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# Why do soliton equations come in hierarchies?

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In this article, an identity satisfied by the so-called recursion operator is derived. The identity generates by itself an infinite sequence of Lax pairs, thus ensuring the complete integrability of the corresponding hierarchy of nonlinear evolution equations. It is also shown that this identity yields the familiar property that the squares of eigenfunctions of the associated linear spectral problem satisfy the linearized version of the respective soliton equation.

## I. INTRODUCTION

A partial answer to the question in the title can be found in the classical Ablowitz–Kaup–Newell–Segur (AKNS) article[1] as well as other articles describing the algebraic structure of some hierarchies of soliton equations. They (implicitly) attribute the existence of a hierarchy to the interplay between a linear operator $L$ and a class of linear operators $A_\lambda$ (depending polynomially on $\lambda$) which are chosen in a suitable way to represent $(L,A)$ pairs for some sequence of nonlinear evolution equations (NEE).

In the framework of Hamiltonian formalism, the answer is much more explicit, namely, the presence of a bi-Hamiltonian structure (see, e.g., Ref. 2). However, the latter is still rather complicated, containing a number of requirements in itself.

In the present article, we answer that question from a different perspective, attributing the existence of a hierarchy to a single identity satisfied by the respective recursion operator $\Lambda$. Actually, this property of $\Lambda$ was obtained as a result of an attempt to simplify the proof and clarify the underlying ideas for some results in Ref. 3 and 4.

The contents of this article are as follows.

In Sec. I we introduce two well-known hierarchies of soliton equations (considered in Refs. 3 and 4, respectively) together with their associated linear spectral problems, and show in Sec. II that the respective recursion operators $\Lambda$ satisfy the same(!) operator identity. In Sec. III we use this identity to derive in a unified way the corresponding hierarchies of Lax pairs $(\Lambda,B)$, thus guaranteeing the soliton type of the generated nonlinear evolution equations. Then, in Sec. IV, we obtain as a natural consequence that the eigenfunctions of $\Lambda$ satisfy the linearized versions of the corresponding NEE.

All these results can be found in Ref. 5. Also, Secs. III and IV rederive in a simpler form some results of Refs. 3 and 4, as mentioned above.

First we consider the following generalization of the time-independent Schrödinger equation

$$-\psi''(x,\lambda)+\left[\sum_{r=0}^{N-1}\lambda^r v_r(x)\right]\psi(x,\lambda)=\lambda^N\psi(x,\lambda),\quad \lambda\in\mathbb{C},\quad v_r\in\mathfrak{S},\quad '=\frac{\partial}{\partial x},\tag{1}$$

where $\mathfrak{S}$ is the Schwartz space of functions $v:\mathbb{R}\to\mathbb{C}$ and $N$ is a natural number.

It is known[2] that the spectral problem (1) is associated with a hierarchy of nonlinear systems of evolution equations

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$$v_t=\Omega(\Lambda)v_x, \tag{2}$$

where

$$\Omega(\mu)=a_0\mu^n+a_1\mu^{n-1}+\cdots+a_n$$

and

$$v(x,t)=\begin{pmatrix} v_0\\ v_1\\ \vdots\\ v_{N-1}\end{pmatrix},\quad \Lambda=\begin{pmatrix} 0 & 0 & \cdots & 0 & (-\frac{1}{4}\partial_{xxx}+j(v_0))\partial_x^{-1}\\ 1 & 0 & \cdots & 0 & j(v_1)\partial_x^{-1}\\ 0 & 1 & \ddots & \vdots & j(v_2)\partial_x^{-1}\\ \vdots & \ddots & \ddots & 0 & \vdots\\ 0 & \cdots & 0 & 1 & j(v_{N-1})\partial_x^{-1}\end{pmatrix},$$

with

$$j(v_r)=v_r\partial_x+\tfrac{1}{2}v_{r,x},\quad \partial_x^{-1}=\int_{-\infty}^{x} dx.$$

$\Lambda$ is called a recursion operator. For $N=1$, Eq. (1) reduces to the Schrödinger equation and Eq. (2) yields the Korteweg–de Vries (KdV) hierarchy of nonlinear equations. In that case

$$\Lambda=(-\tfrac{1}{4}\partial_{xxx}+v\partial_x+\tfrac{1}{2}v_x)\partial_x^{-1}$$

and the KdV equation can be obtained when $\Omega(\mu)=4\mu$.

