Aspects of Supersymmetry

Bei Jia

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Eric R. Sharpe, Chair
Patrick Huber
Djordje Minic
Tatsu Takeuchi

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(ABSTRACT)

This thesis is devoted to a discussion of various aspects of supersymmetric quantum field theories in four and two dimensions. In four dimensions, $\mathcal{N}=1$ supersymmetric quantum gauge theories on various four-manifolds are constructed. Many of their properties, some of which are distinct to the theories on flat spacetime, are analyzed. In two dimensions, general $\mathcal{N}=(2,2)$ nonlinear sigma models on S^2 are constructed, both for chiral multiplets and twisted chiral multiplets. The explicit curvature coupling terms and their effects are discussed. Finally, $\mathcal{N}=(0,2)$ gauged linear sigma models with nonabelian gauge groups are analyzed. In particular, various dualities between these nonabelian gauge theories are discussed in a geometric content, based on their Higgs branch structure.

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Contents

Chapter 1

Introduction

This thesis is devoted to a discussion of various aspects of supersymmetric quantum field theories in four and two dimensions. The material of this thesis is based on the author's papers [1].

One of the major advances in physics of the past century was the understanding of the importance of symmetries. A symmetry of a given physical system is an invariance of this system under certain transformations. For example, the laws of classical Newtonian mechanics are invariant under time reversal, i.e. replacing t by $-t$ in the equations of Newtonian mechanics. As another example, our current model of fundamental particles is given by the Standard Model, which is governed by gauge symmetry.

Supersymmetry is a symmetry relating two different fields: bosonic fields and fermionic fields. Given a supersymmetric system, one can show (in generic cases) that every bosonic quantum state of the system is paired with a fermionic state with the same mass. The idea of such a symmetry actually dates back to as early as the 1970s, through advances in both quantum field theory and then newly discovered string theory [2].

This idea puts bosons and fermions, which are the only two types of particles in our universe, on equal footing, so it provides a further understanding of these particles. Therefore, supersymmetry has been the playground for theorists who would like to find the fundamental laws of nature, as well as for model builders who would like to find new physics beyond the Standard Model.

On the theoretical side, this powerful symmetry has enabled theorists to analyze systems with supersymmetry to a depth that can hardly be reached by other means. For example, many nonrenormalization theorems were proven using supersymmetry, thus providing perturbatively exact results in many supersymmetric quantum field theories in various dimensions. What's even more intriguing is that supersymmetric quantum field theories often provide surprisingly deep insights and computational methods for pure mathematics, such as computing Gromov-Witten invariants and exploring mirror symmetry.

On the practical side, phenomenologists have been using supersymmetry to extend the Standard Model, hoping to explain questions like the hierarchy problem or the naturalness problem. There are numerous supersymmetric models of fundamental particles to the current date. Unfortunately, all of the experimental results so far, such as the newest LHC results, show no direct sign of supersymmetry at our current level of energy. This is disappointing since superpersymmetry, as beautiful and powerful as it is, may not be part of the fundamental laws of nature. However, it is still too early to tell.

This thesis is devoted to a discussion of some theoretical aspects of supersymmetric quantum field theories on various spacetimes, and some relation to string compactification. In the remaining of this introductory chapter, some basic setup and technics will be laid out, as an attempt to make this thesis more self-contained. In the first section, $\mathcal{N} = 1$ supersymmetry in four dimensions is reviewed, as a prototype of supersymmetric systems. Then in the next section, $\mathcal{N} = (2,2)$ supersymmetry and, more generally, $\mathcal{N} = (0,2)$ supersymmetry in two dimensions are discussed.

After the introduction, in chapter 2 we turn to the study of $\mathcal{N} = 1$ supersymmetric field theories on curved background spacetime. In particular, in section 2.1 we discuss $\mathcal{N} = 1$ nonlinear sigma models (NLSMs) on various spaces such as the four dimensional anti-de Sitter space AdS_4. Based on this, in section 2.2 supersymmetric gauge theories coupled to these nonlinear sigma models are constructed and analyzed. In section 2.3, a general principle called the background prinriple, which relates quantum anomalies to classical constraints, is presented and generalized.

Chapter 3 contains three major parts, in which we move on to two dimensional theories. In section 3.1, we construct and discuss $\mathcal{N} = (2, 2)$ supersymmetric nonlinear sigma models on the two dimensional sphere S^2, including general couplings to the curvature of S^2. This is done for both chiral multiplets and twisted chiral multiplets. Then in section 3.2, we discuss some general aspects of dynamical supersymmetry breaking in (0,2) theories. Finally in section 3.3, we focus on basic properties of nonabelian (0,2) gauge theories, as well as geometric understanding of various dualities between them.

Finally, in the last chapter, a summery is given, together with some discussions and future directions. We also include some appendices, to supplement the main text with more background and detailed computations.

1.1 $\mathcal{N} = 1$ supersymmetry in four flat dimensions

A system is said to be supersymmetric if there are some operators, which satisfy certain *supersymmetry algebra*, that generate a symmetry of this system. The prototype of a supersymmetry algebra is the four-dimensional $\mathcal{N} = 1$ supersymmetry algebra, which contains the four-dimensional Poincaré algebra as a subalgebra[1]. The generators are usually labelled

[1] We will mainly follow the notation of [3].

as $Q_\alpha, \bar{Q}_{\dot{\beta}}$, with the following product rules:

$$\begin{aligned} \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} &= 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu, \\ \{Q_\alpha, Q_\beta\} &= \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0, \\ [P_\mu, Q_\alpha] &= [P_\mu, \bar{Q}_{\dot{\alpha}}] = 0, \end{aligned} \tag{1.1}$$

where $\alpha, \dot{\beta}$ are spinor indices of opposite chiralities, σ^μ are the Pauli matrices:

$$\begin{aligned} \sigma^0 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned} \tag{1.2}$$

and P_μ is the momentum operator. These generators $Q_\alpha, \bar{Q}_{\dot{\beta}}$ are usually called supersymmetry *charges*. In the present case, there are four real supercharges.

One can then construct various representations of this supersymmetry algebra.

Example 1.1. One particular representation of fundamental importance is called *chiral multiplet*:

$$\Phi = (\phi, \psi, F), \tag{1.3}$$

where ϕ is a complex bosonic field, ψ is a Weyl fermionic field, and F is a bosonic auxiliary field. They are related to each other by the following supersymmetry transformations:

$$\begin{aligned} \delta_\xi \phi &= \sqrt{2}\xi\psi, \\ \delta_\xi \psi &= i\sqrt{2}\sigma^\mu\bar{\xi}\partial_\mu\phi + \sqrt{2}\xi F, \\ \delta_\xi F &= i\sqrt{2}\bar{\xi}\bar{\sigma}^\mu\partial_\mu\psi. \end{aligned} \tag{1.4}$$

Example 1.2. Another important representation is called *vector multiplet*:

$$V = (A^a_\mu, \lambda^a, \bar{\lambda}^a, D^a), \tag{1.5}$$

under the Wess-Zumino gauge, where A^a_μ is a gauge field, λ^a is a fermionic field called gaugino, and D^a is a bosonic auxiliary field, all of which belong to the adjoint representation of some Lie algebra. They are related to each other by the following supersymmetry transformations:

$$\begin{aligned} \delta_\xi A^a_\mu &= -i\bar{\lambda}^a \bar{\sigma}^\mu \xi + i\bar{\xi}\bar{\sigma}^\mu \lambda^a, \\ \delta_\xi \lambda^a &= \sigma^{\mu\nu}\xi F^a_{\mu\nu} + i\xi D^a, \\ \delta_\xi D^a &= -\xi\sigma^\mu \mathcal{D}_\mu \bar{\lambda}^a - \mathcal{D}_\mu \lambda^a \sigma^\mu \bar{\xi}, \end{aligned} \tag{1.6}$$

where $F^a_{\mu\nu}$ is the corresponding gauge field strength, and $\mathcal{D}_\mu \lambda^a = \partial_\mu \lambda^a - t^{abc} A^b_\mu \lambda^c$ is the gauge covariant derivative, with t^{abc} being the structure constant of the gauge Lie algebra.

There is a more convenient way of organizing various supersymmetric multiplets, called the *superspace* formulation. In our current case, we can construct the four-dimensional $\mathcal{N} = 1$ superspace by adding four fermionic coordinates $\theta_\alpha, \bar{\theta}_{\dot{\beta}}$ to the usual bosonic coordinates x^μ on $\mathbb{R}^{1,3}$. Mathematically, this space is usually labeled as $\mathbb{R}^{1,3|4}$. Then a supersymmetric multiplet can be viewed as a function on this superspace. First, let's define the following derivatives on superspace:

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial \theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_\mu, \\ \bar{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu, \end{aligned} \tag{1.7}$$

which satisfy the following anticommutation relations:

$$\begin{aligned} \{D_\alpha, \bar{D}_{\dot{\beta}}\} &= -2i\sigma^\mu_{\alpha\dot{\beta}} \partial_\mu, \\ \{D_\alpha, D_\beta\} &= \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0. \end{aligned} \tag{1.8}$$

One can easily check that these $D's$ anticommute with those $Q's$.

Definition 1.3. A *superfield* is defined as a function on superspace, generally as a power series of the fermionic coordinates:

$$\begin{aligned} F(x^\mu, \theta, \bar{\theta}) =& \phi(x) + \theta\psi(x) + \bar{\theta}\chi(x) \\ &+ \theta\theta m(x) + \bar{\theta}\bar{\theta} n(x) + \theta\sigma^\mu \bar{\theta} A_\mu(x) \\ &+ \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\zeta(x) + \theta\theta\bar{\theta}\bar{\theta} d(x). \end{aligned} \tag{1.9}$$

Example 1.4. A *chiral superfield* Φ is a superfield satisfying

$$\bar{D}_{\dot{\alpha}}\Phi = 0. \tag{1.10}$$

Let $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$. Then Φ has expansion

$$\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y), \tag{1.11}$$

which corresponds to a chiral multiplet.

Example 1.5. A *vector superfield* V is a superfield satisfying

$$V = V^\dagger. \tag{1.12}$$

Under the Wess-Zumino gauge[2], V has expansion

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x), \tag{1.13}$$

which corresponds to a vector multiplet.

Now let's construct supersymmetric quantum field theories using these various superfields.

Example 1.6. Let $K(\Phi^i, \bar{\Phi}^{\bar{\jmath}})$ be a real function of some chiral superfields Φ^i (and their conjugates). Let $W(\Phi^i)$ be a holomorphic function of these fields. Then one can show that the most general supersymmetric Lagrangian one can write down for chiral superfields is of the form:

$$\mathcal{L}_\Phi = \int d^2\theta d^2\bar{\theta} K(\Phi^i, \bar{\Phi}^{\bar{\jmath}}) + \left[\int d^2\theta W(\Phi^i) + \text{ h.c.}\right]. \tag{1.14}$$

In terms of components, this is the same as

$$\begin{aligned}\mathcal{L}_\Phi =& - g_{i\bar{\jmath}}\partial_\mu\phi^i\partial^\mu\bar{\phi}^{\bar{\jmath}} - ig_{i\bar{\jmath}}\bar{\psi}^{\bar{\jmath}}\bar{\sigma}^\mu D_\mu\psi^i + g_{i\bar{\jmath}}F^i\bar{F}^{\bar{\jmath}} \\ &- \frac{1}{2}g_{i\bar{\jmath},\bar{k}}F^i\bar{\psi}^{\bar{\jmath}}\bar{\psi}^{\bar{k}} - \frac{1}{2}g_{j\bar{\imath},k}\bar{F}^{\bar{\imath}}\psi^j\psi^k + \frac{1}{4}g_{i\bar{\jmath},k\bar{l}}\psi^i\psi^k\bar{\psi}^{\bar{\jmath}}\bar{\psi}^{\bar{l}},\end{aligned} \tag{1.15}$$

[2]See [3] for details of the Wess-Zumino gauge.

where

$$\begin{aligned} g_{i\bar{\jmath}} &= \partial_i \partial_{\bar{\jmath}} K(\phi, \bar{\phi}), \\ D_\mu \psi^i &= \partial_\mu \psi^i + \Gamma^i_{jk} \partial_\mu \phi^j \psi^k. \end{aligned} \tag{1.16}$$

One might realize that this is precisely the construction of a Kähler manifold (let's call it M): K is the Kähler potential, and $g_{i\bar{\jmath}}$ is the corresponding Kähler metric on M. As such, this model is usually called the $\mathcal{N} = 1$ supersymmetric *nonlinear sigma model (NLSM)* with *target space* M.

Example 1.7. For a given vector superfield V, one can construct a chiral superfield which contains the gauge invariant field strength, as follows:

$$W_\alpha = -\frac{1}{4} \bar{D} \bar{D} D_\alpha V. \tag{1.17}$$

One can show that W_α is gauge invariant. Then the most general gauge invariant Lagrangian for a vector superfield is

$$\begin{aligned} \mathcal{L}_V &= \int d^2\theta \ \mathrm{Tr}(W^\alpha W_\alpha + \text{ h.c.}), \\ &= -\frac{1}{4} F_{\mu\nu a} F^{\mu\nu a} - i \lambda^a \sigma^\mu \mathcal{D}_\mu \bar{\lambda}^a + \frac{1}{2} (D^a)^2. \end{aligned} \tag{1.18}$$

Clearly this Lagrangian contains the usual Yang-Mills Lagrangian, as well as coupling to gauginos valued in the adjoint representation of the gauge group. This model is usually called $\mathcal{N} = 1$ *supersymmetric Yang-Mills theory (SYM).*

Example 1.8. The next natural question is to consider couplings between chiral superfields and vector superfields. Based on the above dicussion, the most general way of doing this is to gauge some isometries on a Kähler maniflod, which is the target space of a nonlinear sigma model. The resulting Lagrangian is

$$\begin{aligned} \mathcal{L} = &- g_{i\bar{\jmath}} \mathcal{D}_\mu \phi^i \mathcal{D}^\mu \bar{\phi}^{\bar{\jmath}} - i g_{i\bar{\jmath}} \bar{\psi}^{\bar{\jmath}} \bar{\sigma}^\mu \mathcal{D}_\mu \psi^i - i \bar{\lambda}^a \bar{\sigma}^\mu \mathcal{D}_\mu \lambda^a - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} - \frac{1}{2} (D^a)^2 \\ &+ \sqrt{2} g_{i\bar{\jmath}} (X^{ia} \bar{\psi}^{\bar{\jmath}} \bar{\lambda}^a + \bar{X}^{\bar{\jmath} a} \psi^i \lambda^a) - \frac{1}{2} \mathcal{D}_i W_j \psi^i \psi^j - \frac{1}{2} \bar{\mathcal{D}}_{\bar{\imath}} \bar{W}_{\bar{\jmath}} \bar{\psi}^{\bar{\imath}} \bar{\psi}^{\bar{\jmath}} \\ &- g^{i\bar{\jmath}} W_i \bar{W}_{\bar{\jmath}} + \frac{1}{4} \mathcal{R}_{i\bar{\jmath}k\bar{l}} \psi^i \psi^k \bar{\psi}^{\bar{\jmath}} \bar{\psi}^{\bar{l}}, \end{aligned} \tag{1.19}$$

where we have set the gauge coupling constant to 1, and integrated out the auxiliary fields F^i. Here X^i are components of a holomorphic Killing vector field on the target space, generating the gauge isometry. All derivatives are properly covariant:

$$\begin{aligned}\mathcal{D}_\mu\phi^i &= \partial_\mu\phi^i - A^a_\mu X^{ia},\\ \mathcal{D}_\mu\psi^i &= \partial_\mu\psi^i + \Gamma^i_{jk}\mathcal{D}_\mu\phi^j\psi^k - A^a_\mu\partial_j X^{ia}\psi^j,\\ \mathcal{D}_\mu\lambda^a &= \partial_\mu\lambda^a - t^{abc}A^b_\mu\lambda^c.\end{aligned} \tag{1.20}$$

An especially important special case, when the target space is simply $\mathbb{C}^n$ with its natural Kähler potential $\sum|\phi^i|^2$, is called *gauged linear sigma model (GLSM).*

1.2 (0,2) and (2,2) supersymmetry in two flat dimensions

1.2.1 (0,2) supersymmetry

Now that we have some experience with four-dimensional $\mathcal{N}=1$ superspace $\mathbb{R}^{1,3|4}$, let's start the discussion of (0,2) supersymmetry[3] in two dimensions by constructing its own version of superspace: (0,2) superspace $\mathbb{R}^{1,1|2}$. The bosonic coordinates on $\mathbb{R}^{1,1}$ are x^0, x^1. Judging from the notation, there are two right-moving real supercharges $Q_+, \bar{Q}_+$, corresponding to two Grassmann coordinates $\theta^+, \bar{\theta}^+$.

Analogous to the four-dimensional case, here one can define derivatives on this superspace:

$$\begin{aligned}D_+ &= \frac{\partial}{\partial\theta_+} - i\bar{\theta}^+(\partial_0+\partial_1),\\ \bar{D}_+ &= -\frac{\partial}{\partial\bar{\theta}_+} + i\theta^+(\partial_0+\partial_1).\end{aligned} \tag{1.21}$$

Then one can construct various superfields as functions on this superspace.

[3] One can read [4, 5] for an extensive discussion on two-dimensional (0,2) supersymmetry. We will mostly follow notations of [4].

Example 1.9. A *chiral superfield* is defined by

$$\bar{D}_+\Phi = 0. \tag{1.22}$$

It has expansion in terms of fermionic coordinates:

$$\Phi = \phi + \sqrt{2}\theta^+\psi_+ - i\theta^+\bar{\theta}^+(\partial_0 + \partial_1)\phi, \tag{1.23}$$

where ϕ is a complex scalar field, while ψ_+ is a right-moving spin-$\frac{1}{2}$ field. Notice that this is roughly "half" of a four-dimensional $\mathcal{N} = 1$ chiral superfield.

Example 1.10. One can construct a so called *Fermi superfield* Λ by mimicking the other "half" of a four-dimensional $\mathcal{N} = 1$ chiral superfield. It is defined by

$$\bar{D}_+\Lambda = \sqrt{2}E, \tag{1.24}$$

where E is a chiral superfield. Λ has an expansion in terms of fermionic coordinates:

$$\Lambda = \lambda_- - \sqrt{2}\theta^+ G - i\theta^+\bar{\theta}^+(\partial_0 + \partial_1)\lambda_- - \sqrt{2}\bar{\theta}^+ E, \tag{1.25}$$

where λ_- is a left-moving spin-$\frac{1}{2}$ field, and G is a scalar auxiliary field. In all of those cases we care about, E is a holomorphic function of some other chiral superfields Φ_i, meaning it has expansion

$$E(\Phi_i) = E(\phi_i) + \sqrt{2}\theta^+\frac{\partial E}{\partial \phi_i}\psi_{i+} - i\theta^+\bar{\theta}^+(\partial_0 + \partial_1)E(\phi_i). \tag{1.26}$$

Notice that E naturally provides interactions between fields.

Example 1.11. One can obtain a (0,2) *vector superfiled* by dimensionally reducing a four-dimensional $\mathcal{N} = 1$ vector superfield. The result, under the Wess-Zumino gauge, has the following form:

$$V = A_0 - A_1 - 2i\theta^+\bar{\lambda}_- - 2i\bar{\theta}^+\lambda_- + 2\theta^+\bar{\theta}^+ D, \tag{1.27}$$

where A_0, A_1 are the components of the two-dimensional gauge field, λ_- is the left-moving gaugino, and D is a real scalar auxiliary field.

Analogous to the four-dimensional case, here one can define a gauge invariant superfield strength:

$$\Upsilon = -\lambda_- - i\theta^+(D - F_{01}) - i\bar{\theta}^+(\mathcal{D}_0 + \mathcal{D}_1)\lambda_-, \tag{1.28}$$

where $F_{01} = \partial_0 A_1 - \partial_1 A_0 - i[A_0, A_1]$ is the gauge field strength, and we have defined gauge covariant derivatives

$$\mathcal{D}_m = \partial_m - iA_m, \quad m = 1, 2. \tag{1.29}$$

Now we are ready to write down Lagrangians that govern the dynamics of these (0,2) superfields.

Example 1.12. From our experience with four-dimensional $\mathcal{N} = 1$ nonlinear sigma models, we would naturally guess that the theory containing (0,2) chiral superfields will be a nonlinear sigma model with some Kähler manifold M being the target space. It is indeed the case, but with more structure, namely a holomorphic vector bundle $E \to M$, whose sections are the left-moving fermions. The most general Lagrangian for chiral multiplets is

$$\begin{aligned} \mathcal{L}_\Phi = &- g_{i\bar{j}}\partial_m\phi^i\partial^m\bar{\phi}^{\bar{j}} - ig_{i\bar{j}}\bar{\psi}_+^{\bar{j}}(D_0 - D_1)\psi_+^i - ih_{a\bar{b}}\bar{\lambda}_-^{\bar{b}}(D_0 + D_1)\lambda_-^a \\ &+ F_{a\bar{b}i\bar{j}}\lambda_-^a\bar{\lambda}_-^{\bar{b}}\psi_+^i\bar{\psi}_+^{\bar{j}}, \end{aligned} \tag{1.30}$$

where $g_{i\bar{j}}$ is the Kähler metric on M, $h_{a\bar{b}}$ is a Hermitian metric on E, and $F_{a\bar{b}i\bar{j}}$ is the curvature of E corresponding to $h_{a\bar{b}}$. Notice that we have integrated out auxiliary fields. This model is usually called *heterotic nonlinear sigma model.*

Example 1.13. Similar to the four-dimensional $\mathcal{N} = 1$ case, the gauge invariant Lagrangian for (0,2) vector superfields can be constructed from gauge invariant superfield strength:

$$\begin{aligned} \mathcal{L}_V &= \int d^2\theta \ \mathrm{Tr}(\bar{\Upsilon}\Upsilon), \\ &= \frac{1}{2}F_{01}^2 + i\bar{\lambda}_-^a(D_0 + D_1)\lambda_-^a + \frac{1}{2}(D^a)^2, \end{aligned} \tag{1.31}$$

where a is an gauge index, i.e. labelling the basis of the adjoint representation of the gauge group.

Example 1.14. Since we will talk about (0,2) GLSM's in more detail later, let's write down the general structure of such theories. The Lagrangian for a chiral superfield Φ, in some representation of the gauge group, is

$$\mathcal{L}_{\text{ch}} = -|D_m\phi|^2 + i\bar{\psi}_+(D_0 - D_1)\psi_+ - i\sqrt{2}\bar{\phi}\lambda_-\psi_+ + i\sqrt{2}\bar{\psi}_+\bar{\lambda}_-\phi + D|\phi|^2, \tag{1.32}$$

where we have suppressed all gauge indices.

The most general Lagrangian of a charged Fermi superfield Λ is

$$\mathcal{L}_{\text{f}} = i\bar{\lambda}_-(D_0 + D_1)\lambda_- + |G|^2 - |E|^2 - \left(\bar{\lambda}_-\partial_i E\psi_{+,i} + \bar{\partial}_{\bar{\imath}}\bar{E}\bar{\psi}_{+,\bar{\imath}}\lambda_-\right). \tag{1.33}$$

Given some chiral superfields J^a, which are holomorphic functions of some chiral superfields Φ^i, one can write down a (0,2) superpotential

$$L_J = -G_a J^a + \lambda_{-,a}\psi_{+,i}\partial_i J^a - h.c. \tag{1.34}$$

provided that the following constraint is satisfied:

$$E_a J^a = 0, \tag{1.35}$$

where $\bar{\mathcal{D}}_+\Lambda_{-,a} = \sqrt{2}E_a$, with

$$\bar{\mathcal{D}}_+ = \frac{\partial}{\partial\theta^+} - i\bar{\theta}^+(\mathcal{D}_0 + \mathcal{D}_1) \tag{1.36}$$

being a gauge covariant derivative on the (0,2) superspace.

Finally, one can write down the Fayet-Iliopoulos D-term

$$\mathcal{L}_{\text{FI}} = i\frac{t}{2}(D - iF_{01}) + h.c., \tag{1.37}$$

where $t = ir + \frac{\theta}{2\pi}$ is the complexified FI parameter.

1.2.2 (2,2) supersymmetry

One important special case of (0,2) supersymmetry can be obtained by demanding the existence of two left-moving supercharges $Q_-, \bar{Q}_-$, in addition to the present two right-moving supercharges. The resulting supersymmetry is called (2,2) supersymmetry. Consequently, the corresponding (2,2) superspace is $\mathbb{R}^{1,1|4}$, parameterized by $(x_0, x_1, \theta^\pm, \bar{\theta}^\pm)$.

There is another way of obtaining two-dimensional (2,2) supersymmetry, by simply dimensionally reducing the four-dimensional $\mathcal{N} = 1$ supersymmetry down to two dimensions. This way, all of the four real supercharges get preserved. As such, two-dimensional (2,2) superfields have a even closer relation to four-dimensional $\mathcal{N} = 1$ superfields.

Example 1.15. A (2,2) *chiral superfield* Φ is defined by[4]

$$\bar{D}_\pm \Phi = 0, \tag{1.38}$$

which has the following expension:

$$\Phi = \phi + \sqrt{2}\theta\psi + \theta\theta F. \tag{1.39}$$

Note this is the same as the expansion of a four-dimensional $\mathcal{N} = 1$ chiral superfield. Here $\psi = (\psi_-, \psi_+)$ is a two-dimensional Dirac spinor with two chiral components ψ_-, ψ_+. Consequently, a (2,2) supersymmetric nonlinear sigma model governs maps $\phi: \mathbb{R}^{1,1} \to M$ where M is a Kähler target space. Its Lagrangian has the exactly same form as (1.15). In the language of (0,2) supersymmetry, here our vector bundle E is simply the holomorphic tangent bundle of M.

Example 1.16. For two dimensional (2,2) theories, there is one special type of superfields that only arise in two dimensions, called *twisted chiral superfields.* (See also [66][section 12.2] for another comparison of ordinary and twisted chiral multiplets.) Let $\mathcal{P}$ be such a

[4] We will slightly abuse notation to denote (2,2) and (0,2) superfields with the same notation.

twisted chiral superfield. The defining equation for $\mathcal{P}$ is then

$$\bar{D}_+\mathcal{P} = D_-\mathcal{P} = 0. \tag{1.40}$$

One can do this essentially because a four-dimensional Weyl spinor contains the same degree of freedom as two two-dimensional Weyl spinors. A generic twisted chiral superfield has components fields $(\rho^i, \bar{\chi}^i_+, \chi^i_-, G^i)$, where the ρ^i is a bosonic field, $\bar{\chi}^i_+, \chi^i_-$ are Wyel fermionic fields, and G^i is a bosonic axillary field. The Lagrangian for a theory containing only twisted chiral multiplets on $\mathbb{R}^2$ is of the form

$$\begin{aligned}\mathcal{L}_T =& g_{i\bar{\jmath}}\partial_m\rho^i\partial^m\bar{\rho}^{\bar{\jmath}} + 2ig_{i\bar{\jmath}}\bar{\chi}^{\bar{\jmath}}_-\nabla_{\bar{z}}\chi^i_- + 2ig_{i\bar{\jmath}}\chi^{\bar{\jmath}}_+\nabla_z\bar{\chi}^i_+ + g_{i\bar{\jmath}}G^i\bar{G}^{\bar{\jmath}}\\ &- G^i\left(g_{i\bar{\jmath},\bar{k}}\bar{\chi}^{\bar{\jmath}}_-\chi^{\bar{k}}_+ - \mathcal{W}_i\right) - i\mathcal{W}_{ij}\chi^i_-\bar{\chi}^j_+\\ &- \bar{G}^{\bar{\imath}}\left(g_{\bar{\imath}j,k}\chi^j_-\bar{\chi}^k_+ - \bar{\mathcal{W}}_{\bar{\imath}}\right) - i\bar{\mathcal{W}}_{\bar{\imath}\bar{\jmath}}\bar{\chi}^{\bar{\imath}}_-\chi^{\bar{\jmath}}_+\\ &- g_{i\bar{\jmath},k\bar{l}}\bar{\chi}^i_+\chi^{\bar{\jmath}}_+\chi^k_-\bar{\chi}^{\bar{l}}_-,\end{aligned} \tag{1.41}$$

with supersymmetry transformations

$$\begin{aligned}\delta\rho^i &= i\bar{\zeta}_+\chi^i_- + i\zeta_-\bar{\chi}^i_+,\\ \delta\bar{\rho}^{\bar{\imath}} &= i\bar{\zeta}_-\chi^{\bar{\imath}}_+ + i\zeta_+\bar{\chi}^{\bar{\imath}}_-,\\ \delta\bar{\chi}^i_+ &= -2\bar{\zeta}_-\bar{\partial}\rho^i - \bar{\zeta}_+G^i,\\ \delta\chi^i_- &= -2\zeta_+\partial\rho^i + \zeta_-G^i,\\ \delta\chi^{\bar{\imath}}_+ &= -2\zeta_-\bar{\partial}\bar{\rho}^{\bar{\imath}} - \zeta_+\bar{G}^{\bar{\imath}},\\ \delta\bar{\chi}^{\bar{\imath}}_- &= -2\bar{\zeta}_+\partial\bar{\rho}^{\bar{\imath}} + \bar{\zeta}_-\bar{G}^{\bar{\imath}},\\ \delta G^i &= 2i(\zeta_+\tilde{\nabla}_z\bar{\chi}^i_+ - \bar{\zeta}_-\tilde{\nabla}_{\bar{z}}\chi^i_-),\\ \delta\bar{G}^{\bar{\imath}} &= 2i(\bar{\zeta}_+\tilde{\nabla}_z\chi^{\bar{\imath}}_+ - \zeta_-\tilde{\nabla}_{\bar{z}}\bar{\chi}^{\bar{\imath}}_-).\end{aligned} \tag{1.42}$$

They can be obtained by taking the limit of $r \to \infty$ in the corresponding theory on S^2 discussed in section 3.1.2. If we integrate out the auxiliary fields G^i, $\bar{G}^i$, the lagrangian takes the simpler form

$$\begin{aligned}\mathcal{L}_T =& g_{i\bar{\jmath}}\partial_m\rho^i\partial^m\bar{\rho}^{\bar{\jmath}} + 2ig_{i\bar{\jmath}}\bar{\chi}_-^{\bar{\jmath}}\nabla_{\bar{z}}\chi_-^i + 2ig_{i\bar{\jmath}}\chi_+^{\bar{\jmath}}\nabla_z\bar{\chi}_+^i + R_{i\bar{\jmath}k\bar{\ell}}\bar{\chi}_+^i\chi_+^{\bar{\jmath}}\chi_-^k\bar{\chi}_-^{\bar{\ell}} \\ &- g^{i\bar{\jmath}}\mathcal{W}_i\bar{\mathcal{W}}_{\bar{\jmath}} - i\chi_-^i\bar{\chi}_+^j D_i\partial j\mathcal{W} - i\bar{\chi}^{\bar{\imath}}\chi_+^{\bar{\jmath}} D_{\bar{\imath}}\partial_{\bar{\jmath}}\bar{\mathcal{W}}\end{aligned} \tag{1.43}$$

with supersymmetry transformations

$$\begin{aligned}\delta\rho^i &= i\bar{\zeta}_+\chi_-^i + i\zeta_-\bar{\chi}_+^i, \\ \delta\bar{\rho}^{\bar{\imath}} &= i\bar{\zeta}_-\chi_+^{\bar{\imath}} + i\zeta_+\bar{\chi}_-^{\bar{\imath}}, \\ \delta\bar{\chi}_+^i &= -2\bar{\zeta}_-\bar{\partial}\rho^i + \bar{\zeta}_+\left(\Gamma^i_{jk}\chi_-^j\bar{\chi}_+^k + g^{i\bar{\jmath}}\bar{\mathcal{W}}_{\bar{\jmath}}\right), \\ \delta\chi_-^i &= -2\zeta_+\partial\rho^i + \zeta_-\left(\Gamma^i_{jk}\chi_-^j\bar{\chi}_+^k + g^{i\bar{\jmath}}\bar{\mathcal{W}}_{\bar{\jmath}}\right), \\ \delta\chi_+^{\bar{\imath}} &= -2\zeta_-\bar{\partial}\bar{\rho}^{\bar{\imath}} + \zeta_+\left(\Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}}\bar{\chi}_-^{\bar{\jmath}}\chi_+^{\bar{k}} + g^{\bar{\imath}j}\mathcal{W}_j\right), \\ \delta\bar{\chi}_-^{\bar{\imath}} &= -2\bar{\zeta}_+\partial\bar{\rho}^{\bar{\imath}} + \bar{\zeta}_-\left(\Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}}\bar{\chi}_-^{\bar{\jmath}}\chi_+^{\bar{k}} + g^{\bar{\imath}j}\mathcal{W}_j\right).\end{aligned} \tag{1.44}$$

For contrast, compare the Lagrangian for ordinary chiral superfields we discussed earlier. Modulo signs and irrelevant factors, the Lagrangian for a theory of purely twisted chiral multiplets is the same as the action for a theory of purely ordinary chiral multiplets – the difference between the two lies in the supersymmetry transformations.

Example 1.17. A (2,2) *vector superfield* has component fields $(v_m, \sigma, \bar{\sigma}, \lambda, \bar{\lambda}, D)$, which are derived by dimensionally reducing a four-dimensional $\mathcal{N} = 1$ vector superfield. In particular, $\sigma = (A_2 - iA_3)/\sqrt{2},\ \bar{\sigma} = (A_2 + iA_3)/\sqrt{2}$ are from components of the four-dimensional gauge field A_μ.

The corresponding superfield strength, which will be denoted as Σ, however, is a bit different from the four-dimensional case. It is an example of a twisted chiral superfield that we defined earlier.

One can write down the Lagrangian of a vector superfield:

$$\mathcal{L}_V = \frac{1}{2}(v_{01}^a)^2 + \frac{1}{2}(D^a)^2 + i\bar{\lambda}_+^a(D_0 - D_1)\lambda_+^a + i\bar{\lambda}_-^a(D_0 + D_1)\lambda_-^a - |\partial_m\sigma^a|^2. \tag{1.45}$$

Gauged linear sigma models with (2,2) supersymmetry are a class of important examples of two-dimensional gauge theories. Here we discuss the Lagrangians of various fields that defines such a model, which contains (2,2) chiral superfiel Φ^i and a vector superfield V.[5] It can be easily derived from (1.19) for the case $M = \mathbb{C}^n$, by dimensional reduction. Alternatively, it can be obtained from a (0,2) GLSM discussed in Example 1.14 in the following steps:

- Relate left and right-moving chiral fermions, i.e. combine them into Dirac spinors;
- Add right-moving parts of (2,2) guaginos, as well as scalar fields σ^a, to obtain the whole (2,2) vector multiplet;
- Set $J_i = \partial_i W(\Phi)$, and $E_i = i\sqrt{2}(\Sigma|_{\theta_-=\bar{\theta}|_-=0})(\Phi_i|_{\theta_-=\bar{\theta}|_-=0})$.

[5]One can also include generic twisted chiral superfields. However, we will not discuss them here.

Chapter 2

$\mathcal{N} = 1$ Supersymmetry on Four-manifolds

Historically, most discussions of rigidly supersymmetric nonlinear sigma models have focused on Minkowski spacetime. Recently, rigidly supersymmetric nonlinear sigma models on some nontrivial spacetime manifolds have been discussed by several groups [6–20]. (See also references contained therein for older literature on this subject.) These have interesting new properties, different from the traditional Minkowski spacetime models, essentially because one must add additional terms to the action to take into account the curvature of the spacetime manifold.

One way to derive those extra terms in the action is to manually add extra terms consistent with the requirements imposed by (rigid) supersymmetry [6, 8]. Another approach [6, 7] is to start with a supergravity theory in four dimensions, then decouple gravity in order to obtain a theory that is rigidly supersymmetric. Demanding that the supersymmetry variation of the gravitino vanishes then constrains the possible spacetime manifolds. The solutions of these constraining equations generate two classes of spacetime geometries, including AdS_4

and S^4 (after Wick rotation to a Euclidean metric) in one class, and $S^3 \times \mathbb{R}$ in a second class that requires a covariantly constant vector field.

These theories have many interesting properties that are different from the Minkowski spacetime case. For instance [6, 7], the target spaces of the supersymmetric nonlinear sigma models on spacetimes such as AdS_4 and S^4 must be noncompact Kähler manifolds, with exact Kähler forms. Furthermore, the Lagrangian depends only on certain combination of the Kähler potential and the superpotential – neither alone is physically meaningful.

In this chapter we construct $N = 1$ rigidly supersymmetric gauged nonlinear sigma models and gauge theories on nontrivial four-dimensional spacetime manifolds, by starting with $\mathcal{N} = 1$ supergravity and decoupling gravity, following the methods of [6, 7]. Just as target spaces of rigidly supersymmetric ungauged theories are constrained, we find analogous constraints in gauge theories. For example, just as the Fayet-Iliopoulos parameter is constrained in $N = 1$ supergravity [21–23], we find a constraint on the Fayet-Iliopoulos parameter in rigidly supersymmetric theories, which guarantees that the Kähler form on the quotient space is exact. Just as in $N = 1$ supergravity in four dimensions, the superpotential is a section of a line bundle [24], we interpret the superpotential in these rigidly supersymmetric theories as a section of an affine bundle. Just as in $N = 1$ supergravity [21, 22], where the Fayet-Iliopoulos parameter was determined by the group action on the Bagger-Witten line bundle, here too the Fayet-Iliopoulos parameter can be understood in terms of lifts to the affine bundle.

We should mention that some analogous results were obtained in linearized supergravity theories obtained by coupling a rigidly supersymmetric theory to gravity. In such theories, for *e.g.* couplings involving the Ferrara-Zumino multiplet, one also often sees that Kähler forms are exact and Fayet-Iliopoulos parameters vanish [25, 26], just as we describe here for rigidly supersymmetric theories on *e.g.* AdS_4. (As observed in [23], however, one should distinguish supergravities obtained by coupling a rigid theory to gravity, from more general supergravity theories. For example, in generic heterotic Calabi-Yau compactifications to four dimensions and N=1 supergravity, it is widely believed that the Bagger-Witten line bundle is nontrivial, and so such supergravities cannot be obtained by coupling a rigid theory in the fashion above.)

We start in section 2.1 with a review of those rigidly supersymmetric nonlinear sigma models constructed in [6–8], using the superspace formulation to make the story more compact. We also give the interpretation of the superpotential as a section of certain affine bundle over the target space. In section 2.2 we construct supersymmetric gauged sigma models and gauge theories, from which we derive some constraints on the theory. We find that the Fayet-Iliopoulos parameter has to vanish in these theories, which has the effect of enforcing that Kähler forms on quotient spaces be exact. We also provide some mathematical background about affine bundles and equivariant structures on affine bundles in an appendix.

In passing, since one of the spaces we study will be AdS_4, we should mention that, just as in previous papers [6, 7], we shall ignore the role of boundary conditions. See *e.g.* [27–30] for an overview of boundary conditions in AdS_4, and *e.g.* [31–33] for information on how such boundary conditions can restrict chiral matter representations.

2.1 Review of rigidly susy sigma models on curved superspace

There are several ways of deriving rigid supersymmetric nonlinear sigma model from supergravtiy. For example, we can decouple gravity in the weak coupling limit to get supersymmetric nonlinear sigma model on AdS_4 [6]. On the other hand, it was noted in [7] that the auxiliary fields b_μ and M from the $\mathcal{N} = 1$ supergravity multiplet could be used to determine the geometry of spacetime, therefore generating two classes of spacetime geometries. The idea is to start with $\mathcal{N} = 1$ supergravity Lagrangian, then set the gravitino to zero to completely remove the dynamics of gravity, and make the auxiliary fields b_μ and M from the supergravity multiplet satisfy certain constraining equations to make sure we have $\mathcal{N}=1$ supersymmetry, as well as the ability to perform a modified Kähler transformation with the resulting Lagrangian invariant.

Let us review the approach of [7], as we shall apply it to gauge theories in the next section. We start with the $N=1$ chiral supergravity Lagrangian in superspace [3]:

$$\mathcal{L}=\frac{1}{\kappa^2}\int d^2\Theta\, 2\mathcal{E}\left[\frac{3}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}}-8R)\exp(-\frac{\kappa^2}{3}K(\Phi^i,\bar{\Phi}^{\bar{\imath}}))+\kappa^2 W(\Phi^i)\right]+h.c. \tag{2.1}$$

Then we remove the effect of gravity. First, we need to expand in κ^2, then only keep the terms that are independent of κ. We get

$$\mathcal{L}=\int d^2\Theta\, 2\mathcal{E}\left[-\frac{1}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}}-8R)K(\Phi^i,\bar{\Phi}^{\bar{\imath}})+W(\Phi^i)\right]+h.c. \tag{2.2}$$

As observed in [7], in order for consistency of the method above, we must demand that the spacetime be such that the supersymmetry variation of the gravitino vanishes, as we have truncated it. The off-shell supersymmetry variation of the gravitino is of the form [3, 7]

$$\begin{aligned}\delta\Psi_\mu^\alpha&=-2\nabla_\mu\zeta^\alpha+\frac{i}{3}\left(M(\epsilon\sigma_\mu\overline{\zeta})^\alpha+3b_\mu\zeta^\alpha+2b^\nu(\zeta\sigma_{\nu\mu})^\alpha\right),\\ \delta\overline{\Psi}_{\mu\dot\alpha}&=-2\nabla_\mu\overline{\zeta}_{\dot\alpha}-\frac{i}{3}\left(\overline{M}(\zeta\sigma_\mu)_{\dot\alpha}+3b_\mu\overline{\zeta}_{\dot\alpha}+2b^\nu(\overline{\zeta}\overline{\sigma}_{\nu\mu})_{\dot\alpha}\right),\end{aligned} \tag{2.3}$$

and demanding that it vanishes implies constraints of the form [7][equ'n (2.11)]

$$\begin{gathered}Mb_\mu=\bar{M}b_\mu=0,\quad \nabla_\mu b_\nu=0,\quad \partial_\mu M=\partial_\mu\bar{M}=0,\quad W_{\mu\nu\kappa\lambda}=0,\\ R_{\mu\nu}=-\tfrac{2}{9}\left(b_\mu b_\nu-g_{\mu\nu}b_\rho b^\rho\right)+\tfrac{1}{3}g_{\mu\nu}M\bar{M},\end{gathered}$$

where M, $\bar{M}$, b_μ are auxiliary fields in the N=1 supergravity multiplet, and $W_{\mu\nu\kappa\lambda}$ is the Weyl tensor. According to [7], there are two classes of solutions to the equations above, namely:

1. $b_\mu=0$, constant M, $\bar{M}$,

2. $M = \bar{M} = 0$, b_μ a covariantly-constant vector.

