge when the semigroups possess suitable holomorphy properties.

Section VII consists of some concluding remarks.

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II. PRELIMINARIES

Let $J=[-T,T]$, $T>0$, denote a compact subinterval of the real line and $V=\otimes_{s\in J}\mathscr{H}(s)$ the infinite tensor product Hilbert space, where $\mathscr{H}(s)=\mathscr{H}$ for each $s\in J$ and $\mathscr{H}$ denotes a fixed abstract separable Hilbert space. Here $L[\mathscr{H}]$ and $L[V]$ denote the bounded linear operators on the respective spaces. Here $L[\mathscr{H}(s)]$ is defined by

$$L[\mathscr{H}(s)]=\left\{B(s)=\overline{\underset{T\geqslant r>s}{\otimes} I_r\otimes\widetilde{B}\otimes\left(\underset{s>r\geqslant -T}{\otimes} I_r\right)}\,\middle|\,\widetilde{B}\in L[\mathscr{H}]\right\}\tag{2.1}$$

where I_r is the identity operator, and $L^{\#}[V]$ is the uniform closure of the algebra generated by the family: $\{L[\mathscr{H}(s)]|s\in J\}$.

Definition 2.1: We say that $\phi=\otimes_s\phi_s$ is *equivalent* to $\psi=\otimes_S\psi_s$ and write $\phi\simeq\psi$ if and only if

$$\sum_s|\langle\phi_s,\psi_s\rangle_s-1|<\infty\,,\tag{2.2}$$

where $\langle\ ,\ \rangle_s$ denotes the inner product on $\mathscr{H}(s)$. It is to be understood that the sum is meaningful only if at most a countable number of terms are different from zero. The following result is due to von Neumann,[13] but see Guichardet[14] for a simplified proof.

Theorem 2.1: The above relation is an equivalence relation V. If we let V_ϕ denote the closure of the linear spin of all $\psi\simeq\phi$, then (1) ψ not equivalent to ϕ implies $V_\psi\cap V_\phi=\{0\}$; and (2) if we replace J by $\bar{J}\subset J$, where $\bar{J}$ is a countable dense subset, in our definition of V [i.e., $V=\hat{\otimes}_{s\in\bar{J}}(s)$], then V is a separable Hilbert space.

Let $\mathbb{P}_\phi$ be the projection from V onto V_ϕ.

Theorem 2.2[13]: For all $T\in L^{\#}[V]$, the restriction of T to V_ϕ is a bounded linear operator, and

$$\mathbb{P}_\phi T=T\mathbb{P}_\phi\,.\tag{2.3}$$

Let $\mathbb{C}[V]$ denote the set of closable linear operators on V.

Definition 2.2: An exchange operator $E[t,t']$ is a linear operator defined on $\mathbb{C}[V]$ for pairs $t,t'\in J$ such that

(1) $E[t,t']$ maps $\mathbb{C}[\mathscr{H}(t')]$ onto $\mathbb{C}[\mathscr{H}(t)]$,
(2) $E[t,s]\,E[s,t']=E[t,t']$,
(3) $E[t,t']\,E[t',t]=I$,
(4) if $s\neq t,t'$, then $E[t,t']A(s)=A(s)$, for all $A(s)\in\mathbb{C}[\mathscr{H}(s)]$.

It should be noted that $E[t,t']$ is linear in the sense that whenever the sum of two closable operators is defined and closable, then $E[t,t']$ maps in the appropriate manner (see Gill[15]). In particular, $E[t,t']$ restricted to $L^{\#}[V]$ is a Banach algebra isomorphism and $E[t,t']\,E[s,s']=E[s,s']\,E[t,t']$ for distinct pairs (t,t') and (s,s') in J.

Theorem 2.3: If $F=\Pi_{n=1}^{\infty}E[\tau_n,s_n]$, $\{(\tau_n,s_n)\in J\times J\,|n\in\mathbb{N}\}$ then F is a Banach algebra isomorphism on $L^{\#}[V]$ and

(1) $\|F\|_{\#}=1$,
(2) $F^{-1}=F$.

Proof: As $\|E[\tau,s]\|_{\#}=1$, F is a convergent product of algebra isomorphisms and $\|F\|_{\#}\leqslant\|E[\tau_n,s_n]\|_{\#}=1$. On the other hand, $1=\|I\|_{\#}=\|F(I)\|_{\#}\leqslant\|F\|_{\#}\|I\|_{\#}$, so that $\|F\|_{\#}=1$. Since $E[\tau_n,s_n]E[s_n,\tau_n]=I$ and exchange operators for distinct pairs commute, we see that $F^2=I\Rightarrow F^{-1}=F$.

Definition 2.3: A chronological morphism (or c-morphism) on $L^{\#}[V]$ is any (Banach) algebra isomorphism F on $L^{\#}[V]$ composed of products of exchange operators such that

(1) $\|F\|_{\#}=1$,
(2) $F^{-1}=F$.

Definition 2.4: Let $\{\widetilde{H}(t)|t\in J\}\subset\mathbb{C}[\mathscr{H}]$ denote a family of densely defined closed self-adjoint operators on $\mathscr{H}$, then the corresponding time-ordered version in $\mathbb{C}[V]$ is defined by

$$H(t)=\overline{\underset{T\geqslant s>t}{\otimes} I_s\otimes\widetilde{H}(t)\otimes\left(\underset{t>s\geqslant -T}{\otimes} I_s\right)}\,.\tag{2.4}$$

Definition 2.5: A family $\{H(t)|t\in J\}\subset\mathbb{C}[V]$ is said to be *chronologically continuous (or c-continuous) in the strong sense at* t_0 if there exists an exchange operator $E[t_0,t]$ such that

$$\lim_{t\to t_0}\|E[t_0,t]H(t)\phi-H(t_0)\phi\|=0\,,\tag{2.5}$$

where $\phi\in\otimes_{s\in J}\mathscr{D}[\mathscr{H}(s)]$.

Definition 2.6: The family $\{H(t)|t\in J\}$ is said to be chronologically differentiable (or c-differentiable) in the strong sense at t_0 if there exists an operator $DH(t_0)$ and an exchange operator $E(t_0,t)$ such that

$$\lim_{t\to t_0}\left\|\frac{E(t_0,t)H(t)\phi-H(t_0)\phi}{t-t_0}-DH(t_0)\phi\right\|=0\,,$$

for all $\phi\in\otimes_{s\in J}\mathscr{D}(H(s))$.

Theorem 2.4: Suppose the family of operators $\{\widetilde{H}(t)|t\in J\}$ have a common domain. Then the corresponding family $\{H(t)|t\in J\}$ is strongly c-continuous iff $\{\widetilde{H}(t)|t\in J\}$ is strongly continuous.

Proof: See Gill.[15]

III. INTEGRALS AND EVOLUTIONS

In the following discussion, all operators of the form $\{\widetilde{A}(t)|t\in J\}$ are closed infinitesimal generators of contraction semigroups, while $\{\widetilde{H}(t)|t\in J\}$ are strongly continuous densely defined linear operators with a common domain, and generate unitary groups. The corresponding operators of the form $\{A(t)|t\in J\}$ [resp. $\{H(t)|t\in J\}$] are the time-ordered versions. Define $A^z(t)$ by

$$A^z(t)=\exp\{zA(t)\}-I/z\tag{3.1}$$

and recall that $\exp\{A^z(t)\}$ is a linear contraction and $\text{s-lim}_{z\downarrow 0}A^z(t)=A(t)$ (strong limit). Similar results hold for $H^z(t)$, with z replaced by iz in (3.1).

Definition 3.1: An integral approximate on $L^{\#}[V]$ is a family of operators of the form $\{Q_\lambda^z[t,-T]$ with $-T\leqslant t\leqslant T,\lambda>0\}$, where

$$Q_\lambda^z[t,-T]=e^{-2\lambda T}\sum_{n=0}^{\infty}\frac{(2\lambda T)^n}{n!}\sum_{j=1}^{k(n)}\Delta t_jA^z(\tau_j)\,.$$

For each $n,k=k(n)\geqslant n$ and $\{\mathbb{P}_k=\{-T=t_1<t_2\cdots<t_k=t\},n,k\in\mathbb{N}\}$ is a family of partitions of $[-T,t]$ such that $\lim_{n\to\infty}|\mathbb{P}_k|=0$ and we take $\tau_j\in[t_{j-1},t_j)$.

Definition 3.2: Let $\{Q_\lambda^z[t,-T]\}$ and $\{\bar{Q}_\lambda^z[t,-T]\}$ be any two families of integral approximates. We say Q_λ^z is c-

equivalent to $\overline{Q}{}^z_\lambda$ and write $Q^z_\lambda \overset{c}{\simeq} \overline{Q}{}^z_\lambda$ (in the uniform sense) if and only if there exists a c-morphism $F = F[Q^z_\lambda, \overline{Q}{}^z_\lambda]$ such that

$$\lim_{\lambda\to\infty} \|Q^z_\lambda[t,-T] - F\overline{Q}{}^z_\lambda[t,-T]\| = 0. \tag{3.2}$$

Theorem 3.1: The relation $\overset{c}{\simeq}$ is an equivalence relation on the set of all integral approximates on $L^\#[V]$.