For $N=2$, Eq. (1) becomes a quadratic pencil (see, e.g., Ref. 6) and then Eq. (2) yields the Jaulent–Miodek hierarchy.[7]

Here we see that $\Lambda$ alone determines the soliton equations (2). In fact, we can even write formally $v_x=\Lambda(0)$ assuming $\partial_x^{-1}(0)=2$. All this suggests that $\Lambda$ is not arbitrary but has some property which determines the soliton nature of the equations in (2).

The second hierarchy of NEE that we consider is the AKNS hierarchy[1]

$$\begin{pmatrix} v_1\\ v_2\end{pmatrix}_t=\Omega(\widetilde{\Lambda})\begin{pmatrix} v_1\\ v_2\end{pmatrix}_x,\quad \widetilde{\Lambda}=i\begin{pmatrix} \frac{1}{2}\partial_x-v_1\partial_x^{-1}v_2 & -v_1\partial_x^{-1}v_1\\ v_2\partial_x^{-1}v_2 & -\frac{1}{2}\partial_x+v_2\partial_x^{-1}v_1\end{pmatrix},\quad v_1\in\mathfrak{S} \tag{3}$$

associated with the AKNS (or Zakharov–Shabat) system

$$i\begin{pmatrix} \partial_x & -v_1\\ v_2 & -\partial_x\end{pmatrix}\psi=\lambda\psi,\quad \psi=\begin{pmatrix} \psi_1\\ \psi_2\end{pmatrix}.$$

Here again $\binom{v_1}{v_2}_x=\widetilde{\Lambda}\binom{-2iv_1}{2iv_2}=\widetilde{\Lambda}^2\binom{0}{0}$, assuming $\partial_x^{-1}(0)=2$.

## II. RECURSION OPERATOR IDENTITY

*Definition 1*: For any linear operator $\Phi$ depending on a function $u$ we define

$$(\delta\Phi)_w=\frac{d}{d\epsilon}\Phi(u+\epsilon w)\,|_{\epsilon=0}.$$

*Definition 2:* For any linear operator $\Phi_u$ depending linearly on $u$ we define $\bar{\Phi}_w$ by

$$\bar{\Phi}_w u = \Phi_u w.$$

**Theorem:** *Both recursion operators introduced in Sec. I satisfy the following operator identity:*

$$(\delta\Lambda)_{\Lambda w} - \Lambda(\delta\Lambda)_w = \overline{\delta\Lambda}_w \Lambda - \Lambda \overline{\delta\Lambda}_w. \tag{4}$$

*Proof:* According to the definitions above

$$(\delta\Lambda)_w = \begin{pmatrix} 0 & \cdots & 0 & j(w_0)\partial_x^{-1} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & j(w_{N-1})\partial_x^{-1} \end{pmatrix}, \quad \overline{(\delta\Lambda)}_w = w_{N-1} + \tfrac{1}{2}(\partial_x^{-1} w_{N-1})\partial_x,$$

where $w = (w_0, w_1, ..., w_{N-1})^{\mathrm{T}}$, T—transposition.

Let the vector-function $\mathscr{A}(w,u)$ be defined as

$$\mathscr{A}(w,u) = (\overline{\delta\Lambda}_w \Lambda - \Lambda \overline{\delta\Lambda}_w) u.$$

Then Eq. (4) is equivalent to $\mathscr{A}(w,u) = \mathscr{A}(u,w)$. In order to show it we calculate the $r$th component ($r=0, 1, ..., N-1$) of $\mathscr{A}(w,u)$. For $r>0$

$$\begin{aligned}
&[w_{N-1} + \tfrac{1}{2}(\partial_x^{-1} w_{N-1})\partial_x][u_{r-1} + j(v_r)(\partial_x^{-1} u_{N-1})] - \{[w_{N-1} + \tfrac{1}{2}(\partial_x^{-1} w_{N-1})\partial_x] u_{r-1} \\
&\quad + j(v_r)\partial_x^{-1}[w_{N-1} + \tfrac{1}{2}(\partial_x^{-1} w_{N-1})\partial_x] u_{N-1}\} \\
&= \tfrac{1}{4} v_{r,xx}(\partial_x^{-1} u_{N-1})(\partial_x^{-1} w_{N-1}) + \tfrac{1}{2} v_{r,x}[u_{N-1}(\partial_x^{-1} w_{N-1}) + w_{N-1}(\partial_x^{-1} u_{N-1}) \\
&\quad + 2\partial_x^{-1}(u_{N-1} w_{N-1})],
\end{aligned}$$

which is obviously symmetric with respect to the interchange $u \leftrightarrow w$.