In the first case, if we Wick rotate to Euclidean space, it can be argued from the existence of spinors ζ in the gravitino variation [35] that the spacetime metric either has constant sectional curvature, or is Ricci-flat and self-dual or anti-self-dual. To see this, note in this case we have Killing spinor equations

$$\begin{aligned} \nabla_\mu \zeta^\alpha &= \frac{i}{6} M (\epsilon \sigma_\mu \bar{\zeta})^\alpha, \\ \nabla_\mu \bar{\zeta}_{\dot{\alpha}} &= \frac{-i}{6} \bar{M} (\zeta \sigma_\mu)_{\dot{\alpha}}, \end{aligned} \tag{2.4}$$

Taking the covariant derivative of the first equation, then together with the second equation we can get

$$\nabla_\nu \nabla_\mu \zeta^\alpha = \frac{1}{36} M \bar{M} (\sigma_\mu \zeta \bar{\sigma}_\nu)^\alpha, \tag{2.5}$$

then it follows that

$$[\nabla_\mu, \nabla_\nu] \zeta^\alpha = R_{\mu\nu\rho\sigma} (\zeta \sigma^{\rho\sigma})^\alpha = \frac{1}{9} M \bar{M} (\zeta \sigma_{\mu\nu})^\alpha. \tag{2.6}$$

Similarly, we can find an analogous equation from the second Killing spinor equation above

$$[\nabla_\mu, \nabla_\nu] \bar{\zeta}_{\dot{\alpha}} = R_{\mu\nu\rho\sigma} (\bar{\zeta} \bar{\sigma}^{\rho\sigma})_{\dot{\alpha}} = \frac{1}{9} M \bar{M} (\bar{\zeta} \bar{\sigma}_{\mu\nu})_{\dot{\alpha}}, \tag{2.7}$$

Suppose we have $\zeta^\alpha \neq 0$ as well as $\bar{\zeta}_{\dot{\alpha}} \neq 0$, then we see

$$R_{\mu\nu\rho\sigma} \sigma^{\rho\sigma} \propto g_{\mu\rho} g_{\nu\sigma} \sigma^{\rho\sigma}, \tag{2.8}$$

which is equivalent to

$$R_{\mu\nu\rho\sigma} \sigma^{\rho\sigma} \propto g_{\mu\sigma} g_{\nu\rho} \sigma^{\sigma\rho} = -g_{\mu\sigma} g_{\nu\rho} \sigma^{\rho\sigma}. \tag{2.9}$$

Therefore, from linear independence of the σ's, we see that

$$R_{\mu\nu\rho\sigma} \propto g_{\mu\rho} g_{\upsilon\sigma} - g_{\mu\sigma} g_{\nu\rho}, \tag{2.10}$$

which means the spacetime has constant sectional curvature, *i.e.* it is a space form. Similarly, if $\zeta^\alpha = 0$ (or $\bar{\zeta}_{\dot{\alpha}} = 0$), it follows that the spacetime is Ricci-flat and self-dual (or anti-self-dual). In the second case, it can be similarly argued [35] that the spacetime metric is a product of a line and a metric of nonnegative constant sectional curvature.

2.1.1 $M = \bar{M}$ =constant, $b_\mu = 0$

Now, let us specialize to the first case, in which M, $\bar{M}$ are nonzero constants, and b_μ vanishes. With this choice the above superspace Lagrangian can be written in an interesting form

$$\mathcal{L} = \int d^2\Theta\, 2\mathcal{E}\Big[-\frac{1}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)\big(K(\Phi^i, \bar{\Phi}^{\bar{\imath}}) - \frac{3}{2M}W(\Phi^i) - \frac{3}{2\bar{M}}\bar{W}(\bar{\Phi}^{\bar{\imath}})\big)\Big] + h.c., \tag{2.11}$$

from which we can clearly see that this nonlinear sigma model Lagrangian depends explicitly on the combination of the Kähler potential and the superpotential

$$K \;-\; \frac{3}{2M}W \;-\; \frac{3}{2\bar{M}}\bar{W}, \tag{2.12}$$

which suggests that K and W alone are not physically meaningful; it is only the combination above that is physically meaningful. This makes the modified Kähler transformation mentioned above apparent, which can be derived as following: if we perform super-Kähler transformation to the superspace Kähler potential

$$K(\Phi, \bar{\Phi}) \mapsto K(\Phi, \bar{\Phi}) + F(\Phi) + \bar{F}(\bar{\Phi}) \tag{2.13}$$

then in order to make the superspace Lagrangian (2.11) invariant, we must transform the superspace superpotential $W(\Phi)$ accordingly, which leads the the following transformation of the superpotential

$$W(\Phi) \mapsto W(\Phi) + \frac{2}{3}MF(\Phi). \tag{2.14}$$

The Lagrangian is invariant under the combination of these two transformation (the modified Kähler transformation) [1]. This superspace transformation leads exactly to what was observed by others [6–8] that the nonlinear sigma model action is invariant under the Kähler transformation of the target space X

$$K(\phi, \bar{\phi}) \mapsto K(\phi, \bar{\phi}) + f(\phi) + \bar{f}(\bar{\phi}) \tag{2.15}$$

supplemented by the following transformation of the superpotential

$$W(\phi) \mapsto W(\phi) + \frac{2}{3} M f(\phi). \tag{2.16}$$

Let U_α and U_β be two open subsets of the target space X. Then across $U_\alpha \cap U_\beta$ we have

$$\begin{aligned} K_\alpha &\mapsto K_\beta + f_{\alpha\beta} + \bar{f}_{\alpha\beta}, \\ W_\alpha &\mapsto W_\beta + \frac{2}{3} M f_{\alpha\beta}, \end{aligned} \tag{2.17}$$

which is a clear indication that the superpotential W is not a function globally on X, but rather is a section of a rank 1 affine bundle[2] $(\mathcal{O}, A)$ over X, whose line bundle part is trivially $\mathcal{O}$, while the $\mathcal{O}$-torsor A is determined by the geometry of the spacetime and the Kähler transformation of the target space. Then the combination (2.12) should be interpreted as a pairing between sections of affine bundles and their dual bundles, which is globally well-defined and invariant under Kähler transformation of the target space.

Physically, the transformations above mean that there is not a well-defined global function W that we can think of as the superpotential, as suggested by the appearance of the combination (2.12). We can combine (K_α, W_α) on patches into

$$\left(K'_\alpha \equiv K_\alpha - \frac{3}{2M} W_\alpha - \frac{3}{2\bar{M}} \bar{W}_\alpha, 0 \right),$$

[1]Note in supergravity, the superspace Lagrangian is invariant under the combined super-Kähler and super-Weyl transformations, the latter of which is a transformation of the superspace superpotential [3] which indicates the fact that the superpotential is a holomorphic section of a line bundle over the target space [3, 24]

[2]See appendix A for a discussion of affine bundles.

and then perform another Kähler transformation to

$$\left(K'_\alpha - \frac{3}{2M} f_\alpha - \frac{3}{2\bar{M}} \bar{f}_\alpha, f_\alpha \right),$$

thus replacing W_α by f_α, for any holomorphic function f_α on the patch. Only the combination of K and W is physically meaningful.

One consequence of this phenomenon is that the target space X is necessarily noncompact [6–8]. Equation (2.17) not only requires the Čech cocycle $(\delta f)_{\alpha\beta\gamma} = 0$ on all triple overlaps, but also requires that the Čech cocycle be trivial ($f_{\alpha\beta}$ is a Čech coboundary), therefore the Kähler form of X must be cohomologically trivial, which leads to the noncompactness of X, as well as the existence of a globally defined Kähler potential.

Example 2.1. **AdS$_4$**

To describe the spacetime AdS$_4$ spacetime, we set

$$\begin{aligned} M = \bar{M} &= -\frac{3}{2r}, \\ b_\mu &= 0, \end{aligned} \tag{2.18}$$

where r can be interpreted as the radius of the AdS$_4$ curvature, with the scalar curvature given by $\mathcal{R} = \frac{15}{2r^2}$. [3] The resulting superspace Lagrangian is

$$\mathcal{L} = \int d^2\Theta \, 2\mathcal{E} \Big[-\frac{1}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)\big(K(\Phi^i, \bar{\Phi}^{\bar{i}}) + rW(\Phi^i) + r\bar{W}(\bar{\Phi}^{\bar{i}})\big)\Big] + h.c. \tag{2.19}$$

When expanded in components, we get the off-shell Lagrangian on AdS$_4$ described in [6, 7],

[3]Two notes on conventions. First, we are working in mostly-plus metric conventions in this chapter, same as [3], and in those conventions, typically the AdS$_4$ curvature is negative, not positive. The reason it is positive above is that we are following the conventions of [3], in which the spin connection has an atypical sign [3][equ'n (17.12)], and which results in the AdS$_4$ curvature being positive instead of negative. We would like to thank the authors of [6] for explaining this to us. Second, the curvature is related to other descriptions as follows. If we describe AdS$_4$ as the hypersurface $-u^2 - v^2 + x^2 + y^2 + z^2 = -\alpha^2$ in $\mathbb{R}^{2,3}$, then its curvature is $\mathcal{R} = -\frac{12}{\alpha^2}$.

namely

$$\begin{aligned}\mathcal{L} = &- g_{i\bar{\jmath}}\partial_\mu\phi^i\partial^\mu\bar{\phi}^{\bar{\jmath}} - ig_{i\bar{\jmath}}\bar{\chi}^{\bar{\jmath}}\bar{\sigma}^\mu\mathcal{D}_\mu\chi^i + g_{i\bar{\jmath}}F^i\bar{F}^{\bar{\jmath}} - F^i(\frac{1}{2}g_{i\bar{\jmath},\bar{k}}\bar{\chi}^{\bar{\jmath}}\bar{\chi}^{\bar{k}} - \frac{1}{r}(K_i + rW_i)) \\ &- \bar{F}^{\bar{\imath}}(\frac{1}{2}g_{j\bar{\imath},k}\chi^j\chi^k - \frac{1}{r}(K_{\bar{\imath}} + r\bar{W}_{\bar{\imath}})) - \frac{1}{2r}(K_{ij} + rW_{ij})\chi^i\chi^j - \frac{1}{2r}(K_{\bar{\imath}\bar{\jmath}} + r\bar{W}_{\bar{\imath}\bar{\jmath}})\bar{\chi}^{\bar{\imath}}\bar{\chi}^{\bar{\jmath}} \\ &+ \frac{1}{4}g_{i\bar{\jmath},k\bar{l}}\chi^i\chi^k\bar{\chi}^{\bar{\jmath}}\bar{\chi}^{\bar{l}} + \frac{3}{r^2}(K + rW + r\bar{W}),\end{aligned} \tag{2.20}$$

where $\mathcal{D}_\mu\chi^i = \partial_\mu\chi^i - \omega_\mu\chi^i + \Gamma^i_{jk}\mathcal{D}_\mu\phi^j\chi^k$, with ω_μ the spin connection on the AdS$_4$ spacetime. Note in this way we can also recover the supersymmetry transfromation of the chiral multiplet on AdS$_4$, simply by setting the gravitino to zero in the supergravity transformation of the chiral multiplet from chiral supergravity, which leads to

$$\begin{aligned}\delta_\zeta\phi^i &= \sqrt{2}\zeta\chi^i, \\ \delta_\zeta\chi^i &= \sqrt{2}F^i\zeta + i\sqrt{2}\sigma^\mu\bar{\zeta}\partial_\mu\phi^i, \\ \delta_\zeta F^i &= -\frac{\sqrt{2}}{2r}\zeta\chi^i + i\sqrt{2}\bar{\zeta}\bar{\sigma}^\mu\mathcal{D}_\mu\chi^i,\end{aligned} \tag{2.21}$$

where the supersymmetry parameter ζ should satisfy the Killing spinor equations

$$\begin{aligned}(\nabla_\mu\zeta)^\alpha &= \frac{i}{2r}(\bar{\zeta}\bar{\sigma}_\mu)^\alpha, \\ (\nabla_\mu\bar{\zeta})_{\dot{\alpha}} &= \frac{i}{2r}(\zeta\sigma_\mu)_{\dot{\alpha}}.\end{aligned} \tag{2.22}$$

Example 2.2. S^4

Next, let us Wick rotate to a Euclidean spacetime. Consider the case of S^4, where

$$\begin{aligned}M = \bar{M} &= -\frac{3i}{2r}, \\ b_\mu &= 0.\end{aligned} \tag{2.23}$$

The resulting Euclidean Lagrangian in components is

$$\begin{aligned}\mathcal{L} =& g_{i\bar{\jmath}}\partial_\mu\phi^i\partial^\mu\bar{\phi}^{\bar{\jmath}} + ig_{i\bar{\jmath}}\bar{\chi}^{\bar{\jmath}}\bar{\sigma}^\mu\mathcal{D}_\mu\chi^i - g_{i\bar{\jmath}}F^i\bar{F}^{\bar{\jmath}} + F^i(\frac{1}{2}g_{i\bar{\jmath},\bar{k}}\bar{\chi}^{\bar{\jmath}}\bar{\chi}^{\bar{k}} - \frac{i}{r}(K_i - irW_i)) \\ &+ \bar{F}^{\bar{\imath}}(\frac{1}{2}g_{j\bar{\imath},k}\chi^j\chi^k - \frac{i}{r}(K_{\bar{\imath}} - ir\bar{W}_{\bar{\imath}})) + \frac{i}{2r}(K_{ij} - irW_{ij})\chi^i\chi^j + \frac{i}{2r}(K_{\bar{\imath}\bar{\jmath}} - ir\bar{W}_{\bar{\imath}\bar{\jmath}})\bar{\chi}^{\bar{\imath}}\bar{\chi}^{\bar{\jmath}} \\ &- \frac{1}{4}g_{i\bar{\jmath},k\bar{l}}\chi^i\chi^k\bar{\chi}^{\bar{\jmath}}\bar{\chi}^{\bar{l}} + \frac{3}{r^2}(K - irW - ir\bar{W}).\end{aligned} \tag{2.24}$$

(Note in this case the pertinent combination of K, W, is $K - irW - ir\bar{W}$, and because the two terms with W are not complex conjugates, one could debate whether the symmetry mixing K and W should properly be termed a Kähler transformation.) As discussed in [7], this action is not real.

In an analogous fashion as of the AdS$_4$ case, we can find the supersymmetry transformations of this Euclidean theory on S^4

$$\begin{aligned}\delta_\zeta \phi^i &= \sqrt{2}\zeta\chi^i,\\ \delta_\zeta \chi^i &= \sqrt{2}F^i\zeta + i\sqrt{2}\sigma^\mu\bar{\zeta}\partial_\mu\phi^i,\\ \delta_\zeta F^i &= -\frac{\sqrt{2}i}{2r}\zeta\chi^i + i\sqrt{2}\bar{\zeta}\bar{\sigma}^\mu\mathcal{D}_\mu\chi^i,\end{aligned} \tag{2.25}$$

Now the Killing spinor equations become

$$\begin{aligned}(\nabla_\mu\zeta)^\alpha &= -\frac{1}{2r}(\bar{\zeta}\bar{\sigma}_\mu)^\alpha,\\ (\nabla_\mu\bar{\zeta})_{\dot{\alpha}} &= -\frac{1}{2r}(\zeta\sigma_\mu)_{\dot{\alpha}}.\end{aligned} \tag{2.26}$$

2.1.2 $M = \bar{M} = 0$, $b_\mu \neq 0$

So far we have only reviewed spacetimes corresponding to nonzero M and vanishing b_μ. The second class of solutions of the auxiliary fields M and b_μ found in [7], corresponding to a different class of spacetime geometries, are given by $M = \bar{M} = 0$ with b_μ a covariantly constant vector.

Example 2.3. $S^3 \times \mathbb{R}$

The spacetime $S^3 \times \mathbb{R}$ is consistent with the choices

$$\begin{aligned}M = \bar{M} = b_i &= 0,\\ b_0 &= -\frac{3}{r}.\end{aligned} \tag{2.27}$$

The corresponding component Lagrangian is

$$\begin{aligned}\mathcal{L} =& - g_{i\bar{j}}\partial_\mu\phi^i\partial^\mu\bar{\phi}^{\bar{j}} - ig_{i\bar{j}}\bar{\chi}^{\bar{j}}\bar{\sigma}^\mu\mathcal{D}_\mu\chi^i + g_{i\bar{j}}F^i\bar{F}^{\bar{j}} - F^i(\frac{1}{2}g_{i\bar{j},\bar{k}}\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{k}} - W_i)\\ &- \bar{F}^{\bar{\imath}}(\frac{1}{2}g_{j\bar{\imath},k}\chi^j\chi^k - \bar{W}_{\bar{\imath}}) - \frac{1}{2}W_{ij}\chi^i\chi^j - \frac{1}{2}\bar{W}_{\bar{\imath}\bar{j}}\bar{\chi}^{\bar{\imath}}\bar{\chi}^{\bar{j}} + \frac{1}{4}g_{i\bar{j},k\bar{l}}\chi^i\chi^k\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{l}}\\ &+ \frac{i}{r}(K_i\partial_0\phi^i - K_{\bar{\imath}}\partial_0\bar{\phi}^{\bar{\imath}}) + \frac{1}{2r}g_{i\bar{j}}\chi^i\sigma_0\bar{\chi}^{\bar{j}},\end{aligned} \tag{2.28}$$

where the last line contains some new terms that are different from the familiar Minkowski spacetime model. Note these extra terms vanishes at the limit $r \to \infty$, so this theory reduces to the Minkowski case as expected. (See for example [36,37] for further discussions of rigidly supersymmetric theories on this spacetime.)

In this model the supersymmetry transformations are

$$\begin{aligned}\delta_\zeta\phi^i &= \sqrt{2}\zeta\chi^i,\\ \delta_\zeta\chi^i &= \sqrt{2}F^i\zeta + i\sqrt{2}\sigma^\mu\bar{\zeta}\partial_\mu\phi^i,\\ \delta_\zeta F^i &= \sqrt{2}\bar{\zeta}^{\dot{\alpha}}(i\mathcal{D}_{\alpha\dot{\alpha}}\chi^\alpha - \frac{1}{6}b_{\alpha\dot{\alpha}}\chi^\alpha),\end{aligned} \tag{2.29}$$

where the supersymmetry parameter ζ must satisfy [7]

$$\begin{aligned}(\nabla_0\zeta)_\alpha + \frac{i}{r}\zeta_\alpha &= 0,\\ 2(\nabla_i\zeta)_\alpha - \frac{i}{r}(\sigma_i\bar{\sigma}_0\zeta)_\alpha &= 0.\end{aligned} \tag{2.30}$$

In these cases there is no shift symmetry combining the Kähler potential and superpotential into a single quantity; the resulting Lagrangian is already Kähler invariant. Consequently, many of the conventional powerful methods from theories on Minkowski spacetime, such as holomorphy arguments, can be applied here.

2.2 Rigidly susy gauge theory on curved superspace

Now we apply the method of the last section to the $N = 1$ gauged supergravity Lagrangian in superspace [3]

$$\begin{aligned}\mathcal{L} = \frac{1}{\kappa^2}\int d^2\Theta\, 2\mathcal{E}\Big[&\frac{3}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)\exp\big(-\frac{\kappa^2}{3}[K(\Phi^i,\bar{\Phi}^{\bar{\imath}}) + \Gamma(\Phi^i,\bar{\Phi}^{\bar{\imath}},V)]\big)\\ &+ \frac{\kappa^2}{16g^2}W^{(a)}W^{(a)} + \kappa^2 W(\Phi^i)\Big] + h.c.,\end{aligned} \tag{2.31}$$

where in Wess-Zumino gauge

$$\begin{aligned}\Gamma &= V^{(a)}D^{(a)} + \frac{1}{2}g_{i\bar{\jmath}}X^{i(a)}X^{\bar{\jmath}(b)}V^{(a)}V^{(b)},\\ W_\alpha &= -\frac{1}{4}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)(\mathcal{D}_\alpha V - \frac{1}{2}[V,\mathcal{D}_\alpha V]).\end{aligned} \tag{2.32}$$

$X^{(a)}$ are the holomorphic Killing vectors on the target space X extended to superfields. We now study the two different classes of spacetime geometries separately.

2.2.1 $M = \bar{M} =$ constant, $b_\mu = 0$

Removing the dynamics of gravity and setting the background fields to $M = \bar{M} =$ constant, $b_\mu = 0$ to generate the first class of spacetime geometries, we find in superspace

$$\begin{aligned}\mathcal{L} &= \int d^2\Theta\, 2\mathcal{E}\Big[-\frac{1}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)(K(\Phi^i,\bar{\Phi}^{\bar{\imath}}) + \Gamma(\Phi^i,\bar{\Phi}^{\bar{\imath}},V)) + \frac{1}{16g^2}W^{(a)}W^{(a)} + W(\Phi^i)\Big] + h.c.\\ &= \int d^2\Theta\, 2\mathcal{E}\Big[-\frac{1}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)\big[\big(K(\Phi^i,\bar{\Phi}^{\bar{\imath}}) - \frac{3}{2M}W(\Phi^i) - \frac{3}{2\bar{M}}\bar{W}(\bar{\Phi}^{\bar{\imath}})\big) + \Gamma(\Phi^i,\bar{\Phi}^{\bar{\imath}},V)\big]\\ &\qquad + \frac{1}{16g^2}W^{(a)}W^{(a)}\Big] + h.c.\end{aligned} \tag{2.33}$$

To obtain a gauge invariant Lagrangian from (2.33), we need to impose some constraints. Since in both superspace and in components the Lagrangian only depends on the combination of the Kähler potential and the superpotential in (2.12), namely,

$$K - \frac{3}{2M}W - \frac{3}{2\bar{M}}\bar{W},$$

it is natural to start with the gauge transformation of this globally well-defined combination. We apply the superspace gauge transformations [3]

$$\begin{aligned} \delta\Big(K - \frac{3}{2M}W - \frac{3}{2\bar{M}}\bar{W}\Big) &= \Lambda^{(a)}F^{(a)} + \bar{\Lambda}^{(a)}\bar{F}^{(a)} - i[\Lambda^{(a)} - \bar{\Lambda}^{(a)}]D^{(a)}, \\ \delta\Gamma &= i[\Lambda^{(a)} - \bar{\Lambda}^{(a)}]D^{(a)}, \end{aligned} \tag{2.34}$$

where $\Lambda^{(a)}$ are the gauge transformation parameters extended to superfields, and

$$F^{(a)} = X^{(a)}\Big(K - \frac{3}{2M}W - \frac{3}{2\bar{M}}\bar{W}\Big) + iD^{(a)} \tag{2.35}$$

is a holomorphic function of the superfields Φ^i. Applying these gauge transformations to the superspace Lagrangian, we get

$$\delta\mathcal{L} = \int d^2\Theta\, 2\mathcal{E}\Big[-\frac{1}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)(\Lambda^{(a)}F^{(a)} + \bar{\Lambda}^{(a)}\bar{F}^{(a)})\Big] + h.c. \tag{2.36}$$

We demand that the superspace Lagrangian (2.33) be invariant under these gauge transformations. Using the fact that when the gravitational fields are decoupled, the vielbein superfield $2\mathcal{E}$ has the form

$$2\mathcal{E} = (1 - \Theta\Theta\bar{M})e,$$

where e is the veilbein, we are led to the constraint that M times the lowest component of the gauge transformation $\Lambda^{(a)}F^{(a)}$ must vanish. In other words, we have the constraint equations

$$MF^{(a)}(\phi^i) = 0. \tag{2.37}$$

In all our examples in this class of spacetime geometries, we have $M \sim \frac{1}{r}$, where r is some constant characteristic radius of spacetime. Therefore the constraint is really

$$F^{(a)}(\phi^i) = 0. \tag{2.38}$$

Note that this is well defined globally, since the combination (2.12) is well defined globally. Also note that these constraints reduce to the flat Minkowski spacetime case in the limit $r \to \infty$, in which we have no constraint on $F^{(a)}$ and the superpotential is gauge invariant.

We should point out that we are implicitly giving up gauge-invariance of W, since it is not physically meaningful. Only the linear combination (2.12), namely,

$$K \ - \ \frac{3}{2M}W - \ \frac{3}{2\bar{M}}\bar{W},$$

is physically meaningful. If we were to separately demand that W be gauge-invariant, then the resulting constraint we would obtain would only make sense for those special Kähler transformations that leave W invariant – which is to say, none of them. More explicitly, if we were to treat K and W separately to analyze their individual gauge transformations using the first line of (2.33), then the gauge invariance of (2.33) leads us to

$$\begin{aligned} &\delta W(\phi) = -M\epsilon^{(a)}F'^{(a)}, \\ &MF'^{(a)} = 0, \end{aligned} \tag{2.39}$$

where $F'^{(a)} = X^{(a)}K + iD^{(a)}$ which is not invariant under Kähler transformations. Now the superpotential is gauge invariant, and we have the constraint $F'^{(a)} = 0$. However, in this case to make sense of the constraint $F'^{(a)} = 0$ globally, we need to use the globally defined Kähler potential (whose existence is guaranteed by the trivial Kähler class on the target space) and demand that no Kähler transformation is allowed, which is exactly what we have been seeing. Therefore, physically we should work with the combination (2.12). We should note that in this case, in the limit $r \to \infty$ which leads to the flat Minkowski spacetime, we recover the familiar gauge invariance of the superpotential as expected, since there will be no appearance of the combination (2.12) in the Lagrangian (2.33), which is reduced to the flat Minkowski spacetime Lagrangian.

Mathematically, there is another way of understanding the constraint $F^{(a)} = 0$. Recall that the superpotential is a section of an affine bundle $(\mathcal{O}, A)$ over the target space X, therefore we must lift the action of the gauge group to an action on this affine bundle. Comparing gauge transformations of the superpotential (2.39) with equation (A.2) in the Appendix, we see that we can describe the infinitesimal group action as an infinitesimal lift to the affine bundle, described by

$$\begin{aligned} \lambda &= 1, \\ \mu &= -M\epsilon^{(a)}F'^{(a)}. \end{aligned} \tag{2.40}$$

Thus the lifting property (A.3) requires for example $2F'^{(a)} = F'^{(a)}$ or simply $F'^{(a)} = 0$, *i.e.* the fact that the superpotential is a section of the affine bundle $(\mathcal{O}, A)$ puts exactly the same constraint on the geometry of X as derived from gauge invariance.

Now let us discuss the implications of the constraint

$$F^{(a)} \;=\; X^{(a)}\left(K \;-\; \frac{3}{2M}W \;-\; \frac{3}{2\bar{M}}\bar{W}\right) \;+\; iD^{(a)} \;=\; 0 \tag{2.41}$$

in detail. Let us begin with the definition of $D^{(a)}$, namely

$$\begin{aligned} \partial_i D^{(a)} &= -iX^{(a)\bar{\jmath}}\partial_{\bar{\jmath}}\partial_i K, \\ \partial_{\bar{\jmath}} D^{(a)} &= iX^{(a)i}\partial_i\partial_{\bar{\jmath}} K. \end{aligned}$$

Integrating the equations above, we find that the most general solution for $D^{(a)}$ is given by

$$D^{(a)} \;=\; -iX^{(a)\bar{\jmath}}\partial_{\bar{\jmath}}K' \;+\; C,$$

where K' is any Kahler potential (*i.e.* $K' = K + f + \overline{f}$ for any holomorphic function f), and C is a constant. Thus the constraint that $F^{(a)} = 0$ is fixing $C = 0$ (and also partially fixing K'). Physically, this is setting the Fayet-Iliopoulos parameter to zero.

Let us outline some examples, to understand the implication of this.

Example 2.4. Let the target space X be $\mathbb{C}^n$, with the standard Kähler potential $K = \sum_i |z_i|^2$, and consider an isometry group $U(1)^k$, in which each $U(1)$ acts by phases on the z_i as

$$z_i \;\mapsto\; \lambda^{Q_i^a} z_i.$$

Then the holomorphic Killing vectors are given by

$$X^{i(a)} = iQ_i^a z_i, \quad X^{\bar{\imath}(a)} = -iQ_i^a \bar{z}_{\bar{\imath}}. \tag{2.42}$$

Then the constraining equations tell us

$$D^{(a)} = - \sum_i Q_i^a |z_i|^2. \tag{2.43}$$

For example, if there is only one $U(1)$ and all the $Q_i = 1$, then we are describing a projective space of zero radius. If there is only one $U(1)$, $n = 4$, two charges are $+1$ and two charges are -1, then we are describing a conifold with zero-size small resolution. [4]

In the examples above, we saw that the quotient had Kähler form of trivial cohomology class, as expected – after all, the ungauged theory is only defined on spaces with trivial Kähler class, so one expects the moduli spaces of the gauge theories to have the same property.

More generally, it is straightforward to check that the constraint $F^{(a)} = 0$ ensures that the cohomology class of the Kähler form on the quotient is always trivial. Briefly, the point is that $D^{(a)} = 0$ if and only if

$$X^{(a)\bar{\jmath}} \partial_{\bar{\jmath}} K' \; = \; 0$$

which ensures that K' is gauge-invariant[5] and so descends to the symplectic quotient, where it becomes a globally defined Kähler potential, whose second derivative is (manifestly) the descent of the restriction of the Kähler form on the original space.

Thus, we see that the constraint $F^{(a)} = 0$ forces the quotient space to admit a globally-

[4]Note such zero-size effects imply strong coupling in the nonlinear sigma model. In this chapter we only consider classical actions, not quantum physics.

[5]Gauge-invariance of a form ω, at least infinitesimally, is the statement that for a vector field

$$X^{(a)} \; = \; X^{(a)i} \partial_i \; + \; X^{(a)\bar{\imath}} \partial_{\bar{\imath}}$$

the Lie derivative $L_{X^{(a)}} \omega = 0$. For the function K',

$$L_{X^{(a)}} K' \; = \; X^{(a)i} \partial_i K' \; + \; X^{(a)\bar{\imath}} \partial_{\bar{\imath}} K'$$

whose vanishing follows immediately from $F^{(a)} = 0$. For the Kähler form ω, gauge-invariance $L_{X^{(a)}} \omega = 0$ is easily checked to be a consequence of the fact that the $X^{(a)}$ are Killing vectors.

defined Kähler potential, as we would naively expect from properties of ungauged sigma models.

Let us now apply these general argument to some examples from section 2.1.

Example 2.5. **AdS$_4$**

In components, the superspace Lagrangian (2.33) gives the Lagrangian on AdS$_4$ spacetime

$$\begin{aligned}\mathcal{L} =& - g_{i\bar{j}}\mathcal{D}_\mu\phi^i\mathcal{D}^\mu\bar{\phi}^{\bar{j}} - ig_{i\bar{j}}\bar{\chi}^{\bar{j}}\bar{\sigma}^\mu\mathcal{D}_\mu\chi^i - i\bar{\lambda}^{(a)}\bar{\sigma}^\mu\mathcal{D}_\mu\lambda^{(a)} - \frac{1}{4}F^{(a)}_{\mu\nu}F^{\mu\nu(a)} - \frac{1}{2}D^{(a)2} \\ &+ \sqrt{2}g_{i\bar{j}}(X^{i(a)}\bar{\chi}^{\bar{j}}\bar{\lambda}^{(a)} + \bar{X}^{\bar{j}(a)}\chi^i\lambda^{(a)}) - \frac{1}{2r}\mathcal{D}_i(K_j + rW_j)\chi^i\chi^j - \frac{1}{2r}\bar{\mathcal{D}}_{\bar{i}}(K_{\bar{j}} + r\bar{W}_{\bar{j}})\bar{\chi}^{\bar{i}}\bar{\chi}^{\bar{j}} \\ &- \frac{1}{r^2}g^{i\bar{j}}(K_j + rW_j)(K_{\bar{j}} + r\bar{W}_{\bar{j}}) + \frac{3}{r^2}(K + rW + r\bar{W}) + \frac{1}{4}\mathcal{R}_{i\bar{j}k\bar{l}}\chi^i\chi^k\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{l}},\end{aligned} \tag{2.44}$$

where we have set the gauge coupling to one, and

$$\begin{aligned}\mathcal{D}_\mu\phi^i &= \partial_\mu\phi^i - A^{(a)}_\mu X^{i(a)}, \\ \mathcal{D}_\mu\chi^i &= \partial_\mu\chi^i - \omega_\mu\chi^i + \Gamma^i_{jk}\mathcal{D}_\mu\phi^j\chi^k - A^{(a)}_\mu\partial_j X^{i(a)}\chi^j, \\ \mathcal{D}_\mu\lambda^{(a)} &= \partial_\mu\lambda^{(a)} - \omega_\mu\lambda^{(a)} - f^{abc}A^{(b)}_\mu\lambda^{(c)},\end{aligned} \tag{2.45}$$

are the gauge-covariant derivatives, with f^{abc} being the structure constants of the gauge group G, and ω_μ being the spin connection on the AdS$_4$ spacetime. Similar to the ungauged theory, the supersymmetry transformations can be derived using the gauged supergravity transformations in [3]

$$\begin{aligned}\delta_\zeta\phi^i &= \sqrt{2}\zeta\chi^i, \\ \delta_\zeta\chi^i &= \sqrt{2}F^i\zeta + i\sqrt{2}\sigma^\mu\bar{\zeta}\mathcal{D}_\mu\phi^i, \\ \delta_\zeta F^i &= -\frac{\sqrt{2}}{2r}\zeta\chi^i + i\sqrt{2}\bar{\zeta}\bar{\sigma}^\mu\mathcal{D}_\mu\chi^i + 2iT^{(a)}\phi^i\bar{\zeta}\bar{\lambda}^{(a)}, \\ \delta_\zeta A^{(a)}_\mu &= i(\zeta\sigma_\mu\bar{\lambda}^{(a)} + \bar{\zeta}\bar{\sigma}_\mu\lambda^{(a)}), \\ \delta_\zeta\lambda^{(a)} &= F^{(a)}_{\mu\nu}\sigma^{\mu\nu}\zeta - iD^{(a)}\zeta \\ \delta_\zeta D^{(a)} &= -\zeta\sigma^\mu\mathcal{D}_\mu\bar{\lambda}^{(a)} - \mathcal{D}_\mu\lambda^{(a)}\sigma^\mu\bar{\zeta}.\end{aligned} \tag{2.46}$$

Note that this Lagrangian reduces to the flat Minkowski spacetime case when $r \to \infty$ as expected. Also note that in this Lagrangian the gaugino is not coupled to any part of the affine bundle $(\mathcal{O}, A)$, or equivalently, the lifting of the gauge group action to this line bundle is trivial. This should be compared to the case of $N = 1$ supergravity: there, the gaugino is a section of the Bagger-Witten line bundle, so that the gauge group action lifts to this line bundle nontrivially, which leads to the quantization of the Fayet-Iliopoulos parameter [21,22].

Example 2.6. S^4

Let us now Wick rotate to Euclidean space, and consider the case that the spacetime is S^4, as a related example. Using the values of M, $\bar{M}$, b_μ in equation (2.23), the Euclidean Lagrangian is

$$\begin{aligned}\mathcal{L} =& g_{i\bar{j}}\mathcal{D}_\mu\phi^i\mathcal{D}^\mu\bar{\phi}^{\bar{j}} + ig_{i\bar{j}}\bar{\chi}^{\bar{j}}\bar{\sigma}^\mu\mathcal{D}_\mu\chi^i + i\bar{\lambda}^{(a)}\bar{\sigma}^\mu\mathcal{D}_\mu\lambda^{(a)} + \frac{1}{4}F^{(a)}_{\mu\nu}F^{\mu\nu(a)} + \frac{1}{2}D^{(a)2} \\ & - \sqrt{2}g_{i\bar{j}}(X^{i(a)}\bar{\chi}^{\bar{j}}\bar{\lambda}^{(a)} + \bar{X}^{\bar{j}(a)}\chi^i\lambda^{(a)}) + \frac{i}{2r}\mathcal{D}_i(K_j - irW_j)\chi^i\chi^j + \frac{i}{2r}\bar{\mathcal{D}}_{\bar{\imath}}(K_{\bar{j}} - ir\bar{W}_{\bar{j}})\bar{\chi}^{\bar{\imath}}\bar{\chi}^{\bar{j}} \\ & - \frac{1}{r^2}g^{i\bar{j}}(K_j - irW_j)(K_{\bar{j}} - ir\bar{W}_{\bar{j}}) + \frac{3}{r^2}(K - irW - ir\bar{W}) - \frac{1}{4}\mathcal{R}_{i\bar{j}k\bar{l}}\chi^i\chi^k\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{l}}.\end{aligned} \tag{2.47}$$

The supersymmetry transformations are

$$\begin{aligned}\delta_\zeta\phi^i &= \sqrt{2}\zeta\chi^i, \\ \delta_\zeta\chi^i &= \sqrt{2}F^i\zeta + i\sqrt{2}\sigma^\mu\bar{\zeta}\mathcal{D}_\mu\phi^i, \\ \delta_\zeta F^i &= -\frac{\sqrt{2}i}{2r}\zeta\chi^i + i\sqrt{2}\bar{\zeta}\bar{\sigma}^\mu\mathcal{D}_\mu\chi^i + 2iT^{(a)}\phi^i\bar{\zeta}\bar{\lambda}^{(a)}, \\ \delta_\zeta A^{(a)}_\mu &= i(\zeta\sigma_\mu\bar{\lambda}^{(a)} + \bar{\zeta}\bar{\sigma}_\mu\lambda^{(a)}), \\ \delta_\zeta\lambda^{(a)} &= F^{(a)}_{\mu\nu}\sigma^{\mu\nu}\zeta - iD^{(a)}\zeta, \\ \delta_\zeta D^{(a)} &= -\zeta\sigma^\mu\mathcal{D}_\mu\bar{\lambda}^{(a)} - \mathcal{D}_\mu\lambda^{(a)}\sigma^\mu\bar{\zeta}.\end{aligned} \tag{2.48}$$

Using our method above, the gauge invariance of the Lagrangian leads to the following constraining equation

$$F^{(a)} = X^{(a)}(K - irW - ir\bar{W}) + iD^{(a)} = 0. \tag{2.49}$$

(As in the ungauged theory, since the W terms in $K - irW - ir\bar{W}$ are not complex conjugates, one might debate whether the symmetry transformation relating K, W should be called a Kähler transformation.) This constraint effectively makes $D^{(a)}$ complex, in line with the general observations in [7] on how terms breaking superconformal invariance on S^4 are complex. The real part of the constraint implies the Fayet-Iliopoulos parameter should vanish.

2.2.2 $M = \bar{M} = 0$, $b_\mu \neq 0$

For the other class of spacetime geometries which are determined by having nonzero b_μ and $M = \bar{M} = 0$, the situation is quite different.

Example 2.7. $S^3 \times \mathbb{R}$

The superspace Lagrangian in this case is

$$\mathcal{L} = \int d^2\Theta\Big[-\frac{1}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}}-8R)(K(\Phi^i,\bar{\Phi}^{\bar{\imath}})+\Gamma(\Phi^i,\bar{\Phi}^{\bar{\imath}},V))+\frac{1}{16g^2}W^{(a)}W^{(a)}+W(\Phi^i)\Big]+h.c., \tag{2.50}$$

where now the "chiral density" superfield $\mathcal{E}$ has the property $2\mathcal{E} = 1$. Demanding (2.50) be invariant under the superspace gauge transformations

$$\begin{aligned} \delta K &= \Lambda^{(a)}F^{(a)} + \bar{\Lambda}^{(a)}\bar{F}^{(a)} - i[\Lambda^{(a)} - \bar{\Lambda}^{(a)}]D^{(a)}, \\ \delta\Gamma &= i[\Lambda^{(a)} - \bar{\Lambda}^{(a)}]D^{(a)}, \end{aligned} \tag{2.51}$$

where $F^{(a)} = X^{(a)}K + iD^{(a)}$ (same as in $N = 1$ supergravity), we are led to the result that the superpotential is gauge invariant with no further constraints on the theory, just as ordinary supersymmetric gauge theories on Minkowski spacetime. After eliminating the auxiliary fields, we find the component Lagrangian

$$\begin{aligned} \mathcal{L} = &- g_{i\bar{\jmath}}\mathcal{D}_\mu\phi^i\mathcal{D}^\mu\bar{\phi}^{\bar{\jmath}} - ig_{i\bar{\jmath}}\bar{\chi}^{\bar{\jmath}}\bar{\sigma}^\mu\mathcal{D}_\mu\chi^i - i\bar{\lambda}^{(a)}\bar{\sigma}^\mu\mathcal{D}_\mu\lambda^{(a)} - \frac{1}{4}F^{(a)}_{\mu\nu}F^{\mu\nu(a)} - \frac{1}{2}D^{(a)2} \\ &+ \sqrt{2}g_{i\bar{\jmath}}(X^{i(a)}\bar{\chi}^{\bar{\jmath}}\bar{\lambda}^{(a)} + \bar{X}^{\bar{\jmath}(a)}\chi^i\lambda^{(a)}) - \frac{1}{2}(\mathcal{D}_iW_j)\chi^i\chi^j - \frac{1}{2}(\bar{\mathcal{D}}_{\bar{\imath}}\bar{W}_{\bar{\jmath}})\bar{\chi}^{\bar{\imath}}\bar{\chi}^{\bar{\jmath}} \\ &- g^{i\bar{\jmath}}W_j\bar{W}_{\bar{\jmath}} + \frac{1}{4}\mathcal{R}_{i\bar{\jmath}k\bar{l}}\chi^i\chi^k\bar{\chi}^{\bar{\jmath}}\bar{\chi}^{\bar{l}} + \frac{i}{r}(K_i\partial_0\phi^i - K_{\bar{\imath}}\partial_0\bar{\phi}^{\bar{\imath}}) + \frac{1}{2r}g_{i\bar{\jmath}}\chi^i\sigma_0\bar{\chi}^{\bar{\jmath}}, \end{aligned} \tag{2.52}$$

which, just as the ungauged case in (2.28), is almost the same as the Minkowski spacetime case, with some extra terms coming from the fact that the spacetime is curved. Note these extra terms vanish in the limit $r \to \infty$, so this theory reduces to the Minkowski case as expected. (See for example [36,37] for further discussions of rigidly supersymmetric theories on this spacetime.)

The supersymmetry transformations of this model on $S^3 \times \mathbb{R}$ are

$$\begin{aligned}
\delta_\zeta \phi^i &= \sqrt{2}\zeta\chi^i, \\
\delta_\zeta \chi^i &= \sqrt{2}F^i\zeta + i\sqrt{2}\sigma^\mu\bar{\zeta}\partial_\mu\phi^i, \\
\delta_\zeta F^i &= \sqrt{2}\bar{\zeta}^{\dot{\alpha}}(i\mathcal{D}_{\alpha\dot{\alpha}}\chi^\alpha - \frac{1}{6}b_{\alpha\dot{\alpha}}\chi^\alpha) \\
\delta_\zeta A_\mu^{(a)} &= i(\zeta\sigma_\mu\bar{\lambda}^{(a)} + \bar{\zeta}\bar{\sigma}_\mu\lambda^{(a)}), \\
\delta_\zeta \lambda^{(a)} &= F_{\mu\nu}^{(a)}\sigma^{\mu\nu}\zeta - iD^{(a)}\zeta, \\
\delta_\zeta D^{(a)} &= -\zeta\sigma^\mu\mathcal{D}_\mu\bar{\lambda}^{(a)} - \mathcal{D}_\mu\lambda^{(a)}\sigma^\mu\bar{\zeta}.
\end{aligned} \tag{2.53}$$

2.3 Background principle

The authors of [6] proposed a "background principle": if a rigid $N = 1$ theory on Minkowski spacetime can be quantum-mechanically coupled to $N = 1$ supergravity in a consistent way, then it should behave smoothly under deformation from Minkowski spacetime to AdS as classical theories. If a theory can be consistently coupled to gravity, then it should also be possible to consistently formulate it in a nontrivial background metric. In particular, cancellation of (quantum) anomalies in supergravity couplings is often tied to (classical) consistency conditions in rigid theories.