Proof: Reflexivity is obvious. To prove symmetry, we note that

$$\|Q^z_\lambda - F\overline{Q}{}^z_\lambda\| = \|F^{-1}Q^z_\lambda - \overline{Q}{}^z_\lambda\|$$

since $\|F\| = 1$, and $F = F^{-1}$. Hence $Q^z_\lambda \overset{c}{\simeq} \overline{Q}{}^z_\lambda$ implies $\overline{Q}{}^z_\lambda \overset{c}{\simeq} Q^z_\lambda$. To prove transitivity, suppose F_1 and F_2 exist such that

$$\lim_{\lambda\to\infty} \|Q^z_\lambda - F_1\overline{Q}{}^z_\lambda\| = 0, \quad \lim_{\lambda\to\infty} \|\overline{Q}{}^z_\lambda - F_2\overline{\overline{Q}}{}^z_\lambda\| = 0.$$

Setting $F = F_1F_2$ we have

$$\begin{aligned}\|Q^z_\lambda - F\overline{\overline{Q}}{}^z_\lambda\| &= \|Q^z_\lambda - F_1\overline{Q}{}^z_\lambda + F_1\overline{Q}{}^z_\lambda - F_1F_2\overline{\overline{Q}}{}^z_\lambda\| \\ &\leqslant \|Q^z_\lambda - F_1\overline{Q}{}^z_\lambda\| + \|\overline{Q}{}^z_\lambda + F_2\overline{\overline{Q}}{}^z_\lambda\|,\end{aligned}$$

hence $\lim_{\lambda\to\infty} \|Q^z_\lambda - F\overline{\overline{Q}}{}^z_\lambda\| = 0$, so that $Q^z_\lambda \overset{c}{\simeq} \overline{\overline{Q}}{}^z_\lambda$.

Here $Q^z[t,-T] = \text{s-lim}_{\lambda\to\infty} Q^z_\lambda[t,-T]$ is called the *time-ordered integral operator* associated with the family $\{A^z(t)|t\in J\} \subset L^\#[V]$ if the above limit exists.

Theorem 3.2 (existence): For the family $\{H^z(t)|t\in J\}$ we have (1) $\text{s-lim}_{\lambda\to\infty} Q^z_\lambda[t,-T] = Q^z[t,-T]$ exists and

$$Q^z[t,-T] = Q^z[t,s] + Q^z[s,-T], \quad -T \leqslant s < t,$$

(2) $\text{s-lim}_{z\downarrow 0} Q^z[t,-T] = Q[t,-T]$ exists, is a densely defined generator of a unitary group on V, and

$$Q[t,-T] = Q[t,s] + Q[s,-T],$$

and

(3) $\underset{\lambda\to\infty}{\text{s-lim}}[\underset{z\downarrow 0}{\text{s-lim}}\, Q^z_\lambda[t,-T]] = \underset{z\downarrow 0}{\text{s-lim}}[\underset{\lambda\to\infty}{\text{s-lim}}\, Q^z_\lambda[t,-T]]$.

Proof: See Gill,[16] Theorems (1.1) and (1.2).

From now on, our results assume that we are working with the family $\{H^z(t)|t\in J\}$.

Theorem 3.3: Let $Q^z_\lambda[t,-T]$ and $\overline{Q}{}^z_\lambda[t,-T]$ be two integral approximates with the same family of partitions but different points $\tau_j, s_j \in [t_{j-1}, t_j)$ ("place values"). Then $Q^z_\lambda \overset{c}{\simeq} \overline{Q}{}^z_\lambda$ (in the strong sense).

Proof: Define

$$F = \prod_{n=1}^{\infty}\left(\prod_{j=1}^{n} E[\tau_j, s_j]\right)$$

so that

$$F\overline{Q}{}^z_\lambda = e^{-2\lambda T}\sum_{n=0}^{\infty}\frac{(2\lambda T)^n}{n!}\sum_{j=1}^{k}\Delta t_j E[\tau_j, s_j] H^z(s_j). \tag{3.3}$$

By Theorem 2.3, we see that F is a c-morphism and

$$\begin{aligned}&\|Q^z_\lambda\phi - F\overline{Q}{}^z_\lambda\phi\| \\ &\quad\leqslant e^{-2\lambda T}\sum_{n=0}^{\infty}\frac{(2\lambda T)^n}{n!} \\ &\qquad\times\sum_{j=1}^{k}\Delta t_j\|H^z(\tau_j)\phi - E[\tau_j, s_j]H^z(s_j)\phi\|.\end{aligned}$$

We now note that strong c-continuity of $H(t)$ (cf. definition 2.5) implies strong c-continuity of $H^z(t)$ so, given $\epsilon > 0$, there exists $\delta > 0$ such that $|\tau - s| < \delta$ implies for $\phi\in V$, $\|H^z(\tau)\phi - E[\tau, s]H^z(s)\phi\| < \epsilon/(t+T)$. Now, choose N so large that $n \geqslant N$ implies $|\mathbb{P}_k| < \delta$, then

$$\begin{aligned}\|Q^z_\lambda\phi - F\overline{Q}{}^z_\lambda\phi\| &\leqslant e^{-2\lambda T}\sum_{n=0}^{N-1}\frac{(2\lambda T)^n}{n!}\sum_{j=1}^{k}\Delta t_j\|H^z(\tau_j)\phi - E[\tau_j, s_j]H^z(s_j)\phi\| \\ &\quad + e^{-2\lambda T}\sum_{n=N}^{\infty}\frac{(2\lambda T)^n}{n!}\sum_{j=1}^{k}\Delta t_j\|H^z(\tau_j)\phi - E[\tau_j, s_j]H^z(s_j)\phi\| \\ &\leqslant e^{-2\lambda T}\sum_{n=0}^{N-1}\frac{(2\lambda T)^n}{n!}\sum_{j=1}^{k}\Delta t_j\|H^z(\tau_j)\phi - E[\tau_j, s_j]H^z(s_j)\phi\| + \left(e^{-2\lambda T}\sum_{n=N}^{\infty}\frac{(2\lambda T)^n}{n!}\right)\epsilon \\ &< e^{-2\lambda T}\sum_{n=0}^{N-1}\frac{(2\lambda T)^n}{n!}\sum_{j=1}^{k}\Delta t_j\|H^z(\tau_j)\phi - E[\tau_j, s_j]H^z(s_j)\phi\| + \epsilon.\end{aligned}$$

If we now let $\lambda\to\infty$, we obtain $\lim_{\lambda\to\infty}\|Q^z_\lambda\phi - F\overline{Q}{}^z_\lambda\phi\| < \epsilon$. Since ϵ was arbitrary we are done. ||

Let us note that in Theorem 3.3 it is not necessary to require that $s_j, \tau_j \in [t_{j-1}, t_j)$. It suffices to assume that for n sufficiently large, $|s_j - \tau_j| < \delta$, $1 \leqslant j \leqslant k(n)$ [i.e., $\lim_{n\to\infty}|s_j - \tau_j| = 0$ $\forall_j$, $1 \leqslant j \leqslant k(n)$].

Let $\overline{Q}{}^z_\lambda$ and $\overline{\overline{Q}}{}^z_\lambda$ be two integral approximates generated from arbitrary families of partitions $\{\overline{\mathbb{P}}_{l_1}\}$, $\{\overline{\overline{\mathbb{P}}}_{l_2}\}$ with respective place values $\overline{\tau}_l \in (\overline{t}_{l-1}, \overline{t}_l)$, $1 \leqslant l \leqslant l_1(n)$, and $\overline{\overline{\tau}}_l \in [\overline{\overline{t}}_{l-1}, \overline{\overline{t}}_l)$, $1 \leqslant l \leqslant l_z(n)$. Define a new family of partitions $\mathbb{P}_k = \overline{\mathbb{P}}_{l_1} \cup \overline{\overline{\mathbb{P}}}_{l_2}$ and integral approximate Q^z_λ with $\tau_j \in [t_{j-1}, t_j)$.

Since $\overline{\mathbb{P}}_{l_1} \subset \mathbb{P}_k$,

$$\overline{Q}{}^z_\lambda = e^{-2\lambda T}\sum_{n=0}^{\infty}\frac{(2\lambda T)^n}{n!}\sum_{l=1}^{l_1}\overline{\Delta t_l}\, H^z(\overline{\tau}_l)$$

may be reindexed to give

$$\overline{Q}{}^z_\lambda = e^{-2\lambda T}\sum_{n=0}^{\infty}\frac{(2\lambda T)^n}{n!}\sum_{j=1}^{k}\Delta t_j H^z(s_j), \tag{3.4}$$

where $s_j = \bar{\tau}_l$ for $\bar{t}_{l-1} \leqslant t_{j-1} < t_j \leqslant \bar{t}_l$. Thus $\bar{Q}^z_\lambda$ and Q^z_λ have the same family of partitions, but different place values.

Theorem 3.4: $\bar{Q}^z_\lambda \overset{c}{\simeq} \bar{\bar{Q}}^z_\lambda$.

