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An analysis is presented of third-order resonant interactions in capillary-gravity waves. When dissipation is accounted for, the theory predicts the amplitude of the third harmonic near the resonant frequency and at the early stage of the interaction. A more detailed account of the viscous effects is needed when these effects dominate the interaction.

I. INTRODUCTION

The rippling observed on small progressing gravity waves has been investigated since the beginning of the century. Harrison[1] and Wilton[2] found nonlinear resonances to occur at a denumerable set of critical wavenumbers $k=(\rho g/nT)^{1/2}$, where g is the gravity acceleration, ρ and T are the density and surface tension of the liquid, and n is any integer greater than unity. The first two correspond to wavelengths of 2.44 and 2.99 cm in deep water. These resonances occur whenever waves having different wavenumbers travel with approximately the same phase speed.

McGoldrick[3] experimentally examined third- and higher-order resonances and observed that peak interactions occur slightly off the exact resonant frequency. Admitting that the algebra required to analyze these waves is too cumbersome, he used a model equation (Bretherton's equation) to qualitatively explain the results of his experiments. Nayfeh[4] used the method of multiple scales[5] to analyze the third-harmonic resonant case when the frequencies are exactly in the ratio of three to one. Kaup[6] obtained a solution for a three-wave problem.

In this investigation, we extend the results of Nayfeh[4] to examine the third-harmonic resonant case when the frequencies are not exactly commensurable and to give a quantitative comparison of the analytical results with the experimental results of McGoldrick.[3]

II. PROBLEM FORMULATION

We introduce a Cartesian coordinate system whose x axis lies in the undisturbed free surface and whose y axis is directed away from the liquid. The governing equation for the velocity potential ϕ is

$$\nabla^2\phi=0 \tag{1}$$

in the region

$$-\infty<x<\infty,\quad -\infty<y\le\eta(x,t)\ .$$

The surface elevation is given by $\eta(x,t)$. Here, lengths and time are made dimensionless by using $k_c=(\rho g/T)^{1/2}$ and $(gk_c)^{-1/2}$, as reference quantities, where ρ is the liquid density, T is the surface tension, and g is the body acceleration directed toward the liquid. The boundary conditions at the interface are

$$\eta-\phi_t=\eta_{xx}(1+\eta_x^2)^{-3/2}-\tfrac{1}{2}(\phi_x^2+\phi_y^2)\ \text{at}\ y=\eta\ , \tag{2}$$

$$\eta_t+\phi_y=\eta_x\phi_x\ \text{at}\ y=\eta\ , \tag{3}$$

for deep water

$$\nabla\phi\to0\ \text{as}\ y\to-\infty\ . \tag{4}$$

Equations (1)–(4) are solved by using the method of multiple scales.[5] We introduce the "slow scale" $T_2=\epsilon^2t$ and the "long scale" $X_2=\epsilon^2x$, where ϵ is proportional to the steepness ratio of the waves. Moreover, we seek a uniform third-order expansion in the form

$$\eta(x,t)=\sum_{n=1}^{3}\epsilon^n\eta_n(X_0,X_2,T_0,T_2)+O(\epsilon^4)\ , \tag{5}$$

$$\phi(x,y,t)=\sum_{n=1}^{3}\epsilon^n\phi_n(y,X_0,X_2,T_0,T_2)+O(\epsilon^4)\ . \tag{6}$$

Here, T_0 and X_0 are the usual time and length scales, t and x.

Substituting Eqs. (5) and (6) into Eqs. (1)–(4) and equating coefficients of like power of ϵ, we obtain Eqs. (2.9)–(2.17) of Nayfeh,[4] the only difference being the presence of k^2 in his equations, due to the different nondimensionalization.

The solution of the $O(\epsilon)$ problem gives the dispersion relation

$$\omega^2=k^3+k\ . \tag{7}$$

Since we are interested in the third-harmonic resonant case (i.e., $k^2\approx1/3$), we let the first-order solution contain both the fundamental and its resonating third harmonic; that is,

$$n_1=A_1(X_2,T_2)\exp(i\theta)+A_3(X_2,T_2)\exp(3i\theta+i\Gamma)+\text{c.c.}\ , \tag{8}$$

$$\phi_1=i(\omega_1/k_1)A_1(X_2,T_2)\exp(i\theta+k_1y)+i(\omega_3/k_3)A_3(X_2,T_2)\exp(3i\theta+k_3y+i\Gamma)+\text{c.c.}, \tag{9}$$

where $\theta=k_1X_0-\omega_1T_0$ is a fast varying phase, the A_n are the complex amplitudes, c.c. stands for the complex conjugate of the preceding terms, and

$$\Gamma=(k_3-3k_1)X_0-(\omega_3-3\omega_1)T_0=\sigma_kX_2-\sigma_\omega T_2\ . \tag{10}$$