In this section we will observe that the same ideas also apply to gauge theories, and also trivially extend them to all four-manifolds of the first type discussed in this chapter (for which $M = \overline{M}$ constant, $b_\mu = 0$), not just AdS_4.

Let us begin by reviewing the ungauged case, discussed in [6]. It was observed in [6] that in ungauged theories, the purely classical constraint on AdS of having a cohomologically trivial Kähler form prevents the appearance of gravitional anomaly when one couples the rigid theory to supergravity. In detail, start with the with the six-form anomaly polynomial of $N = 1$ supergravity coupled to an ungauged nonlinear sigma model [21]:

$$\begin{aligned} P_{local} =& \phi^* ch_3(X) - \frac{1}{24} p_1(\Sigma) \phi^* c_1(X) \\ &+ \phi^* c_1(L) \left(\phi^* ch_2(X) + \frac{21-n}{24} p_1(\Sigma) \right) \\ &+ \frac{1}{2} \phi^* \left(c_1(L)^2 c_1(X) \right) + \frac{n+3}{6} \phi^* c_1(L)^3. \end{aligned} \tag{2.54}$$

where Σ denotes the four-dimensional spacetime, L denotes the Kähler (Bagger-Witten) line bundle over the target space X (the moduli space of the supergravity), $\phi : \Sigma \to X$ denotes the map defining a vev of the bosons of the theory, and n is the complex dimension of the target space X. This anomaly polynomial decomposes as a sum

$$P_{local} = P_{global} + \Delta P, \tag{2.55}$$

where

$$P_{global} = \phi^* ch_3(X) - \frac{1}{24} p_1(\Sigma) \phi^* c_1(X), \tag{2.56}$$

is the anomaly polynomial of the rigid nonlinear sigma model, and

$$\begin{aligned} \Delta P =& \phi^* c_1(L) \Bigg[\left(\phi^* ch_2(X) + \frac{21-n}{24} p_1(\Sigma) \right) \\ &+ \frac{1}{2} \phi^* \big(c_1(L) c_1(X) \big) + \frac{n+3}{6} \phi^* c_1(L)^2 \Bigg]. \end{aligned} \tag{2.57}$$

If the Kähler form is cohomologically trivial, then $c_1(L) = 0$. Thus, if the rigid theory on AdS$_4$ is classically consistent, then coupling to supergravity does not change the anomaly: if the rigid theory is anomaly-free, then so is the theory coupled to supergravity.

In passing, let us make the trivial observation that the computation above, which [6] originally only applied to AdS$_4$, also applies to the other four-manifolds of the first type discussed in this chapter (in which $M = \overline{M}$ is constant, $b_\mu = 0$).

It remains to ask whether our construction of $N = 1$ gauge theory satisfies this principle. To show this, let's start with the six-form anomaly polynomial of a $N = 1$ supergravity coupled to a gauged nonlinear sigma model, in which we gauge some global symmetry G of the target space X. We denote this anomaly polynomial as P^G_{local} to distinguish from the ungauged case above. From [21],

$$\begin{aligned} P^G_{local} =& \phi^* ch_3(T_{vert}\mathcal{M}) - \frac{1}{24} p_1(\Sigma)\phi^* c_1(T_{vert}\mathcal{M}) \\ &+ \phi^* c_1(\mathcal{L}) \left(\phi^* ch_2(T_{vert}\mathcal{M}) + \frac{21 - n + dim(G)}{24} p_1(\Sigma) \right) \\ &+ \frac{1}{2}\phi^* \left(c_1(\mathcal{L})^2 c_1(T_{vert}\mathcal{M}) \right) + \frac{n + 3 - dim(G)}{6} \phi^* c_1(\mathcal{L})^3. \end{aligned} \tag{2.58}$$

In the expression above, ϕ is no longer a map $\Sigma \to X$. Instead, to define ϕ, we first pick a principal G bundle over our four-dimensional spacetime Σ, call it P. (The path integral of the gauge theory sums over P's.) Define

$$\mathcal{M} \equiv (P \times X)/G$$

which is a bundle over Σ with fiber X. Then, ϕ is a section of $\mathcal{M}$, *i.e.* a map $\phi : \Sigma \to \mathcal{M}$ behaving well with respect to the projection $\mathcal{M} \to \Sigma$. Finally, the line bundle $\mathcal{L}$ is defined by taking the pullback of $L \to X$ to $P \times X$, and then using the G-equivariant structure to descend to $\mathcal{M} = (P \times X)/G$, *i.e.* schematically, $\mathcal{L} = (\pi_X^* L)/G$.

It is worth emphasizing at this point that the Fayet-Iliopoulos parameters of the supergravity theory are encoded implicitly in the expression above. Specifically, they are encoded in $c_1(\mathcal{L})$. That Chern class manifestly contains information about $c_1(L)$ on X, and in addition, it also contains information about the choice of G-equivariant structure on L. That equivariant structure encodes the Fayet-Iliopoulos parameters in the supergravity, as described in [21].

It will be useful later to understand this in more detail, so let us consider a simple example. Suppose that the supergravity moduli space X is a point, so that any Bagger-Witten line bundle L is automatically trivial, and $c_1(L) = 0$. The choice of G-equivariant structure is then simply a one-dimensional representation of G, *i.e.* an action of G on the one-dimensional fiber $\mathbb{C}$. In this case, $\mathcal{M} = \Sigma$ and $\mathcal{L}$ is then the line bundle associated to the principal bundle P via that representation. If that representation is nontrivial, then that associated bundle $\mathcal{L} \to \Sigma$ will vary as P varies, for general Σ.

In particular, it will be important later to note that the only way to ensure that $c_1(\mathcal{L}) = 0$ for all choices of Σ and P's is if both $L \to X$ is trivial, *and* the G-equivariant structure on L is also trivial[6] (Fayet-Iliopoulos parameters vanish).

In passing, we should note that the fact that in supergravity, Fayet-Iliopoulos parameters appear in anomalies, has been discussed elsewhere in the literature, see for example [40, 41] for an excellent description and overview.

Now, let us return to the background principle and the discussion of anomalies. Much as in [6], in the gauged case the supergravity anomaly decomposes as

$$\begin{aligned} P^G_{local} &= P^G_{global} + \Delta P^G \\ P^G_{global} &= \phi^* ch_3(T_{vert}\mathcal{M}) - \frac{1}{24} p_1(\Sigma) \phi^* c_1(T_{vert}\mathcal{M}) \\ \Delta P^G &= \phi^* c_1(\mathcal{L}) \Bigg[\left(\phi^* ch_2(T_{vert}\mathcal{M}) + \frac{21 - n + dim(G)}{24} p_1(\Sigma) \right) \\ &\quad + \frac{1}{2} \phi^* \big(c_1(\mathcal{L}) c_1(T_{vert}\mathcal{M}) \big) + \frac{n + 3 - dim(G)}{6} \phi^* c_1(\mathcal{L})^2 \Bigg], \end{aligned} \tag{2.59}$$

where P^G_{global} is the anomaly of the rigid G-gauged nonlinear sigma model.

As before, ΔP^G is proportional to $c_1(\mathcal{L})$. As we noted earlier, to guarantee that $c_1(\mathcal{L})$ vanish, we must require not only that the $L \to X$ be trivial (*i.e.* that the target space X has a cohomologically trivial Kähler form), but also that the Fayet-Iliopoulos parameters vanish.

We saw earlier in section 2.2.1 that in rigidly supersymmetric gauge theories on four-manifolds of the first type, including AdS$_4$, we must classically require both that X have a

[6] For a nontrivial bundle, there is no meaningful notion of a 'trivial' equivariant structure, but in the special case that the bundle is trivial, there is a canonical 'trivial' equivariant structure.

cohomologically trivial Kähler form and that the Fayet-Iliopoulos parameters vanish. Thus, these classical constraints prevent anomalies when coupling to supergravity, consistent with the background principle.

Example 2.8. $\mathbb{C}P^n$ model The $\mathbb{C}P^n$ model can be described either by a nonlinear sigma model with target space $\mathbb{C}P^n$, or by a $U(1)$ gauge theory with $n+1$ chiral superfields of charge 1. Since $\mathbb{C}P^n$ is compact with a cohomologically nontrivial Kähler form, this theory cannot be coupled to gravity in a consistent way, as discussed above. Similarly, the four-dimensional gauge theory is anomalous. Note, on the other hand, in the example from section 2.2.1 we showed that our constraint $F^{(a)} = 0$ requires the target space to have zero radius, *i.e.* not really a $\mathbb{C}P^n$ anymore. Therefore the constraint $F^{(a)} = 0$ is consistent with the background principle, albeit perhaps trivially so.

Chapter 3

(0,2) and (2,2) Supersymmetry on Two-manifolds

3.1 (2,2) Nonlinear Sigma Models on S^2

In recent years, supersymmetric localization techniques have been applied to theories of this form to obtain quantum mechanically exact results such as the partition function [17, 18, 44–48]. Recently these methods were applied to two-dimensional gauged linear sigma models (GLSMs) to derive exact expressions for partition functions [17, 18], which has quickly led to some interesting new computational methods and results for Gromov-Witten invariants [49–51] and the Seiberg-Witten Kähler potential [52], as well as other insights into older results [53]. As part of that work, the papers [17, 18] worked out curvature couplings in two-dimensional linear sigma models whose target spaces are vector spaces.

In this section, we return to [17, 18] to re-examine general rigidly supersymmetric nonlinear sigma models with potential and work out curvature couplings, for more general target spaces (and with $U(1)_R$ actions described by more general holomorphic Killing vectors), for both ordinary and twisted chiral supermultiplets, on constant-curvature (round) two-sphere worldsheets, so as to give some insight into the rather complicated linear actions of [17, 18].

We begin in section 3.1.1 by working out general nonlinear sigma models with potential for ordinary chiral supermultiplets on round two-spheres. To add a superpotential in a theory of ordinary chiral multiplets, one must extend the flat-space $U(1)_R$ symmetry by a holomorphic Killing vector, which generates *e.g.* a curvature-dependent potential term in the action. We also discuss why two-dimensional theories of this sort do not have constraints on the Kahler form on the target space, unlike typical behavior in four dimensional theories.

In section 3.1.2, we perform analogous analyses for twisted chiral multiplets. Here, the curvature couplings have a different form than for ordinary chiral multiplets. For example, although one can extend the $U(1)_R$ of the twisted chiral theory by a holomorphic Killing vector field, that same field can be re-absorbed into the auxiliary field of the multiplet, and so its presence is optional. Moreover, these same couplings naively break a duality between the flat-worldsheet chiral and twisted chiral theories.

We conclude in section 3.1.3 with a discussion of topological twists in the presence of such curvature couplings. Appendix C further discusses the relationship between ordinary and twisted chiral multiplets, and their topological twists.

3.1.1 Ordinary chiral supermultiplets

In this section, we will discuss curvature couplings for ordinary chiral multiplets on an S^2 with a constant curvature metric, the 'round' S^2.

The rigid $\mathcal{N} = (2, 2)$ supersymmetry algebra on S^2 with Euclidean signature is [17, 18]

$$OSp(2|2) \;\cong\; SU(2|1),$$

whose bosonic subalgebra is $SU(2) \times U(1)_R$. The $SU(2)$ factor represents the isometries of S^2, while the $U(1)_R$ factor is the vector-like R-symmetry, which is now contained in the supersymmetry algebra rather than being an outer automorphism of it.

Our spinor notation will follow [18][p. 47], [3]. Spinors are multiplied as

$$\lambda\psi = \lambda^\alpha \varepsilon_{\alpha\beta}\psi^\beta = \psi\lambda,$$

where $\varepsilon_{21} = -\varepsilon_{12} = 1$, $\varepsilon_{11} = -\varepsilon_{22} = 0$. The two-dimensional matrices $(\gamma_m)_\alpha^{\ \beta}$, which are generators of the two-dimensional complex Clifford algebra, are given by the Pauli matrices in local frame coordinates: $\gamma_m = \sigma_m$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

with

$$\gamma_3 = -i\sigma_1\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With our notation, the explicit form of the supersymmetry algebra $SU(2|1)$ is [18]:

$$\begin{aligned} &\{Q_\alpha, \bar{Q}_\beta\} = \gamma^m_{\alpha\beta} J_m - \frac{1}{2}\varepsilon_{\alpha\beta} R \quad && [R, Q_\alpha] = Q_\alpha \quad && [R, \bar{Q}_\alpha] = -\bar{Q}_\alpha \\ &[J_m, J_n] = i\varepsilon_{mnl} J_l && [J_m, Q_\alpha] = -\frac{1}{2}\gamma^{\beta}_{m\alpha} Q_\beta && [J_m, \bar{Q}_\alpha] = -\frac{1}{2}\gamma^{\beta}_{m\alpha}\bar{Q}_\beta \end{aligned} \tag{3.1}$$

where Q_α and $\bar{Q}_\alpha$ are the supersymmetry generators, J_m are the generators of the $SU(2)$ isometry of S^2, and R is the generator of the $U(1)_R$ symmetry.

The space of Killing spinors on S^2 is four-dimensional [54]; a useful basis of this space consists of two Killing spinors called positive Killing spinors and two Killing spinors called negative Killing spinors [17]. Let's denote the positive Killing spinors as $\zeta, \bar{\zeta}$, which are independent from each other and satisfy the same Killing spinor equation

$$\begin{aligned} \nabla_m \zeta - \frac{i}{2r}\gamma_m \zeta = 0, \\ \nabla_m \bar{\zeta} - \frac{i}{2r}\gamma_m \bar{\zeta} = 0. \end{aligned} \tag{3.2}$$

Some useful spinor identities include:

$$\lambda\psi \;=\; +\psi\lambda, \quad \lambda\gamma_m\psi = \lambda^\alpha(\gamma_m)_\alpha^\beta\psi_\beta \;=\; -\psi\gamma_m\lambda, \quad \gamma_m\gamma_n = g_{mn} + i\varepsilon_{mn}\gamma_3, \quad \gamma^m\gamma^n\gamma_m = 0,$$

$$(\psi_1\psi_2)\psi_3 \;+\; (\psi_2\psi_3)\psi_1 \;+\; (\psi_3\psi_1)\psi_2 \;=\; 0.$$

An Euclidean $\mathcal{N} = (2,2)$ chiral multiplet in two dimensions contains components

$$(\phi^i, \bar{\phi}^{\bar{\imath}}, \psi^i, \bar{\psi}^{\bar{\imath}}, F^i, \bar{F}^{\bar{\imath}}).$$

The chiral fields $\phi^i, \bar{\phi}^{\bar{\imath}}$ parametrize the target space M, which is a Kähler manifold. As observed in [17, 18], the $U(1)_R$ charges of the chiral fields ϕ^i enter the definition of the Lagrangian of the linear sigma models on S^2. To construct nonlinear sigma models on a worldsheet S^2, one needs to use the holomorphic Killing vector $X = X^i\partial_i$ corresponding to the $U(1)_R$ symmetry, which should be interpreted as an isometry on the target space M of the nonlinear sigma model.

The general Lagrangian governing $\mathcal{N} = (2,2)$ nonlinear sigma models on S^2 is

$$\begin{aligned}\mathcal{L} \;&= g_{i\bar{\jmath}}\partial_m\phi^i\partial^m\bar{\phi}^{\bar{\jmath}} \;-\; ig_{i\bar{\jmath}}\bar{\psi}^{\bar{\jmath}}\gamma^m\nabla_m\psi^i \;+\; g_{i\bar{\jmath}}F^i\bar{F}^{\bar{\jmath}} \;-\; F^i(\frac{1}{2}g_{i\bar{\jmath},\bar{k}}\bar{\psi}^{\bar{\jmath}}\bar{\psi}^{\bar{k}} - W_i)\\ &- \;\bar{F}^{\bar{\imath}}(\frac{1}{2}g_{j\bar{\imath},k}\psi^j\psi^k - \bar{W}_{\bar{\imath}}) \;-\; \frac{1}{2}W_{ij}\psi^i\psi^j \;-\; \frac{1}{2}\bar{W}_{\bar{\imath}\bar{\jmath}}\bar{\psi}^{\bar{\imath}}\bar{\psi}^{\bar{\jmath}} \;+\; \frac{1}{4}g_{i\bar{\jmath},k\bar{\ell}}(\psi^i\psi^k)(\bar{\psi}^{\bar{\jmath}}\bar{\psi}^{\bar{\ell}})\\ &- \;\frac{1}{4r^2}g_{i\bar{\jmath}}X^iX^{\bar{\jmath}} \;+\; \frac{i}{4r^2}K_iX^i \;-\; \frac{i}{4r^2}K_{\bar{\imath}}X^{\bar{\imath}} \;-\; \frac{i}{2r}g_{i\bar{\jmath}}\bar{\psi}^{\bar{\jmath}}\psi^j\nabla_jX^i,\end{aligned} \tag{3.3}$$

where[1] r is the radius of S^2, K is the Kähler potential of the target space M, W is the superpotential, and

$$\begin{aligned}\nabla_m\psi^i &= \widetilde{\nabla}_m\psi^i + \Gamma^i_{jk}(\partial_m\phi^j)\psi^k,\\ \nabla_jX^i &= \partial_jX^i + \Gamma^i_{jk}X^k,\end{aligned} \tag{3.4}$$

[1]As an aside, terms closely analogous to the curvature-dependent terms in the Lagrangian above have been discussed in two-dimensional theories in a different context in [55].

with Γ^i_{jk} the Christoffel symbols on the target space M, and $\widetilde{\nabla}_m$ denotes the pure worldsheet spin connection covariant derivative. Integrating out the auxiliary fields yields

$$\begin{aligned}\mathcal{L} &= g_{i\bar{\jmath}}\partial_m\phi^i\partial^m\bar{\phi}^{\bar{\jmath}} \;-\; ig_{i\bar{\jmath}}\bar{\psi}^{\bar{\jmath}}\gamma^m\nabla_m\psi^i \;+\; R_{i\bar{\jmath}k\bar{\ell}}(\psi^i\psi^k)(\bar{\psi}^{\bar{\jmath}}\bar{\psi}^{\bar{\ell}}) \\ &\quad -\; g^{i\bar{\jmath}}W_i\bar{W}_{\bar{\jmath}} \;-\; \frac{1}{2}\nabla_i\partial_jW\psi^i\psi^j \;-\; \frac{1}{2}\nabla_{\bar{\imath}}\partial_{\bar{\jmath}}\bar{W}\bar{\psi}^{\bar{\imath}}\bar{\psi}^{\bar{\jmath}} \\ &\quad -\; \frac{1}{4r^2}g_{i\bar{\jmath}}X^iX^{\bar{\jmath}} \;+\; \frac{i}{4r^2}K_iX^i \;-\; \frac{i}{4r^2}K_{\bar{\imath}}X^{\bar{\imath}} \;-\; \frac{i}{2r}g_{i\bar{\jmath}}\bar{\psi}^{\bar{\jmath}}\psi^j\nabla_jX^i.\end{aligned} \tag{3.5}$$

This Lagrangian reduces the the usual $\mathcal{N} = (2,2)$ nonlinear sigma model on flat two-dimensional space when $r \to \infty$, as one would expect.

There are two conditions on the data above. First, the Kähler potential is invariant under the isometry defined by the holomorphic Killing vector X, which means that on each coordinate patch,

$$\mathcal{L}_XK \;=\; X^iK_i \;+\; X^{\bar{\imath}}K_{\bar{\imath}} \;=\; 0 \tag{3.6}$$

and across coordinate patches, $K \mapsto K + f(\phi) + \bar{f}(\bar{\phi})$, the f's are constrained to obey

$$\sum_i X^i\partial_i f(\phi) \;=\; 0. \tag{3.7}$$

On a Kähler manifold with a holomorphic isometry X, for data on a good open cover, one can always choose K's, f's to obey these constraints [56].

The second condition says that the superpotential W is homogeneous of degree 2 under the vector-like $U(1)_R$ symmetry, meaning

$$2W \;-\; iX^iW_i \;=\; 0 \tag{3.8}$$

(up to an additive constant). In particular, if $X = 0$, then the superpotential must vanish (up to a constant).

The Lagrangian above is invariant under the following supersymmetry transformations

$$\begin{aligned}
\delta\phi^i &= \zeta\psi^i, \\
\delta\bar{\phi}^{\bar{\imath}} &= \bar{\psi}^{\bar{\imath}}\bar{\zeta}, \\
\delta\psi^i &= i\gamma^m\bar{\zeta}\partial_m\phi^i - \frac{i}{2r}\bar{\zeta}X^i + \zeta F^i, \\
\delta\bar{\psi}^{\bar{\imath}} &= i\gamma^m\zeta\partial_m\bar{\phi}^{\bar{\imath}} + \frac{i}{2r}\zeta X^{\bar{\imath}} + \bar{\zeta}\bar{F}^{\bar{\imath}}, \\
\delta F^i &= i\bar{\zeta}\gamma^m\tilde{\nabla}_m\psi^i + \frac{i}{2r}\bar{\zeta}\psi^j\partial_j X^i, \\
\delta\bar{F}^{\bar{\imath}} &= i\zeta\gamma^m\tilde{\nabla}_m\bar{\psi}^{\bar{\imath}} - \frac{i}{2r}\zeta\bar{\psi}^{\bar{\jmath}}\partial_{\bar{\jmath}}X^{\bar{\imath}}.
\end{aligned} \tag{3.9}$$

(up to total derivatives) provided the Killing spinor equations (3.2) are satisfied, together with the constraints (3.6), (3.8).

In the special case that $X = 0$ (and hence $W = 0$), the Lagrangian and supersymmetry transformations are identical to those on flat space. One can show that the flat-space Lagrangian is invariant under supersymmetry transformations defined by a Killing spinor appropriate for S^2, not just a constant spinor. Thus, the lagrangian with $X = 0$ is consistent with supersymmetry on both S^2 and $\mathbb{R}^2$, as one would expect.

In the linear case, *i.e.* when the target space $M = \mathbb{C}$ with the trivial Kähler potential $K = \bar{\phi}\phi$ and the $U(1)_R$ holomorphic Killing vector $X = -iq\phi\frac{d}{d\phi}$, the Lagrangian (3.3) reduces to the Lagrangian of the chiral multiplet in [17, 18] (where q is the $U(1)_R$ charge of the chiral field ϕ). The full gauged linear sigma model in [17, 18] can also be obtained by applying a decoupling gravity procedure analogous to the one in [6, 7] to the coupled theory of (2,2) gauged linear sigma model and (1,1) supergravity model in [57].

From the supersymmetry transformations above, we can get some insight into the constraint on W. Specifically, note that with the Killing spinor condition, the supersymmetry variation of the auxiliary field can be written

$$\delta F^i \;=\; i\bar{\zeta}\gamma^m\tilde{\nabla}_m\psi^i \;-\; \frac{1}{2}(\nabla_m\bar{\zeta})\gamma^m\psi^j\partial_j X^i.$$

If $\partial_j X^i = -2i\delta^i_j$, then δF^i is a total derivative. For example, in the linear case above, this is the statement that when $q = 2$, δF is a total derivative. Ultimately that factor of two is the reason why supersymmetry requires that W be homogeneous of degree two under the action of X.

Now, let us describe the vector $U(1)_R$ symmetry explicitly. Infinitesimally, the action of this global $U(1)_R$ symmetry on the fields is given by

$$
\begin{aligned}
\delta\phi^i &= \epsilon X^i, \\
\delta\phi^{\bar\imath} &= \epsilon X^{\bar\imath}, \\
\delta\psi^i &= \epsilon\left(\partial_j X^i \ - \ i\delta^i_j\right)\psi^j, \\
\delta\psi^{\bar\imath} &= \epsilon\left(\partial_{\bar\jmath} X^{\bar\imath} \ + \ i\delta^{\bar\imath}_{\bar\jmath}\right)\psi^{\bar\jmath}, \\
\delta F^i &= \epsilon\left(\partial_j X^i \ - \ 2i\delta^i_j\right)F^j - \frac{\epsilon}{2}\partial_k\partial_j X^i\psi^j\psi^k, \\
\delta F^{\bar\imath} &= \epsilon\left(\partial_{\bar\jmath} X^{\bar\imath} \ + \ 2i\delta^{\bar\imath}_{\bar\jmath}\right)F^{\bar\jmath} - \frac{\epsilon}{2}\partial_{\bar k}\partial_{\bar\jmath} X^{\bar\imath}\psi^{\bar\jmath}\psi^{\bar k},
\end{aligned}
\tag{3.10}
$$

where ϵ is a real constant parametrizing the global $U(1)_R$. Our Lagrangian (3.3) is invariant under this symmetry.

We should observe that even for (2,2) supersymmetric theories on $\mathbb{R}^2$ instead of S^2, the $U(1)_R$ symmetry sometimes involves an action on bosons, and hence involves a holomorphic Killing vector field X. In fact, the explicit transformations (3.10) also applies to the usual $\mathcal{N} = (2,2)$ nonlinear sigma models on $\mathbb{R}^2$. In general, a global symmetry of a nonlinear sigma model on any spacetime should act on the bosonic fields as a Killing vector on the target space. We believe that (3.10) should hold for any two-dimensional nonlinear sigma models with a vector $U(1)_R$ symmetry, regardless of the two-dimensional spacetime they are defined on.

In the case of four-dimensional rigidly supersymmetric theories on spacetimes such as S^4 and AdS_4, supersymmetry imposes constraints on the theory (see *e.g.* [6–8], as well as chapter 2), such as a constraint that the Kähler form on the target space be cohomologically-trivial.

In two dimensional theories, on the other hand, we have found no analogous constraint.

Mechanically, one way to understand this lack of constraints on two-dimensional theories is to think of a two-dimensional theory as a dimensional reduction of a four-dimensional theory on $\mathbb{R}^2 \times \Sigma_2$ (for Σ_2 a two-manifold). Such four dimensional theories were unconstrained by supersymmetry; constraints only existed in four dimensions when all four spacetime directions were 'wrapped up' nontrivially in the topology, when none were flat. Another more abstract way to think about this in the context of the decoupling procedure is as follows[2]. In four dimensional supergravities, the Fayet-Iliopoulos parameter[3], the curvature of the Bagger-Witten line bundle, and so forth are weighted by inverse factors of the four-dimensional Planck mass. The decoupling limit of [6, 7] involves sending the Planck mass to infinity, which necessarily truncates those terms, and leaves one with a rigidly supersymmetric theory in which Fayet-Iliopoulos parameters vanish and Kähler forms are exact. By contrast, in two dimensions, the "Planck mass" is dimensionless. Hence, the procedure of decoupling gravity in two dimensions is a formal way of obtaining rigid supersymmetric theories from supergravity theories, with no further constraints on the target space geometry. Thus, one should not be surprised to find no constraints on the target space geometry in two dimensional cases.

3.1.2 Twisted chiral supermultiplets

In two dimensions, there is another $\mathcal{N} = (2, 2)$ supermultiplet known as the twisted chiral multiplet. The field content of a twisted chiral multiplet is the same as that of an ordinary chiral multiplet:

$$(\rho, \bar{\rho}, \chi, \bar{\chi}, G, \bar{G}).$$

[2]We would like to thank I. Melnikov for making this observation.

[3]See [34, 58–60] for a recent discussion of the Fayet-Iliopoulos parameter in supergravity, and how old obstruction issues summarized in *e.g.* [61] can be circumvented.

The fields $\rho^i, \bar{\rho}^{\bar{\imath}}$ are bosons describing maps into a target space $\tilde{M}$, which is required[4] to be a Kähler manifold. The fields G, $\bar{G}$ are auxiliary fields.

The fields $\chi, \bar{\chi}$ are Dirac spinors. In a twisted chiral multiplet, their components mix holomorphic and antiholomorphic target space indices:

$$\chi = \left(\begin{array}{c} \chi_-^i \\ \chi_+^{\bar{\imath}} \end{array} \right), \quad \bar{\chi} = \left(\begin{array}{c} \bar{\chi}_-^{\bar{\imath}} \\ \bar{\chi}_+^i \end{array} \right). \tag{3.11}$$

In this section, we will work with Killing spinors ζ, $\bar{\zeta}$ obeying

$$\begin{aligned} \nabla_m \zeta &= \frac{i}{2r} \gamma_m \zeta, \\ \nabla_m \bar{\zeta} &= -\frac{i}{2r} \gamma_m \bar{\zeta}, \end{aligned} \tag{3.12}$$

a slightly different convention than we used for ordinary chiral multiplets.

Although the flat-worldsheet action of a twisted chiral multiplet is identical to that of an ordinary chiral multiplet, the curvature couplings to a superpotential are of a very different form. In the Killing spinor convention (3.12), the most general $\mathcal{N} = (2,2)$ Lagrangian for twisted chiral multiplets on a round S^2 is

$$\begin{aligned} \mathcal{L}_T = & \; g_{i\bar{\jmath}} \partial_m \rho^i \partial^m \bar{\rho}^{\bar{\jmath}} \; + \; 2i g_{i\bar{\jmath}} \bar{\chi}_-^{\bar{\jmath}} \nabla_{\bar{z}} \chi_-^i \; + \; 2i g_{i\bar{\jmath}} \chi_+^{\bar{\jmath}} \nabla_z \bar{\chi}_+^i \; + \; g_{i\bar{\jmath}} G^i \bar{G}^{\bar{\jmath}} \\ & - \; G^i \left(i g_{i\bar{\jmath},\bar{k}} \bar{\chi}_-^{\bar{\jmath}} \chi_+^{\bar{k}} \; - \; \mathcal{W}_i \right) \; - \; i \mathcal{W}_{ij} \chi_-^i \bar{\chi}_+^j \\ & - \; \bar{G}^{\bar{\imath}} \left(i g_{\bar{\imath}j,k} \chi_-^j \bar{\chi}_+^k \; - \; \bar{\mathcal{W}}_{\bar{\imath}} \right) \; - \; i \bar{\mathcal{W}}_{\bar{\imath}\bar{\jmath}} \bar{\chi}_-^{\bar{\imath}} \chi_+^{\bar{\jmath}} \\ & + \; g_{i\bar{\jmath},k\bar{l}} \bar{\chi}_+^i \chi_+^{\bar{\jmath}} \chi_-^k \bar{\chi}_-^{\bar{l}} \\ & + \; \frac{i}{r} \mathcal{W} \; - \; \frac{i}{r} \bar{\mathcal{W}}, \end{aligned} \tag{3.13}$$

where, as in the case of ordinary chiral multiplets, $g_{i\bar{\jmath}}$ is the Kähler metric on $\tilde{M}$, and $\mathcal{W}$ is the twisted superpotential. (For notational simplicity, we have chosen to write the spinors in the lagrangian above in chiral components.) Notice that the twisted superpotential

[4] We do not attempt to consider H-flux backgrounds in this section.

$\mathcal{W}$ is coupled to the curvature of S^2, unlike the superpotential W of the ordinary chiral multiplets. Furthermore, unlike ordinary chiral multiplets, no holomorphic Killing vector is needed to define the superpotential. (It is straightforward to check that a curvature coupling of this form is incompatible with the supersymmetry of the ordinary chiral multiplets.) This Lagrangian generalizes[5] that given in [53][equ'ns (4.2), (4.5)] for flat target spaces. It also reduces to the flat $\mathbb{R}^2$ lagrangian when $r \to \infty$, as expected.

The fermions couple to the following bundles:

$$\begin{aligned} \bar{\chi}_+^i &\in \Gamma_{C^\infty}\left(K_\Sigma^{1/2} \otimes \rho^* T^{1,0}\tilde{M}\right), & \chi_-^i &\in \Gamma_{C^\infty}\left(\overline{K}_\Sigma^{1/2} \otimes \left(\rho^* T^{0,1}\tilde{M}\right)^*\right), \\ \chi_+^{\bar{\imath}} &\in \Gamma_{C^\infty}\left(K_\Sigma^{1/2} \otimes \left(\rho^* T^{1,0}\tilde{M}\right)^*\right), & \bar{\chi}_-^{\bar{\imath}} &\in \Gamma_{C^\infty}\left(\overline{K}_\Sigma^{1/2} \otimes \rho^* T^{0,1}\tilde{M}\right). \end{aligned}$$

where Σ is the worldsheet (here, S^2), K_Σ and $\overline{K}_\Sigma$ are the holomorphic and antiholomorphic canonical bundles.

The above Lagrangian is invariant under the following supersymmetry transformations:

$$\begin{aligned} \delta\rho^i &= i\bar{\zeta}_+\chi_-^i + i\zeta_-\bar{\chi}_+^i, \\ \delta\bar{\rho}^{\bar{\imath}} &= i\bar{\zeta}_-\chi_+^{\bar{\imath}} + i\zeta_+\bar{\chi}_-^{\bar{\imath}}, \\ \delta\bar{\chi}_+^i &= -2\bar{\zeta}_-\bar{\partial}\rho^i - \bar{\zeta}_+ G^i, \\ \delta\chi_-^i &= -2\zeta_+\partial\rho^i + \zeta_- G^i, \\ \delta\chi_+^{\bar{\imath}} &= -2\zeta_-\bar{\partial}\bar{\rho}^{\bar{\imath}} - \zeta_+\bar{G}^{\bar{\imath}}, \\ \delta\bar{\chi}_-^{\bar{\imath}} &= -2\bar{\zeta}_+\partial\bar{\rho}^{\bar{\imath}} + \bar{\zeta}_-\bar{G}^{\bar{\imath}}, \\ \delta G^i &= 2i(\zeta_+\tilde{\nabla}_z\bar{\chi}_+^i - \bar{\zeta}_-\tilde{\nabla}_{\bar{z}}\chi_-^i), \\ \delta\bar{G}^{\bar{\imath}} &= 2i(\bar{\zeta}_+\tilde{\nabla}_z\chi_+^{\bar{\imath}} - \zeta_-\tilde{\nabla}_{\bar{z}}\bar{\chi}_-^{\bar{\imath}}), \end{aligned} \tag{3.14}$$

provided that the Killing spinor equations (3.12) are satisfied.

In order to check supersymmetry, it is useful to write down the Killing spinor equations (3.12)

[5] We have absorbed the weight Δ in [53] in field redefinitions; later in this section we shall give an alternative form of the lagrangian in which that weight reappears, in terms of a vector Y.

in chiral components:

$$\begin{aligned} \nabla_z \zeta_- &= 0, & \nabla_{\bar{z}} \zeta_- &= \tfrac{i}{2r}\zeta_+, \\ \nabla_z \zeta_+ &= \tfrac{i}{2r}\zeta_-, & \nabla_{\bar{z}} \zeta_+ &= 0, \end{aligned}$$

$$\begin{aligned} \nabla_z \bar{\zeta}_- &= 0, & \nabla_{\bar{z}} \bar{\zeta}_- &= -\tfrac{i}{2r}\bar{\zeta}_+, \\ \nabla_z \bar{\zeta}_+ &= -\tfrac{i}{2r}\bar{\zeta}_-, & \nabla_{\bar{z}} \bar{\zeta}_+ &= 0. \end{aligned}$$

Note that the supersymmetry transformation of the auxiliary field above is not a total derivative, even though the theory is supersymmetric and contains a superpotential, which is very different from the behavior of ordinary chiral multiplets. We will shortly describe how one can couple a holomorphic Killing vector field, which could be used to make δG^i a total derivative, but unlike the case of ordinary chiral multiplets, it is not necessary in order to couple a superpotential. One suspects that this may be linked to the existence of superfield representations on spheres, but we will not speculate further in that direction.

For $\mathcal{N} = (2,2)$ nonlinear sigma models on $\mathbb{R}^2$, the Lagrangian of twisted chiral multiplets can be obtained by a simple "twist" from the Lagrangian of ordinary chiral multiplets, just by dualizing the tangent bundle TM to T^*M on the right-movers. However, notice that for a nonzero superpotential, the above Lagrangian for twisted chiral fields on S^2 cannot be obtained from the Lagrangian of chiral fields on S^2, given in equation (3.3), in a similar fashion, because of different couplings to the curvature of S^2.

To compare to the lagrangian for twisted chiral multiplets given in [53], one performs a slight field redefinition. Specifically, redefine G^i to be $G^i + (i/r)Y^i$ for Y a vector field. Then, the lagrangian becomes

$$
\begin{aligned}
\mathcal{L}_T \ = & \, g_{i\bar{\jmath}}\partial_m\rho^i\partial^m\bar{\rho}^{\bar{\jmath}} \ + \ 2ig_{i\bar{\jmath}}\bar{\chi}_-^{\bar{\jmath}}\nabla_{\bar{z}}\chi_-^i \ + \ 2ig_{i\bar{\jmath}}\chi_+^{\bar{\jmath}}\nabla_z\bar{\chi}_+^i \ + \ g_{i\bar{\jmath}}\left(G^i + \frac{i}{r}Y^i\right)\left(\bar{G}^{\bar{\jmath}} - \frac{i}{r}Y^{\bar{\jmath}}\right) \\
& - \ \left(G^i + \frac{i}{r}Y^i\right)\left(ig_{i\bar{\jmath},\bar{k}}\bar{\chi}_-^{\bar{\jmath}}\chi_+^{\bar{k}} \ - \ \mathcal{W}_i\right) \ - \ i\mathcal{W}_{ij}\chi_-^i\bar{\chi}_+^j \\
& - \ \left(\bar{G}^{\bar{\imath}} - \frac{i}{r}Y^{\bar{\imath}}\right)\left(ig_{\bar{\imath}j,k}\chi_-^j\bar{\chi}_+^k \ - \ \bar{\mathcal{W}}_{\bar{\imath}}\right) \ - \ i\bar{\mathcal{W}}_{\bar{\imath}\bar{\jmath}}\bar{\chi}_-^{\bar{\imath}}\chi_+^{\bar{\jmath}} \\
& + \ g_{i\bar{\jmath},k\bar{l}}\bar{\chi}_+^i\chi_+^{\bar{\jmath}}\chi_-^k\bar{\chi}_-^{\bar{l}} \\
& + \ \frac{i}{r}\mathcal{W} \ - \ \frac{i}{r}\bar{\mathcal{W}},
\end{aligned}
\tag{3.15}
$$

with, assuming Y^i is chosen holomorphic, supersymmetry transformations

$$
\begin{aligned}
\delta\rho^i &= i\bar{\zeta}_+\chi_-^i + i\zeta_-\bar{\chi}_+^i, \\
\delta\bar{\rho}^{\bar{\imath}} &= i\bar{\zeta}_-\chi_+^{\bar{\imath}} + i\zeta_+\bar{\chi}_-^{\bar{\imath}}, \\
\delta\bar{\chi}_+^i &= -2\bar{\zeta}_-\bar{\partial}\rho^i - \bar{\zeta}_+\left(G^i + \frac{i}{r}Y^i\right), \\
\delta\chi_-^i &= -2\zeta_+\partial\rho^i + \zeta_-\left(G^i + \frac{i}{r}Y^i\right), \\
\delta\chi_+^{\bar{\imath}} &= -2\zeta_-\bar{\partial}\bar{\rho}^{\bar{\imath}} - \zeta_+\left(\bar{G}^{\bar{\imath}} - \frac{i}{r}Y^{\bar{\imath}}\right), \\
\delta\bar{\chi}_-^{\bar{\imath}} &= -2\bar{\zeta}_+\partial\bar{\rho}^{\bar{\imath}} + \bar{\zeta}_-\left(\bar{G}^{\bar{\imath}} - \frac{i}{r}Y^{\bar{\imath}}\right), \\
\delta G^i &= 2i(\zeta_+\tilde{\nabla}_z\bar{\chi}_+^i - \bar{\zeta}_-\tilde{\nabla}_{\bar{z}}\chi_-^i) - \frac{i}{r}\left(i\bar{\zeta}_+\chi_-^j + i\zeta_-\bar{\chi}_+^j\right)\partial_j Y^i, \\
\delta\bar{G}^{\bar{\imath}} &= 2i(\bar{\zeta}_+\tilde{\nabla}_z\chi_+^{\bar{\imath}} - \zeta_-\tilde{\nabla}_{\bar{z}}\bar{\chi}_-^{\bar{\imath}}) + \frac{i}{r}\left(i\bar{\zeta}_-\chi_+^{\bar{\jmath}} + i\zeta_+\bar{\chi}_-^{\bar{\jmath}}\right)\partial_{\bar{\jmath}}Y^{\bar{\imath}},
\end{aligned}
\tag{3.16}
$$

provided that the Killing spinor equations (3.12) are satisfied. To recover [53][equ'ns (4.2), (4.5)] on flat target spaces, take $Y^i = -i\Delta\rho^i$.

3.1.3 Topological twists

When the superpotential vanishes (and, for ordinary chiral multiplets, $X = 0$), there are no curvature couplings, and the theories admit the same topological twists discussed in *e.g.* [62].

When the superpotential is nonzero, this story is more complicated. Sometimes those curvature couplings are incompatible with the topological twist (which can sometimes be alleviated by twisting bosons as in *e.g.* [63, 64]). However, a more fundamental problem is that even if one can consistently twist the theory, the curvature terms induced by the superpotential break the BRST symmetry, in the sense that the action is no longer BRST closed. Those terms are compatible with supersymmetry only so long as the supersymmetry parameters ζ, $\bar{\zeta}$ obey the Killing spinor equation, and serve to 'mop up' the curvature-dependent terms that result from using the Killing spinor equations. Since in a topological field theory the BRST transformations are parametrized by a scalar, the Killing spinor equations are not relevant, and so the curvature-dependent terms in the action are extraneous, breaking the BRST symmetry.

However, just because we cannot always twist, does not imply that topological field theories do not exist. For example, the papers [63, 64] describe examples of two-dimensional topological field theories with nonzero superpotential, on two-spheres and other two-dimensional worldsheets. These topological field theories were *not* obtained by topological twisting of a two-dimensional theory with curvature couplings. Instead, they were obtained by twisting a flat-worldsheet theory. The result continues to make sense on two-spheres and other two-dimensional worldsheets because the BRST transformations are parametrized by a scalar; the action neither needs nor contains curvature-dependent terms.

Thus, to summarize, if the superpotential vanishes (and $X = 0$), then topological twistings exist. If the superpotential is nonzero, there still exist topological field theories, but they are not obtained by twisting an action that includes curvature-dependent terms.

3.2 Dynamical supersymmetry breaking

As is well-known, the Witten index Tr $(-)^F$ (for (0,2) theories, Tr $(-)^{F_R}$) is a measure of the possibility of dynamical supersymmetry breaking: if it vanishes, dynamical supersymmetry breaking is unobstructed.

Any operator that commutes with the fermion number operator can be used to give a refinement of the Witten index, a graded version with the property that vanishing of each separate graded component is a necessary condition for supersymmetry breaking. An example of such a refinement is the elliptic genus, which was utilized in [88] as such a refined Witten index to check for supersymmetry breaking.