For $r=0$ we have the expression above plus an additional term

$$\begin{aligned}
&[w_{N-1} + \tfrac{1}{2}(\partial_x^{-1} w_{N-1})\partial_x](-\tfrac{1}{4} u_{N-1,xx}) - (-\tfrac{1}{4}\partial_{xx})[w_{N-1} + \tfrac{1}{2}(\partial_x^{-1} w_{N-1})\partial_x] u_{N-1} \\
&= \tfrac{1}{4}(u_{N-1} w_{N-1})_{xx} + \tfrac{1}{8} u_{N-1,x} w_{N-1,x}.
\end{aligned}$$

This is also symmetric. Therefore, $\mathscr{A}(w,u) = \mathscr{A}(u,w)$.

In a similar way for $\tilde{\Lambda}$ we obtain

$$(\delta\tilde{\Lambda})_w = i\left[\begin{pmatrix} -v_1 \\ v_2 \end{pmatrix}\partial_x^{-1}(w_2 \;\; w_1) + \begin{pmatrix} -w_1 \\ w_2 \end{pmatrix}\partial_x^{-1}(v_2 \;\; v_1)\right],$$

$$\overline{(\delta\tilde{\Lambda})}_w = i\left[\begin{pmatrix} -v_1 \\ v_2 \end{pmatrix}\partial_x^{-1}(w_2 \;\; w_1) + (\partial_x^{-1}(v_2 w_1 + v_1 w_2))\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right].$$

Then

$$
\begin{aligned}
\mathscr{A}(w,u) &= (\overline{\delta\widetilde{\Lambda}}_w\widetilde{\Lambda} - \widetilde{\Lambda}\overline{\delta\widetilde{\Lambda}}_w)u \\
&= -\left\{\frac{1}{2}\begin{pmatrix} v_{1,x} \\ v_{2,x} \end{pmatrix}[\partial_x^{-1}(w_2u_1+w_1u_2)] + \frac{1}{2}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}(w_2u_1+w_1u_2) + \frac{1}{2}\begin{pmatrix} v_2u_1w_1 \\ v_1u_2w_2 \end{pmatrix}\right. \\
&\quad + \frac{1}{2}\begin{pmatrix} v_1\partial_x^{-1}(u_1w_{2,x}+w_1u_{2,x}) \\ v_2\partial_x^{-1}(u_2w_{1,x}+w_2u_{1,x}) \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}[\partial_x^{-1}(v_2u_1+v_1u_2)][\partial_x^{-1}(v_2w_1+v_1w_2)] \\
&\quad \left. + \begin{pmatrix} -v_1 \\ v_2 \end{pmatrix}\partial_x^{-1}[(v_2w_1-v_1w_2)\partial_x^{-1}(v_2u_1+v_1u_2)+(v_2u_1-v_1u_2)\partial_x^{-1}(v_2w_1+v_1w_2)]\right\}.
\end{aligned}
$$

This is also symmetric with respect to $u \leftrightarrow w$ and therefore $\mathscr{A}(w,u) = \mathscr{A}(u,w)$.

*Remark:* If $\Lambda$ satisfies the identity (4) then so does $\Lambda^n$ for any integer $n \geqslant 1$ due to the following corollary (and generalization) of Eq. (4):

$$
(\delta\Lambda^n)_{\Lambda^k w} - \Lambda^k(\delta\Lambda^n)_w = \overline{(\delta\Lambda^k)}_w\Lambda^n - \Lambda^n\overline{(\delta\Lambda^k)}_w, \quad n \geqslant 1, \ k \geqslant 1.
$$

## III. LAX PAIRS

Using the above theorem we can easily derive a Lax pair for Eqs. (2) and (3). Consider the sequence of identities

$$
\begin{gathered}
(\delta\Lambda)_{v_x} = \partial_x\Lambda - \Lambda\partial_x, \\
(\delta\Lambda)_{\Lambda v_x} - \Lambda(\delta\Lambda)_{v_x} = \overline{\delta\Lambda}_{v_x}\Lambda - \Lambda\overline{\delta\Lambda}_{v_x}, \\
(\delta\Lambda)_{\Lambda^2 v_x} - \Lambda(\delta\Lambda)_{\Lambda v_x} = \overline{\delta\Lambda}_{\Lambda v_x}\Lambda - \Lambda\overline{\delta\Lambda}_{\Lambda v_x}, \\
\vdots \\
(\delta\Lambda)_{\Lambda^n v_x} - \Lambda(\delta\Lambda)_{\Lambda^{n-1} v_x} = \overline{\delta\Lambda}_{\Lambda^{n-1} v_x}\Lambda - \Lambda\overline{\delta\Lambda}_{\Lambda^{n-1} v_x},
\end{gathered}
\tag{5}
$$