Proof: We first show that $\bar{Q}^z_\lambda \overset{c}{\simeq} Q^z_\lambda$. From the above remarks, it suffices to show that $|\tau_j - s_j| \to 0$, $n \to \infty$. To see this, recall that $\tau_j \in [t_{j-1}, t_j)$ and $s_j = \bar{\tau}_l$ for $\bar{t}_{l-1} \leqslant t_{j-1} < t_j \leqslant \bar{t}$, hence $|\tau_j - s_j| < \overline{\Delta t_l} \to 0$ as $n \to \infty$. Therefore $\bar{Q}^z_\lambda \overset{c}{\simeq} Q^z_\lambda$ by Theorem 3.3. The same argument with $\bar{Q}^z_\lambda$ replaced by $\bar{\bar{Q}}^z_\lambda$ shows that $\bar{\bar{Q}}^z_\lambda \overset{c}{\simeq} Q^z_\lambda$. We now use the transitivity of $\overset{c}{\simeq}$ to conclude that $\bar{Q}^z_\lambda \overset{c}{\simeq} \bar{\bar{Q}}^z_\lambda$. ||

Definition 3.3: A time-ordered integral operator is said to be *chronologically unique* (or *c-unique*) if every integral approximate is c-equivalent.

Let $Q[t, -T] = \text{s-lim}_{z\downarrow 0} Q^z[t, -T]$.

Theorem 3.5: (1) $Q^z[t, -T]$ is c-unique.

(2) $Q[t, -T]$ is a generator of a unitary group (densely defined and closed).

Proof: (1) is clear; (2) is in Gill.[16]

The uniqueness property in part (1) of this theorem is an important feature of our theory. There are path integrals which depend upon the choice of partition. See Ref. 17 for a discussion.

Theorem 3.6: $U^z[t, -T] = \exp\{-iQ^z[t, -T]\}$ satisfies

(1) $U^z[t, -T] = U^z[t,s]U^z[s, -T], \quad -T \leqslant s \leqslant t,$

(2) $i\dfrac{\partial U^z[t, -T]}{\partial t} = H^z(t)U^z[t, -T]$,

(3) $U[t, -T] = \underset{z\downarrow 0}{\text{s-lim}}\, U^z[t, -T]$

$= \exp\{-iQ[t, -T]\}$

satisfies

$U[t,s]U[s, -T] = U[t, -T], \quad -T \leqslant s \leqslant t,$

(4) $i\dfrac{\partial U[t, -T]}{\partial t} = H(t)U[t, -T]$.

Proof: See Gill.[16] The derivatives are in the strong chronological sense. This theorem allows us to give a complete solution to the Cauchy problem. Recall that if $\phi_0 \in D(\tilde{H}(t)) \subset \mathscr{H}$ for $t \in J$, then the initial value problem

$$i\frac{\partial f(t)}{\partial t} = \tilde{H}(t)f(t), \quad f(-T) = \phi_0,$$

has a unique solution $f(t)$ provided a few additional assumptions are made. For a direct proof with explicit statements of the required additional assumptions, see Tanabe.[18] We prove a similar result in the Hilbert space V with no additional assumptions.

Theorem 3.7: Let $\phi_s = \phi_0$, $\|\phi_0\| = 1$, $s \in J$, and set $\phi = \otimes_s \phi_s$. Then $\phi(t) = U(t, -T)\phi$ is the c-unique solution to

$$i\frac{\partial \phi(t)}{\partial t} = H(t)\phi(t), \quad \phi(-T) = \phi,$$

where the derivatives are interpreted in the strong chronological sense.

Proof: Follows from Theorems 3.5 and 3.6.

IV. OPERATOR ALGEBRAS

Let us recall from Theorem 2.1 that if we replace J by $\bar{J} \subset J$, where $\bar{J}$ is a dense subset and construct $\bar{V} = \hat{\otimes}_{s \in \bar{J}} \mathscr{H}(s)$ then $\bar{V}_\phi$ (the closure of the linear span of all $\psi \simeq \phi$) is a separable (Hilbert) subspace. The next theorem is quite interesting in view of the fact that $\bar{V}$ and V are not related as spaces.

Theorem 4.1: $L^\#[\bar{V}] \subset L^\#[V]$ (i.e., is an injection into).

Proof: From (2.1), it is easy to see that $L[\mathscr{H}(s)]$ is a closed subalgebra of $L^\#[V]$ for each $s \in J$ (a detailed proof is in von Neumann[13]). This is also true for each $s \in \bar{J}$, so the result follows trivially, since $L^\#[\bar{V}]$ is generated by $\{L[\mathscr{H}(s)] | s \in \bar{J}\}$, and $L[\mathscr{H}(s)] \subset L^\#[V]$, $s \in \bar{J}$.

Let us note that the existence and uniqueness of $Q^z[t, -T]$ and $U[t, -T]$ do not change if we restrict $\{\tau_j | 1 \leqslant j \leqslant k(n), n \in \mathbb{N}\}$, to lie in $\bar{J}$ in defining Q^z_λ and U_λ. This means that the following holds.

Theorem 4.2:

(1) $Q^z[t, -T]$ and $U[t, -T]$ belong to $L^\#[\bar{V}]$,

$$(2)\ Q^z[t, -T]|_{\bar{V}_\phi} \in L[\bar{V}_\phi], \tag{4.1}$$

$$(3)\ U[t, -T]|_{\bar{V}_\phi} \in L[\bar{V}_\phi]. \tag{4.2}$$

Proof: (1) is obvious while (2) and (3) follows from Theorem 2.3.

The above result shows that both $U[t, -T]$ and $Q[t, -T]$ are well defined (and the same operators as in V_ϕ) when restricted to $\bar{V}_\phi$, which is a separable Hilbert space. This means that all of standard quantum theory can be formulated in our setting.

We now turn to some other important properties of $L^\#[V]$. First, let us establish some notation. If $\{\tilde{B}(t), t \in J\}$ denotes an arbitrary family of opertors in $L[\mathscr{H}]$, the operator $\Pi_{t \in J}\tilde{B}(t)$ (when defined) is understood in its natural order:

$$\prod_{T \geqslant t \geqslant -T} \tilde{B}(t). \tag{4.3}$$

It is easy to see that every operator A in $L^\#[V]$ that depends on a countable number of elements in J may be written as

$$A = \sum a_i \prod_{k=1}^{n_i} A_i(t_k), \tag{4.4}$$

where $A_i(t_k) \in L[\mathscr{H}(t_k)]$, $t_1, t_2, \ldots, t_{n_i}$ for all i. Define dT: $L^\#[V] \to L[\mathscr{H}]$ by

$$dT[A] = \sum_{i=1}^{\infty} a_i \prod_{n_i \geqslant k \geqslant 1} \tilde{A}_i(t_k). \tag{4.5}$$

Lemma 4.2: The map dT is a bounded linear map which is surjective but not injective.

Proof: The proof is trivial. To see that dT is not injective, note that (for example) $dT[E[t,s]A(s)] = dT[A(s)]$ yet $A(s) \in L[\mathscr{H}(s)]$ while $E[t,s]A(s) \in L[\mathscr{H}(t)]$ so that these operators are not equal when $t \neq s$.

From Theorem 2.2, we know that the algebras $L[\mathscr{H}(t)]$ and $L[\mathscr{H}]$ are isomorphic as Banach algebras so that for each $t\in J$, there exists an isomorphism $t\theta$: $L[\mathscr{H}]\to L[\mathscr{H}(t)]$. Now $t\theta^{-1}$: $L[\mathscr{H}(t)]\to L[\mathscr{H}]$; and since $L[\mathscr{H}(t)]$ is a closed subalgebra of $L^{\#}[V]$, we know that dT restricted to $L[\mathscr{H}(t)]$ is an algebra homomorphism.

Theorem 4.3: $dT_{L[\mathscr{H}(t)]} = t\theta^{-1}$.

Proof: It is clear that $t\theta^{-1}[A(t)] = \widetilde{A}(t)$ and $dT[A(t)] = \widetilde{A}(t)$, $A(t)\in L[\mathscr{H}(t)]$, so we need only show that dT is injective when restricted to $L[\mathscr{H}(t)]$. If $A(t)$ and $B(t)$ belong to $L[\mathscr{H}(t)]$ and $dT[A(t)] = dT[B(t)]$, then $\widetilde{A}(t) = \widetilde{B}(t)$ (by definition of dT) so that $A(t) = B(t)$ by definition of $L[\mathscr{H}(t)]$.

Definition 4.1: The map dT is called the *disentanglement morphism*.

Definition 4.2: The quadruple $(\{t\theta\,|\,t\in J\}, L[\mathscr{H}], dT, L^{\#}[V])$, is called a *Feynman–Dyson algebra* (FD *algebra*) over $\mathscr{H}$ for the parameter set J.

We now show that the FD algebra is universal for time ordering in the following sense.

Theorem 4.4: Given any family $\{\widetilde{B}(t)\,|\,t\in J\}\in(L[\mathscr{H}])^J$ there is a unique family $\{B(t)\,|\,t\in J\}\subset L^{\#}[V]$ such that the following conditions hold.

(1) $B(t)\in L[\mathscr{H}(t)]$, $\quad t\in J$.

(2) $dT[B(t)] = \widetilde{B}(t)$, $\quad t\in J$.