Let us quickly review the application of elliptic genera to supersymmetry breaking. For a heterotic nonlinear sigma model describing a space X with holomorphic vector bundle $\mathcal{E}$ satisfying

$$\mathrm{ch}_2(TX) \;=\; \mathrm{ch}_2(\mathcal{E}), \quad c_1(TX) \;\equiv\; \pm c_1(\mathcal{E}) \bmod 2$$

the elliptic genus[6]

$$\mathrm{Tr}_{\mathrm{R,R}}(-)^F q^{L_0} \overline{q}^{\overline{L}_0} \tag{3.17}$$

is well-defined, and given by [93][equ'n (31)]

$$(-)^{r/2} q^{+(1/12)(r-n)} \int_X \hat{A}(TX) \wedge \mathrm{ch}\Bigg((\det \mathcal{E})^{+1/2} \wedge_{-1} (\mathcal{E}^*) \cdot \bigotimes_{k=1,2,3,\cdots} S_{q^k}((TX)^{\mathbb{C}}) \bigotimes_{k=1,2,3,\cdots} \wedge_{-q^k} \left((\mathcal{E})^{\mathbb{C}} \right) \Bigg), \tag{3.18}$$

or equivalently

$$(-)^{r/2} q^{+(1/12)(r-n)} \int_X \hat{A}(TX) \wedge \mathrm{ch}\Bigg((\det \mathcal{E})^{-1/2} \wedge_{-1} (\mathcal{E}) \cdot \bigotimes_{k=1,2,3,\cdots} S_{q^k}((TX)^{\mathbb{C}}) \bigotimes_{k=1,2,3,\cdots} \wedge_{-q^k} \left((\mathcal{E})^{\mathbb{C}} \right) \Bigg), \tag{3.19}$$

[6]This particular elliptic genus is sometimes known as the "Witten genus."

where r is the rank of $\mathcal{E}$, n is the dimension of X, and

$$\begin{aligned} S_q(TX) &= 1 + qTX + q^2\mathrm{Sym}^2(TX) + q^3\mathrm{Sym}^3(TX) + \cdots , \\ \wedge_q(\mathcal{E}) &= 1 + q\mathcal{E} + q^2 \wedge^2 (\mathcal{E}) + q^3 \wedge^3 (\mathcal{E}) + \cdots , \end{aligned}$$

and the $\mathbb{C}$ superscripts indicate complexifications, *i.e.* $\mathcal{E}^{\mathbb{C}} \cong \mathcal{E} \oplus \overline{\mathcal{E}} \cong \mathcal{E} \oplus \mathcal{E}^*$. By using the fact that

$$(-)^F = (-)^{F_R}(-)^{F_L} = (-)^{F_R}(-)^{(J_L)_0}$$

we can see explicitly that the genus above is a refinement of the Witten index for (0,2) supersymmetry. It has been graded via operators (L_0, J_L mod 2) that commute with the right-moving fermion number. In order for (0,2) supersymmetry to hold, a necessary condition is that every graded component of the index above must vanish.

If we have a nonanomalous symmetry, then in principle we can use it to further grade or refine the index above. For example, in the special case that X is Calabi-Yau and the bundle $\mathcal{E}$ has trivial determinant, there is a nonanomalous left $U(1)$ current J_L, and for the corresponding nonlinear sigma model we can define[7]

$$\mathrm{Tr}_{\mathrm{R,R}}(-)^F y^{(J_L)_0} q^{L_0} \overline{q}^{\overline{L}_0}, \tag{3.20}$$

which is given by [96, 97][8]

$$(-)^{r/2} q^{+(1/12)(r-n)} y^{+r/2} \int_X \hat{A}(TX) \wedge \mathrm{ch}\Bigg((\det \mathcal{E})^{+1/2} \wedge_{-1} \left(y^{-1}\mathcal{E}^* \right) \\ \cdot \bigotimes_{k=1,2,3,\cdots} S_{q^k}((TX)^{\mathbb{C}}) \bigotimes_{k=1,2,3,\cdots} \wedge_{-q^k} \left((y\mathcal{E})^{\mathbb{C}} \right) \Bigg), \tag{3.21}$$

[7]Occasionally some references, including this section, will consider elliptic genera with general y and anomalous J_L. For example, in checking dualities we will often compare elliptic genera with general y even if J_L is anomalous, though when we do we will remark on the relevance of more general y. See also [94, 95] for related discussions in different contexts.

[8]The conventions used here differ slightly from those of [97]. To convert, z should be identified with $-y^{-1}$.

or equivalently

$$(-)^{r/2} q^{+(1/12)(r-n)} y^{-r/2} \int_X \hat{A}(TX) \wedge \mathrm{ch}\Bigg((\det \mathcal{E})^{-1/2} \wedge_{-1} (y\mathcal{E}) \cdot \bigotimes_{k=1,2,3,\cdots} S_{q^k}((TX)^{\mathbb{C}}) \bigotimes_{k=1,2,3,\cdots} \wedge_{-q^k} \left((y\mathcal{E})^{\mathbb{C}} \right) \Bigg), \tag{3.22}$$

This reduces to the earlier expressions in the special case that $y = +1$. If only a finite subgroup of the left $U(1)$ above is nonanomalous, then one can make sense of the expressions above for a finite number of values of y. We shall see this in examples later. (See also [98][section 5] for a related discussion of constraints on y.)

Now, for the moment, let us return to the general, non-Calabi-Yau, case, to obtain some quick measures of potential (0,2) supersymmetry breaking from the leading term in the elliptic genus, following the spirit of [96]. Let us first compute the index above on the (2,2) locus where $\mathcal{E} = TX$. Using the fact that $S_q(\mathcal{E}) = \wedge_{-q}(\mathcal{E})^{-1}$ for any vector bundle $\mathcal{E}$, we see that on the (2,2) locus, the Witten genus reduces to

$$\int_X \hat{A}(TX) \wedge \mathrm{ch}\Bigg((\det TX)^{+1/2} \wedge_{-1} (T^*X) \Bigg),$$

which is independent of q. (Since this amounts to a topological field theory partition function on the (2,2) locus, the q-independence is not surprising.) Furthermore, for any bundle $\mathcal{E}$, it is straightforward to show that

$$\mathrm{ch}\left((\det \mathcal{E})^{+1/2} \otimes \wedge_{-1}(\mathcal{E}^*) \right) \; = \; c_r(\mathcal{E}) \; + \; (\text{higher degree}),$$

where $\mathcal{E}$ has rank r, so that on the (2,2) locus,

$$\mathrm{Tr}_{R,R} (-)^{F_R} (-)^{F_L} q^{L_0} \overline{q}^{\overline{L}_0} \; \propto \; \chi(X).$$

Thus, we recover the standard result that on the (2,2) locus, the Witten index is given by the Euler characteristic.

Off the (2,2) locus, the q dependence does not drop out. We can get a quick measure of supersymmetry breaking by examining the first graded component, namely

$$
\begin{aligned}
& (-)^{r/2} q^{+(1/12)(r-n)} \int_X \hat{A}(TX) \wedge \mathrm{ch}\left((\det \mathcal{E})^{+1/2} \wedge_{-1}(\mathcal{E}^*) \right) \\
= \; & (-)^{r/2} q^{+(1/12)(r-n)} \left\{ \begin{array}{cc} 0 & r > n, \\ \int_X c_r(\mathcal{E}) & r = n, \\ \cdots & r < n. \end{array} \right.
\end{aligned} \tag{3.23}
$$

As this is only one graded component of an infinite series, it is merely a rather primitive check of supersymmetry breaking.

For later computational purposes, let us rewrite the expression above in a few more forms. In the special case that $\det \mathcal{E}^* \cong K_X$, so that the theory admits an A/2 twist, we can use the fact that

$$
\hat{A}(TX) \; = \; \mathrm{td}(TX) \exp\left(-\frac{1}{2} c_1(TX) \right)
$$

to write the elliptic genus (3.18) in the form

$$
\begin{aligned}
& (-)^{r/2} q^{+(1/12)(r-n)} \int_X \mathrm{td}(TX) \wedge \mathrm{ch} \Bigg(\wedge_{-1}(\mathcal{E}^*) \\
& \qquad \cdot \bigotimes_{k=1,2,3,\cdots} S_{q^k}((TX)^{\mathbb{C}}) \bigotimes_{k=1,2,3,\cdots} \wedge_{-q^k}\left((\mathcal{E})^{\mathbb{C}} \right) \Bigg),
\end{aligned}
$$

from which we read off the leading term

$$
(-)^{r/2} q^{+(1/12)(r-n)} \int_X \mathrm{td}(TX) \wedge \mathrm{ch}\left(\wedge_{-1}(\mathcal{E}^*) \right) \; = \; (-)^{r/2} q^{+(1/12)(r-n)} \sum_{s=0}^{r} (-)^s \chi\left(\wedge^s \mathcal{E}^* \right), \tag{3.24}
$$

which can be used as a crude test for dynamical supersymmetry breaking.

Alternatively, in the special case that $\det \mathcal{E} \cong K_X$, so that the theory admits a B/2 twist, we can write the elliptic genus (3.19) in the form

$$
\begin{aligned}
& (-)^{r/2} q^{+(1/12)(r-n)} \int_X \mathrm{td}(TX) \wedge \mathrm{ch} \Bigg(\wedge_{-1}(\mathcal{E}) \\
& \qquad \cdot \bigotimes_{k=1,2,3,\cdots} S_{q^k}((TX)^{\mathbb{C}}) \bigotimes_{k=1,2,3,\cdots} \wedge_{-q^k}\left((\mathcal{E})^{\mathbb{C}} \right) \Bigg),
\end{aligned}
$$

from which we read off the leading term

$$(-)^{r/2} q^{+(1/12)(r-n)} \int_X \mathrm{td}(TX) \wedge \mathrm{ch}\left(\wedge_{-1}(\mathcal{E})\right) \; = \; (-)^{r/2} q^{+(1/12)(r-n)} \sum_{s=0}^{r} (-)^s \chi\left(\wedge^s \mathcal{E}\right),$$

which again can be used as a crude test for possible supersymmetry breaking.

As a consistency check, let us apply this in the special case of a deformation of the (2,2) supersymmetric $\mathbb{CP}^n$ model discussed in [88, 99]. This deformation involved decoupling the σ field, resulting in a (0,2) theory which dynamically broke supersymmetry, as could be seen from the one-loop correction to the Fayet-Iliopoulos parameter. (It should be noted that removing the σ field from (2,2) GLSM's has long been known to result in ill-behaved theories [100], so this result is not surprising.) Geometrically, decoupling the σ field corresponds to replacing the tangent bundle of $\mathbb{P}^n$ with an extension, specifically the extension given by the Euler sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \oplus^{n+1} \mathcal{O}(1) \longrightarrow T\mathbb{P}^n \longrightarrow 0$$

as the role of the σ field is to realize the cokernel above. Thus, the new gauge bundle is $\oplus^{n+1}\mathcal{O}(1)$. Since the rank is greater than the dimension of the space, our primitive supersymmetry index above suggests that supersymmetry may be broken, which is consistent with the results of [88, 99].

In this example, the anomalous axial $U(1)$ is well-known to have a nonanomalous $\mathbb{Z}_{n+1}$ subgroup, which suggests that we may be able to form a more refined index by taking y to be an $(n+1)$th root of unity, not necessarily $+1$. As this model admits an A/2 twist, one can repeat earlier analyses to get that for more general y, the elliptic genus should have leading term

$$(-)^{r/2} q^{(r-n)/12} y^{+r/2} \sum_{s=0}^{r} (-y^{-1})^s \chi\left(\wedge^s \mathcal{E}^*\right)$$

for $\mathcal{E} = \oplus^{n+1}\mathcal{O}(1)$. From the Bott formula [101][p. 8]

$$h^q\left(\mathbb{P}^n, \mathcal{O}(k)\right) \;=\; \begin{cases} \begin{pmatrix} n+k \\ k \end{pmatrix} & q = 0, k \geq 0, \\ \begin{pmatrix} -k-1 \\ -k-1-n \end{pmatrix} & q = n, k \leq -n-1, \\ 0 & \text{else,} \end{cases}$$

we have that

$$\chi(\mathcal{O}) \;=\; 1, \;\;\; \chi\left(\wedge^{n+1}\mathcal{E}^*\right) \;=\; (-)^n,$$

and $\chi(\wedge^s\mathcal{E}^*)$ vanishes for $s \neq 0, n+1$. Thus, the leading term in the elliptic genus is

$$(-)^{r/2} q^{(r-n)/12} y^{+r/2}\left(1 \,+\, (-y^{-1})^{n+1}(-)^n\right) \;=\; (-)^{r/2} q^{(r-n)/12} y^{+r/2}\left(1 \,-\, y^{-n-1}\right),$$

which vanishes for y an $(n+1)$th root of unity. Thus, even the refined index is consistent with supersymmetry breaking.

In fact, it is straightforward to show using the methods of [85] that the entire elliptic genus for the (0,2) $\mathbb{CP}^n$ model above, obtained by omitting the σ field, vanishes identically, a stronger sign of supersymmetry breaking. The point is that since there is no superpotential and no corresponding analogue of an R-symmetry, the contribution from each (0,2) chiral multiplet,

$$i\frac{\eta(q)}{\theta_1(q,x)}$$

cancels the contribution from the corresponding (0,2) Fermi multiplet,

$$i\frac{\theta_1(q,x)}{\eta(q)}$$

leaving one with only the contribution from the $U(1)$ gauge multiplet,

$$\frac{2\pi\eta(q)^2}{i},$$

which has no pole and hence no residues.

More generally, it will be shown in [102] that singular loci on the (0,2) moduli space, where in the GLSM E's vanish, often correspond to points where worldsheet supersymmetry is dynamically broken. Such loci correspond to (singular) rank-changing transitions, and so in general terms is consistent with our quick-and-dirty computation above.

Now, let us return to Calabi-Yau's. The nonlinear sigma model for a Calabi-Yau has additional symmetries when $\det \mathcal{E}$ is trivial, namely both J_R and J_L are separately nonanomalous, so the elliptic genus admits a finer grading. Demonstrating supersymmetry breaking, for example, now requires not only vanishing of the separate coefficients of powers of q, but also the vanishing of the separate coefficients of powers of y. The leading contribution to the elliptic genus in this case was computed in [96] (compare also equation (3.22)) to be proportional to

$$q^{+(1/12)(r-n)} y^{-r/2} \int_X \mathrm{td}(TX) \wedge \mathrm{ch}\left(\wedge_{-1}(y\mathcal{E})\right).$$

Reference [96] defined

$$\begin{aligned} \chi_y(\mathcal{E}) &\equiv \int_X \mathrm{td}(TX) \wedge \mathrm{ch}\left(\wedge_{-1}(y\mathcal{E})\right), \\ &= \sum_{i=0}^{r} (-y)^i \chi(\wedge^i \mathcal{E}). \end{aligned}$$

so that the leading term in the elliptic genus is

$$q^{+(1/12)(r-n)} y^{-r/2} \chi_y(\mathcal{E}), \tag{3.25}$$

(In passing, an index of this form was independently suggested, from more abstract considerations of (0,2) analogues of Morse theory and supersymmetry, in [103][section 6.4].)

Results of computations of χ_y can be found in [96]. One result which we shall occasionally use, and so repeat here, is that on a Calabi-Yau 3-fold (so that $n = 3$), when the gauge bundle has $c_1(\mathcal{E}) = 0$,

$$\chi_y(\mathcal{E}) = \begin{cases} 0 & r < 3 \\ -\tilde{\chi}(\mathcal{E}) y (1+y)(1-y)^{r-3} & r \geq 3 \end{cases}$$

for

$$\tilde{\chi}(\mathcal{E}) \equiv \frac{1}{2} \int_X c_3(\mathcal{E}).$$

We shall apply this result explicitly later to double-check computations of elliptic genera.

In passing, note that in the special case $y = +1$, for $r > 3$ this vanishes, in agreement with the general result (3.23). However, for Calabi-Yau's, y is not required to be $+1$, and so vanishing of the elliptic genus is a stronger constraint on Calabi-Yau models than it is for nonlinear sigma models on other Kähler manifolds.

So far, we have only discussed elliptic genera in nonlinear sigma models, whereas the bulk of this section is concerned with GLSM's. However, in weakly-coupled two dimensional theories, we are not missing any information. After all, as gauge fields in two dimensions are not dynamical, in weak coupling regimes it is physically sensible to integrate them out and work with the resulting lower-energy nonlinear sigma model. Any dynamical supersymmetry breaking in such models should happen below the scale at which the nonlinear sigma model description becomes relevant, and necessarily reflects properties of the underlying geometry and heterotic gauge bundle, and not the GLSM gauge field.

We would like to conclude this section with a few comments on dynamical supersymmetry breaking in models associated to Calabi-Yau's. First, note that because the nonlinear sigma model has additional conserved currents, the elliptic genus admits a finer grading, and so, just at the level of the index, a vanishing index requires further constraints than non-Calabi-Yau cases, suggesting that supersymmetry breaking in (0,2) models associated to Calabi-Yau's may be comparatively rare relative to supersymmetry breaking in (0,2) nonlinear sigma models on other Kähler manifolds. This observation is certainly consistent with existing lore in the field.

Furthermore, there is an additional subtlety, namely that even in cases in which the index vanishes, there are indirect reasons to believe that supersymmetry might still not be broken.

Specifically, we are thinking of the old work [104, 105], which argued, essentially by an index computation, that worldsheet instanton effects should destabilize (0,2) theories. A few years after those papers were written, it was discovered in a succession of papers (see *e.g.* [106–109]) that although index computations permit it, when one actually sums up all of the worldsheet instantons in theories derived from GLSM's, the sum vanishes, and the theory is not destabilized. Thus, although it was permitted by an index theory result, no destabilization actually happens. The mathematical reasons for this result are, in our opinion, not especially well-understood, but we mention this as a caution that to convincingly demonstrate supersymmetry breaking in Calabi-Yau models built from GLSM's requires more than just demonstrating that an index vanishes.

3.3 (0,2) Nonabelian GLSM's

Over the last few years we have seen significant advances in our understanding of gauged linear sigma models [4], ranging from GLSM's for different geometries (see *e.g.* [68–72]), new understandings of GLSM phases [**?**, 51, 73–80], through more recent applications of supersymmetric localization [18, 81] to new computations of Gromov-Witten invariants [49, 50, 82] and elliptic genera (see *e.g.* [83–87]), and new dualities (see *e.g.* [88–90]), among many other advances, too numerous to comprehensively list here.

The purpose of this section is to work out some basic aspects of some (0,2) nonabelian GLSM's, which have been studied comparatively rarely. In this section we will be primarily concerned with weak-coupling limits of GLSM's, with clear relations to large-radius geometries[9].

[9]Not all phases of GLSM's flow to nonlinear sigma models; many phases are related to various Landau-Ginzburg models. In this section, however, we are primarily interested in phases of GLSM's which do flow to nonlinear sigma models.

In two dimensions, gauge fields do not have propagating degrees of freedom, which simplifies certain analyses. For many purposes, gauge fields can be treated as Lagrange multipliers and integrated out. When what is left is a weakly coupled nonlinear sigma model, questions about the GLSM can often usefully be turned into questions about geometry. One of our interests in this section lies in applying such ideas to two-dimensional dualities. After all, if one can argue that two different GLSM's RG flow to the same weakly-coupled nonlinear sigma model, then in principle one has shown that they have the same IR limit, establishing a two-dimensional analogue of a Seiberg-like duality. Such IR matching implies matching Higgs moduli spaces, chiral rings, and global symmetries, which in higher dimensions are used as indirect tests[10] of a common RG IR endpoint, rather than as consequences of a known IR matching. We will use such geometric identifications in theories flowing to weakly-coupled nonlinear sigma models to make several predictions for dualities in two-dimensional (2,2) and (0,2) theories, predictions checked by *e.g.* comparing elliptic genera.

Another of our interests lies in understanding string compactifications, in this section including (0,2) versions of Pfaffian constructions, and when bending GLSM's above to such purposes, determining whether the lower-energy nonlinear sigma model has a nontrivial IR fixed point is usually the significant complication. For example, in a heterotic nonlinear sigma model on a Calabi-Yau, if the gauge bundle is not stable, there is not expected to be a nontrivial RG fixed point, a nontrivial SCFT associated to that bundle, but checking stability is extremely complicated, even moreso when working in a UV GLSM. In this section, in discussing Calabi-Yau examples in which existence of a nontrivial fixed point is possible, we will use recent advances to compute central charges as a check for existence of such a fixed point. In addition, we will also discuss the possibility of dynamical supersymmetry breaking, which has recently been discussed in the (0,2) literature.

[10]Such tests should be applied with care; for example, examples were given in [91] of different SCFT's with matching chiral rings.

We begin in section 3.3.1 by describing some basic aspects of (0,2) theories which are utilized later. We begin with a abstract overview of bundles in GLSM's. We also discuss the role of spectators in fixing technical issues with understanding RG flow of Fayet-Iliopoulos parameters.

In section 3.3.2 we discuss some toy (0,2) GLSM's on Grassmannians, as basic examples and warm-ups for later constructions. We relate gauge anomaly cancellation to cohomological conditions on Chern classes in Grassmannians, discuss the details of several examples, and also work through some dualities in these models, concluding with an outline of some tests of those dualities and a discussion of supersymmetry breaking in those toy examples.

In section 3.3.3 we outline some constructions of nonabelian (0,2) theories corresponding to complete intersections in Grassmannians and affine Grassmannians, some dualities that should be obeyed in such constructions, and outline tests of those dualities and supersymmetry breaking, computed via elliptic genera. In section 3.3.4 discuss (0,2) models on Pfaffians.

We then turn to a mathematically-oriented study of dualities in two-dimensional nonabelian GLSM's. The heart of our discussion is the observation that if two weakly-coupled theories are believed to RG flow to nonlinear sigma models on the same space, then by definition, they have the same Higgs moduli space, the same chiral ring, and the same global symmetries, which in four dimensions are typical criteria for identifying dualities.

There are a few known examples of Seiberg-like dualities in two-dimensional (2,2) theories with nonabelian gauge groups. (For abelian gauge groups, there are numerous examples of duality, perhaps most prominently including mirror symmetry, as well as more recent examples such as the (0,2) gerbe dualities in [92].) The prototype for nonabelian examples is encapsulated mathematically in two presentations of the same Grassmannian: the Grassmannian $G(k, n)$ of k-planes in $\mathbb{C}^n$ is the same as the Grassmannian $G(n-k, n)$ of $n-k$ planes in $\mathbb{C}^n$, which becomes a statement relating universality classes of $U(k)$ gauge theories with n chiral superfields in the fundamental representation to $U(n-k)$ gauge theories also with n chiral superfields in the fundamental representation. We discuss how this generalizes mathematically to dualities in theories with both fundamentals and antifundamentals in section 3.3.5, and describe how physical dualities can be understood as relating different presentations of nonlinear sigma models on the same space. Our approach has the advantage that it applies to generic weakly-coupled (2,2) and (0,2) theories, in which flavor symmetries are explicitly broken by choices of superpotentials (holomorphic maps), so *e.g.* 't Hooft anomaly matching is of little utility. We also similarly use geometry to make a prediction for a duality between Grassmannians $G(2,n)$ and certain Pfaffians, realizing a mathematical equivalence.

We then turn to dualities in (0,2) theories. We begin with discussion of a duality between (0,2) theories describing a space X with bundle $\mathcal{E}$, and (0,2) theories describing the same space but with dual bundle $\mathcal{E}^*$, in section 3.3.6. This duality has been considered by others, as we discuss, but is neither well-known nor thoroughly justified; our purpose is both to advertise its existence and give additional justifications.

In section 3.3.7 we then turn to dualities in nonabelian (0,2) theories. Such dualities have only been rarely considered. One recent example was discussed in [88], involving a triality between two-dimensional (0,2) theories with unitary gauge groups and matter in (anti)fundamental representations. We review it from a mathematical perspective in this section.

In section 3.3.8 we return to the study of Pfaffians, and outline how some of the dualities just discussed can illuminate the relationship between the PAX and PAXY constructions of GLSM's for Pfaffians.

Finally in section 3.3.9 we formally consider dualities in open and heterotic strings with more general representations. We describe how dualities should work for open strings, and argue that dualities for more general (0,2) models will often not exist.

In an attempt to make this section reasonably self-contained, we have also included appendices discussing technical aspects of the relation between GLSM's and cohomology, Schur polynomials (used to compute relations between cohomology classes on Grassmannians), and also describe our conventions for representations of $U(k)$ and summarize pertinent properties. Two final appendices give details of elliptic genus computations whose results are summarized and utilized in the main text.

Overall, this section discusses several different dualities:

- A nonabelian/abelian duality, relating the nonabelian GLSM for $G(2,4)$ to the abelian GLSM for $\mathbb{P}^5[2]$ and its (0,2) cousins, in section 3.3.2.
- Another geometric duality, relating $G(2,n)$ and Pfaffian constructions, is discussed in section 3.3.5.
- Generalizations of the $G(k,n) \leftrightarrow G(n-k,n)$ duality relating $U(k)$ and $U(n-k)$ gauge groups are discussed in sections 3.3.5, 3.3.7, 3.3.8, and 3.3.9.
- A nongeometric duality, relating (0,2) theories on spaces X with bundle $\mathcal{E}$ to (0,2) theories on the same space but with dual bundle, is discussed in section 3.3.6.

The first three have an essentially mathematical understanding; part of our point is to apply known mathematics to understand existing dualities between weakly-coupled theories and propose new relationships.

3.3.1 General features of nonabelian (0,2) constructions

Overview of bundles on Grassmannians

Let us briefly define some notation we shall use throughout this section. Briefly, all bundles in a (0,2) GLSM, abelian or nonabelian, are ultimately built from bundles defined by representations of the gauge group. In a GLSM with gauge group $U(1)$, say, all bundles are built as kernels, cokernels, or cohomologies of monads built from bundles defined by $U(1)$ charges. Nonabelian GLSM's are very similar.

In this section, Grassmannians will form an important prototype for many constructions, so let us specialize to that case. A (2,2) GLSM for a Grassmannian $G(k,n)$ of k-planes in $\mathbb{C}^n$ is built as a $U(k)$ gauge theory with n fundamentals [110].

Given a representation ρ of $U(k)$, we will let $\mathcal{O}(\rho)$ denote the corresponding bundle on a Grassmannian. (We will use the same notation in related contexts, such as for affine Grassmannians.) In the special case of a $U(1)$ gauge theory, a representation is defined by a set of charges, so the description above specializes to give *e.g.* line bundles of the form $\mathcal{O}(n)$ on projective spaces.

In principle, not every bundle on a Grassmannian is of the form $\mathcal{O}(\rho)$ for ρ a representation of $U(k)$, just as not every bundle on a projective spaces is a line bundle. Instead, bundles of the form $\mathcal{O}(\rho)$ define a subset of a special class of bundles, known as homogeneous bundles. A homogeneous bundle is defined by a representation of $U(k) \times U(n-k)$; bundles defined solely by representations of $U(k)$ form what we shall sometimes call special homogeneous bundles.

Some simple examples are provided by the universal subbundle S and universal quotient bundle Q on $G(k,n)$. S is rank k, Q is rank $n-k$, and they are related by the short exact sequence

$$0 \longrightarrow S \longrightarrow \mathcal{O}^n \longrightarrow Q \longrightarrow 0. \tag{3.26}$$

On a projective space, $S = \mathcal{O}(-1)$, and $Q = T \otimes \mathcal{O}(-1)$, where T denotes the tangent bundle. In our notation above, $S = \mathcal{O}(\overline{\mathbf{k}})$, *i.e.* S is a special homogeneous bundle defined by the antifundamental representation of $U(k)$. The universal quotient bundle is homogeneous but not special homogeneous; it is defined by the antifundamental representation of $U(n-k)$.

Bundles associated to more general representations of $U(k)$ can be built by expressing the representation as a sum or tensor product of copies of the antifundamental and its dual, and then summing or tensoring together copies of S in the same fashion. For example:

$$\mathcal{O}\left(\overline{\mathbf{k}} \otimes \overline{\mathbf{k}}\right) \;=\; S \otimes S, \;\; \mathcal{O}\left(\overline{\mathbf{k}} \otimes \mathbf{k}\right) \;=\; S \otimes S^*, \;\; \mathcal{O}\left(\mathbf{k} \oplus \mathbf{k}\right) \;=\; S^* \oplus S^*, \;\; \mathcal{O}\left(\mathrm{Sym}^n \mathbf{k}\right) \;=\; \mathrm{Sym}^n S^*,$$

and so forth.

The prototype for many dualities in $U(k)$ gauge theories in two dimensions is defined by the relationship $G(k,n) = G(n-k,n)$: a $U(k)$ gauge theory with n fundamental chirals is in the same universality class as a $U(n-k)$ gauge theory with n fundamental chirals. An observation that will be key for many of our later observations is that under the interchange above,

$$\left(S \;\longrightarrow\; G(k,n)\right) \;=\; \left(Q^* \;\longrightarrow\; G(n-k,n)\right),$$

i.e. the interchange $G(k,n) \leftrightarrow G(n-k,n)$ also exchanges the universal subbundle S with the dual of the universal quotient bundle Q. Although Q is not special homogeneous, it is related to S via the three-term exact sequence (3.26), and so Q can be constructed indirectly, as we shall see in examples.

Weak coupling limits and spectators

In two-dimensional theories at low energies, the strength of the coupling is effectively determined by the Fayet-Iliopoulos parameter, which is additively renormalized at one-loop, by the sums of the charges of the bosons:

$$\Delta r_{1-\mathrm{loop}} \;\propto\; \sum_{\mathrm{bosons}} Q_i.$$

In a conventional (2,2) GLSM, it is well-known that vanishing of this renormalization is equivalent to the Calabi-Yau condition, and furthermore the signs are such that positively-curved spaces shrink under RG flow, and negatively-curved spaces expand, precisely as one would expect.

In a (0,2) GLSM describing a Calabi-Yau, it is often the case that the sums of the charges of the bosonic chiral superfields is nonzero. However, as observed in [111], that does not imply that the Fayet-Iliopoulos parameter necessarily runs: one can add 'spectators' to the theory to cancel out charge sums. Let us briefly review how this works in abelian GLSM's. Let Q_α denote the sums of the charges of the bosonic chiral superfields with respect to the αth $U(1)$. Then, we add two fields to the theory, a bosonic chiral superfield X of $U(1)$ charges $-Q_\alpha$ and a Fermi superfield Ω of $U(1)$ charges $+Q_\alpha$, together with a (0,2) superpotential

$$W = m_s \Omega X,$$

where m_s is a constant, defining the mass of the spectators. Thanks to the addition of X, the sum of the $U(1)$ charges of the bosonic chiral superfields now vanishes, so that the Fayet-Iliopoulos parameter is not renormalized. As we have added matching chiral and Fermi superfields, anomaly matching is unaffected, and since the superpotential effectively makes both X and Ω massive, of mass m_s, they do not contribute to the IR behavior of the theory.

Thus, after adding spectators, at scales $\Lambda > m_s$ the Fayet-Iliopoulos parameter becomes an RG invariant, and so it can be tuned to any desired value, such as a weak coupling limit in which geometric descriptions are valid. Below the scale m_s, if the theory is sufficiently close to a nonlinear sigma model on a Calabi-Yau, the rest of the RG flow should typically be determined by the mathematical properties of the theory.

The discussion above was outlined for abelian cases; however, we can also follow exactly the same procedure in nonabelian (0,2) GLSM's formally associated to Calabi-Yau geometries. For every $U(1)$ factor in the gauge group, there is a Fayet-Iliopoulos parameter, and one can apply the same procedure above to add spectators to understand weak coupling limits.

In (0,2) GLSM's formally describing spaces which are not Calabi-Yau, the sum of the boson charges no longer matches the sum of the fermion charges. We can again add spectators to cancel the sum of the boson charges, which has the effect of cancelling the one-loop renormalization of Fayet-Iliopoulos parameters at scales above m_s. However, it is less clear how the theory behaves at scales below m_s, after the spectators have been integrated out. Even if one used a small m_s to tune the theory to a weak coupling regime, below the scale set by m_s the sigma model coupling would surely begin running again.

In this section we are primarily concerned with understanding geometric interpretations in weak-coupling regimes. Therefore, implicitly we will add spectators as needed.

3.3.2 Examples on ordinary Grassmannians

Two-dimensional (2,2) GLSM's for Grassmannians have been discussed in [74, 110], and for flag manifolds in [76]. Briefly, the Grassmannian $G(k, n)$ is constructed via a $U(k)$ gauge theory with n chiral superfields in the fundamental representation.

Two-dimensional (0,2) theories describing bundles on $G(k, n)$ can be built from $U(k)$ gauge theories with n (0,2) chiral superfields in the fundamental and suitable matter to describe the gauge bundle. These form the prototype for other constructions: understanding (0,2) Grassmannian constructions is essential to understand (0,2) Pfaffian constructions, for example, and will also be important in our analysis of dualities.

In this section, we will outline some general aspects of (0,2) GLSM's and their relation to cohomology and bundles on Grassmannians, as simple toy models to illustrate various phenomena.

Anomaly cancellation and Chern classes

We shall begin by considering anomaly cancellation in nonabelian (0,2) models, and its relation to cohomology of the underlying space. In two-dimensional gauge theories, anomaly cancellation requires, schematically,

$$\sum_{R_{\mathrm{left}}} \mathrm{tr}(T^a T^b) = \sum_{R_{\mathrm{right}}} \mathrm{tr}(T^a T^b). \tag{3.27}$$

More concretely, in terms of the Casimirs discussed in appendix D, we have the following conditions:

$$\sum_{R_{\mathrm{left}}} \dim(R_{\mathrm{left}})\mathrm{Cas}_2(R_{\mathrm{left}}) = \sum_{R_{\mathrm{right}}} \dim(R_{\mathrm{right}})\mathrm{Cas}_2(R_{\mathrm{right}}), \tag{3.28}$$

$$\sum_{R_{\mathrm{left}}} \dim(R_{\mathrm{left}})(\mathrm{Cas}_1(R_{\mathrm{left}}))^2 = \sum_{R_{\mathrm{right}}} \dim(R_{\mathrm{right}})(\mathrm{Cas}_1(R_{\mathrm{right}}))^2. \tag{3.29}$$

(The first condition is the $u(k)^2$ gauge anomaly condition, the second the $u(1)^2$ condition; there is no $u(1) - su(k)$ condition, as elements of the Lie algebra of $su(k)$ are traceless.) Note for $SU(n)$ gauge theories, the second condition is automatically satisfied, because of the fact that $\mathrm{Cas}_1(R) = 0$ for any representation R of $SU(n)$.

For example, consider a (0,2) GLSM with right-moving chiral superfields Φ, P, left-moving fermi superfields Λ, Γ, and (left-moving) gauginos. The gauge anomaly cancellation conditions are given by

$$\begin{aligned}
\sum_{R_\Lambda} \dim(R_\Lambda)\mathrm{Cas}_2(R_\Lambda) &+ \dim(\mathrm{adj})\mathrm{Cas}_2(\mathrm{adj}) \\
&= \sum_{R_\Phi} \dim(R_\Phi)\mathrm{Cas}_2(R_\Phi) + \sum_{R_P} \dim(R_P)\mathrm{Cas}_2(R_P), \\
\sum_{R_\Lambda} \dim(R_\Lambda)(\mathrm{Cas}_1(R_\Lambda))^2 &+ \dim(\mathrm{adj})(\mathrm{Cas}_1(\mathrm{adj}))^2 \\
&= \sum_{R_\Phi} \dim(R_\Phi)(\mathrm{Cas}_1(R_\Phi))^2 + \sum_{R_P} \dim(R_P)(\mathrm{Cas}_1(R_P))^2.
\end{aligned} \tag{3.30}$$

In principle, anomaly cancellation in the UV GLSM implies

$$\mathrm{ch}_2(E) = \mathrm{ch}_2(TX) \tag{3.31}$$

in the IR NLSM on the space X, and in general is slightly stronger than the IR condition (see for example [112][section 6.5] for examples of anomalous GLSM's associated mathematically to anomaly-free IR geometries).

In appendix B we review the cohomology of the Grassmannian $G(k,n)$ of k-planes in $\mathbb{C}^n$. Briefly,

$$H^2(G(k,n),\mathbb{Z}) \;=\; \mathbb{Z}, \quad H^4(G(k,n),\mathbb{Z}) \;=\; \mathbb{Z}^2.$$

The generators of the cohomology are given by Schubert cycles which are defined by certain Young diagrams. For example, we use $\sigma_{\square}$ to denote the generator of H^2, and each of $\sigma_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$, $\sigma_{\square\square}$, $\sigma_{\square}^2$ describe elements of H^4, related by

$$\sigma_{\square}^2 \;=\; \sigma_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \;+\; \sigma_{\square\square}.$$

Furthermore, as discussed in appendix D, the Chern classes are determined by the Casimirs: for any given representation λ,

$$c_1(\mathcal{O}(\lambda)) \;=\; \frac{d_\lambda \mathrm{Cas}_1(\lambda)}{k}\sigma_{\square}, \tag{3.32}$$

$$\begin{aligned} \mathrm{ch}_2(\mathcal{O}(\lambda)) \;&=\; (1/2)c_1(\mathcal{O}(\lambda))^2 \;-\; c_2(\mathcal{O}(\lambda)), \\ &=\; d_\lambda \mathrm{Cas}_2(\lambda)\left[-\frac{1}{k^2-1}\sigma_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \;+\; \frac{1}{2k(k+1)}\sigma_{\square}^2\right] \\ &\qquad + d_\lambda \mathrm{Cas}_1(\lambda)^2\left[\frac{1}{k(k^2-1)}\sigma_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \;+\; \frac{1}{2k(k+1)}\sigma_{\square}^2\right], \end{aligned} \tag{3.33}$$

where d_λ is the dimension of representation λ.

Let us apply this to heterotic geometries, and check that the Casimir conditions above imply the mathematical matching of Chern classes and characters. Specifically, consider a bundle

$\mathcal{E}$ defined by the kernel

$$0 \longrightarrow \mathcal{E} \longrightarrow \oplus_i \mathcal{O}(\lambda_i) \longrightarrow \oplus_a \mathcal{O}(\lambda_a) \longrightarrow 0. \tag{3.34}$$

This is defined by a set of Fermi superfields Λ in the representations λ_i, chiral superfields P in representations dual to λ_a, and a (0,2) superpotential encoding the second nontrivial map. Mathematically, using the additivity properties of Chern characters, we have

$$\begin{aligned} \mathrm{ch}_2(\mathcal{E}) &= \mathrm{ch}_2\left(\oplus_i \mathcal{O}(\lambda_i)\right) - \mathrm{ch}_2\left(\oplus_a \mathcal{O}(\lambda_a)\right), \\ &= \sum_i \mathrm{ch}_2\left(\mathcal{O}(\lambda_i)\right) - \sum_a \mathrm{ch}_2\left(\mathcal{O}(\lambda_a)\right). \end{aligned}$$

The tangent bundle of the Grassmannian $G(k,n)$ is defined as the cokernel

$$0 \longrightarrow S^* \otimes S \longrightarrow S^* \otimes \mathcal{O}^n \longrightarrow S^* \otimes Q = TG(k,n) \longrightarrow 0,$$

and so

$$\begin{aligned} \mathrm{ch}_2(TG(k,n)) &= \mathrm{ch}_2\left(S^* \otimes \mathcal{O}^n\right) - \mathrm{ch}_2\left(S^* \otimes S\right), \\ &= n\,\mathrm{ch}_2\left(S^*\right) - \mathrm{ch}_2\left(S^* \otimes S\right). \end{aligned}$$

The anomaly-cancellation condition is given by

$$\mathrm{ch}_2(TG(k,n)) = \mathrm{ch}_2(\mathcal{E}),$$

which is equivalent to

$$\sum_i \mathrm{ch}_2\left(\mathcal{O}(\lambda_i)\right) + \mathrm{ch}_2\left(S^* \otimes S\right) = \sum_a \mathrm{ch}_2\left(\mathcal{O}(\lambda_a)\right) + n\,\mathrm{ch}_2\left(S^*\right).$$

Writing ch_2 in terms of Casimirs as in equation (3.33) above, we see that the mathematical anomaly-cancellation condition above is satisfied if and only if the physical gauge anomaly constraints (3.30) are satisfied, as expected.

Now, let us turn to the A/2 pseudo-topological field theory. As discussed in [112, 113], for a gauge bundle $\mathcal{E}$ over a space X, in addition to the anomaly-cancellation condition one must also impose the constraint

$$\wedge^{\rm top}\mathcal{E}^* \;\cong\; K_X,$$

which implies $c_1(\mathcal{E}) = c_1(TX)$. For the gauge bundle defined by (3.34) over $X = G(k,n)$, this constraint becomes

$$c_1\left(\oplus_i \mathcal{O}(\lambda_i)\right) \;-\; c_1\left(\oplus_a \mathcal{O}(\lambda)a)\right) \;=\; c_1\left(S^* \otimes \mathcal{O}^n\right) \;-\; c_1\left(S^* \otimes S\right),$$

which can easily be checked to be equivalent to the statement

$$\sum_{R_\Lambda} \dim(R_\Lambda){\rm Cas}_1(R_\Lambda) + \dim({\rm adj}){\rm Cas}_1({\rm adj}) = \sum_{R_\Phi} \dim(R_\Phi){\rm Cas}_1(R_\Phi) + \sum_{R_P} \dim(R_P){\rm Cas}_1(R_P),$$

or more simply,

$$\sum_{R_{\rm left}} \dim(R_{\rm left}){\rm Cas}_1(R_{\rm left}) \;=\; \sum_{R_{\rm right}} \dim(R_{\rm right}){\rm Cas}_1(R_{\rm right}).$$

Examples

In table 3.1 we list examples of bundles $\mathcal{E}$ of the form

$$0 \;\longrightarrow\; \mathcal{E} \;\longrightarrow\; \oplus^m \mathcal{O}(\lambda_{A1}, \lambda_{B1}) \;\longrightarrow\; \oplus^n \mathcal{O}(\lambda_{A2}, \lambda_{B2}) \;\longrightarrow\; 0$$

on $G(2,4)$, satisfying anomaly cancellation. For simplicity we have chosen to focus on bundles defined by kernels; however, nonabelian (0,2) GLSM's can also be used to describe cokernels and cohomologies of monads. As those constructions are simple generalizations, we omit their discussion.

In table 3.1 we have used the notation $\mathcal{O}(\lambda_1, \lambda_2)$ to indicate a vector bundle on $G(2,4)$ defined by the (λ_1, λ_2) representation of $U(2)$ $(\lambda_1 \geq \lambda_2)$. See appendix D for our conventions.

Looking at the D-term constraints in these theories, we see a potential issue that may sometimes arise[11]. Schematically, if we let X's denote the chiral superfields defining $G(2,4)$

[11] The issue presented here is more subtle for Grassmannians, as the FI parameter will run, but the same

m	$(\lambda_{A1}, \lambda_{B1})$	n	$(\lambda_{A2}, \lambda_{B2})$	rank
5	(-2, -2)	2	(3, 3)	3
3	(-1, -1)	1	(1, 1)	2
4	(0, -1)	1	(1, -1)	5
2	(1, -2)	1	(2, -2)	3
5	(2, 2)	2	(3, 3)	3

Table 3.1: Anomaly-free examples on $G(2,4)$.

and P's denote the chiral superfields corresponding to the third terms in the short exact sequence defining the gauge bundle $\mathcal{E}$, then they are schematically of the form

$$XX^\dagger \; - \; P^\dagger P \; = \; r_{U(2)} I.$$

If the $P^\dagger P$ term is negative-definite, then for $r \gg 0$, we get that the X's are not all zero, and so we have a Grassmannian as usual. If the $P^\dagger P$ term does not have that property, then the D term implies a weaker condition, and so it is no longer clear that the geometry described, semiclassically, is a Grassmannian. A closely related issue also arises in abelian GLSM's: the total space of $\mathcal{O}(-1) \to \mathbb{P}^n$ is easy to describe with a collection of $n+1$ chiral superfields of charge $+1$ and one of charge -1, but the total space of $\mathcal{O}(+1) \to \mathbb{P}^n$ cannot be similarly described in GLSM's, as adding an extra chiral superfield of charge $+1$ would merely increase the size of the projective space. We will largely ignore this potential problem for the time being, but it will crop up occasionally in our discussion.