which follows from Eq. (4) with the exception of the first one. We add them after multiplying the $k$th identity by $\Lambda^{n+1-k}$ on the left

$$
(\delta\Lambda)_{\Lambda^n v_x} = D_n\Lambda - \Lambda D_n, \quad \text{with} \quad D_n = \overline{\delta\Lambda}_{\Lambda^{n-1} v_x} + \Lambda\overline{\delta\Lambda}_{\Lambda^{n-2} v_x} + \cdots + \Lambda^{n-1}\overline{\delta\Lambda}_{v_x} + \Lambda^n\partial_x. \tag{6}
$$

If $v$ evolves according to Eq. (2) then Eq. (6) implies

$$
\Lambda_t \equiv (\delta\Lambda)_{v_t} = D\Lambda - \Lambda D, \quad \text{with} \quad D = \sum_{i=0}^{n} a_i D_{n-i}. \tag{7}
$$

The operator $D$ has also a representation $B$ as a polynomial of $\Lambda$ with coefficients on the left, namely, we will show that $D_n = B_n$, where

$$
B_n = (\overline{\delta\Lambda} - \delta\Lambda)_{\Lambda^{n-1} v_x} + (\overline{\delta\Lambda} - \delta\Lambda)_{\Lambda^{n-2} v_x}\Lambda + \cdots + (\overline{\delta\Lambda} - \delta\Lambda)_{v_x}\Lambda^{n-1} + \partial_x\Lambda^n. \tag{8}
$$

For $n=1$ it follows from the first identity in the system (5). Suppose that $D_n = B_n$ for some integer $n > 0$, then [see Eqs. (6), (8)]

$$B_{n+1}=(\overline{\delta\Lambda}-\delta\Lambda)_{\Lambda^n v_x}+B_n\Lambda=\overline{\delta\Lambda}_{\Lambda^n v_x}-(D_n\Lambda-\Lambda D_n)+B_n\Lambda=\overline{\delta\Lambda}_{\Lambda^n v_x}+\Lambda D_n=D_{n+1}.$$

In Refs. 3 and 4, only the $B$-form of these operators was obtained.

## IV. SOLUTIONS OF THE LINEARIZED SOLITON EQUATIONS

Suppose that the function $v(x,t)$ evolves in time according to Eq. (2), and therefore Eq. (7) holds as well. Then, after differentiating $\Lambda G=\lambda G$ [$G(x,t,\lambda)$ is an eigenfunction of $\Lambda$] with respect to $t$ we find that

$$\Lambda(G_t-BG)=\lambda(G_t-BG),$$

i.e., $G_t-BG$ is also an eigenfunction of $\Lambda$. In Refs. 3 and 4 it was shown that for any $\lambda\in\mathbb{C}$ there are eigenfunctions $G$ of $\Lambda$ for which $G_t-BG=0$. As a consequence, they appeared to be solutions to the linearized version

$$G_t=(\delta(\Omega(\Lambda)v_x))_G$$

of the soliton equation (2) due to the following identity:

$$(\delta(\Lambda^n v_x))_G=B_nG. \tag{9}$$

Here we will give a shorter proof of Eq. (9) based on the representation (6).

For $n=0$ we have $B_0=D_0=\partial_x$, and $(\delta(v_x))_G=G_x$, so Eq. (9) holds. Let Eq. (9) be true for any integer $n\geqslant 0$, then

$$(\delta(\Lambda^{n+1}v_x))_G=(\delta(\Lambda))_G\Lambda^n v_x+\Lambda(\delta(\Lambda^n v_x))_G=(\overline{\delta\Lambda})_{\Lambda^n v_x}G+\Lambda D_nG=D_{n+1}G.$$

For completeness, we will mention here that $G=v_x$ is also a solution for the linearized equation above [to see this, just differentiate Eq. (2) with respect to $x$] and therefore can be expanded in the eigenfunctions of $\Lambda$. This expansion provides the familiar analogy between the inverse scattering transform and the Fourier transform.

As a topic of future research we may consider the possibility of deriving other known properties of the soliton equations (2) and (3) from the identity (4), such as a Hamiltonian formalism, first integrals, Lax pairs of the usual type, Bäcklund transformations, etc.

Also, it is not clear if (and in what form) the above scheme can be applied to other known hierarchies of soliton equations.

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