(3) For an arbitrary family $\{\{\tau_j\,|\,1\leqslant j\leqslant n\}\,|\,n\in N\}$, $\tau_j\in J$ (distinct) the map

$$\mathop{X}_{n=1}^{\infty}(\widetilde{B}(\tau_n),\dots,\widetilde{B}(\tau_1))\to\sum_{n=1}^{\infty}a_n\prod_{n\geqslant j\geqslant 1}\widetilde{B}(\tau_j)$$

from

$$\mathop{X}_{n=1}^{\infty}\left\{\mathop{X}_{j=1}^{n}L[\mathscr{H}]\right\}\text{ to }L[\mathscr{H}],$$

has a unique factorization through $L^{\#}[V]$ so that

$$\sum_{n=0}^{\infty}a_n\prod_{n\geqslant j\geqslant 1}\widetilde{B}(\tau_j)$$

corresponds to

$$\sum_{n=0}^{\infty}a_n\prod_{j=1}^{n}B(\tau_j)\,.$$

Here we naturally assume that $\{a_n\}$ is such that

$$\sum_{n=0}^{\infty}a_n\prod_{n\geqslant j\geqslant 1}\widetilde{B}(\tau_j)\in L[\mathscr{H}]\,.$$

Proof: $B(t) = t\theta[\widetilde{B}(t)]$, $\forall t\in J$, gives (1). By Theorem 4.3 we have $dT[B(t)] = t\theta^{-1}[B(t)] = \widetilde{B}(t)$ which gives (2). To prove (3), note that

$$\Theta:\ \mathop{X}_{n=1}^{\infty}\left\{\mathop{X}_{j=1}^{n}L[\mathscr{H}]\right\}\to\mathop{X}_{n=1}^{\infty}\mathop{X}_{j=1}^{n}L[\mathscr{H}(\tau_j)]$$

defined by

$$\Theta[X_{n=1}^{\infty}(\widetilde{A}_n,\widetilde{A}_{n-1},\dots,\widetilde{A}_1)]$$
$$= X_{n=1}^{\infty}(\tau_n\theta[\widetilde{A}_n],\ \tau_{n-1}\theta[\widetilde{A}_{n-1}],\dots,\tau_1\theta[\widetilde{A}_1])$$

is one-to-one and onto $(\tau_j\theta[\widetilde{A}_j] = A(\tau_j)\in L[\mathscr{H}(\tau_j)])$. The map

$$\mathop{X}_{n=1}^{\infty}(B(\tau_n),\dots,B(\tau_1))\to\sum_{n=1}^{\infty}a_n\prod_{j=1}^{\infty}B(\tau_j)\in L^{\#}[V]$$

factors through the tensor algebra $\oplus_{n=1}^{\infty}\{\otimes_{j=1}^{n}L[\mathscr{H}(\tau_j)]\}$ via the universal property of that object (Hu,[19] p. 179). We now note that $\oplus_{n=1}^{\infty}\{\otimes_{j=1}^{n}L[\mathscr{H}(\tau_j)]\}\subset L^{\#}[V]$. In diagram form we have

$$\begin{array}{ccc}
\mathop{X}\limits_{n=1}^{\infty}(\widetilde{B}(\tau_n),\dots,\widetilde{B}(\tau_1))\in\mathop{X}\limits_{n=1}^{\infty}\left\{\mathop{X}\limits_{j=1}^{n}L[\mathscr{H}]\right\} & \xrightarrow{f} & \sum\limits_{n=1}^{\infty}a_n\prod\limits_{n\geqslant j\geqslant 1}\widetilde{B}(\tau_j)\in L[\mathscr{H}]\\
\theta\big\downarrow & & \big\uparrow dT\\
\mathop{X}\limits_{n=1}^{\infty}(B(\tau_n),\dots,B(\tau_1))\in\mathop{X}\limits_{n=1}^{\infty}\left\{\mathop{X}\limits_{j=1}^{n}L[\mathscr{H}(\tau_j)]\right\} & \xrightarrow{f_\otimes} & \sum\limits_{n=1}^{\infty}a_n\prod\limits_{j=1}^{n}B(\tau_j)\in L^{\#}[V]
\end{array}$$

so that $dT\circ f_\otimes\circ\theta = f$.

Example 1: Let

$$A(t) = \overline{\bigotimes_{T\geqslant\tau>t}I_\tau\otimes\widetilde{A}\otimes\left(\bigotimes_{t>\tau\geqslant -T}I_\tau\right)}\,,$$

$$B(s) = \overline{\bigotimes_{T\geqslant\tau>s}I_\tau\otimes\widetilde{B}\otimes\left(\bigotimes_{s>\tau\geqslant -T}I_\tau\right)}\,,$$

where $\widetilde{A}$ and $\widetilde{B}$ are bounded on $\mathscr{H}$. If $s<t$, then by Lemma 2.3 in Ref. 15, we have

$$A(t)B(s) = B(s)A(t)$$
$$= \overline{\bigotimes_{T\geqslant\tau>t}I_\tau\otimes\widetilde{A}\otimes\left(\bigotimes_{t>\tau>s}I_\tau\right)\otimes\widetilde{B}\otimes\left(\bigotimes_{s>\tau\geqslant -T}I_\tau\right)}\,,$$

so that $dT[A(t)B(s)] = dT[B(s)A(t)] = \widetilde{A}\widetilde{B}$, while

$$dT[A(t)B(s) - B(t)A(s)]$$
$$= dT[A(t)B(s) - A(s)B(t)] = \widetilde{A}\widetilde{B} - \widetilde{B}\widetilde{A}\,.$$

Example 2: Let $\{\widetilde{H}(t)\,|\,t\in J\}$ be strongly continuous (with common dense domain), and suppose this family generates a product integral (Dollard and Friedman[20]). Choose any family $\{\mathbb{P}_n\,|\,n\in\mathbb{N}\}$ of partitions such that

$$\lim_{n\to\infty}\prod_{j=1}^{n}\exp\{-i\Delta t_j\widetilde{H}(\tau_j)\} = \widetilde{U}[t,-T]\,,$$

then $\lim_{\lambda\to\infty}\widetilde{U}_\lambda[t,-T] = \widetilde{U}[t,-T]$, where

$$\widetilde{U}_\lambda[t,-T]$$
$$= e^{-2\lambda T}\sum_{n=0}^{\infty}\frac{(2\lambda T)^n}{n!}\prod_{j=1}^{n}\exp\{-i\Delta t_j\widetilde{H}(\tau_j)\}\,.$$

This follows from the fact that Borel summability is regular. For the same family $\{\mathbb{P}_n\,|\,n\in\mathbb{N}\}$, construct

$$U_\lambda [t, -T] = e^{-2\lambda T} \sum_{n=0}^{\infty} \frac{(2\lambda T)^n}{n!} \exp\left\{ -i \sum_{j=1}^{n} \Delta t_j H(\tau_j) \right\}.$$

As in Ref. 16, we see that $U[t, -T] = \lim_{\lambda\to\infty} U_\lambda [t, -T]$ exists in $L^{\#}[V]$. Furthermore, $dT\{U[t, -T]\} = dT \lim_{\lambda\to\infty} U_\lambda [t, -T] = \lim_{\lambda\to\infty} dT\{U_\lambda [t, -T]\} = \widetilde{U}[t, -T]$. We can interchange limits since dT is a closed linear operator on $L^{\#}[V]$. It should be noted that the above limit can exist even if the standard product integral does not. This result will be discussed in a subsequent paper (see Gill and Zachary[21]).

V. APPLICATIONS TO THE CONSTRUCTION OF PATH INTEGRALS

In the present section we consider time-ordered operators in more detail, and discuss the proposition that there exists a one-to-one correspondence between path integrals and semigroups which are integral operators defined by a kernel. We apply our formulation of time-ordered operators to the discussion of path integrals of the type first considered by Feynman.[8] There have been many approaches to the mathematical construction of time-ordered operators and path integrals in recent years. We will not be using any of these approaches, so we content ourselves with offering the following admittedly incomplete list of references[7,9,10,22–25] from which the reader can trace these developments.

Let us consider the time-independent self-adjoint generator $\widetilde{H}_0$ of a unitary group defined on $\mathscr{H}$ in terms of a transition kernel $\widetilde{K}$ which satisfies the Chapman–Kolmogorov equation.

If we replace the operator $\widetilde{H}_0$ by its time-ordered version $\{H_0(t)\colon t\in J\}$, we induce a natural family of kernels $K(x(t),t; y(s),s)$ via Theorem 3.2. To see this, note that

$$\begin{aligned}
\overline{U}_n(t, -T)\phi_0 &= \exp\left[-i \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} E(\tau_j,\tau) H_0(\tau)\, d\tau \right] \phi_0 \\
&= \prod_{j=1}^{n} \overline{\left[\left(\bigotimes_{t\geqslant s>\tau_j} I_s \right) \otimes \exp[-i(t_j - t_{j-1})\widetilde{H}_0] \otimes \left(\bigotimes_{\tau_j>s\geqslant -T} I_s \right) \right]} \phi_0 \\
&= \prod_{j=1}^{n} \overline{\left[\left(\bigotimes_{t\geqslant s>\tau_j} I_s \right) \otimes \int_{\mathbb{R}^k} \widetilde{K}(x_j,t_j;x_{j-1},t_{j-1})\, dx_{j-1} \otimes \left(\bigotimes_{\tau_j>s\geqslant -T} I_s \right) \right]} \phi_0 \\
&= \prod_{j=1}^{n} \int_{\mathbb{R}^k} \mathbb{K}_{\tau_j}(x_j,t_j;x_{j-1},t_{j-1})\, dx_{j-1}\, \phi_0,
\end{aligned} \tag{5.1}$$

where $\phi_0 = \otimes_{s\in J} \phi(s)$, $J = [-T,T]$. In (5.1), $x_j = x(t_j)$ and the index τ_j on $\mathbb{K}$ is used to indicate the time at which $\mathbb{K}$ acts. Combination of (5.1) with Theorem 3.2 shows that $\overline{U}_\lambda(t, -T)$ may be represented in the form

$$\overline{U}_\lambda (t, -T)\phi_0 = e^{-2\lambda T} \sum_{n=0}^{\infty} \frac{(2\lambda T)^n}{n!} \times \prod_{j=1}^{n} \int_{\mathbb{R}^k} \mathbb{K}_{\tau_j}(x_j,t_j;x_{j-1},t_{j-1})\, dx_{j-1}\, \phi_0. \tag{5.2}$$

Since $\overline{U}_\lambda(t, -T)$ exists as a well-defined bounded operator, and

$$\lim_{\lambda\to\infty} \overline{U}_\lambda (t, -T) = U_0(t, -T)$$

exists in the uniform operator topology, $U_0(t, -T)$ has a natural representation as an operator-valued path integral:

$$U_0(t, -T) = \int_{\mathscr{X}(t,-T)} \mathbb{K}(x(t),t;x(s),s)\, \mathscr{D}[x(s)], \tag{5.3}$$

where $\mathscr{X}(t, -T) = \mathbb{R}^{k(t,-T)}$ denotes the set of *all* functions from $[t, -T]$ to $\mathbb{R}^k$. In (5.3) we have used a formal "functional measure" notation, although a measure generally does not exist, as we discuss in more detail below.