Example 3.1. Let us describe the maps and superpotentials in these nonabelian (0,2) models. In the first entry in table 3.1, we have a map $\mathcal{O}(-2,-2)^5 \to \mathcal{O}(3,3)^2$. The elements of this

issue will apply more generally to other, Calabi-Yau cases, so we present it here as a prototype for later discussions.

map are provided by sections of

$$\mathcal{O}(5,5) \;=\; (\det S^*)^5 \,.$$

A section of $\det S^*$ is a baryon constructed from the chiral superfields defining the Grassmannian $G(2,4)$, *i.e.* an operator of the form

$$B_{ij} \;\equiv\; \epsilon_{ab}\phi_i^a\phi_j^b,$$

where in this case $i,j \in \{1,\cdots,4\}$. Therefore, the maps in the bundle in the first entry in table 3.1 are provided by degree five polynomials in the B^{ij}, and the (0,2) superpotential is then of the form

$$W \;=\; \Lambda_\alpha p_\gamma f_5^{\alpha\gamma}(B_{ij}),$$

where Λ's are Fermi superfields in representation $(-2,-2)$, p's are chiral superfields in the representation dual to $(3,3)$, and f_5 is a degree five polynomial.

The second and fifth entries in table 3.1 are very similar: the maps are between powers of $\det S^*$, and so are polynomials in the A^{ij}, of degree 2 in the second entry and of degree 1 in the fifth entry.

Example 3.2. The third entry in the table is more interesting. The bundles

$$\mathcal{O}(0,-1) \;=\; S \;=\; (\det S^*)^{-1} \otimes S^*,$$

$$\mathcal{O}(1,-1) \;=\; (\det S^*)^{-1} \otimes \mathcal{O}(2,0) \;=\; (\det S^*)^{-1} \otimes \mathrm{Sym}^2 S^* \;=\; (\det S^*)^{-1} \otimes K_{\square\square} S^*,$$

(where for any Young diagram T we use $K_T S^*$ to indicate a tensor product of copies of S^* built in the fashion indicated by T), so we need to describe explicitly maps

$$S \;=\; (\det S^*)^{-1} S^* \;\longrightarrow\; (\det S^*)^{-1}\mathrm{Sym}^2 S^*.$$

In principle, if Λ^a couples to S, then such maps are of the form

$$\Lambda^a\phi_i^b \;+\; \Lambda^b\phi_i^a,$$

where the ϕ_i^a are sections of S^* corresponding to the chiral superfields used to describe the underlying Grassmannian. If we let p_{ab} denote the chiral superfield in the representation dual to $(1,-1)$, then the (0,2) superpotential for this case is of the form

$$W \;=\; \left(\Lambda_n^a \phi_i^b \;+\; \Lambda_n^b \phi_i^a\right) f^{in} p_{ab},$$

where the f^{in} are constants.

Example 3.3. The fourth entry is also nontrivial. Here the pertinent bundles are

$$\mathcal{O}(1,-2) \;=\; (\det S^*)^{-2} \otimes \mathcal{O}(3,0) \;=\; (\det S^*)^{-2} \otimes \mathrm{Sym}^3 S^* \;=\; (\det S^*)^{-2} \otimes K_{\square\square\square} S^*,$$

$$\mathcal{O}(2,-2) \;=\; (\det S^*)^{-2} \otimes \mathcal{O}(4,0) \;=\; (\det S^*)^{-2} \otimes \mathrm{Sym}^4 S^* \;=\; (\det S^*)^{-2} \otimes K_{\square\square\square\square} S^*,$$

so we need to describe explicitly maps

$$(\det S^*)^{-2} \otimes \mathrm{Sym}^3 S^* \;\longrightarrow\; (\det S^*)^{-2} \otimes \mathrm{Sym}^4 S^*.$$

We can build such maps in much the same form as for the third entry. If Λ^{abc} couples to $(\det S^*)^{-2} \otimes \mathrm{Sym}^3 S^*$, then the needed map is of the form

$$\Lambda^{abc} \phi_i^d \;+\; (\text{symmetric permutations of } a, b, c, d),$$

and so the (0,2) superpotential for this model is of the form

$$W \;=\; \left(\Lambda_n^{abc} \phi_i^d \;+\; (\text{perm's})\right) f^{ni} p_{abcd},$$

where p_{abcd} is a chiral superfield in the representation dual to $(2,-2)$, and f^{ni} are constants as before.

Abelian/nonabelian duality to projective space

In four dimensions, a Seiberg-like duality between an abelian and a nonabelian gauge theory seems impossible, as only one of the two could be asymptotically free. In two dimensions, however, since the gauge field does not describe a propagating degree of freedom, more exotic possibilities exist, including dualities between abelian and nonabelian gauge theories.

One such example was discussed implicitly in [85][section 4.6], as part of their discussion of the duality between GLSM's for the Grassmannians $G(k,n)$ and $G(n-k,n)$. In the special case $k=1$, this relates $G(1,n)=\mathbb{P}^{n-1}$, described by an abelian gauge theory, to $G(n-1,n)$, described by a $U(n-1)$ gauge theory. Elliptic genera of these two theories were compared in [85][section 4.6], as were elliptic genera for more general values of k, and found to match exactly as one would expect. (Note that (0,2) dualities built on the equivalence $G(k,n)=G(n-k,n)$ will be described in section 3.3.7.)

We propose that additional dualities of analogous forms should also exist. For example, the Grassmannian $G(2,4)$ has the unusual property[12] that it is the same as a quadric hypersurface in $\mathbb{P}^5$ (see for example [76] and references therein), which lends itself to a natural proposal for another duality between (2,2) supersymmetric abelian and nonabelian gauge theories, a duality between the GLSM's for these two presentations of the same space. In weak coupling regimes, because the geometries described are identical, one immediately, trivially, has a matching between Higgs phases, chiral rings, and global symmetries, which in four dimensions would typically be sufficient to demonstrate the existence of the duality. However, to be thorough, in appendix E, we also check that global symmetries and elliptic genera match, consistent with the proposed duality.

Now, we would also like to use similar mathematical ideas to make predictions for dualities between (0,2) theories describing gauge bundles on the spaces above, and to do so, we need to relate bundles on these dual mathematical descriptions. For example, the universal subbundle and quotient bundle on $G(2,4)$ correspond to the two spinor bundles on $\mathbb{P}^5[2]$ [116, 117]. To systematically compare (0,2) GLSM's on $G(2,4)$ to (0,2) GLSM's on $\mathbb{P}^5[2]$, the essential ingredient is to compare the restriction of $\mathcal{O}(1)\to\mathbb{P}^5$ to the hypersurface, to a

[12] We will discuss generalizations of this duality in section 3.3.5.

bundle on $G(2,4)$. Now, sections of the restriction of $\mathcal{O}(1)$ are just homogeneous coordinates, *i.e.*

$$B_{ij} \; = \; \phi_i^a \phi_j^b \epsilon_{ab} \text{ on } G(2,4).$$

The homogeneous coordinates above are sections of $\det S^*$ on $G(2,4)$, so the restriction of $\mathcal{O}(1)$ to $\mathbb{P}^5[2]$ is equivalent to the line bundle $\det S^*$ on $G(2,4)$. As a consistency check, note that both have $c_1 = 1$. Given this dictionary, from a (0,2) model on $\mathbb{P}^5[2]$, in principle one could build (0,2) models on $G(2,4)$. The converse, building (0,2) GLSM's for $\mathbb{P}^5[2]$ from those for $G(2,4)$, could in principle be done using the fact that the universal subbundle on $G(2,4)$ maps to a spinor bundle, and the (0,2) GLSM on $G(2,4)$ will build bundles from tensor products, duals, and so forth of the universal subbundle.

If two weakly-coupled (0,2) GLSM's describe the same geometry and gauge bundle, then as before, Higgs phases, chiral rings, and global symmetries all immediately match, which is the reason we claim a duality.

Let us work through a few specific examples. Consider the first entry in table 3.1. This describes the gauge bundle

$$0 \longrightarrow \mathcal{E} \longrightarrow \oplus^5 (\det S^*)^{-2} \longrightarrow \oplus^2 (\det S^*)^3 \longrightarrow 0$$

on $G(2,4)$, which from our analysis above is the same as the (0,2) GLSM on $\mathbb{P}^5[2]$ with gauge bundle described as

$$0 \longrightarrow \mathcal{E} \longrightarrow \oplus^5 \mathcal{O}(-2) \longrightarrow \oplus^2 \mathcal{O}(3) \longrightarrow 0.$$

The abelian (0,2) model on $\mathbb{P}^5[2]$ is anomaly-free, just as its dual on $G(2,4)$. The map $\mathcal{O}(-2)^5 \to \mathcal{O}(3)^2$ is defined by homogeneous polynomials of degree 5, just as in the analysis of the bundle on $G(2,4)$. In a little more detail, we can identify the six baryons B_{ij} on $G(2,4)$ with homogeneous coordinates z_{ij} on $\mathbb{P}^5$, and thereby build maps on $\mathbb{P}^5[2]$ from maps on $G(2,4)$. For example,

$$B_{12} B_{13} (B_{24})^3 \; \mapsto \; z_{12} z_{13} (z_{24})^3.$$

That said, the baryons B^{ij} satisfy some additional consistency conditions, more than just homogeneous coordinates, which are encoded in the quadric hypersurface condition. In this fashion, we can construct a (0,2) GLSM on $\mathbb{P}^5[2]$ from the first entry in table 3.1.

Conversely, given a homogeneous polynomial p on $\mathbb{P}^5$ of degree n, we can construct a section of $(\det S^*)^n$ on $G(2,4)$, by mapping $z_{ij} \mapsto B_{ij}$. Some of the terms will drop out after making the identification, because the B_{ij}'s satisfy an algebraic equation encoded in the quadric hypersurface. Put another way, if the relation between the B_{ij}'s is encoded in a quadric q, then to find the remainder after mapping $z_{ij} \mapsto B_{ij}$ we divide:

$$p \;=\; mq \;+\; r,$$

where m is some degree $n-2$ polynomial and r is a degree n polynomial. After mapping to $G(2,4)$, the factor mq vanishes automatically, since by definition $q(B_{ij}) = 0$, leaving one just with the homogeneous polynomial $r = r(B_{ij})$.

The second and fifth entries in table 3.1 are very similar. The second entry corresponds to the gauge bundle

$$0 \longrightarrow \mathcal{E} \longrightarrow \oplus^3\mathcal{O}(-1) \longrightarrow \mathcal{O}(1) \longrightarrow 0$$

on $\mathbb{P}^5[2]$, and the fifth entry corresponds to the gauge bundle

$$0 \longrightarrow \mathcal{E} \longrightarrow \oplus^5\mathcal{O}(2) \longrightarrow \oplus^2\mathcal{O}(3) \longrightarrow 0$$

on $\mathbb{P}^5[2]$. Both of these define anomaly-free abelian (0,2) gauge theories.

We have outlined above how to convert (0,2) GLSM's between $G(2,4)$ and $\mathbb{P}^5[2]$, implicitly using the fact that the GLSM for $G(2,4)$ is built from special homogeneous bundles, *i.e.* bundles defined by $U(2)$ representations, whereas even a general homogeneous bundle would require a $U(2)\times U(2)$ representation, and analogous properties of (0,2) GLSM's for complete intersections in projective spaces. To map a general bundle, one not expressed in terms of a three-term sequence in which the other terms are of the form above, would in principle be more complicated.

Examples of this latter form are provided by the third and fourth entries in table 3.1. Here, we are not aware of a three-term sequence describing symmetric powers of the spinor bundle on $\mathbb{P}^5[2]$, hence we do not understand how to map those (0,2) GLSM's on $G(2,4)$ to (0,2) GLSM's on $\mathbb{P}^5[2]$.

Supersymmetry breaking and checks of dualities

The purpose of this section has been to give basic toy examples to illustrate features of the technology of nonabelian (0,2) GLSM's, not to give viable compactification candidates. Nevertheless, for completeness, in this subsection we will check both dualities and supersymmetry breaking in the examples in table 3.1, by computing elliptic genera. In particular, we will see the following interesting results:

- Although the left $U(1)$ is anomalous, we will formally compute elliptic genera for all y, and will discover that for general y, elliptic genera of duals match. In principle, as only $y = +1$ is physically meaningful, such a matching is not necessary. We are not currently sure how to interpret this. Perhaps, for example, the methods we use implicitly make a gauge choice, and the same gauge choice is being applied to both genera in each pair. In any event, it is an intriguing test of duality.
- At $y = +1$, we will see evidence that the elliptic genera all vanish, and in particular, both genera of dual pairs vanish, suggesting that supersymmetry is broken, and is broken for both elements of the pair. This is consistent with our expectations: at weak coupling in two dimensions, since the gauge field is not dynamical, whether supersymmetry breaks should be a function of the low-energy nonlinear sigma model, independent of the details of the presentation of the UV GLSM.

First, a general observation on the entries in that table. The second, third, and fourth entries obey

$$\det \mathcal{E} \;\cong\; K_X$$

and so, for example, admit B/2 twists. The fifth entry obeys

$$\det \mathcal{E}^* \;\cong\; K_X$$

and so, for example, admits an A/2 twist. The first entry obeys neither condition, but does satisfy $c_1(\mathcal{E}) = c_1(TX)$ mod 2, hence we can at least define and compute elliptic genera.

Of these examples, the first, second, and fifth entries in table 3.1 admit abelian duals, so we will focus on those.

Example 3.4. Following the methods in [85] and appendix F, the elliptic genus for the first entry in table 3.1 is a residue of

$$-2\pi^2 i\eta(q)^7 \frac{\theta_1(q, x_1 x_2^{-1})\theta_1(q, x_1^{-1} x_2)\theta_1(q, y x_1^{-2} x_2^{-2})}{\theta_1(q, x_1)^4 \theta_1(q, x_2)^4 \theta_1(q, y^{-1} x_1^{-3} x_2^{-3})}.$$

The elliptic genus of the abelian dual is a residue of

$$2\pi i\eta(q)^4 \frac{\theta_1(q, x^{-2})\theta_1(q, y x^{-2})^5}{\theta_1(q, x)^6 \theta_1(q, y^{-1} x^{-3})^2}.$$

The first few terms in power series in q for both of these elliptic genera match, and are given by

$$\begin{aligned} & q^{-1/12}\frac{1}{y^{3/2}(y-1)}\left(336 + 1559y + 2460y^2 + 1559y^3 + 336y^4\right) \\ &\quad + q^{11/12}\frac{1}{y^{5/2}(y-1)}\left(-4025 - 7137y + 7157y^2 + 20510y^3 + 7157y^4 - 7137y^5 - 4025y^6\right) \\ &\quad + q^{23/12}\frac{1}{y^{7/2}(y-1)}\big(15203 - 23272y - 91869y^2 + 31081y^3 + 168964y^4 + 31081y^5 \\ &\qquad\qquad\qquad\qquad -91869y^6 - 23272y^7 + 15203y^8\big) \;+\; \mathcal{O}\left(q^{35/12}\right). \end{aligned}$$

Although only the special case $y = +1$ is physically meaningful, it is at least an intriguing test of dualities that these series match for more general y, something that we will also see in the other examples in this subsection. This might reflect a gauge choice implicit in [85], which matches for both computations. We leave the precise understanding of this matching for anomalous cases for future work.

Now, let us turn to the physically meaningful case $y = +1$. Judging from the expressions above, it would appear naively that the elliptic genus must diverge at $y = +1$; however, one should be careful, as limits and residues do not commute. For a simple example, consider $f(z, u) = 1/(z + u)$. For this function,

$$\begin{aligned} \mathrm{Res}_{u=0} \left(\mathrm{Lim}_{z \to 0} f(z, u) \right) &= 1, \\ \mathrm{Lim}_{z \to 0} \left(\mathrm{Res}_{u=0} f(z, u) \right) &= 0. \end{aligned}$$

In particular, for both of the elliptic genera above, if we first take $y = +1$ and then compute the residue, we find that the residue vanishes. That computation, at $y = +1$, is a bit too naive, as the pole intersections are, in the language of [85], nonprojective, and so correct version of the Jeffrey-Kirwan residue could be more complicated.

That said, we can also independently compute the leading term in the elliptic genus. From equation (3.18), the leading term is proportional to

$$\int_X \hat{A}(TX) \wedge \mathrm{ch} \left((\det \mathcal{E})^{+1/2} \wedge_{-1} (\mathcal{E}^*) \right)$$

In general,

$$\hat{A}(TX) \; = \; 1 \; - \; \frac{1}{24} \left(c_1(TX)^2 - 2 c_2(TX) \right) \; + \; (\text{degree } 8)$$

and for a rank 3 bundle,

$$\mathrm{ch} \left((\det \mathcal{E})^{+1/2} \wedge_{-1} (\mathcal{E}^*) \right) \; = \; c_3(\mathcal{E}) \; + \; (\text{degree } 5)$$

hence the leading term vanishes, in agreement with the extremely naive computation at $y = +1$ above.

If the elliptic genus does in fact vanish at $y = +1$, it suggests that supersymmetry may be broken dynamically. It is important to note that both of the elliptic genera should vanish – supersymmetry does not break in one and remain unbroken in the other. This is because at weak coupling in two dimensions, since gauge fields have no propagating degrees of freedom, whether supersymmetry breaks is a function of the low-energy nonlinear sigma model, independent of the details of the presentation of the UV GLSM. The fact that both the elliptic genera vanish is a (weak) check of the claimed duality.

Later, in discussing Calabi-Yau compactifications, we will see closely related abelian-nonabelian (0,2) dualities in which supersymmetry is not broken.

Example 3.5. Following the methods in [85] and appendix F, the elliptic genus for the second entry in table 3.1 is a residue of

$$\frac{(2\pi)^2}{2}\eta(q)^8 \frac{\theta_1(q, x_1 x_2^{-1})\theta_1(q, x_2 x_1^{-1})\theta_1(q, y x_1^{-1} x_2^{-1})^3}{\theta_1(q, x_1)^4 \theta_1(q, x_2)^4 \theta_1(q, y^{-1} x_1^{-1} x_2^{-1})}.$$

The elliptic genus of the abelian dual is a residue of

$$-2\pi\eta(q)^5 \frac{\theta_1(q, x^{-2})\theta_1(q, yx^{-1})}{\theta_1(q, x)^6 \theta_1(q, yx^{-1})}.$$

The first few terms in power series in q for both of these elliptic genera match, and are given by

$$\begin{aligned} & q^{-1/6}\frac{(1+y)^4}{y(y-1)^2} \;+\; q^{5/6}\frac{1}{2y^2(y-1)^2}\left(-36 + 68y^2 + 64y^3 + 68y^4 - 36y^6\right) \\ & \quad + q^{11/6}\frac{(1+y)^2}{y^3(y-1)^2}\left(57 - 360y + 661y^2 - 660y^3 + 661y^4 - 360y^5 + 57y^6\right) \;+\; \mathcal{O}\left(q^{17/6}\right). \end{aligned}$$

As before, the fact that these expressions match for $y \neq +1$ is an intriguing test of duality.

Now, let us turn to the physically meaningful special case $y = +1$. As before, limits and residues do not commute. For both elliptic genera, taking the limit $y \to +1$ and then evaluating the residue, one finds that naively, ignoring subtleties due to non-projective intersections, both of the elliptic genera vanish for $y = +1$, suggesting that supersymmetry may be broken dynamically. As before, both of the genera vanish: any supersymmetry breaking that occurs, must happen below the scale at which the nonlinear sigma model becomes a pertinent description.

Now, let us compare the result above to the prediction of section 3.2. This model satisfies $\det \mathcal{E} \cong K_X$, so the leading term in the elliptic genus is predicted to be proportional to

$$q^{(r-n)/12} \sum_{s=0}^{r} (-)^s \chi(\wedge^s \mathcal{E}).$$

It is straightforward to compute that in this example,

$$\chi(\mathcal{O}) \; = \; 1, \quad \chi(\mathcal{E}) \; = \; 2, \quad \chi(\wedge^2 \mathcal{E}) \; = \; 1,$$

hence

$$\sum_{s=0}^{r} (-)^s \chi\left(\wedge^s \mathcal{E}\right) \; = \; 0,$$

in agreement with the naive direct computations.

Example 3.6. For the fifth entry in table 3.1, the elliptic genus is a residue of

$$-\frac{i}{2}(2\pi)^2 \eta(q)^7 \frac{\theta_1(q, x_1 x_2^{-1}) \theta_1(q, x_2 x_1^{-1}) \theta_1(q, y + x_1^2 x_2^2)^5}{\theta_1(q, x_1)^4 \theta_1(q, x_2)^4 \theta_1(q, y^{-1} x_1^{-3} x_2^{-3})^2}.$$

The elliptic genus of the abelian dual is a residue of

$$+2\pi i \eta(q)^4 \frac{\theta_1(q, x^{-2}) \theta_1(q, y x^2)^5}{\theta_1(q, x)^6 \theta_1(q, y^{-1} x^{-3})^2}.$$

The first few terms in power series in q of these two elliptic genera match perfectly:

$$\begin{aligned} &\frac{q^{-1/12}}{y^{3/2}(y-1)} \left(1 - y + 10y^2 - y^3 + y^4\right) \; + \; \frac{q^{11/12}}{y^{3/2}(y-1)} \left(-17 + 12y + 30y^2 + 12y^3 - 17y^4\right) \\ &\quad + \frac{q^{23/12}}{y^{7/2}(y-1)} \left(98 - 207y - 54y^2 + 216y^3 - 56y^4 + 216y^5 - 54y^6 - 207y^7 + 98y^8\right) \\ &\quad + \mathcal{O}\left(q^{35/12}\right). \end{aligned}$$

As before, the fact that the two genera match in this form is an interesting check of duality; however, only the special case $y = +1$ is physically meaningful.

As before, limits and residues do not commute. For both elliptic genera, taking the limit $y \to +1$ and then evaluating the residue, one finds that both of the elliptic genera vanish for $y = +1$. Ignoring as before subtleties in non-projective intersections, this suggests that supersymmetry may be broken dynamically. As before, both of the genera vanish: any supersymmetry breaking that occurs, must happen below the scale at which the nonlinear sigma model becomes a pertinent description.

Now, let us compare to the predictions of section 3.2. This model satisfies $\det \mathcal{E}^* \cong K_X$, so the leading term in the elliptic genus is predicted to be

$$q^{(r-n)/12} \sum_{s=0}^{r} (-)^s \chi(\wedge^s \mathcal{E}^*).$$

It is straightforward to compute that

$$\chi(\mathcal{O}) \; = \; 1, \;\; \chi(\mathcal{E}^*) \; = \; 0, \;\; \chi(\wedge^2 \mathcal{E}^*) \; = \; 0, \;\; \chi(\wedge^3 \mathcal{E}^*) \; = \; 1,$$

hence the leading term vanishes, matching our naive computation above.

3.3.3 Calabi-Yau and related examples

Examples on $G(2,4)[4]$

To build a (0,2) GLSM for a complete intersection, we follow a pattern similar to that in abelian (0,2) GLSM's: for each hypersurface $\{G_a = 0\}$ (degree d_a) in the complete intersection, we add a Fermi superfield Γ^a, charged under $\det U(k)$ with charge $-kd_a$ (*i.e.* couples to bundle $(\det S^*)^{-d_a} = \mathcal{O}(-d_a, -d_a)$), and a (0,2) superpotential term

$$W \; = \; \Gamma^a G_a(\phi).$$

Integrating out the auxiliary field in Γ^a forces the vacua to lie along $\{G_a = 0\}$. The reason for the charge assignments lies in how the polynomials G_a are defined. Specifically, these are functions of baryons in the $U(k)$ theory (*i.e.* homogeneous coordinates in the Plücker embedding),

$$B_{i_1 \cdots i_k} \; = \; \epsilon_{a_1 \cdots a_k} \phi^{a_1}_{i_1} \cdots \phi^{a_k}_{i_k},$$

which each have $\det U(k)$ charge k.

In this language, the Calabi-Yau condition for a complete intersection of hypersurfaces in $G(k,n)$ is that the sum of the degrees of the hypersurfaces equals n:

$$\sum_a d_a \; = \; n.$$

In table 3.2 we list anomaly-free examples of bundles $\mathcal{E}$ of the form

$$0 \; \rightarrow \; \mathcal{E} \; \rightarrow \; \oplus^{m_1} \mathcal{O}(\lambda_{A1}, \lambda_{B1}) \oplus^{m_2} \mathcal{O}(\lambda_{A2}, \lambda_{B2}) \; \rightarrow \; \oplus^{n_1} \mathcal{O}(\lambda_{A3}, \lambda_{B3}) \oplus^{n_2} \mathcal{O}(\lambda_{A4}, \lambda_{B4}) \; \rightarrow \; 0$$

on $G(2,4)[4]$ with $c_1(\mathcal{E}) = 0$. For bundles of the form above,

$$\begin{aligned} c_1(\mathcal{E}) \; &= \; m_1 c_1(\mathcal{O}(\lambda_{A1}, \lambda_{B1})) \; + \; m_2 c_1(\mathcal{O}(\lambda_{A2}, \lambda_{B2})) \\ &\qquad - n_1 c_1(\mathcal{O}(\lambda_{A3}, \lambda_{B3})) \; - \; n_2 c_1(\mathcal{O}(\lambda_{A4}, \lambda_{B4})), \\ &\propto \; d_{(\lambda_{A1}, \lambda_{B1})} \mathrm{Cas}_1(\lambda_{A1}, \lambda_{B1}) \; + \; d_{(\lambda_{A2}, \lambda_{B2})} \mathrm{Cas}_1(\lambda_{A2}, \lambda_{B2}) \\ &\qquad - d_{(\lambda_{A3}, \lambda_{B3})} \mathrm{Cas}_1(\lambda_{A3}, \lambda_{B3}) \; - \; d_{(\lambda_{A4}, \lambda_{B4})} \mathrm{Cas}_1(\lambda_{A4}, \lambda_{B4}). \end{aligned}$$

Let us examine carefully the first entry in table 3.2. The field content of the (0,2) GLSM pertinent to anomalies is as follows:

- 1 Fermi superfield in representation (1,0) (for the middle term defining $\mathcal{E}$),
- 5 Fermi superfields in representation (2,1) (for the middle term defining $\mathcal{E}$),
- 1 Fermi superfield Γ in representation (-4,-4) (for the hypersurface),
- 1 left-moving gaugino in the adjoint,

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$	rank
1	(1, 0)	5	(2, 1)	1	(3, 1)	2	(3, 2)	5
3	(1, 1)	5	(1, 1)	1	(2, 2)	2	(3, 3)	5
5	(1, 1)	5	(2, 0)	4	(2, 2)	2	(3, 0)	8
2	(1, 1)	5	(2, 2)	1	(3, 3)	3	(3, 3)	3
5	(1, 1)	2	(2, 2)	1	(3, 3)	2	(3, 3)	4
2	(2, 1)	5	(2, 2)	2	(3, 2)	2	(3, 3)	3

Table 3.2: Anomaly-free examples on $G(2,4)[4]$.

- 4 chiral superfields in the fundamental (1,0) (defining the Grassmannian),
- 1 chiral superfield in the dual of (3,1) (corresponding to the last term defining $\mathcal{E}$),
- 2 chiral superfields in the dual of (3,2) (corresponding to the last term defining $\mathcal{E}$).

It is straightforward to check that this field content is anomaly-free, and defines a theory with $c_1(\mathcal{E}) = 0$.

Maps are given in the same fashion as discussed earlier for bundles on $G(2,4)$. For example, maps $\mathcal{O}(1,0) \to \mathcal{O}(3,1)$ are of the form

$$\Lambda^a \ \mapsto \ (\epsilon_{bc}\phi_i^b \phi_j^c) \left(\Lambda^a \phi_k^b \ + \ \Lambda^b \phi_k^a \right)$$

maps $\mathcal{O}(1,0) \to \mathcal{O}(3,2)$ are of the form

$$\Lambda^a \ \mapsto \ (\epsilon_{bc}\phi_i^b \phi_j^c)^2 \Lambda^a$$

and so forth, leading to superpotential terms of the form discussed previously.

Furthermore, just as in section 3.3.2, some of the examples above can be rewritten as examples in the (0,2) GLSM for $\mathbb{P}^5[2]$. For example, a complete intersection $G(2,4)[d_1, \cdots, d_n]$ is the same as the complete intersection

$$\mathbb{P}^5[2, d_1, \cdots, d_n]$$

and at least sometimes it is possible to map the bundles, consistent with the structure of (0,2) GLSM's. For example, the second entry in table 3.2 corresponds to the bundle

$$0 \longrightarrow \mathcal{E} \longrightarrow \oplus^8 \mathcal{O}(1) \longrightarrow \mathcal{O}(2) \oplus^2 \mathcal{O}(3) \longrightarrow 0$$

over $\mathbb{P}^5[2,4]$, which is easily realized as an anomaly-free abelian (0,2) GLSM. Similarly, the fourth entry in table 3.2 corresponds to the anomaly-free bundle

$$0 \longrightarrow \mathcal{E} \longrightarrow \oplus^2 \mathcal{O}(1) \oplus^5 \mathcal{O}(2) \longrightarrow \oplus^4 \mathcal{O}(3) \longrightarrow 0$$

on $\mathbb{P}^5[2,4]$, and the fifth entry in table 3.2 corresponds to the anomaly-free bundle

$$0 \longrightarrow \mathcal{E} \ \oplus^5 \mathcal{O}(1) \oplus^2 \mathcal{O}(2) \longrightarrow \oplus^3 \mathcal{O}(3) \longrightarrow 0$$

on $\mathbb{P}^5[2,4]$.

In appendix F we work through the details of computations of elliptic genera for the three nonabelian examples above and their abelian duals. In each case, the elliptic genera of the proposed duals match, consistent with geometric expectations. For the second entry in table 3.2, the first few terms in the q-expansion of the elliptic genus are shown to be

$$\begin{aligned} &72\left(-y^{-1/2}+y^{+1/2}\right)^2\left(y^{-1/2}+y^{+1/2}\right)q^{1/6} \\ &\quad -72\left(-y^{-1/2}+y^{+1/2}\right)^2\left(y^{-1/2}+y^{+1/2}\right)^3\left(y^{-1}-1+y\right)q^{7/6} \\ &\quad +72\left(-y^{-1/2}+y^{+1/2}\right)^2\left(y^{-7/2}-y^{-3/2}+2y^{-1/2}+2y^{+1/2}-y^{+3/2}+y^{+7/2}\right)q^{13/6} \ + \ \mathcal{O}\left(q^{19/6}\right). \end{aligned}$$

For the fourth entry in table 3.2, the first few terms in the q-expansion are shown to be

$$\begin{aligned} &88y^{-1/2}(1+y) \ - \ 88y^{-5/2}\left(1-y^2-y^3+y^5\right)q \\ &\quad -88y^{-7/2}\left(1+y\right)\left(-1+y^3\right)^2q^2 \ - \ 88y^{-7/2}\left(-1+y\right)^2\left(1+y\right)^3\left(1+y+y^2\right) \ + \ \mathcal{O}\left(q^4\right). \end{aligned}$$

For the fifth entry in table 3.2, the first few terms in the q-expansion are shown to be

$$80\left(y - y^{-1}\right) q^{1/12} \; - \; 80\left(-y^{-3} + y^{-1} - y + y^{3}\right) q^{13/12} \\ - 80\left(-y^{-3} + 2y^{-1} - 2y + y^{3}\right) q^{25/12} \; + \; \mathcal{O}\left(q^{37/12}\right).$$

In each case, the leading term is independently checked. Note that in none of these cases do the elliptic genera vanish, hence we do not expect supersymmetry breaking in any of these cases.

Affine Grassmannians

Let us next consider some anomaly-free examples formally associated to the affine Grassmannian over $G(k, n)$. This Grassmannian is defined by an $SU(k)$ gauge theory with n chiral multiplets in the fundamental representation. The ordinary Grassmannian is defined by a $U(k)$ gauge theory with the same matter. (See for example [76][section 2.5] for more information on affine and weighted Grassmannians.)

Since the gauge group is $SU(k)$ rather than $U(k)$, there is no continuously-variable Fayet-Iliopoulos parameter, and hence no way to take a weak coupling large-radius limit in this theory, making any discussion of geometry rather suspect. Nevertheless, recently there has been interest in *e.g.* GLSM's for non-Kähler compactifications [68–72] in which the overall radius is also fixed, so with an eye towards possible applications to such examples, we include here a short discussion of $SU(k)$ GLSM's. For simplicity, we will characterize them in geometric terms, though as already noted, geometry should be applied with care here.

In table 3.3 we list some anomaly-free examples with gauge bundle of the form

$$0 \;\to\; \mathcal{E} \;\to\; \oplus^{m_1}\mathcal{O}(\lambda_{A1}, 0) \oplus^{m_2} \mathcal{O}(\lambda_{A2}, 0) \;\to\; \oplus^{n}\mathcal{O}(\lambda_{A3}, 0) \;\to\; 0$$

on affine $G(2, 4)$. We compute that for each of the examples in the table, the elliptic genus vanishes identically, which we take as an indication of possible dynamical supersymmetry breaking in these toy models. As these models have no weak coupling large-radius limit, we are not surprised, but we list them here regardless as toy examples of the technology.

m_1	λ_{A1}	m_2	λ_{A2}	n	λ_{A3}	rank
4	3	—	—	2	4	6
1	3	3	4	2	5	7
5	2	—	—	2	3	7

Table 3.3: Examples on the affine Grassmannian over $G(2,4)$.

For example, the first entry in the table involves maps of the form

$$\yng(3) \longrightarrow \yng(4),$$

which are given by, schematically,

$$\Lambda^{abcd} \mapsto \Lambda^{abcd}\phi_i^e + \text{(symmetric permutations)}.$$

This would be given physically by a (0,2) superpotential of the form

$$W = f^i \left(\Lambda^{abcd}\phi_i^e + \text{perm's} \right) p_{abcde},$$

where f^i's are constants and p_{abcde} is a chiral superfield in the representation (dual to)

$$\yng(4).$$

Note that in general, in $SU(2)$ theories, there will be more possible maps than in $U(2)$ theories, because one is not constrained by gauge invariance under the overall $U(1)$. For example, in an $SU(2)$ gauge theory with matter in fundamental representations, we can define a map

$$\yng(2) \longrightarrow \yng(1)$$

by, schematically,

$$\Lambda^{ab} \mapsto \Lambda^{ab}\Phi_i^c \epsilon_{bc},$$

where Λ^{ab} couples to $\yng(2)$. This works because $\epsilon_{ab}\Phi^a$ is the dual of Φ^a in an $SU(2)$ theory. This is not true in a $U(2)$ gauge theory, and there, a map of the form above would not respect the $\det U(2)$ charges. Phased another way, our proposed map sends

$$\yng(2) \longrightarrow \yng(2,1).$$

As representations of $SU(2)$,

$$\yng(2,1) \cong \yng(1),$$

but in $U(2)$, the representation $(2,1) \neq (1,0)$.

3.3.4 Pfaffian constructions

Review of (2,2) constructions

The paper [73] gave two constructions of (2,2) GLSM's associated to a given Pfaffian variety, denoted the PAX and PAXY models. Schematically, for an $n \times n$ matrix A, each entry a homogeneous function over some toric variety V, each construction defines a Pfaffian variety given by the locus on V where the

$$\operatorname{rank} A \leq k$$

for some k.

In the PAX model, in addition to the gauge-theoretic data defining the toric variety, one adds a $U(n-k)$ gauge theory with two chiral superfields P, X, where X transforms as n copies of the fundamental[13] of $U(n-k)$ and P as n copies of the antifundamental of $U(n-k)$,

[13] To make our (0,2) conventions cleaner, we have made a trivial convention flip with respect to [73], in that P and X are defined in opposite representations.

together with a (2,2) superpotential

$$W = \mathrm{tr}\, PAX \tag{3.35}$$

from which the model derives its name. P and X also have charges under the abelian gauge symmetry defining the toric variety, so in effect, the model describes a superpotential over a bundle with fibers that are the total spaces of

$$S^{\oplus n} \longrightarrow G(n-k,n) \tag{3.36}$$

fibered over the given toric variety. All charges are required to be such that the superpotential (3.35) is neutral.

The (2,2) GLSM above has two phases, which are closely related. The D-terms give a constraint of the form

$$XX^\dagger - P^\dagger P = rI,$$

where r is a Fayet-Iliopoulos parameter associated to the overall $U(1)$. Without loss of generality, we shall take $r \gg 0$. The F-terms give constraints of the form

$$AX = 0, \quad PA = 0, \quad P(dA)X = 0.$$

The first constraint defines the variety

$$Z \equiv \{(\phi, x) \,|\, A(\phi)x = 0\},$$

which is our desired (resolution of a) Pfaffian. (The constraint forces X to describe $n-k$ null eigenvectors of A, and so only has solutions when the rank of A is bounded by k.) Under a smoothness assumption, the second two F-term constraints imply $P = 0$, as discussed in [73][section 3.2]. Thus, we expect that this theory flows at low energies to a nonlinear sigma model on Z. Nearly an identical analysis applies when $r \ll 0$, except that the roles of X and P are reversed.

In passing, let us work out the Calabi-Yau condition in a PAX model of the form above. First, note that the fibers (3.36) are already Calabi-Yau, so we merely need a constraint on charges of the abelian gauge symmetries defining the underlying toric variety.

Specifically, the space will be Calabi-Yau if the sum of the $U(1)$ charges vanishes, for each $U(1)$ defining the underlying toric variety. For example, suppose the underlying toric variety is a projective space, $\mathbb{P}^m$ for some m. Let p_i denote the $U(1)$ of the i-th fundamental in P, and x_i the $U(1)$ charge of the i-th antifundamental in X. Then the Calabi-Yau condition can be succinctly stated as the condition

$$\sum_i (n-k) p_i \, + \, \sum_i (n-k) x_i \, + \, m+1 \, = \, 0,$$

where we have used the fact that the fundamentals and antifundamentals both have dimension $n-k$.

In the (2,2) PAXY model, given the GLSM for the underlying toric variety, one insteads adds a $U(k)$ gauge theory with n fundamentals $\tilde{X}$, n antifundamentals $\tilde{Y}$, and an $n \times n$ matrix of neutral chiral superfields $\tilde{P}$, together with a (2,2) superpotential

$$W \, = \, \mathrm{tr} \, \tilde{P} \left(A \, - \, \tilde{Y} \tilde{X} \right). \tag{3.37}$$

Here also, $\tilde{P}$, $\tilde{X}$, $\tilde{Y}$ are charged under the abelian gauge symmetry defining the underlying toric variety, with charges such that the superpotential (3.37) is gauge invariant.

For a PAXY model over $\mathbb{P}^m$, as before, the Calabi-Yau condition would be

$$k \sum_i x_i \, + \, k \sum_i y_i \, + \, k \sum_i p_i \, + \, m \, + \, 1 \, = \, 0,$$

where we have used the fact that the fundamentals and antifundamentals have dimension k.

The PAX and PAXY models look different, but for a given Pfaffian, are equivalent to one another, as we shall review in section 3.3.8.

More general (0,2) examples

To understand (0,2) models on Pfaffians, let us begin by rewriting the (2,2) PAX and PAXY models in (0,2) language.

Let us begin with the (0,2) PAX model. Let X, Λ_X, denote the (0,2) chiral, Fermi superfields associated to the (2,2) superfield X, all describing n copies of the fundamental, and let P, Λ_P denote the (0,2) chiral, Fermi superfields associated to the (2,2) superfield P, describing n copies of the antifundamental. Let Φ, Λ_Φ denote the (0,2) chiral, Fermi superfields associated to the (2,2) Φ defining the underlying toric variety. This decomposition of the (2,2) theory also gives rise to an adjoint-valued (0,2) chiral Σ, originating in the (2,2) gauge multiplet.

Then, the (0,2) theory is a $U(n-k)$ gauge theory with fields P, Λ_P, X, Λ_X, Φ, Λ_Φ, obeying

$$\overline{D}_+ \Lambda_P \propto \Sigma P$$

(and similarly for other Fermi superfields), and with (0,2) superpotential

$$W = \mathrm{tr}\left(\Lambda_P A(\Phi) X \,+\, P A(\Phi) \Lambda_X \,+\, P \frac{\partial A(\Phi)}{\partial \Phi^\alpha} \Lambda_\Phi^\alpha X \right).$$

Intuitively, for $r \gg 0$, one can interpret Λ_P as acting as a Lagrange multiplier, forcing $AX = 0$, and Λ_X, Λ_Φ^α as describing the fermions in which the gauge bundle lives.

Given the structure above, we can read off the monad whose cohomology defines the tangent bundle of the Pfaffian:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}^r \oplus (S^* \otimes S) \stackrel{*_1}{\longrightarrow} & \oplus_{a,\alpha} \mathcal{O}((0,0), q_{a,\alpha}) \oplus_i \mathcal{O}((1,0), x_{a,i}) \\ & \stackrel{*_2}{\longrightarrow} \oplus_i \mathcal{O}((1,0), -p_{a,i}) \longrightarrow 0, \end{aligned} \tag{3.38}$$

where

$$*_1 = \left[\begin{array}{cc} q_{a,\alpha}\Phi^\alpha & 0 \\ x_a X & X \end{array} \right], \quad *_2 = \left[\frac{\partial A}{\partial \Phi^\alpha} X, A \right].$$

As a consistency check, note that the composition of the two maps above has the form

$$*_2 *_1 \;=\; \left[q_{a,\alpha} \Phi^\alpha \frac{\partial A}{\partial \Phi^\alpha} X \;+\; x_a AX, AX \right] \;=\; [p_a AX, AX],$$

which vanishes on the Pfaffian, as expected. The monad above is determined by the field theory, as follows. The $\mathcal{O}^r \oplus S^* \otimes S$ is determined by the gauginos; the other terms are determined by remaining fermions.

Note that the Calabi-Yau condition implied by the monad above is of the form

$$-(n-k) \sum_i p_{a,i} \;=\; (n-k) \sum_i x_{a,i} \;+\; \sum_\alpha q_{a,\alpha}$$

for each a, which specializes to the Calabi-Yau condition discussed previously in (2,2) models.