Substituting this solution into the $O(\epsilon^2)$ problem, we obtain

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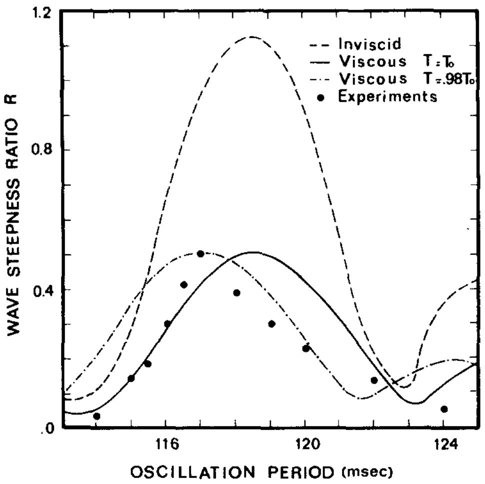

FIG. 1. The ratio, R, of the steepness of the third harmonic 40 cm from the wavemaker to the steepness of the fundamental near the wavemaker.

$$\eta_2 = k_1[4(A_1^2 + 4\bar{A}_1 A_3 \exp i\Gamma)\exp(2i\theta) - 8A_1A_3\exp(i\Gamma + 4i\theta) - \tfrac{12}{5}A_3^2\exp(2i\Gamma + 6i\theta)] + \text{c.c.} \quad (11)$$

$$\phi_2 = i\omega\{3(A_1^2 + 4\bar{A}_1 A_3\exp i\Gamma)\exp[2(i\theta + k_1y)] - 12A_1A_3 \times \exp[i\Gamma + 4(i\theta + k_1y) - \tfrac{27}{5}A_3^2\exp[2i\Gamma + 6(i\theta + k_1y)]\} + \text{c.c.} \quad (12)$$

We note that for deep water and $k^2 \approx \frac{1}{3}$, the second harmonic is $O(\epsilon^2)$ only, while for shallow water, it is $O(\epsilon)$.

Substituting the solutions of the $O(\epsilon)$ and $O(\epsilon^2)$ problems into the $O(\epsilon^3)$ problem, we obtain inhomogeneous equations whose particular solutions contain secular terms. Elimination of the secular terms leads to the following solvability conditions:

$$\frac{\partial A_1}{\partial T_2} + \omega_1' \frac{\partial A_1}{\partial X_2} = i\frac{3^{1/2}}{108\omega_1} \times[-77\bar{A}_1A_1^2 - 369\bar{A}_1^2A_3\exp(i\Gamma) - 138A_1A_3\bar{A}_3], \quad (13)$$

$$\frac{\partial A_3}{\partial T_2} + \omega_3' \frac{\partial A_3}{\partial X_2} = i\frac{3^{1/2}}{108\omega_1} \times[-205A_1^3\exp(-i\Gamma) - 230A_1\bar{A}_1A_3 + 261A_3^2\bar{A}_3], \quad (14)$$

where the ω_i' are the group velocities. These two hyperbolic first-order equations govern the modulation of the amplitudes and the phases of the interacting harmonics. In the case of either space or time modulating waves, Eqs. (13) and (14) become ordinary differential equations whose exact solution can be expressed in quadrature.[4]

III. VISCOUS EFFECTS

Before attempting any comparison with the experiments, we shall incorporate the effects of viscosity which, for relatively short waves, play a dominant role after some distance of propagation. If the surface is clean and the wave-Reynolds number $R = \omega/\nu k^2$ is large, the influence of viscous dissipation is to attenuate infinitesimal waves exponentially with distance from the generating source, the modulous decay (logarithmic decrement) being given by $\Delta = 2\nu k^2/\omega'$ per unit distance. McGoldrick[7] reports an order of magnitude variation in Δ, depending on the cleanliness of the liquid surface. Taking these factors into account, letting $A_n = \frac{1}{2}a_n\exp(i\beta_n)$, and separating real and imaginary parts in Eqs. (13) and (14), we obtain

$$\frac{da_1}{dX_2} = \frac{41(3)^{1/2}}{48\omega_1\omega_1'}a_1^2a_3\sin\gamma - \Delta_1a_1, \quad (15)$$

$$\frac{da_3}{dX_2} = \frac{41(3)^{1/2}}{144\omega_1\omega_3'}a_1^3\sin\gamma - \Delta_3a_3, \quad (16)$$