In recent years many authors have attempted to bypass the difficulty that Feynman-type path integrals cannot generally be written in terms of countably additive measures,[25] as is the case for its closest relative, the Wiener integral. In the present paper we take the point of view that integration theory, as contrasted with measure theory, is the appropriate vehicle to be considered for a theory of path integration. An essential ingredient in our approach is the idea that it is possible to define path integrals by giving up the requirement of the existence of a countably additive measure. This idea has a precursor in the theory of integration in Euclidean spaces. *That is, it is possible to define a consistent theory of integration, which generalizes Lebesgue integration, in which the integrals are finitely additive, but are generally not countably additive.*[26] Indeed, Henstock[27] has already discussed the Feynman integral from this point of view.

Returning now to our discussion of (5.3), we note that many authors have sought to restrict consideration to continuous functions in the definition of path integrals. The best known example is undoubtedly the Wiener integral.[28] However, the fact that we must see $\mathscr{X}(t, -T)$ follows naturally from the time-ordered operator calculus, and such a restriction is probably neither possible nor desirable in our theory. This means that our approach does not encourage attempts at the standard measure theoretic formulations with countably additive measures. In previous work by one of us,[15] the Riemann-complete (generalized Riemann) integral of Henstock and Kurzweil[26] was employed, because the time-ordered integrals need not be absolutely integrable, even in the bounded operator case. These issues will be studied in greater depth at another time. We note in passing that this failure of absolute integrability also plays an important role in the path integral theory of Albeverio and Høegh-Krohn,[23] and also in more recent developments (see, e.g., Ref. 24). Our theory, to be discussed in the remainder of the present section, allows for more general Hamiltonians.

Before proceeding to a discussion of these results, we pause to discuss some examples. The first one is well known—the familiar Laplacian operator. Our purpose in discussing it here is to show how our theory works in a familiar case.

Let $\widetilde{H}_0 = -\Delta/2$ so that $H_0(t) = -\Delta_t/2$, where the subscript t indicates the time slot at which this operator is to be evaluated. We have

$$\widetilde{K}(x,t;y,s) = (2\pi i(t-s))^{-k/2} \exp[i|x-y|^2/2(t-s)] .$$

In this case it is easy to see that

$$\begin{aligned}\overline{U}_n(t,-T)\phi_0 &= \prod_{j=1}^{n} \int_{\mathbf{R}^k} \exp\left[\frac{i(t_j-t_{j-1})}{2}\left|\frac{x_j-x_{j-1}}{t_j-t_{j-1}}\right|^2(\tau_j)\right] \\ &\quad\times \frac{dx_{j-1}\,\phi_0}{[2\pi i(t_j-t_{j-1})]^{k/2}} \qquad (5.4)\\ &= \int_{\mathbf{R}^{kn}} \exp\left[i\sum_{j=1}^{n}\frac{1}{2}(t_j-t_{j-1})\left|\frac{x_j-x_{j-1}}{t_j-t_{j-1}}\right|^2(\tau_j)\right] \\ &\quad\times \prod_{j=1}^{n}\frac{dx_{j-1}}{[2\pi i(t_j-t_{j-1})]^{k/2}}\phi_0 . \qquad (5.5)\end{aligned}$$

By analogy with the definition of $H_0(t)$ given above, the (τ_j) are used to remind us that the corresponding functions in (5.4) and (5.5) are not ordinary exponentials because they have a specific time slot at which they are evaluated. This is our version of the occurrence of expansionals in the usual approach.[3,5] Using (5.5) with (5.2), we have

$$\begin{aligned}\overline{U}^0_\lambda(t,-T)\phi_0 &= e^{-2\lambda T}\sum_{n=0}^{\infty}\frac{(2\lambda T)^n}{n!} \\ &\quad\times\int_{\mathbf{R}^{kn}} \exp\left[i\sum_{j=1}^{n}\frac{1}{2}(t_j-t_{j-1})\left|\frac{x_j-x_{j-1}}{t_j-t_{j-1}}\right|^2(\tau_j)\right] \\ &\quad\times\prod_{j=1}^{n} D(x_{j-1})\phi_0 ,\end{aligned}$$

where $D(x_{j-1}) = (2\pi i(t_j-t_{j-1}))^{-k/2}\,dx_{j-1}$. This means that $U^0(t,-T)$ may be represented by

$$U^0(t,-T)\phi = \int_{\mathscr{X}(t,-T)} \exp\left[\frac{1}{2} i\int_{-T}^{t}\left|\frac{dx}{ds}\right|^2 ds\right] \times \prod_{t>s>-T} D(x(s))\phi_0 .$$

As our second example, we consider the operator $\widetilde{H} = \sqrt{-\Delta+\omega^2}$. It was shown by Pursey[29] that the Bargmann–Wigner equation for a relativistic particle of any physically allowed spin value $s = 0,\frac{1}{2},1,\frac{3}{2},\ldots$ is unitarily equivalent to the equation defining the Cauchy problem for this square root operator. Foldy and Wouthuysen[30] showed that this operator is nonlocal with effective spatial extension equal to a Compton wavelength. Our interest here is to show that it is an integral operator defined by a kernel $\widetilde{K}$.

The method of pseudodifferential operators can be used to show that a kernel exists and, under reasonable conditions, can provide a phase space representation as we discuss in detail more general operators later in this section. However, if we desire a direct representation, then other methods are required. In our case, we have found that the method of fractional powers of operator semigroups allows us to solve the problem in a simple manner. By using results on pp. 281 and 302 of Ref. 31, p. 260 of Ref. 32, and p. 498 of Ref. 11, it can be shown that the semigroup generated by the closure of $\sqrt{-\Delta+\omega^2}$, $T(\tau)$, can be written in the form

$$T(\tau)\phi(x) = \frac{i\omega^2}{2\pi^2\tau}\int_{\mathbf{R}^3}\frac{K_2[\omega\tau\sqrt{|x-y|^2/\tau^2-1}]}{|x-y|^2/\tau^2-1}\phi(y)dy , \qquad (5.6)$$

where $K_2(\cdot)$ denotes the modified Bessel function of the third kind of order 2. It is clear that $T(\tau)$ is holomorphic. From (5.6) we see that we have an example of a semigroup with a kernel that is not of the form

$$\exp\left[i\frac{m}{2}\left|\frac{x_j-x_{j-1}}{t_j-t_{j-1}}\right|^2(t_j-t_{j-1})\right] . \qquad (5.7)$$

Since (5.7) is appropriate for the nonrelativistic regime, we cannot expect it to have general validity. However, if the argument of the Bessel function is large, we should expect the kernel in (5.6) to approximate (5.7) when $|(x_j-x_{j-1})/(t_j-t_{j-1})|$ is small compared to unity (= speed of light). Since $K_2(z) \sim \sqrt{\pi/2}\, e^{-z}/z$ for large argument, we see that we may approximate the kernel in (5.6) by (using $\sqrt{v^2-1} \to i\sqrt{1-v^2}$)

$$\begin{aligned}\widetilde{K}(x_j,t_j;x_{j-1},t_{j-1}) &\cong \frac{i\omega^2}{2\pi^2(t_j-t_{j-1})} \\ &\quad\times\sqrt{\frac{\pi}{2}}\frac{\exp[-i\omega(t_j-t_{j-1})\sqrt{1-v^2}]}{\sqrt{i\omega(t_j-t_{j-1})\sqrt{1-v^2}(1-v^2)}} ,\end{aligned}$$

where $v = |(x_j-x_{j-1})/(t_j-t_{j-1})|$. Now, letting $v\to 0$ in the denominator and approximating the square root in the numerator, we obtain

$$\begin{aligned}\widetilde{K}(x_j,t_j;x_{j-1},t_{j-1}) &\cong +i\left(\frac{\omega}{2\pi i(t_j-t_{j-1})}\right)^{3/2}\exp[-i\omega(t_j-t_{j-1})] \\ &\quad\times\exp\left[i\frac{\omega}{2}\left|\frac{x_j-x_{j-1}}{t_j-t_{j-1}}\right|^2(t_j-t_{j-1})\right] . \qquad (5.8)\end{aligned}$$

Thus we see that the kernel in (5.6) reduces to the nonrelativistic limit except for the extra phase factor which corresponds to a rest mass term in the standard approaches. It is important to realize, however, that two distinct assumptions are required to obtain (5.8). The first corresponds to observations far removed from the particle, while the second involves the nonrelativistic approximation. In order to see the effect of the first assumption, we need only note that for small z,

$$K_2(z) \sim 2z^{-2} . \qquad (5.9)$$

It is also of interest to investigate the limit $\omega \to 0$ corresponding to a massless particle. In this case we replace $K_2(z)$ by (5.9) to obtain

$$\widetilde{K}(x_j,t_j;x_{j-1},t_{j-1}) \cong \frac{+i}{\pi^2(t_j-t_{j-1})^3}\left[1-\left|\frac{x_j-x_{j-1}}{t_j-t_{j-1}}\right|^2\right]^{-2}. \quad (5.10)$$

It is very interesting to note that both (5.8) and (5.10) are propagators for unitary groups.