A (0,2) deformation of the tangent bundle of the Pfaffian would be described by a theory with the same matter content, but (0,2) superpotential

$$W \;=\; \mathrm{tr} \left(\Lambda_P A(\Phi) X \;+\; PA(\Phi) \Lambda_X \;+\; P \left(\frac{\partial A(\Phi)}{\partial \Phi^\alpha} \;+\; G_\alpha(\Phi) \right) \Lambda_\Phi^\alpha X \right),$$

where

$$q_{a,\alpha} \Phi^\alpha G_\alpha \;=\; 0$$

for each a. This is described by a monad of the same form as in equation (3.38), but with maps

$$*_1 \;=\; \left[\begin{array}{cc} q_{a,\alpha}\Phi^\alpha & 0 \\ x_a X & X \end{array} \right], \quad *_2 \;=\; \left[\left(\frac{\partial A}{\partial \Phi^\alpha} X \;+\; G_\alpha \right), A \right].$$

A more general (0,2) model over a Pfaffian, describing a bundle built as a kernel, based on the PAX model, can be built as follows. First, to build the Pfaffian itself, we will need a $U(n-k)$ gauge theory, n chiral superfields in the fundamental, forming an $n \times (n-k)$ matrix denoted X, and n Fermi superfields in the antifundamental, forming an $n \times (n-k)$ matrix of Fermi superfields denoted Λ_0. Then, to describe a bundle $\mathcal{E}$ as a kernel, say,

$$0 \longrightarrow \mathcal{E} \longrightarrow \oplus_\beta \mathcal{O}((\lambda_{\beta 1}, \lambda_{\beta 2}), q_{a,\beta}) \xrightarrow{F_\beta^\gamma} \oplus_\gamma \mathcal{O}((\lambda_{\gamma 1}, \lambda_{\gamma 2}), q_{a,\gamma}) \longrightarrow 0,$$

we add a set of Fermi superfields Λ^β in the $(\lambda_{\beta 1}, \lambda_{\beta 2})$ representation of $U(n-k)$ and with charges $q_{a,\beta}$ under the abelian gauge symmetry $U(1)^r$ defining the toric variety, along with a set of chiral superfields P_γ in the $U(n-k)$ representation dual to $(\lambda_{\gamma 1}, \lambda_{\gamma 2})$ and with charges $-q_{a,\gamma}$ under the abelian gauge symmetry defining the toric variety. In addition, we have a (0,2) superpotential

$$W \;=\; \mathrm{tr}\, \left(\Lambda_0 A(\Phi) X \;+\; \Lambda^\beta F^\gamma_\beta(\Phi) P_\gamma \right).$$

Of course, all representations must be chosen to satisfy gauge anomaly cancellation for this $U(n-k) \times U(1)^r$ gauge theory. (Given the kernel construction above, GLSM's for bundles built as cokernels and as cohomologies of monads are very straightforward, and so for brevity are omitted.)

Let us briefly check the space of vacua in this theory. From D-terms for $U(2)$ we have a constraint of the form

$$X X^\dagger \;+\; \sum_\gamma P_\gamma^\dagger P_\gamma \;=\; r I$$

so, for suitable bundle representations, as discussed previously in section 3.3.2, the X's are not all zero. From the F terms we get the constraint

$$A X \;=\; 0,$$

which describes the underlying Pfaffian variety. So long as the nontrivial map determined by F^γ_β is surjective, the P_γ chiral superfields will all become massive, leaving us with a gauge bundle contained within the associated Fermi superfields, as expected.

To get a bundle with $c_1(\mathcal{E}) = 0$, we impose the conditions

$$\sum_\beta d_{\lambda\beta} \mathrm{Cas}_1(\lambda_{\beta 1}, \lambda_{\beta 2}) \;=\; \sum_\gamma d_{\lambda\gamma} \mathrm{Cas}_1(\lambda_{\gamma 1}, \lambda_{\gamma 2}),$$

$$\sum_\beta q_{a,\beta} \;=\; \sum_\gamma q_{a,\gamma}.$$

So far we have outlined (0,2) versions of the PAX model. Let us now briefly outline analogues for the PAXY model. Here, if we start with the (2,2) model and write it in (0,2) language, following the same convention as previously for the PAX model, we are led to a $U(k)$ gauge theory with (0,2) chiral superfields $\tilde{P}$, $\tilde{X}$, $\tilde{Y}$, Φ^α, (0,2) Fermi superfields $\Lambda_{\tilde{P}}$, $\Lambda_{\tilde{X}}$, $\Lambda_{\tilde{Y}}$, Λ_Φ^α, and a (0,2) superpotential of the form

$$W \,=\, \mathrm{tr}\left(\Lambda_{\tilde{P}}\left(A \,-\, \tilde{Y}\tilde{X}\right) \,+\, \tilde{P}\left(\frac{\partial A}{\partial \Phi^\alpha}\Lambda_\Phi^\alpha \,-\, \Lambda_{\tilde{Y}}\tilde{X} \,-\, \tilde{Y}\Lambda_{\tilde{X}}\right)\right). \tag{3.39}$$

A (0,2) theory describing a deformation of the tangent bundle is defined by the superpotential

$$W \,=\, \mathrm{tr}\left(\Lambda_{\tilde{P}}\left(A \,-\, \tilde{Y}\tilde{X}\right) \,+\, \tilde{P}\left(\left(\frac{\partial A}{\partial \Phi^\alpha} \,+\, G_\alpha\right)\Lambda_\Phi^\alpha \,-\, \Lambda_{\tilde{Y}}\tilde{X} \,-\, \tilde{Y}\Lambda_{\tilde{X}}\right)\right). \tag{3.40}$$

Now, consider a (0,2) theory describing a gauge bundle $\mathcal{E}$, given as a kernel

$$0 \longrightarrow \mathcal{E} \longrightarrow \oplus_\beta \mathcal{O}((\lambda_{\beta 1}, \lambda_{\beta 2}), q_{a,\beta}) \xrightarrow{F_\beta^\gamma} \oplus_\gamma \mathcal{O}((\lambda_{\gamma 1}, \lambda_{\gamma 2}), q_{a,\gamma}) \longrightarrow 0 \tag{3.41}$$

over the Pfaffian. We can describe this following the PAXY pattern as follows. Given the abelian gauge theory for the toric variety, we add a $U(k)$ gauge theory with

- n chiral superfields in the fundamental, forming a matrix $\tilde{X}$,
- n chiral superfields in the antifundamental, forming a matrix $\tilde{Y}$,
- an $n \times n$ matrix of neutral Fermi superfields Λ_0,
- a set of Fermi superfields Λ^β in the $(\lambda_{\beta 1}, \lambda_{\beta 2})$ representation of $U(k)$, with charges $q_{a,\beta}$ under the abelian gauge symmetry defining the toric variety,
- a set of chiral superfields P_γ in the $U(k)$ representation dual to $(\lambda_{\gamma 1}, \lambda_{\gamma 2})$ and with charges $-q_{a,\gamma}$ under the abelian gauge symmetry defining the toric variety,
- and finally a (0,2) superpotential

$$W \,=\, \mathrm{tr}\left(\Lambda_0\left(A(\Phi) \,-\, \tilde{Y}\tilde{X}\right) \,+\, \Lambda^\beta F_\beta^\gamma(\Phi) P_\gamma\right).$$

Note that although the data defining the bundle is formally very similar to that in the PAX construction, the representations given in the short exact sequence (3.41) are representations of $U(k)$, whereas the representations given in the analogue for the PAX construction are representations of $U(n-k)$. The relationship between such representations will be discussed in section 3.3.8, but is not particularly simple.

Examples

Listed in table 3.4 are some examples of (0,2) models on Pfaffians. The Pfaffians themselves are all constructed via the (0,2) PAX model for gauge bundle kernels, as Pfaffians of a 4×4 matrix A, defined as the locus where the rank of A is less than or equal to 2. Hence, we have a $U(4-2) = U(2)$ gauge theory. The Pfaffians are subvarieties of $\mathbb{P}^7$, so for the PAX construction we have fibered

$$S^{\oplus 4} \longrightarrow G(2,4)$$

over $\mathbb{P}^7$, with the fibering defined by the statement that the n antifundamentals[14] X have $U(1)$ charge 0 and the fundamentals Λ_0 have $U(1)$ charge -1. The chiral superfields defining $\mathbb{P}^7$ have charge 1, and the entries of the matrix A are of degree 1. It is straightforward to check that the resulting Pfaffian is Calabi-Yau, from the criteria given earlier, and applying the methods of *e.g.* [76] we see that these are 3-folds.

Table 3.4 lists data for bundles over the total space of the $(S^4 \to G(2,4))$-bundle over $\mathbb{P}^7$. We have restricted to bundles built as kernels. (More general cases are straightforward, and

[14] Our conventions in the table are flipped relative to the earlier discussion: X is here a set of antifundamentals rather than fundamentals, and Λ_0 is a set of fundamentals rather than antifundamentals. The choice is arbitrary.

so are left as exercises.) Bundles are kernels of the form

$$0 \longrightarrow \mathcal{E} \longrightarrow \oplus^{m_1}\mathcal{O}((\lambda_{A1}, \lambda_{B1}), Q_1) \oplus^{m_2} \mathcal{O}((\lambda_{A2}, \lambda_{B2}), Q_2) \\ \longrightarrow \oplus^{n_1}\mathcal{O}((\lambda_{A3}, \lambda_{B3}), Q_3) \oplus^{n_2} \mathcal{O}((\lambda_{A4}, \lambda_{B4}), Q_4) \longrightarrow 0.$$

For each special homogeneous bundle appearing, we give both a representation of $U(2)$ and also a charge under the $U(1)$ defining the $\mathbb{P}^7$. Conventions are such that $U(2)$ representation $(\lambda_{Ai}, \lambda_{Bi})$ has $\mathbb{P}^7$ $U(1)$ charge Q_i, a fact we have indicated above in subscripts. All of the examples in table 3.4 have $c_1(\mathcal{E}) = 0$.

Example 3.7. For completeness, let us describe the first example in table 3.4 in detail. It describes a theory containing charged left-moving fermions as:

- 5 Fermi superfields in the ((0,0),-1), for part of the gauge bundle,
- 2 Fermi superfields in the ((2,2),0), for part of the gauge bundle,
- Λ_0: 4 Fermi superfields in the ((1,0),-1),
- 1 $U(2) \times U(1)$ gaugino,

and charged right-moving fermions as:

- X: 4 chiral superfields in the ((0,-1),0),
- 2 chiral superfields in the dual of ((2,2),-1), for part of the gauge bundle,
- 1 chiral superfield in the dual of ((1,-1),-1), for part of the gauge bundle,
- 8 chiral superfields in the ((0,0),+1), describing homogeneous coordinates on $\mathbb{P}^7$.

There are several gauge anomaly cancellation conditions that must be obeyed: the Cas_2 condition and $(\mathrm{Cas}_1)^2$ conditions for $U(2)$ gauge anomaly cancellation, plus a q^2 condition for solely the extra $U(1)$ for $\mathbb{P}^7$, plus a mixed $U(1) - U(1)$ condition involving products of the general form $q\mathrm{Cas}_1$.

In passing, let us comment on the possible existence of a duality to an abelian description. Since the nonabelian gauge theory in the PAX model describes, in part, $G(2,4)$, one might hope to use its duality to $\mathbb{P}^5[2]$ to find an equivalent abelian model. Unfortunately, to do so, we would need a dual description of the universal subbundle on $G(2,4)$. On $\mathbb{P}^5[2]$, this is a spinor bundle for which no simple three-term sequence construction is expected. Thus, we do not expect there to exist a dual abelian description of any of the theories described in this section.

We will discuss dualities between PAX and PAXY models in section 3.3.8.

3.3.5 Mathematics of duality in (2,2) theories

In the next few sections, we will analyze dualities between two dimensional (2,2) and (0,2) theories. We focus on weakly-coupled theories RG flowing to nonlinear sigma models. In some cases, we can understand dualities as relating different presentations of the same mathematical geometry. In such a case, where we can identify RG endpoints, a duality is immediate (and as an immediate consequence, one can identify Higgs moduli spaces, chiral rings, and global symmetries). Analyses of this form will not apply to every theory, only to weakly coupled theories with a clear relationship to geometry, and moreover even in weakly coupled theories we will later see examples of physical dualities not of this form.

Although such a mathematical approach does not apply to every theory, it can be useful for suggesting nonobvious dualities, especially in theories with no flavor symmetries. The latter are generic in Calabi-Yau compactification, where *e.g.* superpotentials typically break most if not all flavor symmetries.

We shall first discuss the two-dimensional analogue of Seiberg duality for (2,2) $U(k)$ gauge theories with both fundamentals and antifundamentals [81]. In particular, although the relation between $G(k,n)$ and $G(n-k,n)$ is well-known, it is perhaps less well-known that Seiberg duality itself has an equally simple mathematical description, only slightly generalizing the $G(k,n)$, $G(n-k,n)$ relation, which we shall outline in that section.

$U(k)$ gauge theories with fundamentals and antifundamentals

In this section we will give a geometric understanding of the duality in $(2,2)$ $U(k)$ gauge theories with both fundamentals and antifundamentals described in [81]. This is a prototype for many other dualities we shall discuss in this section. It will also serve as a useful caution: such mathematical dualities are only applicable to weakly-coupled physical theories. In particular, in the present case we will see there is a chain of mathematical equivalences, but only some of those mathematical equivalences correspond to relations between weakly coupled theories and are physically meaningful, as we shall discuss.

Consider a two-dimensional (2,2) GLSM with gauge group $U(k)$, n multiplets in the fundamental representation, and A multiplets in the antifundamental representation. This GLSM has two geometric phases, describing:

- Tot $\left(S^A \to G(k,n)\right)$, and
- Tot $\left(S^n \to G(k,A)\right)$.

(In addition, as observed in [118], there will be discrete Coulomb vacua in general, but as they will not play an essential role in our discussion, we omit their details.)

Mathematically, the first phase is equivalent to

$$\mathrm{Tot}\ \left((Q^*)^A \ \longrightarrow\ G(n-k,n)\right)$$

as discussed in section 3.3.1.

Physically, the Q^* must be realized indirectly, from the fact that

$$0 \longrightarrow Q^* \longrightarrow \mathcal{O}^n \longrightarrow S^* \longrightarrow 0.$$

Specifically, for each Q^* one wishes to implement, one must add chiral superfields corresponding to $\mathcal{O}^n$ and the dual of S^*, together with a suitable superpotential. For example, the phase

$$\mathrm{Tot}\ \left((Q^*)^A \longrightarrow G(n-k,n) \right)$$

above arises in the GLSM with gauge group $U(n-k)$, n chiral superfields Φ in the fundamental representation, nA neutral chiral superfields Γ (A copies of $\mathcal{O}^n$), and A chiral superfields P in the antifundamental representation (A copies of the dual of S^*), together with the superpotential

$$W = \Gamma \Phi P.$$

Note in passing that building a (2,2) GLSM to realize the total space of $Q^{\oplus A}$, rather than $(Q^*)^{\oplus A}$, would be more problematic. Formally, one could build each Q as a cokernel, by adding chiral superfields corresponding to $\mathcal{O}^n$ and the dual of S. However, chiral superfields corresponding to S^* are in the fundamental representation, and so physically are indistinguishable from the chiral superfields defining the Grassmannian – the result physically would be a larger Grassmannian, rather than a bundle on the Grassmannian. A closely related problem exists in abelian GLSM's: although it is straightforward to build a (2,2) GLSM describing the total space of the line bundle $\mathcal{O}(-1)$ on $\mathbb{P}^n$, by adding a chiral superfield of opposite charge from the rest, if instead one adds a chiral superfield of the same charge, the result is a larger projective space, and not the line bundle $\mathcal{O}(+1)$ on $\mathbb{P}^n$.

Now, let us return to the case at hand. In the case $n = A$, which is the case that the space is a noncompact Calabi-Yau, the data above is the same as the data given by *e.g.* [82] to describe the GLSM dual to the $U(k)$ GLSM at top (the neutral chiral superfields Γ being their mesons M, for example), closely following the pattern of Seiberg duality in four dimensions. In effect, we are using mathematics to give a purely geometric understanding of Seiberg duality, by studying what in four dimensions would be the classical Higgs branch.

In the case $n \neq A$, this pattern results in a chain of mathematical dualities between GLSM's:

$$
\begin{tikzcd}
S^A \longrightarrow G(k,n) \arrow[r, dashed, no head] \arrow[d, leftrightarrow, "="'] & S^n \longrightarrow G(k,A) \\
(Q^*)^A \longrightarrow G(n-k,n) \arrow[r, dashed, no head] & (Q^*)^n \longrightarrow G(n-k,A) \arrow[d, leftrightarrow, "="] \\
S^A \longrightarrow G(A-n+k,n) \arrow[r, dashed, no head] \arrow[d, leftrightarrow, "="'] & S^n \longrightarrow G(A-n+k,A) \\
(Q^*)^A \longrightarrow G(2n-A-k,n) \arrow[r, dashed, no head] & (Q^*)^n \longrightarrow G(2n-A-k,A) \arrow[d, leftrightarrow, "="] \\
S^A \longrightarrow G(2A-2n+k,n) \arrow[r, dashed, no head] \arrow[d, leftrightarrow, "="'] & S^n \longrightarrow G(2A-2n+k,A) \\
(Q^*)^A \longrightarrow G(3n-2A-k,n) \arrow[r, dashed, no head] & (Q^*)^n \longrightarrow G(3n-2A-k,A) \arrow[d, leftrightarrow, "="] \\
S^A \longrightarrow G(3A-3n+k,n) \arrow[r, dashed, no head] & S^n \longrightarrow G(3A-3n+k,A)
\end{tikzcd}
$$

and so forth. Horizontal rows correspond to the phases of a single GLSM; vertical arrows indicate mathematical dualities. We made the arbitrary decision to run the dualities in one direction; one could also continue in the opposite direction vertically, and it is straightforward to check that a very similar pattern of dualities occurs in that direction. Note that if $A = n$, then the sequence of GLSM's above is 2-periodic.

Now, physics restricts which of the mathematical dualities above is physically meaningful. The issue revolves around renormalization group flow. In the special case that $A = n$, all the spaces appearing above are Calabi-Yau, the Fayet-Iliopoulos parameter is a renormalization-group invariant number. In other cases, however, the Fayet-Iliopoulos parameter will flow.

Briefly, the space

$$\mathrm{Tot}\left(S^A \longrightarrow G(k,n)\right)$$

is positively-curved (and so will shrink) if $A < n$, and is negatively-curved (and so will expand) if $A > n$. Note that the two phases of the non-Calabi-Yau GLSM's have opposite-signed curvature: since the Fayet-Iliopoulos parameter can only flow in one direction, if one limit is positively-curved, the other limit must be negatively-curved, and that is consistent with the mathematics.

Now, in a GLSM, there is a weakly-coupled UV phase in which the Higgs branches are closely identified with geometry. As one flows to the IR in non-Calabi-Yau GLSM's, however, the theory develops isolated Coulomb vacua [118]. For example, in the supersymmetric $\mathbb{P}^n$ model, these form the $n+1$ vacua in the asymptotic IR limit of the theory. Strictly speaking, those Coulomb vacua must be taken into account, and so a purely geometric description of dualities, one that ignores Coulomb vacua as we have done, is potentially misleading in the IR.

Thus, the geometric dualities we have outlined need only correspond to physical dualities in the weakly-coupled UV phases. If $A < n$, say, that means one should expect there to be a physical duality between the UV GLSM phases, of the form

$$\begin{tikzcd}
S^A \longrightarrow G(k,n) \arrow[r, dashed, no head] \arrow[d, leftrightarrow, "="'] & S^n \longrightarrow G(k,A) \\
(Q^*)^A \longrightarrow G(n-k,n) \arrow[r, dashed, no head] & (Q^*)^n \longrightarrow G(n-k,A)
\end{tikzcd}$$

but the mathematical duality on the other side of the diagram need not translate to anything physical. If $A > n$, the opposite mathematical duality should be physical.

This duality is discussed in gauge theories in [81][section 7.1]. As each GLSM has the same number of fundamentals (n) and antifundamentals (A), checking anomaly matching is straightforward. They show S^2 partition functions match for $n > A+1$; their particular expressions for the cases $n = A, A+1$ do not match, but it is believed [121] that the partition functions differ merely by a Kähler transformation in those cases, and so describe equivalent theories. (The paper [82] conjectures differently.) Later work [85][section 4.6.1] shows elliptic genera match more generally. Based on the relationship between the geometries, we conjecture that the theories match in general.

So far we have discussed (2,2) dualties for the total spaces of essentially two bundles on $G(k, n)$, and Whitney sums thereof: S and Q^*. It is not clear whether more general bundles can be dualized. The problem is to relate a more general representation of $U(k)$ to representations of $U(n-k)$; as we shall discuss in section 3.3.9, although one can find long exact sequences relating them, and those can be realized in open strings, it is not currently known how to realize those long exact sequences in closed-string (2,2) or (0,2) theories, so barring the existence of additional surprising physical relationships, it is natural to conjecture that more general bundles cannot be dualized.

A proposed duality involving Pfaffians

Proceeding in the same spirit, it is possible to formulate additional proposals for dualities between GLSM's, motivated by mathematics. In this subsection we focus on one particular example in (2,2) GLSM's, relating a Grassmannian $G(2, n)$ of 2-planes in $\mathbb{C}^n$ to a determinantal variety.

Mathematically ([116], [122][chapter 9]), $G(2, n)$ is the rank 2 locus of the $n \times n$ matrix

$$A(z_{ij}) = \begin{bmatrix} z_{11} = 0 & z_{12} & z_{13} & \cdots \\ z_{21} = -z_{12} & z_{22} = 0 & z_{23} & \cdots \\ z_{31} = -z_{13} & z_{32} = -z_{23} & z_{33} = 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

over

$$\mathbb{P}^{\binom{n}{2}-1}$$

where the $z_{ij} = -z_{ji}$ are homogeneous coordinates on that projective space. In the special case that $n = 4$, the rank 2 locus is determined by the quadric condition

$$z_{12}z_{34} \; - \; z_{13}z_{24} \; + \; z_{14}z_{23} \; = \; 0$$

yielding a duality between $G(2,4)$ and $\mathbb{P}^5[2]$ which we have already discussed. For more general n, the dual cannot be described as a hypersurface, but instead is a determinantal variety which can be built using the methods of [73].

A PAX model for the dual is given by a (2,2) $U(n-2) \times U(1)$ gauge theory with matter content

- $n!/(2!(n-k)!)$ chiral superfields Φ, neutral under $U(n-2)$ but charge $+1$ under the $U(1)$, corresponding to homogeneous coordinates on the projective space,
- n chiral superfields in the fundamental of $U(n-2)$, neutral under the $U(1)$, which we label X,
- n copies of the antifundamental of $U(n-2)$, charge -1 under the $U(1)$, which we label P,
- and a superpotential $W = \operatorname{tr} PA(\Phi)X$.

Alternatively, a PAXY model for the dual is given by a (2,2) $U(2) \times U(1)$ gauge theory with matter content

- $n!/(2!(n-k)!)$ chiral superfields Φ, neutral under $U(n-2)$ but charge $+1$ under the $U(1)$, corresponding to homogeneous coordinates on the projective space,

- n fundamentals of $U(2)$, neutral under $U(1)$, which we label $\tilde{X}$,
- n antifundamentals of $U(2)$, charge $+1$ under $U(1)$, which we label $\tilde{Y}$,
- an $n \times n$ matrix of chiral superfields $\tilde{P}$, neutral under $U(2)$, charge -1 under $U(1)$,
- and a superpotential $W = \mathrm{tr}\, \tilde{P} \left(A(\Phi) - \tilde{Y}\tilde{X} \right)$.

As these theories admit weakly-coupled phases describing the same geometries, we propose that there is a physical Seiberg-like duality relating them. The universal subbundle and quotient bundle are realized as [116] the image and cokernel, respectively, of the matrix A. More examples of analogous forms can also be constructed, and we leave their analyses for future work.

3.3.6 Invariance of (0,2) under gauge bundle dualization

In this section we will propose that physical (0,2) theories are invariant under dualizing the gauge bundle, *i.e.* a (0,2) theory on space X with bundle $\mathcal{E}$ defines the same universality class as that for the same space X with dual bundle $\mathcal{E}^*$. We will use this later to help simplify our description of other (0,2) dualities.

This particular duality has been discussed previously in pseudo-topological field theories in [123], as we will review later, and has also been previously considered by [124, 125]. It has also been used implicitly in [88]. However, we are not aware of published checks of this duality in physical non-topological theories.

It is extremely straightforward to show that this satisfies some basic tests, such as leaving massless spectra invariant. However, to show that this is true of an entire physical theory, one must also check, for example, that massive states are also invariant under this operation, as are worldsheet instanton effects. We will check such details in the next several subsections.

Initial checks

Let us begin by considering the worldsheet lagrangian for a two-dimensional (0,2) theory[15] [128][equ'n (7)]:

$$\frac{1}{2}\left(g_{\mu\nu} + iB_{\mu\nu}\right)\partial\phi^\mu \overline{\partial}\phi^\nu + \frac{i}{2} g_{\mu\nu}\psi_+^\mu D_{\overline{z}}\psi_+^\nu + \frac{i}{2} h_{\alpha\beta}\lambda_-^\alpha D_z \lambda_-^\beta + F_{i\overline{\jmath}a\overline{b}}\psi_+^i \psi_+^{\overline{\jmath}}\lambda_-^a \lambda_-^{\overline{b}}$$
$$+ h^{a\overline{b}} F_a \overline{F}_{\overline{b}} + \psi_+^i \lambda_-^a D_i F_a + \psi_+^{\overline{\imath}} \lambda_-^{\overline{b}} D_{\overline{\imath}} \overline{F}_{\overline{b}}$$
$$+ h_{a\overline{b}} E^a \overline{E}^{\overline{b}} + \psi_+^i \lambda_-^{\overline{a}} \left(D_i E^b\right) h_{\overline{a}b} + \psi_+^{\overline{\imath}} \lambda_-^a \left(D_{\overline{\imath}} \overline{E}^{\overline{b}}\right) h_{a\overline{b}}.$$

In the expression above, $(E^a) \in \Gamma(\mathcal{E})$ and $(F_a) \in \Gamma(\mathcal{E}^*)$, and act as the (0,2) analogues of a superpotential. They are subject to the constraint

$$\sum_a E^a(\phi) F_a(\phi) \; = \; 0.$$

If we exchange $\mathcal{E} \rightarrow \mathcal{E}^*$, simultaneously exchanging E^a and F_a, it is straightforward to check that the lagrangian above is invariant. For example, under the bundle interchange described, λ_-^a is exchanged with $h_{a\overline{b}}\lambda_-^{\overline{b}}$, which leaves kinetic terms invariant and is needed to make sense of the $E^a \leftrightarrow F_a$ exchange. Under the same interchange, the curvature $F \mapsto -F$; however, when combined with the $\lambda_-^a \leftrightarrow \lambda_-^{\overline{b}}$ exchange, the four-fermi term is left invariant.

Given that the classical action remains invariant, classically the theories are identical, but there could be (and in fact are) subtleties involving regularizations, so let us perform additional checks.

As another check, note that anomaly cancellation conditions are invariant under this dualization: $\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(\mathcal{E}^*)$. When this duality appears in UV GLSM's, this becomes a statement that gauge anomaly cancellation conditions are invariant under dualizing matter representations.

As another check, consider massless spectra in heterotic Calabi-Yau compactifications. As discussed in [129], the massless spectra are computed by sheaf cohomology groups of the

[15] The expression given corrects some minor typos in the lagrangian written in [128].

form

$$H^\bullet(X, \wedge^\bullet \mathcal{E}), \quad H^\bullet(X, \mathrm{End}\,\mathcal{E}),$$

and it is straightforward to check that these groups are invariant under $\mathcal{E} \leftrightarrow \mathcal{E}^*$ (for bundles of trivial determinant, as is typical in Calabi-Yau compactification). Physical properties are determined by the gradings; the effect seems to merely be to exchange particles and antiparticles, a trivial operation.

As another consistency check, dualization of the gauge bundle preserves stability. One way to see this is directly in the Donaldson-Uhlenbeck-Yau equation:

$$g^{i\bar{\jmath}} F_{i\bar{\jmath}} \;=\; 0.$$

Dualization of the bundle sends $F \mapsto -F$, so the original bundle will satisfy Donaldson-Uhlenbeck-Yau if and only if the dual bundle also does. In terms of Mumford stability [130], [131][lemma II.1.2.4], dualization gives a one-to-one correspondence between saturated subsheaves of $\mathcal{E}^*$ and quotient torsion-free sheaves of $\mathcal{E}$, which preserves slope inequalities.

Elliptic genera

Let us compare elliptic genera for (0,2) theories with complex vector bundles $\mathcal{E}$ and $\mathcal{E}^*$, using the expressions in and notation of [97]. (In this section, we only consider complex vector bundles; we make no claims about invariance under duality for *e.g.* real vector bundles.) For example, the elliptic genera of nonlinear sigma models with left-movers in an NS sector [97][equ'n (5)] are of the form

$$\begin{aligned} & \mathrm{Tr}\ (-)^{F_R} \exp(i\gamma (J_L)_0) q^{L_0} \overline{q}^{\overline{L}_0} \\ & \quad = \; q^{-(1/24)(2n+r)} \int_X \mathrm{Td}(TX) \wedge \mathrm{ch} \left(\bigotimes_{k=1,2,3,\cdots} S_{q^k}((TX)^{\mathbb{C}}) \bigotimes_{k=1/2,3/2,5/2,\cdots} \wedge_{q^k} \left((z\mathcal{E})^{\mathbb{C}} \right) \right), \end{aligned}$$

where $z = \exp(i\gamma)$,

$$(z\mathcal{E})^{\mathbb{C}} \;=\; z\mathcal{E} \oplus \overline{z}\overline{\mathcal{E}},$$

and other notation follows [97]. Note from the expression above that $(z\mathcal{E})^{\mathbb{C}}$ is invariant under the exchange $\mathcal{E} \leftrightarrow \mathcal{E}^*$, so long as one simultaneously exchanges $z \leftrightarrow \overline{z} = z^{-1}$, the twist on the left-movers. As a result, the elliptic genus above is automatically invariant under the exchange.

For a heterotic nonlinear sigma model with left-moving fermions in an R sector, the elliptic genus

$$\mathrm{Tr}_{\mathrm{R,R}}(-)^{F_R} \exp\left(i\gamma(J_L)_0\right) q^{L_0}\overline{q}^{\overline{L}_0}$$

is given by [97][equ'n (6)]

$$\begin{aligned} & q^{+(1/12)(r-n)} \\ & \quad \cdot \int_X \hat{A}(TX) \wedge \mathrm{ch}\Bigg(z^{-r/2} \left(\det \mathcal{E}\right)^{+1/2} \wedge_1 \left(z\mathcal{E}^*\right) \\ & \qquad \cdot \bigotimes_{k=1,2,3,\cdots} S_{q^k}((TX)^{\mathbb{C}}) \bigotimes_{k=1,2,3,\cdots} \wedge_{q^k} \left((z^{-1}\mathcal{E})^{\mathbb{C}}\right)\Bigg). \end{aligned}$$

Here, invariance under the interchange $\mathcal{E} \leftrightarrow \mathcal{E}^*$, $\gamma \leftrightarrow -\gamma$ is a consequence of the observations above plus the fact that

$$z^{-r/2} \left(\det \mathcal{E}\right)^{+1/2} \wedge_1 \left(z\mathcal{E}^*\right) \;=\; z^{+r/2} \left(\det \mathcal{E}\right)^{-1/2} \wedge_1 \left(z^{-1}\mathcal{E}\right). \tag{3.42}$$

Now, let us turn to (0,2) nonlinear sigma models with potential. The NS sector elliptic genus of a theory describing a cokernel $\mathcal{E}'$ of an injective map

$$0 \longrightarrow \mathcal{F}_1 \stackrel{\tilde{E}}{\longrightarrow} \mathcal{F}_2 \longrightarrow \mathcal{E}' \longrightarrow 0$$

is given by [97][equ'n (21)]

$$\begin{aligned} & q^{-(1/24)(+2n-r_1+r_2)} \\ & \quad \cdot \int_B \mathrm{Td}(TB) \wedge \mathrm{ch}\Bigg(\bigotimes_{k=1,2,3\cdots} S_{q^k}\left((TB)^{\mathbb{C}}\right) \bigotimes_{k=1/2,3/2,\cdots} S_{-q^k}\left((z^{-1}\mathcal{F}_1)^{\mathbb{C}}\right) \\ & \qquad \bigotimes_{k=1/2,3/2,\cdots} \wedge_{q^k}\left((z^{-1}\mathcal{F}_2)^{\mathbb{C}}\right)\Bigg). \end{aligned}$$

This should be compared to the NS sector elliptic genus of a theory describing a kernel of a surjective map

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{F}_1 \xrightarrow{F_a} \mathcal{F}_2 \longrightarrow 0,$$

which is given by [97][equ'n (24)]

$$\begin{aligned} & q^{-(1/24)(2n-r_2+r_1)} \\ & \quad \cdot \int_B \mathrm{Td}(TB) \wedge \mathrm{ch} \left(\bigotimes_{k=1,2,3,\cdots} S_{q^k}\left((TB)^{\mathbb{C}}\right) \bigotimes_{k=1/2,3/2,\cdots} S_{-q^k}\left((z\mathcal{F}_2^*)^{\mathbb{C}}\right) \right. \\ & \qquad \left. \bigotimes_{k=1/2,3/2,\cdots} \wedge_{q^k}\left((z\mathcal{F}_1^*)^{\mathbb{C}}\right) \right). \end{aligned}$$

The duality we are checking dualizes the sequences, so we must compare elliptic genera with

$$\mathcal{F}_2 \leftrightarrow \mathcal{F}_1^*$$

exchanged at the same time as $z \leftrightarrow z^{-1}$. It is straightforward to check that this operations maps the two elliptic genera into one another, and so these elliptic genera are compatible with the proposed duality.

Now, let us compare the R sector elliptic genera. For gauge bundles realized as cokernels as above, the R sector elliptic genus is given by [97][equ'n (22)]

$$\begin{aligned} & q^{-(1/24)(2n+2r_1-2r_2)} \\ & \quad \cdot \int_B \mathrm{Td}(TB) \wedge \mathrm{ch} \left(z^{+r_2/2} \wedge_1 (z^{-1}\mathcal{F}_2) z^{+r_1/2} \wedge_1 (z^{-1}\mathcal{F}_1) \right. \\ & \qquad \cdot (\det \mathcal{F}_2)^{-1/2} (\det \mathcal{F}_1)^{-1/2} \\ & \qquad \cdot \bigotimes_{k=1,2,3,\cdots} S_{q^k}((TB)^{\mathbb{C}}) \bigotimes_{k=0,1,2,\cdots} S_{-q^k}((z^{-1}\mathcal{F}_1)^{\mathbb{C}}) \\ & \qquad \left. \cdot \bigotimes_{k=1,2,3,\cdots} \wedge_{q^k}((z^{-1}\mathcal{F}_2)^{\mathbb{C}}) \right), \end{aligned}$$

and the R sector elliptic genus for a gauge bundle realized as a kernel is[16] [97][equ'n (25)]

$$
\begin{aligned}
& q^{-(1/24)(2n+2r_2-2r_1)} \\
& \quad \cdot \int_B \mathrm{Td}(TB) \wedge \mathrm{ch}\Bigg(z^{+r_1/2} \wedge_1 (z^{-1}\mathcal{F}_1) z^{-r_2/2} \wedge_1 (z\mathcal{F}_2^*) \\
& \qquad \cdot (\det \mathcal{F}_1)^{-1/2} (\det \mathcal{F}_2)^{1/2} \\
& \qquad \cdot \bigotimes_{k=1,2,3,\cdots} S_{q^k}((TB)^{\mathbb{C}}) \bigotimes_{k=0,1,2,\cdots} S_{-q^k}((z\mathcal{F}_2^*)^{\mathbb{C}}) \\
& \qquad \cdot \bigotimes_{k=1,2,3,\cdots} \wedge_{q^k}((z\mathcal{F}_1^*)^{\mathbb{C}}) \Bigg).
\end{aligned}
$$

As before, to compare, we must exchange

$$\mathcal{F}_1 \leftrightarrow \mathcal{F}_2^*$$

as well as $z \leftrightarrow z^{-1}$. It is straightforward to check that the expressions above are indeed exchanged under this operation, which implies that the elliptic genus is invariant under $\mathcal{E}' \leftrightarrow \mathcal{E}'^*$.

For completeness, if the gauge bundle is given by the cohomology of the short complex

$$0 \longrightarrow \mathcal{F}_0 \xrightarrow{\tilde{E}^a} \mathcal{F}_1 \xrightarrow{\tilde{F}_a} \mathcal{F}_2 \longrightarrow 0,$$

then the NS sector elliptic genus is given by

$$
\begin{aligned}
& q^{-(1/24)(2n-r_2-r_0+r_1)} \\
& \quad \cdot \int_B \mathrm{Td}(TB) \wedge \mathrm{ch}\Bigg(\bigotimes_{k=1,2,3,\cdots} S_{q^k}\left((TB)^{\mathbb{C}}\right) \\
& \qquad \cdot \bigotimes_{k=1/2,3/2,\cdots} S_{-q^k}\left((z\mathcal{F}_2^*)^{\mathbb{C}}\right) \bigotimes_{k=1/2,3/2,\cdots} S_{-q^k}\left((z^{-1}\mathcal{F}_0)^{\mathbb{C}}\right) \\
& \qquad \cdot \bigotimes_{k=1/2,3/2,\cdots} \wedge_{q^k}\left((z^{-1}\mathcal{F}_1)^{\mathbb{C}}\right) \Bigg).
\end{aligned}
$$

[16]The expression given above corrects a minor typo in [97][equ'n (25)], in the first version on the arXiv, which incorrectly listed a $(\det \mathcal{F}_2^*)^{1/2}$ which should have been a $(\det \mathcal{F}_2)^{1/2}$.

In order for the elliptic genus to be invariant under $\mathcal{E}' \leftrightarrow \mathcal{E}'^*$ would require invariance of the expressions above under

$$\mathcal{F}_0 \leftrightarrow \mathcal{F}_2^*, \quad \mathcal{F}_1 \leftrightarrow \mathcal{F}_1^*, \quad z \leftrightarrow z^{-1},$$

and it is straightforward to check that the expression above is indeed so invariant.

The R sector elliptic genus is given by [97][equ'n (27)]

$$\begin{aligned} & q^{-(1/24)(2n+2r_0+2r_2-2r_1)} \\ & \quad \cdot \int_B \mathrm{Td}(TB) \wedge \mathrm{ch}\Bigg(z^{+r_1/2} \wedge_1 (z^{-1}\mathcal{F}_1) z^{+r_0/2} \wedge_1 (z^{-1}\mathcal{F}_0) z^{-r_2/2} \wedge_1 (z\mathcal{F}_2^*) \\ & \qquad \cdot (\det \mathcal{F}_1)^{-1/2} (\det \mathcal{F}_0)^{-1/2} (\det \mathcal{F}_2)^{+1/2} \\ & \qquad \cdot \bigotimes_{k=1,2,3,\cdots} S_{q^k}((TB)^{\mathbb{C}}) \bigotimes_{k=0,1,2,\cdots} S_{-q^k}((z^{-1}\mathcal{F}_0)^{\mathbb{C}}) \\ & \qquad \cdot \bigotimes_{k=0,1,2,\cdots} S_{-q^k}((z\mathcal{F}_2^*)^{\mathbb{C}}) \bigotimes_{k=1,2,3,\cdots} \wedge_{q^k}((z^{-1}\mathcal{F}_1)^{\mathbb{C}}) \Bigg). \end{aligned}$$

In order for the elliptic genus to be invariant under $\mathcal{E}' \leftrightarrow \mathcal{E}'^*$ would require invariance of the expressions above under

$$\mathcal{F}_0 \leftrightarrow \mathcal{F}_2^*, \quad \mathcal{F}_1 \leftrightarrow \mathcal{F}_1^*, \quad z \leftrightarrow z^{-1},$$

and it is straightforward to check that the expression above is indeed so invariant, using (3.42).

Worldsheet instantons

Worldsheet instanton corrections in this context were discussed in [123], which argued for a simple relation between the A/2 and B/2 models:

$$\mathrm{A}/2(X, \mathcal{E}) \; = \; \mathrm{B}/2(X, \mathcal{E}^*),$$

or more precisely, there existed regularizations (compactifications of the moduli space of worldsheet instantons) compatible with the statements above. (For more information on worldsheet instantons in heterotic strings, see for example [112, 132–134] and references therein.)

One of the corners specifically explored in [123] is the special case relating the ordinary B model on X (the B/2 model on $(X, \mathcal{E} = TX)$ to the A/2 model on $(X, \mathcal{E}^* = T^*X)$). Specifically, a worldsheet instanton such that $\phi^*TX \cong \phi^*T^*X$, as arises in genus zero if the normal bundle is $\mathcal{O} \oplus \mathcal{O}(-2)$, seems to provide a potential contradiction: the B model does not receive worldsheet instanton corrections, but the A/2 model typically will receive worldsheet instanton corrections. It was observed in [123] that in such cases, in simple examples, there were two moduli space compactifications, one reproducing B model results, the other reproducing A/2 model results. Thus, so long as the regularization is exchanged consistent with the theory, the worldsheet instanton counting was consistent.

In any event, it is believed that the A/2 and B/2 models are exchanged when the gauge bundle is dualized, consistent with the interpretation of flipping the sign of a left $U(1)$ symmetry.

Reducible gauge bundles

If the gauge bundle is reducible, then we conjecture that the (0,2) QFT's remain isomorphic after dualizing the various factors separately.

Much of our analysis in the rest of this section applies with little change, for example:

- Massless spectra in Calabi-Yau compactifications are invariant under dualizing factors separately.
- Elliptic genera are invariant (so long as the vector bundle is complex, which we have

assumed throughout).

- As there are now several left $U(1)$ symmetries, there are potentially several analogues of the A/2 and B/2 models, involving different sets of twists on left-moving fermions, and with different compatibility conditions generalizing the A/2 condition $\det \mathcal{E}^* \cong K_X$. If multiple twists exist, the duality here should exchange them.

In the examples we shall encounter in section 3.3.7, there is another way of thinking about this in the UV GLSM. In those examples, the duality is applied to Fermi superfields which are not coupled via a superpotential or other supersymmetry transformations to the other matter fields. The theory appears invariant under dualizing the representation of those Fermi superfields, which implies an IR duality of the form discussed here.

One point that is more subtle, however, involves the role of stability. The stability condition shows up in worldsheet beta functions, and so is necessary to have a nontrivial IR conformal fixed point. Dualizing one of the factors will flip the sign of the slope of that factor, likely destabilizing the bundle. However, because stability only enters via beta functions, we need only be concerned with its role in Calabi-Yau compactifications, and in such compactification, if the gauge bundle is reducible, each factor will have vanishing slope, hence the slopes are unaffected by dualizing factors. Each factor must still be stable, but as previously discussed, a bundle is stable if and only if its dual bundle is also stable. For a more extensive discussion of compactifications on reducible gauge bundles in the context of stability, see for example [135–138].

Example of (0,2) dual to (2,2)

For completeness, let us give an example of a nonabelian (0,2) GLSM which, assuming the conjectured duality is correct, will RG flow to a (2,2) GLSM, specifically to the (2,2) GLSM for the Grassmannian $G(k, n)$.