$$\frac{d\beta_1}{dX_2} = \frac{(3)^{1/2}}{432\omega_1\omega_1'}(77a_1^2 + 369a_1a_3\cos\gamma + 138a_3^2), \quad (17)$$

$$\frac{d\beta_2}{dX_2} = \frac{(3)^{1/2}}{720\omega_1\omega_3'}\left(205\frac{a_1^3}{a_3}\cos\gamma + 230a_1^2 - 261a_3^2\right), \quad (18)$$

where

$$\gamma = \beta_3 - 3\beta_1 + \Gamma. \quad (19)$$

Analytical solutions of the system of equations (15)–(19) are not available, in general. However, they can easily be integrated numerically using any of the standard forward integration methods. We used a Runge–Kutta routine with a variable step and an error control provided by IBM's scientific subroutine package.

IV. RESULTS AND DISCUSSION

Caution has been exercised to reproduce the initial conditions of McGoldrick.[3] He varied the frequency of the wavemaker and plotted the response as a function of the period; σ_k is related to the period by

$$\sigma_k = 3\Delta\omega\left(\frac{1}{\omega_3'} - \frac{1}{\omega_1'}\right), \quad (20)$$

where $\Delta\omega$ is the difference between the operating frequency and resonant frequency. Figure 1 shows a comparison between the analytical results and the experimental results of McGoldrick.[3] By neglecting viscous dissipation one overpredicts the maximum amplitude of the third harmonic by about 100%. Including the viscous effects leads to a good prediction of the maximum amplitude but not of the period at which it occurs. Figure 1 shows that maximum interaction occurs at a period slightly less than the exact-resonant value, which is 119.5 msec.

These calculations were based on the static value for surface tension. However, it is well known that the "dynamic" surface tension is less than the "static" one. Furthermore, McGoldrick[7] reported large differences in the measured surface tension and viscosity of water from their nominal values due to the dirtiness of the water surface. Our numerical results show that the period corresponding to the maximum interaction is very sensitive to the value of the surface tension. In fact, Fig. 1 shows that basing the analytical results on a value of the surface tension that is about 2% less than the static value leads to an agreement with the experimentally observed period for maximum interaction.

It should be noted that although the analytical model

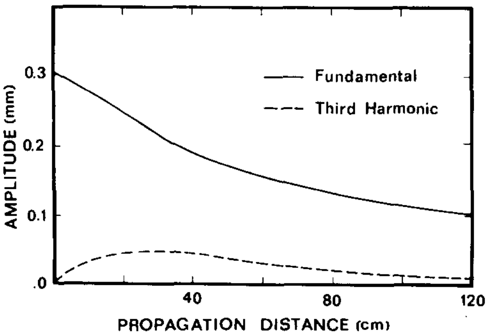

FIG. 2. The amplitude of the fundamental and its third harmonic, versus the distance of propagation at period 119.25 msec.

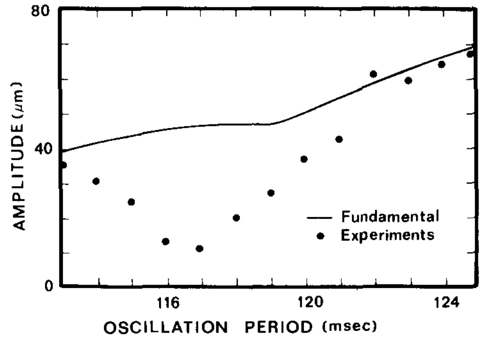

FIG. 3. The amplitude of the fundamental as a function of wavemaker oscillation period at 120 cm from the wavemaker.

can be made to predict the maximum amplitude and the period of interaction, by choosing a value for the surface tension, it cannot predict the amplitude at periods below the one corresponding to the maximum amplitude.

Figure 2 shows the variation of the amplitudes of the two harmonics with distance. Unfortunately, McGoldrick did not repeat his measurements at different axial stations. Figure 2 shows the initial growth of the third harmonic and the subsequent decay of both harmonics as the viscous effects dominate the nonlinear interaction. These results are in qualitative agreement with McGoldrick's earlier experiments.[7] Measurements at different axial stations seem to be very important because the amplitude of the third harmonic changes considerably over relatively short distances.

Figure 3 shows the variation in the amplitude of the fundamental with period, 120 cm away from the wavemaker. To compare the analytical and experimental results we need to estimate the value of Δ, because the nominal values overpredict the amplitude at exact resonance by a factor of about 3. In Fig. 3, the values of Δ were chosen to reproduce the experimental results away from resonance. Figure 3 shows that the analysis overpredicts the amplitude of the fundamental in the region near resonance by a factor of about 2, which implies that the assumption that the viscosity acts independently from the nonlinear interactions is not valid when its effects are dominant.

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