In order to describe path integrals for more general situations than covered thus far in the present section, we consider the case of two families of self-adjoint time-ordered operators $\{H_0(t): t\in J\}$ and $\{H_1(t): t\in J\}$ with respective domains D_0 and D_1 which are dense in V_ϕ. It is assumed that both families are strongly c-continuous generators of unitary groups. Consider a partition P_n of $[-T,t]$ as in Definition 3.1 and let $\tau_j,s_j\in[t_{j-1},t_j)$. We then define

$$U_n(t,-T) = \exp\left[\sum_{j=1}^n (t_j-t_{j-1})\{H_0(\tau_j)+H_1(s_j)\}\right],$$

$$U_n^0(t,-T) = \exp\left[\sum_{j=1}^n (t_j-t_{j-1})H_0(\tau_j)\right],$$

$$U_n^1(t,-T) = \exp\left[\sum_{j=1}^n (t_j-t_{j-1})H_1(s_j)\right].$$

Since we do not assume any relationship between D_0 and D_1, $U_n(t,-T)$ is well defined except when $\tau_j = s_j$ for some j. In the contrary case we have

$$U_n(t,-T) = U_n^0(t,-T)U_n^1(t,-T) = U_n^1(t,-T)U_n^0(t,-T).$$

Now, defining $U_\lambda(t,-T)$, $U_\lambda^0(t,-T)$, and $U_\lambda^1(t,-T)$ by combining the notations of Theorems 3.2 and 3.6, we have the following theorem.

Theorem 5.4[16]:

(1) $\lim_{\lambda\to\infty} U_\lambda(t,-T) = U(t,-T)$ exists a.s.,

(2) $U(t,-T) = U^1(t,-T)U^0(t,-T) = U^0(t,-T)U^1(t,-T)$ a.s.

By specializing the partition P_n by choosing $t_j - t_{j-1} = 1/n$, $1\leqslant j\leqslant n$, we have

$$U_\lambda(t,-T) = e^{-2\lambda T}\sum_{n=0}^\infty \frac{(2\lambda T)^n}{n!}\left[\prod_{j=1}^n \exp\left\{\frac{1}{n}H_0(\tau_j)\right\}\right] \times\left[\prod_{j=1}^n \exp\left\{\frac{1}{n}H_1(s_j)\right\}\right].$$

This is reminiscent of the Trotter–Kato product formula,[31,33] but is more general due to our weak restrictions on the two self-adjoint operator families and our use of the Borel summability procedure. For example, it is not necessary to assume that H_0+H_1 is self-adjoint as in Ref. 33. This means that, in particular, it is not necessary to assume that one of the operators, $\widetilde{H}_1$ say, is small in some sense relative to the other, $\widetilde{H}_0$. The fact that Theorem 5.4 does not depend on the domains is anticipated by the work of Chernoff[34] on the "generalized additivity" of generators of semigroups arising from Trotter–Kato-type product formulas. This author has given an extensive discussion of these formulas for quite arbitrary domains. See also Kato[35] for a discussion of the case of two positive self-adjoint operators on a Hilbert space when the intersection of their domains may be arbitrary.

We remind the reader at this point that the Trotter–Kato formula is one of the standard methods for formal derivations of Feynman's formula for the nonrelativistic time-evolution operator.[23,36] Similarly, Theorem 5.4 is the basis for our treatment of the Feynman integral which, however, is completely rigorous.

We now discuss the results in Theorem 5.4 from a slightly different point of view. We see from this theorem that

$$U(t,-T) = \exp\left[-i\int_{-T}^t \{H_0(\tau)+H_1(\tau)\}\,d\tau\right]$$

exists a.e. and

$$U(t,-T) = \lim_{\lambda\to\infty} \overline{U}_\lambda(t,-T),$$

where

$$\overline{U}_\lambda(t,-T) = e^{-2\lambda T}\sum_{n=0}^\infty \frac{(2\lambda T)^n}{n!} \times\exp\left[-i\sum_{j=1}^n\int_{t_{j-1}}^{t_j}\{E(\tau_j,\tau)H_0(\tau) + E(s_j,\tau)H_1(\tau)\}\,d\tau\right] \quad (5.11)$$

with $\tau_j,s_j\in[t_{j-1},t_j)$. If we use (5.5), the exponent in (5.11) can be replaced by

$$i\sum_{j=1}^n\left\{(t_j-t_{j-1})\left|\frac{x_j-x_{j-1}}{t_j-t_{j-1}}\right|^2(\tau_j) - \int_{t_{j-1}}^{t_j}E(s_j,\tau)H_1(x(\tau),\tau)d\tau\right\}.$$

Taking limits, we have

$$U(t,-T) = \iint_{\mathscr{P}(t,-T)}\exp\left[i\int_{-T}^t\left\{\frac{1}{2}\left|\frac{dx}{ds}\right|^2 - H_1(x(s),s)\right\}ds\right]\prod_{t>s>-T}D(x(s)). \quad (5.12)$$

It is clear that our conditions on the family $\widetilde{H}_1(x,s)$ are sufficiently general to cover most cases of interest in nonrelativistic quantum theory. We can now write (5.12) in the form originally envisioned by Feynman, namely,

$$U(t,-T) = \int_{\mathscr{P}(t,-T)}\exp\left[i\int_{-T}^t L(\dot{x}(s),x(s),s)ds\right]\times\prod_s D(x(s)),$$

where $x(s) = dx/ds$ and

$$L(\dot{x}(s),x(s),s) = \frac{1}{2}\left|\frac{dx}{ds}\right|^2 - H_1(x(s),s)$$

denotes the Lagrangian.

We now generalize the representation (5.12) by considering more general choices for the operator $\widetilde{H}_0(\tau)$. For these operators we choose the class of hypoelliptic pseudodifferential operators studied by Shishmarev.[10] In this way, we are able to derive a representation for $U(t,-T)$ analogous to (5.12) which will include cases useful for studies in relativistic quantum mechanics such as, e.g., perturbations of the square root operator studied earlier.

Let $\widetilde{H}(x,p)$ denote a $k\times k$ matrix operator $[\widetilde{H}_{ij}(x,p)]$, $i,j=1,2,...,k$, whose components are pseudodifferential operators with symbols $h_{ij}(x,\eta)\in C^\infty(\mathbb{R}^N\times\mathbb{R}^N)$ and we have, for any multi-indices α and β,

$$|h_{ij(\beta)}^{(\alpha)}(x,\eta)|\leqslant C_{\alpha\beta}(1+|\eta|)^{m-\xi|\alpha|+\delta|\beta|}, \qquad (5.13)$$

where

$$h_{ij(\beta)}^{(\alpha)}(x,\eta)=\partial^\alpha p^\beta h_{ij}(x,\eta)$$

with $\partial_l=\partial/\partial\eta_l$, and $p_l=(1/i)(\partial/\partial x_l)$. The multi-indices are defined in the usual manner by $\alpha=(\alpha_1,...,\alpha_N)$ for integers $\alpha_j\geqslant 0$, and $|\alpha|=\Sigma_{j=1}^N\alpha_j$, with similar definitions for β. The notation for derivatives is $\partial^\alpha=\partial_1^{\alpha_1}\cdots\partial_N^{\alpha_N}$ and $p^\beta=p_1^{\beta_1}\cdots p_N^{\beta_N}$. Here, m, β, and δ are real numbers satisfying $0\leqslant\delta<\xi$. Equation (5.13) states that each $h_{ij}(x,\eta)$ belongs to the symbol class[37] $S_{\xi,\delta}^m$.

Let $h(x,\eta)=[h_{ij}(x,\eta)]$ be the matrix-valued symbol for $\widetilde{H}(x,p)$, and let $\lambda_1(x,\eta),...,\lambda_k(x,\eta)$ denote its eigenvalues. If $|\cdot|$ denotes a norm in the space of $k\times k$ matrices, we suppose that the following conditions are satisfied by $h(x,\eta)$: For $|\eta|>c_0>0$ and $x\in\mathbb{R}^N$ we have

(1) $|h_{(\beta)}^{(\alpha)}(x,\eta)|\leqslant C_{\alpha\beta}|h(x,\eta)|(1+|\eta|)^{-\xi|\alpha|+\delta|\beta|}$

(hypoellipticity),

(2) $\lambda_0(x,\eta)=\max_{1\leqslant j\leqslant k}\operatorname{Re}\lambda_j(x,\eta)<0$,

(3) $\dfrac{|h(x,\eta)|}{|\lambda_0(x,\eta)|}=O((1+|\eta|)^{(\xi-\delta)/(2k-\epsilon)}),\quad \epsilon>0$.