Specifically, consider a (0,2) GLSM on $G(k,n)$ for gauge bundle $\mathcal{E} = T^*G(k,n) = S \otimes Q^*$:

$$0 \longrightarrow \mathcal{E} \longrightarrow S \otimes \mathcal{O}^n \longrightarrow S \otimes S^* \longrightarrow 0.$$

This is described by the (0,2) $U(k)$ gauge theory with the following matter content:

- n chiral superfields Φ in the fundamental representation,
- 1 chiral superfield P in the adjoint representation,
- n Fermi superfields Γ in the antifundamental representation,

plus a (0,2) superpotential of the form

$$W \;=\; \Gamma P \Phi.$$

It is straightforward to check that this nonabelian (0,2) GLSM satisfies anomaly cancellation. From our conjectures above, it should be in the same universality class as the (2,2) GLSM for $G(k,n)$.

Relation to (0,2) mirror symmetry

Depending upon how one defines (0,2) mirror symmetry (see *e.g.* [139–141] for some recent reviews), the duality we have just discussed might be considered an example. After all, the duality we have discussed has the properties that it flips the sign of a left-moving $U(1)$ (in Calabi-Yau examples), it rotates sheaf cohomology groups, and exchanges the A/2 and B/2 models, in precisely the same fashion as one would expect of (0,2) mirror symmetry.

On the other hand, when this duality acts on a (2,2) A-twisted theory on a space X, for example it generates the B/2 model on (X, T^*X) rather than a (2,2) B-twisted theory on the ordinary mirror Y. So, it does not specialize to ordinary mirror symmetry, but then again, we do not expect (0,2) mirror symmetry for most (0,2) theories to be related easily to ordinary mirror symmetry. Only when the gauge bundle is a deformation of the tangent bundle is such a relation possible.

The most conservative description of how this duality relates to (0,2) mirrors is encapsulated in the following diagram:

$$\begin{array}{ccc} \mathrm{B}/2(X,\mathcal{E}^*) & = & \mathrm{A}/2(Y,\mathcal{F}^*) \\ \| & & \| \\ \mathrm{A}/2(X,\mathcal{E}) & = & \mathrm{B}/2(Y,\mathcal{F}). \end{array}$$

In this diagram, horizontal lines indicate ordinary (0,2) mirrors, and vertical lines indicate the duality discussed here. For example, the (0,2) theory defined by $(X,\mathcal{E})$ is (0,2) mirror – in the conventional sense – to $(Y,\mathcal{F})$.

Another possibility is that the notion of (0,2) mirrors might be much more general than previously considered. Much of (0,2) mirror symmetry is motivated by the example of ordinary mirror symmetry, which is a relation between single pairs of spaces, hence many workers have long thought of (0,2) mirrors as also being relations between single pairs of spaces and bundles. However, it is also possible that a given (0,2) theory might admit a variety of different (0,2) mirrors – the family of dualities might be much more complicated than previously considered. Perhaps the duality discussed in this section should be interpreted as an indication of such a more complicated structure. We leave this issue for future work.

3.3.7 Mathematics of Gadde-Gukov-Putrov triality

In this section we will describe[17] the Gadde-Gukov-Putrov triality [88] from a mathematical perspective, as an example of a nontrivial (0,2) duality.

We begin by working through the mathematical dualities one encounters in their picture, *i.e.* relating $G(k,n)$ to $G(n-k,n)$, with suitable gauge bundles. We shall find a twelve-step

[17]We have been told this will also be discussed in [142].

duality formally; however, not all of the bundles appearing admit a (0,2) GLSM description. This can be fixed by applying physical dualities between (0,2) theories with dual gauge bundles, at which point this will effectively truncate to a three-step duality, their triality.

To begin, consider the bundle

$$S_k^{\oplus A} \oplus (Q_{n-k}^*)^{\oplus B} \longrightarrow G(k,n).$$

Under the relation $G(k,n) = G(n-k,n)$, the bundles are related as follows:

$$\begin{aligned} S_k &\leftrightarrow Q_k^*, \\ Q_{n-k}^* &\leftrightarrow S_{n-k}, \end{aligned}$$

so we see that the bundle above is the same as

$$(Q_k^*)^{\oplus A} \oplus S_{n-k}^{\oplus B} \longrightarrow G(n-k,n).$$

Now, the bundles Q^* above cannot be realized directly in the GLSM, but they can be realized indirectly, in mathematics as kernels:

$$0 \longrightarrow Q_{n-k}^* \longrightarrow \mathcal{O}^n \longrightarrow S_k^* \longrightarrow 0,$$

and in physics by adding a set of n neutral Fermi fields and a chiral superfield transforming in the antifundamental[18], together with a (0,2) superpotential.

For example, ignoring anomalies for the moment, the bundle

$$S_k^{\oplus A} \oplus (Q_{n-k}^*)^{\oplus B} \longrightarrow G(k,n)$$

is realized physically by a $U(k)$ gauge theory containing

- n chiral superfields Φ_i each in the fundamental representation of $U(k)$,

[18] The chiral superfield should couple to the dual of the bundle appearing in the third term, *i.e.* to S_k, which means it corresponds to the antifundamental.

- B chiral superfields P^i each in the antifundamental representation of $U(k)$,
- A Fermi superfields in the antifundamental representation of $U(k)$,
- nB neutral Fermi superfields Γ,
- a (0,2) superpotential $\Gamma\Phi P$.

Gauge anomaly cancellation constrains the values of A, B, k, n. For simplicity, we will use the decomposition $u(k) \cong su(k) \oplus u(1)$ and work out anomaly cancellation in terms of the constituent summands. With the benefit of hindsight, to cancel gauge anomalies, we add Fermi superfields Ω transforming only under $\det U(k)$, then the theory above contains the following matter fields, charged under $su(k) \oplus u(1)$:

	type	multiplicity	$su(k)$	$u(1)$
Φ	chiral	n	$\mathbf{k}$	1
P	chiral	B	$\overline{\mathbf{k}}$	-1
Γ	Fermi	nB	$\mathbf{1}$	0
Ψ	Fermi	A	$\overline{\mathbf{k}}$	-1
λ	fermion	1	ad	0
Ω	Fermi	2	$\mathbf{1}$	k

Using the indices given in appendix D, the $su(k)^2$ gauge anomaly is

$$nk\frac{k^2-1}{k} \,+\, Bk\frac{k^2-1}{k} \,=\, Ak\frac{k^2-1}{k} \,+\, (k^2-1)(2k),$$

and the $u(1)^2$ gauge anomaly is

$$nk \,+\, Bk \,=\, Ak \,+\, 2k^2,$$

which imply the following constraint:

$$2k \,=\, n \,+\, B \,-\, A.$$

The theory describing the same bundle in the dual description, namely

$$(Q_k^*)^{\oplus A} \oplus S_{n-k}^{\oplus B} \longrightarrow G(n-k, n),$$

is a $U(n-k)$ gauge theory containing

- n chiral superfields $\tilde{\Phi}_i$ each in the fundamental representation of $U(n-k)$,
- A chiral superfields $\tilde{P}^i$ each in the antifundamental representation of $U(n-k)$,
- B Fermi superfields in the antifundamental representation of $U(n-k)$,
- nA neutral Fermi superfields $\tilde{\Gamma}$,
- a (0,2) superpotential $\tilde{\Gamma}\tilde{\Phi}\tilde{P}$.

It is straightforward that adding a pair of Fermi superfields Ω, each of charge $n-k$, cancels the gauge anomaly so long as the same constraint from before, namely

$$2k = n + B - A$$

is obeyed. More generally, it is straightforward to check that in all the duality frames discussed before, a pair of Ω's can be added to cancel anomalies, subject to the same constraint as above, so henceforward we will omit the Ω's and take the constraint as given.

Returning to the physical realization of the bundle

$$S_k^{\oplus A} \oplus (Q_{n-k}^*)^{\oplus B} \longrightarrow G(k, n),$$

it is straightforward to see that the (0,2) theory describing this phase has a second distinct Kähler phase describing the bundle

$$(S_k^*)^{\oplus A} \oplus (Q_{n-k}^*)^{\oplus n} \longrightarrow G(k, B),$$

essentially obtained by flipping the interpretation of fundamental and antifundamental representations. (Note that the interpretation of the Fermi superfields describing the S factor also therefore flips, so here we have S^* rather than S in the gauge bundle.) This second Kähler phase also has a dual description, and in this fashion we can construct a chain of dualities.

The first few steps of this chain of dualities are as follows:

$$\begin{array}{ccc}
S^A \oplus (Q^*)^{2k+A-n} \to G(k,n) & \text{------} & (S^*)^A \oplus (Q^*)^n \to G(k, 2k+A-n) \\
\updownarrow = & & \\
(Q^*)^A \oplus S^{2k+A-n} \to G(n-k,n) & \text{-----} & (Q^*)^n \oplus (S^*)^{2k+A-n} \to G(n-k, A) \\
& & \updownarrow = \\
(S^*)^n \oplus Q^A \to G(A-n+k, 2k+A-n) & \text{---} & S^n \oplus Q^{2k+A-n} \to G(A-n+k, A).
\end{array}$$

Horizontal (dashed) lines indicate different Kähler phases; vertical lines indicate mathematical dualities between descriptions of the same object. Formally, if one were to continue for a total of six steps, one would get to a GLSM with the same Grassmannians as the first line, but dual bundles.

We ran into a potential problem in section 3.3.5 in describing chains of dualities of the form above, defined by RG flow and the existence of Coulomb vacua in certain phases. Although these models have FI parameters that will certainly RG flow, there is no σ field in these models, hence no Coulomb vacua to obstruct dualities as in section 3.3.5.

A second problem is less trivial. Specifically, the geometries indicated on the third line above cannot be realized in (0,2) GLSM's. The problem is that the gauge bundle on the third line involves copies of Q. To realize Q as part of the gauge bundle in a (0,2) GLSM, we would need to realize it as the cokernel in a short exact sequence of a form previously described, and to do so, we would need chiral superfields in representations corresponding to the dual of S. This is a problem – such chiral superfields would then be in the same representation as those defining the underlying Grassmannian, so instead of building a bundle, one would build a larger Grassmannian. We discussed an analogous difficulty in (2,2) GLSM's in section 3.3.5; as discussed there, the issue here is the analogue of trying to build a (2,2) GLSM for the total space of $\mathcal{O}(+1) \to \mathbb{P}^n$ – the chiral superfield for the fibers, has the same charges as those appearing in the base, so the obvious GLSM would instead describe $\mathbb{P}^{n+1}$.

Instead, we can dualize the gauge bundle, as in section 3.3.6. Doing so, and using a dashed vertical arrow to indicate a physical isomorphism which is not also a mathematical equivalence, we are led to the duality chain

$$\begin{tikzcd}
S^A \oplus (Q^*)^{2k+A-n} \to G(k,n) \arrow[r, dash, dashed] \arrow[d, leftrightarrow, "="'] & (S^*)^A \oplus (Q^*)^n \to G(k, 2k+A-n) \\
(Q^*)^A \oplus S^{2k+A-n} \to G(n-k,n) \arrow[r, dash, dashed] & (Q^*)^n \oplus (S^*)^{2k+A-n} \to G(n-k, A) \arrow[d, leftrightarrow, dashed, "\cong"] \\
S^n \oplus (Q^*)^A \to G(A-n+k, 2k+A-n) \arrow[r, dash, dashed] \arrow[d, leftrightarrow, "="'] & (S^*)^n \oplus (Q^*)^{2k+A-n} \to G(A-n+k, A) \\
(Q^*)^n \oplus S^A \to G(k, 2k+A-n) \arrow[r, dash, dashed] & (Q^*)^{2k+A-n} \oplus (S^*)^A \to G(k,n) \arrow[d, leftrightarrow, dashed, "\cong"] \\
S^{2k+A-n} \oplus (Q^*)^n \to G(n-k, A) \arrow[r, dash, dashed] \arrow[d, leftrightarrow, "="'] & (S^*)^{2k+A-n} \oplus (Q^*)^A \to G(n-k,n) \\
(Q^*)^{2k+A-n} \oplus S^n \to G(k+A-n, A) \arrow[r, dash, dashed] & (Q^*)^A \oplus (S^*)^n \to G(k+A-n, 2k+A-n) \arrow[d, leftrightarrow, dashed, "\cong"] \\
S^A \oplus (Q^*)^{2k+A-n} \to G(k,n) \arrow[r, dash, dashed] & (S^*)^A \oplus (Q^*)^n \to G(k, 2k+A-n).
\end{tikzcd}$$

After six steps we have returned to our starting point, but in fact one can do better. If we were to dualize the S factors in the gauge bundle in the fourth line, applying the duality discussed in section 3.3.6, then the diagram above would reduce to

$$\begin{array}{ccc}
S^A \oplus (Q^*)^{2k+A-n} \to G(k,n) & - - - - - - - & (S^*)^A \oplus (Q^*)^n \to G(k, 2k+A-n) \\
\updownarrow = & & \\
(Q^*)^A \oplus S^{2k+A-n} \to G(n-k,n) & - - - - - - & (Q^*)^n \oplus (S^*)^{2k+A-n} \to G(n-k, A) \\
& & \updownarrow \cong \\
S^n \oplus (Q^*)^A \to G(A-n+k, 2k+A-n) & - - - & (S^*)^n \oplus (Q^*)^{2k+A-n} \to G(A-n+k, A) \\
\cong \updownarrow & & \\
(Q^*)^n \oplus (S^*)^A \to G(k, 2k+A-n) & - - - - - - - & (Q^*)^{2k+A-n} \oplus S^A \to G(k,n).
\end{array}$$

The fourth line is now identical to the first, except that the Fayet-Iliopoulos parameter has been reversed. In this fashion we can understand this as a triality symmetry, as described in [88].

As in section 3.3.5, we have only described the duality chain moving in one direction; one could also move vertically.

More degenerate examples exist with shorter periodicities. For example, if $n = 2k$, $A = 2k$, then we have

$$\begin{array}{ccc}
S^{2k} \oplus (Q^*)^{2k} \to G(k, 2k) & - - - & (S^*)^{2k} \oplus (Q^*)^{2k} \to G(k, 2k) \\
\updownarrow = & & \\
(Q^*)^{2k} \oplus S^{2k} \to G(k, 2k) & - - - & (Q^*)^{2k} \oplus (S^*)^{2k} \to G(k, 2k).
\end{array}$$

In effect, the (0,2) GLSM defined by

$$S^{2k} \oplus (Q^*)^{2k} \to G(k, 2k)$$

is self-dual.

3.3.8 Relation between models of Pfaffians

In section 3.3.4 we reviewed the construction of (2,2) GLSM's for Pfaffian varieties, and also extended those constructions to (0,2) GLSM's. For any given Pfaffian and bundle, there were a pair of constructions, known as the PAX and PAXY models. Reference [73] described how the (2,2) PAX and PAXY models were related.

In this section, we will use (2,2) and (0,2) nonabelian gauge theory dualities to update the discussion of [73][section 3.4], and also extend to (0,2) cases.

(2,2) GLSM's

Let us begin by rewriting the analysis of [73][section 3.4] utilizing the two-dimensional analogue of Seiberg duality introduced in [81] and reviewed in section 3.3.5.

Briefly, begin with the PAX model. Here one has, in addition to the data defining a toric variety and a matrix A defined over that toric variety, a $U(n-k)$ gauge theory, a set of n fundamentals encoded in an $n \times (n-k)$ matrix P, a set of n antifundamentals encoded in an $n \times (n-k)$ matrix X, and a superpotential of the form

$$W \; = \; \mathrm{tr}\, PAX.$$

Now, let us apply the duality of [81]. The dual theory will be a $U(k)$ gauge theory, with an $n \times n$ matrix of neutral mesons $\tilde{P}$, related to the charged matter of the original theory by

$$\tilde{P} \; = \; XP,$$

as well as a new set of n fundamentals, encoded in a $n \times k$ matrix $\tilde{X}$, a new set of n antifundamentals, encoded in an $n \times k$ matrix $\tilde{Y}$, and, just from the duality, a superpotential

$$W' \; = \; \mathrm{tr}\, \tilde{X} \tilde{P} \tilde{Y},$$

closely following the pattern of four-dimensional Seiberg duality [143]. If we combine the duality contribution with the original superpotential written in dual variables, we find that the complete superpotential for the theory dual to the PAX model is

$$W \; = \; \mathrm{tr}\, \tilde{P} \left(A(\Phi) \; + \; \tilde{Y} \tilde{X} \right).$$

After a trivial field redefinition, this becomes

$$W \;=\; \mathrm{tr}\, \tilde{P} \left(A(\Phi) \;-\; \tilde{Y}\tilde{X} \right),$$

which exactly matches the PAXY model.

Thus, we see that the (2,2) PAX and PAXY models are related by a simple application of the duality discussed in [81] and section 3.3.5.

(0,2) generalizations

Now, let us apply analogous ideas to compute the dual of a more general (0,2) PAX model. We shall begin by studying how deformations of the tangent bundle in the PAX model map to analogous deformations in the PAXY model. Recall that deformations of the tangent bundle are described by a (0,2) PAX model fields as described in section 3.3.4 and with superpotential

$$W \;=\; \mathrm{tr} \left(\Lambda_P A(\Phi) X \;+\; P A(\Phi) \Lambda_X \;+\; P \left(\frac{\partial A(\Phi)}{\partial \Phi^\alpha} \;+\; G_\alpha(\Phi) \right) \Lambda_\Phi^\alpha X \right).$$

Since the deformation is encoded in the superpotential, the fields themselves are the same as in the (2,2) GLSM, so we can apply essentially the same duality as in the (2,2) case, albeit re-expressed in terms of (0,2) superfields. Thus, the dual gauge theory will be a $U(k)$ gauge theory (plus another abelian factor, which will go along for the ride), with

1. an $n \times n$ matrix of neutral (meson) chiral superfields $\tilde{P}$ and Fermi superfields $\Lambda_{\tilde{P}}$, related to fields of the original theory by

$$\tilde{P} \;=\; XP, \;\; \Lambda_{\tilde{P}} \;=\; \Lambda_X P \;+\; X \Lambda_P,$$

2. a new set of n fundamentals , encoded in a $n \times k$ matrix $\tilde{X}$,

3. a new set of n antifundamentals, encoded in a $n \times k$ matrix $\tilde{Y}$,

4. a superpotential term

$$W' \;=\; \mathrm{tr}\left(\Lambda_{\tilde{P}} \tilde{X} \tilde{Y} \;+\; \tilde{P} \Lambda_{\tilde{X}} \tilde{Y} \;+\; \tilde{P} \tilde{X} \Lambda_{\tilde{Y}} \right).$$

When the new superpotential term is added to the previous superpotential expressed in terms of the dual variables, namely

$$\mathrm{tr}\left(\Lambda_{\tilde{P}} A(\Phi) \;+\; \tilde{P} \left(\frac{\partial A(\Phi)}{\partial \Phi^{\alpha}} \;+\; G_{\alpha}(\Phi) \right) \Lambda_{\Phi}^{\alpha} \right),$$

we get the full (0,2) superpotential of the dual theory:

$$W \;=\; \left(\Lambda_{\tilde{P}} A(\Phi) \;+\; \tilde{P} \left(\frac{\partial A(\Phi)}{\partial \Phi^{\alpha}} \;+\; G_{\alpha}(\Phi) \right) \Lambda_{\Phi}^{\alpha} \;+\; \Lambda_{\tilde{P}} \tilde{X} \tilde{Y} \;+\; \tilde{P} \Lambda_{\tilde{X}} \tilde{Y} \;+\; \tilde{P} \tilde{X} \Lambda_{\tilde{Y}} \right).$$

Modulo absorbing signs into trivial field redefinitions, this is the same as the PAXY theory for the deformation off the (2,2) locus given in equation (3.40).

Thus, we see the duality between PAX and PAXY models extends to deformations off the (2,2) locus.

Now, let us consider an example of a more general case, a gauge bundle given as a kernel. We follow the same conventions as in section 3.3.4. In other words, to build the Pfaffian itself, we will need a $U(n-k)$ gauge theory, n chiral superfields in the fundamental, forming an $n \times (n-k)$ matrix denoted X, and n Fermi superfields in the antifundamental, forming an $n \times (n-k)$ matrix of Fermi superfields denoted Λ_0. If the gauge bundle $\mathcal{E}$ is given as a kernel of the form

$$0 \;\longrightarrow\; \mathcal{E} \;\longrightarrow\; \oplus_{\beta} \mathcal{O}((\lambda_{\beta 1}, \lambda_{\beta 2}), q_{a,\beta}) \;\xrightarrow{F_{\beta}^{\gamma}}\; \oplus_{\gamma} \mathcal{O}((\lambda_{\gamma 1}, \lambda_{\gamma 2}, q_{a,\gamma}) \;\longrightarrow\; 0,$$

then we add a set of Fermi superfields Λ^{β} in the $(\lambda_{\beta 1}, \lambda_{\beta 2})$ representation of $U(n-k)$ and with charges $q_{a,\beta}$ under the abelian gauge symmetry $U(1)^r$ defining the toric variety, along with a set of chiral superfields P_{γ} in the $U(n-k)$ representation dual to $(\lambda_{\gamma 1}, \lambda_{\gamma 2})$ and with charges $-q_{a,\gamma}$ under the abelian gauge symmetry defining the toric variety. In addition, we have a (0,2) superpotential

$$W \;=\; \mathrm{tr}\left(\Lambda_0 A(\Phi) X \;+\; \Lambda^\beta F_\beta^\gamma(\Phi) P_\gamma \right).$$

However, we quickly run into a problem. The gauge bundle $\mathcal{E}$ above is defined in terms of representations of $U(n-k)$, in the PAX model. However, in the PAXY model, the gauge bundle is defined in terms of representations of $U(k)$. Now, it is possible to write down long exact sequences relating representations of one to the other, as we shall discuss in section 3.3.9, but as we shall discuss there, to be relevant for (0,2) constructions, we must restrict to duals involving three-term sequences, which are comparatively rare. Thus, we do not expect to be able to construct PAXY duals of most (0,2) PAX models, and also conversely. This is a special case of a more general obstruction we shall discuss in section 3.3.9.

3.3.9 More general bundles and obstructions to duality

So far, we have discussed dualities for closed string (2,2) and (0,2) $U(k)$ gauge theories with matter in fundamental and antifundamental representations. In this section, we will discuss more general matter representations. We will discuss how arbitrary matter representations can be dualized in open strings, and also discuss obstructions to duality for more general matter representations in closed string (2,2) and (0,2) theories.

Duality for $U(k)$ gauge theories in open strings

The key to our deliberations so far has been that the bundles S_k, Q_{n-k} over the Grassmannian $G(k,n)$ are the same as the bundles Q_k^*, S_{n-k}^* over the Grassmannian $G(n-k,n)$. In each case, the universal subbundle is defined by the antifundamental representation (in conventions in which the matter defining the Grassmannian itself is in the fundamental representation), and the universal quotient bundle is built as the cokernel in a short exact sequence, which can be realized physically.

In open strings, the Chan-Paton factors couple to complexes of bundles defined by representations of the gauge group, so in open strings on a GLSM for $G(k,n)$, the Chan-Paton factors are defined by complexes of bundles defined by $U(k)$ representations.

Suppose we start with Chan-Paton factors coupling to a single bundle $\mathcal{O}(\rho)$, defined by some representation ρ of $U(k)$. In the notation of appendix D, if the representation is defined by

$$\rho \;\equiv\; (\lambda_1, \lambda_2, \cdots, \lambda_k),$$

where each $\lambda_i \geq \lambda_{i+1}$, then we can construct $\mathcal{O}(\rho)$ from suitable tensor products of powers of S and S^*. Schematically,

$$\mathcal{O}(\rho) \;=\; K_\rho(S) \otimes (\det S^*)^{\lambda_k}$$

where K_ρ is the tensor product defined by the $SU(k)$ Young diagram associated to ρ.

In the dual $U(n-k)$ gauge theory, the Chan-Paton factors in principle should couple to

$$K_\rho(Q^*) \otimes (\det Q)^{\lambda_k}.$$

However, the bundle $Q \to G(n-k,n)$ is not given directly by a representation of $U(n-k)$. Instead, it is always possible to find a long exact sequence of bundles defined by representations of $U(n-k)$ that 'resolves' the bundle above, and so we can replace the bundle above by its resolution. The resolution then gives well-defined Chan-Paton factors in the dual gauge theory, which in principle must result in an open string in the same universality class as the original heterotic string.

As a consistency check, note that the tangent bundle of the Grassmannian is the cokernel of

$$\{0 \longrightarrow S^\vee \otimes S \longrightarrow S^\vee \otimes \mathcal{O}^n\} \;=\; S^\vee \otimes \{0 \longrightarrow S \longrightarrow \mathcal{O}^n\},$$

which is precisely $S^\vee$ tensored with the dual of the complex representing the dual S, and hence is manifestly symmetric under the duality $G(k,n) \leftrightarrow G(n-k,n)$, which is very satisfying.

Let us consider a less trivial example, namely the bundle $\mathcal{O}(\mathbf{k} \otimes \mathbf{k})$. This is dual to the tensor product of two copies of the complex $\{S_{n-k} \to \mathcal{O}^n\}$ on $G(n-k,n)$. In general, give two chain complexes $P.$, $Q.$, we can define a complex $P \otimes Q$ by taking [144][chapter 2.7]

$$(P \otimes Q)_n \;=\; \bigoplus_{p+q=n} P_p \otimes Q_q,$$

with differential $d \otimes 1 + (-)^p \otimes d$. In the present case, this yields the complex

$$\mathcal{O}^{n^2} \;\longrightarrow\; \oplus_1^2 (S^*_{n-k})^{\oplus n} \;\longrightarrow\; S^*_{n-k} \otimes S^*_{n-k},$$

which we claim is the open string dual to the Chan-Paton bundle $\mathcal{O}(\mathbf{k}\otimes\mathbf{k})$ in the $U(k)$ gauge theory corresponding to $G(k,n)$ (in the bulk of the open string). (As a check, note that the rank is $n^2 - 2n(n-k) + (n-k)^2 = k^2$, as expected.)

For another example, suppose instead the Chan-Paton bundle in the $U(k)$ gauge theory corresponding to $G(k,n)$ was given by the bundle $\wedge^p S \to G(k,n)$ for some $p > 1$. Under the duality, $\wedge^p S \mapsto \wedge^p Q^*$. However, $\wedge^p Q^*$ can not be resolved by a three-term sequence involving only bundles defined by representations of $U(k)$. Instead, it can be resolved as

$$0 \;\to\; \wedge^p Q^* \;\to\; \wedge^p \mathcal{O}^n \;\to\; S^* \otimes \wedge^{p-1}\mathcal{O}^n \;\to\; \cdots \;\to\; \mathrm{Sym}^{p-1} S^* \otimes \mathcal{O}^n \;\to\; \mathrm{Sym}^p S^* \;\to\; 0.$$

Thus, in the dual gauge theory, Chan-Paton factors describing the complex

$$\wedge^p \mathcal{O}^n \;\longrightarrow\; S^* \otimes \wedge^{p-1}\mathcal{O}^n \;\longrightarrow\; \cdots \;\longrightarrow\; \mathrm{Sym}^{p-1} S^* \otimes \mathcal{O}^n \;\longrightarrow\; \mathrm{Sym}^p S^*$$

over $G(n-k,n)$ should be in the same universality class as the original Chan-Paton bundle

$$\wedge^p S$$

over $G(k,n)$ in the original gauge theory.

Obstructions to duality in (0,2) theories

Now, let us apply the same ideas to (0,2) and closed-string (2,2) theories. In a (0,2) theory, we can talk about dualizing the gauge bundle; in a closed-string (2,2) theory describing the total space of the bundle, we can talk about dualizing to another closed-string theory describing the same bundle.

In both cases, there is a potential obstruction to making sense of the duality, lying in the fact that in each case, it is not currently known how to realize longer than three-term complexes.

For example, in a (0,2) GLSM, the gauge bundle can be realized as a kernel, cokernel, or as the cohomology of a three-term monad, but it is not currently known how to physically realize sequences longer than three terms.

As a result, for example, although we can certainly write down a (0,2) GLSM describing the bundle $\wedge^p S \to G(k,n)$ for $p > 1$, it is not known at present how to realize its dual over $G(n-k,n)$ in a (0,2) GLSM, because it involves a complex of length greater than three, barring the use of a physical duality that does not correspond to a mathematical one.

Similarly in (2,2) nonabelian GLSM's, we can describe target spaces that are total spaces of any bundle defined by a $U(k)$ representation over $G(k,n)$, for example, but unless the dual is defined in terms of a three-term sequence, we do not currently know how to describe it with superpotentials and so forth, and so we cannot currently describe it.

For this reason, we conjecture[19] that (0,2) and closed string (2,2) GLSM's describing bundles over $G(k,n)$ corresponding to representations of $U(k)$ other than fundamentals, antifundamentals, and adjoints, do not have Seiberg-like duals. Of course, new physical relationships, unmotivated by mathematics, could easily modify that conclusion.

[19]We hesitate to formulate this as a no-go theorem, as we are reminded of the old saying, "Never trust a no-go theorem until after some counterexamples are known."

By contrast, in open string theories there is no such restriction, and we expect all Chan-Paton factors in corresponding open string GLSM's to have duals.

Q_1	m_1	$(\lambda_{A1}, \lambda_{B1})$	Q_2	m_2	$(\lambda_{A2}, \lambda_{B2})$	Q_3	n_1	$(\lambda_{A3}, \lambda_{B3})$	Q_4	n_2	$(\lambda_{A4}, \lambda_{B4})$	rank
-1	5	(0, 0)	0	2	(2, 2)	-1	2	(2, 2)	-1	1	(1, -1)	2
-1	4	(1, -1)	3	1	(2, 2)	1	1	(2, 2)	-2	1	(2, -2)	7
0	5	(1, 0)	4	2	(2, 2)	4	1	(2, 1)	0	2	(2, 0)	4
-2	2	(0, 0)	1	4	(1, 1)	3	1	(1, 1)	-1	1	(2, 0)	2
-1	4	(0, 0)	0	4	(1, 1)	-1	1	(1, 1)	-1	1	(2, 0)	4
-2	2	(0, 0)	0	5	(1, 1)	-2	2	(1, 1)	0	1	(2, 0)	2
-3	1	(0, -1)	3	5	(1, 0)	2	2	(1, 1)	5	1	(2, -1)	6
-2	5	(0, 0)	0	2	(1, 1)	-2	2	(1, 1)	-2	1	(1, -1)	2
-2	5	(0, 0)	1	1	(1, 1)	-3	1	(1, 1)	-2	1	(1, -1)	2
-2	4	(1, -1)	5	2	(1, 1)	3	2	(1, 1)	-4	1	(2, -2)	7
-1	5	(1, 0)	5	1	(2, 1)	2	2	(2, 0)	-3	2	(1, 0)	2
0	4	(1, 0)	2	1	(2, 1)	0	2	(2, 0)	2	1	(1, 0)	2
0	4	(1, 0)	2	1	(2, 2)	0	2	(2, 0)	2	1	(0, 0)	2
-4	5	(0, 0)	-1	2	(1, 0)	-3	2	(1, 0)	-4	1	(1, -1)	2
-4	5	(0, 0)	0	1	(1, 0)	-4	1	(1, 0)	-4	1	(1, -1)	2
-1	3	(1, -1)	0	4	(1, 0)	-1	1	(0, 0)	-1	2	(2, -1)	8
-4	1	(1, -1)	-1	2	(1, 0)	-4	1	(0, 0)	-3	1	(2, -1)	2
1	2	(0, -1)	4	1	(1, -1)	4	1	(0, 0)	3	1	(1, -2)	2
0	4	(0, -1)	1	3	(1, -1)	1	1	(0, 0)	1	2	(1, -2)	8
-2	1	(-2, -2)	0	4	(0, -1)	-2	1	(0, 0)	0	2	(0, -2)	2
-4	5	(0, 0)	-1	1	(2, -1)	-3	1	(2, -1)	-4	1	(1, -1)	2
0	1	(0, -1)	4	5	(0, 0)	4	1	(1, -1)	4	1	(0, -1)	2
1	2	(0, -1)	4	5	(0, 0)	4	1	(1, -1)	3	2	(0, -1)	2
-1	1	(-1, -1)	2	5	(0, 0)	2	1	(1, -1)	3	1	(-1, -1)	2
0	2	(-1, -1)	2	5	(0, 0)	2	1	(1, -1)	2	2	(-1, -1)	2

Table 3.4: Anomaly-free (0,2) models on Pfaffians inside $\mathbb{P}^7$.

Chapter 4

Summary and Discussion

In this thesis, we discussed various supersymmetric quantum field theories on various manifolds. We started with an introduction to some basic material about supersymmetry and its representations, with focus on four and two dimensional theories. In four dimensions, we constructed $\mathcal{N} = 1$ supersymmetric gauge theories coupled to $\mathcal{N} = 1$ supersymmetric nonlinear sigma models, with spacetime being various four-manifolds. Some interesting new properties of these theories were discussed, such as the vanishing of the Fayet-Iliopoulos parameters in these theories, explicit curvature coupling terms in their Lagrangians, and generalization and application the background principle.

There are many interesting questions that arise from this discussion. Our analyze was mainly restricted to classical interpretations. It would be very interesting to explore more about quantum properties of these four dimensional gauge theories, such as computing their partition functions by using supersymmetric localization. Furthermore, it would be intersting to look for brane construction of these theories, from string theories or M-theory, to extend our understandings of both quantum field theories on curved spaces, as well as branes wrapping general manifolds.

In two dimensions, we first constructed the Lagrangians of $\mathcal{N} = (2,2)$ nonlinear sigma models on S^2, with explicit curvature coupling terms respecting supersymmetry. We did this for both chiral multiplets and twisted chiral multiplets. In the future, it would be very interesting to calculate partition functions of these nonlinear sigma models. This is difficult because the defining properties of nonlinear sigma models: they govern maps from spacetime to target spaces. As such, the bosonic fields are not sections of vector bundles, so naive application of index theorems is not possible. One would need to generalize index theorems to cases other than operators on vector bundles. This is very challenging mathematically.

Then we moved on to a discussion of $\mathcal{N} = (0,2)$ nonabelian gauge theories on $\mathbb{R}^{1,1}$, focusing on the geometries arising from Higgs branches of these theories. Based on this, we proposed and analyzed various dualities between different gauge theories, in the sense that they flow to the same infrared fixed points, based on geometric constructions. We tested these dualities by various means, such as computing their elliptic genuses. We also discussed pertinent dynamical supersymmetry breaking.

Two-dimensional $\mathcal{N} = (0,2)$ nonabelian gauge theories are just beginning to be explored by physicists. Some progress has been made, but there are much more that's need to be done. How doe we, in general, use branes from string theory and M-theory to construct these gauge theories? How do we understand these dualities in the context of string theory? Are there more interesting dualities, such as a continues duality group like the $SL(2,\mathbb{Z})$ duality group of four-dimensional $\mathcal{N} = 4$ syper Yang-Mills theory? Or even more generally, are there any analogue Gaiotto dualities between these two-dimensional theories that arise from four-manifolds with decorations? We leave these interesting questions to future study.

Appendix A

Affine bundles and equivariant structures

In this section we introduce the concept of affine bundle, as well as the equivariant structure on affine bundles. First we give the definition of affine space and affine space morphism [38]

Definition A.1. Let V be a vector space over some field k. An *affine space* modeled on V is a set A, together with a map $t : V \times A \to A$ defined by $t(v, a) = v + a$, which is a free transitive action of V (as an Abelian group under addition) on A.

Intuitively, an affine space is a vector space without origin. Clearly if we fix an element $a \in A$ to be the origin, the above definition makes A into a vector space over the field k, which is isomorphic to V.

Example A.2. A vector space V is naturally an affine space modeled over itself, with the map $t : V \times V \to V$ given by the natural addition operation.

Definition A.3. Let V and V' be two vector spaces over the same field k. Let A and A' be affine spaces modeled on V and V' respectively, with corresponding maps $t : V \times A \to A$

and $t' : V' \times A' \to A'$. An *affine space morphism* between A and A' is a map $\varphi : A \to A'$ and a linear transformation $\tau : V \to V'$ such that

$$t'(\tau(v), \varphi(a)) = \varphi(t(v, a)), \forall a \in A, v \in V. \tag{A.1}$$

Then an affine bundle on a topological space X can be defined as follows.

Definition A.4. Let A be an affine space modeled on a vector space V. An *affine bundle* (V, A) over X is a fiber bundle $\pi : (V, A) \to X$, defined by the following data: each point $x \in X$ has a neighborhood U and a U-isomorphism $\varphi : U \times A \to \pi^{-1}(U)$ such that the restriction $x \times A \to \pi^{-1}(x)$ is an affine space isomorphism.

In other words, V is an ordinary bundle and A is a V-torsor.

Next we consider the equivariant structure on affine bundles. We restrict our attention to the case of trivial affine line bundle $\pi : X \times \mathbb{A}^1 \to X$, which shows up in our discussion of supersymmetric theories on AdS_4. Let G be a group acting on X, then the G-action on the affine bundle $\pi : X \times \mathbb{A}^1 \to X$ is given by [39]

$$g(x, a) = (gx, \lambda_g a + \mu_g(x)), \forall g \in G, x \in X, a \in \mathbb{A}^1, \tag{A.2}$$

where λ and μ are functions on $G \times X$ such that for any $g, h \in G$

$$\begin{aligned} \lambda_{gh} &= \lambda_g \cdot \lambda_h, \lambda_e = 1, \\ \mu_{gh}(x) &= \lambda_g \cdot \mu_h(x) + \mu_g(hx), \mu_e = 0. \end{aligned} \tag{A.3}$$

Let $s : X \to X \times \mathbb{A}^1$ be an G-equivariant section of this affine bundle, defined by $x \to (x, \sigma(x))$ where $\sigma \in \mathcal{O}(X)$. Then from the above result we can see σ satisfies [39]

$$\sigma(gx) = \lambda_g \cdot \sigma(x) + \mu_g(x). \tag{A.4}$$

In chapter 2 we see that the superpotential of supersymmetric theories on AdS_4 is a section of the trivial affine bundle $\pi : X \times \mathbb{C}^1 \to X$. Hence if we consider supersymmetric gauge theory on AdS_4, we will have to lift the gauge group action to this affine bundle, which is given by the gauge transformation of the superpotential.

Appendix B

GLSMs and cohomology

In this appendix we relate GLSM operators to cohomology, focusing in particular on GLSM's for Grassmannians.

Consider a GLSM that is described by gauging the action of some Lie group G on a vector space V, supersymmetrically. We claim that the cohomology ring seen by the GLSM is the G-equivariant cohomology of V, *i.e.*,

$$H^*_G(V) \;=\; H^*_G(\mathrm{pt}) \;=\; H^*(BG),$$

and that this cohomology ring is realized by operators build from the adjoint-valued scalars σ in the two-dimensional gauge multiplet.

Let us work through the details a little more. Suppose we have an abelian GLSM (assumed without a superpotential), which describes, in one Kähler phase, a toric variety

$$\frac{V - E}{(\mathbb{C}^\times)^k},$$

where V is a vector space, and E the exceptional set for that phase.

The cohomology seen by the GLSM is the $(\mathbb{C}^\times)^k$-equivariant cohomology of the vector space V – equivalently, the $U(1)^k$-equivariant cohomology, as $G_{\mathbb{C}}$-equivariant cohomology is the same as G-equivariant cohomology. Using the inclusion $V - E \hookrightarrow V$, the equivariant cohomology of V can be pulled back to the equivariant cohomology of $V - E$, which (assuming there are no fixed points) descends to the ordinary cohomology of the toric variety. (This is a special case of the Kirwan surjectivity theorem, valid for rational coefficients.) This can be done for every exceptional set, and so the equivariant cohomology of V defines something universal for all phases of the GLSM.

We can compute the $(\mathbb{C}^\times)^k$-equivariant cohomology of V by using the fact that V is contractible; the result is just $H^*(BU(1) \times \cdots \times BU(1))$, ($k$ copies of $BU(1)$), which is the polynomial ring in k variables:

$$H^*_{(\mathbb{C}^\times)^k}(V) \; = \; \mathbb{C}[x_1, \cdots, x_k],$$

independent of the dimension of the vector space V. Physically, each x_i corresponds to a σ_i in the vector supermultiplet.

In principle something closely analogous should happen in nonabelian GLSM's. All of the analysis above applies, except that the equivariant cohomology itself now has different values. To this end, recall

- $H^*(BSU(n), \mathbb{Z}) = \mathbb{Z}[c_2, c_3, \cdots, c_n]$,
- $H^*(BU(n), \mathbb{Z}) = \mathbb{Z}[c_1, c_2, \cdots, c_n]$,

where c_i has degree $2i$, and corresponds to a Chern class.

In more detail, the cohomology ring of $BGL(k) = BU(k) = G(k, \infty)$ is discussed in [29][section 16]: it is the ring of symmetric polynomials in k indeterminates.

We can relate the formal structures above to physics as follows. Recall the Cartan model of equivariant cohomology [150][section 10.7] is the multiplet

$$\begin{aligned} dA &= \psi, \\ d\psi &= -D_A \sigma, \\ d\sigma &= 0. \end{aligned}$$

The field σ is a Lie-algebra-valued scalar; see also [150][sections 10.9-10.10]. As discussed in [151][section 3.6], this structure is realized in GLSM's. The generators of the equivariant cohomology rings above correspond to operators of the form $\mathrm{Tr}\,\sigma^k$ for various k.

It is a standard result (see *e.g.* [152][chapter 1.5], [153][chapter 8]) that the integral homology of the Grassmannian $G(k,n)$ has no torsion and is freely generated by cycles in one-to-one correspondence with Young diagrams (unlabelled Young tableaux), specified by a sequence of k positive integers $d_1, \cdots, d_k$, where

$$n - k \ \geq \ d_1 \ \geq \ d_2 \ \geq \ \cdots \ \geq \ d_k \ \geq \ 0$$

(*i.e.* d_i is the number of boxes on row i) and where the Young diagram above corresponds to a cycle of real codimension

$$2 \sum_i d_i.$$

For example,

$$H^2(G(k,n), \mathbb{Z}) \ = \ \mathbb{Z}$$

corresponds to

$$\yng(1).$$

Similarly,

$$H^4(G(k,n), \mathbb{Z}) \ = \ \mathbb{Z}^2$$

corresponds to

$$\yng(2), \ \yng(1,1).$$

Intersection theory on these (Schubert) cycles, cup products on the cohomology, are determined in the same way as representations of $GL(k)$. Schur polynomials provide the link between Young diagrams and symmetrized polynomials that were used earlier to describe the cohomology of the Grassmannian, in terms of equivariant cohomology.

Let us work through some examples in more detail. In general terms, H^2 is generated by $\mathrm{Tr}\,\sigma$; H^4 is generated by $\mathrm{Tr}\,\sigma^2$ and $(\mathrm{Tr}\,\sigma)^2$, and so forth. For example, for three indeterminates, we have the Schur polynomials[1]

$$
\begin{aligned}
\left(s_{\square}(x_1, x_2, x_3)\right)^2 &= x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3, \\
s_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(x_1, x_2, x_3) &= x_1x_2 + x_1x_3 + x_2x_3, \\
s_{\square\square}(x_1, x_2, x_3) &= x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_1x_3,
\end{aligned}
$$

from which we see that

$$
\left(s_{\square}\right)^2 = s_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} + s_{\square\square},
$$

so that there are only two independent quantities. Clearly if we associate $\mathrm{Tr}\,\sigma$ to $\square$, then $(\mathrm{Tr}\,\sigma)^2$ is associated to $\begin{smallmatrix}\square\\\square\end{smallmatrix} + \square\square$. This is the sum of Sym^2 and Alt^2, so it makes perfect sense that the result is just the square. Similarly, looking at the indeterminates as eigenvalues,

$$
\begin{aligned}
\mathrm{Tr}\,\sigma^2 &= x_1^2 + x_2^2 + x_3^2, \\
&= s_{\square\square}(x_1, x_2, x_3) - s_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(x_1, x_2, x_3),
\end{aligned}
$$

so we see that $\mathrm{Tr}\,\sigma^2$ is associated to $\square\square - \begin{smallmatrix}\square\\\square\end{smallmatrix}$.