We assume that $\widetilde{H}(x,p)$ is a self-adjoint generator of a unitary group, so that

$$U(t,0)\psi_0(x)=\exp[-it\widetilde{H}(x,p)]\psi_0(x)=\psi(x,t)$$

solves the Cauchy problem

$$i\frac{\partial\psi}{\partial t}=H(x,p)\psi,\psi(x,0)=\psi_0(x)\,. \qquad (5.14)$$

Definition 5.1: We say that $Q(x,t,\eta,0)$ is a *symbol for the Cauchy* problem (5.14) if $\psi(x,t)$ may be represented as

$$\psi(x,t)=(2\pi)^{-N/2}\int_{\mathbb{R}^N}e^{i(x,\eta)}Q(x,t,\eta,0)\hat{\psi}_0(\eta)d\eta\,. \qquad (5.15)$$

It suffices to assume that ψ_0 belongs to the Schwartz space $\mathscr{S}(\mathbb{R}^N)$, which is contained in the domain of $\widetilde{H}(x,p)$, in order that (5.15) makes sense.

Following Shishmarev,[10] and using the theory of Fourier integral operators, we define an operator-valued kernel for $U(t,0)$ by

$$\widetilde{K}(x,t;y,0)=(2\pi)^{-N/2}\int_{\mathbb{R}^N}e^{i(x-y,\eta)}Q(x,t,\eta,0)d\eta,$$

so that

$$U(t,0)\psi_0(x)=\psi(x,t)=\int_{\mathbb{R}^N}\widetilde{K}(x,t;y,0)\psi_0(y)\,dy\,. \qquad (5.16)$$

The following results are due to Shishmarev.[10]

Theorem 5.5: Suppose $\widetilde{H}(x,p)$ is a self-adjoint generator of a strongly continuous unitary group with a domain which is dense in $L^2(\mathbb{R}^N)$ and contains $\mathscr{S}(\mathbb{R}^N)$, such that conditions (1)–(3) are satisfied. Then there exists precisely one symbol $Q(x,t,\eta,0)$ for the Cauchy problem (5.14).

Theorem 5.6: Suppose one replaces condition (3) in Theorem 5.5 by the condition

(3′) $\dfrac{|h(x,\eta)|}{|\lambda_0(x,\eta)|}=O((1+|\eta|)^{(\xi-\delta)/(3k-1-\epsilon)})$,

$\epsilon>0,\quad |\eta|>c_0$.

Then the symbol $Q(x,t,\eta,0)$ of the Cauchy problem (5.14) has the following asymptotic behavior as $t\to 0$:

$$Q(x,t,\eta,0)=\exp[-ith(x,\eta)]+o(1)\,,$$

uniformly for $x,\eta\in\mathbb{R}^N$.

Now, using Theorem 5.6 we see that under the strengthened condition (3′) the kernel $\widetilde{K}(x,t;y,0)$ satisfies

$$\widetilde{K}(x,t;y,0)=\int_{\mathbb{R}^N}\exp[i\{(x-y,\eta)-th(x,\eta)\}]\frac{d\eta}{(2\pi)^N}+\int_{\mathbb{R}^N}\exp[i(x-y,\eta)]\frac{d\eta}{(2\pi)^N}\,o(1)\,.$$

We now apply the results discussed earlier in this section to construct the path integral associated with $\widetilde{H}(x,p)$. The group property of $U(t,0)$ insures that $\widetilde{K}$ has the reproducing property expressed by the Chapman–Kolmogorov equation. In our time-ordered version, we obtain

$$K_\tau(x,t;y,0)=\int_{\mathbb{R}^N(\tau)}\exp[i\{(x-y,\eta)-th_\tau(x,\eta)\}]\times\frac{d\eta}{(2\pi)^N}+o(1)\,.$$

This representation leads to the Feynman phase space version of the path integral.

We can now obtain more general path integrals than (5.12) by replacing (5.5) by (5.16). It follows from Theorems 5.4–5.6 that path integrals exist which are generalizations of (5.12). These new path integrals correspond, of course, to Hamiltonian operators which are perturbations of the operators described in Theorems 5.5 and 5.6, rather than to Hamiltonians which are perturbations of Laplacians. These path integrals constitute a very large class which contain most integrals of interest in mathematical physics.

VI. PERTURBATION EXPANSIONS

In this section we discuss the Feynman–Dyson operator calculus for $U(t,-T)$. It is shown that the corresponding perturbation expansions do not converge in general, but are "asymptotic in the sense of Poincaré" in the sense used in the theory of semigroups.[11] On the other hand, if we assume that the semigroups possess certain holomorphy properties, then the perturbation series converge. Previous investigations of these perturbation expansions have been confined to the interaction representation in the framework of nonrelativistic scattering by time-dependent potentials[38] and external field problems in quantum field theory.[39]

Our results of this section pertaining to the asymptotic nature of these perturbation expansions affirms a well-known conjecture of Dyson[12] made in the context of the special case of the renormalized perturbation expansions in quantum electrodynamics on the basis of a simple physical argument. Although presently, many people *believe* quantum electrodynamics should be formulated in a Hilbert space with an indefinite metric (see, e.g., Ref. 40 and the works cited therein), Dyson made no such assumptions. In our concluding remarks to this section, we make explicit our basic assumptions and argue that they certainly cover conditions that physicists believe QED should satisfy.

Consider the infinite tensor product Hilbert space $V = \hat{\otimes}_{s \in J} \mathcal{H}(s)$ of Sec. II, where $J = [-T, T]$, $\mathcal{H}(s) = \mathcal{H}$ for each $s \in J$, and $\mathcal{H}$ denotes a fixed abstract separable Hilbert space. For a family $\{\hat{H}(t): t \in J\}$ of densely defined strongly continuous self-adjoint operators on $\mathcal{H}$, the corresponding time-ordered family $\{H(t): t \in J\}$ is defined on V by (2.4). Let $U(t, -T)$ denote the corresponding time-evolution operator whose existence is guaranteed by Theorem 3.2.

Let

$$Q(t, -T) = -i \int_{-T}^{t} H(s) ds$$

denote the time-ordered integral of the family $\{-iH(t): t \in J\}$. Then the closure of $Q(t, -T)$, which we will also denote by $Q(t, -T)$, generates the strongly c-continuous unitary group $U(t, -T) = \exp[Q(t, -T)]$ on V. We also have the following.

Theorem 6.1: Suppose $\phi \in D(H^N(s))$ for $-T \leqslant s \leqslant t$. Then $U(t, -T)\phi$ can be written in the form

$$U(t, -T)\phi = \sum_{k=0}^{N-1} \frac{1}{k!} (Q(t, -T))^k \phi + R_N(t, -T)\phi, \tag{6.1}$$

with the following representations for the remainder term:

$$R_N(t, -T)\phi = \int_0^1 dv (1-v)^{N-1} \exp[vQ(t, -T)] \times \frac{(Q(t, -T))^N}{(N-1)!} \phi, \tag{6.2}$$

and

$$R_N(t, -T)\phi = (-i)^N \int_{-T}^{t} d\tau_N \cdots \int_{-T}^{\tau_2} d\tau_1 \times H(\tau_N) \cdots H(\tau_1) U(\tau_1, -T)\phi. \tag{6.3}$$

Proof: It follows from a result of Hille and Phillips (Ref. 11, p. 354) that (6.1) holds with the remainder term given by (6.2). The equality of the latter with (6.3) is a consequence of the following result, which establishes a Fubini-type theorem for the Feynman–Dyson operator calculus.

Lemma 6.1: For any $N = 1, 2, \ldots$, we have

$$\frac{1}{N!} \left[\int_{-T}^{t} H(\tau) d\tau \right]^N = \int_{-T}^{t} d\tau_N \int_{-T}^{\tau_N} d\tau_{N-1} \cdots \int_{-T}^{\tau_2} d\tau_1 \, H(\tau_N) \cdots H(\tau_1).$$

Proof: Recall that the bounded operators

$$H_z(\tau) = [\exp(zH(\tau)) - I]/z, \quad z > 0,$$

converge as $z \downarrow 0$ to $H(\tau)$ on $D(H(\tau))$ uniformly in τ on compact sets. We can therefore, without loss in generality, assume that $H(\tau)$ is bounded for each τ. The proof can then be completed by a bounded operator version of the usual integration by parts procedure for functions.

In the remainder of this section we discuss the problem of approximating the various terms in the expansion (6.1). For this purpose we use the form (6.2) for the remainder term.

Using the fact that $Q(t, -T)$ generates the strongly c-continuous unitary group $U(t, -T)$, we find from the theory of semigroups[11,41] that

$$P_z(t, -T) = (\exp[zQ(t, -T)] - I)/z, \quad z > 0,$$

converges to $Q(t, -T)$ on $D(Q(t, -T))$ as $z \downarrow 0$. More generally, we have the following.