We will argue next that all cohomology of $G(k, n)$ can be constructed from operators of the form

$$
\mathrm{Tr}\,\sigma^k = \sum_i x_i^k
$$

in this form.

[1] See appendix C for a short introduction to Schur polynomials.

In general, the dimension of $H^{2\bullet}(G(k,n),\mathbb{Z})$ is the same as the number of gauge-invariant independent σ polynomials of degree $\bullet$ in σ. To see this, we define an isomorphism between Young diagrams and σ polynomials as follows. First, to a Young diagram $(d_1,\cdots,d_k)$, one can associate

$$\prod_i \operatorname{Tr}\sigma^{d_i}.$$

(Note that this association is not intended to relate representations of elements of cohomology, but rather is merely meant to be used in a set-theoretic counting.) Conversely, given a gauge-invariant σ polynomial

$$\prod_i \left(\operatorname{Tr}\sigma^i\right)^{a_i},$$

where j is the highest power of σ appearing in a trace, the largest j such that $a_j \neq 0$, we associate a Young diagram defined by

$$(d_1,\cdots,d_k) \;=\; \left(\underbrace{j,\cdots,j,}_{a_j}\underbrace{j-1,\cdots,j-1,}_{a_{j-1}}\cdots\underbrace{1,\cdots,1}_{a_1}\right).$$

As a consistency check, the σ polynomial should contribute to cohomology in degree

$$2\sum_i i a_i$$

and the Young diagram indicated should contribute to cohomology in the same degree. It is straightforward to check that these two maps are inverses of one another, and so we see that the dimension of the cohomology of $G(k,n)$ is the same as the number of gauge-invariant σ polynomials of the same degree.

.

Appendix C

Schur polynomials

Since Schur polynomials are not often encountered in the physics literature, in this appendix we briefly review some of their pertinent properties.

Briefly, Schur polynomials are polynomials in k variables associated to Young diagrams (unlabelled Young tableaux) describing representations of $SL(k)$ or $SU(k)$. Such a Young diagram can be characterized by a sequence of k positive integers $d_1, \cdots, d_k$, where

$$d_1 \;\geq\; d_2 \;\geq\; \cdots \;\geq\; d_k \geq\; 0$$

and d_i gives the number of boxes in row i of the Young diagram.

Define

$$a_{(d_1,\cdots,d_k)}(x_1,\cdots,x_k) \;=\; \det \left[\begin{array}{cccc} x_1^{d_1} & x_2^{d_1} & \cdots & x_k^{d_1} \\ x_1^{d_2} & x_2^{d_2} & \cdots & x_k^{d_2} \\ \vdots & \vdots & & \vdots \\ x_1^{d_k} & x_2^{d_k} & \cdots & x_k^{d_k} \end{array} \right],$$

then the Schur polynomial corresponding to the Young diagram defined by $(d_1, \cdots, d_k)$ is

$$s_{(d_1,\cdots,d_k)}(x_1,\cdots,x_k) \;=\; \frac{a_{(d_1+k-1,d_2+k-2,\cdots,d_k+0)}(x_1,\cdots,x_k)}{a_{(k-1,k-2,\cdots,0)}(x_1,\cdots,x_k)}.$$

(For a different perspective on the Schur polynomials, compare the characters given in [150][equ'n (4.5)].)

For example, it is straightforward to compute that

$$s_{\ydiagram{1}}(x_1, x_2, x_3) \;=\; s_{(1,0,0)}(x_1, x_2, x_3) \;=\; x_1 \;+\; x_2 \;+\; x_3,$$

$$s_{\ydiagram{1,1}}(x_1, x_2, x_3) \;=\; s_{(1,1,0)}(x_1, x_2, x_3) \;=\; x_1x_2 \;+\; x_1x_3 \;+\; x_2x_3,$$

$$s_{\ydiagram{2}}(x_1, x_2, x_3) \;=\; s_{(2,0,0)}(x_1, x_2, x_3) \;=\; x_1^2 \;+\; x_2^2 \;+\; x_3^2 \;+\; x_1x_2 \;+\; x_2x_3 \;+\; x_1x_3,$$

$$s_{\ydiagram{2,1}}(x_1, x_2, x_3) \;=\; s_{(2,1,0)}(x_1, x_2, x_3) \;=\; x_1^2(x_2 + x_3) \;+\; x_1(x_2 + x_3)^2 \;+\; x_2x_3(x_2 + x_3),$$

$$s_{\ydiagram{2,2}}(x_1, x_2, x_3) \;=\; s_{(2,2,0)}(x_1, x_2, x_3) \;=\; x_1^2x_2^2 \;+\; x_1^2x_2x_3 \;+\; x_1^2x_3^2 \;+\; x_2^2x_1x_3 \;+\; x_2^2x_3^2 \;+\; x_3^2x_1x_2.$$

Appendix D

Representations of $U(k)$

The representation theory of $SU(k)$ is certainly well-known; however, representations of $U(k)$ can be more complicated, because of the possibility of tensoring in powers of the determinant. In this appendix, we give our conventions for describing representations of $U(k)$.

Any irreducible unitary representation of $U(k)$ is given by a k-tuple of ordered integers [154][sections 19-22]

$$\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k), \;\; \lambda_i \in \mathbb{Z}, \;\; \forall i. \tag{D.1}$$

This is the highest weight of the corresponding representation. For completeness, here are a few examples [154][sections 19-22]:

- The defining fundamental representation of $U(k)$ has highest weight $(1, 0, \cdots, 0)$, while its conjugate, the antifundamental representation, has highest weight $(0, \cdots, 0, -1)$.
- The exterior product representation on $\wedge^{\ell}\mathbb{C}^k$ has highest weight $(1, 1, \cdots, 1, 0, 0, \cdots, 0)$ (ℓ 1's). In particular, the determinant representation has highest weight $(1, 1, \cdots, 1)$.
- The adjoint representation of $U(k)$ is reducible: $ad = (1, 0, \cdots, 0) \otimes (0, 0, \cdots, -1) = (1, 0, \cdots, 0, -1) \oplus (0, 0, \cdots, 0)$.

Below are some frequently used formulas for $U(k)$ representations [155][chapter 5]:

- The dimension of λ is given by [155][eq (4.56)]

$$d_\lambda = \prod_{i<j} \frac{l_i - l_j}{l_i^0 - l_j^0} = \prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j - i}, \tag{D.2}$$

  where $l_i^0 = k - i$, and $l_i = \lambda_i + k - i$, with $i, j = 1, \cdots, k$.

- The eigenvalue of the first Casimir operator on λ is [155][eq. (5.24), table 5.1]

$$\mathrm{Cas}_1(\lambda) = \sum_i \lambda_i, \tag{D.3}$$

- The eigenvalue of the second Casimir operator on λ is [155][eq. (5.24), table 5.1]

$$\mathrm{Cas}_2(\lambda) = \sum_i \lambda_i(\lambda_i + k + 1 - 2i). \tag{D.4}$$

In terms of bundles on $G(k,n)$ of the form $\mathcal{O}(\lambda)$ for some representation λ, it is straightforward to show that

$$c_1(\mathcal{O}(\lambda)) \; = \; \frac{d_\lambda \mathrm{Cas}_1(\lambda)}{k} \sigma_{\square}, \tag{D.5}$$

where $\sigma_{\square}$ denotes the Schubert cycle generating $H^2(G(k,n), \mathbb{Z})$, which is one-dimensional, and

$$\begin{aligned} \mathrm{ch}_2(\mathcal{O}(\lambda)) \; &= \; (1/2) c_1(\mathcal{O}(\lambda))^2 \; - \; c_2(\mathcal{O}(\lambda)), \\ &= \; d_\lambda \mathrm{Cas}_2(\lambda) \left[-\frac{1}{k^2-1} \sigma_{\substack{\square \\ \square}} \; + \; \frac{1}{2k(k+1)} \sigma_{\square}^2 \right] \\ &\qquad + d_\lambda \mathrm{Cas}_1(\lambda)^2 \left[\frac{1}{k(k^2-1)} \sigma_{\substack{\square \\ \square}} \; + \; \frac{1}{2k(k+1)} \sigma_{\square}^2 \right], \end{aligned} \tag{D.6}$$

where $\sigma_{\substack{\square \\ \square}}$ and $\sigma_{\square\square}$ generate

$$H^4(G(k,n), \mathbb{Z}) \; = \; \mathbb{Z}^2$$

and

$$\sigma_{\square}^2 \; = \; \sigma_{\substack{\square \\ \square}} \; + \; \sigma_{\square\square},$$

as we demonstrated in appendix C.

As a consistency check, recall that the bundle $\wedge^p S^* \to G(k,n)$ has rank

$$\begin{pmatrix} k \\ p \end{pmatrix}$$

and

$$c_1\left(\wedge^p S^*\right) \;=\; \begin{pmatrix} k-1 \\ p-1 \end{pmatrix} \sigma_{\Box},$$

and these are both consistent with the formulas above for the representation

$$(1, 1, \cdots, 1, 0, \cdots, 0)$$

(p 1's) of $U(k)$, which defines the bundle $\wedge^p S^*$. We list here results for a few other cases, which can also be used to check the general formulas above. For $p = 1$ [156], [157][prop. 3.5.5],

$$c_2(S^*) \;=\; \sigma_{\substack{\Box\\\Box}}, \quad \mathrm{ch}_2(S^*) \;=\; (1/2)\sigma_{\Box}^2 \;-\; \sigma_{\substack{\Box\\\Box}},$$

and in fact $c_i(S^*)$ is given by the Schubert cycle associated to the Young diagram with i vertical boxes. In the special case $p = 2$,

$$c_2(\wedge^2 S^*) \;=\; \begin{pmatrix} k-1 \\ 2 \end{pmatrix} \sigma_{\Box}^2 \;+\; (k-2)\sigma_{\substack{\Box\\\Box}},$$

which one can use to show that for the representation $(2, 0, \cdots, 0)$,

$$\mathrm{rk}\,\mathrm{Sym}^2 S^* \;=\; \frac{k(k+1)}{2}, \quad c_1(\mathrm{Sym}^2 S^*) \;=\; (k+1)\sigma_{\Box},$$

$$\mathrm{ch}_2(\mathrm{Sym}^2 S^*) \;=\; \frac{k+3}{2}\sigma_{\Box}^2 \;-\; (k+2)\sigma_{\substack{\Box\\\Box}},$$

and one can also compute that

$$c_2(\wedge^3 S^*) \;=\; \frac{k(k-1)(k-2)(k-3)}{8}\sigma_{\Box}^2 \;+\; \frac{(k-2)(k-3)}{2}\sigma_{\substack{\Box\\\Box}},$$

$$\mathrm{ch}_2(\wedge^3 S^*) \;=\; \frac{(k-1)(k-2)}{4}\sigma_{\square}^2 \;-\; \frac{(k-2)(k-3)}{2}\sigma_{\substack{\square\\\square}}.$$

At the level of Lie algebras, $u(k) \cong su(k) \oplus u(1)$. Therefore, given a representation λ of $u(k)$, we can get an irreducible representation of $su(k) \oplus u(1)$: the representation of $su(k)$ is given by the Young diagram $(\lambda_1 - \lambda_k \geq \lambda_2 - \lambda_k \geq \cdots \geq 0)$, and the representation of $u(1)$ is given by the integer $\mathrm{Cas}_1(\lambda)$.

For completeness, the eigenvalue of an $su(k)$ second Casimir operator on the $su(k)$ representation $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0)$ is given by [155][eq. (5.24), table 5.1]

$$\mathrm{Cas}_2(\lambda) = \sum_i \left(\lambda_i - \frac{\sum_i \lambda_i}{k} \right) \left(\lambda_i - \frac{\sum_i \lambda_i}{k} + 2k - 2i \right). \tag{D.7}$$

For example:

$$\begin{aligned} \mathrm{Cas}_2(ad) &= 2k, \\ \mathrm{Cas}_2(1,0,\cdots,0) &= (k^2-1)/k. \end{aligned}$$

As a consistency check, [158][equ'n (2.18)] lists an index for $su(2)$ representations defined by Young diagrams with n boxes:

$$I_2(n) \;=\; \frac{1}{6} n(n+1)(n+2),$$

where

$$\mathrm{Tr}\left(T_R^a T_R^b \right) \;=\; I_2(R) \delta^{ab}.$$

It is straightforward to check that

$$I_2(n) \;=\; \frac{d_{(n,0)} \mathrm{Cas}_2(n,0)}{\dim su(2)},$$

where $d_{(n,0)} = n+1$ and

$$\mathrm{Cas}_2(n,0) \;=\; (n/2)(n/2+4-2) + (-n/2)(-n/2+4-4) \;=\; (1/2)n(n+2).$$

Appendix E

Checks of (2,2) abelian/nonabelian duality

In this appendix, we shall use compare elliptic genera as a check of the duality between the (2,2) GLSM's for $G(2,4)$ and $\mathbb{P}^5[2]$ proposed in the text. As discussed earlier, as the two GLSM's have weak-coupling limits describing the same geometry, they have, by construction, the same IR limit, making checks of elliptic genera somewhat unnecessary. Nevertheless, to be thorough, in this appendix we will verify that elliptic genera match.

To fix notation, for a (2,2) supersymmetric gauge theory with global symmetry K, the elliptic genus is the quantity

$$Z_{T^2}(\tau, u) := \mathrm{Tr}_{\mathrm{RR}}(-1)^F q^{L_0} \overline{q}^{\overline{L}_0} y^J \prod_a x_a^{K_a} \tag{E.1}$$

where F is the fermion number operator, and $q = e^{2\pi i\tau}$ on our T^2 defined by τ. In addition, we define $x_a = e^{2\pi i u_a}$ coming from fugacities u_a of global and gauge symmetries, and $y = e^{2\pi i z}$ coming from the fugacity of the left-moving $U(1)$ R-symmetry J.

In [84, 85], this index was computed for general (2,2) gauge theories in two dimensions. In particular, they derived

$$Z_{T^2}(\tau, u, \xi) = \frac{1}{|W|} \sum_{u_* \in \mathfrak{M}^*_{\text{sing}}} \text{JK-Res}(Q(u_*, \eta)) Z_{\text{1-loop}}. \tag{E.2}$$

(See [84, 85] for notation.)

Note in passing that since these GLSM's are not Calabi-Yau, the left-moving R-symmetry J is anomalous, so in principle we can only expect a physically unambiguous result for special values of y. Nevertheless, we will compute for general values of y and find matching, a strong check. (For related analyses in different contexts see for example [94, 95]).

Let's use this index to test the abelian/nonabelian dualtiy between the GLSM on $G(2, 4)$ and the GLSM on $\mathbb{P}^5[2]$. The elliptic genus of the GLSM on $G(2, 4)$ was computed in [84, 85], so let's compute the elliptic genus of the GLSM on $\mathbb{P}^5[2]$. The GLSM is a $U(1)$ gauge theory with 6 chiral superfields Φ^i with charge 1, a chiral superfield P with charge -2, and a superpotential $W = PG(\Phi)$ where $G(\Phi)$ is a generic polynomial of degree 2.

The 1-loop determinant coming from the Φ's is

$$Z_\Phi = \left(\frac{\theta_1(q, y^{-1}x)}{\theta_1(q, x)} \right)^6, \tag{E.3}$$

since Φ^i has R-charge 0. The 1-loop determinant coming from P is

$$Z_P = \frac{\theta_1(q, x^{-2})}{\theta_1(q, yx^{-2})}, \tag{E.4}$$

since P has R-charge 2. Finally, the 1-loop determinant coming from the vector multiplet is

$$Z_V = \frac{2\pi\eta(q)^3}{\theta_1(q, y^{-1})} du. \tag{E.5}$$

Then, applying the methods of [84,85], we recover the elliptic genus (in the geometric phase):

$$Z_{T^2}(q, z) = \frac{\eta(q)^3}{i\theta_1(q, y^{-1})} \oint_{u=0} du \left(\frac{\theta_1(q, y^{-1}x)}{\theta_1(q, x)} \right)^6 \frac{\theta_1(q, x^{-2})}{\theta_1(q, yx^{-2})}. \tag{E.6}$$

One can use Mathematica to evaluate this integral. In the limit $z \to 0$, one finds $Z_{T^2}(q, z \to 0) = 6$, independent of the value of q, which matches precisely the corresponding computation for the GLSM describing $G(2,4)$ (or the Euler characteristic of $G(2,4)$), given in [85][equ'n (4.43)].

Now, to properly compare elliptic genera, let us take into account the action of the $G(2,4)$ symmetries on $\mathbb{P}^5[2]$. Let z_{ij} denote homogeneous coordinates on $\mathbb{P}^5$, which are related to the fundamentals ϕ_i^a defining $G(2,4)$ as the baryons

$$z_{ij} \;=\; \epsilon_{ab}\phi_i^a \phi_j^b.$$

Now, one of symmetries of $G(2,4)$ used in [85] in computing the elliptic genus is the rescaling symmetry

$$\phi_i^a \;\mapsto\; e^{2\pi i \xi_i}\phi_i^a,$$

from which we read off that on $\mathbb{P}^5[2]$, we should have the symmetry

$$z_{ij} \;\mapsto\; e^{2\pi i(\xi_i + \xi_j)} z_{ij}.$$

A generic quadric would break rescaling symmetries of this form, but in the present case, we are interested in a quadric which is a linear combination of $z_{12}z_{34}$, $z_{13}z_{24}$, $z_{14}z_{23}$, and so it is preserved by the symmetry. With this in mind, we can now read off the flavored elliptic genus of $\mathbb{P}^5[2]$, taking into account this symmetry:

$$Z_{T^2}(q, z, \xi_i) \;=\; \frac{2\pi\eta(q)^3}{\theta_1(q, y^{-1})} \oint du \left(\prod_{i,j} \frac{\theta_1\left(q, y^{-1} x e^{2\pi i(\xi_i+\xi_j)}\right)}{\theta_1\left(q, x e^{2\pi i(\xi_i+\xi_j)}\right)} \right) \frac{\theta_1\left(q, x^{-2} e^{2\pi i(-\xi_1-\xi_2-\xi_3-\xi_4)}\right)}{\theta_1\left(q, y x^{-2} e^{2\pi i(-\xi_1-\xi_2-\xi_3-\xi_4)}\right)}.$$

The residues are computed at six poles, at the locations

$$u \;=\; -\xi_i \;-\; \xi_j$$

for $i \neq j$. For example, the residue at $u = -\xi_1 - \xi_2$ is given by

$$\frac{\theta_1\left(q, y^{-1} e^{2\pi i(\xi_1-\xi_3)}\right)}{\theta_1\left(q, e^{2\pi i(\xi_1-\xi_3)}\right)} \frac{\theta_1\left(q, y^{-1} e^{2\pi i(\xi_1-\xi_4)}\right)}{\theta_1\left(q, e^{2\pi i(\xi_1-\xi_4)}\right)} \frac{\theta_1\left(q, y^{-1} e^{2\pi i(\xi_2-\xi_3)}\right)}{\theta_1\left(q, e^{2\pi i(\xi_2-\xi_3)}\right)} \frac{\theta_1\left(q, y^{-1} e^{2\pi i(\xi_2-\xi_4)}\right)}{\theta_1\left(q, e^{2\pi i(\xi_2-\xi_4)}\right)}.$$

Each residue precisely corresponds to a term in the expression for the flavored elliptic genus for $G(2,4)$ given in [85][equ'n (4.42)]. Thus, we see that the flavored elliptic genus of the abelian GLSM for $\mathbb{P}^5[2]$ precisely matches that of the nonabelian GLSM for $G(2,4)$ computed in [85], as expected from the proposed duality.

So far we have used a $(\mathbb{C}^\times)^4$ symmetry group common to both $G(2,4)$ and $\mathbb{P}^5[2]$. More generally, there is a global $GL(4,\mathbb{C})$ symmetry acting linearly on the four fundamentals defining $G(2,4)$. Under this symmetry,

$$\phi_i^a \mapsto V_i^j \phi_j^a$$

and so

$$z_{ij} \mapsto V_i^{i'} V_j^{j'} z_{i'j'}$$

(transforming in the $\wedge^2 \mathbf{4}$ representation, in other words). Furthermore, the quadric hypersurface in $\mathbb{P}^5$ is invariant. Specifically, the hypersurface polynomial

$$z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23}$$

transforms to

$$V_1^i V_2^j V_3^k V_4^m \left(z_{ij}z_{km} - z_{ik}z_{jm} + z_{im}z_{jk} \right) = (\det V)(z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23}),$$

where we have used the fact that

$$z_{ij}z_{km} - z_{ik}z_{jm} + z_{im}z_{jk}$$

is completely antisymmetric in all its indices.

Appendix F

(0,2) elliptic genera in Calabi-Yau duals

In this appendix we will outline the computation of some (0,2) elliptic genera, to check for dynamical supersymmetry breaking and as evidence of dualities. We will follow the conventions of [85].

F.1 Second entry

We will begin with the second entry in table 3.2. This describes a bundle $\mathcal{E}$ on the Calabi-Yau hypersurface $G(2,4)[4]$, given by

$$0 \longrightarrow \mathcal{E} \longrightarrow \oplus^8 \mathcal{O}(1,1) \longrightarrow \mathcal{O}(2,2) \oplus^2 \mathcal{O}(3,3) \longrightarrow 0$$

The field content and corresponding contributions to the elliptic genus are as follows:

- 4 chiral multiplets each in the fundamental of $U(2)$

$$\left(i \frac{\eta(q)}{\theta_1(q,x_1)} \right)^4 \left(i \frac{\eta(q)}{\theta_1(q,x_2)} \right)^4,$$

- 1 Fermi multiplet in the $(-4,-4)$ representation of $U(2)$, enforcing the hypersurface condition:
$$i\frac{\theta_1(q, x_1^{-4}x_2^{-4})}{\eta(q)},$$
- 8 Fermi multiplets in the $(1,1)$ representation of $U(2)$, partially defining the gauge bundle:
$$\left(i\frac{\theta_1(q, yx_1x_2)}{\eta(q)}\right)^8,$$
- 1 chiral multiplet in the $(-2,-2)$ representation of $U(2)$, partially defining the gauge bundle:
$$i\frac{\eta(q)}{\theta_1(q, y^{-1}x_1^{-2}x_2^{-2})},$$
- 2 chiral multiplets in the $(-3,-3)$ representation of $U(2)$, partially defining the gauge bundle:
$$\left(i\frac{\eta(q)}{\theta_1(q, y^{-1}x_1^{-3}x_2^{-3})}\right)^2,$$
- and finally the $U(2)$ gauge field contributes
$$\left(\frac{2\pi\eta(q)^2}{i}\right)^2 i\frac{\theta_1(q, x_1x_2^{-1})}{\eta(q)} i\frac{\theta_1(q, x_2x_1^{-1})}{\eta(q)} du_1du_2.$$

In this particular example, the sum of the charges of the chiral superfields vanishes without any spectators. The dual, on the other hand, will contain spectators, but as we shall argue there, spectators cancel out of elliptic genus computations.

In the expressions above we have implicitly used a left-moving $U(1)$ symmetry, under which the Fermi multiplets defining the gauge bundle have charge $+1$ and the chiral multiplets defining the gauge bundle have charge -1.

Assembling these components gives an elliptic genus of the form

$$\frac{1}{2}\frac{(2\pi)^2\eta(q)^4}{(2\pi i)^2}\oint du_1du_2\frac{\theta_1(q, x_1x_2^{-1})\theta_1(q, x_2x_1^{-1})\theta_1(q, x_1^{-4}x_2^{-4})\theta_1(q, yx_1x_2)^8}{\theta_1(q, x_1)^4\theta_1(q, x_2)^4\theta_1(q, y^{-1}x_1^{-2}x_2^{-2})\theta_1(q, y^{-1}x_1^{-3}x_2^{-3})^2}.$$

(The overall factor of 1/2 is from the Weyl group of $SU(2)$.) Poles lie along the hypersurfaces $\{u_1 = 0\}$, $\{u_2 = 0\}$, $\{z + 2(u_1 + u_2) = 0\}$, $\{z + 3(u_1 + u_2) = 0\}$. The intersection of these hypersurfaces is projective[1]. Let us work in a geometric phase, specified by $\eta = (1, 1)$. The only pole in the corresponding chamber is at the origin, so we compute the repeated residue there.

Expanding the genus above in a power series in q, the first few terms are

$$
\begin{aligned}
&72\left(-y^{-1/2}+y^{+1/2}\right)^2\left(y^{-1/2}+y^{+1/2}\right)q^{1/6}\\
&\quad -72\left(-y^{-1/2}+y^{+1/2}\right)^2\left(y^{-1/2}+y^{+1/2}\right)^3\left(y^{-1}-1+y\right)q^{7/6}\\
&\quad +72\left(-y^{-1/2}+y^{+1/2}\right)^2\left(y^{-7/2}-y^{-3/2}+2y^{-1/2}+2y^{+1/2}-y^{+3/2}+y^{+7/2}\right)q^{13/6} \;+\; \mathcal{O}\left(q^{19/6}\right).
\end{aligned}
$$

As described in section 3.3.3, the example above is mathematically equivalent to an abelian GLSM describing a bundle $\mathcal{E}$ on $\mathbb{P}^5[2,4]$, given by

$$
0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(1)^8 \longrightarrow \mathcal{O}(2) \oplus \mathcal{O}(3)^2 \longrightarrow 0.
$$

The field content and corresponding contributions to the elliptic genus are as follows:

- 6 chiral multiplets each of charge $+1$:

$$
\left(i\frac{\eta(q)}{\theta_1(q,x)} \right)^6,
$$

- 8 Fermi multiplets Λ_α each of charge $+1$:

$$
\left(i\frac{\theta_1(q,yx)}{\eta(q)} \right)^8,
$$

- one chiral multiplet p_1 of charge -2, helping to form the gauge bundle:

$$
i\frac{\eta(q)}{\theta_1(q,y^{-1}x^{-2})},
$$

[1] The multiplicity of the θ_1^2 in the denominator does not count.

- two chiral multiplets $p_{2,3}$ of charge -3, helping to form the gauge bundle:

$$\left(i \frac{\eta(q)}{\theta_1(q, y^{-1}x^{-3})} \right)^2,$$

- one Fermi multiplet Γ_1 of charge -2, enforcing a hypersurface condition:

$$i \frac{\theta_1(q, x^{-2})}{\eta(q)},$$

- one Fermi multiplet Γ_2 of charge -4, enforcing a hypersurface condition:

$$i \frac{\theta_1(q, x^{-4})}{\eta(q)},$$

- one chiral multiplet of charge $+2$, one of the spectators:

$$i \frac{\eta(q)}{\theta_1(q, x^{+2})},$$

- one Fermi multiplet of charge -2, one of the spectators:

$$i \frac{\theta_1(q, x^{-2})}{\eta(q)},$$

- and finally the $U(1)$ gauge field contributes

$$\frac{2\pi\eta(q)^2}{i} du.$$

This theory has a (0,2) superpotential of the form

$$W \; = \; \Lambda_\alpha p_a F^{\alpha a}(\phi) \; + \; \Gamma_1 G_2(\phi) \; + \; \Gamma_2 G_4(\phi)$$

(plus a term for spectators). This theory has a nonanomalous global symmetry acting on the fermions, under which the left-moving fermions λ_α have charge $+1$ and the chiral multiplets p_a have charge -1. We implicitly used this global symmetry to flavor the elliptic genus contributions above, as this is the symmetry defining the variable y.

Using the identity [85][equ'n (A.5)],

$$\theta_1(q,x) \;=\; -\theta_1(q,x^{-1}),$$

it is straightforward to see that the contribution from the spectators cancel out. This is a (0,2) analogue of an observation in [85][section 2.1], that in (2,2) supersymmetry, a pair of chiral multiplets in conjugate representations of the gauge group and with R-charges obeying $R_1+R_2 = 2$ will cancel out of the elliptic genus, reflecting the fact that with those R-charges, there can be a superpotential term pairing them up to become massive.

Putting this together, we get the elliptic genus

$$\begin{aligned}
&\frac{1}{2\pi i}\frac{2\pi\eta(q)^2}{i}\oint_{u=0} du \left(i\frac{\eta(q)}{\theta_1(q,x)}\right)^6 \left(i\frac{\theta_1(q,yx)}{\eta(q)}\right)^8 i\frac{\eta(q)}{\theta_1(q,y^{-1}x^{-2})} \left(i\frac{\eta(q)}{\theta_1(q,y^{-1}x^{-3})}\right)^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot i\frac{\theta_1(q,x^{-2})}{\eta(q)} i\frac{\theta_1(q,x^{-4})}{\eta(q)} \\
&= \; -\frac{\eta(q)}{i}\oint_{u=0} du \frac{\theta_1(q,yx)^8\theta_1(q,x^{-2})\theta_1(q,x^{-4})}{\theta_1(q,x)^6\theta_1(q,y^{-1}x^{-2})\theta_1(q,y^{-1}x^{-3})^2}.
\end{aligned}$$

Expanding this genus in a power series in q, we compute the same first few terms as in the proposed dual:

$$\begin{aligned}
&72\left(-y^{-1/2}+y^{+1/2}\right)^2\left(y^{-1/2}+y^{+1/2}\right)q^{1/6} \\
&\quad -72\left(-y^{-1/2}+y^{+1/2}\right)^2\left(y^{-1/2}+y^{+1/2}\right)^3\left(y^{-1}-1+y\right)q^{7/6} \\
&\quad +72\left(-y^{-1/2}+y^{+1/2}\right)^2\left(y^{-7/2}-y^{-3/2}+2y^{-1/2}+2y^{+1/2}-y^{+3/2}+y^{+7/2}\right)q^{13/6} \;+\; \mathcal{O}\left(q^{19/6}\right).
\end{aligned}$$

Thus, we have good evidence that the proposed (0,2) duals are, in fact, dual, consistent with the fact that weakly-coupled limits describe the same geometry and gauge bundle.

For completeness, let us also compare the leading term above to what one would expect from the general analysis of [96]. Recall equation (3.25) says the leading term in the elliptic genus on a Calabi-Yau 3-fold, for a rank 5 bundle, is given by

$$q^{(5-3)/12}y^{-5/2}(-)\tilde{\chi}(\mathcal{E})y(1+y)(1-y)^{5-3} \;=\; -\tilde{\chi}(\mathcal{E})q^{+1/6}y^{-5/2}y(1-y-y^2+y^3).$$

It is straightforward to compute in this case that $\tilde{\chi}(\mathcal{E}) = -72$, and a bit of algebra suffices to demonstrate that the leading term above matches the prediction of [96].

F.2 Fourth entry

We now compute the elliptic genus of the fourth entry in table 3.2 and compare to the elliptic genus of the proposed abelian dual.

The fourth entry describes the bundle

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(1,1)^2 \oplus \mathcal{O}(2,2)^5 \longrightarrow \mathcal{O}(3,3)^4 \longrightarrow 0$$

on the Calabi-Yau threefold $G(2,4)[4]$.

The field content and corresponding contributions to the elliptic genus are as follows:

- 4 chiral multiplets each in the fundamental of $U(2)$
$$\left(i\frac{\eta(q)}{\theta_1(q,x_1)} \right)^4 \left(i\frac{\eta(q)}{\theta_1(q,x_2)} \right)^4,$$
- 1 Fermi multiplet in the $(-4,-4)$ representation of $U(2)$, enforcing the hypersurface condition:
$$i\frac{\theta_1(q,x_1^{-4}x_2^{-4})}{\eta(q)},$$
- 2 Fermi multiplets in the $(1,1)$ representation of $U(2)$, forming part of the gauge bundle:
$$\left(i\frac{\theta_1(q,yx_1x_2)}{\eta(q)} \right)^2,$$
- 5 Fermi multiplets in the $(2,2)$ representation of $U(2)$, forming part of the gauge bundle:
$$\left(i\frac{\theta_1(q,yx_1^2x_2^2)}{\eta(q)} \right)^5,$$

- 4 chiral multiplets in the $(-3,-3)$ representation of $U(2)$, forming part of the gauge bundle:

$$\left(i\frac{\eta(q)}{\theta_1(q, y^{-1}x_1^{-3}x_2^{-3})} \right)^4,$$

- and finally the $U(2)$ gauge field contributes

$$\left(\frac{2\pi\eta(q)^2}{i} \right)^2 i\frac{\theta_1(q, x_1x_2^{-1})}{\eta(q)} i\frac{\theta_1(q, x_2x_1^{-1})}{\eta(q)} du_1 du_2.$$

(We omit spectators, as they do not contribute.)

Putting this together, we get the elliptic genus

$$\frac{1}{2}\frac{(2\pi)^2}{(2\pi i)^2}\eta(q)^6 \oint du_1 du_2 \frac{\theta_1(q, x_1^{-4}x_2^{-4})\theta_1(q, yx_1x_2)^2\theta_1(q, yx_1^2x_2^2)^5\theta_1(q, x_1x_2^{-1})\theta_1(q, x_1^{-1}x_2)}{\theta_1(q, x_1)^4\theta_1(q, x_2)^4\theta_1(q, y^{-1}x_1^{-3}x_2^{-3})^4}.$$

This has poles along the hypersurfaces $\{u_1 = 0\}$, $\{u_2 = 0\}$, $\{-z - 3u_1 - 3u_2 = 0\}$, which have projective intersections. Proceeding as before, we compute the residue at $u_1 = u_2 = 0$. Expanding in a power series in q, the first few terms are

$$\begin{aligned} &88y^{-1/2}(1+y) \;-\; 88y^{-5/2}\left(1 - y^2 - y^3 + y^5\right) q \\ &\quad - 88y^{-7/2}\left(1+y\right)\left(-1+y^3\right)^2 q^2 \;-\; 88y^{-7/2}\left(-1+y\right)^2\left(1+y\right)^3\left(1+y+y^2\right) \;+\; \mathcal{O}\left(q^4\right). \end{aligned}$$

The abelian dual to this GLSM describes the bundle

$$0 \longrightarrow \mathcal{O}(1)^2 \oplus \mathcal{O}(2)^5 \longrightarrow \mathcal{O}(3)^4 \longrightarrow 0$$

on $\mathbb{P}^5[2,4]$.

The field content and corresponding contributions to the elliptic genus are as follows:

- 6 chiral multiplets each of charge $+1$:

$$\left(i\frac{\eta(q)}{\theta_1(q, x)} \right)^6,$$

- 1 Fermi multiplet of charge -2, enforcing a hypersurface condition:

$$i\frac{\theta_1(q,x^{-2})}{\eta(q)},$$

- 1 Fermi multiplet of charge -4, enforcing a hypersurface condition:

$$i\frac{\theta_1(q,x^{-4})}{\eta(q)},$$

- 2 Fermi multiplets of charge $+1$, forming part of the gauge bundle:

$$\left(i\frac{\theta_1(q,yx)}{\eta(q)}\right)^2,$$

- 5 Fermi multiplets of charge $+2$, forming part of the gauge bundle:

$$\left(i\frac{\theta_1(q,yx^2)}{\eta(q)}\right)^5,$$

- 4 chiral multiplets of charge -3, forming part of the gauge bundle:

$$\left(i\frac{\eta(q)}{\theta_1(q,y^{-1}x^{-3})}\right)^4,$$

- and finally the $U(1)$ gauge field contributes

$$\frac{2\pi\eta(q)^2}{i}du.$$

Putting this together, we get the elliptic genus

$$-\frac{2\pi}{2\pi i}\eta(q)^3\oint du\frac{\theta_1(q,x^{-2})\theta_1(q,x^{-4})\theta_1(q,yx)^2\theta_1(q,yx^2)^5}{\theta_1(q,x)^6\theta_1(q,y^{-1}x^{-3})^4}.$$

We compute the residue at $u=0$ and, expanding in a power series in q, get the same result as for the dual:

$$\begin{aligned}&88y^{-1/2}(1+y)\ -\ 88y^{-5/2}\left(1-y^2-y^3+y^5\right)q\\&\quad -\ 88y^{-7/2}\left(1+y\right)\left(-1+y^3\right)^2q^2\ -\ 88y^{-7/2}\left(-1+y\right)^2\left(1+y\right)^3\left(1+y+y^2\right)\ +\ \mathcal{O}\left(q^4\right).\end{aligned}$$

This is a good check that the proposed (0,2) duals are, in fact, dual, consistent with the fact that weakly-coupled limits describe the same geometry and gauge bundle.

For completeness, let us also compare the leading term above to what one would expect from the general analysis of [96]. Recall equation (3.25) says that the leading term in the elliptic genus on a Calabi-Yau 3-fold, for a rank 3 bundle, is given by

$$q^{(3-3)/12} y^{-3/2} (-) \tilde{\chi}(\mathcal{E}) y (1+y) \;=\; -\tilde{\chi}(\mathcal{E}) y^{-1/2} (1+y).$$

It is straightforward to compute in this case that $\tilde{\chi}(\mathcal{E}) = -88$, and so the leading term computed above is consistent with the predictions of [96].

F.3 Fifth entry

Next, we shall compare elliptic genera for the example given in the fifth entry in table 3.2), and that of its abelian dual.

The fifth entry is the GLSM for the bundle

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(1,1)^5 \oplus \mathcal{O}(2,2)^2 \longrightarrow \mathcal{O}(3,3)^3 \longrightarrow 0$$

on the Calabi-Yau $G(2,4)[4]$.

The field content and corresponding contributions to the elliptic genus are as follows:

- 4 chiral multiplets each in the fundamental of $U(2)$
$$\left(i \frac{\eta(q)}{\theta_1(q,x_1)} \right)^4 \left(i \frac{\eta(q)}{\theta_1(q,x_2)} \right)^4,$$
- 1 Fermi multiplet in the $(-4,-4)$ representation of $U(2)$, enforcing the hypersurface condition:
$$i \frac{\theta_1(q, x_1^{-4} x_2^{-4})}{\eta(q)},$$

- 5 Fermi multiplets in the $(1,1)$ representation of $U(2)$, partially defining the gauge bundle:

$$\left(i\frac{\theta_1(q, yx_1x_2)}{\eta(q)} \right)^5,$$

- 2 Fermi multiplets in the $(2,2)$ representation of $U(2)$, partially defining the gauge bundle:

$$\left(i\frac{\theta_1(q, yx_1^2x_2^2)}{\eta(q)} \right)^2,$$

- 3 chiral multiplets in the $(-3,-3)$ representation of $U(2)$, partially defining the gauge bundle:

$$\left(i\frac{\eta(q)}{\theta_1(q, y^{-1}x_1^{-3}x_2^{-3})} \right)^3,$$

- and finally the $U(2)$ gauge field contributes

$$\left(\frac{2\pi\eta(q)^2}{i} \right)^2 i\frac{\theta_1(q, x_1x_2^{-1})}{\eta(q)} i\frac{\theta_1(q, x_2x_1^{-1})}{\eta(q)} du_1 du_2.$$

Putting this together, we get the elliptic genus

$$-\frac{i}{2}\frac{(2\pi)^2}{(2\pi i)^2}\eta(q)^5 \oint du_1 du_2 \frac{\theta_1(q, x_1^{-4}x_2^{-4})\theta_1(q, yx_1x_2)^5\theta_1(q, yx_1^2x_2^2)^2\theta_1(q, x_1x_2^{-1})\theta_1(q, x_1^{-1}x_2)}{\theta_1(q, x_1)^4\theta_1(q, x_2)^4\theta_1(q, y^{-1}x_1^{-3}x_2^{-3})^3}.$$

Expanding as before in a series in q, the first few terms of the elliptic genus above are given by

$$\begin{aligned} &80\left(y - y^{-1}\right) q^{1/12} \;-\; 80\left(-y^{-3} + y^{-1} - y + y^3\right) q^{13/12} \\ &\quad - 80\left(-y^{-3} + 2y^{-1} - 2y + y^3\right) q^{25/12} \;+\; \mathcal{O}\left(q^{37/12}\right). \end{aligned}$$

The proposed abelian dual describes the bundle

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(1)^5 \oplus \mathcal{O}(2)^2 \longrightarrow \mathcal{O}(3)^3 \longrightarrow 0$$

on the Calabi-Yau $\mathbb{P}^5[2,4]$.

The field content and corresponding contributions to the elliptic genus are as follows:

- 6 chiral multiplets each of charge +1:

$$\left(i \frac{\eta(q)}{\theta_1(q,x)} \right)^6,$$

- one Fermi multiplet Γ_1 of charge -2, enforcing a hypersurface condition:

$$i \frac{\theta_1(q, x^{-2})}{\eta(q)},$$

- one Fermi multiplet Γ_2 of charge -4, enforcing a hypersurface condition:

$$i \frac{\theta_1(q, x^{-4})}{\eta(q)},$$

- 5 Fermi multiplets of charge +1, partially defining the gauge bundle:

$$\left(i \frac{\theta_1(q, yx)}{\eta(q)} \right)^5,$$

- 2 Fermi multiplets of charge +2, partially defining the gauge bundle:

$$\left(i \frac{\theta_1(q, yx^2)}{\eta(q)} \right)^2,$$

- 3 chiral multiplets of charge -3, partially defining the gauge bundle:

$$\left(i \frac{\eta(q)}{\theta_1(q, y^{-1}x^{-3})} \right)^3,$$

- and finally the $U(1)$ gauge field contributes

$$\frac{2\pi \eta(q)^2}{i} du.$$

Assembling these pieces, we find that the elliptic genus is given by

$$\eta(q)^2 \oint du \frac{\theta_1(q, x^{-2}) \theta_1(q, x^{-4}) \theta_1(q, yx)^5 \theta_1(q, yx^2)^2}{\theta_1(q,x)^6 \theta_1(q, y^{-1}x^{-3})^3},$$

and expanding in a power series in q, we find the same expression as in the dual theory:

$$\begin{aligned} &80 \left(y - y^{-1} \right) q^{1/12} \; - \; 80 \left(-y^{-3} + y^{-1} - y + y^3 \right) q^{13/12} \\ &\quad - 80 \left(-y^{-3} + 2y^{-1} - 2y + y^3 \right) q^{25/12} \; + \; \mathcal{O}\left(q^{37/12} \right). \end{aligned}$$

For completeness, let us also compare the leading term above to what one would expect from the general analysis of [96]. Recall equation (3.25) says that the leading term in the elliptic genus on a Calabi-Yau 3-fold, for a rank 4 bundle, is given by

$$q^{(4-3)/12} y^{-4/2} (-) \tilde{\chi}(\mathcal{E}) y (1+y)(1-y) \; = \; -\tilde{\chi}(\mathcal{E}) q^{+1/12} y^{-1} \left(1 - y^2 \right) .$$

It is straightforward to compute in this case that $\tilde{\chi}(\mathcal{E}) = -80$, and so the leading term computed above is consistent with the predictions of [96].

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