Lemma 6.2: Fix some $r \in \{1, 2, \ldots\}$ and take $f \in D(\{Q(t, -T)\}^r)$. Then

$$\operatorname*{s-lim}_{z \downarrow 0} \{P_z(t, -T)\}^r f = \{Q(t, -T)\}^r f.$$

Proof: From p. 99 of Ref. 41 we have

$$(P_z^r - Q^r)\phi = \frac{1}{r!} \sum_{j=1}^{r} (-1)^{r-j} j^r \binom{r}{j} \times \left\{ \frac{r!}{(jz)^r} \left[e^{jzQ}\phi - \sum_{k=0}^{r-1} \frac{(jz)^k}{k!} Q^k \phi \right] - Q^r \phi \right\},$$

so that

$$\|(P_z^r - Q^r)\phi\| \leqslant \sup_{0 \leqslant u \leqslant rz} \|(e^{uQ} - I) Q^r \phi\|,$$

from which the proof readily follows.

Let us now define the bounded operators

$$U_z(t, -T) = \exp[P_z(t, -T)] = \sum_{k=0}^{N-1} \frac{[P_z(t, -T)]^k}{k!} + R_N^z,$$

where

$$R_N^z(t, -T) = \int_0^1 dv (1-v)^{N-1} \exp[vP_z(t, -T)] \times \frac{[P_z(t, -T)]^N}{(N-1)!}. \tag{6.4}$$

The boundedness of these operators follows from the estimates,

$$\|\{P_z(t, -T)\}^r\| \leqslant (2/z)^r, \quad r = 1, 2, \ldots, \tag{6.5}$$

which are, in turn, consequences of the fact that $Q(t, -T)$ generates a contractive semigroup.

Now we have the following Theorem.

Theorem 6.2:

(a) $\operatorname*{s-lim}_{z \downarrow 0} U_z(t, -T) = U(t, -T)$,

(b) $\operatorname*{s-lim}_{z \downarrow 0} R_N^z(t, -T)\phi = R_N(t, -T)\phi, \quad \phi \in D(\{Q(t, -T)\}^N)$.

Proof: (a) follows from the fact that $U(t, -T)$ is a strongly c-continuous unitary group on V and Hille's first exponential formula (see, e.g., Ref. 41, Theorem 1.2.2).

To prove (b) we write, using (6.2) and (6.4),

$$(R_N - R_N^z)\phi = \int_0^1 dv \, \frac{(1-v)^{N-1}}{(N-1)!} \left[e^{vQ}Q^N - e^{vP_z}P_z^N\right]\phi$$

so that

$$\begin{aligned}\|(R_N - R_N^z)\phi\| &\leqslant \frac{1}{N!} \sup_{v\in[0,1]} \|(e^{vQ}Q^N - e^{vP_z}P_z^N)\phi\| \\ &\leqslant \frac{1}{N!} \sup_{v\in[0,1]} \left[\|(e^{vQ} - e^{vP_z})Q^N\phi\| \right. \\ &\quad \left. + \|e^{vP_z}(Q^N - P_z^N)\phi\|\right] . \end{aligned} \tag{6.6}$$

For the first term on the right-hand side of (6.6) we use the fact that, by Theorem 1.2.2 of Ref. 41,

$$\|(e^{vQ} - e^{vP_z})Q^N\phi\| \to 0 \quad \text{as } z\downarrow 0 ,$$

for $\phi\in D(Q^N)$ uniformly with respect to $v\in[0,1]$. The vanishing of the remaining term in (6.6) as $z\downarrow 0$ follows from Lemma 6.2 coupled with the estimate

$$\|\exp[vP_z(t, -T)]\| \leqslant 1 , \tag{6.7}$$

which in turn follows from Hille's first exponential formula and the fact that $U(t, -T)$ is unitary. ||

We see from (6.4) that R_N^z is a bounded operator, and we find with the help of (6.5) and (6.7),

$$\|R_N^z\| \leqslant (1/N!)(2/z)^N .$$

Now, using this estimate and Theorem 6.2, we obtain an estimate for the remainder term of the perturbation series:

$$\begin{aligned}\|R_N\phi\| &\leqslant \|R_N^z\phi\| + \|(R_N - R_N^z)\phi\| \\ &\leqslant (1/N!)(2/z)^N\|\phi\| + \epsilon ,\end{aligned} \tag{6.8}$$

where, for N fixed and given $\epsilon > 0$, we choose $z_0 > 0$ sufficiently small that

$$\|(R_N - R_N^z)\phi\| < \epsilon , \quad \phi\in D(Q^N) ,$$

for $z < z_0$. However, it does not follow from the estimate (6.8) that $R_N\phi \to 0$ as $N\to\infty$ because z_0 cannot be chosen independently of N. Thus the perturbation series does not converge.

It does follow from the above results, however, that the perturbation expansion is "asymptotic in the sense of Poincaré." Compare the definition of this concept on p. 487 of Ref. 11 with Theorem 2.2.13 of Ref. 41.

We can use techniques similar to those discussed in the present section to obtain results for the perturbation series for the scattering operator, since $\lim_{T\to\infty} U_\lambda[T,t] = U[\infty,t]$ and $\lim_{T\to\infty} U_\lambda[t, -T] = U[t, -\infty]$; $S[\infty, -\infty] = U[\infty,t]U[t, -\infty]$.

We now make a few remarks concerning the convergence of the perturbation expansions when the corresponding semigroup is holomorphic. The semigroup that we have been considering is $U(t, -T) = \exp\{Q(t, -T)\}$, which we now rewrite in the form

$$U(t, -T) = \exp[\tau\{Q(t, -T)/\tau\}]$$

in terms of a parameter τ. We say that $U(t, -T)$ is *holomorphic* if, as a function of τ, it can be continued into a neighborhood of unity in the complex τ-plane (compare with Ref. 32, p. 254). It then follows from the general theory of semigroups that the perturbation series (6.1) converges. The proof is similar to that of Theorem 1.1.11 in Ref. 41.

In conclusion, it is important to note that our only assumptions are (1) $\widetilde{H}(t) = \int_{\mathbb{R}} \widetilde{H}(t,\mathbf{x})d\mathbf{x}$ is the generator of a unitary group on $\mathscr{H}$ for each t [where $\widetilde{H}(t,\mathbf{x})$ is the field energy density on $\mathbb{R}^n$]; (2) the set of operators $\{\widetilde{H}(t)\,|\,t\in J\}$ is strongly continuous with common dense domain; and (3) $\mathscr{H}$ is a separable Hilbert space. It could be argued that the assumption of a common dense domain for the Hamiltonians is too strong for any formulation of QED; however, this assumption is not necessary for our theory to apply. This will be taken up at a later time when we consider applications to nonlinear formulations.

VII. CONCLUDING REMARKS

In this paper we have used an algebraic approach to time-ordered operators based upon von Neumann's infinite tensor product Hilbert spaces to define path integrals which appear to include most cases of interest in mathematical physics. We have proved that there exists a one-to-one correspondence between path integrals and semigroups which are integral operators defined by a kernel. The reproducing property of the kernel is a consequence of the semigroup property.

The generality of our construction is intimately connected with the fact that our tensor product Hilbert spaces are constructed using an *abstract* separable Hilbert space as a base. This allows application to many different physical problems according to different choices of this base Hilbert space. We will consider some of these applications in future work.

We have shown that our treatment is a generalization of the customary approach to time-ordered operators and path integration by means of product integrals. Moreover, when Hamiltonians which are sums of two parts (in a certain well-defined sense) are considered, our results do not depend upon the domains of the latter operators.

We have also shown that our approach leads to unique solutions to the Cauchy problem for Schrödinger equations with time-dependent Hamiltonians. This is clearly of interest for mathematics as well as physics, since one is concerned here with linear time-evolution equations.

We have advanced the point of view that it is unnatural to try to force path integrals into a description by means of countably additive measures. The viewpoint has been expressed that the theory of integration, rather than measure theory, is the appropriate vehicle for a *general* formulation of path integration. Thus, although path integrals can be written in terms of countably additive measures in certain special cases, this is not the situation in general.

We have also discussed perturbation expansions for time-evolution operators. It has been shown that these expansions generally do not converge, but are asymptotic in a certain well-defined sense. On the other hand, these series converge when the semigroups possess suitable holomorphy properties. It should also be noted that our approach shows that the general belief expressed in Ref. 39, to the effect that the Dyson expansion can only hold with $H(t)$ bounded, is not quite correct (see p. 283 of that reference).

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[15]T. L. Gill, Trans. Am. Math. Soc. **266**, 161 (1981). On p. 163, the last line of definition 1.6 should read "...where $\{J_t\}$ is a fixed family of duality maps for $\{\phi_t\}$."

[16]T. L. Gill, Trans. Am. Math. Soc. **279**, 617 (1983). On p. 617, paragraph 3 should read: "...We assume that unless otherwise stated, all operators of the form $\widetilde{A}(s)$ are strongly space continuous operators in the sense of definition 2.6 in I, while... ." On p. 618, Theorem 1.1 should read (1) $\text{s-lim}_{\lambda\to\infty} Q^z_\lambda[t,0] = \text{s-lim}_{\lambda\to\infty} \overline{Q}^z_\lambda[t,0]$ exists and $Q^z[t,0] = Q^z[t,s] + Q^z[s,0]$, $0 \leqslant s < t$. On p. 619, the eighth line from the bottom of the page should read $\lambda \geqslant \lambda_2$. The series in the seventh line from the bottom should be truncated for sufficiently large N so that z_0 in the sixth line from the bottom may be chosen independent of n and τ_j. On p. 630, at the end of the second paragraph the last line should read: "In our approach we assume no special domain relationship between domains other than that required for strong space continuity."

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